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GENERALIZED HYERS-ULAM STABILITY OF DERIVATIONS IN PROPER LIE CQ^*-ALGEBRAS

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Abstract. In this paper, we obtain the general solution and the generalized Hyers-Ulam stability for the following functional equation

$$f\Big(\frac{\sum_{i=1}^{m} x_i}{m}\Big) + \sum_{\substack{i=1\\ i\neq j}}^{m} f\Big(\frac{x_j - x_i}{m}\Big) = f(x_j).$$

This is applied to investigate derivations and their stability in proper Lie CQ^*-algebras.

1. Introduction and preliminaries

Ulam [42] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these problems was the following question concerning the stability of homomorphisms.

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Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies the inequality

$$d(f(x * y), f(x) \diamond f(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $T : G_1 \to G_2$ with

$$d(f(x), T(x)) < \epsilon$$

for all $x \in G_1$?

If the answer is affirmative, we say that the equation of homomorphism $T(xy) = T(x)T(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

Hyers [18] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \leq \epsilon.$$

Hyers' theorem was generalized by Aoki [3] for additive mappings and independently by Th.M. Rassias [36] for linear mappings by considering an *unbounded Cauchy difference*. In 1994, a generalization of Th.M. Rassias' theorem was obtained by Găvruta [15]. J.M. Rassias [31]-[34] generalized Hyers result. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [11]-[13], [20], [24]-[28],[30], [37]-[39]). We also refer the readers to the books [1], [10], [19], [21] and [37].

We recall some basic facts concerning quasi $*$-algebras.

Definition 1.1. Let A be a linear space and A_0 be a $*$-algebra contained in A as a subspace. We say that A is a *quasi $*$-algebra* over A_0 if

(*i*) the right and left multiplications of an element of A and an element of A_0 are always defined and linear;

(*ii*) $x_1(x_2a) = (x_1x_2)a$, $(ax_1)x_2 = a(x_1x_2)$ and $x_1(ax_2) = (x_1a)x_2$ for all $x_1, x_2 \in A_0$ and all $a \in A$;

(*iii*) an involution $*$, which extends the involution of A_0, is defined in A with the property $(ab)^* = b^*a^*$, whenever the multiplication is defined.

Quasi $*$-algebras [22, 23] arise in natural way as completions of locally convex $*$-algebras whose multiplication is not jointly continuous; in this case one has to deal with topological quasi $*$-algebras.

A quasi $*$-algebra (A, A_0) is called *topological* if a locally convex topology τ on A is given such that:

(*i*) the involution $a \mapsto a^*$ is continuous for each $a \in A$,

(*ii*) the mappings $a \mapsto ab$ and $a \mapsto ba$ are continuous for each $a \in A$ and $b \in A_0$,

(*iii*) A_0 is dense in $A[\tau]$.

Throughout this paper, we suppose that a locally convex quasi $*$-algebra (A, A_0) is complete. For an overview on partial $*$-algebra and related topics we refer to [2].

In a series of papers [4], [5], [6], [7] many authors have considered a special class of quasi $*$-algebras, called proper CQ^*-algebras, which arise as completions of C^*-algebras. They can be introduced in the following way:

Definition 1.2. Let A be a Banach module over the C^*-algebra A_0 with involution $*$ and C^*-norm $\|.\|_0$ such that $A_0 \subset A$. We say that (A, A_0) is a *proper CQ^*-algebra* if

(*i*) A_0 is dense in A with respect to its norm $\|.\|$;

(*ii*) $(ab)^* = b^*a^*$ whenever the multiplication is defined;

(*iii*) $\|y\|_0 = \max\{ \sup_{a \in A, \|a\| \leq 1} \|ay\|, \sup_{a \in A, \|a\| \leq 1} \|ya\| \}$ for all $y \in A_0$.

A proper CQ^*-algebra (A, A_0) is said to have a unit e if there exists an element $e \in A_0$ such that $ae = ea = a$ for all $a \in A$. In this paper we will always assume that the proper CQ^*-algebra under consideration have an identity.

Definition 1.3. A proper CQ^*-algebra (A, A_0), endowed with a bilinear multiplication $[,] : (A \times A_0) \cup (A_0 \times A) \to A$, called the bracket, which satisfies two simple properties:

(*i*) $[x_1, x_2] = -[x_2, x_1]$ for all $(x_1, x_2) \in (A \times A_0) \cup (A_0 \times A)$;

(*ii*) $[x_1, [x_2, x_3]] = [[x_1, x_2], x_3] + [x_1, [x_2, x_3]]$ for all $x_1, x_2, x_3 \in A_0$

is called a *proper Lie CQ^*-algebra.*

Definition 1.4. Let (A, A_0) be a proper Lie CQ^*-algebras. A $\mathbb{C}$-linear mapping $\delta : A_0 \to A$ is called a *Lie derivation* if

$$\delta([z, x]) = [\delta(z), x] + [z, \delta(x)]$$

for all $x, z \in A_0$ (see [28]).

Throughout this paper, we assume that m and j are fixed positive integers with $m \geq 2$.

In this paper, we obtain the general solution and the generalized Ulam-Hyers stability for the following functional equation

$$f\big(\frac{\sum_{i=1}^{m} x_i}{m}\big) + \sum_{\substack{i=1 \\ i \neq j}}^{m} f\big(\frac{x_j - x_i}{m}\big) = f(x_j) \tag{1.1}$$

where m is a fixed positive integer with $m \geq 2$. This is applied to investigate derivations and their stability on proper Lie CQ^*-algebras.

2. Solution of functional equation (1.1)

Throughout this section, let both X and Y be real vector spaces. We here present the general solution of (1.1).

Theorem 2.1. *A mapping $f : X \to Y$ satisfies (1.1) if and only if the mapping $f : X \to Y$ is additive.*

We first assume that the mapping $f : X \to Y$ satisfies (1.1). Setting $x_j = x$ and $x_i = 0$ for all $1 \leq i \leq m$ and $i \neq j$ in (1.1), we get

$$f\left(\frac{x}{m}\right) = \frac{1}{m} f(x) \tag{2.1}$$

for all $x \in X$. Setting $x_j = x$, $x_{j+1} = y$ and $x_i = 0$ for $i \neq j, j+1$ in (1.1) and using (2.1), we get

$$f\left(\frac{x+y}{m}\right) + f\left(\frac{x-y}{m}\right) = \frac{2}{m} f(x) \tag{2.2}$$

for all $x, y \in X$. Replacing x and y by mx and my in (2.2), we get

$$f(x+y) + f(x-y) = 2f(x) \tag{2.3}$$

for all $x, y \in X$. Setting $y = x$ in (2.3), we get

$$f(2x) = 2f(x) \tag{2.4}$$

for all $x \in X$. Replacing x by $\frac{x+y}{2}$ and y by $\frac{x-y}{2}$ in (2.3), and using (2.4) we get

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$. So the mapping $f : X \to Y$ is additive.

Conversely, let the mapping $f : X \to Y$ be additive. By a simple computation, one can show that the mapping f satisfies the functional equation (1.1).

3. Stability of derivation on proper Lie CQ^*-algebras

Throughout this section, assume that (A, A_0) is a proper Lie CQ^*-algebra with C^*-norm $\|.\|_{A_0}$ and norm $\|.\|_A$. For convenience, we use the following abbreviation for a given mapping $f : \underbrace{A_0 \times A_0 \times ... \times A_0}_{m-\text{times}} \to A$

$$D_\mu f(x_1, ..., x_m) := f\left(\frac{\sum_{i=1}^m \mu x_i}{m}\right) + \sum_{\substack{i=1\\ i\neq j}}^m f\left(\frac{\mu x_j - \mu x_i}{m}\right) - \mu f(x_j)$$

for all $x_1, \cdots, x_m \in A_0$, where $\mu \in \mathbb{T}^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$.

We will use the following lemma:

Lemma 3.1. [29] *Let $f : A_0 \to A$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in A_0$ and all $\mu \in \mathbb{T}^1$. Then the mapping f is $\mathbb{C}$-linear.*

Theorem 3.2. *Let $\varphi : \underbrace{A_0 \times A_0 \times ... \times A_0}_{m-times} \to [0, \infty)$ and $\psi : A_0 \times A_0 \to [0, \infty)$ be mappings such that*

$$\lim_{n\to\infty} \frac{1}{m^n}\varphi(m^n x_1, ..., m^n x_m) = 0, \tag{3.1}$$

$$\lim_{n\to\infty} \frac{1}{m^{2n}}\psi(m^n x_1, m^n x_2) = 0, \tag{3.2}$$

$$\widetilde{\varphi_j}(x) := \sum_{i=1}^{\infty} \frac{1}{m^i}\varphi(0, ..., \underbrace{m^i x}_{j\,th}, ...0) < \infty \tag{3.3}$$

for all $x, x_1, \cdots, x_m \in A_0$. Suppose that $f : A_0 \to A$ is a mapping such that

$$\|D_\mu f(x_1, ..., x_m)\|_A \le \varphi(x_1, ..., x_m), \tag{3.4}$$

$$\big\|f([x_1, x_2]) - [f(x_1), x_2] - [x_1, f(x_2)]\big\|_A \le \psi(x_1, x_2) \tag{3.5}$$

for all $x_1, \cdots, x_m \in A_0$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique Lie derivation $\delta : A_0 \to A$ such that

$$\|f(x) - \delta(x)\|_A \le \widetilde{\varphi_j}(x) \tag{3.6}$$

for all $x \in A_0$.

Letting $\mu = 1$, $x_j = mx$ and $x_i = 0$ for all $1 \le i \le m$ with $i \neq j$ in (3.4) , we get

$$\|f(mx) - mf(x)\|_A \le \varphi(0, ..., \underbrace{mx}_{j\,th}, ...0) \tag{3.7}$$

for all $x \in A_0$. Replacing x by $m^n x$ in (3.7) and dividing both sides of (3.7) by m^{n+1}, we get

$$\Big\|\frac{1}{m^{n+1}}f(m^{n+1}x) - \frac{1}{m^n}f(m^n x)\Big\|_A \le \frac{1}{m^{n+1}}\varphi(0, ..., \underbrace{m^{n+1}x}_{j\,th}, ..., 0) \tag{3.8}$$

for all $x \in A_0$ and all non-negative integers n. Hence

$$\begin{aligned} \Big\|\frac{1}{m^{n+1}}f(m^{n+1}x) - \frac{1}{m^k}f(m^kx)\Big\|_A &\le \sum_{i=k}^{n}\Big\|\frac{1}{m^{i+1}}f(m^{i+1}x) - \frac{1}{m^i}f(m^ix)\Big\|_A \\ &\le \sum_{i=k+1}^{n+1}\frac{1}{m^i}\varphi(0,...,\underbrace{m^ix}_{j\,th},...,0) \end{aligned} \tag{3.9}$$

for all $x \in A_0$ and all non-negative integers n and k with $n \ge k$. Therefore, we conclude from (3.3) and (3.9) that the sequence $\{\frac{1}{m^n}f(m^nx)\}_n$ is a Cauchy sequence in A for all $x \in A_0$. Since A is complete, the sequence $\{\frac{1}{m^n}f(m^nx)\}_n$ converges in A for all $x \in A_0$. So one can define the mapping $\delta : A_0 \to A$ by

$$\delta(x) := \lim_{n\to\infty}\frac{1}{m^n}f(m^nx) \tag{3.10}$$

for all $x \in A_0$. Letting $k = 0$ and passing the limit $n \to \infty$ in (3.9), we get (3.6). Now, we show that δ is a $\mathbb{C}$-linear mapping. It follows from (3.1), (3.4) and (3.10) that

$$\begin{aligned} \|D_1\delta(x_1,...,x_m)\|_A &= \lim_{n\to\infty}\frac{1}{m^n}\big\|D_1f(m^nx_1,...,m^nx_m)\big\|_A \\ &\le \lim_{n\to\infty}\frac{1}{m^n}\varphi(m^nx_1,...,m^nx_m) = 0 \end{aligned}$$

for all $x_1,\cdots,x_m \in A_0$. So the mapping δ satisfies (1.1). By Theorem 2.1, the mapping δ is additive.

Letting $x_j = mx$ and $x_i = 0$ for all $1 \le i \le m$ with $i \ne j$ in (3.4), we get

$$\|mf(\mu x) - \mu f(mx)\|_A \le \varphi(0,...,\underbrace{mx}_{j\,th},...,0) \tag{3.11}$$

for all $x \in A_0$. Replacing x by m^nx in (3.11) and dividing both sides of (3.11) by m^{n+1}, we get

$$\begin{aligned} &\Big\|\frac{1}{m^n}f(\mu m^nx) - \frac{\mu}{m^{n+1}}f(m^{n+1}x)\Big\|_A \\ &\le \frac{1}{m^{n+1}}\varphi(0,...,\underbrace{m^{n+1}x}_{j\,th},...,0) \end{aligned} \tag{3.12}$$

for all $x \in A_0$ and all non-negative integers n. Passing the limit $n \to \infty$ in (3.12) and using (3.1) and (3.10), we get

$$\delta(\mu x) = \mu\delta(x)$$

for all $\mu \in \mathbb{T}^1$ and for all $x \in A_0$. So by Lemma 3.1, we infer that the mapping $\delta : A_0 \to A$ is $\mathbb{C}$-linear. To prove the uniqueness of δ, let $\delta' : A_0 \to A$ be another additive mapping satisfying (3.6). It follows from (3.6) and (3.10) that

$$\begin{aligned} \|\delta(x) - \delta'(x)\|_A &= \lim_{n\to\infty}\frac{1}{m^n}\big\|f(m^nx) - \delta'(m^nx)\big\|_A \\ &\le \lim_{n\to\infty}\frac{1}{m^n}\widetilde{\varphi_j}(m^nx) = 0 \end{aligned}$$

for all $x \in A_0$. So $\delta = \delta'$.

It follows from (3.2), (3.5) and (3.10) that

$$\begin{aligned}&\left\|\delta([x_1, x_2]) - [\delta(x_1), x_2] - [x_1, \delta(x_2)]\right\|_A \\ &= \lim_{n\to\infty} \frac{1}{m^{2n}} \left\|f(m^{2n}[x_1, x_2]) - [f(m^n x_1), m^n x_2] - [m^n x_1, f(m^n x_2)]\right\|_A \\ &\le \lim_{n\to\infty} \frac{1}{m^{2n}} \psi(m^n x_1, m^n x_2) = 0\end{aligned}$$

for all $x_1, x_2 \in A_0$. So

$$\delta([x_1, x_2]) = [\delta(x_1), x_2] + [x_1, \delta(x_2)]$$

for all $x_1, x_2 \in A_0$. Hence the mapping $\delta : A_0 \to A$ is a unique Lie derivation satisfying (3.6).

Corollary 3.3. *Let $\delta, \alpha_1, \alpha_2, s_1, s_2, \{\theta_i\}_{i=1}^m$ and $\{r_i\}_{i=1}^m$ be non-negative real numbers such that $0 < s_1, s_2 < 2$, and $0 < r_i < 1$ for all $1 \le i \le m$. Suppose that $f : A_0 \to A$ is a mapping such that*

$$\|D_\mu f(x_1, ..., x_m)\|_A \le \delta + \sum_{i=1}^m \theta_i \|x_i\|_{A_0}^{r_i},$$

$$\|f([x_1, x_2]) - [f(x_1), x_2] - [x_1, f(x_2)]\|_A \le \delta + \alpha_1 \|x_1\|_{A_0}^{s_1} + \alpha_2 \|x_2\|_{A_0}^{s_2},$$

for all $x_1, \cdots, x_m \in A_0$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique Lie derivation $\delta : A_0 \to A$ such that

$$\|f(x) - \delta(x)\|_A \le \frac{\delta}{m-1} + \gamma(x)$$

for all $x \in A_0$, where

$$\gamma(x) := \min_{1 \le i \le m} \Big\{ \frac{\theta_i m^{r_i}}{m - m^{r_i}} \|x\|_{A_0}^{r_i} \Big\}.$$

Corollary 3.4. *Let $\delta, \alpha_1, \alpha_2, \alpha_3, s_1, s_2$ and $\{r_i\}_{i=1}^m$ be non-negative real numbers such that $s_1 + s_2 < 2$ and $0 < \sum_{i=1}^m r_i < 1$ for all $1 \le i \le m$. Suppose that $f : A_0 \to A$ is a mapping such that*

$$\|D_\mu f(x_1, ..., x_m)\|_A \le \delta + \sum_{i=1}^m \|x_i\|_{A_0}^{r_i} + \prod_{i=1}^m \|x_i\|_{A_0}^{r_i},$$

$$\|f([x_1, x_2]) - [f(x_1), x_2] - [x_1, f(x_2)]\|_A \le \delta + \alpha_1 \|x_1\|_{A_0}^{s_1} + \alpha_2 \|x_2\|_{A_0}^{s_2} + \alpha_3 \|x_1\|_{A_0}^{s_1} \|x_2\|_{A_0}^{s_2},$$

for all $x_1, \cdots, x_m \in A_0$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique Lie derivation $\delta : A_0 \to A$ such that

$$\|f(x) - \delta(x)\|_A \le \frac{\delta}{m-1} + \tau(x)$$

for all $x \in A_0$, *where*

$$\tau(x) := \min_{1 \le i \le m} \{ \frac{m^{r_i}}{m - m^{r_i}} \|x\|_{A_0}^{r_i} \}.$$

Note that the mixed "product-sum" function was introduced by J. M. Rassias in 2008-09 ([8, 9, 16, 17, 40, 41]).

Theorem 3.5. *Let* $\Phi : \underbrace{A_0 \times A_0 \times ... \times A_0}_{m-times} \to [0, \infty)$ *and* $\Psi : A_0 \times A_0 \to [0, \infty)$ *be mappings such that*

$$\lim_{n\to\infty} m^n \Phi\left(\frac{x_1}{m^n}, ..., \frac{x_m}{m^n}\right) = 0,$$

$$\lim_{n\to\infty} m^{2n} \Psi\left(\frac{x_1}{m^n}, \frac{x_2}{m^n}\right) = 0, \tag{3.13}$$

$$\widetilde{\Phi_j}(x) := \sum_{i=0}^{\infty} m^i \Phi(0, ..., \underbrace{\frac{x}{m^i}}_{j\,th}, ..., 0) < \infty$$

for all $x, x_1, \cdots, x_m \in A_0$. *Suppose that* $f : A_0 \to A$ *is a mapping such that*

$$\|D_\mu f(x_1, ..., x_m)\|_A \le \Phi(x_1, ..., x_m),$$

$$\big\|f([x_1, x_2]) - [f(x_1), x_2] - [x_1, f(x_2)]\big\|_A \le \Psi(x_1, x_2)$$

for all $x_1, \cdots, x_m \in A_0$ *and all* $\mu \in \mathbb{T}^1$. *Then there exists a unique Lie derivation* $\delta : A_0 \to A$ *such that*

$$\|f(x) - \delta(x)\|_A \le \widetilde{\Phi_j}(x) \tag{3.14}$$

for all $x \in A_0$.

Similarly to the proof of Theorem 3.2, we have

$$\|f(mx) - mf(x)\|_A \le \Phi(0, ..., \underbrace{mx}_{j\,th}, ...0) \tag{3.15}$$

for all $x \in A_0$. Replacing x by $\frac{x}{m^{n+1}}$ in (3.15) and multiplying both sides of (3.15) to m^n, we get

$$\left\|m^{n+1} f\left(\frac{x}{m^{n+1}}\right) - m^n f\left(\frac{x}{m^n}\right)\right\|_A \le m^n \Phi\big(0, ..., \underbrace{\frac{x}{m^n}}_{j\,th}, ...0)$$

for all $x \in A_0$ and all non-negative integers n. Hence

$$\begin{aligned} \left\|m^{n+1} f\left(\frac{x}{m^{n+1}}\right) - m^k f\big(\frac{x}{m^k}\big)\right\|_A &\le \sum_{i=k}^{n} \left\|m^{i+1} f\left(\frac{x}{m^{i+1}}\right) - m^i f\big(\frac{x}{m^i}\big)\right\|_A \\ &\le \sum_{i=k}^{n} m^i \Phi\big(0, ..., \underbrace{\frac{x}{m^i}}_{j\,th}, ...0) \end{aligned} \tag{3.16}$$

for all $x \in A_0$ and all non-negative integers n and k with $n \ge k$. Therefore the sequence $\{m^n f\big(\frac{x}{m^n}\big)\}$ is a Cauchy sequence in A for all $x \in A_0$. Since A is complete, the sequence $\{m^n f\big(\frac{x}{m^n}\big)\}$ converges in A for all $x \in A_0$. So one can define the mapping $\delta : A_0 \to A$ by

$$\delta(x) := \lim_{n\to\infty} m^n f\big(\frac{x}{m^n}\big)$$

for all $x \in A_0$. Letting $k = 0$ and passing the limit $n \to \infty$ in (3.16), we get (3.14).

The rest of the proof is similar to the proof of Theorem 3.2.

Corollary 3.6. *Let $\alpha_1, \alpha_2, s_1, s_2, \{\theta_i\}_{i=1}^m$ and $\{r_i\}_{i=1}^m$ be non-negative real numbers such that $s_1, s_2 > 2$ and $r_i > 1$ for all $1 \le i \le m$. Suppose that $f : A_0 \to A$ is a mapping such that*

$$\|D_\mu f(x_1, ..., x_m)\|_A \le \sum_{i=1}^m \theta_i \|x_i\|_{A_0}^{r_i},$$

$$\|f([x_1, x_2]) - [f(x_1), x_2] - [x_1, f(x_2)]\|_A \le \alpha_1 \|x_1\|_{A_0}^{s_1} + \alpha_2 \|x_2\|_{A_0}^{s_2},$$

for all $x_1, \cdots, x_m \in A_0$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique Lie derivation $\delta : A_0 \to A$ such that

$$\|f(x) - \delta(x)\|_A \le \Gamma(x)$$

for all $x \in A_0$, where

$$\Gamma(x) := \min_{1\le i\le m} \Big\{ \frac{\theta_i m^{r_i}}{m^{r_i} - 1} \|x\|_{A_0}^{r_i} \Big\}.$$

Corollary 3.7. *Let $\alpha_1, \alpha_2, \alpha_3, s_1, s_2$ and $\{r_i\}_{i=1}^m$ be non-negative real numbers such that $s_1, s_2 > 2$ and $r_i > 1$ for all $1 \le i \le m$. Suppose that $f : A_0 \to A$ is a mapping such that*

$$\|D_\mu f(x_1, ..., x_m)\|_A \le \sum_{i=1}^m \|x_i\|_{A_0}^{r_i} + \prod_{i=1}^m \|x_i\|_{A_0}^{r_i},$$

$$\|f([x_1, x_2]) - [f(x_1), x_2] - [x_1, f(x_2)]\|_A \le \alpha_1 \|x_1\|_{A_0}^{s_1} + \alpha_2 \|x_2\|_{A_0}^{s_2} + \alpha_3 \|x_1\|_{A_0}^{s_1} \|x_2\|_{A_0}^{s_2},$$

for all $x_1, \cdots, x_m \in A_0$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique Lie derivation $\delta : A_0 \to A$ such that

$$\|f(x) - \delta(x)\|_A \le \Delta(x)$$

for all $x \in A_0$, where

$$\Delta(x) := \min_{1\le i\le m} \Big\{ \frac{m^{r_i}}{m^{r_i} - m} \|x\|_{A_0}^{r_i} \Big\}.$$

4. SUBADDITIVE MAPPING AND STABILITY OF EQ. (1.1)

Next, using some idea of [35], we are going to establish other theorems about the stability of Eq. (1.1)

We call that a subadditive mapping is a mapping $\varphi : A \to B$, having a domain A and a codomain $(B, \le)$ that are both closed under addition, with the following property:

$$\varphi(x + y) \le \varphi(x) + \varphi(y)$$

for all $x, y \in X$. Now we say that a mapping $\varphi : X \to Y$ is contractively subadditive if there exists a constant L with $0 < L < 1$ such that

$$\varphi(x+y) \leq L[\varphi(x) + \varphi(y)]$$

for all $x, y \in X$. Therefore φ satisfies the following properties $\varphi(mx) \leq mL\varphi(x)$ and so $\varphi(m^n x) \leq (mL)^n \varphi(x)$, for all $x \in X$ and all positive integer $m \geq 2$.
Similarly, we say that a mapping $\varphi : A \to B$ is expansively superadditive if there exists a constant L with $0 < L < 1$ such that

$$\varphi(x+y) \geq \frac{1}{L}[\varphi(x) + \varphi(y)]$$

for all $x, y \in X$. Therefor φ satisfies the following properties $\varphi(x) \leq \frac{L}{m}\varphi(mx)$ and so $\varphi(\frac{x}{m^n}) \leq (\frac{L}{m})^n \varphi(x)$, for all $x \in X$ and all positive integer $m \geq 2$.

Theorem 4.1. *Let* $\varphi : \underbrace{A_0 \times A_0 \times ... \times A_0}_{m-times} \to [0, \infty)$ *be a contractively subadditive with the constant* L *and* $\psi : A_0 \times A_0 \to [0, \infty)$ *be a mapping such that*

$$\lim_{n\to\infty} \frac{1}{m^{2n}} \psi(m^n x_1, m^n x_2) = 0, \tag{4.1}$$

for all $x_1, x_2 \in A_0$. *Suppose that* $f : A_0 \to A$ *is a mapping such that*

$$\|D_\mu f(x_1, ..., x_m)\|_A \leq \varphi(x_1, ..., x_m), \tag{4.2}$$

$$\big\|f([x_1, x_2]) - [f(x_1), x_2] - [x_1, f_1(x_2)]\big\|_A \leq \psi(x_1, x_2) \tag{4.3}$$

for all $x_1, \cdots, x_m \in A_0$ *and all* $\mu \in \mathbb{T}^1$. *Then there exists a unique Lie derivation* $\delta : A_0 \to A$ *such that*

$$\|f(x) - \delta(x)\|_A \leq \frac{L}{1-L}\varphi(0, ..., \underbrace{x}_{j\,th}, ...0) \tag{4.4}$$

for all $x \in X$.

Letting $\mu = 1$, $x_j = mx$ and $x_i = 0$ for all $1 \leq i \leq m$ with $i \neq j$ in (4.2) , we get

$$\|f(mx) - mf(x)\|_A \leq \varphi(0, ..., \underbrace{mx}_{j\,th}, ...0) \tag{4.5}$$

for all $x \in A_0$.

Replacing x by $m^n x$ in (4.5) and dividing both sides of (4.5) by m^{n+1}, we get

$$\begin{aligned} \Big\|\frac{1}{m^{n+1}} f(m^{n+1}x) - \frac{1}{m^n} f(m^n x)\Big\|_A &\leq \frac{1}{m^{n+1}}\varphi(0, ..., \underbrace{m^{n+1}x}_{j\,th}, ..., 0) \\ &\leq \frac{(mL)^{n+1}}{m^{n+1}}\varphi(0, ..., \underbrace{x}_{j\,th}, ..., 0) \\ &\leq L^{n+1}\varphi(0, ..., \underbrace{x}_{j\,th}, ..., 0) \end{aligned} \tag{4.6}$$

for all $x \in A_0$ and all non-negative integers n. Hence

$$\begin{aligned}\left\|\frac{1}{m^{n+1}}f(m^{n+1}x)-\frac{1}{m^k}f(m^kx)\right\|_A &\le \sum_{i=k}^{n}\left\|\frac{1}{m^{i+1}}f(m^{i+1}x)-\frac{1}{m^i}f(m^ix)\right\|_A \\ &\le \sum_{i=k+1}^{n+1} L^i\varphi(0,...,\underbrace{x}_{j\,th},...,0)\end{aligned} \tag{4.7}$$

for all $x \in A_0$ and all non-negative integers n and k with $n \ge k$. Therefore, we conclude from and (4.7) that the sequence $\{\frac{1}{m^n}f(m^nx)\}$ is a Cauchy sequence in A for all $x \in A_0$. Since A is complete, the sequence $\{\frac{1}{m^n}f(m^nx)\}$ converges in A for all $x \in A_0$. So one can define the mapping $\delta : A_0 \to A$ by

$$\delta(x) := \lim_{n\to\infty}\frac{1}{m^n}f(m^nx) \tag{4.8}$$

for all $x \in A_0$. Letting $k = 0$ and passing the limit $n \to \infty$ in (4.7), we get (4.4). Now, we show that δ is a $\mathbb{C}$-linear mapping. It follows from (4.8) that

$$\begin{aligned}\|D_1\delta(x_1,...,x_m)\|_A &= \lim_{n\to\infty}\frac{1}{m^n}\left\|D_1f(m^nx_1,...,m^nx_m)\right\|_A \\ &\le \lim_{n\to\infty}\frac{1}{m^n}\varphi(m^nx_1,...,m^nx_m) \\ &\le \lim_{n\to\infty}L^n\varphi(x_1,...,x_m)=0\end{aligned}$$

for all $x_1,\cdots,x_m \in A_0$. So the mapping δ satisfies (1.1). By Lemma 2.1, the mapping δ is additive.

Letting $x_j = mx$ and $x_i = 0$ for all $1 \le i \le m$ with $i \ne j$ in (4.2), we get

$$\|mf(\mu x)-\mu f(mx)\|_A \le \varphi(0,...,\underbrace{mx}_{j\,th},...,0) \tag{4.9}$$

for all $x \in A_0$. Replacing x by m^nx in (4.9) and dividing both sides of (4.9) by m^{n+1}, we get

$$\begin{aligned}&\left\|\frac{1}{m^n}f(\mu m^nx)-\frac{\mu}{m^{n+1}}f(m^{n+1}x)\right\|_A \\ &\le \frac{1}{m^{n+1}}\varphi(0,...,\underbrace{m^{n+1}x}_{j\,th},...,0)\end{aligned} \tag{4.10}$$

for all $x \in A_0$ and all non-negative integers n. Passing the limit $n \to \infty$ in (4.10) and using (4.8), we get

$$\delta(\mu x) = \mu\delta(x)$$

for all $\mu \in \mathbb{T}^1$ and for all $x \in A_0$. So by Lemma 3.1, we infer that the mapping $\delta : A_0 \to A$ is $\mathbb{C}$-linear. To prove the uniqueness of δ, let $\delta' : A_0 \to A$ be another additive mapping satisfying (4.4). It follows from (4.8) that

$$\begin{aligned}\|\delta(x)-\delta'(x)\|_A &= \lim_{n\to\infty}\frac{1}{m^n}\left\|f(m^n x)-\delta'(m^n x)\right\|_A \\ &\le \lim_{n\to\infty}\frac{L^{n+1}}{1-L}\varphi(0,...,\underbrace{x}_{j\,th},...0)=0\end{aligned}$$

for all $x\in A_0$. So $\delta=\delta'$.

The rest of the proof is similar to the proof of Theorem 3.2.

Corollary 4.2. *Let θ be non-negative real number and $f: A_0\to A$ be a mapping for which*

$$\|D_\mu f(x_1,...,x_m)\|_A\le\theta$$

$$\left\|f([x_1,x_2])-[f(x_1),x_2]-[x_1,f_1(x_2)]\right\|_A\le\theta$$

for all $x_1,...,x_m\in A_0$. Then there exists a unique Lie derivation $\delta: A_0\to A$ such that

$$\|f(x)-\delta(x)\|_A\le\theta \tag{4.11}$$

for all $x\in A_0$.

The proof follows from Theorem 4.1 by taking

$$\varphi(x_1,...,x_m):=\theta$$

for all $x_1,...,x_m\in A_0$.

Replacing contractively subadditive by expansively superadditive in Theorem 4.1, one can obtain the following theorem:

Theorem 4.3. *Let $\varphi:\underbrace{A_0\times A_0\times...\times A_0}_{m-times}\to[0,\infty)$ be a expansively superadditive with the constant L and $\psi: A_0\times A_0\to[0,\infty)$ be a mapping such that*

$$\lim_{n\to\infty} m^{2n}\psi\left(\frac{x_1}{m^n},\frac{x_2}{m^n}\right)=0, \tag{4.12}$$

for all $x_1,x_2\in A_0$. Suppose that $f: A_0\to A$ is a mapping such that

$$\|D_\mu f(x_1,...,x_m)\|_A\le\varphi(x_1,...,x_m), \tag{4.13}$$

$$\left\|f([x_1,x_2])-[f(x_1),x_2]-[x_1,f_1(x_2)]\right\|_A\le\psi(x_1,x_2) \tag{4.14}$$

for all $x_1,\cdots,x_m\in A_0$ and all $\mu\in\mathbb{T}^1$. Then there exists a unique Lie derivation $\delta: A_0\to A$ such that

$$\|f(x)-\delta(x)\|_A\le\frac{1}{1-L}\varphi(0,...,\underbrace{x}_{j\,th},...0) \tag{4.15}$$

for all $x\in X$.

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