

ON γ -SEMI- θ -CLOSED SETS AND LOCALLY γ -S-REGULAR SPACES

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ABSTRACT. We continue studying the properties and characterizations of γ -semi- θ -closed sets, γ -s-closed relative to a space X and locally γ -s-regular spaces defined and discussed by B. Ahmad and S. Hussain in 2008 and S. Hussain and B. Ahmad in 2007 and 2009.

1. INTRODUCTION

In 1969, Signal and Arya [22] defined a new separation axiom called almost regularity which is weaker than regularity. It has been shown in [17], that for Hausdorff spaces, this axiom occupies a position between Urysohn's separation axiom and T_3 -axiom. Maheswari and Prasad [18] have defined another axiom called s-regularity which is weaker than regularity (without T_2). In 1982, C. Dorsett [8] defined and investigated a new separation axiom called semi-regular space. It is shown that s-regularity is weaker than semi-regularity. In 1979, S. Kasahara [14] defined an operation α on topological spaces. B. Ahmad and M. Khan [7] defined and study locally s-regular spaces. It is interesting to mention that class of s-regular spaces is a proper subclass of locally s-regular spaces. In 1992 (1993), B. Ahmad and F. U. Rehman [1] [21] introduced the notions of γ -interior, γ -boundary and γ -exterior points in topological spaces. B. Ahmad and S. Hussain further studied the properties of γ -operations in topological spaces in [2] [3]. Recently S. Hussain, B. Ahmad and T. Noiri [12] introduced and discussed γ -semi-open sets in topological spaces. Further B. Ahmad and S. Hussain [5] explored the characterizations of γ -semi-open (closed), γ -semi-closure (interior) and γ -semi-continuous functions. They defined and discussed γ -s-closed spaces and subspaces ([4], [10]) using γ -semi-closure. It is

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known [10] that the concept of γ -s-closed space is a generalization of s-closed space [16]. They defined and discussed the characterizations and properties of γ -semi- θ -closed sets [10], γ -s-closed relative to a space X [4] and Locally γ -s-Regular spaces [4].

In this paper, we continue studying the properties and characterizations of γ -semi- θ -closed sets, γ -s-closed relative to a space X and locally γ -s-regular spaces defined and discussed in [4], [10], [11].

2. PRELIMINARIES

Hereafter X will be represented as a topological space and we shall write a space in place of a topological space for our convenience.

Now we recall some notions defined in [5],[11], [12], [14] and [15]. In [14] an operation $\gamma : \tau \rightarrow P(X)$ is defined as a function from τ to the power set of X such that $V \subseteq V^\gamma$, for each $V \in \tau$, where V^γ denotes the value of γ at V . The operations defined by $\gamma(G) = G$, $\gamma(G) = cl(G)$ and $\gamma(G) = intcl(G)$ are examples of operation γ . Let A be a subset of space X . A point $x \in A$ is said to be γ -interior point [15] of A , if there exists an open nbd N of x such that $N^\gamma \subseteq A$. We denote the set of all such points by $int_\gamma(A)$. Thus

$$int_\gamma(A) = \{x \in A : x \in N \in \tau \text{ and } N^\gamma \subseteq A\}.$$

Note that A is γ -open [15] iff $A = int_\gamma(A)$. A is called γ -closed [21] iff $X - A$ is γ -open. A point $x \in X$ is called a γ -closure point [15] of A , if $U^\gamma \cap A \neq \emptyset$, for each open nbd U of x . The set of all γ -closure points of A is called γ -closure of A and is denoted by $cl_\gamma(A)$. A is called γ -closed, if $cl_\gamma(A) \subseteq A$. Note that $cl_\gamma(A)$ is contained in every γ -closed superset of A . An operation γ on τ is said to be regular [15], if for any open nbds U, V of $x \in X$, there exists an open nbd W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$. An operation γ on τ is said to be open [15], if for every nbd U of each $x \in X$, there exists γ -open set B such that $x \in B$ and $U^\gamma \subseteq B$. A subset A of a space X is said to be a γ -semi-open set [12], if there exists a γ -open set O such that $O \subseteq A \subseteq cl_\gamma(O)$. The set of all γ -semi-open sets is denoted by $SO_\gamma(X)$. A is γ -semi-closed iff $X - A$ is γ -semi-open in X . Note that A is γ -semi-closed iff $int_\gamma(cl_\gamma(A)) \subseteq A$. The intersection of all γ -semi-closed sets containing A is called γ -semi-closure [5] of A and is denoted by $scl_\gamma(A)$. Note that A is γ -semi-closed iff $scl_\gamma(A) = A$. The union of γ -semi-open subsets of A is called γ -semi-interior [5] of A and is denoted by $sint_\gamma(A)$. A is γ -semi-regular [5], if it is both γ -semi-open and γ -semi-closed. The class of all γ -semi-regular sets of X is denoted by $SR_\gamma(A)$. Note that if γ is a regular operation, then the union of γ -semi-regular sets is γ -semi-regular. A space X is said to be γ -s-regular [11], if for any γ -semi-regular set A and $x \notin A$, there exist disjoint γ -open sets U and V such that $A \subseteq U$ and $x \in V$.

3. γ -SEMI- θ -CLOSED SETS.

In [10] a point x of a space X is defined as the γ -semi- θ -closure point of a subset A of X , if $A \cap scl_\gamma(U) \neq \emptyset$, for every $U \in SO_\gamma(X)$ containing x . The set of all γ -semi- θ -closure points of A is called γ -semi- θ -closure of A and is denoted by $s_\gamma cl_\theta(A)$. A subset A is said to be γ -semi- θ -closed, if $A = s_\gamma cl_\theta(A)$. The complement of a γ -semi- θ -closed set is said to be γ -semi- θ -open.

Now we define:

Definition 3.1. A function $f : X \rightarrow Y$ is said to be a γ -semi- θ -closed, if $f(K)$ is γ -semi- θ -closed in Y , for every γ -semi- θ -closed set K of X .

Theorem 3.2. A function $f : X \rightarrow Y$ is γ -semi- θ -closed iff $s_\gamma cl_\theta(f(A)) \subseteq f(s_\gamma cl_\theta(A))$, for every subset A of X , where γ is open.

Proof. Necessity: let f be γ -semi- θ -closed and A any subset of X . Then $f(s_\gamma cl_\theta(A))$ is γ -semi- θ -closed. But $f(A) \subseteq f(s_\gamma cl_\theta(A))$ implies that $s_\gamma cl_\theta(f(A)) \subseteq f(s_\gamma cl_\theta(A))$ [10].

Sufficiency: let $s_\gamma cl_\theta(f(A)) \subseteq f(s_\gamma cl_\theta(A))$, for every subset A of X . Let B be a γ -semi- θ -closed set of X . Then $s_\gamma cl_\theta(f(B)) \subseteq f(s_\gamma cl_\theta(B)) = f(B)$. But $f(B) \subseteq s_\gamma cl_\theta(f(B))$. This proves that $s_\gamma cl_\theta(f(B)) = f(B)$. This gives that $f(B)$ is γ -semi- θ -closed. Hence the proof. \square

Theorem 3.3. Let γ be an open operation. Then the following are equivalent for a function $f : X \rightarrow Y$:

- (1) f is γ -semi- θ -closed.
- (2) $s_\gamma cl_\theta(f(A)) \subseteq f(s_\gamma cl_\theta(A))$, for every subset A of X .
- (3) For every subset B of Y and every γ -semi- θ -open set U of X containing $f^{-1}(B)$, there exists a γ -semi- θ -open set V of Y containing B such that $f^{-1}(V) \subseteq U$.
- (4) For every point $y \in Y$ and every γ -semi- θ -open set U of X containing $f^{-1}(y)$, there exists a γ -semi- θ -open set V of Y containing y such that $f^{-1}(V) \subseteq U$.

Proof. The proof is similar to that of Theorem 5.2 [6] and is thus omitted. \square

Recall [Lemma 1 [4]] that a subset A of a space X is γ -s-closed relative to X iff every cover of A by γ -semi- θ -open sets of X has a finite subcover.

Theorem 3.4. Let $f : X \rightarrow Y$ be a γ -semi- θ -closed function such that $f^{-1}(y)$ is γ -s-closed relative to X , for each point y of Y . If K is γ -s-closed relative to Y , then $f^{-1}(K)$ is γ -s-closed relative to X , where γ is a regular operation.

Proof. Let $\{U_\alpha : \alpha \in I\}$ be any cover of $f^{-1}(K)$ by γ -semi- θ -open sets of X . For each $y \in K$, $f^{-1}(y)$ is γ -s-closed relative to X and by Lemma 1 [4], there exists a finite subset $I(y)$ of I such that $f^{-1}(y) \subseteq \cup\{U_\alpha : \alpha \in I(y)\}$. Let $U(y) = \cup\{U_\alpha : \alpha \in I(y)\}$, then $U(y)$ is γ -semi- θ -open in X [4]. Since f is γ -semi- θ -closed, by Theorem 3.4, there exists a γ -semi- θ -open set $V(y)$ containing y such that $f^{-1}(V(y)) \subseteq U(y)$, since $\{V(y) : y \in K\}$ is a γ -semi- θ -open cover of K . By Lemma 1 [4], there exists a finite subset K_0 of K such that $K \subseteq \cup\{V(y) : y \in K_0\}$. Therefore, we obtain

$$f^{-1}(K) \subseteq \cup\{f^{-1}(V(y)) : y \in K_0\} \subseteq \cup\{U_\alpha(y) : \alpha \in I(y), y \in K_0\}.$$

This shows that $f^{-1}(K)$ is γ -s-closed relative to X . This completes the proof. \square

Corollary 1. Let $f : X \rightarrow Y$ be a γ -semi- θ -closed surjection such that $f^{-1}(y)$ is γ -s-closed relative to X for each point $y \in Y$. If Y is γ -s-closed, then X is γ -s-closed, where γ is a regular operation.

In [4], we defined that a filterbase \mathfrak{S} on X is said to γ -SR-converge to $x \in X$, if for each $V \in SR_\gamma(X)$, there exists $F \in \mathfrak{S}$ such that $F \subseteq V$.

Definition 3.5. A point $x \in X$ is called γ -semi- θ -adherent point of a filterbase \mathfrak{S} in X , if $x \in [s_\gamma ad]_\theta(\mathfrak{S}) = \bigcap\{s_\gamma cl_\theta(F) : F \in \mathfrak{S}\}$.

Definition 3.6. A filterbase \mathfrak{S} is said to be γ -semi- θ -directed towards $S \subseteq X$, if every filterbase subordinate to \mathfrak{S} has a γ -semi- θ -adherent point in S .

Definition 3.7. A function $f : X \rightarrow Y$ is said to be γ -semi- θ -perfect, if for every filterbase \mathfrak{S} in $f(X)$ γ -SR-converges to $y \in Y$, $f^{-1}(\mathfrak{S})$ is γ -semi- θ -directed towards $f^{-1}(y)$.

Theorem 3.8. Every γ -semi- θ -perfect function is γ -semi- θ -closed, where γ is open.

Proof. Let $f : X \rightarrow Y$ be a γ -semi- θ -perfect function and A any subset of X . Let $y \in s_\gamma cl_\theta(f(A))$. Then there exists a filterbase \mathfrak{S} on $f(A)$ which γ -SR-converges to y . Put $\xi = \{f^{-1}(F) \cap A : F \in \mathfrak{S}\}$. Then ξ is a γ -filterbase on X subordinate to the filterbase $f^{-1}(\mathfrak{S})$. Since $f^{-1}(\mathfrak{S})$ is γ -semi- θ -directed towards $f^{-1}(y)$, we have $f^{-1}(y) \cap [s_\gamma ad]_\theta(\xi) \neq \phi$. Therefore, we obtain $y \in f(s_\gamma cl_\theta A)$. By Theorem 3.2, f is γ -semi- θ -closed. Hence the proof. \square

Theorem 3.9. A function $f : X \rightarrow Y$ is γ -semi- θ -perfect iff $[s_\gamma ad]_\theta f(\mathfrak{S}) \subset f([s_\gamma ad]_\theta(\mathfrak{S}))$, for every filterbase \mathfrak{S} in X .

Proof. Necessity: suppose that $f : X \rightarrow Y$ is γ -semi- θ -perfect. Let \mathfrak{S} be a filterbase in X and $y \in [s_\gamma ad]_\theta f(\mathfrak{S})$. Then there exists a filterbase ξ in $f(X)$ which is subordinate to $f(\mathfrak{S})$ and γ -SR-converges to y . Put $H = \{f^{-1}(G) \cap F : F \in \mathfrak{S}, G \in \xi\}$. Then H is filterbase in X subordinate to $f^{-1}(\xi)$. Since f is γ -semi- θ -perfect, $f^{-1}(\xi)$ is γ -semi- θ -directed towards $f^{-1}(y)$. Therefore we have $f^{-1}(y) \cap [s_\gamma ad]_\theta(H) \neq \phi$ and hence $y \in f([s_\gamma ad]_\theta(\mathfrak{S}))$. This proves that $[s_\gamma ad]_\theta f(\mathfrak{S}) \subset f([s_\gamma ad]_\theta(\mathfrak{S}))$. This proves necessity.

Sufficiency: suppose that $[s_\gamma ad]_\theta f(\mathfrak{S}) \subset f([s_\gamma ad]_\theta(\mathfrak{S}))$, for every filterbase \mathfrak{S} in X . We prove that f is γ -semi- θ -perfect. Assume contrary that f is not γ -semi- θ -perfect, then there exists a filterbase \mathfrak{S} in $f(X)$ such that \mathfrak{S} γ -SR-converges to a point $y \in Y$. But $f^{-1}(\mathfrak{S})$ is not γ -semi- θ -directed towards $f^{-1}(y)$. Thus there exists a filterbase ξ in X which is subordinate to $f^{-1}(\mathfrak{S})$ and $f^{-1}(y) \cap [s_\gamma ad]_\theta(\xi) = \phi$. Therefore we have $y \notin [s_\gamma ad]_\theta f(\xi)$ and hence $y \notin s_\gamma cl_\theta(f(G_1))$, for some $G_1 \in \xi$. Then there exists a γ -semi-open set V containing y such that $scl_\gamma(V) \cap f(G_1) = \phi$. Since \mathfrak{S} γ -SR-converges to y and ξ is subordinate to $f^{-1}(\mathfrak{S})$, there exists a $G_2 \in \xi$ such that $f(G_2) \subset scl_\gamma(V)$. Consequently, we obtain $G_1 \cap G_2 = \phi$. This contradicts that ξ is a filterbase. This proves that f is γ -semi- θ -perfect. Hence the proof. \square

4. γ -S-CLOSED RELATIVE TO A SPACE.

In [10] a subset A of a space X is defined to be γ -s-closed relative to X , if for every cover of A by γ -semi-open sets of X , there exists a finite subset I_0 of I such that $A \subseteq \bigcup_{\alpha \in I_0} scl_\gamma(V_\alpha)$.

Theorem 4.1. A subset K of a space X is γ -s-closed relative to X iff $K \cap [s_\gamma ad]_\theta(\mathfrak{S}) \neq \phi$, for every filterbase \mathfrak{S} in K .

Proof. Necessity: suppose that K is γ -s-closed relative to X . Assume that \mathfrak{S} is a filterbase in K such that $K \cap [s_\gamma ad]_\theta(\mathfrak{S}) = \phi$. Then for each $x \in K$, there exists a γ -semi-open set U_x containing x and $F_x \in \mathfrak{S}$ such that $F_x \cap scl_\gamma(U_x) = \phi$. Since K is γ -s-closed relative to X , there exists a finite number of points x_1, x_2, \dots, x_n in K such that $K \subset \bigcup \{scl_\gamma(U_{x_j}) : j = 1, 2, \dots, n\}$. Put $F = \bigcap \{F_{x_j} : j = 1, 2, \dots, n\}$, then we obtain $F \cap K = \phi$. This contradicts that \mathfrak{S} is a filterbase in K . Therefore $K \cap [s_\gamma ad]_\theta(\mathfrak{S}) \neq \phi$. This completes necessity.

Sufficiency: suppose that $K \cap [s_\gamma ad]_\theta(\mathfrak{S}) \neq \phi$, for every filterbase \mathfrak{S} in K . Assume that K is not γ -s-closed relative to X . Then there exists a cover $\{U_\alpha : \alpha \in I\}$ of K be γ -semi-open sets of X such that $K \not\subseteq \bigcup \{scl_\gamma(U_\alpha) : \alpha \in I_o\}$ for every $I_o \in J(I)$, where $J(I)$ denotes the family of all finite subsets of I . Put $F_{I_o} = \bigcap \{K - scl_\gamma(U_\alpha) : \alpha \in I_o\}$, for each $I_o \in J(I)$. Then $\mathfrak{S} = \{F_{I_o} : I_o \in J(I)\}$ is a filterbase in K and $K \cap [s_\gamma ad]_\theta(\mathfrak{S}) = \phi$. This is a contradiction. Therefore, K is γ -s-closed relative to X . This completes the proof. \square

Theorem 4.2. *If $f : X \rightarrow Y$ is a γ -semi- θ -perfect function, then $f^{-1}(K)$ is γ -s-closed relative to X , for every $K \subseteq Y$, γ -s-closed relative to Y . Where γ is open.*

Proof. Let $f : X \rightarrow Y$ be a γ -semi- θ -perfect function and $K \subset Y$ γ -s-closed relative to Y . We prove that $f^{-1}(K)$ is γ -s-closed relative to X . Let \mathfrak{S} be a filterbase in X . Then $\xi = \{f(F) \cap K : F \in \mathfrak{S}\}$ is a filterbase in K subordinate to the filterbase $f(\mathfrak{S})$. By Theorem 4.1, $K \cap [s_\gamma ad]_\theta(\xi) \neq \phi$ and hence we obtain $K \cap [s_\gamma ad]_\theta f(\mathfrak{S}) \neq \phi$. Since f is γ -semi- θ -perfect, therefore by Theorem 3.8, we obtain $K \cap f([s_\gamma ad]_\theta(\mathfrak{S})) \neq \phi$. This gives $f^{-1}(K) \cap [s_\gamma ad]_\theta(\mathfrak{S}) \neq \phi$. Hence by Theorem 4.1, we conclude that $f^{-1}(K)$ is γ -s-closed relative to X . Hence the proof. \square

Theorem 4.3. *Let γ be an open operation. A function $f : X \rightarrow Y$ is γ -semi- θ -perfect iff*

- (1) f is γ -semi- θ -closed, and
- (2) $f^{-1}(y)$ is γ -s-closed relative to X , for each $y \in Y$.

Proof. Necessity: let $f : X \rightarrow Y$ be a γ -semi- θ -perfect function. Then (1) follows from Theorem 3.8 and (2) follow from Theorem 4.2, because each singleton set in Y is γ -s-closed relative to Y [10].

Sufficiency: we suppose on the contrary that conditions (1) and (2) hold but f is not γ -semi- θ -perfect. Then there exists a filterbase \mathfrak{S} in $f(X)$ such that \mathfrak{S} γ -SR-converges to a point $y \in Y$. But $f^{-1}(\mathfrak{S})$ is not γ -semi- θ -directed towards $f^{-1}(y)$. Thus there exists a filterbase ξ in X which is subordinate to $f^{-1}(\mathfrak{S})$ and $f^{-1}(y) \cap [s_\gamma ad]_\theta(\xi) = \phi$. Therefore we have $y \notin [s_\gamma ad]_\theta f(\xi)$ and hence $y \notin s_\gamma cl_\theta(f(G_1))$, for some $G_1 \in \xi$. Then there exists a γ -semi-open set V containing y such that $scl_\gamma(V) \cap f(G_1) = \phi$. Since \mathfrak{S} γ -SR-converges to y and ξ is subordinate to $f^{-1}(\mathfrak{S})$, therefore, there exists a $G_2 \in \xi$ such that $f(G_2) \subset scl_\gamma(V)$. Consequently, we obtain $G_1 \cap G_2 = \phi$. This contradicts that ξ is filterbase. This proves that f is γ -semi- θ -perfect. This completes the proof. \square

5. LOCALLY γ -S-REGULAR SPACES .

In [10] a subset A of a space X is called γ -regular-open, if $A = int_\gamma(cl_\gamma(A))$. The set of all γ -regular-open sets in X is denoted by $RO_\gamma(X, \tau)$. Note that $RO_\gamma(X, \tau) \subseteq \tau_\gamma \subseteq \tau$ [10].

A space X is a γ -extremally disconnected space [10], if $cl_\gamma(U)$ is a γ -open set, for every γ -open set U in X . A space X is said to be locally γ -s-regular [11], if for each point of X has a γ -regular-open nbd which is γ -s-regular subspace of X .

The following Theorem shows that locally γ -s-regularity is a γ -regular-open hereditary property:

Theorem 5.1. *Every γ -regular-open subspace of a locally γ -s-regular space is locally γ -s-regular, where γ is a regular operation.*

Proof. Let Y be a γ -regular-open subspace of a locally γ -s-regular space X and $x \in Y$. Since X is locally γ -s-regular, therefore $x \in X$ has a γ -regular-open set V containing x which is γ -s-regular subspace of X . Since $int_{\gamma_Y}((cl_{\gamma_Y})(A)) = Y \cap int_{\gamma_X}((cl_{\gamma_X})(A))$ [2], for any γ -open set Y of X and any set A of Y . Therefore, there exists a γ -regular-open set U of Y such that $U = Y \cap V$ and $x \in U$. Now we show that U is γ -s-regular subspace of Y . Since V is γ -s-regular, by Theorem 3.4 [11], for γ -semi-regular set V_1 of V containing x , there exists γ -open set W such that $x \in W \subset cl_{\gamma_V}(W) \subset V_1$. Since $V \in RO_\gamma(X)$ and $W \subseteq V$, therefore we have $x \in W \subseteq cl_{\gamma_X}(W) \cap V \subseteq V_1$ [11]. But $cl_{\gamma_X}(W) \subset cl_{\gamma_X}(V_1) = V_1$. This gives $x \in W \subset cl_{\gamma_X}(W) \subset V_1$. Thus we have $x \in W \cap U \subset cl_{\gamma_V}(W) \cap U \subset V_1 \cap U$. Since Y is γ -semi-closed in X [5], therefore we obtain $cl_{\gamma_X}(W \cap U) \subset cl_{\gamma_X}(W) \cap cl_{\gamma_X}(U) = cl_{\gamma_X}(W) \cap U$ or $cl_{\gamma_X}(W \cap U) \subset cl_{\gamma_X}(W) \cap U$. This gives that $x \in W \cap U \subset cl_{\gamma_X}(W \cap U) \subseteq cl_{\gamma_X}(W) \cap U \subseteq V_1 \cap U$ or $x \in W \cap U \subset cl_{\gamma_U}(W \cap U) \subset V_1 \cap U$, where $W \cap U$ is γ -open and $V_1 \cap U$ is γ -semi-regular in U , since γ is regular. Thus, by Theorem 3.4 [11], U is γ -s-regular. This proves that Y is γ -s-regular. This completes the proof. \square

Corollary 2. *Every γ -clopen subspace of a locally γ -s-regular space is locally γ -s-regular, where γ is a regular operation.*

Corollary 3. *Every γ -open subspace of a locally γ -s-regular space is locally γ -s-regular, if X is γ -extremally disconnected, where γ is a regular operation.*

Next we characterize locally γ -s-regular spaces as:

Theorem 5.2. *In a space X , the following are equivalent:*

- (1) X is locally γ -s-regular.
- (2) Every point $x \in X$ has a γ -regular-open set containing x which is a γ -s-regular subspace of X .
- (3) Every point $x \in X$ has a γ -open nbd U of x such that $int_{\gamma_X}(cl_{\gamma_X}(U))$ is a γ -s-regular subspace of X .
- (4) Every point $x \in X$ has a γ -open set U containing x such that $scl_{\gamma_X}(U)$ is a γ -s-regular subspace of X .
- (5) Every point $x \in X$ has a γ -open nbd U such that $s_\gamma cl_\theta(U)$ is a γ -s-regular subspace of X .

Proof. (1) \Rightarrow (2). This is straightforward.

(2) \Rightarrow (3). Let U be γ -open nbd of $x \in X$. Then $int_{\gamma_X}(cl_{\gamma_X}(U))$ is a γ -regular-open set containing x . By (2), $int_{\gamma_X}(cl_{\gamma_X}(U))$ is a γ -s-regular subspace of X . This proves (3).

(3) \Rightarrow (4). Let U be γ -open nbd of a point $x \in X$. Then by Lemma 3.5 [10], $scl_{\gamma_X}(U) = int_{\gamma_X}(cl_{\gamma_X}(U))$. This shows that $scl_{\gamma_X}(U)$ is a γ -regular-open set of X containing x . By (3), $scl_{\gamma_X}(U)$ is γ -s-regular subspace of X . This proves (4).

(4) \Rightarrow (5). Let U be γ -open nbd of x in X . Then by Proposition 3.9 [10], $s_\gamma cl_{\theta_X}(U) = scl_{\gamma_X}(U)$. By (4), $s_\gamma cl_{\theta_X}(U)$ is a γ -s-regular subspace of x . This proves (5).

(5) \Rightarrow (1). Let $x \in X$ and U a γ -regular-open nbd of x . Then by (5), $s_\gamma cl_{\theta_X}(U) = scl_{\gamma_X}(U) = int_{\gamma_X}(cl_{\gamma_X}(U)) = U$ is γ -s-regular subspace of X . This proves that X is locally γ -s-regular. This completes the proof. \square

Theorem 5.3. *A space X is locally γ -s-regular iff for each point $x \in X$, there exists a γ -regular-open set A of X such that $x \in A$ and A is locally γ -s-regular, where γ is regular.*

Proof. Necessity follows from Theorem 5.1.

Sufficiency: Let $x \in X$ and A a γ -regular-open set of X such that $x \in A$ and A is locally γ -s-regular. This gives that there exists a γ -regular-open set U in A such that $x \in U$ and U is γ -s-regular subspace of A . Hence by Lemma 3.5 [10], $U = scl_{\gamma_X}(U) = int_{\gamma_X}(cl_{\gamma_X}(U))$. Thus for each $x \in X$, there exists a γ -regular-open set U in X such that $x \in U$ and U is a γ -s-regular subspace of X . This proves that X is locally γ -s-regular. This completes the proof. \square

In [5] a function $f : X \rightarrow Y$ is defined to be γ -semi-continuous, if for any γ -open B of Y , $f^{-1}(B)$ is γ -semi-open in X . A function $f : X \rightarrow Y$ is defined to be γ -semi-open (respt. γ -semi-closed) [6], if for each γ -open (respt. γ -closed) set U in X , $f(U)$ is γ -semi-open (respt. γ -semi-closed) in Y .

Theorem 5.4. *Let $f : X \rightarrow Y$ be γ -semi-continuous, γ -semi-open and γ -semi-closed preserving surjection. If X is γ -s-regular, then Y is γ -s-regular, where γ is an open operation.*

Proof. Let U be a γ -regular-open set in Y and $y \in U$. Let $x \in f^{-1}(y)$. Then $f^{-1}(U)$ is γ -semi-open in X and $x \in f^{-1}(U)$. Since X is γ -s-regular, therefore by Theorem 3.4 [11], there exists a γ -open set V such that $x \in V \subset cl_{\gamma}(V) \subset f^{-1}(U)$ or $y \in f(V) \subset f(cl_{\gamma}(V)) \subset f(f^{-1}(U)) \subset U$ or $y \in f(V) \subset f(cl_{\gamma}(V)) \subset U$, where $f(V)$ is γ -semi-open and $f(cl_{\gamma}(V))$ is γ -semi-closed [5]. Therefore $cl_{\gamma}(f(V)) \subset f(cl_{\gamma}(V))$ and $y \in f(V) \subset cl_{\gamma}(f(V)) \subset f(cl_{\gamma}(V)) \subset U$ or $y \in f(V) \subset cl_{\gamma}(f(V)) \subset U$. This proves that Y is γ -s-regular [11]. Hence the proof. \square

Theorem 5.5. *Let $f : X \rightarrow Y$ be γ -semi-continuous, γ -semi-open preserving bijection. If X is γ -s-regular, then Y is locally γ -s-regular, where γ is a regular operation.*

Proof. Let U be γ -regular open set in Y such that $y \in U$. Let F be γ -closed set of U such that $y \notin F$. Then there exists a γ -closed set G of Y such that $F = U \cap G$ [2] and $y \notin G$. Then $f^{-1}(G)$ is γ -semi-closed in X and $f^{-1}(y) \notin f^{-1}(G)$. Since X is γ -s-regular, therefore there exists disjoint γ -open sets U_1, U_2 in X such that $f^{-1}(y) \in U_1$, $f^{-1}(G) \subset U_2$ or $y \in f(U_1)$, $G \subset f(U_2)$, where $f(U_1)$ and $f(U_2)$ are γ -semi-open in Y . Clearly $y \in U \cap f(U_1)$ and $F = U \cap G \subseteq U \cap f(U_2)$, where $U \cap f(U_1)$ and $U \cap f(U_2)$ are disjoint γ -semi-open sets in U [5]. This proves that Y is locally γ -s-regular. This completes the proof. \square

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