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A CONVOLUTION IDENTITY FOR EXCHANGEABLE RISKS

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Abstract. In this short note we provide an extension of a convolution identity for exchangeable dependent risks, which is motivated by an elegant proof of Panjer's algorithm derived in Mikosch (2006).

1. Introduction and Results

Let $X_i, 1 \le i \le n$ be independent and identically distributed random variables (risks), and denote $S_n = \sum_{i=1}^n X_i, n > 1$. In an insurance context S_n plays a crucial role for modelling aggregate portfolio losses where $X_i, 1 \le i \le n$ (assumed to be positive) model the losses of some insured portfolio over a specific time period. Since often the number of losses occurring in a particular time interval is itself random, say N, modeling of S_N is also of interest. In order to calculate the distribution function of S_N for particular choice of N, i.e., N being Binomial, Poisson or negative Binomial, the well-known Panjer's recursion algorithm is utilised; an elegant proof of that algorithm is given in Mikosch (2006).

If $X_1 \neq 0$ almost surely, following Mikosch (2006) we find the following equation for the conditional expectation

$$\boldsymbol{E}\left\{\frac{X_i}{S_n}\Big|S_n\right\} = \frac{1}{n}, \quad 1 \le i \le n \tag{1}$$

holds almost surely. Consequently, by removing the conditioning we are left with

$$\boldsymbol{E}\left\{\frac{X_i}{S_n}\right\} = \frac{1}{n}, \quad 1 \le i \le n.$$

Hence, for any $1 \le l < n$

$$\sum_{1 \le i \le l} \boldsymbol{E}\left\{\frac{X_i}{S_n}\right\} = \boldsymbol{E}\left\{\frac{S_l}{S_n}\right\} = \frac{l}{n},$$

or alternatively

$$\boldsymbol{E}\left\{\frac{S_l}{S_l + \sum_{l+1 \le i \le n} X_i}\right\} = \frac{l}{n}. \tag{2}$$

which is the main result of Theorem 2.1 in Mukhopadhyay (2010).

The independence assumption is in diverse applications not tenable. Therefore, it is of some interest to consider the validity of (1) for such instances.

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In the aforementioned paper several examples of dependent risks are provided extending thus the result (2). The reason why the dependence among the risks does not influence the expectation of the ratio S_l/S_n is the special dependence underlying those examples, namely the fact that the Gaussian random vectors with identical and equicorrelated components are exchangeable. We present below a more general result:

Lemma 1. *Let $X_i, 1 \le i \le n$ be exchangeable risks such that $X_i \neq 0$ almost surely. Then both* (1) *and* (2) *hold.*

Proof: The proof follows easily by the exchangeability assumption.

□

Remark: a) Exchangeability of $X_1, \ldots, X_n$ which means that the joint distribution function is invariant to permutation of the indices follows for instance if $X_i, 1 \le i \le n$ are conditionally independent.

b) The assumption that X_1 possesses a probability density function imposed in Theorem 2.1 of Mukhopadhyay (2010) is redundant.

2. Examples

In this section we give two examples of dependent risks $X_i, 1 \le i \le n$ which satisfy the convolution identity (2).

Example 1. Let $Z_1, \ldots, Z_n$ be standard Gaussian random variables with mean 0, variance 1, being further equicorrelated with correlation coefficient $\rho \in (-1, 1)$. Let R be a positive random variable independent of $Z_i, 1 \le i \le n$, and let $X_i = RZ_i, 1 \le i \le n$. The random vector $(X_1, \ldots, X_n)$ has a scale mixture Gaussian distribution. Since $X_i, 1 \le i \le n$ are exchangeable risks the result of Lemma 1 holds for this case.

Example 2. Consider a random vector $(Z_1, \ldots, Z_n)$ such that given $\Theta = \theta$ the conditional survival probability is specified by

$$\boldsymbol{P}\{Z_i > x_i, i = 1, \ldots, n | \Theta = \theta\} = \exp\Big(-\theta\lambda \sum_{1 \le i \le n} x_i\Big), \qquad x_i > 0, i \le n,$$

with some positive constant λ, and Θ being Gamma distributed with parameter $\alpha, 1$ (thus with mean α). Direct calculation shows that $(Z_1, \ldots, Z_n)$ has the multivariate Pareto distribution with survival function

$$\boldsymbol{P}\{Z_1 > x_1, \ldots, Z_d > x_n\} = \Big(1 + \lambda \sum_{1 \le i \le n} x_i\Big)^{-\alpha}, \quad x_i > 0, i \le n.$$

With R as in Example 1, applying Lemma 1 to $X_i = RZ_i, 1 \le i \le n$ we find that again (2) holds for this case.

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