ALBANIAN JOURNAL
OF MATHEMATICS
Volume 5, Number 1, Pages 3–10
ISSN 1930-1235: (2011)

AN UPPER BOUND FOR THE X-RANKS OF POINTS OF $\mathbb{P}^n$ IN POSITIVE CHARACTERISTIC

E. BALLICO

Abstract. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate m-dimensional variety. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ is the minimal cardinality of $S \subset X$ such that $P \in \langle S \rangle$. Here we study the pairs (X, P) such that $r_X(P) \geq n+2-m$, i.e. $r_X(P) = n+2-m$. These pairs exist only in positive characteristic, with X strange and P a strange point of X.

1. Introduction

Fix an integral and non-degenerate variety $X \subseteq \mathbb{P}^n$ defined over an algebraically closed field $\mathbb{K}$. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle \ \ \rangle$ denote the linear span. Hence $r_X(P) = 1$ if and only if $P \in X$. Since X is non-degenerate, the X-ranks are defined and $r_X(P) \leq n+1$ for all $P \in \mathbb{P}^n$. For any integer $r > 0$ let $\sigma_r^0(X) \subseteq \mathbb{P}^n$ denote the the union all $(r-1)$-dimensional linear spaces spanned by r points of X. Let $\sigma_r(X)$ denote the closure of $\sigma_r^0(X)$ in $\mathbb{P}^n$ (sometimes called the $(r-1)$-secant variety of X). The border X-rank of a point $P \in \mathbb{P}^n$ is the minimal integer r such that $P \in \sigma_r(X)$. Each $\sigma_r(X)$ is irreducible. An easy estimate gives that either $\sigma_r(X) = \mathbb{P}^n$ or $\dim(\sigma_{r+1}(X)) > \dim(\sigma_r(X))$ ([1], 1.2). Hence $\sigma_x(X) = \mathbb{P}^n$, where $x := n - \dim(X)$. Moreover, either $\sigma_{r+1}(X) = \mathbb{P}^n$ or $\dim(\sigma_{r+1}(X)) \geq 2 + \dim(\sigma_r(X))$ ([1], Corollary 1.4). Even if $\sigma_x(X) = \mathbb{P}^n$ there may be points with X-rank $> x$. The main concern of this paper is to extend the basic estimate $r_X(P) \leq n - \dim(X)$ made in [15], Proposition 5.1, in characteristic zero to the case $p := \mathrm{char}(\mathbb{K}) > 0$, listing some exceptional pairs (X, P) for which $r_X(P) = n - \dim(X) + 1$ (e.g. take $(n, m, p) = (2, 1, 1)$, as X a smooth conic and as P its strange point ([10], Example IV.3.8.2); in this example every line through P intersects X in a unique point and hence we need 3 points of X to span a linear space containing P).

It is believed that the concept of X-rank may be useful for "real world applications". In the applications when X is a Veronese embedding of $\mathbb{P}^m$ the X-rank is also called the "structured rank" (this is related to the virtual array concept encountered in sensor array processing ([2], [8])). On this topic there was the 2008 AIM workshop Geometry and representation theory of tensors for computer science, statistics and other areas. In [15] a book in preparation is quoted ([14]). Up to now the applied part was toward engineering. All theory was done in characteristic

1991 *Mathematics Subject Classification.* 14N05; 14H50.
Key words and phrases. ranks; strange variety; very strange curve; structured rank.
The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

zero. Our dream is to use these ideas together with specialists of computer algebra for real applications in coding theory. A preliminary step to fulfil this dream is to check the theory at least over an algebraically closed field with positive characteristic. Up to now the only general result on the X-rank (i.e. a result which does not use specific properties of very particular varieties X) is [15], Proposition 5.1. Hence its extension to positive characteristic seemed to be the first step needed to fulfil our dream. The aim of this paper is to prove that [15], Proposition 5.1, is not true in positive characteristic, but that it is "almost always true" and when it is not true it is "almost true" (it fails by +1). We also give a reasonable description of the projective varieties for which it is not true. The embedded variety $X \subseteq \mathbb{P}^n$ is said to be *strange* if there is $O \in \mathbb{P}^n$ such that $O \in T_QX$ (the embedded tangent space in $\mathbb{P}^n$) for all $Q \in X_{reg}$ (or, equivalently, for a general $Q \in X$) ([4]). If X is strange, a point as above is called a *strange point* of X. The set of all strange points of X is either empty or a linear subspace of dimension at most $\dim(X) - 1$ (unless $X = \mathbb{P}^n$). If $\mathrm{char}(\mathbb{K}) = 0$, then X is strange if and only if it is a cone and in this case the set of all strange points is its vertex (with the convention that a linear space is a cone with itself as its vertex). If X is strange with O as one of its strange points, but not a cone with vertex containing O, then $p := \mathrm{char}(\mathbb{K}) > 0$. If p is a large prime, then also $\deg(X)$ must be large (e.g. $\deg(X) \ge p(n-m)$) (see Proposition 3). We first prove the following result.

Theorem 1. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate m-dimensional variety. Fix $P \in \mathbb{P}^n$.*

(a) If P is not a strange point of X, then $r_X(P) \le n + 1 - m$.

(b) If P is a strange point of X, then $r_X(P) \le n + 2 - m$.

See Remark 4 for an example of an integral, non-degenerate and m-dimensional ($m \ge 2$) variety $X \subset \mathbb{P}^n$ with as strange points an $(m-1)$-dimensional linear space V and $r_X(P) = n - m + 2$ for all $P \in V \backslash N$, where N is a hyperplane of V and $N \subset X$.

The proof of Theorem 1 is very elementary. To prove Theorem 1 we just follow the proof of [15], Proposition 5.1 (the case $\mathrm{char}(\mathbb{K}) = 0$ of Theorem 1), analysing the only missing piece in positive characteristic (a use of Bertini's theorem). In the one-dimensional case we are able to improve Theorem 1. A non-degenerate curve $X \subset \mathbb{P}^n$ is said to be *very strange* if its general hyperplane section is not in linearly general position ([18]). A very strange curve is strange ([18], Lemma 1.1).

Definition 1. Let $X \subset \mathbb{P}^n$, $n \ge 2$, be a non-degenerate strange curve and let O be its strange point. Let $\ell_O : \mathbb{P}^n \backslash \{O\} \to \mathbb{P}^{n-1}$ be the linear projection from O and $T \subset \mathbb{P}^{n-1}$ the closure of $\ell_O(Y \backslash \{O\})$. Thus T is non-degenerate and

$$\deg(X) = p^e s \cdot \deg(T) + \mu, \tag{1}$$

where μ is the multiplicity of X at O, while s and p^e are the separable and the inseparable degree of $\ell_O|X$, respectively ([4], Theorem 2.3). Now assume $n \ge 3$, $\mu = 0$ (i.e. $O \notin X$) and $s = 1$. We say that X is *flat* or *flat with respect to its strange point* O or a *flat strange curve* if for any $S \subset X$ such that $\sharp(S) \le n$ we have $\dim(\langle S \rangle) = \dim(\langle \ell_O(S) \rangle)$.

The proofs that $e > 0$ in the set-up of Definition 1 and that (1) holds are given in [4], §2 (see [4], eq. (2.1.1) and Theorem 2.3); the integer p^e is shown to be equal to the intersection multiplicity of T_QX with X at Q, where Q is a general point of X

(the so-called Generic Order of Contact Theorem proved in [9], 3.5, for embedded varieties with arbitrary dimension). See [12] for a very useful survey. For related details, see the proof of Proposition 3.

Notice that if $\mu = 0$, then (1) gives $\deg(X) \equiv 0 \mod p$.

Remark 1. Take the set-up of Definition 1.

(a) Since a strange curve (not a line) has a unique strange point, the point O is uniquely determined by X. Hence we do not need to specify it to check if a strange curve is flat or not.

(b) The assumption $(\mu, s) = (0, 1)$ implies that $\ell_O|X$ is generically injective. Flatness implies that $\ell_O|X$ is injective, but it is far stronger. We have $r_X(O) \geq 2$ if and only if $O \notin X$. We have $r_X(O) \geq 3$ if and only if $O \notin X$ and $\ell_O|X$ injective. If $\mu = 0$, then the flatness of a strange curve is equivalent to $r_X(O) = n+1$ (use that $r_X(P) \leq n+1$ for any $P \in \mathbb{P}^n$ and any non-degenerate reduced subset $X \subset \mathbb{P}^n$ and that for any finite $S \subset X$ we have $\dim(\langle \ell_O(S) \rangle) < \dim(\langle S \rangle)$ if and only if $O \in \langle S \rangle$).

(c) Part (b) shows that the "if" part of the following theorem is just the definition of flatness of a strange curve. It also gives the "only if" part if we first prove that X is a strange point of X with invariants $(\mu, s) = (0, 1)$.

Theorem 2. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate curve and $P \in \mathbb{P}^n$. We have $r_X(P) \geq n+1$ (i.e. $r_X(P) = n+1$) if and only if X is a flat strange curve and P is the strange point of X.*

V. Bayer and A. Hefez gave explicit equations for all plane strange curves in terms of the invariants μ, s and p^e introduced in Definition 1 ([4]). Later we extended the construction to strange varieties with a fixed strange point O, fix integers μ, s, p^e and a fixed image $T \subset \mathbb{P}^{n-1}$ with respect to the linear projection from O ([3]). All strange curves X such that $O \notin X$, $s = 1$ and $\ell_O(X)$ is a rational normal curve (where O is the strange point of X) are flat (Proposition 2). These curves are explicitely described by one equation in a Hirzebruch surface F_{n-1} ([3]). The other flat strange curves are very strange (Proposition 1) and we know only one example of these flat curves (see Example 1, i.e. [18], Example 1.2). See Remark 2 for another reason to say that the flat curves X with $\ell_O(X)$ a rational normal curve are "almost maximally linearly independent from the set-theoretic point of view".

The topic considered in [15] is very active (see also [7], [6], [5] and references therein). We stress that [15] and the other quoted papers are over $\mathbb{C}$: none of their statements and proofs is affected by the examples given here.

2. Proofs and related results

Proof of Theorem 1. If $P \in X$, then $r_X(P) = 1$. Hence to prove parts (a) and (b) we may assume $P \notin X$. First assume $m = 1$. Assume $r_X(P) \geq n+1$. Hence for a general hyperplane H containing P the set $(X \cap H)_{red}$ does not span H. Since X is connected, the cohomology exact sequence of the exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{I}_X(1) \to \mathcal{I}_{X \cap H}(1) \to 0$$

gives that the scheme $X \cap H$ spans H. Thus $X \cap H$ is not reduced. Since $P \notin X$ and H is general among the hyperplanes containing P, $H \cap \mathrm{Sing}(X) = \emptyset$. Hence the non-reducedness of $X \cap H$ and the generality of H implies that X is a strange curve with P as its strange point. In the case $m = 1$ we have $r_X(P) \le n+1$ for all P, because X spans $\mathbb{P}^n$ proving parts (a) and (b) in the case $m = 1$.

Now assume $m \ge 2$ and that Theorem 1 is true for varieties of dimension $m-1$. Assume the existence of $P \in \mathbb{P}^n$ such that $r_X(P) \ge n+2-m$, but P is not a strange point of X. Fix a general hyperplane H containing P. Let $\ell_P : \mathbb{P}^n \backslash \{P\} \to \mathbb{P}^{n-1}$ be the linear projection from P. Since $P \notin X$, $\ell_P|X$ is a finite morphism. Bertini's theorem gives that $X \cap H$ is geometrically integral ([11], part 4) of Th. I.6.3). Fix a general $Q \in (X \cap H)_{reg}$. For general H we may take as Q a general point of X. Hence $P \notin T_QX$. Hence $P \notin (T_QX) \cap H = T_Q(X \cap H)$. Thus P is not a strange point of $X \cap H$. The inductive assumption gives $r_{X\cap H}(P) \le (n-1)-(m-1)+1 = n-m+1$. Since $r_X(P) \le r_{X\cap H}(P)$, we proved part (a) for all m, X, P.

Now assume that P is a strange point of X. Since we proved part (b) in the case $m = 1$, we may assume $m \ge 2$. Fix an integer $k \ge 3$ and a general $Q \in X_{reg}$. Let Y be the intersection of X with a general degree k hypersurface W such that $Q \in W$. The scheme $Y \backslash \{Q\}$ is geometrically integral by the characteristic free version of Bertini's theorem for very ample linear systems on non-complete varieties ([11], part 4) of Th. I.6.3). Since $k \ge 3$, it is easy to find W such that $Y = X \cap W$ is smooth at Q. Hence Y is geometrically integral and $Q \in Y_{reg}$. Since $k \ge 3$, we may find W as above such that $P \notin T_QW$. Hence $P \notin T_QW \cap T_QX = T_QY$. Hence P is not a strange point of Y. Part (a) applied to Y gives $r_X(P) \le r_Y(P) \le n-(m-1)+1$. □

Proof of Theorem 2. By part (c) of Remark 1 it is sufficient to prove the "only if" part. Fix X, P such that $r_X(P) \ge n+1$. The case $m = 1$ of Theorem 1 implies $r_X(P) = n+1$ and that P is a strange point of X. Call μ, s and p^e the invariants of X with respect to the linear projection ℓ_P from P. Since $r_X(P) \ge 2$, $P \notin X$, i.e. $\mu = 0$. Notice that $s = 1$ if and only if $\ell_P|X$ has separablle degree 1, i.e. it is generically injective. Since $r_X(P) \ge 3$, we have $\sharp((X \cap D)_{red}) \le 1$ for every line D such that $P \in D$. Thus $\ell_P|X$ in injective. Thus $s = 1$. As observed in part (c) of Remark 1 if $(\mu, s) = (0, 1)$ and P is the strange point of X, then the definition of flatness is equivalent to $r_X(P) \ge n+1$. □

Proposition 1. *Let $X \subset \mathbb{P}^n$, $n \ge 3$, be a non-degenerate and flat strange curve with O as its strange point. Then either X is very strange or $\ell_O(X)$ is a rational normal curve.*

Proof. Let O be the strange point of X. Set $d := \deg(\ell_O(X))$. If $d = n-1$, then $\ell_O(X)$ is a rational normal curve. Now assume $d \ge n$. By assumption $\mu = 0$ and $s = 1$. Fix a general $S \subset X$ such that $\sharp(S) = n-1$. Hence $\sharp(\ell_O(S)) = n-1$ and $\ell_O(S)$ spans a hyperplane of $\mathbb{P}^{n-1}$. Since $d \ge n$, there is $U \in \ell_O(X)\backslash\ell_O(S)$ such that $U \in \langle \ell_O(S)\rangle$. Fix $V \in X$ such that $\ell_O(V) = U$. Hence $\sharp(S \cup \{V\}) = n$. Since X is flat, $V \in \langle S\rangle$. Since this is true for a general $S \subset X$ such that $\sharp(S) = n-1$, X satisfies the definition of a very strange curve. □

Proposition 2. *Let $X \subset \mathbb{P}^n$, $n \ge 2$, be a non-degenerate and strange curve with O as its strange point and invariants $\mu = 0$ and $s = 1$, i.e. assume $O \notin X$ and that $\ell_O|X$ is generically injective. If either $n = 2$ or $\ell_O(X)$ is a rational normal curve of $\mathbb{P}^{n-1}$ (i.e. if $\deg(X) = (n-1)p^e$, where p^e is the inseparable degree of $\ell_O|X$), then X is flat.*

Proof. Fix $S \subset X$ such that $\sharp(S) \le n$. Let $u : C \to X$ be the normalization map. By assumption $\ell_O(X) \cong \mathbb{P}^1$ (even if $n = 2$). Since $\ell_O|X : X \to T \cong \mathbb{P}^1$ is purely inseparable, $C \cong \mathbb{P}^1$. Since $s = 1$, the morphism $\ell_O|X \circ u : \mathbb{P}^1 \to \mathbb{P}^1$ is purely inseparable. Hence it is injective. Thus the morphism $\ell_O|X$ is injective, not just generically injective. Hence $\sharp(\ell_O(S)) = \sharp(S) \leq n$. Since any n points of a rational normal curve of $\mathbb{P}^{n-1}$ are linearly independent, we get $\dim(\langle \ell_O(S) \rangle) = \sharp(S) - 1$. □

Remark 2. Take X as in Proposition 2. The proof of Proposition 2 gives that every $S \subset X$ such that $\sharp(S) \leq n$ is linearly independent, i.e. X has no codimension 2 multisecant linear subspace from the set-theoretical point of view (but of course every tangent line of X at one of its smooth points contains a length p^e subscheme of X). We stress again that all curves X as in Proposition 2 are explicitely constructed in [3]. The rational normal curves of $\mathbb{P}^n$ are the only integral curves for which no hyperplane contains $n + 1$ points of the curve, i.e. for which the reduction of every codimension 1 linear section is linearly independent.

Example 1. Here we check that the example of a very strange curve given in [18], Example 1.2, is a flat strange curve. Fix an integer $n \geq 3$, a prime p and a p-power q. Here $q = p^e$ is the inseparable degree of the linear projection from the strange point. Fix homogeneous coordinates $x_0, \dots, x_n$ of $\mathbb{P}^n$ and homogeneous coordinates $x_1, \dots, x_n$ of $\mathbb{P}^{n-1}$. Set $A := (0; \dots ; 0; 1; 0)$ and $O := (1; 0; \dots ; 0; 0)$. We recall that every point of the vertex of a cone T is a strange point of T. An integral hypersurface $\{f(x_0, \dots , x_n) = 0\}$ has O as one of its strange points if and only if in each monomial of f with a non-zero coefficient the variable x_0 appears with exponent divisible by p. Let X be the scheme with equations $x_0^q - x_1x_n^{q-1}, x_1^q - x_2x_n^{q-1}, \dots , x_{n-2}^q - x_{n-1}x_n^{q-1}$. The point O is a strange points of the $n-1$ hypersurfaces with these equations (the latter $n-2$ hypersurfaces are cones with vertex containing O). Set $X' := X \cap \{x_n \neq 0\}$. We have $(X \cap \{x_n = 0\})_{red} = \{A\}$. Since X is given by $n - 1$ equations, each irreducible component of X_{red} has dimension at least 1. Hence A is in the closure of X'. Set $t := x_0/x_n$. The scheme $(X')_{red}$ has a rational parametrization

$$t \mapsto (t, t^q, t^{q^2}, \dots , t^{q^{n-1}}), \tag{2}$$

because in X' we have $x_i/x_n = (x_{i-1}/x_n)^q$ for every $i \in \{1, \dots , n - 1\}$. Hence $(X')_{red}$ is integral, smooth, rational and its closure X_{red} in $\mathbb{P}^n$ has O as its strange point. Since $\deg(X_{red}) = q^{n-1}$ and X_{red} is set-theoretically the intersection of $n - 1$ hypersurfaces of degree q, the algebraic set X_{red} is the complete intersection of these hypersurfaces, outside finitely many points. Hence the scheme X is a complete intersection and it is reduced outside finitely many points. Since X is a complete intersection, each local ring $\mathcal{O}_{X,Q}$, $Q \in X_{red}$, is Cohen-Macaulay. Hence X has no embedded component and it is generically reduced. Thus it is reduced. We have $O \notin X$. Set $Y := \ell_O(X) \subset \mathbb{P}^{n-1}$, $Y' := Y \cap \{x_n \neq 0\}$ and $A' := (0; \dots ; 1; 0) = \ell_O(A) \in Y$. Since $\ell_O((t; t^q; \dots ; t^{q^{n-1}}; 1)) = (t^q; \dots ; t^{q^{n-1}}; 1)$ for all $t \in \mathbb{K}$, the curve Y' has a parametrization

$$z \mapsto (z, z^q, \dots , z^{q^{n-2}}), \tag{3}$$

where $z = t^q$. Hence $\ell_O|X' : X' \to Y'$ is injective and purely inseparable with inseparable degree q. Thus X has parameters $(\mu, s, p^e) = (0, 1, q)$. The parametrization (3) shows that Y' is smooth, that Y is strange with $O'' := (1; 0; \dots ; 0; 0)$ as its strange point and that $Y \backslash Y' = \{A'\}$. Fix linearly independent $P_1, \dots , P_n \in X'$ and set $S := \{P_1, \dots , P_n\}$ and $M := \langle S \rangle$. The parametrization (2) shows that $(M \cap X')_{red} = \{P_1 + a_1(P_2 - P_1) + \cdots + a_{n-1}(P_n - P_1)\}$, where each a_i is an arbitrary element of $\mathbb{F}_q$. Since $\sharp((M \cap X')_{red}) = q^{n-1} = \deg(X)$, we get that this is a scheme-theoretic intersection and that $M \cap (X \setminus X') = \emptyset$. Since $M \cap X = (M \cap X')_{red}$ scheme-theoretically and $O \in T_{P_i}X$, we have $O \notin M$, i.e. $\dim(\ell_O(M)) = n-1$. Recall that $X \setminus X' = \{A\}$. Fix $S_1 \subset X$ such that $\sharp(S_1) = n$, $A \in S_1$ and S_1 is linearly independent. Let M_1 be the hyperplane spanned by S_1. Set $S_2 := S_1 \setminus \{A\}$ and write $S_2 := \{P_1, \dots, P_{n-1}\}$. Set $Q_i := \ell_O(P_i)$, $1 \le i \le n-1$. We proved that $\sharp(\ell_O(S_2)) = n-1$, $\ell_O(S_2) \subset Y'$ and that $\ell_O(S_2)$ is linearly independent. Set $M_2 := \langle \ell_O(S_2) \rangle$. Since $A' = \ell_O(A)$, to conclude the proof of the flatness of X it is sufficient to prove $A' \notin M_2$. Let $E \subset \mathbb{P}^{n-1}$ the set $\{Q_1 + a_1(Q_2 - Q_1) + \dots + a_{n-2}(Q_{n-1} - Q_1)\}$, where each a_i is an arbitrary element of $\mathbb{F}_q$. Since $P_1 + a_1(P_2 - P_1) + \dots + a_{n-2}(P_{n-1} - P_1) \in X'$ for all $a_i \in \mathbb{F}_q$, we have $E \subseteq M_2 \cap \ell_O(X')$. Since $\ell_O|X'$ is injective, we have $\sharp(E) = q^{n-2} = \deg(Y)$. Thus $E = M_2 \cap Y$ and $(Y \setminus \ell_O(X')) \cap M_2 = \emptyset$. Since $\{A'\} = Y \setminus \ell_O(X')$, we get $A' \notin M_2$. Thus X is flat.

Remark 3. A theorem of Luiss' says that there is a unique smooth strange curve (if we exclude the lines): a smooth plane conic in characteristic 2 ([13], Proposition 3, or [10], Theorem IV.3.9). If $p = 2$ a smooth plane conic is obviously flat. This example shows that if $n = 2$ and $p = 2$ the ranks of the rational normal curves of $\mathbb{P}^n$ are not as in characteristic zero (see [7], [15], 4.1, or [5], 3.1). This phenomenon does not occur when $n = 3$. Let $C \subset \mathbb{P}^3$ be a rational normal curve. Let $TC := \cup_{Q \in C} T_QC \subset \mathbb{P}^3$ denote the tangent developable of C. If $P \in C$, then $r_C(P) = 1$. If $P \notin TC$, then $r_C(P) = 2$, because $\mathbb{P}^3$ is the secant variety of C ([1], Remark 1.6). Fix $P \in TC \setminus C$, say $P \in T_QC \setminus \{Q\}$ with $Q \in C$. Assume $r_C(P) = 2$ and take $P_1, P_2 \in C$ such that $P_1 \ne P_2$ and $P \in \langle \{P_1, P_2\} \rangle$. Since any length 3 scheme $Z \subset C$ spans a plane, $Q \notin \langle \{P_1, P_2\} \rangle$. Since $P \in T_QC \cap \langle \{P_1, P_2\} \rangle$, the linear space $M := \langle T_QC \cup \{P_1, P_2\} \rangle$ is a plane and $\text{length}(M \cap C) \ge 4$. Since $\deg(C) \ge 3$, we get a contradiction. Hence $r_C(P) \ge 3$. Since C is not strange, Theorem 1 gives $r_C(P) = 3$. Hence the stratification by ranks of C is the same as in characteristic zero.

Fix an integer $m \ge 2$. Here we construct m-dimensional examples of pairs (X, P) such that $r_X(P) = n + 2 - m$, i.e. such that the inequality in part (b) of Theorem 1 is an equality. Just taking cones we get an m-dimensional example from any one-dimensional example with the same codimension in an ambient projective space. This is the only example we know of pairs (X, P) with $m \ge 2$ and $r_X(P) = n+2-m$, i.e. a pair for which part (b) of Theorem 1 is sharp. Are there other examples?

Remark 4. Fix integers $n > m \ge 2$, an $(n-m+1)$-dimensional linear subspace M of $\mathbb{P}^n$ and an $(m-2)$-dimensional linear subspace N of $\mathbb{P}^n$ such that $M \cap N = \emptyset$, i.e. a complementary subspace. For any variety $Y \subset M$ let $C(N, Y) \subset \mathbb{P}^n$ denote the cone with vertex N and Y as its basis. Hence for each $O \in M$ the scheme $C(N, O)$ is an $(m-1)$-dimensional linear subspace of $\mathbb{P}^n$. We claim that $r_{C(N,Y)}(P) = r_Y(O)$ for every $P \in C(N, O) \setminus N$. Fix $P \in C(N, O) \setminus N$. Take an $(n-m+1)$-dimensional linear subspace M' of $\mathbb{P}^n$ such that $P \in M'$ and $N \cap M' = \emptyset$. The linear projection from N induces an isomorphism of pairs $(C(N, Y) \cap M', P) \cong (Y, O)$ as pairs of subvarieties, respectively of M' and of M. Thus $r_{C(N,Y)}(P) \le r_{C(N,Y) \cap M'}(P) = r_Y(O)$. To prove the reverse inequality we fix $P \in C(N, O)$ and $S \subset C(N, Y)$ computing $r_{C(N,Y)}(P)$. The image $S' \subset M$ of the linear projection of S from N is a set such that $\sharp(S') \le \sharp(S) = r_{C(N,Y)}(P)$. Since $O \in \langle S' \rangle$, we get $r_Y(O) \le \sharp(S') \le r_{C(N,Y)}(P)$. Taking as Y a flat curve with strange point O, $X = C(N, Y)$ and

$V = C(N, O)$ we get the existence (for all $n > m \geq 2$) of an integral, non-degenerate and m-dimensional variety $X \subset \mathbb{P}^n$ with as set of its strange points an $(m-1)$-dimensional linear space V and $r_X(P) = n - m + 2$ for all $P \in V \backslash N$, where N is an $(m-2)$-dimensional linear space and $N \subset X$.

Proposition 3. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate m-dimensional variety. Fix $O \in \mathbb{P}^n$ and assume that O is a strange point of X, but that X is not a cone with vertex containing O. Then* $\deg(X) \geq p \cdot (n - m)$.

Proof. Fix $A \in \mathbb{P}^n \backslash \{O\}$ and take any integral quasi-projective variety $E \subseteq \mathbb{P}^n \backslash \{O\}$ such that $A \in E_{reg}$. Set $x := \dim(E)$. The inclusion $j : E \subseteq \mathbb{P}^n$ induces an inclusion between the abstract tangent spaces $\Theta_{E,A}$ of E at A and the abstract tangent space $\Theta_{\mathbb{P}^n,A}$ of $\mathbb{P}^n$ at A. As usual in projective geometry we "complete" these vector spaces $\Theta_{E,A}$ and $\Theta_{\mathbb{P}^n,A}$ to projective spaces, respectively of dimension x and n, and call them T_AE and $T_A\mathbb{P}^n = \mathbb{P}^n$. Since $A \neq 0$, the submersion $\ell_O : \mathbb{P}^n \backslash \{O\} \to \mathbb{P}^{n-1}$ induces a linear surjective map of $\mathbb{K}$-vector spaces $\rho_O(A) : \Theta_{\mathbb{P}^n,A} \to \Theta_{\mathbb{P}^{n-1},\ell_O(A)}$. Since $\rho_O(A)$ is surjective, its kernel is one-dimensional. If we identify $\Theta_A\mathbb{P}^n$ with an affine n-dimensional open subset of $T_A\mathbb{P}^n = \mathbb{P}^n$, then the closure of this kernel is the line $\langle \{O, A\} \rangle$ (in the case $x = 1$, see [13], lines 3–4 of p. 215). Thus the differential of $\ell_O|E$ at A is injective if and only if $O \notin T_AE$. Thus the differential of $\ell_O|E$ at a general point of E is injective if and only if the closure $\overline{E} \subseteq \mathbb{P}^n$ of E is not strange with O as one of its strange points.

Let $T \subset \mathbb{P}^{n-1}$ denote the closure of $\ell_O(X \backslash \{O\})$. Since X is not a cone with vertex containing O, $\ell_O|X \backslash \{O\}$ is a generically finite morphism. Hence $\dim(T) = m$. Since T spans $\mathbb{P}^{n-1}$, we have $\deg(T) \geq n - m$. Since $\ell_O|X \backslash \{O\}$ is generically finite, the function field $K(X)$ of X is a finite extension of the function field $K(T)$. Since O is a strange point of X, this extension of fields is not separable (use the geometric interpretation of $\rho_O(A)$ just given and the differential criterion of separability, i.e. [17], Theorem 26.6, or [16], Th. 59 at p. 191, quoted in [10], Theorem II.8.6). Call p^e, $e \geq 1$, the inseparable degree of this extension of fields. A general fiber of $\ell_O|X \backslash \{O\}$ is a disjoint union of finitely many connected zero-dimensional schemes, each of them with degree p^e. Hence $\deg(X) \geq p^e \cdot \deg(T) \geq p(n - m)$. □

In the set-up of Proposition 3 if $O \in X$, then $\deg(X) > p \cdot (n - m)$. Proposition 3 is very weak, but we are unable to make a substantial improvement of it. In the case of a strange curve X the formula (1) relates $\deg(X)$ to other data. Nothing more can be said in the one-dimensional case. Indeed, the construction of [3] shows that we may take an arbitrary T spanning $\mathbb{P}^{n-1}$ and then find a solution X with arbitrary $e \geq 1$ and $\mu \geq 0$. Formula (1) is very useful to check if a curve X is strange. We observed after Definition 1 that if $\deg(X)/p \notin \mathbb{Z}$, then either X is not strange or its strange point belongs to X. If X is strange, we also see that the image curve T has much lower degree and hence it should be easier.

It seems to be very difficult to construct very strange curves. We know only the examples given in [18]. We expect that if they exist, then they have very large degree, at least p^{n-1} in $\mathbb{P}^n$.

References

[1] B. Ådlandsvik, Joins and higher secant varieties, Math. Scand. 62 (1987), 213–222.

[2] L. Albera, P. Chevalier, P. Comon and A. Ferreol, On the virtual array concept for higher order array processing, IEEE Trans. Sig. Proc., 53(4):1254–1271, April 2005.

[3] E. Ballico, On strange projective curves, Rev. Roum. Math. Pures Appl. 37 (1992), 741–745.
[4] V. Bayer and A. Hefez, Strange plane curves, Comm. Algebra 19 (1991), no. 11, 3041–3059.
[5] A. Bernardi, A. Gimigliano and M. Idà, On the stratification of secant varieties of Veronese varieties via symmetric rank. J. Symbolic. Comput. 46 (2011), 34–55.
[6] J. Buczyński and J. M. Landsberg, Ranks of tensors and a generalization of secant varieties, arXiv:0909.4262v1 [math.AG].
[7] G. Comas and M. Seiguer, On the rank of a binary form, arXiv:math.AG/0112311.
[8] P. Comon, G. Golub, L.-H. Lim, and B. Mourrain, Symmetric tensors and symmetric tensor rank, SIAM Journal on Matrix Analysis Appl., 30(3):1254-1279, 2008.
[9] A. Hefez and S. L. Kleiman, Notes on the duality of projective varieties, Geometry today (Rome, 1984), 143–183, Progr. Math., 60, Birkhäuser Boston, Boston, MA, 1985.
[10] R. Harshorne, Algebraic Geometry, Springer, Berlin, 1977.
[11] J.-P. Jouanolou, Théorèmes de Bertini et applications, Progress in Mathematics, 42. Birkhäuser Boston, Inc., Boston, MA, 1983.
[12] S. L. Kleiman, Tangency and duality, Proceedings of the 1984 Vancouver conference in algebraic geometry, 163–225, CMS Conf. Proc., 6, Amer. Math. Soc., Providence, RI, 1986.
[13] D. Laksov, Indecomposability of restricted tangent bundles, in: Young tableaux and Schur functors in algebra and geometry (Toruń, 1980), pp. 221–247, Astérisque 87–88, Soc. Math. France, Paris, 1981.
[14] J. M. Landsberg and J. Morton, The geometry of tensors: applications to complexity, statistics and engineering, book in preparation.
[15] J. M. Landsberg and Z. Teitler, On the ranks and border ranks of symmetric tensors. Found. Comput. Math. (2010) **10**: 339–366.
[16] H. Matsumura, Commutative Algebra, W. A. Benjamin Co., New York, 1970.
[17] H. Matsumura, Commutative ring theory, Cambridge University Press, Cambridge, 1986.
[18] J. Rathmann, The uniform position principle for curves in characteristic p, Math. Ann. 276 (1987), no. 4, 565–579.

Dept. of Mathematics, University of Trento, 38050 Povo (TN), Italy
E-mail address: `ballico@science.unitn.it`