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THREE-STEP PROJECTION METHODS FOR NONCONVEX VARIATIONAL INEQUALITIES

MUHAMMAD ASLAM NOOR

ABSTRACT. It is well-known that the nonconvex variational inequalities are equivalent to the fixed point problems. We use this equivalent formulation to suggest and analyze some three-step iterative methods for solving the nonconvex variational inequalities. We prove the convergence of the three-step iterative methods under suitable weaker conditions. Several special cases are also discussed. Our method of proof is very simple.

1. Introduction

Variational inequalities theory, which was introduced by Stampacchia [1], provides us with a simple, general and unified framework to study a wide class of problems arising in pure and applied sciences. For the applications, physical formulation, numerical methods and other aspects of variational inequalities, see [1-15] and the references therein. It is worth mentioning that all the research work carried our in this direction assumed that the underlying set is a convex set. In many practical problems, a choice set may not be a convex so that the existing results may not be applicable. In this direction, Noor [8] has introduced and considered a new class of variational inequalities, called nonconvex variational inequalities on the uniformly prox-regular sets. It is well-known that the uniformly prox-regular sets are nonconvex and include the convex sets as a special case, see [3,12]. Using the projection operator, Noor [8] has established the equivalence between the nonconvex variational inequalities and the fixed point problem. This equivalent formation has been used to consider the existence theory as well as to develop some numerical methods for nonconvex variational inequalities. We would like to point that the convergence analysis for the Mann and Ishkawa iterative methods requires that the operator must be strongly monotone and Lipschitz continuous. These conditions are very strict and rule out many applications. To overcome these drawbacks, several modifications of the projection iterative methods have been analyzed in recent years, see [6,9,13,15] and the references therein. Inspired and motivated by the research going on in this interesting and fascinating field, we suggest and analyze three-step iterative methods for solving the nonconvex variational inequalities. Using the technique of Noor [6,13,15], we also consider the convergence criteria of

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three-step iterative method for the partially relaxed strongly monotone operator. It is well known that the partially relaxed strongly monotonicity implies monotonicity, but the converse is not true. This shows that the partially relaxed strongly monotonicity is a weaker condition than monotonicity. We remark that our proof of the convergence analysis is independent of the projection. Consequently, our results represent a refinement of the previously known results. Several special cases are also considered.

2. Basic Concepts

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|.\|$ respectively. Let K be a nonempty closed convex set in H. The basic concepts and definitions used in this paper are exactly the same as in Noor [8].

Definition 2.1. The proximal normal cone of K at $u \in H$ is given by

$$N_K^P(u) := \{\xi \in H : u \in P_K[u+\xi]\}.$$

where

$$P_K[u] = \{u^* \in K : d_K(u) = \|u-u^*\| = \inf_{v\in K}\|v-u\|\}.$$

The proximal normal cone $N_K^P(u)$ has the following characterization.

Lemma 2.1. Let K be a nonempty, closed and convex subset in H. Then $\zeta \in N_K^P(u)$ if and only if there exists a constant $\alpha > 0$ such that

$$\langle\zeta, v-u\rangle \le \alpha\|v-u\|^2, \quad \forall v \in K.$$

Poliquin et al. [14] and Clarke et al [3] have introduced and studied a new class of nonconvex sets, which are called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions.

Definition 2.2. For a given $r \in (0,\infty]$, a subset K_r is said to be normalized uniformly r-prox-regular if and only if every nonzero proximal normal to K_r can be realized by an r-ball, that is,$\forall u \in K_r$ and $0 \neq \xi \in N_{K_r}^P(u)$, one has

$$\langle(\xi)/\|\xi\|, v-u\rangle \le (1/2r)\|v-u\|^2, \quad \forall v \in K_r.$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see [2,3,12]. Obviously, for $r = \infty$, the uniformly prox-regularity of K_r is equivalent to the convexity of K. This class of uniformly prox-regular sets have played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions. It is known that if K_r is a uniformly prox-regular set, then the proximal normal cone $N_{K_r}^P(u)$ is closed as a set-valued mapping.

We now recall the well known proposition which summarizes some important properties of the uniformly prox-regular sets K_r.

Lemma 2.2. Let K be a nonempty closed subset of H, $r \in (0, \infty]$ and set $K_r = \{u \in H : d_K(u) < r\}$. If K_r is uniformly prox-regular, then
(i) $\forall u \in K_r, P_{K_r}(u) \neq \emptyset$.
(ii) $\forall r^{'} \in (0, r)$, P_{K_r} is Lipschitz continuous with constant $\frac{r}{r-r'}$ on $K_{r'}$.

For a given nonlinear operator T, we consider the problem of finding $u \in K_r$ such that

$$\langle Tu, v-u \rangle \geq 0, \quad \forall v \in K_r, \tag{1}$$

which is called the *nonconvex variational inequality,* introduced and studied by Noor [8].

We now give some examples of prox-regular sets to give an idea and applications of the nonconvex variational inequalities (1). These examples are mainly due to Noor [10].

Example 2.1. Let $u = (x, y)$ and $v = (t, z)$ belong to the real Euclidean plane and consider $Tu = (2x, 2(y-1))$. Let $K = \{t^2 + (z-2)^2 \geq 4, \quad -2 \leq t \leq 2, \quad z \geq -2\}$ be a subset of the Euclidean plane. Then one can easily show that the set K is a prox-regular set K_r. It is clear that nonconvex variational inequality (1) has no solution.

Example 2.2. Let $u = (x, y) \in R^2$, $v = (t, z) \in R^2$ and let $Tu = (-x, 1-y)$. Let the set K be the union of 2 disjoint squares, say A and B having respectively, the vertices in the points $(0, 1), (2, 1), (2, 3), (0, 3)$ and in the points $(4, 1), (5, 2), (4, 3), (3, 2)$.
The fact that K can be written in the form:

$$\big\{(t, z) \in R^2 : \max\{|t-1|, |z-2|\} \leq 1\} \cup \{|t-4| + |z-2| \leq 1\}\big\}$$

shows that it is a prox-regular set in R^2 and the nonconvex variational inequality (1) has a solution on the square B. We note that the operator T is the gradient of a strictly concave function. This shows that the square A is redundant.

We note that, if $K_r \equiv K$, the convex set in H, then problem (1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u \rangle \geq 0, \qquad \forall v \in K. \tag{2}$$

Inequality of type (2) is called the *variational inequality*, which was introduced and studied by Stampacchia [1] in 1964. It turned out that a number of unrelated obstacle, free, moving, unilateral and equilibrium problems arising in various branches of pure and applied sciences can be studied via variational inequalities, see [1-13] and the references therein.

If K_r is a nonconvex (uniformly prox-regular) set, then problem (1) is equivalent to finding $u \in K_r$ such that

$$0 \in Tu + N^P_{K_r}(u) \tag{3}$$

where $N^P_{K_r}(u)$ denotes the normal cone of K_r at u in the sense of nonconvex analysis. Problem (3) is called the nonconvex variational inclusion problem associated with nonconvex variational inequality (1). This equivalent formulation plays a crucial and basic part in this paper. We would like to point out this equivalent formulation allows us to use the projection operator technique for solving the nonconvex variational inequalities of the type (1).

Definition 2.3. An operator $T : H \to H$ is said to be *partially relaxed strongly monotone,* iff, there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, z - v\rangle \geq -\alpha\|u - z\|^2, \quad \forall u, v, z \in H.$$

Note that for $z = u$, partially relaxed strongly monotonicity reduces to monotoncity. It is well known that the cocoercivity implies partially relaxed strongly monotonicity, but, the converse is not true, see Noor [6,13].

3. Main Results

It is known [8] that the nonconvex variational inequalities (1) are equivalent to the fixed point problem. We recall this result.

Lemma 3.1[8]. $u \in K_r$ is a solution of the nonconvex variational inequality (1) if and only if $u \in K_r$ satisfies the relation

$$u = P_{K_r}[u - \rho Tu], \tag{4}$$

where $\rho > 0$ is a constant and P_{K_r} is the projection of H onto the uniformly prox-regular set K_r.

Lemma 2.1 implies that (1) is equivalent to the fixed point problem (4). This alternative equivalent formulation is very useful from the numerical and theoretical points of view. Noor [7,8] has used this equivalent formulation to discuss the existence of a solution of the nonconvex variational inequality (1). Using the fixed point formulation (4), we suggest and analyze some iterative methods for solving the nonconvex variational inequality (1).

Algorithm 3.1. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$u_{n+1} = P_{K_r}[u_n - \rho Tu_n], \quad n = 0, 1, \ldots,$$

where $\rho > 0$ is a constant. For the convergence analysis of Algorithm 3.1, see Noor [8].

On can also suggest the following implicit method for solving the nonconvex variational inequality (1) as:

Algorithm 3.2. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$u_{n+1} = P_{K_r}[u_n - \rho Tu_{n+1}], \quad n = 0, 1, \ldots,$$

Noor[7] has studied the convergence analysis of Algorithm 3.2 for the pseudomonotone operator. We remark that Algorithm 3.2 is equivalent to the following iterative method

Algorithm 3.3. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned} u_{n+1} &= P_{K_r}[u_n - \rho T w_n] \\ w_n &= P_{K_r}[u_n - \rho T u_n], \quad n = 0, 1, \dots, \end{aligned}$$

which is known as the extragradient method, see Noor [10].

We now use the technique of updating the solution to rewrite the fixed-point formulation as:

$$\begin{aligned} w &= P_{K_r}[u - \rho T u] \\ y &= P_{K_r}[w - \rho T w] \\ u &= P_{K_r}[y - \rho T y], \end{aligned}$$

where $\rho > 0$ is a constant.

This is another different fixed point formulation of the nonconvex variational inequality (1). This alternative fixed-point formulation enables us to suggest the following iterative methods for solving the nonconvex variational inequality (1).

Algorithm 3.4. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned} w_n &= P_{K_r}[u_n - \rho T u_n] \\ y_n &= P_{K_r}[w_n - \rho T w_n] \\ u_{n+1} &= P_{K_r}[y_n - \rho T y_n], \quad n = 0, 1, 2, ..., \end{aligned}$$

Algorithm 3.4 is called the three-step iterative method and can also be considered as an predictor-corrector methods for solving (1). We rewrite Algorithm 3.4 in the following equivalent form which plays a key role in the analysis of the convergence of Algorithm 3.4.

Algorithm 3.5 For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$\langle \rho T u_n + w_n - u_n, v - w_n \rangle \geq 0, \quad \forall v \in K_r \tag{5}$$

$$\langle \rho T w_n + y_n - w_n, v - y_n \rangle \geq 0, \quad \forall v \in K_r \tag{6}$$

$$\langle \rho T y_n + u_{n+1} - y_n, v - u_{n+1} \rangle \geq 0, \quad \forall v \in K_r \tag{7}$$

We now consider the convergence analysis of Algorithm 3.4 and this is the main motivation of our next result.

Theorem 3.1. Let $u \in K_r$ be a solution of (1) and let u_{n+1} be the approximate solution obtained from Algorithm 3.4. If the operator T is partially relaxed strongly monotone with constant $\alpha > 0$, then

$$\|u_{n+1} - u\|^2 \leq \|y_n - u\|^2 - (1 - 2\alpha\rho)\|u_{n+1} - y_n\|^2 \tag{8}$$

$$\|y_n - u\|^2 \leq \|w_n - u\|^2 - (1 - 2\alpha\rho)\|w_n - y_n\|^2 \tag{9}$$

$$\|w_n - u\|^2 \leq \|u_n - u\|^2 - (1 - 2\alpha\rho)\|w_n - u_n\|^2. \tag{10}$$

Proof. Let $u \in K_r$ be solution of (1). Then

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in K_r. \tag{11}$$

Take $v = w_n$ in (11), we have

$$\langle Tu, w_n - u\rangle \geq 0. \tag{12}$$

Taking $v = u$ in (5) and using (12), we have

$$\langle w_n - u_n, u - w_n\rangle \geq \rho\langle Tu_n - Tu, w_n - u\rangle \geq -\alpha\rho\|u_n - w_n\|^2, \tag{13}$$

since T is partially relaxed strongly monotone with constant $\alpha > 0$.

From (13), we have

$$\|w_n - u\|^2 \leq \|u_n - u\|^2 - (1 - 2\alpha\rho)\|w_n - u_n\|^2,$$

the required result (10).

Now taking $v = u_{n+1}$ in (11), we have

$$\langle Tu, u_{n+1} - u\rangle \geq 0. \tag{14}$$

Taking $v = u$ in (7), we have

$$\langle \rho Ty_n + u_{n+1} - y_n, u - u_{n+1}\rangle \geq 0. \tag{15}$$

From (15), (14) and using the partially relaxed strongly monotonicity T with constant $\alpha > 0$, we have

$$\|u_{n+1} - u\|^2 \leq \|y_n - u\|^2 - (1 - 2\alpha\rho)\|u_{n+1} - y_n\|^2,$$

the required result (8).

Taking $v = y_n$ in (11), we have

$$\langle Tu, y_n - u\rangle \geq 0. \tag{16}$$

Setting $v = u$ in (6), we have

$$\langle \rho Tw_n + y_n - w_n, u - y_n\rangle \geq 0. \tag{17}$$

From (17), (16) and using the partially relaxed strongly monotonicity of T, we have

$$\|y_n - u\|^2 \leq \|w_n - u\|^2 - (1 - \alpha\rho)\|y_n - w_n\|^2,$$

which is the required (9). □

Theorem 3.2. Let $u \in K_r$ be a solution of (1) and let u_{n+1} be the approximate solution obtained from Algorithm 3.4. If H is a finite dimensional space and $0 < \rho < \frac{1}{2\alpha}$, then $\lim_{n\to\infty} u_n = u$.

Proof. Let $\bar{u} \in K_r$ be a solution of (1). Then, the sequences $\{\|u_n - \bar{u}\|\}$ is nonincreasing and bounded and

$$\begin{aligned}
\sum_{n=0}^{\infty}(1 - 2\alpha\rho)\|u_{n+1} - w_n\|^2 &\leq \|y_0 - u\|^2 \\
\sum_{n=0}^{\infty}(1 - 2\alpha\rho)\|w_n - y_n\|^2 &\leq \|w_0 - u\|^2 \\
\sum_{n=0}^{\infty}(1 - 2\alpha\rho)\|w_n - u_n\|^2 &\leq \|u_0 - u\|^2,
\end{aligned}$$

which implies

$$\lim_{n\to\infty}\|u_{n+1} - w_n\| = 0 \quad \lim_{n\to\infty}\|w_n - y_n\| = 0. \quad \lim_{n\to\infty}\|y_n - u_n\| = 0.$$

Thus

$$(18)\quad \lim_{n\to\infty}\|u_{n+1}-u_n\| = |\lim_{n\to\infty}\|u_{n+1}-w_n\|+\lim_{n\to\infty}\|w_n-y_n\|+\lim_{n\to\infty}\|y_n-u_n\| = 0.$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$; there exists a subsequence $\{u_{n_i}\}$ such that $\{u_{n_i}\}$ converges to $\hat{u}$. Replacing u_{n+1} by u_{n_i} in (7), w_n by u_{n_i} in (6), y_n by y_{n_i} in (5) and taking the limits and using (18), we have

$$\langle T\hat{u}, v-\hat{u}\rangle \geq 0, \quad \forall v \in K_r.$$

This shows that $\hat{u} \in K_r$ solves the nonconvex variational inequality (1) and

$$\|u_{n+1}-\hat{u})\|^2 \leq \|u_n-\hat{u}\|^2,$$

which implies that the sequence $\{u_n\}$ has a unique cluster point and $\lim_{n\to\infty} u_n = \hat{u}$, is a solution of (1), the required result. $\square$.

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Mathematics Department, COMSATS Institute of Information Technology, Islamabad, Pakistan

Mathematics Department, College of Science, King Saud University, Riyadh, Saudi Arabia

E-mail address: noormaslam@hotmail.com