ALBANIAN JOURNAL
OF MATHEMATICS
Volume 4, Number 2, Pages 57–65
ISSN 1930-1235: (2010)

ON τ-$\oplus$-SUPPLEMENTED MODULES

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Abstract. Let τ be any preradical and M any module. In [2], Al-Takhman, Lomp and Wisbauer defined τ-supplemented module. In this paper we introduce the (completely) τ-$\oplus$-supplemented modules. It is shown that (1) Any finite direct sum of τ-$\oplus$-supplemented modules is τ-$\oplus$-supplemented. (2) If M is τ-$\oplus$-supplemented module and (D_3) then M is completely τ-$\oplus$-supplemented.

1. Introduction

Throughout this paper R will denote an arbitrary associative ring with identity and all modules will be unitary right R-modules. A functor τ from the category of the right R-modules to itself is called a *preradical* if it satisfies the following properties:

(1) $\tau(M)$ is a submodule of an R-module M,

(2) If $f : M' \to M$ is an R-module homomorphism, then $f(\tau(M')) \subseteq \tau(M)$ and $\tau(f)$ is the restriction of f to $\tau(M')$.

A preradical τ is called a *right exact preradical* if for any submodule K of M, $\tau(K) = \tau(M) \cap K$. But it is well known if K is a direct summand of M, then $\tau(K) = \tau(M) \cap K$ for a preradical.

Let M be an R-module and τ denote a preradical. Like in [2], a submodule $K \leq M$ is called τ-supplement (weak τ-supplement) provided there exists some $U \leq M$ such that $M = U + K$ and $U \cap K \subseteq \tau(K)$ $(U \cap K \subseteq \tau(M))$.

M is called *τ-supplemented* (*weakly τ-supplemented*) if each of its submodules has a τ-supplement (weak τ-supplement) in M. M is called *amply τ-supplemented*, if for all submodules K and L of M with $K + L = M$, K contains a τ-supplement of L in M. Kosan and Harmanci [9] studied supplemented modules relative to torsion theories. Motivated by their work, we study $\oplus$-supplemented modules with respesct to a preradical. Also another work has been done on C_1 modules (see [12]).

A module M is called *τ-lifting* if for every submodule K of M, there is a decomposition $K = A \oplus B$, such that A is a direct summand of M and $B \subseteq \tau(M)$.

In this paper we introduce the (completely) τ-$\oplus$-supplemented modules and investigate some properties of them.

Our paper is organized as follows.

In Section 2, we define the concept of τ-$\oplus$-supplemented module. We call a module M *τ-$\oplus$-supplemented* if every submodule of M has a τ-supplement that is a direct summand of M. Then we show any finite direct sum of τ-$\oplus$-supplemented modules is τ-$\oplus$-supplemented. We also investigate when a direct summand of a τ-$\oplus$-supplemented module is τ-$\oplus$-supplemented.

In Section 3, we call a module M *completely τ-$\oplus$-supplemented* if every direct summand of M is τ-$\oplus$-supplemented and prove if M is τ-$\oplus$-supplemented module and (D_3), then M is completely τ-$\oplus$-supplemented.

The notation $N \leq_d M$ denotes that N is a direct summand of M.

Definition 1.1. For any preradical τ, we call a module M, *τ-$\oplus$-supplemented* if every submodule of M has a τ-supplement that is a direct summand of M.

Theorem 1.2. *For any preradical τ, any finite direct sum of τ-$\oplus$-supplemented modules is τ-$\oplus$-supplemented.*

Proof. Let $M = M_1 \oplus M_2$ where M_1 and M_2 are two τ-$\oplus$-supplemented modules. Let P be any submodule of M. We have $P + M_2 = M_2 \oplus [(P + M_2) \cap M_1]$ and $(P + M_2) \cap M_1$ is a submodule of M_1. Since M_1 is τ-$\oplus$-supplemented, there exists a direct summand K_1 of M_1 such that $[(P + M_2) \cap M_1] + K_1 = M_1$ and $(P + M_2) \cap K_1 \subseteq \tau(K_1)$. We have $(P + K_1) \cap M_2$ is a submodule of M_2, so there exists a direct summand K_2 of M_2 such that $[(P + K_1) \cap M_2] + K_2 = M_2$ and $(P+K_1)\cap K_2 \subseteq \tau(K_2)$. Let $K = K_1 \oplus K_2$, K is a direct summand of M. Moreover $M_1 \leq P + M_2 + K_1$ and $M_2 \leq P + K_1 + K_2$. Hence $M = P + K_1 + K_2 = P + K$. Since $P\cap(K_1+K_2) \leq [(P+K_1)\cap K_2]+[(P+K_2)\cap K_1]$, thus $P\cap(K_1+K_2) \leq [(P+K_1)\cap K_2]+[(P+N_2)\cap K_1]$. As $(P+M_2)\cap K_1 \subseteq \tau(K_1)$ and $(P+K_1)\cap K_2 \subseteq \tau(K_2)$, we have $(P \cap K) \subseteq \tau(K)$. Thus M is τ-$\oplus$-supplemented. □

A nonzero module M is called *completely torsion* if for every proper submodule K of M, $K \subseteq \tau(M)$.

Corollary 1.3. *For any preradical τ, any finite direct sum of completely torsion modules is τ-$\oplus$-supplemented.*

Theorem 1.4. *Let M_i $(1 \leq i \leq n)$ be any finite collection of relatively projective modules. Then for any preradical τ, the module $M = \bigoplus_{i=1}^{n} M_i$ is τ-$\oplus$-supplemented if and only if M_i is τ-$\oplus$-supplemented for each $1 \leq i \leq n$.*

Proof. The sufficiency is proved in Theorem 1.2. Conversely, we only prove M_1 to be τ-$\oplus$-supplemented. Let $A \leq M_1$. Then there exists $B \leq M$ such that $M = A + B$, B is a direct summand of M and $A\cap B \subseteq \tau(B)$. Since $M = A+B = M_1+B$, by [10, Lemma 4.47], there exists $B_1 \leq B$ such that $M = M_1\oplus B_1$. Thus $B = B_1\oplus(M_1\cap B)$. Note that $M_1 = A + (M_1 \cap B)$ and $M_1 \cap B$ is a direct summand of M_1. Therefore $A \cap B = A \cap (M_1 \cap B) \subseteq \tau(B) \cap (M_1 \cap B) = \tau(M_1 \cap B)$. Hence M_1 is τ-$\oplus$-supplemented. □

A factor module of a τ-$\oplus$-supplemented module need not be τ-$\oplus$-supplemented for $\tau = Rad$ (see [6, Examples 2.2 and 2.3]).

Theorem 1.5. *Let M be a τ-$\oplus$-supplemented module for any preradical τ and $X \leq M$. If for every direct summand K of M, $(X + K)/X$ is a direct summand of M/X, then M/X is τ-$\oplus$-supplemented.*

Proof. Let $N/X \leq M/X$. Since M is τ-$\oplus$-supplemented, there exists a direct summand K of M such that $N + K = M$ and $N \cap K \subseteq \tau(K)$. Then $N/X + (K + X)/X = M/X$. By assumption, $(K + X)/X$ is a direct summand of M/X. It is easy to check that $(N/X) \cap ((K + X)/X) \subseteq \tau((K + X)/X)$. □

Let M be a module. Then M is called *distributive* if its lattice of submodules is a distributive lattice, equivalently for submodules K, L, N of M, $N + (K \cap L) = (N + K) \cap (N + L)$ or $N \cap (K + L) = (N \cap K) + (N \cap L)$.
Let M be a module. A submodule X of M is called *fully invariant*, if for every $f \in End(M)$, $f(X) \subseteq X$. The module M is called *duo module*, if every submodule of M is fully invariant. The submodule A of M is called *projection invariant* in M if $f(A) \subseteq A$, for any idempotent $f \in End(M)$.

Corollary 1.6. *Let M be a τ-$\oplus$-supplemented module for any preradical τ.*
(1) *Let $N \leq M$ such that for each decomposition $M = M_1 \oplus M_2$ we have $N = (N \cap M_1) \oplus (N \cap M_2)$. Then M/N is τ-$\oplus$-supplemented. (In particular, this is true for any distributive module). If moreover $N \leq_d M$, then N is τ-$\oplus$-supplemented.*
(2) *Let X be a projection invariant submodule of M. Then M/X is τ-$\oplus$-supplemented. In particular, for every fully invariant submodule A of M, M/A is τ-$\oplus$-supplemented.*

Proof. (1) Let $L/N \leq M/N$. Since M is τ-$\oplus$-supplemented, there exists a direct summand D of M such that $M = L + D$ and $L \cap D \subseteq \tau(D)$. Then $M/N = L/N + (D + N)/N$ and $L/N \cap (D + N)/N = (L \cap (D + N))/N \subseteq \tau((D + N)/N)$. Let $M = D \oplus D'$. By assumption, $N = (N \cap D) \oplus (N \cap D') = (D + N) \cap (D' + N)$. So, $(D + N)/N \oplus (D' + N)/N = M/N$. It follows that M/N is τ-$\oplus$-supplemented.

Now let $N \leq_d M$ and $V \leq N$. Then there exist submodules K and K' of such that $M = K \oplus K' = V + K$ and $V \cap K \subseteq \tau(K)$. Thus $N = V + N \cap K$. By assumption $N \cap K \leq_d N$. Moreover, $V \cap (N \cap K) \subseteq \tau(K)$. Then $V \cap (N \cap K) \subseteq \tau(N \cap K)$. Therefore, N is τ-$\oplus$-supplemented.
(2) Clear by (1). □

Let M be an R-module. By $P_\tau(M)$ we denote the sum of all submodules N of M with $\tau(N) = N$. Since $P_\tau(M)$ is a sum of some submodules of M, itself is a submodule of M.

Corollary 1.7. *Let M be a τ-$\oplus$-supplemented module for any preradical τ. Then $M/P_\tau(M)$ is τ-$\oplus$-supplemented. If moreover $P_\tau(M) \leq_d M$, then $P_\tau(M)$ is τ-$\oplus$-supplemented.*

Proof. By Corollary 1.6(1), it suffices to prove that $P_\tau(M)$ is a fully invariant submodule of M. Let $N \leq M$ such that $N = \tau(N)$ and $f \in End(M)$ and g its restriction to N. But $\tau(N) = N$ and $f(N) = g(N)$, hence $f(N) \subseteq \tau(f(N))$. Thus, $\tau(f(N)) = f(N)$. This implies that $f(N) \subseteq P_\tau(M)$. This completes the proof. □

We recall that a module M is called *semi-Artinian* if every nonzero quotient module of M has nonzero socle. For a module M, we define $Sa(M) = \sum\{U \leq M \mid U semi - Artinian\}$.

Corollary 1.8. *Let M be a τ-$\oplus$-supplemented module for any preradical τ. Then $M/Sa(M)$ is τ-$\oplus$-supplemented. If, moreover, $Sa(M)$ is a direct summand of M, then $Sa(M)$ is also τ-$\oplus$-supplemented.*

Proof. Let $f \in End(M)$ and U a semi-Artinian submodule. Let g be restriction of f to U. Thus $U/Ker(g) \cong g(U)$. Hence $f(U) \cong U/Ker(g)$. But it is easy to check that $U/Ker(g)$ is a semi-Artinian module. Therefore, $f(U)$ is semi-Artinian. This implies that $f(Sa(M)) \subseteq Sa(M)$. Thus $Sa(M)$ is a fully invariant submodule of M. The result follows from Corollary 1.6(1). □

Remark 1.9. If M is a τ-$\oplus$-supplemented module for any preradical τ, then $M/\tau(M)$ is semisimple and hence τ-$\oplus$-supplemented.

Example 1.10. Let M be the Z-module $Z/2Z \oplus Z/8Z$. By [8, Example 10], M is not lifting and it is not τ-lifting. By [5, Theorem 1.4], M is $\oplus$-supplemented and hence τ-$\oplus$-supplemented for $\tau = Rad$.

A τ-lifting module is τ-$\oplus$-supplemented. But the converse does not hold. The following proposition shows that under some assumption it can be true.

Proposition 1.11. *Assume M is τ-$\oplus$-supplemented for any preradical τ such that whenever $M = M_1 \oplus M_2$ then M_1 and M_2 are relatively projective. Then M is τ-lifting.*

Proof. Let $N \leq M$. Since M is τ-$\oplus$-supplemented, there exists a decomposition $M = M_1 \oplus M_2$ such that $M = N + M_2$ and $N \cap M_2 \subseteq \tau(M_2)$ for submodules M_1, M_2 of M. By hypothesis, M_1 is M_2-projective. By [10, Lemma 4.47], we obtain $M = A \oplus M_2$ for some submodule A of M such that $A \leq N$. Then $N = A \oplus (M_2 \cap N)$. So M is τ-lifting by [2, 2.8]. □

Corollary 1.12. *Let M be a τ-$\oplus$-supplemented module for any prerardical τ. If M is projective then M is τ-lifting.*

Now we give a characterization of τ-$\oplus$-supplemented rings.

Theorem 1.13. *Let τ be any preradical. Then the following are equivalent:*
(1) *R is τ-$\oplus$-supplemented;*
(2) *Every finitely generated free R-module is τ-$\oplus$-supplemented;*
(3) *If F is a finitely generated free R-module and N a fully invariant submodule, then F/N is τ-$\oplus$-supplemented.*

Proof. (1) $\Rightarrow$ (2) Let M be a finitely generated free R-module. Then $M \cong \bigoplus_{i=1}^{n} R$. Since any finite direct sum of τ-$\oplus$-supplemented modules is τ-$\oplus$-supplemented, the result follows.

(2) $\Rightarrow$ (3) By (2), F is τ-$\oplus$-supplemented. The result follows from Corollary 1.6(2).

(3) $\Rightarrow$ (1) is clear. □

Lemma 1.14. *Let $M = M_1 \oplus M_2$. Then for any preradical τ, M_2 is τ-$\oplus$-supplemented if and only if for every submodule N/M_1 of M/M_1, there exists a direct summand K of M such that $K \leq M_2$, $M = K + N$ and $N \cap K \subseteq \tau(M)$.*

Proof. Suppose that M_2 is τ-$\oplus$-supplemented. Let $N/M_1 \leq M/M_1$. As M_2 is τ-$\oplus$-supplemented, there exists a decomposition $M_2 = K \oplus K'$ such that $M_2 = (N \cap M_2) + K$ and $N \cap K \subseteq \tau(K)$. Note that $M = (N \cap M_2) + K + M_1$ gives $M = N + K$.

Conversely, suppose that M/M_1 has the stated property. Let H be a submodule of M_2. Consider the submodule $(H \oplus M_1)/M_1 \leq M/M_1$. By hypothesis, there exists a direct summand L of M such that $L \leq M_2$, $M = (L + H) + M_1$ and $L \cap (H + M_1) \subseteq \tau(M)$. By modularity, $M_2 = L + H$. Then $L \cap H \subseteq \tau(L)$. Thus, L is a τ-supplement of H in M_2 and it is a direct summand of M_2. Therefore, M_2 is τ-$\oplus$-supplemented. □

Theorem 1.15. *Let τ be any preradical and M_2 a direct summand of a τ-$\oplus$-supplemented module M such that for every direct summand K of M with $M = K + M_2$, $K \cap M_2$ is a direct summand of M. Then M_2 is τ-$\oplus$-supplemented.*

Proof. Suppose that $M = M_1 \oplus M_2$ and let $N/M_1 \leq M/M_1$. Consider the submodule $N \cap M_2$ of M. Since M is τ-$\oplus$-supplemented, there exists a direct summand K of M such that $M = (N \cap M_2) + K$ and $N \cap M_2 \cap K \subseteq \tau(K)$. Note that $M = N + M_2$. By [7, Lemma 1.2], $M = (K \cap M_2) + N$. Since $M = K + M_2$, $K \cap M_2$ is a direct summand of M by hypothesis. By Lemma 1.14, M_2 is τ-$\oplus$-supplemented. □

Corollary 1.16. *Let M be a τ-$\oplus$-supplemented module for any preradical τ and K a direct summand of M such that M/K is K-projective. Then K is τ-$\oplus$-supplemented.*

Proof. Let L be a direct summand of M with $M = L + K$. Since K is a direct summand of M, $M = K \oplus K_0$ for some submodule K_0 of M. Therefore, K_0 is K-projective. Then by [16, 41.14], there exists a submodule L_0 of L such that $M = L_0 \oplus K$. Now $L = L' \oplus (L \cap K)$ implies that $L \cap K$ is a direct summand of M. By Theorem 1.15, K is τ-$\oplus$-supplemented. □

Corollary 1.17. *Let M be a τ-$\oplus$-supplemented module for any preradical τ and $N \leq_d M$ such that M/N is projective. Then N is τ-$\oplus$-supplemented.*

A submodule N of M is called *small* in M (notation $N \ll M$) if $\forall L \lneq M, L+N \neq M$. A module M is called *hollow* if every proper submodule of M is small in M.

Let M be a module and S denote the class of all small modules. Talebi and Vanaja [13] defined $\overline{Z}(M)$ as follows:

$\overline{Z}(M) = \bigcap\{kerg \mid g \in Hom(M, L), L \in S\}$. The module M is called *cosingular (non-cosingular)* if $\overline{Z}(M) = 0$ ($\overline{Z}(M) = M$). Clearly every non-cosingular module is $\overline{Z}$-$\oplus$-supplemented. Also if R is a non-cosingular ring, then every R-module is $\overline{Z}$-$\oplus$-supplemented by [13, Proposition 2.4].

In [11] for any preradical τ, the authors call a module M, τ-semiperfect if it is satisfies one of the following conditions (see [11, Proposition 2.1]):

(1) For every submodule K of M there exists a decomposition $K = A \oplus B$ such that A is a projective direct summand of M and $B \subseteq \tau(M)$;

(2) For every submodule K of N, there exists a decomposition $M = A \oplus B$ such that A is a projective direct summand of M, $A \leq K$ and $K \cap B \subseteq \tau(M)$.

By this definition every τ-semiperfect module is τ-lifting and hence τ-$\oplus$-supplemented. Also if M is projective we have the following:

τ-semiperfect $\Leftrightarrow$ τ-lifting $\Leftrightarrow$ τ-$\oplus$-supplemented.

A τ-$\oplus$-supplemented module need not be $\oplus$-supplemented and the converse also hold.

Example 1.18. Let K be a field and let $R = \prod_{n\geq 1} K_n$ with $K_n = K$. By [14, Example 4.1(1)] R is not semiperfect. Since R is projective, R is not $\oplus$-supplemented by [5, Lemma 1.2]. Again by [14, Example 4.1(1)], the module R is $\overline{Z}$-semiperfect and so it is $\overline{Z}$-$\oplus$-supplemented.

If R is a DVR (Discrete Valuation Ring) , then by [14, Example 4.1(1)] the R-module R_R is semiperfect and hence $\oplus$-supplemented but it is not $\overline{Z}$-semiperfect and so it is not $\overline{Z}$-$\oplus$-supplemented.

Now we give an equivalent condition for a module to be $\overline{Z}$-$\oplus$-supplemented under some assumptions.

Proposition 1.19. *Let R be a commutative ring and P a projective module with $Rad(P) \ll P$ and P has finite hollow dimension. Then the following are equivalent:*

(1) *P is $\overline{Z}$-$\oplus$-supplemented;*

(2) *$P = P_1 \oplus P_2 \oplus P_3$ with P_1 is $\oplus$-supplemented and $Rad(P_1) = \overline{Z}(P_1)$, P_2 is semisimple and $\overline{Z}(P_3) = P_3$.*

Proof. (1) $\Rightarrow$ (2) By the proof of [14, Corollary 4.3] and since every semiperfect is $\oplus$-supplemented .

(2) $\Rightarrow$ (1) By [14, Corollary 4.3] all P_1, P_2 and P_3 are $\overline{Z}$-semiperfect and hence $\overline{Z}$-$\oplus$-supplemented. Since any finite direct sum of $\overline{Z}$-$\oplus$-supplemented modules is $\overline{Z}$-$\oplus$-supplemented, P is $\overline{Z}$-$\oplus$-supplemented. □

Let $e = e^2 \in R$. Then e is called a *left (right) semicentral idempotent* if $xe = exe$ ($ex = exe$), for all $x \in R$. The set of all left (right) semicentral idempotents is denoted by $S_l(R)$ ($S_r(R)$). A ring R is called *Abelian* if every idempotent is central.

Let M be a module. We consider the following condition.

(D_3) If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of M.

By [10, Lemma 4.6 and Proposition 4.38], every quasi-projective module is (D_3).

Proposition 1.20. *Let M be an R-module such that $End(M)$ is Abelian and $X \leq M$ implies $X = \sum_{i \in I} h_i(M)$ where $h_i \in End(M)$. Then for any preradical τ, M is τ-$\oplus$-supplemented if and only if M is τ-lifting and has (D_3)-condition.*

Proof. The sufficiency is obvious. Conversely, let $X \leq M$, $X = \sum_{i \in I} h_i(M)$ with $h_i(M) \in End(M)$. Since M is τ-$\oplus$-supplemented, there exists a direct summand eM such that $X + eM = M$ and $(X \cap eM) \subseteq \tau(eM)$ for some $e^2 = e \in End(M)$. Since $End(M)$ is Abelian, $(1-e)X = (1-e)M = (1-e)\sum_{i \in I} h_i(M) = \sum_{i \in I} h_i(1-e)(M) \subseteq X$. Therefore $X = (1-e)M \oplus (X \cap eM)$. Hence M is τ-lifting. If $eM + fM = M$ for $e^2 = e$, $f^2 = f \in End(M)$, then $eM \cap fM = efM$ with $(ef)^2 = ef$. So M has (D_3)-condition.

□

Recall that an R-module M is said to be a *multiplication module* if for each $X \leq M$ there exists $A_R \leq R_R$ such that $X = MA$.

Corollary 1.21. *If M satisfies one of the following conditions, then M is τ-lifting if and only if M is τ-$\oplus$-supplemented for any preradical τ.*
(1) *M is cyclic and R is commutative.*
(2) *M is a multiplication module and R is commutative.*

Proof. (1) Assume that M is cyclic and R is commutative. There exists $B_R \leq R_R$ such that $M \cong R/B$. Let $Y/B \leq R/B$, $Y/B = \sum_{i \in I}(y_i R + B) = (\sum_{i \in I} y_i + B)R$ where each $y_i \in Y$. Define $h_i : R/B \to R/B$ by $h_i(r + B) = y_i r + B$, $i \in I$. Then it is easy to check that $h_i \in End_R(R/B)$. Hence $Y/B = \sum_{i \in I} h_i(R/B)$. Since R is commutative, $End_R(R/B)$ is also commutative. By Proposition 1.20, M is τ-lifting.
(2) Assume M is a multiplication module. Let $X \leq M$. Then $X = MA$ for some $A_R \leq R_R$. For each $a \in A$, define $h_\alpha : M \to M$ by $h_\alpha(m) = ma$ for all $m \in M$. Then h_α is an R-homomorphism and $X = MA = \sum_{\alpha \in A} h_\alpha(M)$. Since every multiplication module is a duo module, thus if $e^2 = e \in S = End(M)$, then e,

$1-e \in S_l(S)$. Therefore e is central. So $End(M)$ is Abelian. Again by Proposition 1.20, M is τ-lifting.

□

2. Completely τ-$\oplus$-Supplemented Modules

Definition 2.1. For any preradical τ, we call a module M *completely τ-$\oplus$-supplemented* for any preradical τ if every direct summand of M is a τ-$\oplus$-supplemented.

Theorem 2.2. *Let M be a module with (D_3) and τ a preradical. Then M is τ-$\oplus$-supplemented if and only if M is completely τ-$\oplus$-supplemented.*

Proof. Sufficiency is clear. Conversely, assume that M is τ-$\oplus$-supplemented and K a direct summand of M and A a submodule of K. We show A has a τ-supplement in K that is a direct summand of K. Since M is τ-$\oplus$-supplemented, there exists a direct summand B of M such that $M = A + B$ and $A \cap B \subseteq \tau(B)$. Then $K = A + (K \cap B)$. Furthermore $K \cap B$ is a direct summand of M because M has (D_3). Then $A \cap (K \cap B) = (A \cap B) \cap (K \cap B) \subseteq \tau(B) \cap (K \cap B) = \tau(K \cap B)$. □

A submodule K of M is called *essential* in M (notation $K \leq_e M$) if $K \cap A \neq 0$ for any nonzero submodule A of M.

Proposition 2.3. *Let M be a τ-supplemented module for any preradical τ. Then $M = M_1 \oplus M_2$, where M_1 is semisimple module and M_2 is a module with $\tau(M_2)$ essential in M_2.*

Proof. See [2, 2.2]. □

Recall that a module M has the *Summand Sum Property* (SSP) if the sum of any two direct summand of M is again a direct summand.

Theorem 2.4. (1) *Every τ-lifting module is completely τ-$\oplus$-supplemented for any preradical τ.*

(2) *Let M be a τ-$\oplus$-supplemented module for any preradical τ. If M has the (SSP), then M is completely τ-$\oplus$-supplemented.*

Proof. (1) By [2, 2.10] every direct summand of a τ-lifting module is τ-lifting. The rest is clear.

(2) Assume that M is τ-$\oplus$-supplemented and M has the (SSP). Let N be a direct summand of M. We will show that N is τ-$\oplus$-supplemented. Let $M = N \oplus N'$ for some submodule N' of M. Suppose that A is a direct summand of M. Since M has the (SSP), $A + N'$ is a direct summand of M. Let $M = (A + N') \oplus B$ for some $B \leq M$. Then $M/N' = (A+N')/N' \oplus (B+N')/N'$. Hence by Theorem 1.5, M/N' is τ-$\oplus$-supplemented and so N is τ-$\oplus$-supplemented. □

We give a decomposition of any τ-$\oplus$-supplemented (D_3)-module by the second singular submodule $Z_2(M)$ of M. We will show that if M is τ-$\oplus$-supplemented and $N \leq M$ with M/N projective, then N is τ-$\oplus$-supplemented.

Recall that the *singular submodule* $Z(M)$ of a module M is defined by $Z(M) = \{m \in M \mid mE = 0, E \leq_e R\}$.

The *Goldie torsion submodule (or second singular submodule)* $Z_2(M)$ of M is a submodule of M containing $Z(M)$ such that $Z_2(M)/Z(M)$ is the singular submodule of $M/Z(M)$.

Proposition 2.5. *Let M be a module with (D_3). Suppose that $Z_2(M)$ is τ-coclosed in M. Then for any preradical τ, M is τ-$\oplus$-supplemented if and only if $M = Z_2(M) \oplus K$ for some submodule K of M and, $Z_2(M)$ and K are τ-$\oplus$-supplemented.*

Proof. Sufficiency is clear by Theorem 1.2. Conversely, assume that M is τ-$\oplus$-supplemented. There exist submodules K and K' of M such that $M = K \oplus K' = Z_2(M) + K$ and $Z_2(M) \cap K \subseteq \tau(K)$. Now $Z_2(M) = Z_2(K) \oplus Z_2(K')$. Thus, $M = K \oplus Z_2(K')$ and hence $Z_2(K') = K'$. Note that $Z_2(M) \cap K = Z_2(K) \subseteq \tau(K)$. So, we can obtain that $Z_2(M)/K' \subseteq \tau(M/K')$. Therefore, $Z_2(M) = K'$ because $Z_2(M)$ is τ-coclosed in M. So, $M = K \oplus Z_2(M)$. Clearly K and $Z_2(M)$ are τ-$\oplus$-supplemented. □

Proposition 2.6. *Let M be a τ-supplemented module for any preradical τ. Then $M = M_1 \oplus M_2$, where M_1 is semisimple module and M_2 is a module with $\tau(M_2)$ essential in M_2.*

Proof. See [2, 2.2]. □

Corollary 2.7. *Let M be a τ-$\oplus$-supplemented module for any preradical τ. Then $M = M_1 \oplus M_2$ where M_1 is a semisimple module and M_2 is a module with $\tau(M_2)$ essential in M_2.*

Proof. Since each τ-$\oplus$-supplemented module is τ-supplemented the result follows from Proposition 2.6. □

Proposition 2.8. *Let M be a τ-$\oplus$-supplemented module for a left exact preradical τ. Then $M = M_1 \oplus M_2$ such that $\tau(M_2) = M_2$.*

Proof. Suppose that M is a τ-$\oplus$-supplemented module. There exists a direct summand M_1 of M such that $M = M_1 + \tau(M)$ and $M_1 \cap \tau(M) = \tau(M_1)$ since τ is a left exact preradical and $M = M_1 \oplus M_2$ for some submodule M_2 of M. Then $M = \tau(M_2) \oplus M_1$. Thus $M_2 = \tau(M_2)$. □

Theorem 2.9. *For module M with (D_3) and a left exact preradical τ the following statements are equivalent:*

(1) *M is completely τ-$\oplus$-supplemented;*

(2) *M is τ-$\oplus$-supplemented;*

(3) *$M = M_1 \oplus M_2$, where M_1 is semisimple module and M_2 is a τ-$\oplus$-supplemented module with $\tau(M_2)$ essential in M_2;*

(4) *$M = M_1 \oplus M_2$ such that M_1 is a τ-$\oplus$-supplemented module and M_2 is a τ-$\oplus$-supplemented module with $\tau(M_2) = M_2$.*

Proof. (1) $\Rightarrow$ (2) Clear from definition.

(2) $\Rightarrow$ (1) It follows from Theorem 2.2.

(1) $\Rightarrow$ (3) By Proposition 2.6, $M = M_1 \oplus M_2$, where M_1 is semisimple module and M_2 is module with $\tau(M_2)$ essential in M_2. By (1), M_2 is τ-$\oplus$-supplemented.

(1) $\Rightarrow$ (4) By Proposition 2.8, $M = M_1 \oplus M_2$ such that $\tau(M_2) = M_2$ and M_1, M_2 are τ-$\oplus$-supplemented by (1).

(3) $\Rightarrow$ (2), (4) $\Rightarrow$ (2) follows by Theorem 1.2. □

Lemma 2.10. *Let M be an indecomposable module. Then for any preradical τ, M is completely torsion if and only if M is completely τ-$\oplus$-supplemented.*

Proof. Clear. □

Proposition 2.11. *Let $M = M_1 \oplus M_2$ such that M_1 and M_2 have local endomorphism rings. Then for any preradical τ, M is completely τ-$\oplus$-supplemented if and only if M_1 and M_2 are completely torsion modules.*

Proof. The necessity is clear from Lemma 2.10. Conversely, let K be a direct summand of M. If $K = M$ then by Corollary 1.3, K is τ-$\oplus$-supplemented. Assume $K \neq M$. Then either $K \cong M_1$ or $K \cong M_2$ by [3, Corollary 12.7]. In either case K is τ-$\oplus$-supplemented. Thus M is completely τ-$\oplus$-supplemented. □

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