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A COMMON FIXED POINT THEOREM FOR A FAMILY OF SELFMAPPINGS SATISFYING A GENERAL CONTRACTIVE CONDITION OF OPERATOR TYPE

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Abstract. In this paper, we prove a common fixed point theorem for a family of selfmappings satisfying a general contractive condition of operator type.

1. Introduction

The class of generalized contraction mappings, introduced and studied by Ćirić in [6], is very significant in a fixed point theory. As noted by Gornićki and Rhoades [8], a contractive condition (2.1) on a pair of generalized contractions. Jungck [9] proved a fixed point theorem for commuting maps generalizing the Banach's fixed point and further he [10] introduced more generalizing commutativity, so called compatibility, which is more general than that of weak commutativity defined by Sessa [12]. Lately, Branciari [4] obtained a fixed point results for a single mapping satisfying an analogue of Banach's contraction principle (see [3] and [5]) for an integral type inequality. Rhoades [11] proved two fixed point theorems involving more general contractive conditions. Vijayaraju et al. [13] established a general principle, which maked it possible to proved many fixed point theorems for a pair of maps of integral type. Aliouche [1] gave a common fixed point theorem for selfmappings of a symmetric space under a contractive condition of integral type. Altun and Turkoglu [2] proved a fixed point theorem for mappings satisfying a general contractive of operator type.

The main purpose of this paper is to give a common fixed point theorem for a family of selfmappings satisfying a general contractive condition of operator type.

2. Preliminaries

Let X be a nonempty set and let $\{T_\alpha\}_{\alpha\in J}$ be a family of selfmappings on X and J indexing set. A point $u \in X$ is called a common fixed point for a family

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$\{T_\alpha\}_{\alpha\in J}$ iff for each T_α. The following theorem was given by Ćirić [7] for a family of generalized contraction.

Theorem 1. *Let (X,d) be a complete metric space and let $\{T_\alpha\}_{\alpha\in J}$ be a family of selfmappings of X. If there exists fixed $\beta \in J$ such that for each $\alpha \in J$:*

$$d(T_\alpha x, T_\beta y) \le \lambda \max \left\{ \begin{array}{c} d(x,y), d(x,T_\alpha x), d(y,T_\beta y), \\ \frac{1}{2}\left[d(x,T_\beta y)+d(y,T_\alpha x)\right] \end{array} \right\} \tag{2.1}$$

for some $\lambda = \lambda(\alpha) \in (0,1)$ and all x, $y \in X$, then all T_α have a unique common fixed point, which is a unique fixed point of each T_α, $\alpha \in J$.

The following theorem was given by Branciari [4] was to analyze the existence of fixed points for mappings of f defined on a complete metric space (X,d) satisfing a contractive condition of integral type.

Theorem 2. *Let (X,d) be a complete metric space, $c \in (0,1)$ and $f : X \to X$ be a mapping such that for each x, $y \in X$ one has*

$$\int_0^{d(fx,fy)} \varphi(t)dt \le c \int_0^{d(x,y)} \varphi(t)dt$$

where $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,+\infty)$, non-negative and such that for each $\varepsilon > 0$, $\int_0^\varepsilon \varphi(t)dt > 0$; then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^n x = a$.

The following concept of $O(f\ ;.)$ and its examples was given by Altun and Turkoglu [2].

Let $F([0,\infty))$ be class of all function $f : [0,\infty) \to [0,\infty]$ and let Θ be class of all operators

$$O(\bullet;.) : F([0,\infty)) \to F([0,\infty)),\ \ f \to O(f\ ;.)$$

satisfying the following conditions:

(i) $O(f\ ;t) > 0$ for $t > 0$ and $O(f\ ;0) = 0$,

(ii) $O(f\ ;t) \le O(f\ ;s)$ for $t \le s$,

(iii) $\lim_{n\to\infty} O(f\ ;t_n) = O\left(f\ ;\ \lim_{n\to\infty} t_n\right)$,

(iv) $O(f\ ;\max\{t,s\}) = \max\{O(f\ ;t), O(f\ ;s)\}$ for some $f \in F([0,\infty))$.

Example 1. *If $f : [0,\infty) \to [0,\infty)$ is a Lebesque integrable mapping which is finite integral on each compact subset of $[0,\infty)$, non-negative and such that for each $t > 0$, $\int_0^t f(s)ds > 0$, then the operator defined by*

$$O(f\ ;t) = \int_0^t f(s)ds$$

satisfies the conditions (i)-(iv).

Example 2. *If $f : [0,\infty) \to [0,\infty)$ non-decreasing, continuous function such that $f(0) = 0$ and $f(t) > 0$ for $t > 0$, then the operator defined by*

$$O(f\ ;t) = \frac{f(t)}{1+f(t)}$$

satisfies the conditions (i)-(iv).

Example 3. *If $f:[0,\infty)\to[0,\infty)$ non-decreasing, continuous function such that $f(0)=0$ and $f(t)>0$ for $t>0$, then the operator defined by*

$$O\left(f\ ;t\right)=\frac{f(t)}{1+\ln\left(1+f(t)\right)}$$

satisfies the conditions (i)-(iv).

3. A common fixed point theorem and it's results

Now, we prove a common fixed point theorem for a family of selfmappings satisfying a general contractive condition of operator type in complete metric spaces.

Theorem 3. *Let (X,d) be a complete metric space and $\{T_\alpha\}_{\alpha\in J}$ be a family of selfmappings of X. If there exists a fixed $\beta\in J$ such that for each $\alpha\in J$:*

$$O\left(f;d\left(T_\alpha x,T_\beta y\right)\right)\leq\lambda O\left(f;m(x,y)\right)\tag{3.1}$$

where $O\left(\bullet;.\right)\in\Theta$ and

$$m(x,y)=\max\left\{d(x,y),d(x,T_\alpha x),d(y,T_\beta y),\frac{1}{2}\left[d(x,T_\beta y)+d(y,T_\alpha x)\right]\right\}\tag{3.2}$$

for some $\lambda=\lambda(\alpha)\in(0,1)$ and all x, $y\in X$, then all T_α have a unique common fixed point, which is a unique fixed point of each T_α, $\alpha\in J$.

Proof. Let $\alpha\in J$ and $x\in X$ be arbitrary. Consider a sequence, defined inductively by

$$x_0=x,\ x_{2n+1}=T_\alpha x_{2n},\ x_{2n+2}=T_\beta x_{2n+1},\quad(n\geq 0)\,.$$

For each integer $n\geq 0$, from (3.1),

$$\begin{aligned}O\left(f;d\left(x_{2n+1},x_{2n+2}\right)\right)&=O\left(f;d\left(T_\alpha x_{2n},T_\beta x_{2n+1}\right)\right)\\&\leq\lambda O\left(f;m(x_{2n},x_{2n+1})\right).\end{aligned}\tag{3.3}$$

Using (3.2), we have

$$m(x_{2n},x_{2n+1})=\max\left\{d\left(x_{2n},x_{2n+1}\right),d\left(x_{2n+1},x_{2n+2}\right)\right\}.$$

Substituting into (3.3) and (iv), one obtains

$$\begin{aligned}&O\left(f;d\left(x_{2n+1},x_{2n+2}\right)\right)\\\leq\ &\lambda O\left(f;\max\left\{d\left(x_{2n},x_{2n+1}\right),d(x_{2n+1},x_{2n+2})\right\}\right)\\=\ &\lambda\max\left\{O\left(f;d\left(x_{2n},x_{2n+1}\right)\right),O\left(f;d\left(x_{2n+1},x_{2n+2}\right)\right)\right\}.\end{aligned}\tag{3.4}$$

If $O\left(f;d\left(x_{2n+1},x_{2n+2}\right)\right)\geq O\left(f;d\left(x_{2n},x_{2n+1}\right)\right)$, then from (3.4) we have

$$O\left(f;d\left(x_{2n+1},x_{2n+2}\right)\right)\leq\lambda O\left(f;d\left(x_{2n+1},x_{2n+2}\right)\right)$$

which is a contradiction ($\lambda<1$). Thus $O\left(f;d\left(x_{2n+1},x_{2n+2}\right)\right)<O\left(f;d\left(x_{2n},x_{2n+1}\right)\right)$ and so from (3.4) one obtains

$$O\left(f;d\left(x_{2n+1},x_{2n+2}\right)\right)\leq\lambda O\left(f;d\left(x_{2n},x_{2n+1}\right)\right).$$

Similarly, we get that

$$O\left(f;d\left(x_{2n},x_{2n+1}\right)\right)\leq\lambda O\left(f;d\left(x_{2n-1},x_{2n}\right)\right).$$

Thus, for any $n \geq 1$ we have

$$
\begin{aligned}
O\left(f; d\left(x_n, x_{n+1}\right)\right) &\leq \lambda O\left(f; d\left(x_{n-1}, x_n\right)\right) \\
&\leq \lambda^2 O\left(f; d\left(x_{n-2}, x_{n-1}\right)\right) \\
&\;\;\vdots \\
&\leq \lambda^n O\left(f; d\left(x_0, x_1\right)\right).
\end{aligned} \tag{3.5}
$$

Taking the limit of (3.5), as $n \to \infty$, we have

$$\lim_{n\to\infty} O\left(f; d\left(x_n, x_{n+1}\right)\right) = 0,$$

which, from (i), implies that

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0.$$

Therefore, $\{x_n\}$ is Cauchy sequence. (Similarly, see [2]).

Since X is complete, there is a $p \in X$ such that

$$\lim_{n\to\infty} x_n = p.$$

From (3.1) we have,

$$
\begin{aligned}
O\left(f; d\left(x_{2n+1}, T_\beta p\right)\right) &= O\left(f; d\left(T_\alpha x_{2n}, T_\beta p\right)\right) \\
&\leq \lambda \max \left\{ \begin{array}{c} d(x_{2n}, p), d(x_{2n}, T_\alpha x_{2n}), d(p, T_\beta p), \\ \frac{1}{2}\left[d(x_{2n}, T_\beta p) + d(p, T_\alpha x_{2n})\right] \end{array} \right\}.
\end{aligned}
$$

Taking the limit as $n \to \infty$ we get

$$O\left(f; d\left(p, T_\beta p\right)\right) \leq \lambda O\left(f; d\left(p, T_\beta p\right)\right),$$

which implies that

$$O\left(f; d\left(p, T_\beta p\right)\right) = 0,$$

which from (i), implies that $d\left(p, T_\beta p\right) = 0$; hence $T_\beta p = p$.

Now we show that p is a fixed point of all $\{T_\alpha\}_{\alpha\in J}$. Let $\alpha \in J$ be arbitrary. Then from (3.1) with $x = y = p = T_\beta p$ we have

$$
\begin{aligned}
O\left(f; d\left(T_\alpha p, p\right)\right) &= O\left(f; d\left(T_\alpha p, T_\beta p\right)\right) \leq \lambda(\alpha) O\left(f; m(p, p)\right) \\
&\leq \lambda(\alpha) \max \left\{ \begin{array}{c} O\left(f; d(p, p)\right), O\left(f; d(p, T_\alpha p)\right), O\left(f; d(p, T_\beta p)\right), \\ \frac{1}{2}\left[O\left(f; d(p, T_\beta p)\right) + O\left(f; d(p, T_\alpha p)\right)\right] \end{array} \right\} \\
&= \lambda(\alpha) \max \left\{ O\left(f; d(p, T_\alpha p)\right), \frac{1}{2} O\left(f; d(p, T_\alpha p)\right) \right\}.
\end{aligned}
$$

Therefore, we get

$$O\left(f; d\left(T_\alpha p, p\right)\right) \leq \lambda(\alpha) O\left(f; d(p, T_\alpha p)\right)$$

which implies that

$$O\left(f; d\left(T_\alpha p, p\right)\right) = 0,$$

which, from (i), implies that $d\left(T_\alpha p, p\right) = 0$ or $T_\alpha p = p$. Thus, all T_α have a common fixed point.

Now we prove the uniqueness of the fixed point p. Suppose that q is another a fixed point of T_β. Then it follows, as above, that q is a common fixed point of all $\{T_\alpha\}_{\alpha\in J}$. Thus, from (3.1) we have

$$\begin{aligned} O(f;d(p,q)) &= O(f;d(T_\alpha p,T_\beta q)) \\ &\leq \lambda O(f;m(p,q)) \\ &= \lambda O(f;d(p,q)), \end{aligned}$$

which implies that

$$O(f;d(p,q))=0,$$

which, from (i), implies that $d(p,q)=0$. Hence $p=q$. Thus, p is a unique common fixed point of all $\{T_\alpha\}_{\alpha\in J}$. □

Remark 1. *It is clear that Theorem 3 is a generalization of Theorem 1 in [2].*

Remark 2. *We can have new result, if we combine Theorem 3 and some examples for $O(f;.)$.*

Remark 3. *Theorem 3 is a generalization of Theorem 1, in fact letting $f=I$ (identity map) and $O(f;t)=t$ in* (3.1) *(it is obvious that $O(f\ ;.)\in\Theta$) one has*

$$d(T_\alpha x,T_\beta y)=O(f;d(T_\alpha x,T_\beta y))\leq\lambda O(f;m(x,y))=\lambda m(x,y),$$

thus Ćirić's [6,7] generalized contraction also satisfies.

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