

FULLY INVARIANT τ_M -LIFTING MODULES

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ABSTRACT. Let τ_M be any preradical for $\sigma[M]$ and N any module in $\sigma[M]$. A module N is called τ_M -lifting if for every submodule K of N , there is a decomposition $K = A \oplus B$, such that A is a direct summand of N and $B \subseteq \tau_M(N)$. We call N is (strongly) FI- τ_M -lifting if for every fully invariant submodule K of N , there is a decomposition $K = A \oplus B$, such that A is a (fully invariant) direct summand of N and $B \subseteq \tau_M(N)$. The class of FI- τ_M -lifting modules properly contains the class of τ_M -lifting modules and the class of strongly FI- τ_M -lifting modules. In this paper we investigate whether the class of (strongly) FI- τ_M -lifting modules are closed under particular class of submodules, direct summands and direct sums.

1. INTRODUCTION

Throughout this paper R will denote an arbitrary associative ring with identity and all modules will be unitary right R -modules. Let $M \in \text{Mod-}R$. By $\sigma[M]$ we mean the full subcategory of $\text{Mod-}R$ whose objects are submodules of M -generated modules. For any module M , τ_M will denote a preradical in $\sigma[M]$. We say that A is a τ_M -coessential submodule of B in N if $B/A \subseteq \tau_M(N/A)$. Like in [2], a submodule $K \subseteq N$ is called τ_M -supplement (weak τ_M -supplement) provided there exists some $U \subseteq N$ such that $U + K = N$ and $U \cap K \subseteq \tau_M(K)$ ($U \cap K \subseteq \tau_M(N)$). M is called τ_M -supplemented (weakly τ_M -supplemented) if each of its submodules has a τ_M -supplement (weak τ_M -supplement) in M . M is called *amply* τ_M -supplemented, if for all submodules K and L of N with $K + L = N$, K contains a τ_M -supplement of L in N . A submodule A of N is said to be τ_M -coclosed in N if it has no proper τ_M -coessential submodule in N . According to [2] and [12], a module N is called τ_M -lifting if for every submodule K of N , there is a decomposition $K = A \oplus B$, such that A is a direct summand of N and $B \subseteq \tau_M(N)$. Recall that a submodule K of M is called *fully invariant* (denoted by $K \triangleleft M$) if $\lambda(K) \subseteq K$ for all $\lambda \in \text{End}_R(M)$.

We mainly study (strongly) FI- τ_M -lifting modules in $\sigma[M]$ in this paper. We call N is (strongly) FI- τ_M -lifting if for every fully invariant submodule K of N , there

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is a decomposition $K = A \oplus B$, such that A is a (fully invariant) direct summand of N and $B \subseteq \tau_M(N)$. In Section 1, we show that FI- τ_M -lifting modules are closed under finite direct sums. We prove that if module R_R is FI- τ_M -lifting then R/I has a projective τ_M -cover for every two sided ideal I of R . In Section 2, We show that a direct summand of a strongly FI- τ_M -lifting module is strongly FI- τ_M -lifting and that a finite direct sum of copies of a strongly FI- τ_M -lifting module is strongly FI- τ_M -lifting.

2. FI- τ_M -LIFTING MODULES

Lemma 2.1. *Let X be a τ_M -supplement submodule of N and $K \subseteq X$. Then X/K is a τ_M -supplement submodule of N/K .*

Proof. See [4, 10.12(3)]. □

Lemma 2.2. *Let M be a module. Then:*

- (1) *Any sum or intersection of fully invariant submodules of M is again a fully invariant submodule of M (in fact the fully invariant submodules form a complete modular sublattice of the lattice of submodules of M).*
- (2) *If $X \subseteq Y \subseteq M$ such that Y is a fully invariant submodule of M and X is a fully invariant submodule of Y , then X is a fully invariant submodule of M .*
- (3) *If $M = \bigoplus_{i \in I} X_i$ and S is a fully invariant submodule of M , then $S = \bigoplus_{i \in I} \pi_i(S) = \bigoplus_{i \in I} (X_i \cap S)$, where π_i is the i -th projection homomorphism of M .*
- (4) *If $X \subseteq Y \subseteq M$ such that X is a fully invariant submodule of M and Y/X is a fully invariant submodule of M/X , then Y is a fully invariant submodule of M .*

Proof. (1), (2), (3) See [3, Lemma 1.1]. (4) Let $f : M \rightarrow M$ be a homomorphism. Then $f(X) \subseteq X$. Now, consider the homomorphism $g : M/X \rightarrow M/X$ defined by $g(m + X) = f(m) + X$, ($m \in M$). Then $g(Y/X) \subseteq Y/X$. Clearly, $g(Y/X) = (f(Y) + X)/X$. Therefore $f(Y) \subseteq Y$. □

We note that if $M = \bigoplus_{i=1}^n M_i$ and N is a fully invariant submodule of M , then $N = \bigoplus_{i=1}^n (N \cap M_i)$ and $N \cap M_i$ is a fully invariant submodule of M_i .

Lemma 2.3. *Let $N \in \sigma[M]$. The following are equivalent:*

- (1) *For every submodule K of N , there is a decomposition $K = A \oplus B$, such that A is a direct summand of N and $B \subseteq \tau_M(N)$;*
- (2) *For every submodule K of N , there is a direct summand A of N such that $A \subseteq K$ and $K/A \subseteq \tau_M(N/A)$;*
- (3) *For every submodule K of N , there is a decomposition $N = A \oplus B$ such that $A \subseteq K$ and $B \cap K \subseteq \tau_M(N)$.*

Proof. See [12, Lemma 3.1]. □

A module $N \in \sigma[M]$ is called τ_M -lifting if it satisfies one of the equivalent conditions of Lemma 2.3.

Proposition 2.4. *Let $N \in \sigma[M]$. The following are equivalent:*

- (1) For every fully invariant submodule K of N , there is a decomposition $K = A \oplus B$, such that A is a direct summand of N and $B \subseteq \tau_M(N)$;
- (2) For every fully invariant submodule K of N , there is a direct summand A of N such that $A \subseteq K$ and $K/A \subseteq \tau_M(N/A)$.
- (3) For every fully invariant submodule K of N , there is a decomposition $N = A \oplus B$ such that $A \subseteq K$ and $B \cap K \subseteq \tau_M(N)$.

Proof. (1) \Rightarrow (2) Let K be a fully invariant submodule of N . By hypothesis, there exists a direct summand A of N and $B \subseteq \tau_M(N)$ such that $K = A \oplus B$. Now $N = A \oplus A'$ for some submodule A' of N . Consider the natural epimorphism $\pi : N \rightarrow N/A$. Then $\pi(B) = (B + A)/A = K/A \subseteq \tau_M(N/A)$. Therefore N is FI- τ_M -lifting module.

(2) \Rightarrow (3) By [12, Lemma 3.1].

(3) \Rightarrow (1) Let K be a fully invariant submodule of N . By hypothesis, there is a decomposition $N = A \oplus B$ such that $A \subseteq K$ and $B \cap K \subseteq \tau_M(N)$. Therefore $K = A \oplus (K \cap B)$, as required. \square

A module $N \in \sigma[M]$ is called τ_M -FI-lifting if it satisfies one of the equivalent conditions of Proposition 2.4.

Theorem 2.5. Let $N = \bigoplus_{i=1}^n N_i$ be a direct sum of τ_M -FI-lifting modules. Then N is τ_M -FI-lifting.

Proof. Let $K \leq N$. Then $K = \bigoplus_{i=1}^n (K \cap N_i)$ and $K \cap N_i$ is a fully invariant submodule of N_i . As each N_i is τ_M -FI-lifting we have $K \cap N_i = A_i \oplus B_i$ where A_i is a direct summand of N_i and $B_i \subseteq \tau_M(N_i)$. Put $A = \bigoplus_{i=1}^n A_i$ and $B = \bigoplus_{i=1}^n B_i$. Then $K = A \oplus B$ where A is a direct summand of N and $B = \bigoplus_{i=1}^n B_i \subseteq \bigoplus_{i=1}^n \tau_M(N_i) = \tau_M(\bigoplus_{i=1}^n N_i) = \tau_M(N)$. \square

Corollary 2.6. If N is a finite direct sum of τ_M -lifting modules, then N is τ_M -FI-lifting.

Let $N \in \sigma[M]$. We call an epimorphism $f : P \rightarrow N$ a projective τ_M -cover of N in $\sigma[M]$ if P is projective in $\sigma[M]$ and $\text{Ker}(f) \subseteq \tau_M(P)$.

Theorem 2.7. Let P be a projective module. If P is FI-lifting then P/A has a projective τ_M -cover for every fully invariant submodule A of P .

Proof. Suppose P is a projective FI-lifting module and A is a fully invariant submodule of P . Then $A = X \oplus S$ where X is a direct summand of P and $S \subseteq \tau_M(P)$. Suppose $P = X \oplus Y$. As $S \subseteq \tau_M(P)$, $(X + S)/X \subseteq (X + \tau_M(P))/X \subseteq \tau_M(P/X)$. Hence the natural map $f : P/X \rightarrow P/(X + S) = P/A$ is a projective τ_M -cover. \square

Corollary 2.8. Suppose R is a ring. If module R_R is FI- τ_M -lifting then R/I has a projective τ_M -cover for every two sided ideal I of R .

Proposition 2.9. Let N be a FI- τ_M -lifting module. Then every fully invariant submodule of $N/\tau_M(N)$ is a direct summand.

Proof. Let $K/\tau_M(N)$ be a fully invariant submodule of $N/\tau_M(N)$. Then K is fully invariant submodule by Lemma 2.2. By hypothesis, there is a decomposition $N = N_1 \oplus N_2$ such that $N_1 \subseteq K$ and $K \cap N_2 \subseteq \tau_M(N)$. Thus $N/\tau_M(N) = (K/\tau_M(N)) \oplus ((N_2 + \tau_M(N))/\tau_M(N))$, as required. \square

3. STRONGLY FI- τ_M -LIFTING MODULES

In this section we define strongly FI- τ_M -lifting modules. This class of modules is properly contained in the class of FI- τ_M -lifting modules; but there is no containment relation between the class of strongly FI- τ_M -lifting modules and the class of lifting modules. We show that a direct summand of a strongly FI- τ_M -lifting module is strongly FI- τ_M -lifting and that a finite direct sum of copies of a strongly FI- τ_M -lifting module is strongly FI- τ_M -lifting.

As in Proposition 2.4 we can prove the following.

Proposition 3.1. *Let $N \in \sigma[M]$. The following are equivalent:*

- (1) *For every fully invariant submodule K of N , there is a decomposition $K = A \oplus B$, such that A is a fully invariant direct summand of N and $B \subseteq \tau_M(N)$;*
- (2) *For every fully invariant submodule K of N , there is a fully invariant direct summand A of N such that $A \subseteq K$ and $K/A \subseteq \tau_M(N/A)$.*

A module $N \in \sigma[M]$ is called *strongly FI- τ_M -lifting* if it satisfies one of the equivalent conditions of Proposition 3.1.

Proposition 3.2. *Let N be an FI- τ_M -lifting with $\tau_M(N) = 0$. Then every fully invariant submodule (in particular N) is strongly FI- τ_M -lifting module.*

Proof. Let K be a fully invariant submodule of N . Suppose A is fully invariant in K . Then A is fully invariant in N also (see Lemma 2.2). As N is FI- τ_M -lifting, $A = B \oplus S$ where B is a direct summand of N and $S \subseteq \tau_M(N)$ (see Proposition 2.4). Since $\tau_M(N) = 0$, $S = 0$ and so A is a direct summand of N and hence of K . Thus K is strongly FI-lifting. \square

Theorem 3.3. *A direct summand of a strongly FI- τ_M -lifting module is strongly FI- τ_M -lifting.*

Proof. Let $N = X \oplus Y$ be a strongly FI- τ_M -lifting module. Assume that $S_1 \trianglelefteq X$. Then there exists $S_2 \trianglelefteq Y$ such that $S_1 \oplus S_2 \trianglelefteq M$ [5, Lemma 1.11]. Since N is a strongly FI- τ_M -lifting, $S_1 \oplus S_2 = B \oplus S$ where $S \subseteq \tau_M(N)$ and B is a fully invariant direct summand of N . But $B \trianglelefteq N$ implies that $B = (X \cap B) \oplus (Y \cap B)$ and $X \cap B$ is fully invariant in X . Also $X \cap B$ is a direct summand of N . We have $S_1 = \pi_X(B) + \pi_X(S) = (X \cap B) + \pi_X(S)$ where $\pi_X : N \rightarrow X$ is the projection along Y . As $S \subseteq \tau_M(N)$, $\pi_X(S) \subseteq \tau_M(X)$. By Proposition 3.1, X is a strongly FI- τ_M -lifting module. \square

Proposition 3.4. *Let $N = \bigoplus_{i=1}^n N_i$ and let $N_i \trianglelefteq M$ for all $1 \leq i \leq n$. Then N is strongly FI- τ_M -lifting if and only if N_i is strongly FI- τ_M -lifting, for all $1 \leq i \leq n$.*

Proof. If N is strongly FI- τ_M -lifting then each N_i is so, by Proposition 3.4.

Conversely, suppose $N = \bigoplus_{i=1}^n N_i$ where each N_i is strongly FI- τ_M -lifting and fully invariant in N . Let $K \trianglelefteq N$. Then $K = \bigoplus_{i=1}^n (K \cap N_i)$ and $(K \cap N_i) \trianglelefteq N_i$, for all $1 \leq i \leq n$. As N_i is strongly FI- τ_M -lifting, $K \cap N_i = B_i \oplus S_i$ where B_i is a fully invariant direct summand of N_i and $S_i \subseteq \tau_M(N_i)$ (see Proposition 3.1). Put $B = \bigoplus_{i=1}^n B_i$ and $S = \bigoplus_{i=1}^n S_i$. Then $K = B \oplus S$ where B is a direct summand of N and $S \subseteq \tau_M(N)$. As $B_i \trianglelefteq N_i$ and $N_i \trianglelefteq N$, $B_i \trianglelefteq N$ for all $1 \leq i \leq n$. Hence $B \trianglelefteq N$. Therefore N is strongly FI- τ_M -lifting. \square

Theorem 3.5. *Suppose K is a strongly FI- τ_M -lifting module and $N = \bigoplus_{i=1}^n N_i$ where each $N_i \simeq K$. Then N is a strongly FI- τ_M -lifting module.*

Proof. There exist isomorphisms $f_i : N_1 \rightarrow N_i$ for $i = 2, \dots, n$. If A is a fully invariant submodule of N , then it is easy to see that $A = A_1 \oplus f_2(A_1) \oplus \dots \oplus f_n(A_1)$ where $A_1 = N_1 \cap A$.

As N_1 is strongly FI- τ_M -Lifting and A_1 is a fully invariant submodule of N_1 , we have $A_1 = L_1 \oplus S_1$ where L_1 is a fully invariant submodule of N_1 and $S_1 \subseteq \tau_M(M_1)$ (see Proposition 3.1). Put $L := L_1 \oplus f_2(L_1) \oplus \dots \oplus f_n(L_1)$ and $S := S_1 \oplus f_2(S_1) \oplus \dots \oplus f_n(S_1)$. Then $A = L \oplus S$, L is a fully invariant direct summand of N and $S \subseteq \tau_M(N)$. Hence N is strongly FI- τ_M -lifting. \square

From Proposition 3.3 and Theorem 3.5 we get the following.

Corollary 3.6. *Suppose R is a ring and R_R is strongly FI- τ_M -lifting. Then any finitely generated projective R -module is strongly FI- τ_M -lifting.*

Proposition 3.7. *Let N be a strongly FI- τ_M -lifting module and X a fully invariant submodule of N . If X is indecomposable, then X is strongly FI- τ_M -lifting.*

Proof. Let $K \triangleleft X$ then $K \triangleleft N$. Since N is strongly FI- τ_M -lifting module, $K = B \oplus S$ where $S \subseteq \tau_M(N)$ and B is a fully invariant direct summand of N . Hence $N = B \oplus C$ for some submodule C of N . Since $X \triangleleft N$, then $X = B \oplus (C \cap X)$. But X is indecomposable, therefore $X = B$ and X is direct summand of N . By Theorem 3.3, X is strongly FI- τ_M -lifting. \square

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