

## REGULAR LIFTING OF COVERS OVER AMPLE FIELDS

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ABSTRACT. Let  $K$  be an ample field,  $G$  a finite group, and  $L$  a finite Galois extension of  $K$  such that  $\text{Gal}(L/K)$  is isomorphic to a subgroup of  $G$ . We prove that  $K(x)$  has a Galois extension  $F$  which is regular over  $L$  such that  $\text{Gal}(F/K(x)) \cong G$  and  $F$  has a  $K$ -place  $\varphi$  such that  $\varphi(x) \in K$  and  $\varphi(F) = L \cup \{\infty\}$ .

### 1. INTRODUCTION

Colliot-Thélène [CoT00] uses the technique of Kollár, Miyaoka, and Mori to prove the following result.

**Theorem A:** *Let  $K$  be an ample field of characteristic 0,  $x$  a transcendental element over  $K$ , and  $G$  a finite group. Then there is a Galois extension  $F$  of  $K(x)$  with Galois group  $G$ , regular over  $K$ .*

Here  $K$  is said to be **ample** if every absolutely irreducible curve defined over  $K$  with a  $K$ -rational simple point has infinitely many  $K$ -rational simple points.

In fact, Colliot-Thélène proves a stronger version, still under the assumption that  $K$  is ample and  $\text{char}(K) = 0$ :

**Theorem B:** *Given a Galois extension  $L/K$  with Galois group  $\Gamma$  which is a subgroup of  $G$ , there exist a Galois extension  $F$  of  $K(x)$  with  $\text{Gal}(F/K(x)) \cong G$  and a place  $\varphi$  that fixes the elements of  $K$  and the residue field extension of  $F/K(x)$  under  $\varphi$  is  $L/K$ .*

Case  $\Gamma = G$  of Theorem B means that  $K$  has the arithmetic lifting property of Beckmann and Black [Bla99].

Since the results of Kollár, Miyaoka, and Mori are valid only in characteristic 0, Colliot-Thélène's proof works only in this case. Nonetheless, Theorem A holds in arbitrary characteristic ([Har87, Corollary 2.4] for complete fields, [Pop96, Main Theorem A]; see also [Liu95] and [HaV96]). Theorem B can be deduced for arbitrary characteristic from Théorème 1.1 of [MoB01]. The proof of that paper uses methods of formal patching.

Here we use algebraic patching to prove Theorem B for arbitrary characteristic. In fact, the main ingredient of the proof is almost contained in [HaJ98]. Therefore

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this note can be considered a sequel to [HaJ98]; a large portion of it recalls the situation and facts considered there.

The idea (displayed in our Lemma 3.1) to use the embedding problem  $G \times G \rightarrow G$  in order to obtain the arithmetic lifting property has been used in [Pop99]; we are grateful to F. Pop for making his note available to us.

## 2. EMBEDDING PROBLEMS AND DECOMPOSITION GROUPS

Let  $K/K_0$  be a finite Galois extension with Galois group  $\Gamma$ . Let  $x$  be a transcendental element over  $K$ . Put  $E_0 = K_0(x)$ . Suppose that  $\Gamma$  acts (from the right) on a finite group  $G$ ; let  $\Gamma \ltimes G$  be the corresponding semidirect product and  $\pi: \Gamma \ltimes G \rightarrow \Gamma$  the canonical projection. We call

$$(1) \quad \pi: \Gamma \ltimes G \rightarrow \Gamma = \text{Gal}(K/K_0)$$

a **finite constant split embedding problem**. A **solution** of (1) is a Galois extension  $F$  of  $E_0$  such that  $K \subseteq F$ ,  $\text{Gal}(F/E_0) = \Gamma \ltimes G$ , and  $\pi$  is the restriction map  $\text{res}_K: \text{Gal}(F/E_0) \rightarrow \text{Gal}(K/K_0)$ .

In [HaJ98, Theorem 6.4] we reprove the following result of F. Pop [Pop96]:

**Proposition 2.1.** *Let  $K_0$  be an ample field. Then each finite constant split embedding problem (1) has a solution  $F$  such that  $F$  has a  $K$ -rational place  $\varphi$  such that  $\varphi(x) \in K_0 \cup \{\infty\}$  (in particular,  $F/K$  is regular).*

In this section we show that the proof of Proposition 5.2 in [HaJ98] yields a stronger assertion.

We denote the residue field of a place  $\varphi$  of a field  $F$  by  $\bar{F}_\varphi$ .

**Lemma 2.2.** *Let  $F$  be a solution of (1). Put  $F_0 = F^\Gamma$ . Let  $\varphi: F \rightarrow \widetilde{K_0} \cup \{\infty\}$  be a  $K$ -place with  $\varphi(x) \in K_0 \cup \{\infty\}$ . Assume that  $\varphi$  is unramified in  $F/E_0$  and let  $D_\varphi$  be its decomposition group in  $F/E_0$ . Then  $K \subseteq \bar{F}_\varphi$  and the following assertions are equivalent:*

- (a)  $K = \bar{F}_\varphi$  and  $\Gamma = D_\varphi$ ;
- (b)  $D_\varphi \subseteq \Gamma$ ;
- (c)  $K_0 = \bar{F}_{0,\varphi}$ ;
- (d)  $K = \bar{F}_\varphi$  and  $\varphi(f^\gamma) = \varphi(f)^\gamma$  for each  $\gamma \in \Gamma$  and  $f \in F$  with  $\varphi(f) \neq \infty$ .

*Proof.* Since  $K \subseteq F$ , we have  $K = \bar{K}_\varphi \subset \bar{F}_\varphi$ . Since the inertia group of  $\varphi$  in  $F/E_0$  is trivial, we have an isomorphism  $\theta: D_\varphi \rightarrow \text{Gal}(\bar{F}_\varphi/K_0)$  given by

$$(2) \quad \varphi(f^\gamma) = \varphi(f)^{\theta(\gamma)}, \quad \gamma \in D_\varphi, f \in F, \varphi(f) \neq \infty.$$

Hence,  $|D_\varphi| = [\bar{F}_\varphi : K_0] \geq [K : K_0] = |\Gamma|$ . This gives (a)  $\Leftrightarrow$  (b).

Since  $\varphi$  is unramified over  $E_0$ , the decomposition field  $F^{D_\varphi}$  is the largest intermediate field of  $F/E_0$  mapped by  $\varphi$  into  $K_0 \cup \{\infty\}$ , and hence (b)  $\Leftrightarrow$  (c).

Clearly (d)  $\Rightarrow$  (c). If  $\bar{F}_\varphi = K$ , then  $f^\gamma = \varphi(f^\gamma) = \varphi(f)^{\theta(\gamma)} = f^{\theta(\gamma)}$  for all  $f \in K$  and  $\gamma \in D_\varphi$  (by (2)). Hence,  $\theta(\gamma) = \gamma$  for all  $\gamma \in D_\varphi$ . Applying (2) once more, we have  $\varphi(f^\gamma) = \varphi(f)^{\theta(\gamma)} = \varphi(f)^\gamma$  for each  $f \in F$  with  $\varphi(f) \neq \infty$  and  $\gamma \in D_\varphi$ . Consequently, (a)  $\Rightarrow$  (d). □

**Remark 2.3.** *Let  $K_0$  be an ample field and  $F$  a solution of (1). Suppose  $F$  has a  $K$ -rational place  $\varphi$  unramified over  $E_0$  such that  $\varphi(x) \in K_0 \cup \{\infty\}$  and  $\Gamma$  is the decomposition group of  $\varphi$  in  $F/E_0$ . Then  $F$  has infinitely many such places.*

*Proof.* Indeed, put  $F_0 = F^\Gamma$ . Recall that  $F_0$  is regular over  $K_0$ . By Lemma 2.2,

(a) the assumption is that there is a  $K_0$ -place  $\varphi: F_0 \rightarrow K_0$  unramified over  $K_0(x)$ , and

(b) we have to show that there are infinitely many such places.

But (a)  $\Rightarrow$  (b) is a property of an ample field.  $\square$

**Proposition 2.4.** *Let  $K_0$  be an ample field. Then each finite constant split embedding problem (1) has a solution  $F$  with a  $K$ -rational place  $\varphi$  of  $F$  unramified over  $E_0$  such that  $\varphi(x) \in K_0 \cup \{\infty\}$  and  $\Gamma$  is the decomposition group of  $\varphi$  in  $F/E_0$ .*

*Proof.* Put  $E = K(x) = KK_0(x)$ .

**Part A:** As in the proof of [HaJ98, Theorem 6.4], we first assume that  $K_0$  is complete with respect to a non-trivial discrete ultrametric absolute value  $|\cdot|$ , with infinite residue field and  $K/K_0$  is unramified.

In this case [HaJ98, Proposition 5.2] proves Proposition 2.1. Claim C of that proof shows that, for every  $b \in K_0$  with  $|b| > 1$ ,  $x \rightarrow b$  extends to a  $K$ -homomorphism  $\varphi_b: R \rightarrow K$ , where  $R$  is the principal ideal ring  $K\{\frac{1}{x-c_i} \mid i \in I\}$  and the  $c_i$ 's are properly chosen elements of  $K$ . From there it extends to a  $K$ -place  $\varphi_b: Q \rightarrow K \cup \{\infty\}$  of the  $Q = \text{Quot}(R)$ . Furthermore, [HaJ98, Lemma 1.3(b)] gives an  $E$ -embedding  $\lambda: F \rightarrow Q$ . The compositum  $\varphi = \varphi_b \circ \lambda$  is a  $K$ -rational place of  $F$ . Excluding finitely many  $b$ 's we may assume that  $\varphi$  is unramified over  $E_0$ . To verify that  $\varphi$  satisfies condition (d) of Lemma 2.2, we first recall the relevant facts from [HaJ98].

(a) [HaJ98, Proposition 5.2, Construction B] The group  $\Gamma = \text{Gal}(K/K_0)$  lifts isomorphically to  $\text{Gal}(E/E_0)$ . By the choice of the  $c_i$  we have  $(\frac{1}{x-c_i})^\gamma = \frac{1}{x-c_i^\gamma}$ , for each  $\gamma \in \Gamma$ . It follows that  $\Gamma$  continuously acts on  $R$  in the following way

$$\left(a_0 + \sum_{i \in I} \sum_{n=1}^{\infty} a_{in} \left(\frac{1}{x-c_i}\right)^n\right)^\gamma = a_0^\gamma + \sum_{i \in I} \sum_{n=1}^{\infty} a_{in}^\gamma \left(\frac{1}{x-c_i^\gamma}\right)^n.$$

This action induces an action of  $\Gamma$  on  $Q$ .

(b) [HaJ98, (7) on p. 334] The above mentioned action of  $\Gamma$  on  $Q$  defines an action of  $\Gamma$  on the  $Q$ -algebra

$$N = \text{Ind}_1^G Q = \left\{ \sum_{\theta \in G} a_\theta \theta \mid a_\theta \in Q \right\}$$

in the following way:

$$\left( \sum_{\theta \in G} a_\theta \theta \right)^\gamma = \sum_{\theta \in G} a_\theta^\gamma \theta^\gamma \quad a_\theta \in Q, \gamma \in \Gamma.$$

Furthermore, the field  $F$  is a subring of  $N$  [HaJ98, p. 332] and  $\Gamma$  acts on it by restriction from  $N$  [HaJ98, Proof of Proposition 1.5, Part A].

(c) The embedding  $\lambda: F \rightarrow Q$  is the restriction to  $F$  of the projection

$$\sum_{\theta \in G} a_\theta \theta \mapsto a_1$$

from  $N = \text{Ind}_1^G Q$  onto  $Q$  [HaV96, Proposition 3.4].

(d) The place  $\varphi_b: Q \rightarrow K \cup \{\infty\}$  is induced from the evaluation homomorphism  $\varphi_b: R \rightarrow K$  given by [HaJ98, Remark 3.5]

$$\varphi_b \left( a_0 + \sum_{i \in I} \sum_{n=1}^{\infty} a_{in} \left( \frac{1}{x - c_i} \right)^n \right) = a_0 + \sum_{i \in I} \sum_{n=1}^{\infty} a_{in} \left( \frac{1}{b - c_i} \right)^n.$$

In order to prove condition (d) of Lemma 2.2 it suffices to show that both  $\lambda$  and  $\varphi_b$  are  $\Gamma$ -equivariant.

Let  $f = \sum_{\theta \in G} a_\theta \theta \in F \subseteq N$ . Then, by (b) and (c),

$$\lambda(f^\gamma) = \lambda \left( \sum_{\theta \in G} a_\theta^\gamma \theta^\gamma \right) = a_1^\gamma = \left( \lambda \left( \sum_{\theta \in G} a_\theta \theta \right) \right)^\gamma = \lambda(f)^\gamma.$$

Furthermore, let  $r = a_0 + \sum_{i \in I} \sum_{n=1}^{\infty} a_{in} \left( \frac{1}{x - c_i} \right)^n \in R$ . By (a) and (d),

$$\begin{aligned} \varphi_b(r^\gamma) &= \varphi_b \left( a_0^\gamma + \sum_{i \in I} \sum_{n=1}^{\infty} a_{in}^\gamma \left( \frac{1}{x - c_i^\gamma} \right)^n \right) = a_0^\gamma + \sum_{i \in I} \sum_{n=1}^{\infty} a_{in}^\gamma \left( \frac{1}{b - c_i^\gamma} \right)^n \\ &= \left( a_0 + \sum_{i \in I} \sum_{n=1}^{\infty} a_{in} \left( \frac{1}{b - c_i} \right)^n \right)^\gamma = \varphi_b(r)^\gamma. \end{aligned}$$

Thus  $\varphi_b$  is  $\Gamma$ -equivariant.

**Part B:**  $K_0$  is an arbitrary ample field. As in the proof of [HaJ98, Theorem 6.4] let  $\hat{K}_0 = K_0((t))$  be the field of formal power series in  $t$  over  $K_0$ . Then  $\hat{K} = K\hat{K}_0$  is an unramified extension of  $\hat{K}_0$  with Galois group  $\Gamma$  and infinite residue field.

By Part A,  $\hat{K}_0(x)$  has a Galois extension  $\hat{F}$  which contains  $\hat{K}(x)$ , such that  $\text{Gal}(\hat{F}/\hat{K}_0(x)) = \Gamma \rtimes G$  and the restriction map  $\text{Gal}(\hat{F}/\hat{K}_0(x)) \rightarrow \text{Gal}(K/K_0)$  is the projection  $\pi: \Gamma \rtimes G \rightarrow \Gamma$ . Furthermore, there is  $b \in \hat{K}_0$  such that the place  $x \rightarrow b$  of  $\hat{K}_0(x)$  extends to an unramified  $\hat{K}$ -place  $\hat{\varphi}: \hat{F} \rightarrow \hat{K} \cup \{\infty\}$  and  $\hat{\varphi}(\hat{F}^\Gamma) = \hat{K}_0$ . Put  $m = |G|$ .

Use the Weak Approximation to find  $y \in \hat{F}^\Gamma$  mapped by the  $m$  distinct extensions of  $x \rightarrow b$  to  $\hat{F}^\Gamma$  into  $m$  distinct elements of the separable closure of  $\hat{K}_0$ ; then  $\hat{F}^\Gamma = \hat{K}_0(x, y)$ .

Thus there exist polynomials  $f \in \hat{K}_0[X, Z]$ ,  $g \in \hat{K}_0[X, Y]$ , elements  $z \in \hat{F}$ ,  $y \in \hat{F}^\Gamma$ , and elements  $b, c \in \hat{K}_0$ , such that the following conditions hold:

(3a)  $\hat{F} = \hat{K}_0(x, z)$ ,  $f(x, Z) = \text{irr}(z, \hat{K}_0(x))$ ; we identify  $\text{Gal}(f(x, Z), \hat{K}_0(x))$  with  $\text{Gal}(\hat{F}/\hat{K}_0(x))$ ;

(3b)  $\hat{F}^\Gamma = \hat{K}_0(x, y)$ , whence  $\hat{F} = \hat{K}(x, y)$ , and  $g(x, Y) = \text{irr}(y, \hat{K}_0(x))$ ; therefore  $g(X, Y)$  is absolutely irreducible;

(3c)  $\text{discr}(g(b, Y)) \neq 0$  and  $g(b, c) = 0$ .

All of these objects depend on only finitely many parameters from  $\hat{K}_0$ . Hence, there are  $u_1, \dots, u_n \in \hat{K}_0$  such that the following conditions hold:

(4a)  $F = K_0(\mathbf{u}, x, z)$  is a Galois over  $K_0(\mathbf{u}, x)$ , the coefficients of  $f(X, Z)$  lie in  $K_0[\mathbf{u}]$ ,  $f(x, Z) = \text{irr}(z, K_0(\mathbf{u}, x))$ , and  $\text{Gal}(f(x, Z), K_0(\mathbf{u}, x)) = \text{Gal}(f(x, Z), \hat{K}_0(x))$ ;

(4b) the coefficients of  $g$  lie in  $K[\mathbf{u}]$ ; hence  $g(x, Y) = \text{irr}(y, K_0(\mathbf{u}, x))$ ; furthermore,  $K_0(\mathbf{u}, x, y) = F^\Gamma$ ;

(4c)  $b, c \in K_0[\mathbf{u}]$ ,  $\text{discr}(g(b, Y)) \neq 0$ , and  $g(b, c) = 0$ .

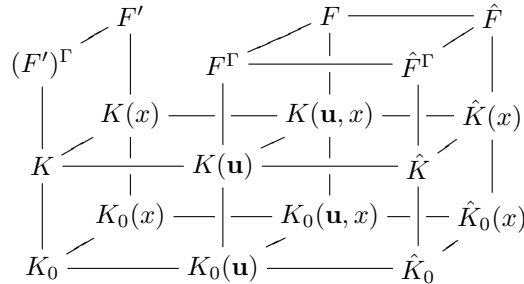
Since  $\hat{K}_0$  has a  $K$ -rational place, namely,  $x \rightarrow 0$ , the field  $\hat{K}_0$  and therefore also  $K_0(\mathbf{u})$  are regular extensions of  $K_0$ . Thus,  $\mathbf{u}$  generates an absolutely irreducible variety  $U = \text{Spec}(K_0[\mathbf{u}])$  defined over  $K_0$ . By Bertini-Noether [FrJ05, Proposition 9.4.3], the variety  $U$  has a nonempty Zariski open subset  $U'$  such that for each  $\mathbf{u}' \in U'$  the  $K_0$ -specialization  $\mathbf{u} \rightarrow \mathbf{u}'$  extends to a  $K$ -homomorphism  $\prime: K[\mathbf{u}, x, z, y] \rightarrow K[\mathbf{u}', x, z', y']$  such that the following conditions hold:

(5a)  $f'(x, z') = 0$ , the discriminant of  $f'(x, Z)$  is not zero, and  $F' = K_0(\mathbf{u}', x, z')$  is the splitting field of  $f'(x, Z)$  over  $K_0(\mathbf{u}', x)$ ; in particular  $F'/K_0(\mathbf{u}', x)$  is Galois;

(5b)  $g'(X, Y)$  is absolutely irreducible and  $g'(x, y') = 0$ ; so  $g'(x, Y) = \text{irr}(y', K(\mathbf{u}', x))$ ; furthermore,  $K_0(\mathbf{u}', x, y') = (F')^\Gamma$ ;

(5c)  $b', c' \in K_0[\mathbf{u}']$  and  $\text{discr}(g'(b', Y)) \neq 0$  and  $g'(b', c') = 0$ .

By assumption,  $K_0$  is ample, so  $K_0$  is existentially closed in  $\hat{K}_0$  [Pop96, Prop. 1.1]. Since  $\mathbf{u} \in U(\hat{K}_0)$ , there is a  $\mathbf{u}' \in U(K_0)$ . Now repeat the end of the proof of [HaJ98, Lemma 6.2] (from “By (5a), the homomorphism . . .”) to conclude that  $F'$  is a solution of (1).



Condition (5c) ensures that the place  $x \rightarrow b'$  of  $K_0(x)$  is unramified in  $(F')^\Gamma$ , hence in  $F'$ , and extends to a  $K_0$ -rational place of  $(F')^\Gamma$ . This ends the proof by Lemma 2.2.

□

## 3. LIFTING PROPERTY OVER AMPLE FIELDS

Consider a subgroup  $\Gamma$  of a finite group  $G$ , let  $\Gamma$  act on  $G$  by the conjugation in  $G$

$$g^\gamma = \gamma^{-1}g\gamma.$$

and consider the semidirect product  $\Gamma \ltimes G$ . To fix notation,

$$\Gamma \ltimes G = \{(\gamma, g) \mid \gamma \in \Gamma, g \in G\}$$

and the multiplication on  $\Gamma \ltimes G$  is defined by

$$(\gamma_1, g_1)(\gamma_2, g_2) = (\gamma_1\gamma_2, g_1^{\gamma_2}g_2).$$

Notice the isomorphism  $\Gamma \ltimes G \cong \Gamma \times G$  given by  $(\gamma, g) \mapsto (\gamma, \gamma g)$  and the epimorphism  $\rho: \Gamma \ltimes G \rightarrow G$  given by  $(\gamma, g) \mapsto \gamma g$ . Let  $N = \text{Ker}(\rho)$ .

**Lemma 3.1.** *Let  $K_0$  be a field,  $K$  a Galois extension of  $K_0$  with Galois group  $\Gamma$ , and  $x$  a transcendental element over  $K_0$ . Assume that (1) has a solution  $\hat{F}$  with a  $K$ -rational place  $\hat{\varphi}$  of  $\hat{F}$  unramified over  $K_0(x)$  such that  $\hat{\varphi}(x) \in K_0 \cup \{\infty\}$  and  $\Gamma$  is the decomposition group of  $\hat{\varphi}$  in  $\hat{F}/K_0(x)$ . Let  $F = \hat{F}^N$  and let  $\varphi$  be the restriction of  $\hat{\varphi}$  to  $F$ . Then*

(6a)  $F$  is a Galois extension of  $K_0(x)$  and  $\text{Gal}(F/K_0(x)) \cong G$ ;

(6b)  $F/K_0$  is a regular extension;

(6c)  $\varphi$  represents a prime divisor  $\mathfrak{p}$  of  $F/K_0$  with decomposition group  $\Gamma$  in  $F/K_0(x)$  and residue field  $K$ .

*Proof.* By assumption,  $\hat{F}$  is a Galois extension of  $K_0(x)$  containing  $K$ , with Galois group  $\Gamma \ltimes G$  such that the restriction  $\text{Gal}(\hat{F}/K_0(x)) \rightarrow \text{Gal}(K/K_0)$  is the projection  $\Gamma \ltimes G \rightarrow \Gamma$ , and  $\hat{F}/K$  is regular. Furthermore,  $\hat{\varphi}: \hat{F} \rightarrow K$  is a  $K$ -place unramified over  $K_0(x)$ , with decomposition group  $\Delta = \{(\gamma, 1) \mid \gamma \in \Gamma\} \cong \Gamma$  in  $\hat{F}/K_0(x)$  and residue field extension  $K/K_0$ . In particular,  $\hat{F}$  is regular over  $K$ .

From the definition of  $F$  we get (6a) and  $\rho(\Delta) = \Gamma \leq G$  is the decomposition group of the restriction  $\varphi: F \rightarrow K$  of  $\hat{\varphi}$  to  $F$ . Since  $|\Delta| = [K : K_0]$ , the residue field of  $\varphi$  is  $K$ . Since  $\Gamma \ltimes G = NG$ , the fields  $F = \hat{F}^N$  and  $K(x) = \hat{F}^G$  are linearly disjoint over  $K_0(x)$ . In addition,  $FK = \hat{F}$  and  $\hat{F}/K$  is regular. Therefore,  $F$  is regular over  $K_0$ .  $\square$

Lemma 3.1 together with Proposition 2.4 and Remark 2.3 yield the following result:

**Theorem 3.2.** *Let  $K_0$  be an ample field,  $G$  a finite group,  $\Gamma$  a subgroup,  $K$  a Galois extension of  $K_0$  with Galois group  $\Gamma$ , and  $x$  a transcendental element over  $K_0$ . Then there is a field  $F$  that satisfies (6a), (6b) and*

(6d) *there are infinitely many prime divisors  $\mathfrak{p}$  of  $F/K_0$  with decomposition group  $\Gamma$  in  $F/K_0(x)$  and residue field  $K$ .*

**Remark 3.3.** *In case of  $\Gamma = G$ , Theorem 3.2 says that an ample field  $K_0$  has the so-called **arithmetic lifting property** of Beckmann-Black [Bla99].*

**Remark 3.4.** *In the special case where  $K$  is a PAC field, it is possible to refine Theorem 3.2. In this case if  $F$  is an arbitrary Galois extension of  $K(x)$  regular over  $K$  and  $L/K$  is a Galois extension with Galois group isomorphic to a subgroup of  $\text{Gal}(F/K(x))$ , there exists a place  $\varphi$  of  $F$  such that the residue field extension of  $F/K(x)$  under  $\varphi$  is  $L$  [Deb99, Remark 3.3]. This stronger property of PAC fields does not hold for an arbitrary ample field  $K$  [CoT00, Appendix].*

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