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SAMPLE EXTREMES OF L_p-NORM ASYMPTOTICALLY SPHERICAL DISTRIBUTIONS

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Abstract. In this paper we deal with the asymptotic behaviour of sample maxima of L_p-norm asymptotically spherical random vectors. If the distribution function of the associated random radius of such random vectors is in the Gumbel of the Weibull max-domain of attraction we show that the normalised sample maxima has asymptotic independent components converging in distribution to a random vector with unit Gumbel or Weibull components. When the associated random radius has distribution function in the Fréchet max-domain we show that the sample maxima has asymptotic dependent components.

1. Introduction

Let $\boldsymbol{X} = (X_1, \dots, X_d)^\top, d \geq 2$, be a random vector in $\mathbb{R}^d, d \geq 2$, defined by

$$\boldsymbol{X} = R\boldsymbol{U}_d, \tag{1}$$

where R is an almost surely positive random variable independent of the random vector $\boldsymbol{U}_d = (U_1, \dots, U_d)^\top$ ($^\top$ stands here for the transpose sign).
Suppose that for some $p \in (0, \infty)$ we have almost surely

$$\sum_{i=1}^{d} |U_i| = 1$$

and the random vector $(U_1, \dots, U_{d-1})^\top$ possesses probability density function

$$p(u_1, \dots, u_{d-1}) \quad = \quad \frac{p^{d-1}\Gamma(d/p)}{2^{d-1}\Gamma(1/p)}\Big(1 - \sum_{i=1}^{d-1} |u_i|^p\Big)^{(1-p)/p}, \quad i \leq d-1,$$

defined for any $u_i \in [-1, 1], i \leq d$ such that $\sum_{i=1}^{d-1} |u_i|^p < 1$, where $\Gamma(\cdot)$ denotes the Gamma function.
Following Gupta and Song (1997) we shall call $\boldsymbol{X}$ with stochastic representation (1) a L_p-norm spherical random vector. In the case $p = 2$ the random vector $\boldsymbol{X}$ reduces to a spherical symmetrical (L_2-norm) random vector with the distribution function invariant with respect to orthogonal transformations in $\mathbb{R}^d$.

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Gupta and Song (1997), Szabłowski (1998) derive the basic distributional properties of L_p-norm spherical random vectors. The asymptotic properties of this class of random vectors have not been investigated so far, in particular in the literature no result is available for the asymptotic behaviour of sample maxima, when considering samples with underling L_p-norm spherical distributions.

As it is the case for the distributional properties, for $p = 2$ several asymptotic properties of spherical random vectors are available with some early works going back to Carnal (1970), Gale (1980), Berman (1982) among several others.
In this paper we show that the basic asymptotic properties of L_2-norm spherical random vectors extend (with minor adjustments) naturally to the general L_p-norm. With motivation from Hashorva (2005) we introduce the class of L_p-norm asymptotically spherical random vectors. We shall show that this new class of random vectors is a natural generalisation of the L_p-norm spherical random vectors with respect to the asymptotic dependence and asymptotic behaviour of sample maxima; thus generalising the results of the aforementioned paper for the L_2-norm setup.

In the next section we shall introduce some notation and provide details on L_p-norm spherical random vectors and investigate their asymptotic dependence.

We then introduce L_p-norm asymptotically spherical random vectors and discuss the main asymptotic properties of this novel class. The proofs of all the results as well as some related results are relegated to Section 5.

2. Preliminaries

We start with presenting the notation and the basic distributional properties of L_p-norm spherical random vectors. For any vector $\boldsymbol{x} = (x_1, \dots, x_d)^\top \in \mathbb{R}^d, d \geq 2$ set $\boldsymbol{x}_I := (x_i, i \in I)^\top$ with I being a non-empty subset of $\{1, \dots, d\}$. We shall write $\boldsymbol{x}_I^\top$ instead of $(\boldsymbol{x}_I)^\top$. Let $\boldsymbol{y} = (y_1, \dots, y_d)^\top$ be another vector in $\mathbb{R}^d$. We define

$$\boldsymbol{x} + \boldsymbol{y} := (x_1 + y_1, \dots, x_d + y_d),$$
$$\boldsymbol{x} > \boldsymbol{y}, \text{ if } x_i > y_i, \quad \forall\, i = 1, \dots, d,$$
$$\boldsymbol{x} \geq \boldsymbol{y}, \text{ if } x_i \geq y_i, \quad \forall\, i = 1, \dots, d,$$
$$\boldsymbol{x} \neq \boldsymbol{y}, \text{ if for some } i \leq d\ x_i \neq y_i,$$
$$\boldsymbol{a}\boldsymbol{x} := (a_1 x_1, \dots, a_d x_d)^\top, \quad c\boldsymbol{x} := (c x_1, \dots, c x_d)^\top, \quad \boldsymbol{a} \in \mathbb{R}^d, c \in \mathbb{R},$$
$$\mathbf{0} := (0, \dots, 0)^\top \in \mathbb{R}^d, \quad \mathbf{1} := (1, \dots, 1)^\top \in \mathbb{R}^d,$$
$$\|\boldsymbol{x}_I\|_p := \Big(\sum_{i \in I} |x_i|^p\Big)^{1/p}, \quad \mathbb{S}_p^{k-1} := \{\boldsymbol{x} \in \mathbb{R}^k : \|\boldsymbol{x}\|_p = 1\}, \quad k \geq 1, p > 0.$$

Note that $\|\cdot\|_p$ is only for $p \in [1, \infty)$ a norm. We still refer to L_p-norm spherical random vectors even when $p \in (0, 1)$.
We shall write $Beta(a, b)$ for the distribution function of a Beta random variable with positive parameters a and b. If Z is a random variable with distribution function G we shall use alternatively the notation $Z \sim G$, and denote by G^{-1} the generalised inverse of G.

Let p be a given positive constant, and let $\boldsymbol{U}_k, k \geq 2$, denote a L_p-norm uniformly distributed random vector on $\mathbb{S}_p^{k-1}$.
Consider $\boldsymbol{X}$ as in (1) with associated random radius $R > 0$ (almost surely) with the distribution function F independent of the random vector $\boldsymbol{U}_d = (U_1, \dots, U_d)^\top$. For $p = 2$ Cambanis et al. (1981) show that for any I, J two non-empty disjoint index

sets such that $I \cup J = \{1, \ldots, d\}$ the random vector $\boldsymbol{X}$ possesses the stochastic representation

$$\boldsymbol{X}_I \stackrel{d}{=} RW_{m,d}\boldsymbol{U}_m, \quad \text{and } \boldsymbol{X}_J \stackrel{d}{=} R(1 - W_{m,d}^2)^{1/2}\boldsymbol{U}_{d-m},$$

where $\boldsymbol{U}_m, \boldsymbol{U}_{d-m}, m := |I|$ are two uniformly distributed random vectors on $\mathbb{S}_2^{m-1}$ and $\mathbb{S}_2^{d-m-1}$, respectively, and

$$W_{m,d}^2 \sim Beta(m/2, (d-m)/2), \quad W_{m,d} > 0,$$

where $\stackrel{d}{=}$ stands for equality of distribution functions of random vectors.
In Gupta and Song (1997) the above stochastic representation is generalised to the L_p-norm spherical random vectors. Referring to Theorem 3.1 therein we have for any $p > 0$ and $\boldsymbol{X}$ defined be (1)

$$\boldsymbol{X}_I \stackrel{d}{=} RW_{m,d,p}\boldsymbol{U}_m, \quad \text{and } \boldsymbol{X}_J \stackrel{d}{=} R(1 - W_{m,d,p}^p)^{1/p}\boldsymbol{U}_{d-m}, \tag{2}$$

with $\boldsymbol{U}_m, \boldsymbol{U}_{d-m}$ two independent L_p-norm uniformly distributed random vectors. Further, $R, W_{m,d,p}, \boldsymbol{U}_m, \boldsymbol{U}_{d-m}$ are mutually independent, and

$$W_{m,d,p}^p \sim Beta(m/p, (d-m)/p), \quad W_{m,d,p} > 0.$$

As shown initially by Berman (1992) the stochastic representation (2) is basic for investigating the asymptotic behaviour of L_2-norm spherical random vectors.
Utilising Berman's results and the ideas Hashorva (2005) discusses the asymptotic dependence and the asymptotic behaviour of sample extremes of asymptotically spherical and elliptical random vectors.
Next we shall consider the asymptotic dependence of L_p-norm spherical random vectors.
In view of the stochastic representation (2) we simply need to investigate the asymptotic dependence of a bivariate L_p-norm spherical random vector.
Let therefore $p > 0$ be fixed and let $\boldsymbol{X} = (X_1, X_2)^\top$ be a L_p-norm bivariate spherical random vector. By (2) both X_1, X_2 have the same distribution and are symmetric about 0. Denote by F the distribution function of the associated random radius R. The simple (well-known) measure of asymptotic dependence between X_1, X_2 is the limit (if it exists)

$$\kappa(X_1, X_2) := \lim_{t \uparrow \omega} \frac{\boldsymbol{P}\{X_1 > t, X_2 > t\}}{\boldsymbol{P}\{X_1 > t\}} \geq 0,$$

with $\omega := \sup\{x : F(x) < 1\}$ the upper endpoint of F. If $\kappa(X_1, X_2) = 0$ then the joint tail probability diminishes faster than the marginal tail probability. For this case we say that X_1 and X_2 are asymptotically independent.

If ω is finite, then $\kappa(X_1, X_2) = 0$ since both X_1, X_2 cannot approach ω simultaneously. We discuss in the following therefore only the case $\omega = \infty$.
Let $a \in (0, \infty)$ and set $c_1^p := \inf\{|x_1|^p + |x_2|^p : x_1 \geq 1, x_2 \geq a\}$. Clearly, c_1 exists and $c_1 > 1$. For any $t > 0, c_0 \in (0, 1)$ we have

$$\begin{aligned}
&\frac{\boldsymbol{P}\{X_1 > t, X_2 > at\}}{\boldsymbol{P}\{X_1 > t\}} \\
&\quad \leq \frac{\boldsymbol{P}\{|X_1|^p + |X_2|^p \geq c_1^p t^p\}}{\boldsymbol{P}\{X_1 > t\}} = \frac{2\boldsymbol{P}\{R \geq c_1 t\}}{\boldsymbol{P}\{|X_1| > t\}} \\
&\quad \leq \frac{2\boldsymbol{P}\{R \geq c_1 t\}}{\boldsymbol{P}\{RW_{1,2,p} > t, W_{1,2,p} < c_0\}} \leq \frac{2\boldsymbol{P}\{R \geq c_1 t\}}{\boldsymbol{P}\{R > t\}\boldsymbol{P}\{W_{1,2,p} < c_0\}}.
\end{aligned} \tag{3}$$

Choosing a $c_0 \in (c_1^{-1}, 1)$ we obtain thus by the above upper bound

$$\kappa(X_1, X_2/a) = \lim_{t\to\infty} \frac{\boldsymbol{P}\{X_1 > t, X_2 > at\}}{\boldsymbol{P}\{X_1 > t\}} \quad = \quad 0, \tag{4}$$

provided that

$$\lim_{t\to\infty} \frac{1 - F(Kt)}{1 - F(t)} = 0 \tag{5}$$

holds for any $K > 1$. (5) means that $1 - F$ is a rapidly varying function. See de Haan (1970) or Resnick (1987) for the main properties of rapidly varying functions. In particular (5) holds if F is in the max-domain of attraction of the unit Gumbel distribution function $\Lambda(x) = \exp(-\exp(-x)), x \in \mathbb{R}$. A necessary and sufficient condition for F to be in the max-domain of attraction of Λ is the existence of a positive scaling function w such that

$$\lim_{t\uparrow\omega} \frac{1 - F(t + x/w(t))}{1 - F(t)} = \exp(-x), \quad \forall x \ge 0. \tag{6}$$

See Leadbetter et al. (1983), Galambos (1987), Resnick (1987), Reiss (1989), Berman (1992), or Falk et al. (2004) for further details.

It follows from the univariate extreme value theory that the two other possible max-domain of attractions for F are the Weibull and the Fréchet ones. For the first case we have for some $\alpha > 0$

$$\lim_{t\to\infty} F^t(\omega + a(t)x) = \exp(-|x|^\alpha) =: \Psi_\alpha(x), \quad \forall x \in (-\infty, 0), \tag{7}$$

with $a(t) := \omega - F^{-1}(1 - 1/t), t > 1$, whereas for the second case

$$\lim_{t\to\infty} F^t(a(t)x) = \exp(-x^{-\alpha}) =: \Phi_\alpha(x), \quad \forall x \in (0, \infty) \tag{8}$$

holds with $a(t) := F^{-1}(1 - 1/t), t > 1$.

If F is in the Weibull max-domain of attraction, then necessarily $\omega < \infty$, consequently (4) holds for any p positive.

Thus in both Gumbel and Weibull cases asymptotic independence of the components is observed for L_p-norm spherical random vectors. Berman (1992) shows that for F in the Fréchet max-domain of attraction $\kappa(X_1, X_2/a)$ is positive for any $a > 0$. It is well-known (see e.g. de Haan (1970) or Kotz and Nadarajah (2005)) that (8) is equivalent with the fact that R is regularly varying with index $\alpha > 0$, i.e.

$$\lim_{t\to\infty} \frac{\boldsymbol{P}\{R > Kt\}}{\boldsymbol{P}\{R > t\}} = K^{-\alpha}, \quad \forall K > 0.$$

Berman (1992) shows further for the case $p = 2$ (see Theorem 12.3.2 therein and Theorem 5.1 below) that also $|X_1|$ is regularly varying with positive index $\alpha > 0$. In Hashorva (2006) (see also Hashorva (2007b)) the converse is proved, i.e. if $|X_1|$ is regularly varying then the associated random radius R is also regularly varying with the same index as $|X_1|$.

We show in the next section that a similar result holds for L_p-norm spherical random vectors with $p > 0$ a given constant. In particular we have

$$\kappa(X_1, X_2/a) > 0, \quad \forall a \in (0, \infty)$$

if X_1 or R is regularly varying with positive index α.

3. Asymptotics of Sample Maxima

Let $\boldsymbol{X}$ be a L_p-norm spherical random vector in $\mathbb{R}^d, d \geq 2$ as in (1) and let further $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n, n \geq 1$ be independent random vectors in $\mathbb{R}^d$ with the same distribution function G as $\boldsymbol{X}$. Denote by F the distribution function of the associated random radius R of $\boldsymbol{X}$, and define the component-wise sample maxima by

$$\boldsymbol{M}_n := (\max_{1\leq j\leq n} X_{j1}, \ldots, \max_{1\leq j\leq n} X_{jd})^\top, \quad n \geq 1.$$

Assuming that F is in the max-domain of attraction of a univariate extreme value distribution function H (abbreviated as $F \in MDA(H)$) Hashorva (2005) derives (when $p = 2$) the convergence in distribution

$$\frac{\boldsymbol{M}_n - b(n)\boldsymbol{1}}{a(n)} \overset{d}{\to} \boldsymbol{Z}, \quad n \to \infty, \tag{9}$$

where the random vector $\boldsymbol{Z} = (Z_1, \ldots, Z_d)^\top$ has distribution function Q which is a product distribution if $H = \Lambda$ or $H = \Psi_\alpha, \alpha > 0$.
If $H = \Phi_\alpha, \alpha > 0$, then $\boldsymbol{Z}$ has dependent components with distribution function Φ_α. Both constants $a(n) > 0, b(n), n \in \mathbb{N}$ are defined in terms of the distribution function of X_1. The convergence in the distribution in (9) is equivalent with

$$\lim_{n\to\infty} G^n(a(n)\boldsymbol{x} + b(n)\boldsymbol{1}) = Q(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{R}^d. \tag{10}$$

As in the univariate setup abbreviate (10) by $G \in MDA(Q)$. Note in passing that (10) implies $G_i \in MDA(Q_i), 1 \leq i \leq d$ with G_i, Q_i the marginal distributions of G and Q, respectively.

In the next two theorems we show that the asymptotic behaviour of the sample maxima is the same (with respect to the limiting distribution Q) for any $p > 0$. We discuss first the case F is in the Gumbel or the Weibull max-domain of attractions.

Theorem 1. *Let $\boldsymbol{X}$ be a L_p-norm spherical random vector in $\mathbb{R}^d, d \geq 2$ with distribution function G defined in* (1) *with $R > 0$ almost surely being independent of $\boldsymbol{U}_d$. Let F be the distribution function of R with the upper endpoint $\omega \in (0, \infty]$.*
i) Assume that $F \in MDA(\Lambda)$ with positive scaling function w. Then (10) *holds where $\boldsymbol{Z}$ has independent components with unit Gumbel distribution and $b(n) := G_1^{-1}(1 - 1/n), a(n) := 1/w(b(n)), n > 1$.*
ii) Suppose that $\omega = 1$ and further $F \in MDA(\Psi_\alpha), \alpha > 0$ holds. Then (10) *holds with $a(n) := 1 - G_1^{-1}(1 - 1/n), n > 1$ and $Z_i, 1 \leq i \leq d$ independent random variables such that $Z_i \sim \Psi_{\alpha+(d-1)/p}$.*

Several examples may illustrate the applicability of Theorem 1.

Example 1. [L_p-norm Kotz Type I] Let $\boldsymbol{X} = (X_1, \ldots, X_d)^\top$ be a random vector in $\mathbb{R}^d, d \geq 2$, with density function given by

$$q(\boldsymbol{x}) := \frac{p^d\Gamma(d/p)r^{(d/p+N)/s}s}{2^d\Gamma^d(1/p)\Gamma((d/p+N)/s)}\|\boldsymbol{x}\|_p^{pN} \exp(-r\|\boldsymbol{x}\|_p^{ps}), \quad \boldsymbol{x} \in \mathbb{R}^d \tag{11}$$

and constants $p, r, s > 0, N \in \mathbb{R} : d + pN > 0$. We refer to $\boldsymbol{X}$ as L_p-norm Kotz Type I random vector. It has stochastic representation (1) where the associated random radius R possesses the density function

$$f(t) = \frac{psr^{(d/p+N)/s}}{\Gamma((d/p+N)/s)}t^{d+pN-1} \exp(-rt^{ps}), \quad t \in (0, \infty).$$

If $s = 1, N = 0, r > 0$, then $\boldsymbol{X}$ possesses a p-generalised Gaussian distribution (see Gordon and Kotz (1973)). It now follows easily that the distribution function F of the density f is in the Gumbel max-domain of attraction with the scaling function

$$w(u) \quad = \quad (1 + o(1))rpsu^{ps-1}, \quad u \to \infty.$$

Next, if $\boldsymbol{X}_n, n \geq 1$ are independent with density function q given in (11), then (9) follows implying further that the sample maxima has asymptotic independent components.

Example 2. [L_p-norm Kotz Type III] We call a random vector $\boldsymbol{X}$ in $\mathbb{R}^d, d \geq 2$ a L_p-norm Kotz Type III spherical random vector if it has stochastic representation (1) where the associated random radius $R > 0$ has asymptotic tail behaviour ($u \to \infty$)

$$\text{(12)} \qquad \boldsymbol{P}\{R > u\} \quad = \quad (1 + o(1))Ku^N \exp(-ru^\delta), \quad K > 0, \delta > 0, N \in \mathbb{R}.$$

It now easily follows that the distribution function F of R is in the Gumbel max-domain of attraction with the positive scaling function

$$w(u) \quad = \quad (1 + o(1))r\delta u^{\delta-1}, \quad u \to \infty.$$

The subvectors of $\boldsymbol{X}$ are all L_p-norm Kotz Type III spherical random vectors. This property is not shared by L_p-norm Kotz Type I spherical random vectors.
In view of Theorem 1 the random vector $\boldsymbol{X}$ has asymptotic independent components, and the maxima of a sample of L_p-norm Kotz Type III spherical random vectors has asymptotic independent components with distribution function attracted by a product distribution with Gumbel marginals.

Example 3. [L_p-norm Pearson Type II] The random vector $\boldsymbol{X}$ in $\mathbb{R}^d, d \geq 2$ has density function (see Example 2.3 of Gupta and Song (1997))

$$q(\boldsymbol{x}) \quad := \quad \frac{p^d \Gamma(d/p + \alpha)}{2^d \Gamma^d(1/p)\Gamma(\alpha)} \Big(1 - \|\boldsymbol{x}\|_p^p\Big)^{\alpha-1}, \quad \boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\|_p < 1, \alpha > 0.$$

$\boldsymbol{X}$ is L_p-norm spherically distributed with random radius R with density function

$$f(t) \quad = \quad \frac{p\Gamma(d/p + \alpha)}{\Gamma(d/p)\Gamma(\alpha)} t^{d-1}(1 - t^p)^{\alpha-1}, \quad t \in (0, 1).$$

It follows that the associated random radius R has the distribution function in the max-domain of attraction of the Weibull distribution Ψ_α. Hence by the above theorem the distribution function of $\boldsymbol{X}$ is in the max-domain of attraction of a product distribution with marginal distributions $\Psi_{\alpha+(d-1)/p}$.

Next, we deal with the case where F is in Fréchet max-domain of attraction. In Hashorva (2005) (see also Hashorva (2007b)) it is shown (considering only the case $p = 2$) that R has distribution function $F \in MDA(\Phi_\alpha), \alpha > 0$ iff X_1 has distribution function in the max-domain of attraction of Φ_α.
Further, it is proved therein that $F \in MDA(\Phi_\alpha)$ implies that $\boldsymbol{X}$ is a regularly varying random vector with index α. Regular variation of random vectors is investigated in details in many recent contributions, see e.g. Basrak (2002), Mikosch (2005). We use the following definitions of regular variation of random vectors.

Definition 1. The random vector $\boldsymbol{X} = (X_1, \ldots, X_d)^\top, d \geq 1$ is regularly varying with index $\alpha > 0$ if there exists a positive sequence $a_n \to \infty$ as $n \to \infty$ and a positive measure μ homogeneous of order α such that the vague convergence in $[-\infty, \infty]^d \setminus \{\mathbf{0}\}$

$$n\boldsymbol{P}\{\boldsymbol{X}/a_n \in \cdot\} \quad \stackrel{v}{\to} \quad \mu(\cdot), \quad n \to \infty \tag{13}$$

holds.

For $\boldsymbol{X}$ a spherical random vector (with respect to L_2-norm) in $\mathbb{R}^d, d \geq 2$ Hashorva (2006) shows that if X_1 is regularly varying with index α then $\boldsymbol{X}$ is regularly varying with the same index α.

We generalise the aforementioned results for the case $p > 0$ (and complete the proof of our previous result). The asymptotic behaviour of the sample maxima is derived in Corollary 3.

Theorem 2. *Let $\boldsymbol{X}$ be as in Theorem 1, and let $R_{i,p} := (\sum_{j=1}^{i} |X_j|^p)^{1/p}, 1 \leq i \leq d$ be the i-th associated random radius of $\boldsymbol{X}$. Then the following statements are equivalent:*
i) $R_{d,p}$ is regularly varying with index $\alpha > 0$.
ii) X_1 is regularly varying with index $\alpha > 0$.
iii) For any $i = 1, \ldots, d$ the random radius $R_{i,p}$ is regularly varying with positive index α. Furthermore

$$\boldsymbol{P}\{R_{i,p} > u\} \quad = \quad (1 + o(1)) C_{i,d,\alpha,p} \boldsymbol{P}\{R_{d,p} > u\}, \quad u \to \infty \tag{14}$$

holds for any $i < d$ where

$$C_{i,d,\alpha,p} \quad := \quad 2\frac{\Gamma(d/p)\Gamma((i+\alpha)/p)}{\Gamma(i/p)\Gamma((d+\alpha)/p)} \in (0, \infty). \tag{15}$$

iv) For any $I \subset \{1, \ldots, d\}$ with $1 \leq |I| = k$ and any Borel set B away from the origin of $\mathbb{R}^k$

$$\lim_{u \to \infty} \frac{\boldsymbol{P}\{(\boldsymbol{X} + \boldsymbol{\mu})_I / u \in B\}}{\boldsymbol{P}\{X_1 > u\}} \quad = \quad C_{1,k,\alpha,p} \int_0^\infty \boldsymbol{P}\{r\boldsymbol{U}_k \in B\}\, d(r^{-\alpha}) \tag{16}$$

holds for any $\boldsymbol{\mu} \in \mathbb{R}^d$.
v) $\boldsymbol{X}$ is regularly varying with index $\alpha > 0$.

We thus immediately obtain the corollary:

Corollary 3. *Let $\boldsymbol{X}, \boldsymbol{U}_d, d \geq 2$ be as in Theorem 1 with distribution function G. Assume that R or X_1 is regularly varying with positive index α. Then $G \in MDA(Q_{d,\alpha,p})$ with $Q_{d,\alpha,p}$ a max-stable distribution function on $(0, \infty)^d$ defined for any $\boldsymbol{x} > \mathbf{0}$ by*

$$Q_{d,\alpha,p}(\boldsymbol{x}) \quad := \quad \exp\Big(-C_{1,d,\alpha,p} \int_0^\infty \boldsymbol{P}\{r\boldsymbol{U}_d \in \mathbb{R}^d \setminus \times_{i=1}^d (-\infty, x_i]\}\, d(r^{-\alpha})\Big). \tag{17}$$

The marginal distributions of $Q_{d,\alpha,p}$ are identical to Φ_α.
Conversely, if $G \in MDA(Q_{d,\alpha,p})$ where $Q_{d,\alpha,p}$ has marginal distributions identical to $\Phi_\alpha, \alpha > 0$, then both X_1 and R_1 are regularly varying with index α.

(17) implies that if $\boldsymbol{Y}$ is a random vector with distribution function $Q_{d,\alpha,p}$ for some $d \geq 2$ and α, p positive constants, then the subvector $\boldsymbol{X}_I$ where I has $k \geq 1$ elements and $I \subset \{1, \ldots, d\}$ has distribution function $Q_{k,\alpha,p}$ which is max-stable. Furthermore, $Q_{k,\alpha,p}$ is not a product distribution for any $k \geq 2$.

We present next an illustrating example:

Example 4. [L_p-norm Pearson Type VII] Define $\boldsymbol{X}$ a L_p-norm spherically distributed random vector in $\mathbb{R}^d, d \geq 2$ as in Example 2.4 of Gupta and Song (1997) with density function given for any $\boldsymbol{x} \in \mathbb{R}^d$ by

$$q(\boldsymbol{x}) \quad := \quad \frac{p^d \Gamma(N)}{2^d \Gamma^d(1/p)\Gamma(N - n/p)} s^{-d/p}\Big(1 + \|\boldsymbol{x}\|_p^p/s\Big)^{-N}, \quad s > 0, N > d/p.$$

If $N = (d+m)/2$ then $\boldsymbol{X}$ has a L_p-norm t-distribution (see Example 2.5 of Gupta and Song (1997)). The associated random radius R has density function

$$f(t) \quad = \quad \frac{p\Gamma(N)}{\Gamma(d/p)\Gamma(N - d/p)} s^{-d/p} t^{d-1}\Big(1 + t^p/s\Big)^{-N}, \quad t \in (0, \infty).$$

In view of Karamata's Theorem (see e.g. Resnick (1987)) the random variable R is regularly varying with index $\alpha := pN - d > 0$. Consequently the marginals $X_i, 1 \leq i \leq d$ are regularly varying with index α. Furthermore, $\boldsymbol{X}$ is a regularly varying random vector with positive index α. The corresponding measure can be easily calculated.

Example 5. [L_p-norm Kotz Type II] We say that a random vector $\boldsymbol{X}$ in $\mathbb{R}^d, d \geq 2$, has L_p-norm Kotz Type II distribution if its density function is given by

$$(18) \quad q(\boldsymbol{x}) := \frac{p^d \Gamma(d/p) r^{d/p+N} s}{2^d \Gamma^d(1/p)\Gamma((d/p+N)/s)} \|\boldsymbol{x}\|_p^{pN} \exp(-r\|\boldsymbol{x}\|_p^{ps}), \quad \boldsymbol{x} \in \mathbb{R}^d,$$

with constants $p > 0, r > 0, s < 0, d/p + N < 0$. Kotz (1975) introduces $\boldsymbol{X}$ with density function as above in the case $p = 2$. Basic properties of $A\boldsymbol{X}$ with $A \in \mathbb{R}^{d \times d}$ a non-singular matrix are discussed in Kotz (2004). It can easily be shown that $\boldsymbol{X}$ has stochastic representation (1) with the random radius R which has distribution function in the Fréchet max-domain of attraction. Consequently, Theorem 2 implies that the components of $\boldsymbol{X}$ are asymptotically dependent and the sample maxima of Kotz Type II random vectors converges in the distribution (after normalisation) to a random vector with dependent Fréchet marginal components.

4. L_p-norm Asymptotically Spherical Random Vectors

L_2-norm asymptotically spherical random vectors are introduced in Hashorva (2005). The crucial asymptotic property of such vectors is that the asymptotic behaviour of the sample extremes can be defined by the asymptotic behaviour of the associated random radius $R_{i,2}, i \leq d$. In this section we introduce the larger class of L_p-norm asymptotically spherical random vectors and show that the asymptotic properties of L_p-norm spherical random vectors still hold under such a general setup.

Definition 2. [L_p-norm asymptotically spherical random vector] Let $\boldsymbol{X}$ be a random vector in $\mathbb{R}^d, d \geq 2$ and let $\omega \in (0, \infty]$ be the upper endpoint of the distribution function of each component $X_i, 1 \leq i \leq d$. For any non-empty subset $I \subset \{1, \ldots, d\}$ set $R_{I,p} := (\sum_{i \in I} |X_i|^p)^{1/p}, p > 0$ and $R := (\sum_{i=1}^d |X_i|^p)^{1/p}$. Assume that $R > 0$ almost surely. If additionally

$$(19) \qquad \lim_{t \uparrow \omega} \frac{\boldsymbol{P}\{X_i > t\}}{\boldsymbol{P}\{|X_i| > t\}} \quad = \quad c_i \in (0, 1]$$

and further for any non-empty index set $I \subset \{1, \ldots, d\}$

$$\lim_{t \uparrow \omega} \frac{\boldsymbol{P}\{RW_{I,p} > t\}}{\boldsymbol{P}\{R_{I,p} > t\}} \quad = \quad d_I \in (0, \infty), \tag{20}$$

where $W_{I,p}^p \sim Beta(\delta_I, \lambda_I), W_{I,p} > 0$ being further independent of R, with δ_I, λ_I positive, then we refer to $\boldsymbol{X}$ as a L_p-norm asymptotically spherical random vector.

In the following we consider for simplicity the case

$$d_I = 1, \quad \forall I \subset \{1, \ldots, d\}.$$

Further we suppose that for two non-empty index sets I, J such that $J \subset I \subset \{1, \ldots, d\}$ we have $\delta_I \geq \delta_J$. For notational simplicity we shall write δ_i, λ_i when $I = \{i\}$.

With the above restrictions we call $\boldsymbol{X}$ a L_p-norm asymptotically spherical random vector with coefficients $\boldsymbol{c}, \delta_I, \lambda_I, I \subset \{1, \ldots, d\}$ where $\boldsymbol{c} = (c_1, \ldots, c_k)^\top \in \mathbb{R}^k$, or shortly a L_p-norm asymptotically spherical random vector.
If $p = 2$ an instance of L_2-norm asymptotically spherical random vector is $\boldsymbol{X}$ a generalised symmetrised Dirichlet random vector introduced in Fang and Fang (1990). The main asymptotic properties derived above for the L_p-norm spherical random vectors can be extended for the more general case L_p-norm asymptotically spherical random vectors. Since both X_1, X_2 and the associated random radius R have by definition the same upper endpoint ω, then $\kappa(X_1, X_2) = 0$ follows in the case $\omega \in (0, \infty)$. We discuss next the case $\omega = \infty$ and R has a rapidly varying survival function.

Theorem 4. *Let $\boldsymbol{X}$ be a L_p-norm asymptotically spherical random vector in $\mathbb{R}^d, d \geq 2$ with coefficients $\boldsymbol{c}, \delta_I, \lambda_I, I \subset \{1, \ldots, d\}$. Let F be the distribution function of $R := (|X_1|^p + \cdots + |X_d|^p)^{1/p}$ with the upper endpoint $\omega \in (0, \infty]$.*
i) If $\omega \in (0, \infty)$ then $\kappa(X_i, X_j) = 0, 1 \leq i < j \leq d$.
ii) If $\omega = \infty$ and F is rapidly varying then for any $a > 0$ we have $\kappa(X_i, X_j/a) = 0, 1 \leq i < j \leq d$.

Let $\boldsymbol{X}$ be a L_p-norm asymptotically spherical random vector with associated random radius R. If the distribution function of R is in the max-domain of attraction of a univariate extreme value distribution, then Theorem 8 below implies that the components of $\boldsymbol{X}$ have distribution function in the same max-domain of attraction. We show next that also the distribution function of $\boldsymbol{X}$ is in the max-domain of attraction of a max-stable distribution function.

Theorem 5. *Let $\boldsymbol{X}, \boldsymbol{X}_n, n \geq 1$ be independent L_p-norm asymptotically spherical random vectors in $\mathbb{R}^d, d \geq 2$ with coefficients $\boldsymbol{c}, \delta_I, \lambda_I, I \subset \{1, \ldots, d\}$ and distribution function G. Denote by $\omega \in (0, \infty]$ the upper endpoint of the distribution function F of the associated random radius $R > 0$ of $\boldsymbol{X}$.*
i) Let $\boldsymbol{Z}$ be a $d-$dimensional random vector with independent unit Gumbel components. If $F \in MDA(\Lambda)$ with the scaling function w, then we have the convergence in the distribution

$$\frac{\boldsymbol{M}_n - \boldsymbol{b}(n)}{\boldsymbol{a}(n)} \quad \stackrel{d}{\to} \quad \boldsymbol{Z}, \quad n \to \infty, \tag{21}$$

provided that $\lambda_i = \lambda > 0, 1 \leq i \leq d$ if $\omega = \infty$, where $\boldsymbol{a}(n), \boldsymbol{b}(n)$ are defined by

$$b_i(n) := G_i^{-1}(1 - 1/n), \quad a_i(n) := 1/w(b_i(n)), \quad 1 \leq i \leq d, n > 1.$$

ii) Assume that $\omega = 1$ *and* $F \in MDA(\Psi_\alpha), \alpha > 0$. *Then we have*

$$\frac{\boldsymbol{M}_n - \mathbf{1}}{\boldsymbol{a}(n)} \stackrel{d}{\to} \boldsymbol{Z}, \quad n \to \infty, \tag{22}$$

where $Z_i, 1 \le i \le d$ *are independent with* $Z_i \sim \Psi_{\alpha+\lambda_i}$, *and* $\boldsymbol{a}(n)$ *has components* $a_i(n) := 1 - G_i^{-1}(1 - 1/n), n > 1, 1 \le i \le d$.

We note in passing that the restriction $\lambda_i = \lambda > 0$ for all $i \le d$ in the above theorem might be redundant. This is the case for instance if $\boldsymbol{X}$ is a L_2-norm generalised symmetrised Dirichlet distribution.
We consider next the case that the associated random radius R is regularly varying.

Theorem 6. *Let* $\boldsymbol{X}$ *be a* L_p*-norm asymptotically spherical random vector as in Theorem 5. If the associated random radius* R *is regularly varying with positive index* α, *then* $X_i, 1 \le i \le d, R_{I,p}, \forall I \subset \{1, \ldots, d\}$ *are regularly varying with index* α *and furthermore*

$$\boldsymbol{P}\{R_{I,p} > u\} = (1 + o(1)) \frac{\Gamma(\delta_I + \lambda_I)\Gamma(\alpha/p + \delta_I)}{\Gamma(\delta_I)\Gamma(\alpha/p + \delta_I + \lambda_I)} \boldsymbol{P}\{R > u\} \tag{23}$$

holds as $u \to \infty$. *Furthermore, if there exists a random vector* $\boldsymbol{U}$ *on* $\mathbb{S}_p^{d-1}$ *independent of* R *such that* $\boldsymbol{X} = R\boldsymbol{U}$, *then we have for any Borel set* $B \subset \mathbb{R}^d$ *away from the origin of* $\mathbb{R}^d$ *and for any vector* $\boldsymbol{\mu} \in \mathbb{R}^d$

$$\lim_{u \to \infty} \frac{\boldsymbol{P}\{(\boldsymbol{X} + \boldsymbol{\mu})/u \in B\}}{\boldsymbol{P}\{X_i > u\}} = C_i \int_0^\infty \boldsymbol{P}\{r\boldsymbol{U} \in B\}\, d(r^{-\alpha}), \tag{24}$$

where

$$C_i := \frac{\Gamma(\delta_i)\Gamma(\alpha/p + \delta_i + \lambda_i)}{c_i \Gamma(\delta_i + \lambda_i)\Gamma(\alpha/p + \lambda_i)} \in (0, \infty).$$

Similarly as in Corollary 3 the distribution function G of $\boldsymbol{X}$ in the above theorem is in the max-domain of attraction of max-stable distribution function with Fréchet marginal distributions which is not a product distribution.

We conclude this section with two illustrating example.

Example 6. [L_2-norm Kotz Type I generalised symmetrised Dirichlet] Let $\boldsymbol{\alpha}$ be a fixed vector in $\mathbb{R}^d, d \ge 2$ with positive components and let N, r, s be positive constants. We refer to a random vector $\boldsymbol{X}$ in $\mathbb{R}^d$ as Kotz Type I generalised symmetrised Dirichlet with parameters $\boldsymbol{\alpha} \in (0, \infty)^d, N \in \mathbb{R}, r > 0, s > 0$ if it possesses the density function

$$h(\boldsymbol{x}) := \frac{r^{(N + \sum_{i \le d} \alpha_i)/s}}{s\Gamma((N + \sum_{i \le d} \alpha_i)/s)} \frac{\Gamma(\sum_{i \le d} \alpha_i)}{\prod_{i=1}^k \Gamma(\alpha_i)} \|\boldsymbol{x}\|_p^{pN} \exp(-r\|\boldsymbol{x}\|^{2s}) \prod_{i=1}^k |x_i|^{2\alpha_i - 1}$$

defined for all $\boldsymbol{x} \in \mathbb{R}^d, d \ge 2$.
In view of the amalgamation property shown in Fang and Fang (1990) (see also Hashorva et al. (2007b)) it follows that $\boldsymbol{X}$ is a L_2-norm asymptotically spherical random vector. The associated random radius R of $\boldsymbol{X}$ is almost surely positive and moreover R^2 is Gamma distributed with parameters $\sum_{i \le d} \alpha_i$ and $1/2$. Hence the distribution function of $\boldsymbol{X}$ is in the max-domain of attraction of a product distribution with marginal distributions Λ.

Example 7. [L_p-norm Kotz Type III asymptotically spherical] Let $\boldsymbol{X}$ be a L_p-norm asymptotically spherical random vector in $\mathbb{R}^d, d \geq 2$, with coefficients $\delta_I, \lambda_I, I \subset \{1, \ldots, d\}$ and distribution function G. We say that $\boldsymbol{X}$ is a L_p-norm Kotz Type III asymptotically spherical random vector if the associated random radius R has asymptotic tail behaviour given by (12). In view of Example 2 and Theorem 5 $G \in MDA(Q)$ with Q a product distribution with unit Gumbel marginal distributions.

5. Related Results and Proofs

The next lemma is presented in the published paper Hashorva et al. (2007) referring to this paper. We give it here for reference purposes.

Lemma 7. *Let X, Y be two independent positive random variables with $Y^p \sim Gamma(a, \lambda), a, \lambda > 0, p > 0$. If X is regularly varying with positive index γ, then we have*

$$\lim_{u \to \infty} \frac{\boldsymbol{P}\{XY > u\}}{\boldsymbol{P}\{Y > u\}} = \frac{\Gamma(a + \gamma/p)}{\lambda^{\gamma/p}\Gamma(a)} \in (0, \infty). \tag{25}$$

Conversely, if the product XY is regularly varying with index $\gamma > 0$, then X is regularly varying with index γ and further (25) *holds.*

Proof of Lemma 7 By Breiman's Lemma (see Breiman (1965)) we have

$$\lim_{u \to \infty} \frac{\boldsymbol{P}\{XY > u\}}{\boldsymbol{P}\{Y > u\}} = \frac{\Gamma(a + \gamma/p)}{\lambda^{\gamma/p}\Gamma(a)} \in (0, \infty).$$

Since X is regularly varying then the first claim follows. We show next the converse. Assume that XY is regularly varying with index $\gamma > 0$. Since $XY = (X^pY^p)^{1/p}, p > 0$ and the fact that X^p is regularly varying iff X is regularly varying it suffices to show the proof for the case $p = 1$. Next, suppose for simplicity that $p = 1, \lambda = 1$. For any $t > 0$ we may write by the independence of X and Y

$$\boldsymbol{P}\{XY > t\} = t^a \int_0^\infty \exp(-tv)\, dG(v),$$

where

$$G(s) := \frac{1}{\Gamma(a)} \int_0^s \boldsymbol{P}\{X > 1/x\} x^{a-1}\, dx, \quad s > 0.$$

The assumption XY is regularly varying with index $\gamma > 0$ means

$$\int_0^\infty \exp(-tv)\, dG(v) = t^{-a-\gamma} L(1/t), \quad t \to \infty, \tag{26}$$

with $L(x)$ such that $\lim_{t \to 0} L(Kt)/L(t) = 1, \forall K > 0$.
In view of Karamata's Tauberian Theorem (Feller (1966), Resnick (1987)) (26) is equivalent with

$$G(t) = \frac{1}{\Gamma(a + \gamma + 1)} t^{a+\gamma} L(t), \quad t \downarrow 0,$$

or equivalently

$$G(1/t) = \frac{1}{\Gamma(a + \gamma + 1)} t^{-a-\gamma} L(1/t), \quad t \to \infty.$$

Consequently

$$\int_0^{1/t} \boldsymbol{P}\{X > 1/x\} x^{a-1}\, dx = \frac{\Gamma(a)}{\Gamma(a + \gamma + 1)} t^{-a-\gamma} L(1/t), \quad t \to \infty.$$

Since $\boldsymbol{P}\{X > x\}x^{-a-1}, x > 0$ decreases monotonically in x for any $a > 0$ we obtain applying the Monotone Density Theorem (Resnick (1987))

$$\boldsymbol{P}\{X > t\}t^{-a-1} = \frac{(a+\gamma+1)\Gamma(a)}{\Gamma(a+\gamma+1)} t^{-a-\gamma-1} L(1/t), \quad t \to \infty,$$

thus the proof follows. □

Theorem 8. *Let Y be a random variable with distribution function H which has the upper endpoint $\omega \in (0, \infty]$ and $H(0) = 0$. Let a, b, τ be positive constants and let $Z_{a,b}$ be a Beta distributed random variable with parameters a, b independent of Y and set $\overline{H}(u) := 1 - H(u), u > 0$.*
i) If $H \in MDA(\Lambda)$ with positive scaling function w then we have as $u \uparrow \omega$

$$\boldsymbol{P}\{Y[1 - Z_{a,b}]^{1/\tau} > u\} = (1 + o(1)) \frac{\Gamma(a+b)}{\Gamma(b)} \left(\frac{\tau}{uw(u)}\right)^a \overline{H}(u). \tag{27}$$

ii) If $H \in MDA(\Phi_\alpha), \alpha > 0$ then $\omega = \infty$ and for $u \to \infty$

$$\boldsymbol{P}\{Y[1 - Z_{a,b}]^{1/\tau} > u\} = (1 + o(1)) \frac{\Gamma(a+b)\Gamma(b+\alpha/\tau)}{\Gamma(b)\Gamma(a+b+\alpha/\tau)} \overline{H}(u). \tag{28}$$

iii) If $H \in MDA(\Psi_\alpha), \alpha > 0$ and $\omega = 1$, then we have

$$\boldsymbol{P}\{Y[1 - Z_{a,b}]^{1/\tau} > u\} = (1 + o(1)) \frac{\Gamma(\alpha+1)\Gamma(a+b)}{\Gamma(b)\Gamma(\alpha+a+1)} (\tau(1-u))^a \overline{H}(u) \tag{29}$$

as $u \to \infty$.

Proof. The proof can be established along the lines of the proof of Theorem 12.3.1, Theorem 12.3.2 and Theorem 12.3.3 of Berman (1992). We give below the sketch of a slightly different proof.
Let $B(y, a, b), y \in [0, 1]$ denote the distribution function of $Z_{a,b}$ and put

$$H_u(s) := H(u + s/w(u))/\overline{H}(u), \quad \overline{H}(u) := 1 - H(u), \quad u \in \mathbb{R}, s > 0.$$

Since $Y > 0$ is independent of $Z_{a,b}$ we have for any $u \in (0, \omega)$

$$\begin{aligned}
&\boldsymbol{P}\{Y(1 - Z_{a,b})^{1/\tau} > u\} \\
&= \int_0^\omega [1 - B((u/s)^\tau, b, a)]\, dH(s) \\
&= \overline{H}(u) \int_0^{w(u)[\omega-u]} [1 - B([1 + s/(uw(u))]^{-\tau}, b, a)]\, dH_u(s).
\end{aligned}$$

The assumption $H \in MDA(\Lambda)$ implies

$$\lim_{u\uparrow\omega} [H_u(t) - H_u(s)] = \exp(-s) - \exp(-t), \quad \forall s, t \in \mathbb{R}, s \geq t,$$

and

$$\lim_{u\uparrow\omega} w(u)[\omega - u] = \infty, \quad \lim_{u\uparrow\omega} uw(u) = \infty.$$

Consequently

$$\lim_{u\uparrow\omega} \left(\frac{uw(u)}{\tau}\right)^a \Big[1 - B([1 + s/(uw(u))]^{-\tau}, b, a)\Big] = \frac{\Gamma(a+b)}{a\Gamma(a)\Gamma(b)} s^a, \quad \forall s \in (0, \infty),$$

hence we obtain further

$$\liminf_{u\uparrow\omega} \left(\frac{uw(u)}{\tau}\right)^a \boldsymbol{P}\{Y(1 - Z_{a,b})^{1/\tau} > u\} \geq \frac{\Gamma(a+b)}{a\Gamma(a)\Gamma(b)} \int_{-\infty}^\infty s^a\, d(\exp(-s)).$$

The same upper bound can be shown for the lim sup of the left hand side above using Lemma 4.3 of Hashorva (2006).
ii) Breiman's Lemma implies as $u \to \infty$

$$\begin{aligned}\boldsymbol{P}\{Y(1-Z_{a,b})^{1/\tau} > u\} &= (1+o(1))\overline{H}(u)\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 x^{\alpha/\tau}x^{b-1}(1-x)^{a-1}\,dx \\ &= (1+o(1))\overline{H}(u)\frac{\Gamma(a+b)\Gamma(b+\alpha/\tau)}{\Gamma(b)\Gamma(a+b+\alpha/\tau)}.\end{aligned}$$

iii) Denote again by $B(y,a,b), y \in (0,1)$ the distribution function of $Z_{a,b}$ and put

$$B_u(y) := \tau_u^{-a} B(y\tau_u, a, b), \quad y \in [0,1], \quad \tau_u := \tau(1-u).$$

We may write for any $u \in (0,1)$

$$\boldsymbol{P}\{Y(1-Z_{a,b})^{1/\tau} > u\} = \tau_u^a \overline{H}(u)\int_0^{1-u^\tau} \frac{\overline{H}(u(1-y\tau_u)^{-1/\tau})}{\overline{H}(u)}\,dB_u(y).$$

Since for any $y \in (0,1)$

$$\lim_{u\uparrow 1}\frac{\overline{H}(u(1-y\tau_u)^{-1/\tau})}{\overline{H}(u)} = (1-y)^\alpha, \quad \text{and} \lim_{u\uparrow 1}\tau_u^{-a}B(y\tau_u,a,b) = \frac{y^a\Gamma(a+b)}{a\Gamma(a)\Gamma(b)}$$

applying Lemma 4.2 of Hashorva (2006) we obtain

$$\lim_{u\uparrow 1}\frac{\boldsymbol{P}\{Y(1-Z_{a,b})^{1/\tau} > u\}}{\tau_u^a[1-H(u)]} = \frac{\Gamma(\alpha+1)\Gamma(a+b)}{\Gamma(b)\Gamma(\alpha+a+1)},$$

hence the proof is complete. □

Proof of Theorem 1 i) Let $V \sim Beta((d-1)/p, 1/p)$ be independent of the associated random radius R. Using (2) we have for any $u > 0$

$$\boldsymbol{P}\{X_i > u\} = \frac{1}{2}\boldsymbol{P}\{R(1-V)^{1/p} > u\}, \quad i = 1,\dots,d. \tag{30}$$

In view of Theorem 8 we obtain taking $\tau = p$

$$\lim_{u\uparrow\omega}\frac{1-G_i(u+x/w(u))}{1-G_i(u)} = \lim_{u\uparrow\omega}\frac{1-F(u+x/w(u))}{1-F(u)} = \exp(-x), \quad \forall x \in \mathbb{R},$$

hence $G_i, 1 \le i \le d$ is in the Gumbel max-domain of attraction with the scaling function w. Since $F \in MDA(\Lambda)$ implies (5), then (4) follows. Consequently the components of the sample maxima are asymptotically independent, hence $G \in MDA(Q)$ with Q a distribution function on $\mathbb{R}^d$ with independent unit Gumbel marginal distributions.
ii) By (29) and (30) it follows that $G_i \in MDA(\alpha + (d-1)/p), 1 \le i \le d$. Since the upper endpoint of F is finite we have that G has all marginal distributions with finite upper endpoint ω. In view of (2) the components of $\boldsymbol{X}$, say $X_i, X_k, i \ne k$ cannot be both extreme (near enough to ω) with a non-zero probability, implying $\kappa(X_1, X_2) = 0$. Hence, the sample maxima has asymptotic independent components. □

Proof of Theorem 2 The assumptions imply that

$$\boldsymbol{X} \stackrel{d}{=} R\boldsymbol{U}_d \stackrel{d}{=} R_{d,p}\boldsymbol{U}_d,$$

with $R_{d,p}$ independent of $\boldsymbol{U}_d$.
$i) \Rightarrow ii)$ In view of (2) we have

$$R_{i,p} \stackrel{d}{=} R_{d,p}(1-V_i)^{1/p},$$

with $V_i \sim Beta((d-i)/p, i/p)$ being further independent of the random radius $R_{d,p}$. Applying Lemma 8 we obtain taking $\tau = p$

$$\boldsymbol{P}\{R_{i,p} > u\} \quad = \quad (1+o(1))\frac{\Gamma(d/p)\Gamma((i+\alpha)/p)}{\Gamma(i/p)\Gamma((d+\alpha)/p)}\boldsymbol{P}\{R_{d,p} > u\}, \quad u \to \infty.$$

In view of (30) X_1 is regularly varying with index $\alpha > 0$ is equivalent with $R_{1,p}$ is regularly varying with the same index α, hence the claim follows.
$iii) \Rightarrow ii)$ Using the fact that $|X_1|^p = R_{1,p}^p$ and X_1 is symmetric about 0 establishes the proof.
$iii) \Rightarrow i)$ Clearly $iii)$ includes $i)$.
$iv) \Rightarrow v) \Rightarrow ii)$ This follows easily by the definition of the regular variation.
$ii) \Rightarrow i)$ Let $\boldsymbol{Z} = (Z_1, \ldots, Z_d)^\top$ be a L_p-norm spherical random vector as in Example 1 being further independent of $\boldsymbol{X}$ and let

$$\widetilde{R}_{i,p} := \Big(\sum_{j=1}^{i} |Z_i|^p\Big)^{1/p}, \quad 1 \le i \le d$$

be the i-th random radius associated with $\boldsymbol{Z}$. By the assumptions it follows easily that $R_{1,p}^p$ is regularly varying with index α/p. Since

$$\widetilde{R}_{i,p}^p \sim Gamma(i/p, 1/p), \quad 1 \le i \le p$$

and $\widetilde{R}_{i,p}$ is independent of $R_{1,p}$ Lemma 7 implies that the product $(\widetilde{R}_{d,p}R_{1,p})^p$ is regular varying with positive index α/p.
Let $V \sim Beta(1/p, (d-1)/p)$ be independent of $\boldsymbol{Z}$ and $\boldsymbol{X}$. Now, the stochastic representation (2) implies

$$(\widetilde{R}_{d,p}R_{1,p})^p \stackrel{d}{=} \widetilde{R}_{d,p}^p(R^pV) \stackrel{d}{=} \widetilde{R}_{d,p}^p V R_{d,p}^p \stackrel{d}{=} \widetilde{R}_{1,p}^p R_{d,p}^p,$$

consequently $\widetilde{R}_{1,p}^p R_{d,p}^p$ is regularly varying with index α/p.
We have $\widetilde{R}_{1,p}^p \sim Gamma(1/p, 1/p)$ with $\widetilde{R}_{1,p}^p$ independent of $R_{d,p}^p = R^p$. Applying again Lemma 7 we have that $R_{d,p}^p$ is regularly varying with positive parameter α/p, thus the proof follows. □

Proof of Theorem 4 i) Let i, j be fixed with $1 \le i < j \le d$. By the definition both X_i and X_j have distribution function with the same upper endpoint ω. Consequently, if $\omega < \infty$ then X_i and X_j cannot be close to ω with non-zero probability, i.e., $\boldsymbol{P}\{X_i > \omega - \varepsilon, X_j > \omega - \varepsilon\} = 0$ for some $\varepsilon > 0$ small enough, hence $\kappa(X_i, X_j) = 0$ for this case.
ii) If $\omega = \infty$ and $1 - F$ is rapidly varying, then in view of condition (20) the associated random radius $R_{\{i,j\},p} := (|X_i|^p + |X_j|^p)^{1/p}$ has a rapidly varying distribution function. Hence, the proofs follows then using (3) and (20). □

Proof of Theorem 5 i) Since the distribution function F is in the max-domain of attraction of Λ and (20) is supposed to hold, then Theorem 8 implies that $X_i, 1 \le i \le d$ has distribution function in the same max-domain of attraction with the scaling function w. If $\omega < \infty$ using further Theorem 4 we get that the sample maxima has asymptotic independent components. If $\omega = \infty$ and $\lambda_i = \lambda >$

$0, 1 \le i \le d$ then Theorem 8 implies the distribution functions of $\boldsymbol{X}_i, 1 \le i \le d$ have the same asymptotic tail behaviour (up to some constant), hence the sample maxima has asymptotic independent components, thus the proof follows easily.
ii) Again using Theorem 8 we have that $X_i, 1 \le i \le d$ has distribution function in the max-domain of attraction of $\Psi_{\alpha+\lambda_i}$. Since $\omega < \infty$ the components then X_i, X_j cannot be both near to ω for any $1 \le i < j \le d$, implying that the sample maxima has independent components, thus the proof is complete. □

PROOF OF THEOREM 6 The proof is similar to the proof of Theorem 2 using further Theorem 8. □

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