J. DIFFERENTIAL GEOMETRY
114 (2020) 305-336

STRANGE DUALITY ON RATIONAL SURFACES

Yao Yuan

Abstract

We study Le Potier's strange duality conjecture on a rational surface. We focus on the case involving the moduli space of rank 2 sheaves with trivial first Chern class and second Chern class 2, and the moduli space of 1-dimensional sheaves with determinant L and Euler characteristic 0. We show the conjecture for this case is true under some suitable conditions on L, which applies to L ample on any Hirzebruch surface $\Sigma_e := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$ except for $e = 1$. When $e = 1$, our result applies to $L = aG + bF$ with $b \geq a + [a/2]$, where F is the fiber class, G is the section class with $G^2 = -1$ and $[a/2]$ is the integral part of $a/2$.

1. Introduction

In this whole paper, X is a rational surface over the complex number $\mathbb{C}$, with K_X the canonical divisor and H the polarization such that the intersection number $K_X.H < 0$. We use the same letter to denote both the line bundles and the corresponding divisor classes, but we write $L_1 \otimes L_2$, L^{-1} for line bundles while $L_1 + L_2$, $-L$ for the corresponding divisor classes. Denote by $L_1.L_2$ the intersection number of L_1 and L_2. $L^2 := L.L$.

Let $K(X)$ be the Grothendieck group of coherent sheaves over X. Define a quadratic form $(u, c) \mapsto \langle u, c\rangle := \chi(u \otimes c)$ on $K(X)$, where $\chi(-)$ is the holomorphic Euler characteristic and $\chi(u \otimes c) = \chi(\mathcal{F} \otimes^L \mathcal{G})$ for any $\mathcal{F}$ of class u, $\mathcal{G}$ of class c and $\otimes^L$ the flat tensor.

For two elements $c, u \in K(X)$ orthogonal to each other with respect to $\langle , \rangle$, we have $M_X^H(c)$ and $M_X^H(u)$ the moduli spaces of H-semistable sheaves of classes c and u respectively. If there are no strictly semistable sheaves of classes c (u, resp.), then over $M_X^H(c)$ ($M_X^H(u)$, resp.) there is a well-defined line bundle $\lambda_c(u)$ ($\lambda_u(c)$, resp.) called determinant line bundle associated to u (c, resp.). If there are strictly semistable sheaves of class u, one needs more conditions on c to get $\lambda_u(c)$ well-defined (see Ch 8 in [11]).

Mathematics Subject Classification. Primary 14D05.

Key words and phrases. Rational surfaces, moduli spaces of sheaves, strange duality.

Received May 1, 2016.

Let $c, u \in K(X)$. Assume both moduli spaces $M_H^X(c)$ and $M_H^X(u)$ are non-empty and the determinant line bundles $\lambda_c(u)$ and $\lambda_u(c)$ are well-defined over $M_H^X(c)$ and $M_H^X(u)$, respectively. According to [**15**] (see [**15**] p.9), if the following ($\bigstar$) is satisfied,

($\bigstar$) *for all H-semistable sheaves $\mathcal{F}$ of class c and H-semistable sheaves $\mathcal{G}$ of class u on X, $\mathrm{Tor}^i(\mathcal{F},\mathcal{G})=0\ \forall\ i\geq 1$, and $H^2(X,\mathcal{F}\otimes\mathcal{G})=0$.*

then there is a canonical map

$$SD_{c,u} : H^0(M_H^X(c),\lambda_c(u))^\vee \to H^0(M_H^X(u),\lambda_u(c)). \tag{1.1}$$

The strange duality conjecture asserts that $SD_{c,u}$ is an isomorphism.

Strange duality conjecture on curves was at first formulated (in [**3**] and [**7**]) and has been proved (see [**16**], [**4**]). Strange duality on surfaces does not have a general formulation so far. There is a special formulation due to Le Potier (see [**15**] or [**6**]). In this paper we choose $u = u_L := [\mathcal{O}_X] - [L^{-1}] + \frac{(L.(L+K_X))}{2}[\mathcal{O}_x]$ with x a single point in X, and $c = c_2^2 := 2[\mathcal{O}_X] - 2[\mathcal{O}_x]$. Then ($\bigstar$) is satisfied and $SD_{c,u}$ is well-defined. We prove the following theorem.

Theorem 1.1 (Corollary 3.15)**.** *Let X be a Hirzebruch surface Σ_e and $L = aG + bF$ with F the fiber class and G the section such that $G^2 = -e$. Then the strange duality map $SD_{c_2^2,u_L}$ as in (1.1) is an isomorphism for the following cases.*

1) $\min\{a,b\} \leq 1$*;*
2) $\min\{a,b\} \geq 2$*, $e \neq 1$, L ample;*
3) $\min\{a,b\} \geq 2$*, $e = 1$, $b \geq a + [a/2]$ with $[a/2]$ the integral part of $a/2$.*

Although strange duality on surfaces is a very interesting problem, there are very few cases known. Our result adds to previous work by the author ([**20**], [**22**]) and others ([**1**], [**5**], [**6**], [**9**], [**17**], [**18**], [**19**]).

Especially, in [**22**] we proved $SD_{c_2^2,u_L}$ is an isomorphism when $X = \mathbb{P}^2$. The limitation of the method in [**22**] is that: we have used Fourier transform on $\mathbb{P}^2$ which does not behave well on other rational surfaces. In this paper we use a new strategy. Actually we show the strange duality map $SD_{c_2^2,u_L}$ is an isomorphism under a list of conditions, and then check that all these conditions are fulfilled for cases in Theorem 1.1. So Theorem 1.1 is an application of our main theorem (Theorem 3.13) to Hirzebruch surfaces and there are certainly more applications to other rational surfaces.

The structure of the paper is arranged as follows. In § 2 we give preliminaries, including some useful properties of $M_X^H(c_2^2)$ (in § 2.1 and § 2.3) and a brief introduction to determinant line bundles and the set-up of strange duality (in § 2.2). § 3 is the main part. In § 3.1 and § 3.2 we prove the strange duality map is an isomorphism under a list of conditions; in § 3.3 we show the main theorem (Theorem 3.13) applies to cases on Hirzebruch surfaces. Although the argument in § 3.3 takes quite much space, the technique used there is essentially a combination of those in [**21**] and [**22**].

Notations. Let $\mathcal{F}$, $\mathcal{G}$ be two sheaves.

- $c_i(\mathcal{F})$ is the i-th Chern class of $\mathcal{F}$;
- $\chi(\mathcal{F})$ is the Euler characteristic of $\mathcal{F}$;
- $h^i(\mathcal{F}) = \dim H^i(\mathcal{F})$;
- $\text{ext}^i(\mathcal{F},\mathcal{G}) = \dim \text{Ext}^i(\mathcal{F},\mathcal{G})$, $\text{hom}(\mathcal{F},\mathcal{G}) = \dim \text{Hom}(\mathcal{F},\mathcal{G})$ and $\chi(\mathcal{F},\mathcal{G}) = \sum_{i\geq 0}(-1)^i\text{ext}^i(\mathcal{F},\mathcal{G})$;
- $Supp(\mathcal{F})$ or $C_{\mathcal{F}}$ is the support of 1-dimensional sheaf $\mathcal{F}$

Acknowledgements. The author was supported by NSFC grant 11301292.

2. Preliminaries

Define $u_L := [\mathcal{O}_X] - [L^{-1}] + \frac{(L.(L+K_X))}{2}[\mathcal{O}_x] \in K(X)$ with L a line bundle on X and x a single point in X. It is easy to check $u_{\mathcal{O}_X} = 0$ and $u_{L_1} + u_{L_2} = u_{L_1\otimes L_2}$. If L is nontrivially effective, i.e. $L \ncong \mathcal{O}_X$ and $H^0(L) \neq 0$, let $|L|$ be the linear system, then u_L is the class of 1-dimensional sheaves supported at curves in $|L|$ and of Euler characteristic 0.

For L nontrivially effective, denote by $M(L,0)$ the moduli space $M_X^H(u_L)$. In fact a sheaf $\mathcal{F}$ of class u_L is semistable (stable, resp.) if and only if $\forall\ \mathcal{F}' \subsetneq \mathcal{F}$, $\chi(\mathcal{F}') \leq 0$ ($\chi(\mathcal{F}') < 0$, resp.). Hence $M(L,0)$ does not depend on the polarization H. We ask $M(\mathcal{O}_X,0)$ to be a single point standing for the zero sheaf.

Let $c_n^r = r[\mathcal{O}_X] - n[\mathcal{O}_x] \in K(X)$ with x a single point on X. Denote by $W(r,0,n)$ the moduli space $M_X^H(c_n^r)$ (but $W(r,0,n)$ might depend on H). In this paper we mainly focus on $W(2,0,2)$ for X a rational surface.

For any L, r, n, u_L and c_n^r are orthogonal with respect to the quadratic form $\langle,\rangle$ on $K(X)$.

2.1. Some basic properties of $W(2,0,2)$.

Definition 2.1. We say the polarization H is **c_2^2-general**, if for any $\xi \in H^2(X,\mathbb{Z}) \cong \text{Pic}(X)$ such that $\xi.H = 0$ and $\xi^2 \geq -2$, we have $\xi = 0$.

Remark 2.2. Since $K_X.H < 0$, $\xi.H = 0 \Rightarrow \xi^2 \leq -2$ for any $0 \neq \xi \in \text{Pic}(X)$. This is because $H^0(\mathcal{O}_X(\pm\xi)) = 0$ by $\xi.H = 0$ and $H^2(\mathcal{O}_X(\pm\xi)) = H^0(\mathcal{O}_X(K_X \mp \xi))^\vee = 0$ by $(K_X \mp \xi).H < 0$, hence $\chi(\mathcal{O}_X(\xi) \oplus \mathcal{O}_X(-\xi)) = 2 + \xi^2 \leq 0$.

Lemma 2.3. *Let $\mathcal{F}$ be an H-semistable sheaf in class c_2^2. If $\mathcal{F}$ is not locally free, then it is strictly semistable and S-equivalent to $\mathcal{I}_x \oplus \mathcal{I}_y$*

with x, y two single points on X. Moreover, if H is c_2^2-general, then $\mathcal{F}$ is H-stable if and only if $\mathcal{F}$ is locally free.

Proof. First assume $\mathcal{F}$ is not locally free, then its reflexive hull $\mathcal{F}^{\vee\vee}$ is locally free of class c_i^2 with $i = 1$ or 0. $H^2(\mathcal{F}^{\vee\vee}) \cong H^2(\mathcal{F}) \cong \mathrm{Hom}(\mathcal{F}, K_X)^\vee = 0$ by $K_X.H < 0$ and the semistability of $\mathcal{F}$. Hence $\dim H^0(\mathcal{F}^{\vee\vee}) \geq \chi(\mathcal{F}^{\vee\vee}) = 2 - i > 0$. Therefore either $\mathcal{F}^{\vee\vee} \cong \mathcal{O}_X^{\oplus 2}$ or $\mathcal{F}^{\vee\vee}$ lies in the following sequence

$$0 \to \mathcal{O}_X \xrightarrow{\jmath} \mathcal{F}^{\vee\vee} \to \mathcal{I}_x \to 0, \tag{2.1}$$

where $\mathcal{I}_x$ is the ideal sheaf of some single point x on X.

If $\mathcal{F}^{\vee\vee}$ lies in (2.1), then we have

$$0 \to \mathcal{F} \to \mathcal{F}^{\vee\vee} \xrightarrow{p} \mathcal{T}_1 \to 0, \tag{2.2}$$

where $\mathcal{T}_1$ is a 0-dimensional sheaf with $\chi(\mathcal{T}_1) = 1$ and hence $\mathcal{T}_1 \cong \mathcal{O}_y$ for some single point $y \in X$. Compose maps $\jmath$ in (2.1) and p in (2.2), the map $p \circ \jmath : \mathcal{O}_X \to \mathcal{T}_1$ is not zero because otherwise $\mathcal{O}_X$ would be a subsheaf of $\mathcal{F}$. Therefore $p \circ \jmath$ is surjective with kernel isomorphic to $\mathcal{I}_y$ which is a subsheaf of $\mathcal{F}$ destabilizing $\mathcal{F}$. Hence $\mathcal{F}$ is not stable and S-equivalent to $\mathcal{I}_x \oplus \mathcal{I}_y$.

If $\mathcal{F}^{\vee\vee} \cong \mathcal{O}_X^{\oplus 2}$, then we have the following exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_X^{\oplus 2} \to \mathcal{T}_2 \to 0,$$

where $\mathcal{T}_2$ is a 0-dimensional sheaf with $\chi(\mathcal{T}_2) = 2$. We also have

$$0 \to \mathcal{O}_x \to \mathcal{T}_2 \to \mathcal{O}_y \to 0,$$

where x, y are two single points on X (it is possible to have $x = y$). Hence we have the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{I}_x & \to & \mathcal{O}_X & \to & \mathcal{O}_x & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{F} & \to & \mathcal{O}_X^{\oplus 2} & \to & \mathcal{T}_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{I}_y & \to & \mathcal{O}_X & \to & \mathcal{O}_y & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & .
\end{array}$$

Hence $\mathcal{F}$ is S-equivalent to $\mathcal{I}_x \oplus \mathcal{I}_y$.

Now assume H is c_2^2-general. We only need to show that any semistable bundle $\mathcal{F}$ of class c_2^2 is stable. If $\mathcal{F}$ is strictly semistable, then we have the following sequence

$$0 \to \mathcal{I}_Z(\xi) \to \mathcal{F} \to \mathcal{I}_W(-\xi) \to 0,$$

where $\xi.H = 0$ and $\mathcal{I}_Z, \mathcal{I}_W$ are ideal sheaves of 0-dimensional subschemes Z, W of X such that the length $len(Z) = len(W) = 1 + \xi^2/2 \geq 0$.

Since H is c_2^2-general, $\xi = 0$ and $\mathcal{I}_Z$ is a subsheaf of $\mathcal{F}$. Hence so is $\mathcal{O}_X$ because $\mathcal{F}$ is locally free, which is a contradiction since $H^0(\mathcal{F}) = 0$ by semistability. Hence $\mathcal{F}$ is stable. The lemma is proved. q.e.d.

Denote by $\mathbf{S}$ the closed subset of $W(2,0,2)$ consisting of non locally free sheaves, then set-theoretically $\mathbf{S}$ is isomorphic to the second symmetric power $X^{(2)}$ of X by Lemma 2.3. $\mathbf{S}$ is of codimension 1 in $W(2,0,2)$. In § 2.3 we will give a scheme-theoretic structure of $\mathbf{S}$ and show that it is a divisor associated to some line bundle.

Remark 2.4. If H is not c_2^2-general, then all strictly semistable vector bundle are S-equivalent to $\mathcal{O}_X(\xi) \oplus \mathcal{O}_X(-\xi)$ with $\xi \in \mathrm{Pic}(X)$, $\xi.H = \xi.K_X = 0$ and $\xi^2 = -2$.

2.2. Determinant line bundles and strange duality. To set up the strange duality conjecture, we briefly introduce so-called determinant line bundles on the moduli spaces of semistable sheaves. We refer to Chapter 8 in [**11**] for more details.

For a Noetherian scheme Y, we denote by $K(Y)$ the Grothendieck groups of coherent sheaves on Y and $K^0(Y)$ be the subgroup of $K(Y)$ generated by locally free sheaves. Then $K^0(X) = K(X)$ since X is smooth and projective.

Let $\mathscr{E}$ be a flat family of coherent sheaves of class c on X parametrized by a noetherian scheme S, then $\mathscr{E} \in K^0(X \times S)$. Let $p : X \times S \to S$, $q : X \times S \to X$ be the projections. Define $\lambda_{\mathscr{E}} : K(X) = K^0(X) \to \mathrm{Pic}(S)$ to be the composition of the following homomorphisms:
(2.3)

$$K^0(X) \xrightarrow{q^*} K^0(X \times S) \xrightarrow{.[\mathscr{E}]} K^0(X \times S) \xrightarrow{R^\bullet p_*} K^0(S) \xrightarrow{det^{-1}} \mathrm{Pic}(S),$$

where q^* is the pull-back morphism, $[\mathscr{F}].[\mathscr{G}] := \sum_i (-1)^i [\mathrm{Tor}^i(\mathscr{F},\mathscr{G})]$, and $R^\bullet p_*([\mathscr{F}]) = \sum_i (-1)^i [R^i p_* \mathscr{F}]$. Proposition 2.1.10 in [**11**] assures that $R^\bullet p_*([\mathscr{F}]) \in K^0(S)$ for any $\mathscr{F}$ coherent and S-flat.

For any $u \in K(X)$, $\lambda_{\mathscr{E}}(u) \in \mathrm{Pic}(S)$ is called the **determinant line bundle** associated to u induced by the family $\mathscr{E}$. Notice that the definition we use here is dual to theirs in [**11**].

Let $S = M_X^H(c)$, then there is in general no such universal family $\mathscr{E}$ over $X \times M_X^H(c)$, and even if it exists, there is ambiguity caused by tensoring with the pull-back of a line bundle on $M_X^H(c)$. Thus to get a well-defined determinant line bundle $\lambda_c(u)$ over $M_X^H(c)$, we need look at the good $GL(V)$-quotient $\Omega(c) \to M_X^H(c)$ with $\Omega(c)$ an open subset of some Quot-scheme and there is a universal quotient $\widetilde{\mathscr{E}}$ over $X \times \Omega(c)$. $\lambda_c(u)$ is then defined by descending the line bundle $\lambda_{\widetilde{\mathscr{E}}}(u)$ over $\Omega(c)$. $\lambda_{\widetilde{\mathscr{E}}}(u)$ descends if and only if it satisfies the "descent condition" (see Theorem 4.2.15 in [**11**]), which implies that u is orthogonal to c with respect to the quadratic form $\langle\ ,\ \rangle$. Hence the homomorphism λ_c is only defined over a subgroup of $K(X)$.

Now we focus on $M(L,0)$ and $W(r,0,n)$. As we have seen, u_L is orthogonal to c_n^r for any L, r, n.

Let $\lambda_{c_n^r}(L)$ be the determinant line bundle associated to u_L over (an open subset of) $W(r,0,n)$. We denote simply by $\lambda_r(L)$ if $r = n$. By checking the descent condition we see that $\lambda_2(L)$ is always well-defined over the stable locus $W(2,0,2)^s$ and $\mathbf{S}$, hence it is well-defined over all $W(2,0,2)$ if H is c_2^2-general. If H is not c_2^2-general, then $\lambda_2(L)$ is well-defined over point $[\mathcal{O}_X(-\xi)\oplus\mathcal{O}_X(\xi)]$ if and only if $\xi.L = 0$. We denote by $W(r,0,n)^L$ the biggest open subset of $W(r,0,n)$ where $\lambda_{c_n^r}(L)$ is well-defined. Notice that the stable locus $W(r,0,n)^s \subset W(r,0,n)^L$. By Remark 2.4, $W(2,0,2)^L = W(2,0,2)^{L\otimes K_X}$.

On the other hand, let $\lambda_L(c_n^r)$ be the determinant line bundle associated to c_n^r over $M(L,0)$, then $\lambda_L(c_n^r)$ is always well-defined over the whole moduli space. We have the following proposition which is analogous to Theorem 2.1 in [**6**].

Proposition 2.5. *(1) There is a canonical section, unique up to scalars,* $\sigma_{c_n^r,u_L} \in H^0(W(r,0,n)^L \times M(L,0), \lambda_{c_n^r}(L)\boxtimes\lambda_L(c_n^r))$ *whose zero set is*

$$\mathtt{D}_{c_n^r,u_L} := \big\{([\mathcal{F}],[\mathcal{G}])\in W(r,0,n)^L\times M(L,0)\ \big|\ h^0(\mathcal{F}\otimes\mathcal{G}) = h^1(\mathcal{F}\otimes\mathcal{G})\neq 0\big\}.$$

(2) The section $\sigma_{c_n^r,u_L}$ *defines a linear map up to scalars*

$$SD_{c_n^r,u_L} : H^0(W(r,0,n)^L, \lambda_{c_n^r}(L))^\vee \to H^0(M(L,0), \lambda_L(c_n^r)). \tag{2.4}$$

(3) Denote by $\sigma_{\mathcal{F}}$ *the restriction of* $\sigma_{c_n^r,u_L}$ *to* $\{\mathcal{F}\}\times M(L,0)$. $\sigma_{\mathcal{F}}$ *only depends (up to scalars) on the S-equivalence class of* $\mathcal{F}$.

(4) If $\sigma_{c_n^r,u_L}$ *is not identically zero, then by assigning* $\mathcal{F}$ *to* $\sigma_{\mathcal{F}}$ *we get a rational map* $\Phi : W(r,0,n)^L \to \mathbb{P}(H^0(M(L,0),\lambda_L(c_n^r)))$. *Similarly we have a rational map* $\Psi : M(L,0) \to \mathbb{P}(H^0(W(r,0,n)^L, \lambda_{c_n^r}(L)))$. *Moreover If the image of* Φ *is not contained in a hyperplane, then* $SD_{c_n^r,u_L}$ *is injective; if the image of* Ψ *is not contained in a hyperplane, then* $SD_{c_n^r,u_L}$ *is surjective.*

Proof. The proof of Theorem 2.1 in [**6**] also applies to our case although the surface may not be $\mathbb{P}^2$. For statement (3) and (4), one can also see Lemma 6.13 and Proposition 6.17 in [**9**]. q.e.d.

The map $SD_{c_n^r,u_L}$ in (2.4) is call the **strange duality map**, and Le Potier's strange duality is as follows (also see Conjecture 2.2 in [**6**])

Conjecture/Question 2.6. If both $W(r,0,n)^L$ and $M(L,0)$ are non-empty, then is $SD_{c_n^r,u_L}$ an isomorphism?

We denote by Θ_L the determinant line bundle associated to $c_0^1 = [\mathcal{O}_X]$ on $M(L,0)$. Then Θ_L has a canonical divisor D_{Θ_L} which consists of sheaves with non trivial global sections. Since λ_L is a group homomorphism, by Proposition 2.8 in [**14**], we have that $\lambda_L(c_n^r) \cong \Theta_L^{\otimes r} \otimes \pi^*\mathcal{O}_{|L|}(n) =: \Theta_L^r(n)$ where $\pi : M(L,0) \to |L|$ sends each sheaf to its support.

In this paper we study the following strange duality map for X a rational surface

(2.5)
$$SD_{2,L} := SD_{c_2^2,u_L} : H^0(W(2,0,2)^L, \lambda_2(L))^\vee \to H^0(M(L,0), \Theta_L^2(2)).$$

2.3. Scheme-theoretic structure of S on $W(2,0,2)$. $\mathbf{S}$ consists of non locally free sheaves in $W(2,0,2)$. Recall we have a good quotient $\rho : \Omega_2 \to W(2,0,2)$. Let $\widetilde{\mathbf{S}} = \rho^{-1}(\mathbf{S})$.

Set-theoretically $\mathbf{S} \cong X^{(2)}$. Let $\Delta \subset X^{(2)}$ be the singular locus and $\Delta \cong X$. Define $\mathbf{S}^o = \mathbf{S} - \Delta$, $W(2,0,2)^o = W(2,0,2)^L - \Delta$, $\widetilde{\mathbf{S}^o} = \rho^{-1}(\mathbf{S}^o)$ and $\Omega_2^o = \rho^{-1}(W(2,0,2)^o)$. Let $\mathscr{F}$ ($\mathscr{F}^o$, resp.) be the universal quotient over $X \times \Omega_2$ ($X \times \Omega_2^o$, resp.). We then have the following proposition due to Abe (see Section 3.4 and Section 5.2 Proposition 3.7 and Proposition 5.2 in [**1**])

Proposition 2.7. *(1) The second Fitting ideal* $\mathtt{Fitt}_2(\mathscr{F}^o)$ *of* $\mathscr{F}^o$ *defines a smooth closed subscheme* $\widehat{\boldsymbol{S}^o}$ *of* $X \times \Omega_2^o$ *supported at the set*
$$\{(x, [q : \mathcal{O}_X(-mH) \otimes V \twoheadrightarrow \mathcal{F}]) |\dim_{k(x)} \mathcal{F}_x \otimes k(x) > 2\} \subset X \times \Omega_2^o.$$
i.e. $\widehat{\boldsymbol{S}^o}$ *consists of points* $(x, [q : \mathcal{O}_X(-mH) \twoheadrightarrow \mathcal{F}])$ *such that* $\mathcal{F}_x$ *is not free.*

(2) We have a surjective map $p_\Omega : \widehat{\boldsymbol{S}^o} \to \widetilde{\boldsymbol{S}^o}$ *induced by the projection* $p_\Omega : X \times \Omega_2 \to \Omega_2$. *We give a scheme structure of* $\widetilde{\boldsymbol{S}^o}$ *by letting its defining ideal be the kernel of* $\mathcal{O}_{\Omega_2^o} \to p_{\Omega *}\mathcal{O}_{\widehat{\boldsymbol{S}^o}}$. *Then* $\widetilde{\boldsymbol{S}^o}$ *is a normal crossing divisor in* Ω_2^o *with* $\widehat{\boldsymbol{S}^o} \to \widetilde{\boldsymbol{S}^o}$ *the normalization.*

(3) The line bundle associated to the divisor $\widetilde{\boldsymbol{S}^o}$ *on* Ω_2^o *is* $\lambda_{\mathscr{F}^o}(u_{K_X^{-1}})$.

Proof. Sheaves in $\widetilde{\mathbf{S}^o}$ are all quasi-bundles (see Definition 2.1 in [**1**]), hence Abe's argument in Section 3.4 in [**1**] gives Statement (1) and (2). Notice that our notations are slightly different from his.

For Statement (3), by Proposition 5.2 in [**1**] we know that $\mathcal{O}_{\Omega_2^o}(\widetilde{\mathbf{S}^o}) \cong \lambda_{\mathscr{F}^o}([K_X])^{-1} \otimes \lambda_{\mathscr{F}^o}([\mathcal{O}_X])^{-1}$. We also see that
$$\lambda_{\mathscr{F}^o}(u_{K_X^{-1}}) \cong \lambda_{\mathscr{F}^o}([K_X])^{-1} \otimes \lambda_{\mathscr{F}^o}([\mathcal{O}_X]).$$
But $\lambda_{\mathscr{F}^o}([\mathcal{O}_X]) \cong \mathcal{O}_{\Omega_2^o}$ since $H^i(\mathcal{F}) = 0$ for $i = 0, 1, 2$ and $\mathcal{F}$ semistable of class c_2^2. Hence the proposition. q.e.d.

Corollary 2.8. *Let* $\boldsymbol{S}$ *have the scheme-theoretic structure as the closure of* $\boldsymbol{S}^o$ *in* $W(2,0,2)$. *Then* $\boldsymbol{S}$ *is a divisor associated to the line bundle* $\lambda_2(K_X^{-1})$ *on* $W(2,0,2)$. *Moreover* $\boldsymbol{S}$ *is an integral scheme.*

Proof. By Proposition 2.7, $\mathbf{S}^o$ is a divisor associated to $\lambda_2(K_X^{-1})$ restricted on $W(2,0,2)^o$. $\mathbf{S}$ is the closure of $\mathbf{S}^o$ in $W(2,0,2)^L$. Since $K_X.H < 0$, $W(2,0,2)$ is normal, Cohen-Macaulay and of pure dimension 5, hence the section given by $\mathbf{S}^o$ extends to a section of $\lambda_2(K_X^{-1})$ on $W(2,0,2)^L$ with divisor $\mathbf{S}$.

We have a morphism $\varphi : X^{(2)} \to \mathbf{S}$ sending (x,y) to $\mathcal{I}_x \oplus \mathcal{I}_y$, which is bijective. Hence $\mathbf{S}$ is irreducible. $\widetilde{\mathbf{S}^o}$ is reduced, hence so are $\mathbf{S}^o$ and $\mathbf{S}$. Thus $\mathbf{S}$ is an integral scheme. q.e.d.

Lemma 2.9. *For any line bundle L, the map $H^0(\boldsymbol{S}, \lambda_2(L)|_{\boldsymbol{S}}) \xrightarrow{\varphi^*} H^0(X^{(2)}, \varphi^*\lambda_2(L))$ induced by $\lambda_2(L)|_{\boldsymbol{S}} \to \varphi_*\varphi^*\lambda_2(L)$ is injective. Moreover $H^0(X^{(2)}, \varphi^*\lambda_2(L)) \cong (H^0(X,L)^{\otimes 2})^{\mathfrak{S}_2} \cong S^2H^0(X,L)$ where $\mathfrak{S}_n$ is the n-th symmetric group.*

Proof. Let $\Delta \subset X^2$ be the diagonal, and $\mathcal{I}_\Delta$ is the ideal sheaf of Δ in X^2. Let $pr_{i,j}$ be the projection from X^n to the product X^2 of the i-th and j-th factors. Then $pr_{1,2}^*\mathcal{I}_\Delta \oplus pr_{1,3}^*\mathcal{I}_\Delta$ gives a family of ideal sheaves on X^3 and induces a morphism $\widetilde{\varphi} : X^2 \to W(2,0,2)$ with image $\mathbf{S}$. $\widetilde{\varphi}$ is $\mathfrak{S}_2$-invariant, hence factors through $X^2 \to X^{(2)}$ and gives the map $\varphi : X^{(2)} \to \mathbf{S}$. The morphism φ is bijective and $\mathbf{S}$ is reduced, hence the map $\varphi^\natural : \mathcal{O}_{\mathbf{S}} \to \varphi_*\mathcal{O}_{X^{(2)}}$ is injective. Hence so is the map $\lambda_2(L)|_{\mathbf{S}} \to \varphi_*\varphi^*\lambda_2(L)$ and therefore $H^0(\mathbf{S}, \lambda_2(L)|_{\mathbf{S}}) \xrightarrow{\varphi^*} H^0(X^{(2)}, \varphi^*\lambda_2(L))$ is injective.

Obviously $H^0(X^{(2)}, \varphi^*\lambda_2(L)) \cong (H^0(X^2, \widetilde{\varphi}^*\lambda_2(L)))^{\mathfrak{S}_2}$. It will suffice to show that $H^0(X^2, \widetilde{\varphi}^*\lambda_2(L)) \cong H^0(X,L)^{\otimes 2}$. By the basic properties (see Lemma 8.1.2 and Theorem 8.1.5 in [**11**]) of the determinant line bundle, we have $\widetilde{\varphi}^*\lambda_2(L) \cong \lambda_{pr_{1,2}^*\mathcal{I}_\Delta \oplus pr_{1,3}^*\mathcal{I}_\Delta}(u_L) \cong \lambda_{pr_{1,2}^*\mathcal{I}_\Delta}(u_L) \otimes \lambda_{pr_{1,3}^*\mathcal{I}_\Delta}(u_L) \cong \lambda_{\mathcal{I}_\Delta}(L)^{\boxtimes 2}$. Obviously $\lambda_{\mathcal{I}_\Delta}(L) \cong L$, so we have

$$H^0(X^2, \widetilde{\varphi}^*\lambda_2(L)) \cong H^0(X, \lambda_{\mathcal{I}_\Delta}(L))^{\otimes 2} \cong H^0(X,L)^{\otimes 2}.$$

Hence the lemma. q.e.d.

The line bundle $L^{\boxtimes n}$ on X^n is $\mathfrak{S}_n$-linearized and descends to a line bundle on $X^{(n)}$, which we denote by $L_{(n)}$. So $\varphi^*\lambda_2(L) \cong L_{(2)}$ on $X^{(2)}$. Denote also by $L_{(n)}$ the pullback of $L_{(n)}$ to $X^{[n]}$ via the Hilbert-Chow morphism, where $X^{[n]}$ is the Hilbert scheme of n-points on X.

3. Main result on $SD_{2,L}$

Let L be a nontrivially effective line bundle. Recall that $SD_{2,L}$ is the following strange duality map as in (2.5):

$$SD_{2,L} : H^0(W(2,0,2)^L, \lambda_2(L))^\vee \to H^0(M(L,0), \Theta_L^2(2)).$$

In this section, we show that under certain conditions $SD_{2,L}$ is an isomorphism (see Theorem 3.13).

On $M(L,0)$ and $W(2,0,2)^L$ we have the following two exact sequences respectively.

$$0 \to \Theta_L(2) \to \Theta_L^2(2) \to \Theta_L^2(2)|_{D_{\Theta_L}} \to 0; \tag{3.1}$$

$$0 \to \lambda_2(L \otimes K_X) \to \lambda_2(L) \to \lambda_2(L)|_{\mathbf{S}} \to 0. \tag{3.2}$$

Notice that $W(2,0,2)^{L\otimes K_X} = W(2,0,2)^L$ and (3.2) is because of Corollary 2.8.

Lemma 3.1. *By taking the global sections of (3.1) and the dual of global sections of (3.2), we have the following commutative diagram*
(3.3)

$$\begin{array}{ccccccc} & H^0(\boldsymbol{S},\lambda_2(L)|_{\boldsymbol{S}})^\vee & \xrightarrow{g_2^\vee} & H^0(\lambda_2(L))^\vee & \xrightarrow{f_2^\vee} & H^0(\lambda_2(L\otimes K_X))^\vee & \longrightarrow 0 \\ & \downarrow{\scriptstyle \alpha_{\boldsymbol{S}}} & & \downarrow{\scriptstyle SD_{2,L}} & & \downarrow{\scriptstyle \beta_D} & \\ 0 \longrightarrow & H^0(\Theta_L(2)) & \xrightarrow[f_L]{} & H^0(\Theta_L^2(2)) & \xrightarrow[g_L]{} & H^0(D_{\Theta_L},\Theta^2(2)|_{D_{\Theta_L}}). & \end{array}$$

Proof. We only need to show that $g_L \circ SD_{2,L} \circ g_2^\vee = 0$. By the definition of $SD_{2,L}$, it is enough to show that the section $\sigma_{c_2^2,L}$ defined in Proposition 2.5 is identically zero on $\mathbf{S} \times D_{\Theta_L}$. Easy to see that $H^0((\mathcal{I}_x \oplus \mathcal{I}_y) \otimes \mathcal{G}) \neq 0$ for all $\mathcal{G} \in M(L,0)$ such that $H^0(\mathcal{G}) \neq 0$, hence $\mathbf{S} \times D_{\Theta_L} \subset \mathtt{D}_{c_2^2,u_L}$ and $\sigma_{c_2^2,L}$ is identically zero on $\mathbf{S} \times D_{\Theta_L}$. The lemma is proved. q.e.d.

3.1. On the map $\alpha_{\mathbf{S}}$.

We introduce the following condition.

Condition (CA). The strange duality map

$$SD_{c_2^1,u_L} : H^0(W(1,0,n),\lambda_{c_2^1}(L))^\vee \to H^0(M(L,0),\Theta_L(2)) \tag{3.4}$$

is an isomorphism.

Remark 3.2. For any $n \geq 1$, $W(1,0,n) \cong X^{[n]}$ and $\lambda_{c_n^1}(L) \cong L_{(n)}$. It is well-known that $H^0(X^{[n]},L_{(n)}) = S^n H^0(X,L)$ for all n and L (see Lemma 5.1 in [**8**]). Therefore **CA** implies $H^0(|L|,\mathcal{O}_{|L|}(2)) \cong H^0(|L|,\pi_*\Theta_L \otimes \mathcal{O}_{|L|}(2))$.

In particular we have $h^0(M(L,0),\Theta_L) = h^0(|L|,\pi_*\Theta_L) = 1$ and D_{Θ_L} is the unique divisor associated to Θ_L.

Lemma 3.3. *If* ***CA*** *is satisfied, then the map $\alpha_{\boldsymbol{S}}$ in (3.3) is an isomorphism. In particular, $g_2^\vee$ is injective.*

Proof. By Lemma 2.9 we have a surjective map

$$\varphi^{*\vee} : H^0(X^{(2)},L_2)^\vee \twoheadrightarrow H^0(\mathbf{S},\lambda_2(L)|_{\mathbf{S}})^\vee.$$

By Proposition 1.2 in [**8**], we have $HC_2^{*\vee} : H^0(X^{[2]},L_2)^\vee \xrightarrow{\cong} H^0(X^{(2)},L_2)^\vee$ where $HC_2 : X^{[2]} \to X^{(2)}$ is the Hilbert-Chow morphism.

To prove the lemma, by **CA** it is enough to show $\alpha_{\mathbf{S}}\circ\varphi^{*\vee}\circ HC_2^{*\vee} = SD_{c_2^1,u_L}$ or equivalently $SD_{2,L}\circ g_2^{\vee}\circ\varphi^{*\vee}\circ HC_2^{*\vee} = f_L\circ SD_{c_2^1,u_L}$.

We have a Cartesian diagram

$$\begin{array}{ccc} \widehat{X^2} & \xrightarrow{\widehat{HC}} & X^2 \\ \Big\downarrow{\scriptstyle\widehat{\mu}} & & \Big\downarrow{\scriptstyle\mu} \\ X^{[2]} & \xrightarrow[HC]{} & X^{(2)}, \end{array} \tag{3.5}$$

where μ is a $\mathfrak{S}_2$-quotient and $\widehat{X^2}$ is the blow-up of X^2 along the diagonal Δ. Then we only need to show

$$\widehat{SD}_L := SD_{2,L}\circ g_2^{\vee}\circ\varphi^{*\vee}\circ HC_2^{*\vee}\circ\widehat{\mu}^{*\vee} = f_L\circ SD_{c_2^1,u_L}\circ\widehat{\mu}^{*\vee} =: \widehat{SD}_R. \tag{3.6}$$

There are two flat families on $X\times\widehat{X^2}$ of sheaves of class c_2^2: $\mathscr{F}^1 := \widehat{HC}^*_X(pr_{1,2}^*\mathcal{I}_\Delta\oplus pr_{1,3}^*\mathcal{I}_\Delta)$ and $\mathscr{F}^2 := \widehat{\mu}_X^*\mathscr{I}_2\oplus q^*\mathcal{O}_X$, where $\widehat{HC}_X := Id_X\times\widehat{HC} : X\times\widehat{X^2}\to X^3$, $\widehat{\mu}_X := Id_X\times\widehat{\mu}$, $q : X\times\widehat{X^2}\to X$ and $\mathscr{I}_2$ is the universal ideal sheaf on $X\times X^{[2]}$.

$\mathscr{F}^i$ induces a section σ_i of $\widehat{\mu}^*\lambda_{c_n^1}(L)\boxtimes\lambda_L(c_2^2)\cong\widehat{\mu}^*L_n\boxtimes\Theta_L^2(2)$ on $\widehat{X^2}\times M(L,0)$. The zero set of σ_i is $\mathtt{D}_i := \{(\underline{x},\mathcal{G})|H^0(\mathscr{F}^i_{\underline{x}}\otimes\mathcal{G})\neq 0\}$. By the definition of $SD_{c_n^r,u_L}$, we see that $\widehat{SD}_L$ is defined by the global section σ_1. On the other hand, the map f_L is defined by multiplying an element in $H^0(\Theta_L)$ defininig the divisor D_{Θ_L}. Therefore $\widehat{SD}_R$ is defined by the global section σ_2. Hence to show (3.6), we only need to show $\mathtt{D}_i$ coincide as divisors for $i=1,2$.

Let $\mathcal{C}\subset X\times|L|$ be the universal curve. Then $\mathcal{C}$ is a divisor in $X\times|L|$. $p_{i,|L|} := p_i\times Id_{|L|} : X^2\times|L|\to X\times|L|$ with p_i the projection to the i-th factor. Denote by $p_M : \widehat{X^2}\times M(L,0)\to M(L,0)$ the projection to $M(L,0)$. Then easy to see that $\mathtt{D}_1=\mathtt{D}_2 = 2p_M^*D_{\Theta_L}+\widehat{HC}^*p_{1,|L|}^*\mathcal{C}+\widehat{HC}^*p_{2,|L|}^*\mathcal{C}$. Hence the lemma. q.e.d.

Corollary 3.4. *If* ***CA*** *is satisfied and moreover $D_{\Theta_L}=\emptyset$ and $H^0(L\otimes K_X^{\otimes n})=0$ for all $n\geq 1$, then the map $SD_{2,L}$ is an isomorphism.*

Proof. By Lemma 3.3, we only need to show that $H^0(\lambda_2(L\otimes K_X))=0$. But $H^0(\lambda_2(L\otimes K_X^{\otimes n})|_{\mathbf{S}})=0$ since $H^0(L\otimes K_X^{\otimes n})=0$ for all $n\geq 1$. Hence $H^0(\lambda_2(L\otimes K_X))\cong H^0(\lambda_2(L\otimes K_X^{\otimes n}))$ for all $n\geq 1$ and hence $H^0(\lambda_2(L\otimes K_X))=0$ because $\lambda_2(K_X^{-1})$ is effective. q.e.d.

Remark 3.5. Assume K_X^{-1} is effective, then for any curve $C\in|K_X^{-1}|$, either $\mathcal{O}_C$ is semistable or C contains an integral subscheme with genus >1. Therefore we have $D_{\Theta_L}=\emptyset\Rightarrow H^0(L\otimes K_X^{\otimes n})=0$ for all $n\geq 1$. This is because otherwise there must be a semistable sheaf of class u_L having nonzero global sections.

Moreover by Proposition 4.1.1 and Corollary 4.3.2 in [**20**], we see that if every curve in $|L|$ does not contain any 1-dimensional subscheme with positive genus and K_X^{-1} is effective, then Corollary 3.4 applies and the strange duality map $SD_{2,L}$ is an isomorphism.

We have a useful lemma as follows.

Lemma 3.6. *If $D_{\Theta_L} \neq \emptyset$, then $L \otimes K_X$ is effective.*

Proof. Let $\mathcal{F} \in D_{\Theta_L}$, then $\mathrm{Ext}^1(\mathcal{F}, K_X) \cong H^1(\mathcal{F})^\vee \neq 0$. Hence there is a non split extension

$$0 \to K_X \to \widetilde{I} \to \mathcal{F} \to 0.$$

If for every proper quotient $\mathcal{F} \twoheadrightarrow \mathcal{F}''$ (i.e. $\mathcal{F} \ncong \mathcal{F}''$) we have $h^1(\mathcal{F}'') = 0$, then $\widetilde{I}$ has to be torsion-free and hence isomorphism to $\mathcal{I}_Z(L \otimes K_X)$ with Z a 0-dimensional subscheme of X. On the other hand $h^0(\widetilde{I}) = h^0(\mathcal{F}) \neq 0$, therefore $H^0(L \otimes K_X) \neq 0$.

If there is a proper quotient $\mathcal{F}_1$ of $\mathcal{F}$ such that $h^1(\mathcal{F}_1) \neq 0$, then we can assume that for every proper quotient $\mathcal{F}_1''$ of $\mathcal{F}_1$ we have $h^1(\mathcal{F}_1'') = 0$. Denote by L_1 the determinant of $\mathcal{F}_1$, then by previous argument $H^0(L_1 \otimes K_X) \neq 0$ and hence $H^0(L \otimes K_X) \neq 0$ because $L \otimes L_1^{-1}$ is effective. q.e.d.

3.2. On the map β_D. In this subsection we assume $D_{\Theta_L} \neq \emptyset$, then by Lemma 3.6 $L \otimes K_X$ is effective. We want to prove that under certain conditions the map β_D is an isomorphism. The main technique and notations are analogous to [**22**].

Let $\ell := L.(L + K_X)/2 = \chi(L \otimes K_X) - 1$ and H_ℓ be the Hilbert scheme of ℓ-points on X which also parametrizes all ideal sheaves $\mathcal{I}_Z$ with colength ℓ, i.e. $len(Z) = \ell$. If $\ell = 0$, we say H_0 is a simple point corresponding to the structure sheaf $\mathcal{O}_X$. Denote by $\mathscr{I}_\ell$ the universal ideal sheaf over $X \times H_\ell$.

From now on by abuse of notation, we always denote by p the projection $X \times M \to M$ and q the projection $X \times M \to X$ for any moduli space M. If we have $Y_1 \times \cdots \times Y_n$ with $n \geq 2$, denote by p_{ij} $(i < j)$ the projection to $Y_i \times Y_j$.

Define

$$Q_1 := Quot_{X \times H_\ell / H_\ell}(\mathscr{I}_\ell \otimes q^*(L \otimes K_X), u_L)$$

and

$$Q_2 := Quot_{X \times H_\ell / H_\ell}(\mathscr{I}_\ell \otimes q^*(L \otimes K_X), u_{L \otimes K_X}).$$

Then Q_1 and Q_2 are the two relative Quot-schemes over H_ℓ parametrizing quotients of class u_L and $u_{L \otimes K_X}$ respectively. Let $\rho_i : Q_i \to H_\ell$ be the projection. Each point $[f_1 : \mathcal{I}_Z(L \otimes K_X) \twoheadrightarrow \mathcal{F}_L] \in Q_1$ ($[f_2 : \mathcal{I}_Z(L \otimes K_X) \twoheadrightarrow \mathcal{F}_{L \otimes K_X}] \in Q_2$, resp.) over $\mathcal{I}_Z \in H_\ell$ must have the kernel K_X ($\mathcal{O}_X$, resp.).

Since $L\otimes K_X$ is effective and X is rational, $H^2(L\otimes K_X)=0$. Hence $h^0(L\otimes K_X)\geq\chi(L\otimes K_X)$. Therefore, for any ideal sheaf $\mathcal{I}_Z$ with colenght ℓ, we have $h^0(\mathcal{I}_Z(L\otimes K_X))\geq 1$ and hence ρ_2 is always surjective. If moreover $L.K_X\leq 0$, then $H^0(\mathcal{I}_Z(L))\neq 0$ and ρ_1 is also surjective.

We write down the following two exact sequences.

$$0\to K_X\to\mathcal{I}_Z(L\otimes K_X)\to\mathcal{F}_L\to 0; \tag{3.7}$$

$$0\to\mathcal{O}_X\to\mathcal{I}_Z(L\otimes K_X)\to\mathcal{F}_{L\otimes K_X}\to 0. \tag{3.8}$$

Notice that if $\mathcal{F}_L$ (resp. $\mathcal{F}_{L\otimes K_X}$) is semi-stable, then (the class of) $\mathcal{F}_L$ (resp. (the class of) $\mathcal{F}_{L\otimes K_X}$) is contained in D_{Θ_L} (resp. $M(L\otimes K_X,0)$).

Let

$$D^o_{\Theta_L}:=\left\{\mathcal{F}_L\in D_{\Theta_L}\;\middle|\;\begin{array}{l}h^1(\mathcal{F}_L)=1,\ h^1(\mathcal{F}_L(-K_X))=0\\ \textit{and } Supp(\mathcal{F}_L) \textit{ is integral.}\end{array}\right\},$$

$$Q^o_1:=\left\{[\mathcal{I}_Z(L\otimes K_X)\overset{f_1}{\twoheadrightarrow}\mathcal{F}_L]\in Q_1\;\middle|\;\begin{array}{l}h^1(\mathcal{F}_L)=1, h^1(\mathcal{F}_L(-K_X))=0\\ \textit{and } Supp(\mathcal{F}_L) \textit{ is integral.}\end{array}\right\},$$

$$M(L\otimes K_X,0)^o:=\left\{\mathcal{F}_{L\otimes K_X}\in M(L\otimes K_X,0)\;\middle|\;\begin{array}{l}h^0(\mathcal{F}_{L\otimes K_X}(K_X))=0\\ \textit{and } Supp(\mathcal{F}_{L\otimes K_X})\\ \textit{is integral.}\end{array}\right\},$$

$$Q^o_2:=\left\{[\mathcal{I}_Z(L\otimes K_X)\overset{f_2}{\twoheadrightarrow}\mathcal{F}_{L\otimes K_X}]\in Q_2\;\middle|\;\begin{array}{l}h^0(\mathcal{F}_{L\otimes K_X}(K_X))=0\\ \textit{and } Supp(\mathcal{F}_{L\otimes K_X})\\ \textit{is integral.}\end{array}\right\}.$$

Let $\mathcal{G}^r_r$ with $r\geq 1$ be a locally free sheaf of class c^r_r on X. We define a line bundle $\mathcal{L}^r:=(det(R^\bullet p_*(\mathscr{I}_\ell\otimes q^*\mathcal{G}^r_r(L\otimes K_X))))^\vee$ over H_ℓ. Then we have the following lemma.

Lemma 3.7. *There are classifying maps $g_1:Q^o_1\to D^o_{\Theta_L}$ and $g_2:Q^o_2\to M(L\otimes K_X,0)^o$, where g_1 is an isomorphism and g_2 is a projective bundle. Moreover $g_1^*\Theta^r_L(r)\cong\rho_1^*\mathcal{L}^r|_{Q^o_1}$ and $g_2^*\Theta^r_{L\otimes K_X}(r)\cong\rho_2^*\mathcal{L}^r|_{Q^o_2}$.*

Proof. The proof is analogous to [22]. See Lemma 4.8, Equation (4.9), (4.10), (4.12) and (4.14) in [22]. q.e.d.

Let $H^o_\ell:=\rho_1(Q^o_1)\cup\rho_2(Q^o_2)$. We introduce some conditions as follows.

Condition (CB). (1) $D^o_{\Theta_L}$ is dense open in D_{Θ_L};

(2) $M(L\otimes K_X,0)$ is of pure dimension and satisfies the "condition S_2 of Serre", and the complement of $M(L\otimes K_X,0)^o$ is of codimension ≥ 2;

(3) $(\rho_1)_*\mathcal{O}_{Q^o_1}\cong\mathcal{O}_{H^o_\ell}$;

(4) Q^o_2 is nonempty and dense open in $\rho_2^{-1}(\rho_2(Q^o_2))$.

Remark 3.8. We say a scheme Y satisfies "condition S_2 of Serre" if $\forall\ y\in Y$ the local ring $\mathcal{O}_y$ has the property that for every prime ideal $\mathfrak{p}\subset\mathcal{O}_y$ of height ≥ 2, we have **depth** $\mathcal{O}_{y,\mathfrak{p}}\geq 2$ (also see Ch II

Theorem 8.22A in [10]). **CB**-(2) implies that for every line bundle $\mathcal{H}$ over $M(L\otimes K_X,0)$, the restriction map $H^0(M(L\otimes K_X,0),\mathcal{H})\hookrightarrow H^0(M(L\otimes K_X,0)^o,\mathcal{H})$ is an isomorphism.

Lemma 3.9. *If* ***CB*** *is satisfied, then we have an injective map for all* $r>0$

$$j_r : H^0(D_{\Theta_L},\Theta_L^r(r)|_{D_{\Theta_L}})\hookrightarrow H^0(M(L\otimes K_X,0),\Theta^r_{L\otimes K_X}(r)).$$

Moreover, $j_2\circ\beta_D = SD_{2,L\otimes K_X}$.

Proof. By **CB**-(1) we have an injection

$$H^0(D_{\Theta_L},\Theta_L^r(r)|_{D_{\Theta_L}})\hookrightarrow H^0(D^o_{\Theta_L},\Theta_L^r(r)|_{D^o_{\Theta_L}}). \tag{3.9}$$

By Lemma 3.7 and **CB**-(3) we have

$$H^0(D^o_{\Theta_L},\Theta_L^r(r)|_{D^o_{\Theta_L}})\xrightarrow{\cong} H^0(Q_1^o,\rho_1^*\mathcal{L}^r|_{Q_1^o})\xrightarrow{\cong} H^0(H_\ell^o,\mathcal{L}^r|_{H_\ell^o}). \tag{3.10}$$

On the other hand ρ_2 is projective and surjective, hence there is a natural injection $\mathcal{O}_{H_\ell}\hookrightarrow(\rho_2)_*\mathcal{O}_{Q_2}$. Hence by **CB**-(4) we have the following injections

$$\begin{array}{ccc} H^0(H_\ell^o,\mathcal{L}^r|_{H_\ell^o}) \hookrightarrow & H^0(\rho_2(Q_2^o),\mathcal{L}^r|_{\rho_2(Q_2^o)}) & \\ & \downarrow & \\ & H^0(\rho_2^{-1}(\rho_2(Q_2^o)),\rho_2^*\mathcal{L}^r) \hookrightarrow & H^0(Q_2^o,\rho_2^*\mathcal{L}^r). \end{array} \tag{3.11}$$

Finally by Lemma 3.7 and **CB**-(2) we have

$$\begin{array}{ccc} H^0(Q_2^o,\rho_2^*\mathcal{L}^r) \xrightarrow{\cong} & H^0(M(L\otimes K_X,0)^o,\Theta^r_{L\otimes K_X}(r)) \\ & \downarrow\cong \\ & H^0(M(L\otimes K_X,0),\Theta^r_{L\otimes K_X}(r)). \end{array} \tag{3.12}$$

The map j_r is obtained by composing all the maps successively in (3.9), (3.10), (3.11) and (3.12).

Now we prove $j_r\circ\beta_D = SD_{2,L\otimes K_X}$. Notice that $\chi(\mathcal{E}\otimes\mathcal{I}_Z(L\otimes K_X)) = h^2(\mathcal{E}\otimes\mathcal{I}_Z(L\otimes K_X))=0$ for all $\mathcal{E}\in W(2,0,2)$ and $\mathcal{I}_Z\in H_\ell$. We then have a determinant line bundle $\lambda_2(\ell)$ (resp. $\lambda_{H_\ell}(c_2^2)$) over $W(2,0,2)^L$ (resp. H_ℓ) associated to $[\mathcal{I}_Z(L\otimes K_X)]$ with $\mathcal{I}_Z\in H_\ell$ (resp. $[\mathcal{E}]$ with $\mathcal{E}\in W(2,0,2)$). Obviously $\lambda_{H_\ell}(c_2^2)=\mathcal{L}^2$. Moreover there is a section $\sigma_{2,\ell}$ of $H^0(W(2,0,2)^L\times H_\ell,\lambda_2(\ell)\boxtimes\mathcal{L}^2)$ vanishing at the points $(\mathcal{E},\mathcal{I}_Z)$ such that $H^0(\mathcal{E}\otimes\mathcal{I}_Z(L\otimes K))\neq 0$. By (3.7), $\lambda_2(L)\cong\lambda_2(\ell)\otimes\lambda_2([K_X])^{-1}\cong\lambda_2(\ell)\otimes\lambda_2(K_X^{-1})$. Hence $\lambda_2(\ell)\cong\lambda_2(L\otimes K_X)$.

The section $\sigma_{2,\ell}$ induces a morphism

$$H^0(W(2,0,2)^L,\lambda_2(L\otimes K_X))^\vee \xrightarrow{SD_{2,\ell}} H^0(H_\ell,\mathcal{L}^2).$$

Composing $SD_{2,\ell}$ with the inclusion $H^0(H_\ell,\mathcal{L}^2)\hookrightarrow H^0(H^o_\ell,\mathcal{L}^2)$, we get $H^0(W(2,0,2)^L,\lambda_2(L\otimes K_X))^\vee \xrightarrow{SD^o_{2,\ell}} H^0(H^o_\ell,\mathcal{L}^2)$. Composing maps in (3.11) and (3.12) and we get

$$H^0(H^o_\ell,\mathcal{L}^2)\xrightarrow{(g_2)_*\circ\rho_2^*} H^0(M(L\otimes K_X,0),\Theta^2_{L\otimes K_X}(2)).$$

We first show that the following diagram commutes.

(3.13)

$$\begin{array}{ccc} H^0(W(2,0,2)^L,\lambda_2(L\otimes K_X))^\vee & \xrightarrow{SD^o_{2,\ell}} & H^0(H^o_\ell,\mathcal{L}^2) \\ & \searrow^{SD_{2,L\otimes K_X}} & \downarrow{\scriptstyle (g_2)_*\circ\rho_2^*} \\ & & H^0(M(L\otimes K_X,0),\Theta^2_{L\otimes K_X}(2)). \end{array}$$

Recall that on $X\times Q^o_2$ there is an exact sequence

$$0\to\mathscr{R}_2\to(id_X\times\rho_2^*)\mathscr{I}_\ell\otimes q^*(L\otimes K_X)\to\mathscr{F}_{L\otimes K_X}\to 0,$$

where $\mathscr{I}_\ell$ is the universal sheaf over $X\times H_\ell$ and $\mathscr{R}_2=p^*\mathcal{R}_2$ with $\mathcal{R}_2$ a line bundle over Q^o_2. For simplicity let $\widetilde{\mathscr{I}_2}:=(id_X\times\rho_2^*)\mathscr{I}_\ell\otimes q^*(L\otimes K_X)$.

Recall the good $PGL(V)$-quotient $\rho:\Omega_2\to W(2,0,2)$ such that there is a universal sheaf $\mathscr{E}$ over $X\times\Omega_2$. Let $\Omega_2^L:=\rho^{-1}(W(2,0,2)^L)$. The map $H^0(\Omega_2^L,\rho^*\lambda_2(L\otimes K_X))^\vee\xrightarrow{\rho^{*\vee}}H^0(W(2,0,2)^L,\lambda_2(L\otimes K_X))^\vee$ is surjective and hence to show that (3.13) commutes it suffices to show

$$SD_{2,L\otimes K_X}\circ\rho^{*\vee}=(g_2)_*\circ\rho_2^*\circ SD_{2,\ell}\circ\rho^{*\vee}. \tag{3.14}$$

Over $X\times\Omega_2^L\times Q^o_2$ we have

$$0\to p_{12}^*\mathscr{E}\otimes p_{13}^*\mathscr{R}_2\to p_{12}^*\mathscr{E}\otimes p_{13}^*\widetilde{\mathscr{I}_2}\to p_{12}^*\mathscr{E}\otimes p_{13}^*\mathscr{F}_{L\otimes K_X}\to 0.$$

By Lemma 2.1.20 in [**11**], we have the following commutative diagram

(3.15)

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & \to & p_{12}^*\mathscr{E}\otimes p_{13}^*\mathscr{R}_2 & \to & p_{12}^*\mathscr{E}\otimes p_{13}^*\widetilde{\mathscr{I}_2} & \to & p_{12}^*\mathscr{E}\otimes p_{13}^*\mathscr{F}_{L\otimes K_X} & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \| & & \\
0 & \to & \mathscr{C}_2 & \to & \mathscr{B}'_2 & \to & \mathscr{A}_2 & \to & p_{12}^*\mathscr{E}\otimes p_{13}^*\mathscr{F}_{L\otimes K_X} & \to & 0 \\
 & & & & \uparrow & & \uparrow & & & & \\
 & & & & \mathscr{B}_2 & \xrightarrow{=} & \mathscr{B}_2 & & & & \\
 & & & & \uparrow & & \uparrow & & & & \\
 & & & & 0 & & \mathscr{C}_2 & & & & \\
 & & & & & & \uparrow & & & & \\
 & & & & & & 0 & & & &
\end{array},$$

where $\mathscr{A}_2$, $\mathscr{B}'_2$, $\mathscr{B}_2$ and $\mathscr{C}_2$ are locally free such that $R^ip_*(\cdot)=0$ for all $i<2$ and $R^2p_*(\cdot)$ locally free over $\Omega_2^L\times Q_2^o$. We have the following commutative diagram

$$(3.16)\qquad \begin{array}{ccccc} R^2p_*\mathscr{C}_2 & \longrightarrow & R^2p_*\mathscr{B}'_2 & \xrightarrow{\nu'_2} & R^2p_*\mathscr{A}_2 \\ =\uparrow & & \eta_2\uparrow & & \uparrow = \\ R^2p_*\mathscr{C}_2 & \longrightarrow & R^2p_*\mathscr{B}_2 & \xrightarrow{\nu_2} & R^2p_*\mathscr{A}_2. \end{array}$$

ν'_2 and ν_2 are surjective because $H^2(\mathcal{F}_{L\otimes K_X}\otimes\mathcal{E})=H^2(\mathcal{I}_Z(L\otimes K_X)\otimes\mathcal{E})=0$ for every $[\mathcal{I}_Z(L\otimes K_X)\twoheadrightarrow\mathcal{F}_{L\otimes K_X}]\in Q_2^o$ and $\mathcal{E}\in\Omega_2^L$. η_2 is an isomorphism because $\mathscr{R}_2$ is a pullback of a line bundle on Q_2^o and $H^1(\mathcal{E})=H^2(\mathcal{E})=0$ for all $\mathcal{E}\in\Omega_2^L$. Denote by $\mathcal{K}_2$ and $\mathcal{K}'_2$ the kernels of ν_2 and ν'_2 respectively. Then we have

$$(3.17)\qquad \begin{array}{ccc} R^2p_*\mathscr{C}_2 & \xrightarrow{\xi_{L\otimes K_X}} & \mathcal{K}'_2 \\ =\uparrow & & \cong\uparrow\eta_2 \\ R^2p_*\mathscr{C}_2 & \xrightarrow[\xi^2_\ell]{} & \mathcal{K}_2. \end{array}$$

The section $det(\xi_{L\otimes K_X})$ induces the map $g_2^*\circ SD_{2,L\otimes K_X}\circ\rho^{*\vee}$ while the section $det(\xi^2_\ell)$ induces the map $\rho_2^*\circ SD^o_{2,\ell}\circ\rho^{*\vee}$. By (3.17) we have $det(\xi_{L\otimes K_X})=det(\eta_2)\cdot det(\xi^2_\ell)$ and hence $det(\xi_{L\otimes K_X})$ and $det(\xi^2_\ell)$ are the same section up to scalars since η_2 is an isomorphism. Hence

$$(3.18)\qquad g_2^*\circ SD_{2,L\otimes K_X}\circ\rho^{*\vee}=\rho_2^*\circ SD^o_{2,\ell}\circ\rho^{*\vee}.$$

(3.18) implies (3.14) because g_2 is a projective bundle and the map $H^0(Q_2^o,g_2^*\Theta^r_{L\otimes K_X}(r))\xrightarrow{(g_2)_*}H^0(M(L\otimes K_X,0)^o,\Theta^r_{L\times K_X}(r))$ is an isomorphism with inverse map g_2^*.

Now we have that (3.13) commutes. To show $j_r\circ\beta_D=SD_{2,L\otimes K_X}$, it suffices to show that the following diagram commutes.

$$(3.19)\qquad \begin{array}{ccc} H^0(\Omega_2^L,\lambda_2(L\otimes K_X))^\vee & \xrightarrow{SD^o_{2,\ell}\circ\rho^{*\vee}} & H^0(H^o_e,\mathcal{L}^2) \\ f^\vee_{2,\Omega}\uparrow & & \uparrow(\rho_1)_*\circ g_1^* \\ H^0(\Omega_2^L,\lambda_2(L))^\vee & \xrightarrow[g_L\circ SD_{2,L}\circ\rho^{*\vee}]{} & H^0(D_{\Theta_L},\Theta^2_L(2)|_{D_{\Theta_L}}). \end{array}$$

In other words, it suffices to show

$$(3.20)\qquad (\rho_1)_*\circ g_1^*\circ g_L\circ SD_{2,L}\circ\rho^{*\vee}=SD^o_{2,\ell}\circ\rho^{*\vee}\circ f^\vee_{2,\Omega}.$$

Recall that on $X\times Q_1^o$ there is an exact sequence

$$0\to\mathscr{R}_1\to(id_X\times\rho_1^*)\mathscr{I}_\ell\otimes q^*(L\otimes K_X)\to\mathscr{F}_L\to 0,$$

where $\mathscr{I}_\ell$ is the universal sheaf over $X \times H_\ell$ and $\mathscr{R}_1 = p^*\mathcal{R}_1 \otimes q^*K_X$ with $\mathcal{R}_1$ a line bundle (actually the relative tautological bundle $\mathcal{O}_{\rho_1}(-1)$) over Q_1^o. Let $\widetilde{\mathscr{I}_1} := (id_X \times \rho_1^*)\mathscr{I}_\ell \otimes q^*(L \otimes K_X)$.

Over $X \times \Omega_2^L \times Q_1^o$ we have

$$0 \to p_{12}^*\mathscr{E} \otimes p_{13}^*\mathscr{R}_1 \to p_{12}^*\mathscr{E} \otimes p_{13}^*\widetilde{\mathscr{I}_1} \to p_{12}^*\mathscr{E} \otimes p_{13}^*\mathscr{F}_L \to 0.$$

Analogously, we have the following commutative diagram

(3.21)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & 0 \longrightarrow & p_{12}^*\mathscr{E} \otimes p_{13}^*\mathscr{R}_1 & \longrightarrow & p_{12}^*\mathscr{E} \otimes p_{13}^*\widetilde{\mathscr{I}_1} & \longrightarrow & p_{12}^*\mathscr{E} \otimes p_{13}^*\mathscr{F}_L & \longrightarrow 0 & \\
 & & \uparrow & & \uparrow & & \uparrow{=} & & \\
0 \longrightarrow \mathscr{C}_1 & \longrightarrow & \mathscr{B}_1' & \longrightarrow & \mathscr{A}_1 & \longrightarrow & p_{12}^*\mathscr{E} \otimes p_{13}^*\mathscr{F}_L & \longrightarrow 0 & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathscr{B}_1 & \xrightarrow{=} & \mathscr{B}_1 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & \mathscr{C}_1 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & & ,
\end{array}$$

where $\mathscr{A}_1$, $\mathscr{B}_1'$, $\mathscr{B}_1$ and $\mathscr{C}_1$ are locally free such that $R^ip_*(\cdot) = 0$ for all $i < 2$ and $R^2p_*(\cdot)$ locally free over $\Omega_2^L \times Q_1^o$. We have the following commutative diagram

(3.22)
$$\begin{array}{ccccc}
R^2p_*\mathscr{C}_1 & \longrightarrow & R^2p_*\mathscr{B}_1' & \xrightarrow{\nu_1'} & R^2p_*\mathscr{A}_1 \\
{=}\uparrow & & \eta_1\uparrow & & \uparrow{=} \\
R^2p_*\mathscr{C}_1 & \longrightarrow & R^2p_*\mathscr{B}_1 & \xrightarrow{\nu_1} & R^2p_*\mathscr{A}_1.
\end{array}$$

ν_1' and ν_1 are surjective because $H^2(\mathcal{F}_L \otimes \mathcal{E}) = H^2(\mathcal{I}_Z(L \otimes K_X) \otimes \mathcal{E}) = 0$ for every $[\mathcal{I}_Z(L \otimes K_X) \twoheadrightarrow \mathcal{F}_L] \in Q_1^o$ and $\mathcal{E} \in \Omega_2^L$. η_1 is a morphism between two vector bundles with same rank with cokernel $R^2p_*(p_{12}^*\mathscr{E} \times p_{13}^*\mathscr{R}_1)$. Since $\mathscr{R}_1 \cong p^*\mathcal{R}_1 \otimes q^*K_X$ with $\mathcal{R}_1$ a line bundle over Q_1^o, $det(\eta_1)$ is the pullback to $\Omega_2^L \times Q_1^o$ of the section of $\lambda_2([K_X]^{-1}) \cong \lambda_2(K_X^{-1})$ defining the subscheme $\widetilde{\mathbf{S}}$.

Denote by $\mathcal{K}_1$ and $\mathcal{K}_1'$ the kernels of ν_1 and ν_1' respectively. Then we have

(3.23)
$$\begin{array}{ccc}
R^2p_*\mathscr{C}_1 & \xrightarrow{\xi_L} & \mathcal{K}_1' \\
{=}\uparrow & & \uparrow\eta_1 \\
R^2p_*\mathscr{C}_1 & \xrightarrow[\xi_\ell^1]{} & \mathcal{K}_1.
\end{array}$$

The section $det(\xi_L)$ induces the map $g_1^*\circ g_L\circ SD_{2,L}\circ\rho^{*\vee}$, the section $det(\xi_\ell^1)$ induces the map $\rho_1^*\circ SD_{2,\ell}^o\circ\rho^{*\vee}$ and multiplying the section $det(\eta_1)$ induces the map $f_{2,\Omega}^\vee$. By (3.23) we have $det(\xi_L)=det(\eta_1)\cdot det(\xi_\ell^1)$ and hence

$$g_1^*\circ g_L\circ SD_{2,L}\circ\rho^{*\vee}=\rho_1^*\circ SD_{2,\ell}^o\circ\rho^{*\vee}\circ f_{2,\Omega}^\vee. \tag{3.24}$$

(3.24) implies (3.20) because by **CB**-(3) the map $H^0(Q_1^o,\rho_1^*\mathcal{L}^r)\xrightarrow{(\rho_1)_*} H^0(H_\ell^o,\mathcal{L}^r)$ is an isomorphism with inverse map ρ_1^*.

The lemma is proved. q.e.d.

Now we want to modify **CB**. Define

$$|L\otimes K_X|':=\left\{C\in|L\otimes K_X|\;\middle|\;\begin{array}{l}\forall\ \textit{integral subscheme } C_1\subset C,\\ \textit{we have } deg(K_X|_{C_1})<0.\end{array}\right\},$$

$$M(L\otimes K_X,0)':=\left\{\mathcal{F}_{L\otimes K_X}\in M(L\otimes K_X,0)\;\middle|\;\begin{array}{l}h^0(\mathcal{F}_{L\otimes K_X}(K_X))=0\\ \textit{and } Supp(\mathcal{F}_{L\otimes K_X})\\ \textit{is in } |L\otimes K_X|'.\end{array}\right\}.$$

$$Q_2':=\left\{[\mathcal{I}_Z(L\otimes K_X)\overset{f_2}{\twoheadrightarrow}\mathcal{F}_{L\otimes K_X}]\in Q_2\;\middle|\;\begin{array}{l}\mathcal{F}_{L\otimes K_X}\ \textit{is semistable},\\ h^0(\mathcal{F}_{L\otimes K_X}(K_X))=0\\ \textit{and } Supp(\mathcal{F}_{L\otimes K_X})\\ \textit{is in } |L\otimes K_X|'.\end{array}\right\}.$$

Let $f_M:\Omega_{L\otimes K_X}\to M(L\otimes K_X,0)$ be the good $PGL(V_{L\otimes K_X})$-quotient with $V_{L\otimes K_X}$ some vector space and $\Omega_{L\otimes K_X}$ an open subscheme of some Quot-scheme. Let $\Omega'_{L\otimes K_X}:=f_M^{-1}(M(L\otimes K_X,0)')$. Notice that $\mathrm{Ext}^2(\mathcal{F}_{L\otimes K_X},\mathcal{F}_{L\otimes K_X})=0$ for $\mathcal{F}_{L\otimes K_X}$ semistable with $Supp(\mathcal{F}_{L\otimes K_X})\in|L\otimes K_X|'$. Hence $\Omega'_{L\otimes K_X}$ is smooth of pure dimension the expected dimension.

Denote by $\mathscr{Q}_{L\otimes K_X}$ the universal quotient over $\Omega_{L\otimes K_X}$. Analogous to [**22**], define $\mathcal{V}':=\mathscr{E}xt_p^1(\mathscr{Q}_{L\otimes K_X}|_{\Omega'_{L\otimes K_X}},q^*\mathcal{O}_X)$ which is locally free of rank $-(L+K_X).K_X$ on $\Omega'_{L\otimes K_X}$. Let $P_2'\subset\mathbb{P}(\mathcal{V}')$ parametrizing torsion free extensions of $\mathscr{Q}_{\mathfrak{s}}$ by $\mathcal{O}_X$ for all $\mathfrak{s}\in\Omega'_{L\otimes K_X}$. Then the classifying map $f'_{Q_2}:P_2'\to Q_2'$ is a principal $PGL(V_{L\otimes K_X})$-bundle (see Lemma 4.7 in [**22**]). We have the following commutative diagram

$$\begin{array}{ccc}P_2' & \xrightarrow{\sigma_2'} & \Omega'_{L\otimes K_X}\\ {\scriptstyle f'_{Q_2}}\downarrow & & \downarrow{\scriptstyle f'_M}\\ Q_2' & \xrightarrow[g_2']{} & M(L\otimes K_X,0)'.\end{array} \tag{3.25}$$

Let $H_\ell':=\rho_1(Q_1^o)\cup\rho_2(Q_2')$. We define **CB′** by keeping **CB**-(1) and replacing **CB**-(2), (3) and (4) by $(2'a)$, $(2'b)$, (3) and $(4')$ as follows.

Condition (CB′). (1) $D^o_{\Theta_L}$ is dense open in D_{Θ_L};

(2′a) $M(L\otimes K_X,0)$ is of pure dimension and satisfies the "condition S_2 of Serre", and the complement of $M(L\otimes K_X,0)'$ is of codimension ≥ 2;
(2′b) The complement of P_2' in $\mathbb{P}(\mathcal{V}')$ is of codimension ≥ 2;
(3′) $(\rho_1)_*\mathcal{O}_{Q_1^\circ}\cong\mathcal{O}_{H'_\ell}$;
(4′) Q_2' is nonempty and dense open in $\rho_2^{-1}(\rho_2(Q_2'))$.

Lemma 3.10. *If* $\boldsymbol{CB}'$ *is satisfied, then there is an injective map for all* $r>0$

$$j_r : H^0(D_{\Theta_L},\Theta_L^r(r)|_{D_{\Theta_L}})\hookrightarrow H^0(M(L\otimes K_X,0),\Theta^r_{L\otimes K_X}(r)),$$

such that $j_2\circ\beta_D = SD_{2,L\otimes K_X}$.

Proof. The only difference from Lemma 3.9 is that the map g_2' is no more a projective bundle. However it is enough to prove $(g_2')_*\mathcal{O}_{Q_2'}\cong\mathcal{O}_{M(L\otimes K_X,0)'}\cong\mathcal{O}_{M(L\otimes K_X,0)}$.

In (3.25) we have f'_{Q_2} a principal $PGL(V_{L\otimes K_X})$-bundle and f'_M a good $PGL(V_{L\otimes K_X})$-quotient. σ_2' is $PGL(V_{L\otimes K_X})$-equivariant and descends to the map g_2'. In order to show $(g_2')_*\mathcal{O}_{Q_2'}\cong\mathcal{O}_{|L\otimes K_X|'}$, we only need to show that $(\sigma_2')_*\mathcal{O}_{P_2'}\cong\mathcal{O}_{\Omega'_{L\otimes K_X}}$.

We have that $(\sigma_2)_*\mathcal{O}_{\mathbb{P}(\mathcal{V}')}\cong\mathcal{O}_{\Omega'_{L\otimes K_X}}$. $\Omega'_{L\otimes K_X}$ is smooth of pure dimension. By $\mathbf{CB}'$-(2′b) the complement of P_2' in $\mathbb{P}(\mathcal{V}')$ is of codimension ≥ 2 and hence $\jmath_*\mathcal{O}_{P_2'}\cong\mathcal{O}_{\mathbb{P}(\mathcal{V}')}$ with $\jmath : P_2'\hookrightarrow\mathbb{P}(\mathcal{V}')$ the embedding. On the other hand $\sigma_2'=\sigma_2\circ\jmath$, hence $(\sigma_2')_*\mathcal{O}_{P_2'}\cong(\sigma_2)_*(\jmath_*\mathcal{O}_{P_2'})\cong(\sigma_2)_*\mathcal{O}_{\mathbb{P}(\mathcal{V}')}\cong\mathcal{O}_{\Omega'_{L\otimes K_X}}$. Hence the lemma q.e.d.

Notice that $\mathbf{CB}$-(2) $\Rightarrow$ $\mathbf{CB}'$-(2′a) if $(L+K_X).K_X<0$. Lemma 3.9 and Lemma 3.10 imply immediately the following proposition.

Proposition 3.11. *If either* $\boldsymbol{CB}$ *or* $\boldsymbol{CB}'$ *is satisfied and* $SD_{2,L\otimes K_X}$ *is an isomorphism, then the map* β_D *in (3.3) is an isomorphism. In particular,* g_L *is surjective.*

Remark 3.12. If $L\cong K_X^{-1}$ then β_D is an isomorphism as long as $\forall\ C\in|L|$, $\mathcal{O}_C$ is stable (which is equivalent to say that C contains no subcurve with genus ≥ 1) and there is a stable vector bundle $\mathcal{E}\in W(2,0,2)$. This is because in this case β_D is a nonzero map between two vector spaces of 1 dimension, hence an isomorphism. β_D is nonzero since $H^0(\mathcal{E}\otimes\mathcal{O}_C)=H^1(\mathcal{E}\otimes\mathcal{O}_C)=0$ for all $C\in|L|$ (also see the proof of Proposition 6.25 in [**9**]).

Combining Lemma 3.3 and Proposition 3.11 we have the following theorem.

Theorem 3.13. *Assume* $\boldsymbol{CA}$, *and assume either* $\boldsymbol{CB}$ *or* $\boldsymbol{CB}'$ *is satisfied, and assume* $SD_{2,L\otimes K_X}$ *is an isomorphism, then* $SD_{2,L}$ *is an isomorphism.*

3.3. Application to Hirzebruch surfaces. Theorem 3.13 applies to a large number of cases on Hirzebruch surface as stated in the following theorem.

Theorem 3.14. *Let* $X = \Sigma_e$ $(e \geq 0) := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$. *Let* F *be the fiber class and* G *the section such that* $G^2 = -e$ *over* X. *Let* $L = aG + bF$. *Then*

(1) ***CA*** *is fulfilled for* L *ample or* $\min\{a, b\} \leq 1$.

(2) If $2 \leq \min\{a, b\} \leq 3$, *then* ***CB′*** *is fulfilled for* L *ample, i.e.* $b > ae$ *for* $e \neq 0$; *or* $a, b > 0$ *for* $e = 0$.

(3) If $\min\{a, b\} \geq 4$, *then* ***CB*** *is fulfilled for both* L *and* $L \otimes K_X$ *ample, i.e.* $b > ae, e > 1$; *or* $b > a + 1, e = 1$; *or* $a, b \geq 4, e = 0$.

Corollary 3.15. *Let* X *be a Hirzebruch surface* Σ_e *and* $L = aG+bF$. *Then the strange duality map* $SD_{2,L}$ *in (2.5) is an isomorphism for the following cases.*

1) $\min\{a, b\} \leq 1$;
2) $\min\{a, b\} \geq 2$, $e \neq 1$, L *ample;*
3) $\min\{a, b\} \geq 2$, $e = 1$, $b \geq a + [a/2]$ *with* $[a/2]$ *the integral part of* $a/2$.

Proof. If $\min\{a, b\} \leq 1$, then every curve in $|L|$ contains no subcurve of positive genus and hence done by Corollary 3.4 and Remark 3.5.

If $\min\{a, b\} \geq 2$ and $e \neq 1$, then L is ample $\Rightarrow L \otimes K_X$ is ample. Therefore by Theorem 3.14 and Theorem 3.13 we can reduce the problem to $L = G + nF$ (or $F + nG$ for $e = 0$), or nF (or mG for $e = 0$) while by Corollary 3.4 and Remark 3.5, $SD_{2,L}$ is an isomorphism in these cases.

If $\min\{a, b\} \geq 2$, $e = 1$ and $b \geq a + [a/2]$, then either both L and $L \otimes K_X$ are ample or L ample and $L \otimes K_X = G + F$ *or* nF. Therefore analogously we are done by Theorem 3.14, Theorem 3.13, Corollary 3.4 and Remark 3.5.

The corollary is proved. q.e.d.

To prove Theorem 3.14, the main task is estimating codimension of some schemes. However we want to use stack language as what we did in [**21**] because it makes the argument clearer and simpler. Therefore, we firstly introduce some stacks as follows, the notations of which are slightly different from [**21**].

Definition 3.16. Let χ and d be two integers.

(1) Let $\mathcal{M}^d(L, \chi)$ be the (Artin) stack parametrizing pure 1-dimensional sheaves $\mathcal{F}$ on X with determinant L, Euler characteristic $\chi(\mathcal{F}) = \chi$ and satisfying either $\mathcal{F}$ is semistable or $\forall \mathcal{F}' \subset \mathcal{F}$, $\chi(\mathcal{F}') \leq d$.

(2) Let $\mathcal{M}(L, \chi)$ ($\mathcal{M}(L, \chi)^s$, resp.) be the substack of $\mathcal{M}^d(L, \chi)$ parametrizing semistable (stable, resp.) sheaves in $\mathcal{M}^d(L, \chi)$.

(3) Let $\mathcal{M}^{int}(L,\chi)$ be the substack of $\mathcal{M}(L,\chi)^s$ parametrizing sheaves with integral supports.

(4) Let $\mathcal{M}^{d,R}(L,\chi)$ be the substack of $\mathcal{M}^d(L,\chi)$ parametrizing sheaves with reducible supports in $\mathcal{M}^d(L,\chi)$. Let $\mathcal{M}^R(L,\chi) = \mathcal{M}^{d,R}(L,\chi)\cap\mathcal{M}(L,\chi)^s$.

(5) Let $\mathcal{M}^{d,N}(L,\chi)$ be the substack of $\mathcal{M}^d(L,\chi)$ parametrizing sheaves with irreducible and non-reduced supports in $\mathcal{M}^d(L,\chi)$. Let $\mathcal{M}^N(L,\chi) = \mathcal{M}^{d,N}(L,\chi)\cap\mathcal{M}(L,\chi)^s$.

(6) Let $\mathcal{C}^d(nL',\chi)$ $(n>1)$ be the substack of $\mathcal{M}^d(nL',\chi)$ parametrizing sheaves $\mathcal{F}$ whose supports are of the form nC with C an integral curve in $|L'|$. $\mathcal{C}(nL',\chi) = \mathcal{C}^d(nL',\chi)\cap\mathcal{M}(L,\chi)^s$.

Lemma 3.17. *Let $X=\Sigma_e$ and $L=aG+bF$ ample with $\min\{a,b\}\geq 2$. Then for all χ and d, $\mathcal{M}^{int}(L,\chi)$ is smooth of dimension L^2, and the complement of $\mathcal{M}^{int}(L,\chi)$ inside $\mathcal{M}^d(L,\chi)$ is of codimension ≥ 2, i.e. of dimension $\leq L^2-2$.*

Proof. Since $L.K_X<0$, $\mathcal{M}^{int}(L,\chi)$ is smooth of dimension L^2. We first estimate the dimension of $\mathcal{C}^d(nL',\chi)$ $(n>1)$. Write $L'=a'G+b'F$. Since $|L'|^{int}\neq\emptyset$, $L'=G$ or F; or $b'\geq a'e$, $e>0$; or $a',b'>0$, $e=0$.

Claim ♣. $\forall\ d,\chi$, $\dim\mathcal{C}^d(nL',\chi)\leq n^2L'^2-\min\{7,-nL'.K_X-1,(n-1)L'^2\}\leq n^2L'^2$ for L' nef and $\dim\mathcal{C}^d(nG,\chi)\leq -n^2$ for $e>0$.

We show Claim ♣. Let

$$\mathcal{T}_m(L,\chi) := \{\mathcal{F}\in\mathcal{M}(L,\chi)|\exists\ x\in X, \textit{such that}\ \dim_{k(x)}(\mathcal{F}\otimes k(x))\geq m\},$$

where $k(x)$ is the residue field of x. Take a very ample divisor $H=G+(e+1)F$ on X. If L' is nef, then $(-jH+K_X).L'<0$ for all $j\geq -1$ and hence $H^1(\mathscr{E}xt^1(\mathcal{F},\mathcal{F})(jH))\cong \mathrm{Ext}^2(\mathcal{F},\mathcal{F}(jH))\cong \mathrm{Hom}(\mathcal{F},\mathcal{F}(K_X-jH))^\vee=0$ for all $j\geq -1$ and $\mathcal{F}\in\mathcal{C}(nL',\chi)$. Therefore by Castelnuovo-Mumford criterion $\mathscr{E}xt^1(\mathcal{F},\mathcal{F})$ is globally generated. Hence by Le Potier's argument in the proof of Lemma 3.2 in [**13**], $\mathcal{C}(nL',\chi)\cap\mathcal{T}_m(nL',\chi)$ is of dimension $\leq n^2L'^2-m^2+2$. Combining Proposition 4.1 and Theorem 4.16 in [**21**], we have

$$\dim\mathcal{C}(nL',\chi)\leq n^2L'^2-\min\{7,n(n-1)L'^2,-nK_X.L'-1\}. \tag{3.26}$$

Let $\mathcal{F}\in\mathcal{C}^d(nL',\chi)\setminus\mathcal{C}(nL',\chi)$. Since $\forall\ \mathcal{F}'\subset\mathcal{F}$, $K_X.c_1(\mathcal{F}')<0$, the proof of Proposition 2.7 in [**21**] applies and $\dim(\mathcal{C}^d(nL',\chi)\setminus\mathcal{C}(nL',\chi))\leq n^2L'^2-(n-1)L'^2$.

Let $e>0$. For every semistable sheaf $\mathcal{F}$ with support nG, the map $\mathcal{F}\xrightarrow{\cdot\delta_G}\mathcal{F}(G)$ is zero because $G^2<0$, where $\delta_G\in H^0(\mathcal{O}_X(G))$ is a function defining the divisor G. Hence $\mathcal{F}$ is a sheaf on G and hence a direct sum of n line bundles over G. Thus $\dim\mathcal{C}(nG,\chi)\leq -n^2$. Let $\mathcal{F}$ be unstable with support G, then take the Harder-Narasimhan filtration of it as follows.

$$0=\mathcal{F}_0\subsetneq\mathcal{F}_1\subsetneq\cdots\subsetneq\mathcal{F}_k=\mathcal{F},$$

with $\mathcal{F}_i/\mathcal{F}_{i-1} \cong \mathcal{O}_G(s_i)^{\oplus n_i}$ such that $s_1 > s_2 > \cdots > s_k$ and $\sum_{i=1}^k n_i = n$. Then

$$\begin{aligned}\mathrm{ext}^2(\mathcal{F}_i/\mathcal{F}_{i-1}, \mathcal{F}_{i-1}) &= \mathrm{hom}(\mathcal{F}_{i-1}, \mathcal{F}_i/\mathcal{F}_{i-1}(K_X)) \\ &\leq \sum_{j<i} \mathrm{hom}(\mathcal{O}_G(s_j)^{\oplus n_j}, \mathcal{O}_G(s_i + (e-2))^{\oplus n_i}) \\ &\leq \sum_{j<i} (e-2) n_i n_j.\end{aligned}$$

By induction assumption $\dim \mathcal{C}^d(\tilde{n}G, \chi) \leq -\tilde{n}^2$ for all $\tilde{n} < n$, then by analogous argument to the proof of Proposition 2.7 in [**21**] we have

$$\begin{aligned}&\dim \mathcal{C}^d(nG, \chi) \\ &\leq \max_{\substack{n_1,\cdots,n_k>0 \\ \sum_i n_i = n}} \{-n^2, -(\sum_{i=1}^{k-1} n_i)^2 - n_k^2 - \sum_{i=0,1} (-1)^i \mathrm{ext}^i(\mathcal{F}_k/\mathcal{F}_{k-1}, \mathcal{F}_{k-1})\} \\ &= \max_{\substack{n_1,\cdots,n_k>0 \\ \sum_i n_i = n}} \{ -n^2, \begin{array}{r} -(\sum_{i=1}^{k-1} n_i)^2 - n_k^2 - \chi(\mathcal{F}_i/\mathcal{F}_{k-1}, \mathcal{F}_{k-1}) \\ +\mathrm{ext}^2(\mathcal{F}_k/\mathcal{F}_{k-1}, \mathcal{F}_{k-1}) \end{array} \} \\ &\leq \max_{\substack{n_1,\cdots,n_k>0 \\ \sum_i n_i = n}} \{-n^2, -(\sum_{i=1}^{k-1} n_i)^2 - n_k^2 - n_k(\sum_{i=1}^{k-1} n_i)e + (e-2)(\sum_{i=1}^{k-1} n_i n_k)\} \\ &= -n^2\end{aligned}$$

Therefore Claim ♣ is proved.

Easy to see $\mathcal{M}^d(L,\chi) \setminus \mathcal{M}^{int}(L,\chi) = \mathcal{M}^{d,R}(L,\chi) \cup \mathcal{M}^{d,N}(L,\chi)$ and $\mathcal{M}^{d,N}(L,\chi) = \cup_{nL'=L} \mathcal{C}^d(nL', \chi)$. Claim ♣ implies that $\mathcal{M}^{d,N}(L,\chi)$ is of codimension ≥ 2 inside $\mathcal{M}^d(L,\chi)$ for L ample. Now we only need to show $\mathcal{M}^{d,R}(L,\chi)$ is of dimension $\leq L^2 - 2$.

Let $\mathcal{G} \in \mathcal{M}^{d,R}(L,\chi)$, then $\mathcal{G}$ admits a filtration as follows.

$$0 = \mathcal{G}_0 \subsetneq \mathcal{G}_1 \subsetneq \cdots \subsetneq \mathcal{G}_l = \mathcal{G},$$

with $\mathcal{S}_i := \mathcal{G}_i/\mathcal{G}_{i-1} \in \mathcal{C}^{d_i}(n_i L_i, \chi_i)$ such that $\sum_{i=1}^{l} n_i L_i = L$, $\sum_{i=1}^{l} \chi_i = \chi$ and $\mathrm{Hom}(\mathcal{S}_i, \mathcal{S}_j) = \mathrm{Ext}^2(\mathcal{S}_i, \mathcal{S}_j) = 0$, $\forall\ i \neq j$. Hence $\mathrm{ext}^1(\mathcal{S}_i, \mathcal{S}_j) = -\chi(\mathcal{S}_i, \mathcal{S}_j) = n_i n_j (L_i.L_j)\ \forall\, i > j$, and $\mathrm{ext}^1(\mathcal{S}_i, \mathcal{G}_{i-1}) = \sum_{j<i} \mathrm{ext}^1(\mathcal{S}_i, \mathcal{G}_{i-1})$.

By analogous argument to the proof of Proposition 2.7 in [**21**], we have

(3.27)

$$\begin{aligned}\dim \mathcal{M}^{d,R}(L,\chi) &\leq \max_{\sum n_i L_i = L} \{\sum_i \dim \mathcal{C}^{d_i}(n_i L_i, \chi_i) + \sum_{j<i} n_i n_j (L_i.L_j)\} \\ &\leq \max_{\substack{\sum n_i L_i = L - a_0 G \\ L_i\ nef,\, a_0 \leq a}} \{\sum_i n_i^2 L_i^2 + \sum_{j<i} n_i n_j (L_i.L_j) - a_0^2 + a_0 G.(L - a_0 G)\}\end{aligned}$$

$$= \max_{\substack{\sum n_i L_i = L - a_0 G \\ L_i \; nef, a_0 \leq a}} \{L^2 - \sum_{j<i} n_i n_j (L_i.L_j) - a_0^2 - a_0 G.L\}$$

$$= L^2 - \min_{\substack{\sum n_i L_i = L - a_0 G \\ L_i \; nef, a_0 \leq a}} \{\sum_{j<i} n_i n_j (L_i.L_j) + a_0^2 + a_0 G.L\}$$

If $a_0 \geq 1$, then $\sum_{j<i} n_i n_j (L_i.L_j) + a_0^2 + a_0 G.L \geq a_0^2 + a_0(b - ea) \geq 2$. If $a_0 = 0$ or $e = 0$, then $\sum_{j<i} n_i n_j (L_i.L_j) \geq 2$ since $\min\{a, b\} \geq 2$ and L_i are all nef. Hence the lemma is proved. q.e.d.

Remark 3.18. Let d, χ be two integers. Claim ♣ and (3.27) also provide an estimate of $\dim \mathcal{M}^d(L, \chi)$ for all L effective. We can see that $\dim \mathcal{M}^d(nG, \chi) = \dim \mathcal{C}^d(nG, \chi) \leq -n^2$ for $e \neq 0$ and $\dim \mathcal{M}^d(nF, \chi) \leq 0$.

Denote by $|L|^{int}$ the open subset of $|L|$ consisting of all integral curves. If L is nef and big, i.e. $|L|^{int} \neq \emptyset$ and $L \neq F, G$, then $L.K_X < 0$ and $\dim \mathcal{M}^{int}(L, \chi)$ is smooth of dimension L^2, and moreover by Claim ♣ and (3.27), $\mathcal{M}^d(L, \chi) \setminus \mathcal{M}^{int}(L, \chi) \leq L^2 - 1$. Hence $\dim \mathcal{M}^d(L, \chi) = \dim \mathcal{M}(L, \chi)^s = L^2$ and $\mathcal{M}(L, \chi)$ is irreducible of expected dimension.

If $|L|^{int} = \emptyset$, $\min\{a, b\} \geq 1$ and $-K_X$ is nef, i.e. $e \leq 2$; then $\mathcal{M}(L, \chi)^s$ is either empty or of smooth of dimension L^2.

If $|L|^{int} = \emptyset$ with $\min\{a, b\} \geq 1$, then $\mathcal{M}^d(L, \chi) = \mathcal{M}^{d,R}(L, \chi)$ and we then have

$$\dim \mathcal{M}^d(L, \chi) \leq \max_{L - a_0 G \; nef} \{(L - a_0 G)^2 + a_0 G(L - a_0 G) - a_0^2\}.$$

Let $\mathcal{F}_L$ be stable with $C_{\mathcal{F}_L} = a_0 G + C'_{\mathcal{F}_L}$ such that G is not a component of $C'_{\mathcal{F}_L}$, let $\mathcal{F}_L^G$ be $\mathcal{F}_L \otimes \mathcal{O}_{a_0 G}$ modulo its torsion. Hence $\mathcal{F}_L^G$ is a quotient of $\mathcal{F}_L$ while $\mathcal{F}_L^G(-C'_{\mathcal{F}_L})$ is a subsheaf of $\mathcal{F}_L$. Hence by stability of $\mathcal{F}_L$, $C'_{\mathcal{F}_L}.G > 0$ and $L - a_0 G$ must be either ample or bF. Hence

$$\dim \mathcal{M}(L, \chi)^s \leq \max_{\substack{L - a_0 G \; ample \\ or \; a_0 = a}} \{(L - a_0 G)^2 + a_0 G(L - a_0 G) - a_0^2\}. \tag{3.28}$$

We can choose an atlas $\Omega^d_{L,\chi} \xrightarrow{\psi} \mathcal{M}^d(L, \chi)$ with $\Omega^d_{L,\chi}$ a subscheme of some Quot-scheme. We also can ask $\psi^{-1}(\mathcal{M}(L, \chi)) =: \Omega_{L,\chi} \xrightarrow{f_M} M(L, \chi)$ to be a good $PGL(V_{L,\chi})$-quotient with $M(L, \chi)$ the coarse moduli space of semistable sheaves. Analogously we define $\Omega^s_{L,\chi}$, $\Omega^{int}_{L,\chi}$, $\Omega^{d,R}_{L,\chi}$, $\Omega^{d,N}_{L,\chi}$ etc. If $\chi = 0$, we write $\Omega^\bullet_L$ instead of $\Omega^\bullet_{L,0}$. Since ψ is smooth, the codimension of $\Omega^\bullet_{L,\chi}$ inside $\Omega^d_{L,\chi}$ is the same as $\mathcal{M}^\bullet(L, \chi)$ inside $\mathcal{M}^d(L, \chi)$. "$\bullet$" stands for "int", "d, R", "d, N" etc.

Let $M^{int}(L, \chi) := \pi^{-1}(|L|^{int})$. Then $M^{int}(L, \chi)$ is a flat family of (compactified) Jacobians over $|L|^{int}$, hence it is connected. $\Omega^{int}_{L,\chi} = f_M^{-1}(M^{int}(L, \chi))$ and $\Omega^{int}_{L,\chi}$ is a principal $PGL(V_L)$-bundle over $M^{int}(L, \chi)$ hence also connected.

We have a corollary to Lemma 3.17 as follows.

Corollary 3.19. *Let $X = \Sigma_e$ and $L = aG + bF$.*

(1) If $\min\{a,b\} \leq 1$, then $M(L,0) \cong |L|$ and $\Theta_L \cong \mathcal{O}_{|L|}$.

(2) If $\min\{a,b\} \geq 2$ and L is nef for $e \neq 1$, ample for $e = 1$, then $M(L,0)$ is integral and normal; $M(L,0) \setminus M^{int}(L,0)$ is of codimension ≥ 2 inside $M(L,0)$; and the dualizing sheaf of $M(L,0)$ is locally free and isomorphic to $\pi^*\mathcal{O}_{|L|}(L.K_X)$. Moreover $\pi_*\Theta_L \cong \mathcal{O}_{|L|}$ and $R^i\pi_*\Theta_L^r = 0$ for all $i, r > 0$.

Proof. If $\min\{a,b\} \leq 1$, then done by Proposition 4.1.1 in [**20**].

Let L be as in (2). There are nonsingular irreducible curves in $|L|$ and the complement of $|L|^{int}$ in $|L|$ is of codimension ≥ 2. Since $L.K_X < 0$, $M^{int}(L,0)$ is smooth and irreducible of dimension $L^2 + 1$. Ω_L^{int} is also smooth, hence irreducible and of expected dimension.

By Lemma 3.17, $\Omega_L^d \setminus \Omega_L^{int}$ is of codimension ≥ 2 inside Ω_L^d, then Ω_L^{int} is dense in Ω_L, hence then Ω_L is of expected dimension and by deformation theory Ω_L is a local complete intersection. On the other hand, Ω_L is smooth in codimension 1, hence normal for local complete intersection. Therefore $M(L,0)$ is integral and normal since Ω_L is.

To show that $M(L,0)\setminus M^{int}(L,0)$ is of codimension ≥ 2, we only need to show $M(L,0)\setminus M(L,0)^s$ is of codimension ≥ 2 with $M(L,0)^s$ the open subset consisting of stable sheaves. By Remark 3.18, $\dim M(L',0)^s = L'^2 + 1$ for L' nef and big, $\dim M(F,0)^s = 1$, $\dim M(nF,0)^s = 0$ for $n > 1$, $\dim M(nG,0) = 0$ for $e > 0$ and finally by (3.28) for $|L'|^{int} = \emptyset$ and $L' \neq nF, mG$,

$$\dim M(L',0)^s \leq \max_{\substack{L-a_0G \text{ ample} \\ or\ a_0=a}} \{(L'-a_0G)^2 + 1 - a_0^2 + a_0G.(L'-a_0G)\}.$$

Hence if $e \neq 0$, then

(3.29)

$$\begin{aligned}
&L^2 + 1 - \dim(M(L,0) \setminus M(L,0)^s) \\
&= L^2 + 1 - \max_{\sum L_i = L} \{\sum_i \dim M(L_i,0)^s\} \\
&\leq L^2 + 1 - \max_{\substack{\sum L_i = L - a_0G \\ L_i' := L_i - a_iG \text{ nef}, \\ a_i \geq 0,\ a_0 \leq a}} \{\sum_i (L_i - a_iG)^2 - a_i^2 + a_iG.(L_i - a_iG) + \#\{L_i\}\} \\
&\leq \min_{\substack{\sum L_i = L - a_0G \\ L_i' := L_i - a_iG \text{ nef}, \\ a_i \geq 0,\ a_0 \leq a}} \left\{ \begin{array}{l} \sum_{j \neq i} (L_i'.L_j') - \#\{L_i'\} + \sum_{i \neq 0} a_i^2 + 2a_0G.L + 1 \\ + \sum_{i \neq 0} a_iG.(L_i' + 2\sum_{j \neq i} L_j') + (a_0^2 - (\sum_{i \neq 0} a_i)^2)e \end{array} \right\}
\end{aligned}$$

$$\leq \min_{\substack{\sum L_i = L - a_0 G \\ L'_i := L_i - a_i G \ nef, \\ a_i \geq 0, \ a_0 \leq a}} \left\{ \begin{array}{c} \sum_{j\neq i} (L'_i.L'_j) - \#\{L'_i\} + \sum_{i\neq 0} a_i^2 + 2a_0 G.L + 1 \\ + \sum_{i\neq 0} a_i G.(L - a_0 G + \sum_{j\neq i} L'_j) + a_0^2 e \end{array} \right\}$$

$$= \min_{\substack{\sum L_i = L - a_0 G \\ L'_i := L_i - a_i G \ nef, \\ a_i \geq 0, \ a_0 \leq a}} \left\{ \begin{array}{c} \sum_{j\neq i} (L'_i.L'_j) - \#\{L'_i\} + \sum_{i\neq 0} a_i^2 + \sum_{i\neq 0, j\neq i} a_i G.L'_j \\ + (\sum_{i\geq 0} a_i) G.L + a_0 G.L + a_0 (\sum_{i\geq 0} a_i) e + 1 \end{array} \right\}$$

We want $\dim(M(L,0) \setminus M(L,0)^s) \leq L^2 - 1$.

Assume $L'_i = n_i F$ for all i, then $\sum_{i\geq 0} a_i = a$. If moreover $a_i = 0$ for $i \neq 0$, then $a_0 = a$ and $-\#\{L'_i\} + 2a_0 G.L + a_0^2 e + 1 = 1 + 2a(b-ae) - b + a^2 e = b(a-1) + a(b-ae) + 1 \geq 5$ since $a, b \geq 2$ and $b > ae$. If $\exists\ a_{k_0} \neq 0$ for $k_0 \neq 0$, then $-\#\{L'_i\} + \sum_{i\neq 0} a_i^2 + \sum_{i\neq 0, j\neq i} a_i G.L'_j + aG.L + 1 \geq a(b-ae) + 1 \geq 3$.

Assume $\exists\ L'_{i_0} \neq nF$, then $L'_{i_0}.L'_j \geq 1$ for L'_j nef hence $\sum_{j\neq i}(L'_i.L'_j) \geq 2(\#\{L'_i\} - 1)$. If $a_0 \geq 1$, then $2a_0 G.L + a_0^2 e \geq 3$ and hence $\sum_{j\neq i}(L'_i.L'_j) - \#\{L'_i\} + 2a_0 G.L + a_0^2 e + 1 \geq 3$. If $a_0 = 0$, then $\#\{L'_i\} \geq 2$ and either $\exists\ L'_{i_0}, L'_{j_0}$, such that $L'_{i_0}.L_j \geq 1$, $L'_{j_0}.L'_j \geq 1$ for L'_j nef; or $\exists\ L'_{i_0}$, such that $L'_{i_0}.L'_j \geq 2$ for L'_j nef; or $\exists\ a_{k_0} \neq 0$ for $k_0 \neq 0$. Then we have

$$\sum_{j\neq i}(L'_i.L'_j) + \sum_{i\neq 0} a_i^2 + \sum_{i\neq 0, j\neq i} a_i G.L'_j + (\sum_{i\geq 0} a_i) G.L \geq 2(\#\{L'_i\} - 1) + 2$$

and

$$\sum_{j\neq i}(L'_i.L'_j) - \#\{L'_i\} + \sum_{i\neq 0} a_i^2 + \sum_{i\neq 0, j\neq i} a_i G.L'_j + (\sum_{i\geq 0} a_i) G.L + 1 \geq \#\{L_i\} + 1 \geq 3.$$

If $e = 0$, then easy to see

(3.30)

$$\begin{aligned} \dim(M(L,0) \setminus M(L,0)^s) &= \max_{\substack{\sum_i L_i = L \\ L_i \ nef}} \{\sum_i L_i^2 + \#\{L_i\}\} \\ &\leq L^2 - \min_{\substack{\sum_i L_i = L \\ L_i \ nef}} \{\sum_{j\neq i} L_i L_j - \#\{L_i\}\} \\ &\leq L^2 - 2. \end{aligned}$$

Therefore the complement of $M(L,0)^s$ inside $M(L,0)$ is of codimension ≥ 3 and hence $M(L,0) \setminus M^{int}(L,0)$ is of codimension ≥ 2.

Because $\Omega_L \setminus \Omega_L^{int}$ is of codimension ≥ 2 and $|L|$ contains smooth curves, sheaves not locally free on their supports form a subset of codimension ≥ 2 inside Ω_L, hence Proposition 4.2.11 in [20] applies and then the dualizing sheaf of $M(L,0)$ is isomorphic to $\pi^*\mathcal{O}_{|L|}(L.K_X)$. Moreover since $M(L,0)$ is normal and integral, and the complement of $|L|^{int}$ inside $|L|$ is of codimension ≥ 2, Theorem 4.3.1 in [**20**] and Proposition 4.3 in [**22**] apply and we obtain that $\pi_*\Theta_L \cong \mathcal{O}_{|L|}$ and $R^i\pi_*\Theta_L^r = 0$ for all $i, r > 0$.

The lemma is proved. q.e.d.

Remark 3.20. Let L be as in Corollary 3.19. Since $\pi_*\Theta_L \cong \mathcal{O}_{|L|}$ and $R^i\pi_*\Theta_L^r = 0$ for all $i, r > 0$, $H^i(\Theta_L(n)) = 0$ for all $i > 0$ and $n \geq 0$. Hence we already know that the map g_L in (3.3) is surjective in this case.

Proof of Statement (1) of Theorem 3.14. By Corollary 3.19, the strange duality map $SD_{c_2^1,u_L}$ in (3.4) is a map between two vector spaces of same dimension, while L is in case (1) of the theorem. The argument proving Corollary 4.3.2 in [**20**] applies and hence $SD_{c_2^1,u_L}$ is an isomorphism. Statement (1) is proved. q.e.d.

To prove Statement (2) and (3), we need to introduce more stacks.

Definition 3.21. For two integers $k > 0$ and i, we define $\mathcal{M}_{k,i}^{int}(L,\chi)$ to be the (locally closed) substack of $\mathcal{M}^{int}(L,\chi)$ parametrizing sheaves $\mathcal{F} \in \mathcal{M}^{int}(L,\chi)$ such that $h^1(\mathcal{F}(-iK_X)) = k$ and $h^1(\mathcal{F}(-nK_X)) = 0$, $\forall\, n > i$. Let $M_{k,i}^{int}(L,\chi)$ be the image of $\mathcal{M}_{k,i}^{int}(L,\chi)$ in $M^{int}(L,\chi)$.

Define $\mathcal{W}_{k,i}^{int}(L,\chi)$ to be the (locally closed) substack of $\mathcal{M}^{int}(L,\chi)$ parametrizing sheaves $\mathcal{F} \in \mathcal{M}^{int}(L,\chi)$ with $h^0(\mathcal{F}(-iK_X)) = k$ and $h^0(\mathcal{F}(-nK_X)) = 0$, $\forall\, n < i$. Let $W_{k,i}^{int}(L,\chi)$ be the image of $\mathcal{W}_{k,i}^{int}(L,\chi)$ in $M^{int}(L,\chi)$.

Remark 3.22. Since $L.K_X < 0$, for fixed χ, $\mathcal{M}_{k,i}^{int}(L,\chi)$ is empty except for finitely many pairs (k,i). We don't define $\mathcal{M}_{k,i}^{d}(L,\chi) \subset \mathcal{M}^d(L,\chi)$ because L may not be K_X-negative (see Definition 2.1 in [**21**]) and the analogous definition may not behave well.

Remark 3.23. By sending each sheaf $\mathcal{F}$ to its dual $\mathscr{E}xt^1(\mathcal{F}, K_X)$, we get an isomorphism $\mathcal{M}_{k,i}^{int}(L,\chi) \xrightarrow{\cong} \mathcal{W}_{k,-i}^{int}(L,-\chi)$.

By Proposition 5.5 and Remark 5.6 in [**21**], we have

Proposition 3.24. *1) $\dim \mathcal{M}_{k,i}^{int}(L,\chi) \leq L^2 + iK_X.L - \chi - k$;*
2) $\dim \mathcal{W}_{k,i}^{int}(L,0) \leq L^2 - iK_X.L + \chi - k$;
3) $\dim M_{k,i}^{int}(L,0) \leq L^2 + 1 + iK_X.L - \chi - k$;
4) $\dim W_{k,i}^{int}(L,0) \leq L^2 + 1 - iK_X.L + \chi - k$.

Corollary 3.25. *Let $X = \Sigma_e$ and $L = aG + bF$ ample with $\min\{a,b\} \geq 2$. Let $D_{\Theta_L}^{int} := D_{\Theta_L} \cap M^{int}(L,0)$. Then* $\dim D_{\Theta_L} \setminus D_{\Theta_L}^{int} \leq$

L^2-2, and $\dim \mathcal{D}_{\Theta_L}\setminus\mathcal{D}^{int}_{\Theta_L}\leq L^2-3$ with $\mathcal{D}_{\Theta_L}$ ($\mathcal{D}^{int}_{\Theta_L}$, resp.) the preimage of D_{Θ_L} ($D^{int}_{\Theta_L}$, resp.) inside $\mathcal{M}(L,0)$.

Proof. We have shown that $M(L,0)\setminus M(L,0)^s$ is of dimension $\leq L^2-2$. Then we only need to show $\dim(\mathcal{D}_{\Theta_L}\setminus\mathcal{D}^{int}_{\Theta_L})\leq L^2-3$. Let $\mathcal{C}_{L_1,\ L_2}$ with $L_1+L_2=L$ be the stack parametring sheaves $\mathcal{F}\in\mathcal{D}_{\Theta_L}$ with supports $C_{\mathcal{F}}=C_{L_1}+C_{L_2}$ such that $C_{L_i}\in|L_i|^{int}$ for $i=1,2$. By (3.26) and (3.27), we only need to show the stacks $\mathcal{C}_{2G+(b-1)F,\ F}$ and $\mathcal{C}_{(a-1)G+(ae+1)F,\ G}$ is of dimension $\leq L^2-3$.

Let $\mathcal{F}\in\mathcal{C}_{2G+(b-1)F,\ F}$. Then we have the following exact sequence

$$0\to\mathcal{F}_1\to\mathcal{F}\to\mathcal{F}_2\to 0, \tag{3.31}$$

where $\mathcal{F}_2$ is the torsion free part of $\mathcal{F}\otimes\mathcal{O}_{C_F}$ and $\mathcal{F}_1\in\mathcal{M}^{int}(2G+(b-1)F,\chi_1)$ with $\chi_1\leq 0$. Notice that $\mathcal{F}_1\otimes\mathcal{O}_X(F)$ is a quotient of $\mathcal{F}$, hence $\chi_1+2\geq 0$. Also $\mathcal{F}_2\otimes\mathcal{O}_X(-2G-(b-1)F)$ is a subsheaf of $\mathcal{F}$ and hence $\mathcal{F}_2\cong\mathcal{O}_{\mathbb{P}^1}$ or $\mathcal{O}_{\mathbb{P}^1}(-1)$. Let $\mathcal{C}^0_{2G+(b-1)F,\ F}\subset\mathcal{C}_{2G+(b-1)F,\ F}$ consist of $\mathcal{F}$ in (3.31) with $H^0(\mathcal{F}_1)=0$. $\mathcal{F}_1\in\bigcup_{i\leq 0}\mathcal{W}^{int}_{k,i}(2G+(b-1)F,\chi_1)$ if $\mathcal{F}\in\mathcal{C}_{2G+(b-1)F,\ F}\setminus\mathcal{C}^0_{2G+(b-1)F,\ F}$. Therefore

$$\begin{aligned}\dim\ &\mathcal{C}_{2G+(b-1)F,\ F}\setminus\mathcal{C}^0_{2G+(b-1)F,\ F}\\ &\leq(2G+(b-1)F).F+\dim\bigcup_{i\leq 0}\mathcal{W}^{int}_{k,i}(2G+(b-1)F,\chi_1)\\ &\leq(2G+(b-1)F)^2-1+\chi_1+2\leq 4b-4e-3=L^2-3.\end{aligned}\tag{3.32}$$

Denote by g_L the arithmetic genus of curves in $|L|$. If $\mathcal{F}\in\mathcal{C}^0_{2G+(b-1)F,F}$, then there is a injection $\mathcal{O}_{C_{\mathcal{F}}}\hookrightarrow\mathcal{F}$ with cokernel $\mathcal{O}_{Z_{\mathcal{F}}}$, where $Z_{\mathcal{F}}$ is a 0-dimensional subscheme of $C_{\mathcal{F}}$ with length g_L-1. We have $\text{ext}^1(\mathcal{O}_Z,\mathcal{O}_C)=\dim Aut(\mathcal{O}_Z)=h^0(\mathcal{O}_Z)=g_L-1$ for all $Z\subset C$. Hence for a fixed curve C and $[Z]\in C^{[g_L-1]}$, there are finitely many possible choices for $\mathcal{F}$ lying in the following sequence

$$0\to\mathcal{O}_C\to\mathcal{F}\to\mathcal{O}_Z\to 0.$$

Hence the fiber of the projection $\mathcal{C}^0_{2G+(b-1)F,F}\to|2G+(b-1)F|\times|F|$ over a curve C is of dimension no larger than

$$\dim C^{[g_L-1]}+\text{ext}^1(\mathcal{O}_Z,\mathcal{O}_C)-\dim Aut(\mathcal{O}_C)\times Aut(\mathcal{O}_Z)=\dim C^{[g_L-1]}-1.$$

Therefore

$$\begin{aligned}&\dim\mathcal{C}^0_{2G+(b-1)F,F}\\ &\leq\dim|2G+(b-1)F|\times|F|-1+\max_{\mathcal{F}\in\mathcal{C}^0_{2G+(b-1)F,F}}\dim C^{[g_L-1]}_{\mathcal{F}}\\ &=3b-3e-1+\max_{\mathcal{F}\in\mathcal{C}^0_{2G+(b-1)F,F}}\dim C^{[g_L-1]}_{\mathcal{F}}\end{aligned}$$

$$= 4b - 4e - 3 + (\max_{\mathcal{F}\in\mathcal{C}^0_{2G+(b-1)F,F}} \dim C^{[g_L-1]}_{\mathcal{F}} - (g_L - 1))$$
$$= L^2 - 3 + (\max_{\mathcal{F}\in\mathcal{C}^0_{2G+(b-1)F,F}} \dim C^{[g_L-1]}_{\mathcal{F}} - (g_L - 1)).$$

The only thing left to prove is $\dim C^{[g_L-1]}_{\mathcal{F}} \leq g_L - 1$ for all $C_{\mathcal{F}}$, and this follows from that $C_{\mathcal{F}}$ only have isolated planner singularities and the result of Iarrobino (Corollary 2 in [**12**]).

Analogously we can show that $\dim \mathcal{C}_{(a-1)G+(ae+1)F,\ G} \leq L^2 - 3$. The corollary is proved. q.e.d.

Proof of Statement (2) and (3) of Theorem 3.14. The proof has 7 steps and we check all conditions in **CB** and **CB′** one by one as follows.

*Step 1: **CB**-(1).*

Since $M(L, 0)$ is integral and D_{Θ_L} is a divisor on it, to show **CB**-(1) it is enough to show $\dim(D_{\Theta_L} \setminus D^o_{\Theta_L}) \leq L^2 - 1$. By Corollary 3.25, it is enough to show $\dim(D^{int}_{\Theta_L} \setminus D^o_{\Theta_L}) \leq L^2 - 1$. By definition

$$(D^{int}_{\Theta_L} \setminus D^o_{\Theta_L}) \subset \bigcup_{\substack{k \geq 2, i = 0 \\ or\ i \geq 1}} M^{int}_{k,i}(L, 0).$$

Therefore we have **CB**-(1) is fulfilled by Proposition 3.24.

*Step 2: **CB**-(2).*

Assume $L = aG + bF$ ample with $\min\{a, b\} \geq 4$. Then Lemma 3.17 applies to $L+K_X = (a-2)G+(b-e-2)F$ and $M(L\otimes K_X, 0)\setminus M^{int}(L\otimes K_X, 0)$ is of codimension ≥ 2. $M(L \otimes K_X, 0)$ satisfies the "condition S_2 of Serre" because it is normal by Corollary 3.19. Hence to prove **CB**-(2) is fulfilled, it is enough to show $M^{int}(L \otimes K_X, 0) \setminus M(L \otimes K_X, 0)^o$ is of dimension $\leq (L + K_X)^2 - 1$. Since we have

$$M^{int}(L \otimes K_X, 0) \setminus M(L \otimes K_X, 0)^o = \bigcup_{i\leq -1} W^{int}_{k,i}(L \otimes K_X, 0),$$

by Proposition 3.24 we have

$$\begin{aligned}\dim M^{int}(L \otimes K_X, 0) \setminus M(L \otimes K_X, 0)^o &\leq (L + K_X)^2 + K_X.(L + K_X)\\ &\leq (L + K_X)^2 - 1.\end{aligned}$$

Hence **CB**-(2).

*Step 3: **CB**-(3).*

To check that **CB**-(3) holds, it is enough to show the following three statements.

1) $\dim Q^o_1 = 2\ell - L.K_X = L^2$;
2) $\rho_1^{-1}(\rho_1(Q^o_1)) \setminus Q^o_1$ is of dimension $\leq 2\ell - L.K_X - 2 = L^2 - 2$;
3) $H_\ell \setminus \rho_1(Q^o_1)$ is of dimension $\leq 2\ell - 2 = L^2 + L.K_X - 2$.

Let $s>0, t\geq 0$, and define

$$Q_1^{s,t}:=\left\{[\mathcal{I}_Z(L\otimes K_X)\overset{f_1}{\twoheadrightarrow}\mathcal{F}_L]\in Q_1\left|h^1(\mathcal{F}_L)=s,\ h^1(\mathcal{F}_L(-K_X))=t\right.\right\},$$
$$H_\ell^{L,s,t}:=\left\{\mathcal{I}_Z\in H_\ell\left|h^1(\mathcal{I}_Z(L\otimes K_X))=s-1,\ h^1(\mathcal{I}_Z(L))=t.\right.\right\}.$$

Then $Q_1^{s,t}=\rho_1^{-1}(H_\ell^{L,s,t})$, $\rho_1(Q_1^o)\subset H_\ell^{L,1,0}$ and $\rho_1^{-1}(\rho_1(Q_1^o))\subset Q_1^{1,0}$. For d large enough, we have the classifying map $Q_1^{s,t}\xrightarrow{\phi_L^{L,s,t}}\mathcal{M}^d(L,0)$. In particular when $s=1$, $\phi_L^{L,1,t}(Q_1^{1,t})\subset\mathcal{M}(L,0)$, hence $\phi_L^{L,1,t}(Q_1^{1,t})\subset\mathcal{D}_{\Theta_L}$. This is because for every $\mathcal{F}\in\mathcal{M}^d(L,0)$, if $h^0(\mathcal{F}_L)=1$ and there is a torsion free extension of $\mathcal{F}_L$ by K_X, then $\forall\ \mathcal{F}'\subsetneq\mathcal{F}$, $h^0(\mathcal{F}')\leq 1$ and $h^1(\mathcal{F}')\geq 1$ hence then $\chi(\mathcal{F}')\leq 0$ and $\mathcal{F}$ is semistable. The fiber of $\phi_L^{L,s,t}$ at $\mathcal{F}_L$ is contained in $\mathrm{Ext}^1(\mathcal{F}_L,K_X)$, and hence

$$\dim Q_1^o=1+dim(\mathcal{D}_{\Theta_L}\cap(\mathcal{M}^{int}(L,0)\setminus\bigcup_{k\geq 2,i=0\ or\ i>0}\mathcal{M}^{int}_{k,i}(L,0)))=L^2.$$

(3.33)
$$\dim(\bigcup_{t\geq 0}Q_1^{1,t})\setminus Q_1^o\leq 1+\dim((\mathcal{D}_{\Theta_L}\setminus\mathcal{D}^{int}_{\Theta_L})\cup\bigcup_{i>0}\mathcal{M}^{int}_{k,i}(L,0))\leq L^2-2,$$

where the last inequality is because of Corollary 3.25 and Proposition 3.24.

By (3.7) $Q_1^{s,t}\cong\mathbb{P}(p_*(\mathscr{I}_\ell\otimes q^*L)|_{H_\ell^{L,s,t}})$, where $p_*(\mathscr{I}_\ell\otimes q^*L)$ is a vector bundle of rank $h^0(\mathcal{I}_Z(L))=t+1-L.K_X$ over $H_\ell^{L,s,t}$. Hence

$$\dim Q_1^{s,t}=\dim H_\ell^{L,s,t}+t-L.K_X. \tag{3.34}$$

Hence (3.33) implies $\dim(\bigcup_{t\geq 0}H_\ell^{L,1,t})\setminus\rho_1(Q_1^o)\leq L^2+L.K_X-2$. Hence we only need to show $\dim H_\ell\setminus(\bigcup_{t\geq 0}H_\ell^{L,1,t})\leq 2\ell-2$, i.e. $\dim H_\ell^{L,s,t}\leq 2\ell-2$ for all $s\geq 2$.

$p_*(\mathscr{I}_\ell\otimes q^*(L\otimes K_X))$ is a vector bundle of rank $h^0(\mathcal{I}_Z(L\otimes K_X))=s$ over $H_\ell^{L,s,t}$. By (3.8) $\mathbb{P}(p_*(\mathscr{I}_\ell\otimes q^*(L\otimes K_X))|_{H_\ell^{L,s,t}})$ is a locally closed subscheme inside Q_2. For d big enough, there is a classifying map

$$\mathbb{P}(p_*(\mathscr{I}_\ell\otimes q^*(L\otimes K_X))|_{H_\ell^{L,s,t}})\xrightarrow{\phi_{L\otimes K_X}^{L,s,t}}\mathcal{M}^d(L\otimes K_X,0).$$

If $s\geq 2$, then the image of $\phi_{L\otimes K_X}^{L,s,t}$ is contained in

$$(\mathcal{M}^d(L\otimes K_X,0)\setminus\mathcal{M}^{int}(L\otimes K_X,0))\cup\bigcup_{\substack{i=0,k=s-1\\ or\ i<0}}\mathcal{W}^{int}_{k,i}(L\otimes K_X,0).$$

The fiber of $\phi_{L\otimes K_X}^{L,s,t}$ at $\mathcal{F}_{L\otimes K_X}$ is contained in $\mathrm{Ext}^1(\mathcal{F}_{L\otimes K_X},\mathcal{O}_X)$. If $\mathcal{F}_{L\otimes K_X}\in\mathcal{W}^{int}_{s-1,0}(L\otimes K_X,0)$, then

$$h^0(\mathcal{F}_{L\otimes K_X}(K_X))=0\text{ and }\mathrm{ext}^1(\mathcal{F}_{L\otimes K_X},\mathcal{O}_X)=-(L+K_X).K_X.$$

If $\mathcal{F}_{L\otimes K_X}\notin\mathcal{W}^{int}_{s-1,0}(L\otimes K_X,0)$, then since $-K_X-G$ is very ample, by (3.8) we have

$$h^0(\mathcal{F}_{L\otimes K_X}(K_X))=h^0(\mathcal{I}_Z(L\otimes K_X^{\otimes 2}))\leq h^0(\mathcal{I}_Z(L\otimes K_X\otimes\mathcal{O}_X(-G)))-1\leq h^0(\mathcal{I}_Z(L\otimes K_X))-1=s-1. \tag{3.35}$$

Hence $\text{ext}^1(\mathcal{F}_{L\otimes K_X},\mathcal{O}_X)\leq s-1-(L+K_X).K_X$. Hence for $s\geq 2$

$$\begin{aligned}&\dim H_\ell^{L,s,t}+s-1=\ \dim\mathbb{P}(p_*(\mathscr{I}_\ell\otimes q^*(L\otimes K_X))|_{H_\ell^{L,s,t}})\\ &\leq\max\left\{\begin{array}{l}\dim\mathcal{W}^{int}_{s-1,0}(L\otimes K_X,0)-(L+K_X).K_X,\\ \dim(\mathcal{M}^d(L\otimes K_X,0)\setminus\mathcal{M}^{int}(L\otimes K_X,0)\cup\bigcup_{i<0}\mathcal{W}^{int}_{k,i}(L\otimes K_X,0))\\ \qquad +s-1-(L+K_X).K_X\end{array}\right\}\\ &\leq\max\{(L+K_X).L-1,\ (L+K_X).L-3+s\}=(L+K_X).L-3+s.\end{aligned} \tag{3.36}$$

Hence $\dim H_\ell^{L,s,t}\leq 2\ell-2$ for all $s\geq 2$. Hence **CB**-(3) is fulfilled.

*Step 4: **CB**-(4).*

CB-(4) can be shown analogously: Q_2^o is obviously nonempty and there is a classifying map $Q_2\xrightarrow{\phi^{L\otimes K_X}_{L\otimes K_X}}\mathcal{M}^d(L\otimes K_X,0)$ with fiber over $\mathcal{F}_{L\otimes K_X}$ contained in $\text{Ext}^1(\mathcal{F}_{L\otimes K_X},\mathcal{O}_X)$.

$\dim\rho_2^{-1}(\rho_2(Q_2^o))\setminus Q_2^o\leq\dim Q_2^o-2$ because

$$\begin{aligned}&\dim\phi^{L\otimes K_X}_{L\otimes K_X}(\rho_2^{-1}(\rho_2(Q_2^o))\setminus Q_2^o)\\ &\leq\dim\mathcal{M}^d(L\otimes K_X,0)\setminus\mathcal{M}^{int}(L\otimes K_X,0)\cup\bigcup_{i\leq -1}\mathcal{W}^{int}_{k,i}(L\otimes K_X,0)\\ &\leq(L+K_X)^2-2,\end{aligned}$$

and

$$\text{ext}^1(\mathcal{F}_{L\otimes K_X},\mathcal{O}_X)=-K_X.(L+K_X),\ \forall\mathcal{F}_{L\otimes K_X}\in\phi^{L\otimes K_X}_{L\otimes K_X}(\rho_2^{-1}(\rho_2(Q_2^o)).$$

Statement (3) is proved.

*Step 5: **CB'**-(3') and **CB'**-(2'a).*

Now we prove Statement (2) of the theorem. We need to check conditions in **CB'** hold. With no loss of generality, we ask $L=aG+bF$ with $b\geq a$. Then in this case $L+K_X=mF$ or $G+nF$ with $n>0$ for $e=0$, and $n\geq 2e-1$ for $e\geq 1$. Then $\mathcal{F}_{L\otimes K_X}$ is semistable $\Leftrightarrow$ $H^0(\mathcal{F}_{L\otimes K_X})=0$. Hence $H'_\ell\subset\bigcup_{t\geq 0}H_\ell^{L,1,t}$ and by (3.33) we have **CB'**-(3'). Notice that (3.33) holds for $L=aG+bF$ ample with $\min\{a,b\}\geq 2$.

Also $M(L\otimes K_X,0)\cong|L\otimes K_X|$ and $M(L\otimes K_X,0)'\cong|L\otimes K_X|'$. Then easy to check **CB'**-(2'a) holds.

Step 6: $\boldsymbol{CB'}$-$(2'b)$.

Now we check $\mathbf{CB}'$-$(2'b)$. First let $L\otimes K_X=G+nF$ with $|G+nF|^{int}\neq\emptyset$. Recall the commutative diagram in (3.25)

$$\begin{array}{ccccc} \mathbb{P}(\mathcal{V}') & \xleftarrow{\supseteq} & P_2' & \xrightarrow{\sigma_2'} & \Omega'_{L\otimes K_X} \\ & & \downarrow{\scriptstyle f'_{Q_2}} & & \downarrow{\scriptstyle f'_M} \\ & & Q_2' & \xrightarrow[g_2']{} & M(L\otimes K_X,0)'. \end{array} \tag{3.37}$$

where $\mathcal{V}'=\mathscr{E}xt^1_p(\mathscr{Q}_{L\otimes K_X}|_{\Omega'_{L\otimes K_X}},q^*\mathcal{O}_X)$ with $\mathscr{Q}_{L\otimes K_X}$ the universal quotient over $\Omega_{L\otimes K_X}$. $\mathcal{V}'$ is locally free of rank $-(L+K_X).K_X$ on $\Omega'_{L\otimes K_X}$. $P_2'\subset\mathbb{P}(\mathcal{V}')$ parametrizing torsion free extensions of $\mathscr{Q}_{\mathfrak{s}}$ by $\mathcal{O}_X$ for all $\mathfrak{s}\in\Omega'_{L\otimes K_X}$ and $f'_{Q_2}:P_2'\to Q_2'$ is the classifying map and also a principal $PGL(V_{L\otimes K_X})$-bundle for some vector space $V_{L\otimes K_X}$.

To show the complement of P_2' inside $\mathbb{P}(\mathcal{V}')$ is of codimension ≥ 2, it is enough to show for every $\mathcal{F}_{L\otimes K_X}\in\mathcal{M}^R(L\otimes K_X,0)$, $H^0(\mathcal{F}_{L\otimes K_X}(K_X))=0$ with support $C_{\mathcal{F}_{L\otimes K_X}}=C^1_{\mathcal{F}_{L\otimes K_X}}\cup C^2_{\mathcal{F}_{L\otimes K_X}}$ such that $C^1_{\mathcal{F}_{L\otimes K_X}}\in|F|$ and $C^2_{\mathcal{F}_{L\otimes K_X}}\in|G+(n-1)F|^{int}$, there is a torsion free extension in $\mathrm{Ext}^1(\mathcal{F}_{L\otimes K_X},\mathcal{O}_X)$. $\forall\ \mathcal{F}'_{L\otimes K_X}\subsetneq\mathcal{F}_{L\otimes K_X}$, $\mathrm{Ext}^1(\mathcal{F}_{L\otimes K_X}/\mathcal{F}'_{L\otimes K_X},\mathcal{O}_X)$ can be view as a subspace of $\mathrm{Ext}^1(\mathcal{F}_{L\otimes K_X},\mathcal{O}_X)$. There is a torsion free extension in $\mathrm{Ext}^1(\mathcal{F}_{L\otimes K_X},\mathcal{O}_X)\Leftrightarrow\mathrm{ext}^1(\mathcal{F}_{L\otimes K_X}/\mathcal{F}'_{L\otimes K_X},\mathcal{O}_X)<\mathrm{ext}^1(\mathcal{F}_{L\otimes K_X},\mathcal{O}_X)$, $\forall\ \mathcal{F}'_{L\otimes K_X}\subsetneq\mathcal{F}_{L\otimes K_X}$. Now we have that $C_{\mathcal{F}_{L\otimes K_X}}=C^1_{\mathcal{F}_{L\otimes K_X}}\cup C^2_{\mathcal{F}_{L\otimes K_X}}$, $C^i_{\mathcal{F}_{L\otimes K_X}}\cong\mathbb{P}^1$ and $deg(K_X|_{C^i_{\mathcal{F}_{L\otimes K_X}}})<0$, for $i=1,2$. Therefore $\forall\ \mathcal{F}'_{L\otimes K_X}\subsetneq\mathcal{F}_{L\otimes K_X}$, either $\mathrm{Ext}^1(\mathcal{F}_{L\otimes K_X}/\mathcal{F}'_{L\otimes K_X},\mathcal{O}_X)=0$ or $\mathrm{Ext}^2(\mathcal{F}_{L\otimes K_X}/\mathcal{F}'_{L\otimes K_X},\mathcal{O}_X)=0$. Hence the map

$$\mathrm{Ext}^1(\mathcal{F}_{L\otimes K_X}/\mathcal{F}'_{L\otimes K_X},\mathcal{O}_X)\hookrightarrow\mathrm{Ext}^1(\mathcal{F}_{L\otimes K_X},\mathcal{O}_X)$$

can not be surjetive. The reason is that $\mathrm{ext}^1(\mathcal{F}_{L\otimes K_X},\mathcal{O}_X)=2n+2-e>0$ and $\mathrm{Ext}^1(\mathcal{F}'_{L\otimes K_X},\mathcal{O}_X)\neq 0$ since $\chi(\mathcal{F}'_{L\otimes K_X}(K_X))<0$.

If $L\otimes K_X=F$, then $\mathbf{CB}'$-$(2'b)$ is obvious. Let $|L\otimes K_X|^{int}=\emptyset$, i.e. $L\otimes K_X=nF$ with $n>1$. In this case $|L\otimes K_X|'=|L\otimes K_X|$. $\Omega'_{L\otimes K_X}=\Omega_{L\otimes K_X}$. In order to show $\dim\mathbb{P}(\mathcal{V}')\setminus P_2'\leq\dim P_2'-2$, it is enough to show for every $\mathcal{F}_{L\otimes K_X}$ semistable, $\mathbb{P}(\mathrm{Ext}^1(\mathcal{F}_{L\otimes K_X},\mathcal{O}_X)\setminus\mathrm{Ext}^1(\mathcal{F}_{L\otimes K_X},\mathcal{O}_X)^{tf})$ is of dimension $\leq -K_X.(L+K_X)-3$, where $\mathrm{Ext}^1(\mathcal{F}_{L\otimes K_X},\mathcal{O}_X)^{tf})$ is the subset parameterizing torsion free extensions. We have

$$\begin{aligned}&\mathrm{Ext}^1(\mathcal{F}_{L\otimes K_X},\mathcal{O}_X)\setminus\mathrm{Ext}^1(\mathcal{F}_{L\otimes K_X},\mathcal{O}_X)^{tf}\\ &=\bigcup_{\mathcal{F}'_{L\otimes K_X}\subsetneq\mathcal{F}_{L\otimes K_X}}\mathrm{Ext}^1(\mathcal{F}_{L\otimes K_X}/\mathcal{F}'_{L\otimes K_X},\mathcal{O}_X).\end{aligned} \tag{3.38}$$

Since $L \otimes K_X = nF$, we have for any $\mathcal{F}'_{L\otimes K_X} \subsetneq \mathcal{F}_{L\otimes K_X}$, either $\mathrm{Ext}^1(\mathcal{F}_{L\otimes K_X}/\mathcal{F}'_{L\otimes K_X}, \mathcal{O}_X) = 0$ or $\mathrm{Ext}^2(\mathcal{F}_{L\otimes K_X}/\mathcal{F}'_{L\otimes K_X}, \mathcal{O}_X) = 0$. Hence we only need to show that $\forall\ \mathcal{F}'_{L\otimes K_X} \subsetneq \mathcal{F}_{L\otimes K_X}$ such that

$$\mathrm{Ext}^1(\mathcal{F}_{L\otimes K_X}/\mathcal{F}'_{L\otimes K_X}, \mathcal{O}_X) \neq 0,$$

we have $\mathrm{ext}^1(\mathcal{F}'_{L\otimes K_X}, \mathcal{O}_X) \geq 2$. It is enough to show $\mathrm{ext}^1(\mathcal{F}'_{L\otimes K_X}, \mathcal{O}_X) \geq 2$ for every $\mathcal{F}'_{L\otimes K_X} \subsetneq \mathcal{F}_{L\otimes K_X}$ with $C_{\mathcal{F}'_{L\otimes K_X}}$ integral. On the other hand $C_{\mathcal{F}'_{L\otimes K_X}} \cong \mathbb{P}^1$ if integral, and also $deg(K_X|_{C_{\mathcal{F}'_{L\otimes K_X}}}) = -2$. Hence $\mathrm{ext}^1(\mathcal{F}'_{L\otimes K_X}, \mathcal{O}_X) = h^1(\mathcal{F}'_{L\otimes K_X}(K_X)) \geq 2$ because $\chi(\mathcal{F}'_{L\otimes K_X}(K_X)) \leq -2$.

Step 7: $\boldsymbol{CB'}$-$(4')$.

$\mathbf{CB'}$-$(4')$ is the last thing left to check.

$$Q'_2 := \left\{ [\mathcal{I}_Z(L\otimes K_X) \overset{f_2}{\twoheadrightarrow} \mathcal{F}_{L\otimes K_X}] \in Q_2 \;\middle|\; \begin{array}{l} \mathcal{F}_{L\otimes K_X} \textit{ is semistable}, \\ h^0(\mathcal{F}_{L\otimes K_X}(K_X)) = 0, \textit{ and} \\ \textit{Supp}(\mathcal{F}_{L\otimes K_X}) \in |L\otimes K_X|' \end{array} \right\}.$$

In this case $\mathcal{F}_{L\otimes K_X}$ is semistable $\Leftrightarrow H^0(\mathcal{F}_{L\otimes K_X}) = 0$. $h^0(\mathcal{I}_Z(L\otimes K_X)) = 1$ for all $\mathcal{I}_Z \in \rho_2(Q'_2)$, hence $\rho_2|_{Q'_2}$ is bijective and hence an isomorphism, therefore $Q'_2 \cong \rho_2^{-1}(\rho_2(Q'_2))$ and $\mathbf{CB'}$-$(4')$ holds.

The proof of Theorem 3.14 is finished. q.e.d.

References

[1] T. Abe, *Deformation of rank 2 quasi-bundles and some strange dualities for rational surfaces*, Duke Math. J. 155 (2010), no. 3, 577–620, MR2738583, Zbl 1208.14034.

[2] T. Abe, *Strange duality for height zero moduli spaces of sheaves on* $\mathbb{P}^2$, Michigan Math. J. 64 (2015), 569–586, MR3394260, Zbl 1359.14038.

[3] A. Beauville, *Vector bundles on curves and generalized theta functions: recent results and open problems.* Current topics in complex algebraic geometry (Berkeley, CA, 1992/93), 17–33, Math. Sci. Res. Inst. Publ., 28, Cambridge Univ. Press, Cambridge, 1995, MR1397056, Zbl 0846.14024.

[4] P. Belkale, *Strange duality and the Hitchin/WZW connection.* J. Differential Geom. Volume 82, Number 2 (2009), 445–465, MR2520799, Zbl 1193.14013.

[5] G. Danila, *Sections de fibré déterminant sur l'espace de modules des faisceaux semi-stable de rang 2 sur le plan projectif*, Ann. Inst. Fourier (Grenoble) 50 (2000), 1323–1374, MR1800122, Zbl 0952.14010.

[6] G. Danila, *Résultats sur la conjecture de dualité étrange sur le plan projectif.* Bull. Soc. Math. France 130 (2002), 1–33, MR1906190, Zbl 1038.14004.

[7] R. Donagi and L.W. Tu, *Theta functions for* $\mathrm{SL}(n)$ *versus* $\mathrm{GL}(n)$, Math. Res. Lett. **1** (1994), no. 3, 345–357, MR1302649, Zbl 0847.14027.

[8] G. Ellingsrud, L. Göttsche, and M. Lehn. *On the cobordism class of the Hilbert scheme of a surface.* J. Algebraic Geom. 10 (2001), no. 1, 81–100. MR1795551, Zbl 0976.14002.

[9] L. Göttsche, and Y. Yuan, *Generating functions for K-theoretic Donaldson invariants and Le Potier's strange duality*, J. Algebraic Geom. 28 (2019), 43–98, MR3875361.

[10] R. Hartshorne, *Algebraic Geometry.* GTM 52, Springer Verlag, New York (1977).

[11] D. Huybrechts, and M. Lehn, *The Geometry of Moduli Spaces of Sheaves.* Friedr. Vieweg & Sohn Verlagsgesellschaft mbH, Braunschweig/Wiesbaden, 1997.

[12] A. Iarrobino, *Punctual Hilbert schemes*, Bull. Amer. Math. Soc. 78 (1972), 819–823, MR0308120, Zbl 0267.14005.

[13] J. Le Potier, *Faisceaux Semi-stables de dimension* 1 *sur le plan projectif*, Rev. Roumaine Math. Pures Appl., 38 (1993), 7–8, 635–678, MR1263210, Zbl 0815.14029.

[14] J. Le Potier, *Faisceaux semi-stables et systèmes cohérents*, Proceedings de la Conference de Durham (July 1993), Cambridge University Press (1995), p. 179–239, MR1338417, Zbl 0847.14005.

[15] J. Le Potier, *Dualité étrange, sur les surfaces,* preliminary version 18.11.05.

[16] A. Marian, and D. Oprea, *The level-rank duality for non-abelian theta functions.* Invent. Math. 168. 225–247 (2007), MR2289865, Zbl 1117.14035.

[17] A. Marian, and D. Oprea, *A tour of theta dualities on moduli spaces of sheaves.* 175–202, Contemporary Mathematics, 465, American Mathematical Society, Providence, Rhode Island (2008), MR2457738, Zbl 1149.14301.

[18] A. Marian, and D. Oprea, *Generic strange duality for K3 surfaces.* Duke Math. J. Volume 162, Number 8 (2013), 1463–1501, MR3079253, Zbl 1275.14037.

[19] A. Marian, and D. Oprea, *On the strange duality conjecture for abelian surfaces.* J. Eur. Math. Soc. (JEMS) 16 (2014), no. 6, 1221–1252, MR3226741, Zbl 1322.14063.

[20] Y.Yuan, *Determinant line bundles on Moduli spaces of pure sheaves on rational surfaces and Strange Duality*, Asian J. Math. Vol. 16, No. 3, pp. 451–478, September 2012, MR2989230, Zbl 1262.14013.

[21] Yao Yuan, *Motivic measures of moduli spaces of 1-dimensional sheaves on rational surfaces*, Commun. Contemp. Math. 20 (2018), no. 3, 1750019, 32 pp., MR3766729.

[22] Yao Yuan, *Moduli spaces of 1-dimensional semi-stable sheaves and Strange duality on* $\mathbb{P}^2$, Adv. Math. 318, 130–157 (2017), MR3689738, Zbl 06769051.

Yau Mathematical Sciences Center
Tsinghua University
100084, Beijing
P. R. China
E-mail address: yyuan@tsinghua.edu.cn