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THE WEYL PROBLEM IN WARPED PRODUCT SPACES

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Abstract

In this paper, we discuss the Weyl problem in warped product spaces. We apply the method of continuity and prove the openness of the Weyl problem. A counterexample is constructed to show that the isometric embedding of the sphere with canonical metric is not unique up to an isometry if the ambient warped product space is not a space form. Then, we study the rigidity of the standard sphere if we fixed its geometric center in the ambient space. Finally, we discuss a Shi-Tam type of inequality for the Schwarzschild manifold as an application of our findings.

1. Introduction

The isometric embedding problem is one of the fundamental issues in differential geometry. Among them, the Weyl problem is a milestone in the development of the theory of nonlinear elliptic partial differential equations, particularly, of the Monge-Ampère type. In 1916, Weyl proposed the following problem: Does every smooth metric on the two dimensional sphere with positive Gauss curvature admit a smooth isometric embedding in the three dimensional Euclidean space? Weyl [47] suggested the method of continuity to solve this problem. He also presented the openness part for the analytic case and established an estimate on the mean curvature of the embedded strictly convex surfaces, which is C^2 a priori estimate for the embedding map. The analytic case was fully solved by Lewy [27]. In 1953, Nirenberg, in his celebrated paper [34], solved the Weyl problem in the smooth case and exhibited a beautiful proof. Alexandrov and Pogorelov [37] used a different approach to solve the problem independently. Pogorelov [38] also generalized Nirenberg's theorem to the hyperbolic space. Furthermore, he considered the problem in Riemannian manifolds [39, 40]. In the 1990s, Weyl's estimate was generalized to the degenerate case with nonnegative Gauss curvature by Guan and Li [17], Hong and Zuily [23] and partially

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by Iaia [25]. Recently, Chang and Xiao [7] and Lin and Wang [28] considered the degenerate case for Pogorelov's theorem in the hyperbolic space.

The isometric embedding problem plays an important role in general relativity. In 1993, Brown and York [6] proposed the following definition of quasi-local mass: Let (Ω, σ) be a compact Riemannian 3-manifold, and suppose its boundary $(\partial\Omega, g)$ has positive Gauss curvature. They define the quantity

$$m_{BY}(\partial\Omega) = \frac{1}{8\pi}\int_{\partial\Omega}(H_0 - H)dV_g, \tag{1.1}$$

where dV_g is the volume form of $\partial\Omega$, H and H_0 are the mean curvature of $\partial\Omega$ in the original Riemannian manifold and the Euclidean space respectively, if it can be isometrically embedded in $\mathbb{R}^3$. Nirenberg's isometric embedding theorem guarantees the existence of such an isometric embedding. In 2002, an important work by Shi and Tam [44] showed that the Brown–York mass is nonnegative.

Liu and Yau [29, 30] introduced the Liu–Yau quasi-local mass

$$m_{LY}(\partial\Omega) = \frac{1}{8\pi}\int_{\partial\Omega}(H_0 - |H|)dV_g, \tag{1.2}$$

where $|H|$ is the Lorentzian norm of the mean curvature vector. They also proved the nonnegativity of their mass. Wang and Yau [48, 49, 50] defined a new quasi-local mass generalizing the Brown–York mass. They have used Pogorelov's work on the isometric embedding of the sphere into the hyperbolic space. The nonnegativity of the Wang–Yau mass is also obtained in [48, 45].

The definitions of the Brown–York mass and the Wang–Yau mass suggest that the isometric embedding of the sphere into model space always plays an important role for the quasi-local mass problem. Thus, an important task is generalizing the Weyl problem to other 3-dimensional ambient spaces which is not a space form. This topic may be helpful for further discussion of the quasi-local mass.

In the present paper, we study the Weyl problem in 3-dimensional warped product spaces. These warped product spaces mean the subset of $\mathbb{R}^n$ with some nontrivial but rotation symmetric metrics. One may write the metric in polar coordinates

$$ds^2 = \frac{1}{f^2(r)}dr^2 + r^2 dS_{n-1}, \tag{1.3}$$

where r is the Euclidean radius and dS_{n-1} is the standard metric of the $n-1$ sphere. f depends only on r, which is called the warping function here. If the warping function takes the form

$$f(r) = \sqrt{1 - \frac{m}{r} + \kappa r^2}, \tag{1.4}$$

where κ and m are constants, then these spaces are called Anti-de Sitter–Schwarzschild (AdS–Sch) spaces. They are special interesting examples in which the quasi-local mass needs to be generalized. If $m = 0$ in (1.4), then the AdS–Sch spaces are space forms.

Here, we present the outline and main results of this paper.

In the first part of this paper, we establish the openness for the strictly convex surface in any 3-dimensional warped product space. We note that Pogorelove [39, 40] has obtained the openness for the strictly convex surface in any 3-dimensional Riemannian space. However, one expects to find another more rigorous and more elementary proof for the openness and existence results. In our argument for the openness part, we observe that a dual relation exists between the linearized isometric embedding system and the homogeneous linearized Gauss-Codazzi system, which is first discovered in our present paper. Then, we use the maximum principle to obtain the uniqueness of solutions to the dual problem. Therefore, the Fredholm theory shows the solvability of the linearized problem. Our proof does not need infinitesimal rigidity of the linearized isometric embedding system. Thus, our argument is considerably different from Pogorelov's proof and Nirenberg's proof.

Let us state the openness theorem. We will always use the notation N to denote the ambient space.

Theorem 1. *Let N be a smooth 3-dimensional warped product space and g be a smooth metric on the sphere $\mathbb{S}^2$. Suppose that $(\mathbb{S}^2, g)$ can be isometrically embedded into N as a closed strictly convex surface. Then, for any $\alpha \in (0,1)$, there exists a positive constant ϵ, depending only on g and α, such that, for any smooth metric $\tilde{g}$ on $\mathbb{S}^2$ satisfying*

$$\|g - \tilde{g}\|_{C^{2,\alpha}(\mathbb{S}^2, Sym(T^*\mathbb{S}^2 \otimes_S T^*\mathbb{S}^2))} < \epsilon,$$

$(\mathbb{S}^2, \tilde{g})$ also can be isometrically embedded into the same space as another closed strictly convex surface, where the notation $C^{2,\alpha}\big(\mathbb{S}^2, Sym\big(T^*\mathbb{S}^2 \otimes T^*\mathbb{S}^2\big)\big)$ means the index $(2,\alpha)$-Hölder space of the covariant symmetric two tensors on $\mathbb{S}^2$.

Here, the metric used to define the norm of the space $C^{2,\alpha}\big(\mathbb{S}^2, \mathrm{Sym}\big(T^*\mathbb{S}^2 \otimes T^*\mathbb{S}^2\big)\big)$ is the standard metric of the sphere. We also note the recent work of Cabrera Pacheco and Miao [41], who have proved the openness near the standard sphere metric in Schwarzschild manifolds with small mass.

Combining the openness with the closeness result of [16, 31], we can obtain some existence results. One of them is the following theorem, which is proposed and proved by Guan and Lu [16] and Lu [31],

Theorem 2. *Suppose a 3-dimensional warped product space N does not have singularity and the warping function f satisfies the assumption (a) and (b). Then for any metric g on $\mathbb{S}^2$, if its Gauss curvature $K > K_0$*

for some constant K_0, $(\mathbb{S}^2, g)$ can be isometric embedded into N as a strictly convex surface.

Here, the meaning of the assumption (a), (b) and the constant K_0 will be explained by (4.21) in section 4. A space with singularity means that it contains some horizon in its interior. If the ambient space is a warped product space, then containing singularities means that the warping function $f(r) = 0$ has solutions, for example, the AdS–Sch spaces. In other words, no singularity means that the space is null homologous.

The second part of this paper discusses the rigidity of the isometric embedding problem in a warped product space. The global rigidity is obtained by Cohn-Vossen for convex surfaces [**8**] in Euclidean spaces. The cases of space forms are still valid [**10**, **18**]. However, in general warped product spaces, the rigidity is not always true.

Theorem 3. *Suppose N is an n-dimensional warped product space. If the function f in (1.3) satisfies*

$$\frac{ff'}{r} + \frac{1 - f^2}{r^2} \neq 0, \ \text{at } r = r_0$$

then level set sphere $r = r_0$ is not rigid. This means that there exists a smooth convex hypersurface that is isometric to the $r = r_0$ sphere but with different second fundamental form. If every r-level set sphere is rigid, then the ambient space must be a space form.

Since the scalar curvature is only determined by its intrinsic metric, the Alexandrov-type theorem for constant scalar curvature hypersurfaces in general warped product spaces is also not true.

In general, strictly convex surfaces in 3-dimensional warped product spaces are not infinitesimally rigid either, because the isometry group of ambient spaces is small. The lack of infinitesimal rigidity yields the global non-rigidity of any strictly convex surfaces in general warped product spaces.

Non-rigidity in general warped product spaces can be explained roughly as follows: The kernel of the linearized problem of the isometric embedding system is six dimensional, but the dimension of the isometry group of the ambient space is less than six in general. Thus, we can move hypersurfaces along a vector field in the kernel of the linearized problem but outside of isometries of the ambient space. Thus, if a certain type of rigidity is expected, then some restrictions have to be imposed. Therefore, we propose the following condition in warped product spaces:

Condition. Suppose that M is a hypersurface in an n dimensional warped product space N and $\vec{r}$ is its position vector field. We require

$$\int_\Sigma \vec{r} dV_g = 0, \tag{1.5}$$

where dV_g is the volume form of M.

Now, we can recover the rigidity of spheres endowed with Einstein metrics and further satisfying the above condition.

Theorem 4. *Suppose that Σ is an $n-1$ dimensional topological sphere and g is an Einstein metric with positive constant scalar curvature on Σ. Suppose that (Σ, g) can be isometrically embedded into an n dimensional warped product space N. If the embedded hypersurface M is σ_2-convex and it satisfies condition (1.5), then M is a slice sphere, i.e., a level set sphere.*

Here, a σ_2-convex hypersurface means that the summation of the product of its any two principal curvatures is positive.

In the third part, we revisit the infinitesimal rigidity in space forms. The infinitesimal rigidity of any embedded closed convex surface in the Euclidean space holds, as obtained initially by Cohn-Vossen [9] and then simplified by Blaschke [4] using Minkowski identities. The corresponding infinitesimal rigidity of any embedded closed strictly convex surface in the hyperbolic space also holds, as discussed and reconfirmed by Lin and Wang [28] recently. Here, we provide an alternative proof for these known results. The infinitesimal rigidities in space forms are summarized in the following theorem.

Theorem 5. *Suppose M is a closed embedded strictly convex surface in a three dimensional space form, then it is infinitesimally rigid. Namely, the set of solutions of the linearized isometric embedding system comes from the Lie algebra of the isometry group of the ambient space.*

In the Euclidean space, we present a new proof of the above theorem using the maximum principle adopted from [19] and [22]. Then, by using the Beltrami map, we extend the infinitesimal rigidity of closed embedded strictly convex surfaces to space forms. Our proofs are different from the classical ones in literature. Furthermore, the openness of the Weyl problem in space forms can be considered as a corollary of the openness of the Weyl problem in the Euclidean space using the Beltrami map.

In the last part, we discuss an application for our theorem. We will prove an inequality similar to Shi-Tam's in Schwarzschild manifold, namely, $\kappa = 0$ in (1.4).

Theorem 6. *Suppose $r > m$ are two positive constants. Suppose Ω is a compact connected 3-dimensional Riemannian manifold with nonnegative scalar curvature. We suppose Σ is its boundary and it is mean convex in Ω. We further assume that the induced metric g on Σ in Ω is in the neighborhood of the canonical metric of the sphere with radius r. Namely, g is closed to the canonical metric on the standard r-Euclidean*

sphere. Then, Σ can be isometrically embedded into the Schwarzschild manifold with mass m as a strictly convex M. Thus, we have the following inequality

$$\frac{1}{8\pi}\int_{\Sigma}(H_0 - H)f(|\vec{r}|)dV_g + \frac{m}{2} \geq 0, \tag{1.6}$$

where H_0, H are the mean curvature of M in the Schwarzschild manifold and Σ in the Riemannian manifold Ω, respectively; $\vec{r}$ is the position vector of M in Schwarzschild manifold; $|\cdot|$ is the norm with respect to the metric of the Schwarzschild manifold; f is the warping function; and dV_g is the volume form of Σ. The equality holds if M is an asymptotic Euclidean ball.

The canonical metric of the sphere is the Euclidean metric induced on the canonical sphere in the Euclidean space. The above theorem may relate to the work of Fan, Shi and Tam [**13**] and Shi, Wang and Wu [**43**].

In section 2, we present a brief review of some basic formulae. In section 3, we solve the linearized system. In section 4, we prove the openness theorem and discuss some existence results. In section 5, we construct some counterexamples, which are non-rigid in any dimensional warped product spaces. In section 6, we present the proof of Theorem 4. In section 7, we revisit the infinitesimal rigidity of the closed embedded strictly convex surface in space forms. The last section proves an inequality of Shi-Tam type and exhibits an example.

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2. Notations and some formulae

For an n-dimensional warped product space, we may rewrite (1.3) as

$$ds^2 = \frac{1}{f^2(r)}dr^2 + r^2\left(\sum_{i=1}^{n-1}\cos^2 u^{i-1}\cdots\cos^2 u^1(du^i)^2\right), \tag{2.1}$$

where $(r, u^1, \cdots, u^{n-1})$ is the polar coordinate. r takes the range $[r_0, r_1]$, where r_0 may be 0 and r_1 may be $+\infty$ depending on specific cases.

In this paper, '$\cdot$' always presents the inner product defined by the metric ds^2 in the ambient space, $|\cdot|$ denotes the norm with respect to ds^2 and D denotes the Levi-Civita connection with respect to the metric ds^2. We will always denote N as an n-dimensional warped product space $(\mathbb{R}^n, ds^2)$ unless otherwise specified. Suppose that X, Y are two vector fields in the warped product space N. The Riemannian curvature tensor of N is defined by

$$\bar{R}(X,Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}.$$

Henceforth, the Greek indices $\alpha, \beta, \gamma, \cdots$ always ranges from 1 to n and the Latin indices $i, j, k, \cdots$ ranges from 1 to $n-1$.

Then, for any frame $\{E_1, E_2, \cdots, E_n\}$ on N, the Ricci curvature and scalar curvature are defined by

$$\bar{R}_{\alpha\beta} = \sum_{\gamma,\delta} \sigma^{\gamma\delta} \bar{R}_{\alpha\gamma\delta\beta}, \quad \bar{R} = \sum_{\alpha,\beta} \sigma^{\alpha\beta} \bar{R}_{\alpha\beta},$$

where $\sigma_{\alpha,\beta}$ is the metric matrix of the metric ds^2 and $\left(\sigma^{\alpha\beta}\right)$ is the inverse matrix of $(\sigma_{\alpha\beta})$, and

$$\bar{R}_{\alpha\gamma\delta\beta} = \bar{R}(E_\alpha, E_\gamma)E_\delta \cdot E_\beta.$$

We define a special orthogonal frame. For $\alpha = 1, \cdots, n$, let

$$\begin{aligned} \tilde{E}_\alpha &= f\frac{\partial}{\partial r}, \text{if } \alpha = 1; \\ \tilde{E}_\alpha &= \frac{1}{r\cos u^{\alpha-1}\cdots\cos u^1}\frac{\partial}{\partial u^{\alpha-1}}, \text{ if } \alpha > 1. \end{aligned} \tag{2.2}$$

With the above frame $\{\tilde{E}_1, \tilde{E}_2, \cdots, \tilde{E}_n\}$, a straightforward calculation shows

$$\begin{aligned} &\bar{R}_{1\gamma 1\gamma} = \frac{ff'}{r}, \text{ for } \gamma \neq 1; \bar{R}_{\alpha\beta\alpha\beta} = \frac{f^2-1}{r^2}, \text{ and } \alpha, \beta \neq 1, \alpha \neq \beta; \\ &\bar{R}_{\alpha\beta\gamma\delta} = 0 \text{ for } \alpha, \beta, \gamma, \delta \text{ taking three different indices.} \end{aligned} \tag{2.3}$$

Accordingly, the scalar curvature is

$$\bar{R} = \frac{2(n-1)ff'}{r} + (n-1)(n-2)\frac{f^2-1}{r^2}, \tag{2.4}$$

where $f' = df/dr$ is the derivative of f.

In the rest of this paper, we use the Einstein summation convention unless otherwise specified.

Suppose M is a hypersurface in N and g is its induced metric. For any two vector fields X, Y defined on the manifold M, the Riemannian curvature tensor of the submanifold M is defined by

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$

where ∇ is the Levi-Civita connection with respect to g.

Suppose $\{e_1, e_2, \cdots, e_{n-1}\}$ is a frame on M. We denote

$$R_{ijkl} = R(e_i, e_j)e_k \cdot e_l,$$

and then the Ricci curvature and the scalar curvature are

$$R_{ij} = g^{lk}R_{ilkj}, \quad R = g^{ij}R_{ij}.$$

We further denote

$$R^l_{ijk} = g^{lp}R_{ijkp}.$$

Suppose $\left\{x^1, x^2, \cdots, x^{n-1}\right\}$ is a local coordinate of M. Using the Ricci identity, for a 1-form $\xi = u_i dx^i$, we have

$$u_{i,jk} - u_{i,kj} = R^l_{jki}u_l, \tag{2.5}$$

where $u_{i,jk}$ represents the second order covariant derivative of u_i with respect to the connection ∇.

We always use $\vec{r}$ to present M's position vector field and ν to be its unit normal vector field of M with a suitable orientation. If M is closed, then we further assume that ν is the exterior normal. We denote κ_i as the i-th principal curvature function of M. Suppose $\{e_1, e_2, \cdots, e_{n-1}\}$ is an orthonormal frame. Let $\bar{R}_{ijij} = \bar{R}(e_i, e_j, e_i, e_j)$. By the Gauss equation, we have

$$\sigma_2(\kappa_1, \cdots, \kappa_{n-1}) = \frac{R}{2} + \sum_{i<j} \bar{R}_{ijij}, \tag{2.6}$$

where $\sigma_2(\kappa_1, \kappa_2, \cdots, \kappa_{n-1}) = \sum_{i<j} \kappa_i\kappa_j$. We let

$$\nu = \sum_\alpha \nu^\alpha \tilde{E}_\alpha,$$

where ν^α is the scalar component of ν with respect to $\tilde{E}_\alpha$. Thus, we have

$$\sum_{i<j} \bar{R}_{ijij} = -\frac{\bar{R}}{2} + \bar{Ric}(\nu, \nu), \quad \bar{Ric}(\nu, \nu) = \sum_\alpha (\nu^\alpha)^2 \bar{R}_{\alpha\alpha}.$$

By (2.3) and (2.4), we have

$$\sum_{i<j} \bar{R}_{ijij} = (n-2)\left[\frac{ff'}{r} + \frac{n-3}{2}\frac{f^2-1}{r^2} - \frac{(\nu^1)^2}{r^2}\left(rff' + 1 - f^2\right)\right]. \tag{2.7}$$

The conformal Killing vector field in N is defined by

$$X = rf(r)\frac{\partial}{\partial r}. \tag{2.8}$$

For any vector field Y, it is well known that

$$D_Y X = fX. \tag{2.9}$$

The squared distance function and support function of M are defined by

$$\rho = \frac{1}{2} X \cdot X = \frac{r^2}{2}, \text{ and } \varphi = X \cdot \nu. \tag{2.10}$$

We defined the second fundamental form of M,

$$h_{ij} = -D_{e_i} e_j \cdot \nu.$$

Note that, if M is a closed strictly convex hypersurface, since ν is assumed to be the out normal, then h_{ij} is positive definite. The Weigarten formula is

$$D_{e_i} \nu = \sum_m h_{im} e_m.$$

Based on (2.9) and the definition of the second fundamental form, the covariant derivatives of ρ with respect to e_i and e_i, e_j are

$$\begin{aligned} \rho_i &= f X \cdot e_i, \\ \rho_{i,j} &= \frac{f_\rho}{f} \rho_i \rho_j + f^2 g_{ij} - h_{ij} f \varphi, \end{aligned} \tag{2.11}$$

where the function f_ρ means

$$f_\rho = \frac{df}{d\rho} = \frac{df}{dr} \frac{dr}{d\rho}.$$

In this paper, an r-geodesic sphere is the set defined by $\{p \in N; \rho(p) = r^2/2\}$. For the r-geodesic sphere in the warped product space, we have

$$\nu = f \frac{\partial}{\partial r}, \quad \varphi = r, \quad \rho = \frac{r^2}{2}, \quad h_{ij} = \frac{f(r)}{r} g_{ij}. \tag{2.12}$$

We always call the r-geodesic sphere the radius r slice sphere or r slice sphere in this paper.

Suppose $\bar{M}$ is an $n-1$ dimensional Riemannian manifold. It can be isometrically embedded into N as the hypersurface M by some isometric embedding map $\vec{r}$, that is $\vec{r} : \bar{M} \to M \subset N$. Since any point in warped product space N corresponds to one vector, we also view $\vec{r}$ as the position vector of M in N if no ambiguity exists. Suppose $\{x^1, x^2, \cdots, x^{n-1}\}$ is a local coordinate of $\bar{M}$. Then, $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^2}, \cdots, \frac{\partial}{\partial x^{n-1}}$ is a local frame on $\bar{M}$. We always denote the push-forward of $\frac{\partial}{\partial x^i}$ in the tangent space of the ambient space by $\vec{r}_i = \frac{\partial \vec{r}}{\partial x^i}$.

As we will consider a differential equation on the sphere, we may need the Sobolev spaces of sections of bundles on $(\mathbb{S}^2, g)$. Suppose E is a vector bundle on $\mathbb{S}^2$, which is endowed with an inner product $\langle \cdot, \cdot \rangle$. The notations $C^{m,\alpha}\left(\mathbb{S}^2, E\right), C^\infty(\mathbb{S}^2, E)$, $L^2(\mathbb{S}^2, E)$ denote the index (m, α) Hölder space, smooth space and L^2 space of sections of the bundle E. Here, the inner product of the L^2 space is

$$\int_{\mathbb{S}^2} \langle \cdot, \cdot \rangle dV_g,$$

m is an integer and $0 < \alpha < 1$ is a real number and not an index. Since the sphere can be presented as some standard unit sphere in the 3-dimensional Euclidean space $\mathbb{R}^3$, there is a standard metric m_S induced by the standard Euclidean metric. If $E = \left(T\mathbb{S}^2\right)^{\otimes p} \otimes \left(T^*\mathbb{S}^2\right)^{\otimes q}$, unless otherwise specified, we always assume that the inner product of E is induced by the metric m_S. Thus, the norm of the Hölder space $C^{m,\alpha}\left(\mathbb{S}^2, E\right)$ is defined by m_S.

3. Linearized problem

In this section, we first study the linearized problem in a 3-dimensional warped product space N, and then we generalize our main results when the ambient space is any 3-dimensional Riemannian manifold. Suppose that $(\mathbb{S}^2, g)$ is a 2-dimensional sphere endowed with the metric g. The isometric embedding problem is to find an embedding map $\vec{r}$ that maps $\mathbb{S}^2$ to $\mathbb{R}^3$ to satisfy the following system:

$$d\vec{r} \cdot d\vec{r} = g.$$

We always denote the image surface of the map $\vec{r}$ by M. The linearized problem of the above system is

$$d\vec{r} \cdot D\tau = q, \tag{3.1}$$

where τ, q are the variation fields of $\vec{r}$ and g, respectively. The variation fields τ, q means that if we have a 1-parameter family of metrics g_t and a 1-parameter family of isometric embeddings $\vec{r}_t$ of $(\mathbb{S}^2, g_t)$ in N such that $g_0 = g, \vec{r}_0 = \vec{r}$, then we let

$$q = \left.\frac{dg_t}{dt}\right|_{t=0}; \tau = \left.\frac{d\vec{r}_t}{dt}\right|_{t=0}.$$

The details to obtain system (3.1) have been written down in [28].

We have to rewrite the system (3.1) on the sphere $\mathbb{S}^2$. Let $\left\{x^1, x^2\right\}$ be a local coordinate. We define a 1-form

$$\xi = \tau \cdot d\vec{r} = u_1 dx^1 + u_2 dx^2,$$

where $u_i = \tau \cdot \vec{r}_i, \phi = \tau \cdot \nu$ and ν is the unit normal vector field of the embedded surface M. The connection D of the ambient space can be decomposed into two components, namely, tangential and normal components $D = \nabla + D^{\perp}$, and then we have

$$\begin{aligned}
& \nabla\xi \\
= \; & du_i \otimes dx^i + u_i \nabla dx^i \\
= \; & \vec{r}_i \cdot D\tau \otimes dx^i + \tau \cdot D\vec{r}_i \otimes dx^i + u_i \nabla dx^i \\
= \; & \vec{r}_i \cdot D_j \tau dx^i \otimes dx^j + \tau \cdot \nabla\vec{r}_i \otimes dx^i + \tau \cdot D^{\perp}_j \vec{r}_i dx^j \otimes dx^i + u_i \nabla dx^i.
\end{aligned}$$

By the Gauss formulas, the definition of the second fundamental form h_{ij} and the definition of the Christoffel symbol Γ^k_{ij} with respect to ∇,

we have

$$D_j^{\perp}\vec{r}_i = h_{ij}\nu, \quad \nabla\vec{r}_i = \Gamma_{ij}^k \vec{r}_k \otimes dx^j.$$

By the Levi-Civita property $\Gamma_{ij}^k = \Gamma_{ji}^k$, we have

$$\begin{aligned} \nabla\xi &= \vec{r}_i \cdot D_j\tau dx^i \otimes dx^j + \phi h_{ij} dx^i \otimes dx^j \\ &\quad + \Gamma_{ij}^k u_k dx^j \otimes dx^i - u_i \Gamma_{ml}^i dx^m \otimes dx^l \\ &= (\vec{r}_i \cdot D_j\tau + \phi h_{ij}) dx^i \otimes dx^j. \end{aligned} \tag{3.2}$$

Based on the previous notations, the system (3.1) is equivalent to the following system:

$$\begin{cases} u_{1,1} &= q_{11} + \phi h_{11} \\ u_{1,2} + u_{2,1} &= 2(q_{12} + \phi h_{12}) \\ u_{2,2} &= q_{22} + \phi h_{22} \end{cases}, \tag{3.3}$$

where the comma indicates the covariant derivative with respect to ∇. Denote the symmetrization operator by Sym. We also denote the cotangent bundle of $\mathbb{S}^2$ by $T^*\mathbb{S}^2$. Then, we define a linear operator

$$\begin{aligned} L_h : T^*\mathbb{S}^2 &\to \mathrm{Sym}(T^*\mathbb{S}^2 \otimes T^*\mathbb{S}^2) \\ \xi &\mapsto \mathrm{Sym}(\nabla\xi) - \frac{\mathrm{tr}_h(\nabla\xi)}{2} h, \end{aligned} \tag{3.4}$$

where tr_h means taking a trace with respect to the positive definite tensor h, i.e., for any $(0,2)$ tensor a, we have $\mathrm{tr}_h(a) = h^{ij}a_{ij}$, where (h^{ij}) is the inverse matrix of (h_{ij}). Thus, (3.3) becomes

$$L_h(\xi) = q - \frac{\mathrm{tr}_h(q)}{2} h, \tag{3.5}$$

because ϕ can be represented by

$$\phi = \frac{\mathrm{tr}_h(\nabla\xi - q)}{2}.$$

It is easy to check that the operator L_h is a strong elliptic operator defined on the closed manifold $\mathbb{S}^2$. As the strong elliptic operator is a Fredholm operator, we know that

$$\mathrm{coker}(L_h) = \ker(L_h^*),$$

where L_h^* is the adjoint operator of L_h. Thus the solvability of the linearized problem is equivalent to show that the kernel of L_h^* is zero. Let's present the framework to clarify our notions. At first we define an inner product space $\mathcal{H}$

$$\mathcal{H} := \left\{ a \in C^\infty\Big(\mathbb{S}^2, \mathrm{Sym}\left(T^*\mathbb{S}^2 \otimes T^*\mathbb{S}^2\right)\Big); \mathrm{tr}_h(a) = 0 \right\}$$

equipped with the inner product

$$(a,b) = \int_{\mathbb{S}^2} K h^{ij} h^{mn} a_{im} b_{jn} dV_g$$

for $a, b \in \mathcal{H}$, where $K = \frac{\det(h)}{\det(g)}$, $\det(g)$, and $\det(h)$ are the determinants of the first and second fundamental forms, respectively, and dV_g is the volume form with respect to the metric g. Thus K is the σ_2 curvature of the surface M in the ambient space N. We also define the inner product in $C^\infty\left(\mathbb{S}^2, T^*\mathbb{S}^2\right)$,

$$(\xi, \eta) = \int_{\mathbb{S}^2} h^{ij} u_i \eta_j dV_g \tag{3.6}$$

for $\xi = u_i dx^i, \eta = \eta_i dx^i \in C^\infty(T^*\mathbb{S}^2)$. Here, $C^\infty(\cdot)$ means the set of smooth sections of the corresponding vector bundle.

According to the definition of adjoint operator, L_h^* is defined by

$$(\xi, L_h^*(a)) = (L_h(\xi), a), \text{ for any } \xi \in C^\infty\left(\mathbb{S}^2, T^*\mathbb{S}^2\right), a \in \mathcal{H}.$$

Then, we have

$$\begin{aligned} & (L_h(\xi), a) \\ = & \int_{\mathbb{S}^2} K h^{ij} h^{mn} u_{i,m} a_{jn} dV_g + \frac{1}{2} \int_{\mathbb{S}^2} K \mathrm{tr}_h(\nabla \xi) h^{ij} h^{mn} h_{im} a_{jn} dV_g \\ = & -\int_{\mathbb{S}^2} (K h^{ij} h^{mn} a_{jn})_{,m} u_i dV_g, \end{aligned}$$

where we have used $h^{ij} a_{ij} = 0$. Thus, the adjoint operator is

$$(L_h^*(a))_k = -(K h^{ij} h^{mn} a_{jn})_{,m} h_{ik}.$$

For any $a \in \mathcal{H}$, $L_h^*(a) = 0$ implies

$$(A_{ij} h^{mn} a_{jn})_{,m} = 0,$$

where A_{ij} is the cofactor of h_{ij} with respect to the matrix (h_{ij}).

For $i = 1$, the preceding equation can be written as

$$\begin{aligned} 0 &= (A_{11} h^{1n} a_{1n} + A_{12} h^{1n} a_{2n})_{,1} + (A_{11} h^{2n} a_{1n} + A_{12} h^{2n} a_{2n})_{,2} \\ &= (-A_{11} h^{2n} a_{2n} + A_{12} h^{1n} a_{2n})_{,1} + (A_{11} h^{2n} a_{1n} - A_{12} h^{1n} a_{1n})_{,2} \\ &= (-h_{22} h^{2n} a_{2n} - h_{12} h^{1n} a_{2n})_{,1} + (h_{22} h^{2n} a_{1n} + h_{12} h^{1n} a_{1n})_{,2} \\ &= -(\delta_{2n} a_{2n})_{,1} + (\delta_{2n} a_{1n})_{,2} \\ &= a_{21,2} - a_{22,1}, \end{aligned}$$

where we have used $h^{ij} a_{ij} = 0$ in the second equality and δ_{ij} is Kronecker's symbol. A similar argument yields

$$a_{11,2} = a_{12,1}.$$

Thus, the tensor a satisfies a homogenous linearized system

$$\begin{cases} h^{ij} a_{ij} = 0 \\ a_{ij,k} = a_{ik,j} \end{cases}. \tag{3.7}$$

In the following, we will prove that (3.7) has only trivial solutions, thereby implying that $\ker(L_h^*) = 0$.

A standard procedure is applied to define the cross product "$\times$" on every tangent vector space of N induced by its inner product "$\cdot$" if N is 3-dimensional. Once we fix an orientation of a given tangent space, for any two tangent vectors $\vec{a}, \vec{b}$, the cross product of $\vec{a}, \vec{b}$ is a unique tangent vector that satisfies the following two properties:

$$\vec{a} \times \vec{b} \cdot \vec{a} = \vec{a} \times \vec{b} \cdot \vec{b} = 0; |\vec{a} \times \vec{b}|^2 = (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2, \tag{3.8}$$

which is a bilinear product for its two variables. It is easy to check that for any three tangent vectors $\vec{a}, \vec{b}, \vec{c}$, we have $\vec{a} \times \vec{b} \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$.

Using the cross product, we define a new $(1,1)$ tensor related to the homogenous linearized system (3.7). We let

$$A_k = g^{ij} a_{ki} \nu \times \vec{r}_j,$$

where ν is the unit normal vector field of the surface M being parallel to $\vec{r}_1 \times \vec{r}_2$. By (3.8), A_k is a tangent vector field. Then, $A_k \otimes dx^k$ is a well defined $(1,1)$ tensor on the sphere $\mathbb{S}^2$. More precisely, using (3.8) and the triple scalar product formula, we obtain

$$\begin{cases} A_1 &= \frac{1}{\sqrt{\det(g)}} (-a_{12}\vec{r}_1 + a_{11}\vec{r}_2) \\ A_2 &= \frac{1}{\sqrt{\det(g)}} (-a_{22}\vec{r}_1 + a_{21}\vec{r}_2) \end{cases}. \tag{3.9}$$

Lemma 7. *For any vector field E in the ambient space N satisfying*

$$d\vec{r} \cdot DE = 0 \tag{3.10}$$

on the given surface M, where M is the image of the embedding map $\vec{r} : \mathbb{S}^2 \to N$, we can define a 1-form on $\mathbb{S}^2$

$$\omega = A_k \cdot E dx^k.$$

Then, ω is a closed form, and is zero.

Proof. Obviously ω is a one form. Then, the differential of ω is

$$\begin{aligned} d\omega &= \partial_j (A_k \cdot E) dx^j \wedge dx^k \\ &= (D_j A_k \cdot E + A_k \cdot D_j E)\, dx^j \wedge dx^k \\ &= \Big[(D_1 A_2 - D_2 A_1) \cdot E + (A_2 \cdot D_1 E - A_1 \cdot D_2 E) \Big] dx^1 \wedge dx^2. \end{aligned} \tag{3.11}$$

By (3.10), we have

$$\vec{r}_1 \cdot D_1 E = 0; \vec{r}_2 \cdot D_2 E = 0; \vec{r}_1 \cdot D_2 E + \vec{r}_2 \cdot D_1 E = 0.$$

Thus, by (3.9), we obtain

$$A_2 \cdot D_1 E - A_1 \cdot D_2 E = \frac{a_{21}}{\sqrt{\det(g)}} \vec{r}_2 \cdot D_1 E + \frac{a_{12}}{\sqrt{\det(g)}} \vec{r}_1 \cdot D_2 E = 0.$$

For convenience, we write

$$A_k = w_k^l \vec{r}_l,$$

where w_k^l are coefficients of the $(1,1)$ tensor Ω, and the relation between a_{ij} and w_k^l is given in (3.9). In detail,

$$w_1^1 = -\frac{1}{\sqrt{\det(g)}}a_{12}, w_1^2 = \frac{1}{\sqrt{\det(g)}}a_{11}, \tag{3.12}$$
$$w_2^1 = -\frac{1}{\sqrt{\det(g)}}a_{22}, w_2^2 = \frac{1}{\sqrt{\det(g)}}a_{21}.$$

Then, we have

$$D_i A_k = \left(w_{k,i}^l + \Gamma_{ik}^m w_m^l\right)\vec{r}_l + w_k^l D_i^\perp \vec{r}_l = \left(w_{k,i}^l + \Gamma_{ik}^m w_m^l\right)\vec{r}_l + w_k^l h_{il}\nu,$$

where $w_{k,i}^l$ is the covariant derivative of w_k^l with respect to $\frac{\partial}{\partial x^i}$. Since $\Gamma_{ik}^m = \Gamma_{ki}^m$, by (3.7) and (3.12), we have

$$\begin{aligned}
& D_1A_2 - D_2A_1 \\
= & -\left(w_{1,2}^l - w_{2,1}^l\right)\vec{r}_l + \left(h_{1l}w_2^l - h_{2l}w_1^l\right)\nu \\
= & \frac{-1}{\sqrt{\det(g)}}((a_{22,1} - a_{12,2})\vec{r}_1 + (a_{11,2} - a_{21,1})\vec{r}_2 \\
& \qquad +(h_{11}a_{22} - h_{12}a_{21} - h_{21}a_{12} + h_{22}a_{11})\nu) \\
= & \frac{-1}{\sqrt{\det(g)}}((a_{22,1} - a_{12,2})\vec{r}_1 + (a_{11,2} - a_{21,1})\vec{r}_2 + \det(h)h^{ij}a_{ij}\nu) \\
= & 0.
\end{aligned}$$

Thus, ω is a closed one form.

Since the first de Rham cohomology of the sphere is trivial, $H_{DR}^1(\mathbb{S}^2) = 0$, there exists a smooth function f on the sphere such that

$$\omega = df = f_k dx^k.$$

Therefore, we have

$$f_k = A_k \cdot E = w_k^l \vec{r}_l \cdot E. \tag{3.13}$$

Similarly, we have

$$\begin{aligned}
\nabla_j\omega &= \partial_j(A_k \cdot E)dx^k + A_k \cdot E\nabla_j dx^k \\
&= (D_jA_k \cdot E + A_k \cdot D_jE)dx^k - \Gamma_{jl}^k A_k \cdot E dx^l \\
&= \left[\left(w_{k,j}^l + \Gamma_{jk}^m w_m^l\right)\vec{r}_l \cdot E + w_k^l h_{jl}\nu \cdot E + w_k^l \vec{r}_l \cdot D_jE\right]dx^k \\
&\quad - \Gamma_{jl}^k w_k^m \vec{r}_m \cdot E dx^l \\
&= \left[w_{k,j}^l \vec{r}_l \cdot E + w_k^l h_{jl}\nu \cdot E + w_k^l \vec{r}_l \cdot D_jE\right]dx^k.
\end{aligned} \tag{3.14}$$

Thus, we obtain

$$f_{k,j} = w_{k,j}^l \vec{r}_l \cdot E + w_k^l h_{jl}\nu \cdot E + w_k^l \vec{r}_l \cdot D_jE.$$

Furthermore, we obtain

(3.15)

$$\begin{aligned} h^{kj}f_{k,j} &= h^{kj}w^l_{k,j}\vec{r}_l\cdot E + w^k_k\nu\cdot E + h^{kj}w^l_k\vec{r}_l\cdot D_jE \\ &= h^{kj}w^l_{k,j}\vec{r}_l\cdot E + \left(\frac{-a_{12}}{\sqrt{\det(g)}} + \frac{a_{21}}{\sqrt{\det(g)}}\right)\nu\cdot E \\ &\quad + h^{1k}w^2_k\vec{r}_2\cdot D_1E + h^{2k}w^1_k\vec{r}_1\cdot D_2E \\ &= h^{kj}w^l_{k,j}\vec{r}_l\cdot E + \left(h^{1k}w^2_k - h^{2k}w^1_k\right)\vec{r}_2\cdot D_1E \\ &= h^{kj}w^l_{k,j}\vec{r}_l\cdot E + \frac{h^{11}a_{11}+h^{12}a_{21}+h^{21}a_{12}+h^{22}a_{22}}{\sqrt{\det(g)}}\vec{r}_2\cdot D_1E \\ &= h^{kj}w^l_{k,j}\vec{r}_l\cdot E. \end{aligned}$$

For $l=1$, we also have

$$\begin{aligned} h^{ij}w^1_{i,j} &= h^{11}w^1_{1,1}+h^{12}w^1_{1,2}+h^{21}w^1_{2,1}+h^{22}w^1_{2,2} \\ &= \frac{1}{\sqrt{\det(g)}}\Big(-h^{11}a_{12,1}-h^{12}a_{12,2}-h^{21}a_{22,1}-h^{22}a_{22,2}\Big) \\ &= -\frac{1}{\sqrt{\det(g)}}\Big(h^{11}a_{11,2}+h^{12}a_{12,2}+h^{21}a_{21,2}+h^{22}a_{22,2}\Big) \\ &= -\frac{1}{\sqrt{\det(g)}}h^{ij}a_{ij,2} \\ &= \frac{1}{\sqrt{\det(g)}}h^{ij}_{,2}a_{ij}. \end{aligned}$$

Similarly, we have

$$h^{ij}w^2_{i,j} = \frac{-1}{\sqrt{\det(g)}}h^{ij}_{,1}a_{ij}.$$

Using the preceding three formulae, we obtain

$$h^{kj}f_{k,j} = \frac{1}{\sqrt{\det(g)}}h^{ij}_{,2}a_{ij}\vec{r}_1\cdot E + \frac{1}{\sqrt{\det(g)}}h^{ij}_{,1}a_{ij}\vec{r}_2\cdot E. \tag{3.16}$$

By Lemma 4 in [19], we obtain

$$\left(w^1_1\right)^2+\left(w^1_2\right)^2+\left(w^2_1\right)^2+\left(w^2_2\right)^2 \le -C\det w, \tag{3.17}$$

where C is a constant only depending on the given surface M. We denote the set $\{P\in M;\det w(P)\neq 0\}$ by U, which is an open subset of M. We conclude that, in U

$$\vec{r}_l\cdot E = \frac{B^m_l f_m}{\det w},$$

where B_l^m is the cofactor of w_l^m. Thus, in U, we have a differential equation satisfied by the function f,

$$h^{kj} f_{k,j} = h^{kj}_{,2} a_{kj} \frac{B_1^m f_m}{\sqrt{\det(g)}\det w} - h^{kj}_{,1} a_{kj} \frac{B_2^m f_m}{\sqrt{\det(g)}\det w}. \tag{3.18}$$

On the other hand, in the set $M \setminus U$, since $\det w = 0$, by (3.17), we have $w_i^j = 0$, which implies $a_{ij} = 0$ for $1 \le i, j \le 2$. Using (3.16), we obtain, in $M \setminus U$

$$h^{kj} f_{k,j} = 0. \tag{3.19}$$

We let

$$A_1^m = \begin{cases} h^{kj}_{,2} a_{kj} \dfrac{B_1^m}{\sqrt{\det(g)}\det w} & \text{if } \det w \neq 0 \\ 0 & \text{if } \det w = 0 \end{cases},$$

$$A_2^m = \begin{cases} -h^{kj}_{,1} a_{kj} \dfrac{B_2^m}{\sqrt{\det(g)}\det w} & \text{if } \det w \neq 0 \\ 0 & \text{if } \det w = 0 \end{cases}.$$

Using the preceding notions and combining (3.18) with (3.19), we have

$$h^{kj} f_{k,j} = A_1^m f_m + A_2^m f_m.$$

Since the coefficients a_{kj} and B_l^m are all linear combinations of w_l^m, the coefficients of the preceding equation are bounded in view of (3.17). The strong maximum principle holds for bounded coefficients of lower order terms, Chapter 3, [**15**]. It implies that f is a constant function on the sphere, which means that

$$\omega = df = 0.$$

q.e.d.

In view of the previous Lemma, we need to find enough special solutions of (3.10).

Lemma 8. *Suppose the coordinate of the Euclidean space* $\mathbb{R}^3$ *is* $\{z^1, z^2, z^3\}$. *Let*

$$I_\alpha = \frac{\partial}{\partial z^\alpha} \times_E \vec{r},$$

where $\times_E$ *is the cross product defined by the Euclidean metric, then every* I_α *satisfies* (3.10) *for* $\alpha = 1, 2, 3$, *i.e.,* $d\vec{r} \cdot DI_\alpha = 0$. *We also have*

$$I_\alpha = \frac{\partial}{\partial z^\alpha} \times X,$$

where X *is the conformal Killing vector of* N *and* $\times$ *is the cross product defined by the metric of the ambient space* N.

Proof. The relationship between the isometric system and its linearized system tells that if we can construct a 1-parameter family of embeddings $\vec{r}_t$ satisfying

$$d\vec{r}_t \cdot d\vec{r}_t = d\vec{r} \cdot d\vec{r},$$

where $\vec{r}$ is the given embedding, then $E = \frac{d\vec{r}_t}{dt}\Big|_{t=0}$ satisfies (3.10). As any 3-dimensional warped product metric is rotationally symmetric, the 3-dimensional orthogonal group $O(3)$ is a subgroup of its isometry group. It is well known that the Lie algebra $o(3)$ of $O(3)$ is the collection of all 3×3 anti-symmetric matrices and I_1, I_2, I_3 can compose a basis of $o(3)$. Thus, for every $I_\alpha, \alpha = 1, 2, 3$, we can always find a 1-parameter family of orthogonal matrices A_t^α, which is a path generated by I_α at the identity matrix I in $O(3)$ using the exponential map. Therefore, we let

$$\vec{r}_t = A_t^\alpha \vec{r}$$

be a 1-parameter family of surfaces in N, and then we have

$$d\vec{r}_t \cdot d\vec{r}_t = A_t^\alpha d\vec{r} \cdot A_t^\alpha d\vec{r} = d\vec{r} \cdot d\vec{r},$$

which implies the first statement.

For the second result, we claim that, in fact, for any tangent vector field $\vec{a}$ of N, we always have

$$\vec{a} \times_E \vec{r} = \vec{a} \times X,$$

where X is the conformal Killing vector defined in (2.8). Suppose the 3-dimensional polar coordinate is (r, u^1, u^2), then we write

$$\vec{a} = \mu \frac{\partial}{\partial r} + \lambda^1 \frac{\partial}{\partial u^1} + \lambda^2 \frac{\partial}{\partial u^2}, \text{ and } \vec{r} = r \frac{\partial}{\partial r}, \tag{3.20}$$

where $\mu, \lambda^1, \lambda^2, r$ are scalar components. The definition of the cross product involves the orientation of the vector space. As the underlying manifold of the 3-dimensional Euclidean space and the 3-dimensional warped product space is the same, we can fix a same orientation for every tangent vector space. Thus, we have

$$\vec{a} \times_E \vec{r} = r \left(\frac{\lambda^1}{\cos u^1} \frac{\partial}{\partial u^2} - \lambda^2 \cos u^1 \frac{\partial}{\partial u^1} \right),$$

which implies

$$\vec{a} \times_E \vec{r} \cdot \vec{a} = \vec{a} \times_E \vec{r} \cdot X = 0.$$

Therefore, $\vec{a} \times_E \vec{r}$ is parallel to $\vec{a} \times X$. We only need to compare their norms,

$$|\vec{a} \times X|^2 = |\vec{a}|^2 |X|^2 - (\vec{a} \cdot X)^2 = r^2 \left((\lambda^2)^2 \cos^2 u^1 + (\lambda^1)^2 \right) = |\vec{a} \times_E \vec{r}|^2.$$

The claim is verified. q.e.d.

We define a set of zero points

$$m_0 = \left\{p \in \mathbb{S}^2; \vec{r}(p) = 0\right\}.$$

As M is a regular surface, m_0 is a finite set by the compactness of the sphere. Furthermore, we have the following fact:

Proposition 9. *The set* $Z = \left\{p \in \mathbb{S}^2 \setminus m_0; \varphi(p) = X \cdot \nu(p) = 0\right\}$ *is a regular curve on the sphere, if the surface* M *is strictly convex.*

Proof. It suffices to check that on Z, $d\varphi \neq 0$. X is the conformal Killing vector field, we have

$$\partial_i \varphi = X \cdot D_i \nu. \text{ and } D_i \nu = -g^{kl} h_{ik} \vec{r}_l.$$

Since $X \cdot \nu = 0$ on Z, we can assume that $X = a^i \vec{r}_i$ for any point $p \in Z$. Then, if $d\varphi = 0$, by the Gauss-Weingarten formulae, we have

$$\partial_i \varphi = -a^k h_{ik} = 0,$$

which implies that $a^k = 0$, namely, $X = 0$. Thus, we know that $\vec{r}(p) = 0$, which contradicts $p \notin m_0$. q.e.d.

We are in a position to prove our main theorems in this section.

Theorem 10. *Suppose that* $(\mathbb{S}^2, g)$ *can be isometrically embedded into a* 3*-dimensional warped product space as a closed strictly convex surface* M *with the embedding map* $\vec{r}$*. For any given* $(0,2)$ *symmetric tensor* q*, there exists a vector field* τ *satisfying the following system:*

$$d\vec{r} \cdot D\tau = q.$$

Proof. We first assume that N is a warped product space. From the previous discussion, it suffices to prove that $\ker(L_h^*) = 0$, which is equivalent to $A_k \otimes dx^k = 0$. By Lemma 7 and 8, we obtain for $\alpha = 1, 2, 3$,

$$A_k \otimes dx^k \cdot I_\alpha = 0, \text{ and } A_k \cdot \nu dx^k = 0.$$

By (3.9), each A_k is a tangent vector field, which implies the above second equality. For any three tangent vector fields a, b, c, we always have

$$(a \times b) \times c = b(a \cdot c) - a(b \cdot c).$$

Thus, for $\alpha \neq \beta$ and $\alpha, \beta = 1, 2, 3$, we have

$$\begin{aligned} I_\alpha \times I_\beta \cdot \nu &= \left(\frac{\partial}{\partial z^\alpha} \times X\right) \times \left(\frac{\partial}{\partial z^\beta} \times X\right) \cdot \nu \\ &= \left(\frac{\partial}{\partial z^\alpha} \times \frac{\partial}{\partial z^\beta} \cdot X\right) \varphi. \end{aligned} \tag{3.21}$$

At a point of M where $X \neq 0$, we can always find two different indices $\alpha_0 \neq \beta_0$ such that $\dfrac{\partial}{\partial z^{\alpha_0}}, \dfrac{\partial}{\partial z^{\beta_0}}$ and X are linearly independent. As the function φ is non-zero on $\mathbb{S}^2 \setminus (Z \cup m_0)$, (3.21) means that $I_{\alpha_0}, I_{\beta_0}$ and ν are linearly independent, which implies that $A_k \otimes dx^k = 0$. Note that

$Z \cup m_0$ is a closed set without interior points. Thus, $A_k \otimes dx^k = 0$ is zero on the entire sphere. We complete the proof for any warped product space. q.e.d.

We now deal with general N. The following theorem and corollary are the only two places in this paper where the assumption of the ambient space is a general Riemannian manifold and not only a warped product space.

Theorem 11. *Suppose that $(\mathbb{S}^2, g)$ can be isometrically embedded into a 3-dimensional Riemannian manifold as a closed strictly convex surface M with the embedding map $\mathcal{F}$. For any given $(0,2)$ symmetric tensor q, there exists a vector field τ satisfying the following system:*

$$d\mathcal{F} \cdot D\tau = q,$$

where $d\mathcal{F}$ means the differential of the map $\mathcal{F}$.

Proof. In the following notations, we still use N to denote the ambient Riemannian manifold. The Riemannian metric is denoted by "$\cdot$" and D is its corresponding Levi-Civita connection. Thus, for a given metric g on $\mathbb{S}^2$, the isometric embedding system is to find an embedding map $\mathcal{F} : \mathbb{S}^2 \to N$ that satisfies

$$d\mathcal{F} \cdot d\mathcal{F} = g,$$

where $d\mathcal{F}$ represents the differential of the map $\mathcal{F}$. We denote the image of $\mathcal{F}$ by M. Then, we derive the linearized problem of the isometric embedding system in a general Riemannian manifold. Suppose that we have a 1-parameter family of metrics g^t on the 2-dimensional sphere $\mathbb{S}^2$ and every $(\mathbb{S}^2, g^t)$ can be isometrically embedded into N by the map $\mathcal{F}(t)$. Let $\mathcal{F}(0) = \mathcal{F}$ and $\tau = \frac{\partial \mathcal{F}_t}{\partial t}\big|_{t=0}$ be the variation vector field. We let $\{x^1, x^2\}$ be a local coordinate of the sphere. We have

$$\begin{aligned} (3.22)\quad & \frac{\partial}{\partial t} g^t_{ij} \\ = \; & \frac{\partial}{\partial t} g^t \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = g^t \left(D_{\frac{\partial}{\partial t}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) + g^t \left(\frac{\partial}{\partial x^i}, D_{\frac{\partial}{\partial t}} \frac{\partial}{\partial x^j} \right) \\ = \; & g^t \left(D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial t}, \frac{\partial}{\partial x^j} \right) + g^t \left(\frac{\partial}{\partial x^i}, D_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial t} \right). \end{aligned}$$

Thus, the preceding equation can be rewritten as

$$d\mathcal{F} \cdot D\tau = \frac{1}{2} \frac{\partial}{\partial t} g^t \Big|_{t=0},$$

which is the same as the linear system (3.1) derived in the warped product space. As the argument of this section before Lemma 8 does not substantially use the warped product structure, we can extend everything in any Riemannian manifolds. Thus, we also have the definition

(3.4) of the linear operator L_h in a Riemannian manifold, where h is the second fundamental form of M.

Let us consider the index of the operator L_h. Since L_h is an elliptic operator defined on the cotangent bundle of the sphere with respect to the surface M, it is a Fredholm operator. Its index is defined by

$$\operatorname{ind}(L_h) = \dim \ker L_h - \dim \operatorname{coker} L_h. \tag{3.23}$$

The index of an elliptic operator only depends on its principal symbol. In the following, we calculate the index of L_h using the homotopic invariance of the index.

We quote Theorem 19.2.2 in the work of Hörmander [24]: Let I be the interval $[0,1]$ on $\mathbb{R}$, and let $I \ni t \mapsto a(t) \in C^\infty(T^*(X); \operatorname{Hom}(\pi^*E, \pi^*F))$ and $I \ni t \mapsto b(t) \in C^\infty(T^*(X); \operatorname{Hom}(\pi^*F, \pi^*E))$ be continuous maps such that $a(t)$ is uniformly bounded in S^m, $b(t)$ is uniformly bounded in S^{-m} and $a(t)b(t) - I$ is uniformly bounded in $S^{-1}(T^*(X); \operatorname{Hom}(\pi^*F, \pi^*E))$ while $b(t)a(t) - I$ is uniformly bounded in $S^{-1}(T^*(X); \operatorname{Hom}(\pi^*E, \pi^*F))$. If $A_0, A_1 \in \Psi^m\left(X; E \otimes \Omega^{\frac{1}{2}}, F \otimes \Omega^{\frac{1}{2}}\right)$ have principal symbols $a(0)$ and $a(1)$, respectively, then it follows that $\operatorname{ind} A_0 = \operatorname{ind} A_1$.

We further explain the above Theorem. X is a compact manifold and E, F are two complex vector bundles on X with the same fiber dimension. $T^*(X)$ is the cotangent bundle of X and the map $\pi : T^*X \to X$ is the canonical projection. Ω is a density line bundle on X. $\Psi^m\left(X; E \otimes \Omega^{\frac{1}{2}}, F \otimes \Omega^{\frac{1}{2}}\right)$ means the set of order m linear elliptic (pseudo) differential operators mapping the sections of $E \otimes \Omega^{\frac{1}{2}}$ to the sections of $F \otimes \Omega^{\frac{1}{2}}$ with coefficients defined on X. If the dimension of X is n, then T^*X can be viewed as $\mathbb{R}^n \times \mathbb{R}^n$ in a local coordinate. Thus, for a real number s, the definition of a section $a \in S^s$ is as follows: In a local coordinate, we can write $a = a(x, \xi)$ where $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$. For all multiple indices α, β, the derivative $a^\alpha_\beta = \partial^\alpha_\xi \partial^\beta_x a(x, \xi)$ has the bound

$$\left|a^\alpha_\beta(x, \xi)\right| \le C_{\alpha,\beta}(1 + |\xi|)^{s-|\alpha|}; \quad x, \xi \in \mathbb{R}^n,$$

where $C_{\alpha,\beta}$ is a positive constant depending on the underlying Riemannian manifold.

Being specific to our case, the underlying manifold X is the 2-dimensional sphere $\mathbb{S}^2$ and the bundle E is the complexification of the cotangent bundle $T^*\mathbb{S}^2$. The bundle F is the complexification of the following bundle:

$$F'_h = \left\{a \in \operatorname{Sym}\left(T^*\mathbb{S}^2 \otimes T^*\mathbb{S}^2\right); \operatorname{tr}_h a = 0\right\}.$$

Now we need to point out two facts. One is that the complexification is not necessary because we can extend our operator to the complexification of our bundles and the complex dimensions of the kernel and cokernel of the complexification are the same as the real dimensions of the original real bundles. Another is that we do not need to consider

the density line bundle in our present case. In the literature, the destiny line bundle is mainly used to define the adjoint operator. In our case, however, the underlying manifold is a Riemannian manifold and the elliptic operator L_h is only composed of the covariant derivatives. Thus, we can use the volume form to define the density line bundle: Suppose we have two different local coordinates for a certain point, $\{x^1, x^2\}$ and $\{y^1, y^2\}$. The metric can be represented by g_{ij} and $\tilde{g}_{ij}$ in the two coordinates. The transition functions of the half density line bundle can be defined by

$$t_{xy} = \frac{(\det g_{ij})^{1/4}}{(\det \tilde{g}_{ij})^{1/4}}.$$

The section of the density line bundle $\left\{(\det g_{ij})^{1/4}\right\}$ will be not involved in the calculation of the dual operator, i.e., no contribution for the integration by part. In the following, we use Hörmander's theorem without the complexification and density bundle.

Let's calculate the principal symbol of L_h. Suppose that $\eta = \eta_i dx^i$ is a unit cotangent vector field (i.e., 1-form), $\xi = u_i dx^i$ is another cotangent vector field and q is a symmetric $(0,2)$ tensor field with $\mathrm{tr}_h q = 0$. Then, the principal symbol of L_h defined by h is a linear operator determined by

$$\begin{aligned} q_{11} &= u_1\eta_1 - \zeta h_{11}, \\ 2q_{12} &= 2q_{21} = u_1\eta_2 + u_2\eta_1 - 2\zeta h_{12}, \\ q_{22} &= u_2\eta_2 - \zeta h_{22}, \end{aligned} \tag{3.24}$$

where $2\zeta = h^{ij}u_i\eta_j$.

The previous calculation shows that the principal symbol only depends on the second fundamental form h. Thus, if we need a homotopy path of the principal symbols, we can consider a 1-parameter family of $(0,2)$ positive definite tensors. Since the sphere can be presented as some standard unit sphere in the 3-dimensional Euclidean space $\mathbb{R}^3$, the standard Euclidean metric g_0 on the standard unit sphere can be viewed as a positive definite section in the bundle $\mathrm{Sym}(T^*\mathbb{S}^2 \otimes T^*\mathbb{S}^2)$. Thus, we can define the following path:

$$h^t = (1-t)h + tg_0, \text{ and } g^t = (1-t)g + tg_0,$$

where $0 \leq t \leq 1$. As g^t, h^t are positive definite, we can define a 1-parameter family of elliptic operators L_t using them

$$L_t\xi = \mathrm{Sym}\left(\nabla^{g^t}\xi\right) - \frac{\mathrm{tr}_{h^t}\left(\nabla^{g^t}\xi\right)}{2}h^t, \text{ for } \xi \in T^*\mathbb{S}^2,$$

where ∇^{g^t} is the Levi-Civita connection with respect to g^t. The image of the map L_t is a rank 2 bundle $F'_{h^t} = \left\{a \in \mathrm{Sym}(T^*\mathbb{S}^2 \otimes T^*\mathbb{S}^2); \mathrm{tr}_{h^t} a = 0\right\}$.

In fact F'_{h^t} is bundle isomorphic to $F'_h = F'_{h^0}$. Let's calculate the bundle isomorphism in a local coordinate $\{x^1, x^2\}$. For any $b^t \in F'_{h^t}$, it means $(h^t)^{ij} b^t_{ij} = 0$. We let section $b \in \mathrm{Sym}(T^*\mathbb{S}^2 \otimes T^*\mathbb{S}^2)$ satisfy $b^t_{ij} = h^t_{im} h^{mn} b_{nj}$ which implies that $h^{ij} b_{ij} = 0$. Therefore, the bundle isomorphism φ^t from F'_h to F'_{h^t} is determined by the matrix $\left(h^t_{im} h^{mn}\right)$ in the local coordinate.

The operators $\left(\varphi^t\right)^{-1} L_t$ have the same image F'_h. Again, let us suppose that ξ is a unit cotangent vector field, u is a cotangent vector field, and q is symmetric $(0,2)$ tensor with $\mathrm{tr}_h q = 0$. We let $q^t = \varphi^t q$. Thus, the principal symbol $a(t)$ of the operators $\left(\varphi^t\right)^{-1} L_t$ is determined by

$$
\begin{aligned}
q^t_{11} &= u_1\eta_1 - \zeta h^t_{11}, \\
2q^t_{12} &= 2q^t_{21} = u_1\eta_2 + u_2\eta_1 - 2\zeta h^t_{12}, \\
q^t_{22} &= u_2\eta_2 - \zeta h^t_{22},
\end{aligned} \tag{3.25}
$$

where $2\zeta = (h^t)^{ij} u_i \eta_j$. The inverse symbol $b(t)$ of $a(t)$ is determined by

$$
\begin{aligned}
u_1 &= \frac{1}{\chi^t}\left[(q^t_{11}h^t_{22} - q^t_{22}h^t_{11})\eta_1 - 2(h^t_{12}q^t_{11} - h^t_{11}q^t_{12})\eta_2\right], \\
u_2 &= \frac{1}{\chi^t}\left[(q^t_{22}h^t_{11} - q^t_{11}h^t_{22})\eta_2 - 2(h^t_{12}q^t_{22} - h^t_{22}q^t_{12})\eta_1\right], \\
\chi^t &= h^t_{22}\eta_1^2 + h^t_{11}\eta_2^2 - 2h^t_{12}\eta_1\eta_2.
\end{aligned} \tag{3.26}
$$

Obviously, since h^t, g^t are uniformly bounded, we have $a(t) \in S^1, b(t) \in S^{-1}$, $a(t)b(t) - I, b(t)a(t) - I \in S^{-1}$ and all of these bounds are uniform, i.e., not depending on t, where the explicit meaning of the notation S^1, S^{-1} can be found in [**24**]. Now, using the invariance of the index along a homotopic path, e.g., Theorem 19.2.2 [**24**], the index of the operator L_h is the same as $(\varphi^1)^{-1} L_1$. It is easy to see that the dimensions of the kernel and cokernel of the operator $(\varphi^1)^{-1} L_1$ are the same as the dimensions of the kernel and cokernel of L_1. The operator L_1 is defined by the standard unit 2-dimensional sphere embedded in the 3-dimensional Euclidean space. Thus, by the infinitesimal rigidity in the Euclidean space, namely Cohn-Vossen Theorem, we have $\mathrm{ind}\,(L_1) = 6$ because the kernel has six dimensions consisting of three rotations and three translations and the cokernel is trivial. Thus, we have $\mathrm{ind}(L_h) = 6$.

We now prove our theorem. In view of (3.23) and $\mathrm{ind}(L_h) = 6$, the kernel of L_h is always nontrivial. Thus, we can find a 1-form ξ satisfying $L_h(\xi) = 0$, which is not identically zero. We claim that the set $S = \{p \in \mathbb{S}^2; \omega|_p = 0\}$ is a closed set with no interior point. In fact, suppose the set includes an open subset. As the equation $L_h(\xi) = 0$ can be viewed as an elliptic differential system of the first order on an open subset of $\mathbb{R}^2$ in a local coordinate, the unique continuation [**2**, **3**] states that $\xi = 0$ on the entire surface, which contradicts that ξ is not identically zero. Thus, we have proved our claim. Let τ be a vector

field that satisfies

$$d\mathcal{F}\cdot D\tau=0,\ \text{and}\ \xi=\tau\cdot d\mathcal{F}.$$

We again suppose that $\{x^1,x^2\}$ is a local coordinate. For any symmetric $(0,2)$ tensor a_{ij} that satisfies the homogenous linearized Gauss-Codazzi system (3.7), using (3.9) and Lemma 7, we have

$$\begin{bmatrix}-a_{12} & a_{11}\\ -a_{22} & a_{21}\end{bmatrix}\begin{pmatrix}\tau\cdot\mathcal{F}_1\\ \tau\cdot\mathcal{F}_2\end{pmatrix}=0. \tag{3.27}$$

Thus, the above algebraic system has nontrivial solution on $\mathbb{S}^2\backslash S$, which implies

$$\det\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}=\det\begin{bmatrix}-a_{12} & a_{11}\\ -a_{22} & a_{21}\end{bmatrix}=0$$

on the entire sphere. As $h^{ij}a_{ij}=0$ and a_{ij} is symmetric, we obtain $a_{ij}=0$. Thus, we have $\ker(L_h^*)=0$, which implies coker $(L_h)=0$. q.e.d.

Although we have two different proofs of the existence theorem for the linearized problem, the benefit of the first one is that it is a purely geometric argument and does not need the unique continuation.

For any 3-dimensional Riemannian manifold N, let TN denote its tangent bundle. An immediate corollary of the above theorem is

Corollary 12. *Suppose that $(\mathbb{S}^2,g)$ can be isometrically embedded into a 3-dimensional Riemannian manifold N as a closed strictly convex surface M by the embedding map $\mathcal{F}$. The dimension of the solution space of the linear homogeneous system*

$$d\mathcal{F}\cdot D\tau=0 \tag{3.28}$$

is always six, where τ is a smooth vector field of the pull-back vector bundle $\mathcal{F}^*TN$, $d\mathcal{F}$ is the differential of the map $\mathcal{F}$ and D is the Levi-Civita connection of N.

Proof. In view of the proof of the previous theorem, the cokernel of the operator L_h is trivial. Thus, we have ind $(L_h)=\dim\ker L_h=6$. q.e.d.

4. Openness and existence results

In this section, we prove the openness for the Weyl problem, namely, Theorem 1. We assume that $(\mathbb{S}^2,g)$ can be isometrically embedded into the ambient space N as a strictly convex surface M by the map $\vec{r}$. Following Nirenberg's approach, for any perturbation metric $\tilde{g}$ closed to the metric g on $\mathbb{S}^2$, we need to find a vector field $\vec{y}$ to satisfy

$$d(\vec{r}+\vec{y})\cdot d(\vec{r}+\vec{y})=\tilde{g}. \tag{4.1}$$

We let $\{x^1, x^2\}$ be a local coordinate of the sphere. As the underlying manifold of the warped product space N is $\mathbb{R}^3$, we have a global coordinate $\{z^1, z^2, z^3\}$ for N such that $\{\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2}, \frac{\partial}{\partial z^3}\}$ is the standard frame of $\mathbb{R}^3$. Note that every $\frac{\partial}{\partial z^\alpha}$ can be viewed as a global vector field in N. We present additional notations. Here, the Greek indices $\alpha, \beta, \gamma, \cdots$ range form 1 to 3 and the Latin indices $i, j, k, \cdots$ range from 1 to 2. The vector fields $\vec{r}$ and $\vec{y}$ can be expressed as linear combinations of $\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2}, \frac{\partial}{\partial z^3}$,

$$\vec{r} = r^\alpha \frac{\partial}{\partial z^\alpha} \text{ and } \vec{y} = y^\beta \frac{\partial}{\partial z^\beta},$$

where r^α, y^β are the scalar components. Let $(\sigma_{\alpha\beta})$ be the metric matrix of the warped product metric ds^2 defined by (1.3). Thus, at points $\vec{r}+\vec{y}$ and $\vec{r}$, $\sigma_{\alpha\beta}$ can be expressed respectively by

$$\sigma_{\alpha\beta}(\vec{r}+\vec{y})\frac{\partial(r^\alpha + y^\alpha)}{\partial x^i}\frac{\partial(r^\beta + y^\beta)}{\partial x^j} = \tilde{g}(\vec{r}+\vec{y})\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \tilde{g}_{ij}(\vec{r}+\vec{y})$$

and

$$\sigma_{\alpha\beta}(\vec{r})\frac{\partial r^\alpha}{\partial x^i}\frac{\partial r^\beta}{\partial x^j} = g(\vec{r})\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij}(\vec{r}).$$

We introduce the following three tensors,

$$\begin{aligned} F_{\alpha\beta\gamma\lambda}(\vec{r},\vec{y}) &= \int_0^1 (1-t)\frac{\partial^2}{\partial z^\gamma \partial z^\lambda}\sigma_{\alpha\beta}(\vec{r}+t\vec{y})dt, \\ G_{\alpha\beta\gamma}(\vec{r},\vec{y}) &= \int_0^1 \frac{\partial}{\partial z^\gamma}\sigma_{\alpha\beta}(\vec{r}+t\vec{y})dt, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} & q_{ij}(\vec{y}, \nabla\vec{y}) \\ = \; & \tilde{g}_{ij}(\vec{r}+\vec{y}) - g_{ij}(\vec{r}) - \sigma_{\alpha\beta}(\vec{r})\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} - F_{\alpha\beta\gamma\lambda}(\vec{r},\vec{y})\frac{\partial r^\alpha}{\partial x^i}\frac{\partial r^\beta}{\partial x^j}y^\gamma y^\lambda \\ & -G_{\alpha\beta\gamma}(\vec{r},\vec{y})y^\gamma\left(\frac{\partial r^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} + \frac{\partial y^\alpha}{\partial x^i}\frac{\partial r^\beta}{\partial x^j} + \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\right), \end{aligned} \tag{4.3}$$

where q is a symmetric $(0,2)$ tensor. Using these notations, we conclude that (4.1) can be rewritten as the following nonlinear system:

$$d\vec{r}\cdot D\vec{y} = q(\vec{y}, \nabla\vec{y}). \tag{4.4}$$

The detailed derivation can be found in [28]. Also note that

$$\begin{aligned} & \|F_{\alpha\beta\gamma\lambda}\|_{C^{m,\alpha}(\mathbb{S}^2, \vec{r}^*T^*N\otimes\vec{r}^*T^*N\otimes\vec{r}^*T^*N\otimes\vec{r}^*T^*N)} \\ & +\|G_{\alpha\beta\gamma}\|_{C^{m,\alpha}(\mathbb{S}^2, \vec{r}^*T^*N\otimes\vec{r}^*T^*N\otimes\vec{r}^*T^*N)} \\ \leq \; & C_{m,\alpha} + \bar{C}_{m,\alpha}\|\vec{y}\|^m_{C^{m,\alpha}(\mathbb{S}^2, \vec{r}^*TN)}, \end{aligned} \tag{4.5}$$

where $C_{m,\alpha}, \bar{C}_{m,\alpha}$ are two constants depending on m, α and M.

Now we define a map Φ on $\mathbb{S}^2$. As in section 3, TN denotes the tangent vector bundle of N. For any given vector field $\vec{z}$ of the pull-back bundle $\vec{r}^*TN$, suppose the image of Φ is the vector field $\vec{y}$ of $\vec{r}^*TN$, namely,

$$\begin{aligned}\Phi : C^{2,\alpha}\left(\mathbb{S}^2, \vec{r}^*TN\right) &\to C^{2,\alpha}\left(\mathbb{S}^2, \vec{r}^*TN\right),\\ \vec{z} &\mapsto \vec{y},\end{aligned}$$

for $0 < \alpha < 1$. We define the vector field $\vec{y}$ as follows. Consider the following system for the unknown vector field τ:

$$d\vec{r} \cdot D\tau = q(\vec{z}, \nabla\vec{z}), \tag{4.6}$$

where the tensor q is defined in (4.3). Theorem 10 states that the above system is always solvable if M is a strictly convex surface. Note that the kernel of the above system is nontrivial. We let the vector field $\vec{y}$ be the solution of the above system being perpendicular to its kernel, i.e., the solutions of the system

$$d\vec{r} \cdot D\tau = 0. \tag{4.7}$$

We explain the meaning of the word "perpendicular" here. Every vector field τ corresponds to a 1-form $\xi = \tau \cdot d\vec{r}$. Thus, any two vector fields perpendicular to each other means that the corresponding 1-forms are perpendicular to each other in the inner product space defined by (3.6). We will prove that $\vec{y}$ is a $C^{2,\alpha}$ vector field in the next several paragraphs.

Suppose $\{x^1, x^2\}$ is a local coordinate of the sphere. We first give the Schauder estimates for the linear elliptic system (3.3). We let $\xi = u_i dx^i$. In (3.3), multiplying h_{12} in the first equation and multiplying h_{11} in the second equation, and then, taking their difference, we have

$$2h_{12}u_{1,1} - h_{11}(u_{1,2} + u_{2,1}) = 2h_{12}q_{11} - 2h_{11}q_{12}. \tag{4.8}$$

Again in (3.3), multiplying h_{11} in the last equation and multiplying h_{11} in the first equation, and then, taking their difference, we have

$$h_{11}u_{2,2} - h_{22}u_{1,1} = h_{11}q_{22} - h_{22}q_{11}. \tag{4.9}$$

Covariant differentiating (4.8) with respect to $\frac{\partial}{\partial x^2}$ and (4.9) with respect to $\frac{\partial}{\partial x^1}$ respectively, and then summing them up, we obtain

$$\begin{aligned}&-h_{22}u_{1,11} + h_{12}u_{1,21} + h_{21}u_{1,12} - h_{11}u_{1,22} + \text{lower order terms}\\ &= (h_{11}q_{22} - h_{22}q_{11})_{,1} + (2h_{12}q_{11} - 2h_{11}q_{12})_{,2},\end{aligned} \tag{4.10}$$

where we have used the Ricci identities $R^l_{211}u_l = u_{1,21} - u_{1,12}$ and $R^k_{122}u_k = u_{2,12} - u_{2,21}$. Similarly, switching 1 and 2 in the above equation, we have

$$\begin{aligned}&-h_{22}u_{2,11} + h_{12}u_{2,21} + h_{21}u_{2,12} - h_{11}u_{2,22} + \text{lower order terms}\\ &= -(h_{11}q_{22} - h_{22}q_{11})_{,2} + (2h_{12}q_{22} - 2h_{22}q_{12})_{,1}.\end{aligned} \tag{4.11}$$

Thus, we obtain a linear elliptic partial differential system (4.10), (4.11) of second order. Using the interior Schauder estimates of elliptic systems [14], we have

$$\begin{aligned}(4.12)\qquad &\|\xi\|_{C^{2,\alpha}(\mathbb{S}^2,T^*\mathbb{S}^2)}\\ &\leq C_{2,\alpha}\left(\|q\|_{C^{1,\alpha}(\mathbb{S}^2,\mathrm{Sym}(T^*\mathbb{S}^2\otimes T^*\mathbb{S}^2))}+\|\xi\|_{C^{0,\alpha}(\mathbb{S}^2,T^*\mathbb{S}^2)}\right),\end{aligned}$$

where the positive constant $C_{2,\alpha}$ only depends on the upper and lower bounds of the principal curvatures κ_1,κ_2 of M and the upper bound of the derivatives of κ_1,κ_2. See Vekua [46] for another argument regarding (4.12).

If ξ is perpendicular to the kernel of the operator L_h, then ξ is the unique solution to the system (3.3) for a given q. Thus, we further have the following estimate by a compactness argument:

$$(4.13)\qquad \|\xi\|_{C^{2,\alpha}(\mathbb{S}^2,T^*\mathbb{S}^2)}\leq C\|q\|_{C^{1,\alpha}(\mathbb{S}^2,\mathrm{Sym}(T^*\mathbb{S}^2\otimes T^*\mathbb{S}^2))}.$$

Therefore, the tangential component of $\vec{y}$ is $C^{2,\alpha}$. The compactness argument is standard in the regularity theory of elliptic PDEs [14, 15], but for the sake of completeness, we include the detail here. If the estimate (4.13) does not holds, then we have a sequence $\{\xi^{(n)}\}_{n=1}^{\infty}$ which satisfies that every $\xi^{(n)}$ is perpendicular to the kernel of L_h in the sense of L^2 product defined by (3.6) and $q^{(n)}$, such that we have

$$\begin{aligned}\lim_{n\to\infty}\|\xi^{(n)}\|_{C^{2,\alpha}(\mathbb{S}^2,T^*\mathbb{S}^2)}&=+\infty,\\ \|q^{(n)}\|_{C^{1,\alpha}(\mathbb{S}^2,\mathrm{Sym}(T^*\mathbb{S}^2\otimes T^*\mathbb{S}^2))}&=1,\end{aligned}$$

and

$$L_h\left(\xi^{(n)}\right)=q^{(n)}-\frac{\mathrm{tr}_h(q^{(n)})}{2}h.$$

We denote

$$\tilde{\xi}^{(n)}=\frac{\xi^{(n)}}{\|\xi^{(n)}\|_{C^{2,\alpha}(\mathbb{S}^2,T^*\mathbb{S}^2)}}.$$

As L_h is a linear operator, we have

$$(4.14)\qquad L_h\left(\tilde{\xi}^{(n)}\right)=\frac{q^{(n)}-\frac{\mathrm{tr}_h(q^{(n)})}{2}h}{\|\xi^{(n)}\|_{C^{2,\alpha}(\mathbb{S}^2,T^*\mathbb{S}^2)}}.$$

Obviously the sequence $\left\{\tilde{\xi}^{(n)}\right\}_{n=1}^{\infty}$ is bounded in $C^{2,\alpha}\left(\mathbb{S}^2,T^*\mathbb{S}^2\right)$. For any $0<\beta<\alpha$, by Sobolv's compact embedding theorem, there exists a subsequence of $\left\{\tilde{\xi}^{(n)}\right\}_{n=1}^{\infty}$, which converges to a ξ^* in the space $C^{2,\beta}\left(\mathbb{S}^2,T^*\mathbb{S}^2\right)$. Without loss of generality, we may assume that the subsequence is the sequence $\left\{\tilde{\xi}^{(n)}\right\}_{n=1}^{\infty}$ itself. By (4.14), we have

$$(4.15)\qquad L_h(\xi^*)=0.$$

As every $\xi^{(n)}$ is perpendicular to $\ker(L_h)$, ξ^* is also perpendicular to $\ker(L_h)$. Therefore, (4.15) implies $\xi^* = 0$. Using Sobolev's embedding theorem, we have

$$\left\|\tilde{\xi}^{(n)}\right\|_{C^{2,\alpha}(\mathbb{S}^2,T^*\mathbb{S}^2)} \leq C \left\|\tilde{\xi}^{(n)}\right\|_{C^{2,\beta}(\mathbb{S}^2,T^*\mathbb{S}^2)},$$

where C is a universal constant. $\xi^* = 0$ implies that the right hand side of the above inequality converges to zero, but it is clear that the left hand side is equal to 1, which is a contradiction.

In view of the system (3.3), if ξ is $C^{2,\alpha}$ and q is smooth enough, we can obtain only $\phi = \vec{y} \cdot \nu$ is $C^{1,\alpha}$, where ν is the unit exterior normal vector field of M. If we would like to define the map Φ well, we need to improve the regularity of ϕ to $C^{2,\alpha}$. We perform this task by modifying Nirenberg's trick [**34**]. Let

$$w = \frac{u_{1,2} - u_{2,1}}{2\sqrt{\det(g)}}, \tag{4.16}$$

where a comma indicates covariant derivatives. w is a globally defined function on $\mathbb{S}^2$ because $2wdV_g = d\xi$. Moreover, w satisfies

$$\begin{aligned} w\sqrt{\det(g)} + q_{12} &= u_{1,2} - h_{12}\phi, \\ q_{21} - w\sqrt{\det(g)} &= u_{2,1} - h_{12}\phi. \end{aligned} \tag{4.17}$$

By Ricci identities (2.5), we have

$$\begin{aligned} R^l_{211}u_l &= u_{1,21} - u_{1,12} = (q_{12} + w\sqrt{\det(g)} + h_{12}\phi)_{,1} - (q_{11} + h_{11}\phi)_{,2}, \\ R^k_{122}u_k &= u_{2,12} - u_{2,21} = (q_{21} - w\sqrt{\det(g)} + h_{21}\phi)_{,2} - (q_{22} + h_{22}\phi)_{,1}. \end{aligned}$$

Thus, we have

$$\begin{aligned} -w_1 &= \frac{1}{\sqrt{\det(g)}}(q_{12,1} - q_{11,2} + (h_{21}\phi)_{,1} - (h_{11}\phi)_{,2} - R^l_{211}u_l), \\ -w_2 &= \frac{1}{\sqrt{\det(g)}}(q_{22,1} - q_{12,2} + (h_{22}\phi)_{,1} - (h_{21}\phi)_{,2} + R^k_{122}u_k). \end{aligned}$$

By the compatibility condition $w_{1,2} = w_{2,1}$, we obtain

$$\begin{aligned} &((\det h)h^{ij}\phi)_{,ij} \\ &= -(R^l_{211}u_l)_{,2} - (R^k_{122}u_k)_{,1} + q_{12,12} - q_{11,22} - q_{22,11} + q_{12,21}, \end{aligned} \tag{4.18}$$

which is an elliptic equation if the second fundamental form h_{ij} is positive definite.

Remark 13. In fact, (4.18) is the linearized Gauss equation. As the tensor q is the linearization of the metric tensor, the Gauss curvature type tensor with respect to q appears on the right hand side.

Using (4.13), (4.18), and Schauder estimates, we obtain the following lemma.

Lemma 14. *For $0 < \alpha < 1$ and any given symmetric $(0,2)$ tensor q, suppose $\vec{y}$ is a solution of the system*

$$d\vec{r} \cdot D\vec{y} = q,$$

which is perpendicular to its kernel. Then, we have the estimate

$$\begin{aligned} (4.19) \qquad & \|\vec{y}\|_{C^{2,\alpha}(\mathbb{S}^2,\vec{r}^*TN)} \\ \leq\ & C\Big(\|q\|_{C^{1,\alpha}(\mathbb{S}^2,T^*\mathbb{S}^2)} \\ & + \left\| \frac{1}{\det(g)} \left(q_{12,12} - q_{11,22} - q_{22,11} + q_{12,21}\right) \right\|_{C^{\alpha}(\mathbb{S}^2)} \Big), \end{aligned}$$

where C is a positive constant depending only on α and the surface M.

By (4.3), in every term of q_{ij} except the term $\tilde{g}_{ij} - g_{ij}$, the vector $\vec{z}$ appears at least twice. For any vector field $\vec{z}$ on M, we let $q = q(\vec{z}, \nabla\vec{z})$ defined by (4.3). As the term $q_{12,12} - q_{11,22} - q_{22,11} + q_{12,21}$ is the linearized Gauss curvature, it does not include any third order derivatives of $\vec{z}$. By (4.3), in every term of q_{ij} except the term $\tilde{g}_{ij} - g_{ij}$, the vector $\vec{z}$ appears at least twice. Thus, by the definition of the map Φ, Lemma 14 implies

$$\begin{aligned} (4.20) \qquad & \|\Phi(\vec{z})\|_{C^{2,\alpha}(\mathbb{S}^2,\vec{r}^*TN)} \\ \leq\ & C\Big(\|g_{ij} - \tilde{g}_{ij}\|_{C^{2,\alpha}(\mathbb{S}^2,\mathrm{Sym}(T^*\mathbb{S}^2\otimes T^*\mathbb{S}^2))} + \|\vec{z}\|^2_{C^{2,\alpha}(\mathbb{S}^2,\vec{r}^*TN)} \\ & + \|\vec{z}\|^3_{C^{2,\alpha}(\mathbb{S}^2,\vec{r}^*TN)} + \|\vec{z}\|^4_{C^{2,\alpha}(\mathbb{S}^2,\vec{r}^*TN)} \Big). \end{aligned}$$

For any number $\delta < 1$, let

$$B_\delta = \left\{ \vec{z} \text{ a vector field defined on the surface } M; \|\vec{z}\|_{C^{2,\alpha}(\mathbb{S}^2,\vec{r}^*TN)} \leq \delta \right\}.$$

If we take a sufficiently small δ, the map Φ is well defined in B_δ by (4.20). We then prove that the map Φ is a contraction map. Suppose $\vec{z}_1, \vec{z}_2 \in B_\delta$, and let

$$\tilde{q} = q(\vec{z}_1, \nabla\vec{z}_1) - q(\vec{z}_2, \nabla\vec{z}_2).$$

By (4.3), we have

$$\begin{aligned} & \tilde{q}_{ij} = \\ & -\sigma_{\alpha\beta}(\vec{r}) \left[\left((z_1^\alpha)_i - (z_2^\alpha)_i \right) \left(z_1^\beta \right)_j + (z_2^\alpha)_i \left(\left(z_1^\beta \right)_j - \left(z_2^\beta \right)_j \right) \right] \\ & - (z_1^\mu - z_2^\mu) \left(\int_0^1 \frac{\partial F_{\alpha\beta\gamma\lambda}}{\partial y^\mu} \left(t\vec{z}_1 + (1-t)\vec{z}_2 \right) dt \right) z_1^\gamma z_1^\lambda r_i^\alpha r_j^\beta \\ & + F_{\alpha\beta\gamma\lambda}(\vec{z}_2) \left(z_1^\gamma + z_2^\gamma \right) \left(z_1^\lambda - z_2^\lambda \right) r_i^\alpha r_j^\beta \\ & - (z_1^\mu - z_2^\mu) \left(\int_0^1 \frac{\partial G_{\alpha\beta\gamma}}{\partial y^\mu} (t\vec{z}_1 + (1-t)\vec{z}_2) dt \right) z_1^\gamma \end{aligned}$$

$$\times\left(r_i^\alpha\left(z_1^\beta\right)_j+r_j^\beta\left(z_1^\alpha\right)_i+\left(z_1^\alpha\right)_i\left(z_1^\beta\right)_j\right)$$
$$+G_{\alpha\beta\gamma}(\vec{z}_2)\left(z_1^\gamma-\vec{z}_2^\gamma\right)\left(r_i^\alpha\left(z_1^\beta\right)_j+r_j^\beta\left(z_1^\alpha\right)_i+\left(z_1^\alpha\right)_i\left(z_1^\beta\right)_j\right)$$
$$+G_{\alpha\beta\gamma}(\vec{z}_2)z_2^\gamma\left(r_i^\alpha\left(z_1^\beta-z_2^\beta\right)_j+r_j^\beta\left(z_1^\alpha-z_2^\alpha\right)_i\right)$$
$$+G_{\alpha\beta\gamma}(\vec{z}_2)z_2^\gamma\left(\left(z_1^\alpha-z_2^\alpha\right)_i\left(z_1^\beta\right)_j^\beta+\left(z_2^\alpha\right)_i\left(z_1^\beta-z_2^\beta\right)_j\right),$$

where z_1^α, z_2^α are the scalar components of $\vec{z}_1, \vec{z}_2$ with respect to $\frac{\partial}{\partial z^\alpha}$, and $\left(z_1^\alpha\right)_i, \left(z_2^\alpha\right)_i$ are the derivatives of z_1^α, z_2^α with respect to $\frac{\partial}{\partial x^i}$, i.e.,

$$\left(z_1^\alpha\right)_i=\frac{\partial z_1^\alpha}{\partial x^i}, \left(z_2^\alpha\right)_i=\frac{\partial z_2^\alpha}{\partial x^i}.$$

Thus, by Lemma 14, we have

$$\|\Phi(\vec{z}_1)-\Phi(\vec{z}_2)\|_{C^{2,\alpha}(\mathbb{S}^2,\vec{r}^*TN)}\leq C\delta\|\vec{z}_1-\vec{z}_2\|_{C^{2,\alpha}(\mathbb{S}^2,\vec{r}^*TN)}.$$

If we choose a sufficiently small δ, then the map Φ is a contraction map that implies the existence of (4.4). We complete the proof of Theorem 1.

Using the openness, we can prove an existence theorem of AdS–Sch spaces. The metrics of AdS–Sch spaces are defined by (1.3), (1.4).

Theorem 15. *For any metric g on $\mathbb{S}^2$, if its Gaussian curvature $K>-\kappa$, where κ is a nonnegative constant, then $(\mathbb{S}^2,g)$ can be isometrically embedded into an AdS–Sch space with the form (1.4). More precisely, $(\mathbb{S}^2,g)$ always can be isometrically embedded into the AdS–Sch space with parameters κ, m as a surface outside of a large ball B_R, where R is the Euclidean radius of the ball.*

Proof. Using the Weyl problem in space forms, we know that, $(\mathbb{S}^2,g)$ can be isometrically embedded into a space form as some surface M. Here, the scalar curvature of the space form is 6κ, which is referred to as κ-space form in the following. We also call the AdS–Sch space with parameters κ, m the (κ,m)-AdS–Sch space in the following. We translate M to the outside of a sufficiently large ball B_R in the κ-space form. As the underlying manifold of the κ-space form and the (κ,m)-AdS–Sch space are the same, we can view M as a surface in the (κ,m)-AdS–Sch space and its induced metric is denoted by g_R. Then, we have

$$\|g-g_R\|_{C^{3,\alpha}(\mathbb{S}^2,\mathrm{Sym}(T^*\mathbb{S}^2\otimes T^*\mathbb{S}^2))}\leq\frac{C}{R},$$

where C is a positive constant not depending on R. The second fundamental forms of M in the κ-space form and in the (κ,m)-AdS–Sch space are denoted by h, h_R, respectively. The unit normal vector fields of M and the Levi-Civita connections on M in the κ-space form and in the (κ,m)-AdS–Sch space are denoted by ν^κ, ν and ∇^κ, ∇, respectively.

Suppose $\{x^1, x^2\}$ is a local coordinate of the sphere and $\vec{r}$ is the position vector of M, then it is not difficult to check that

$$\|\nu - \nu^\kappa\|_{C^{2,\alpha}(\mathbb{S}^2, \vec{r}^* T^* N)} \leq \frac{C}{R}, \text{ and } \|\nabla_i^\kappa \vec{r}_j - \nabla_i \vec{r}_j\|_{C^{2,\alpha}(\mathbb{S}^2, T\mathbb{S}^2)} \leq \frac{C}{R},$$

which implies

$$\|h - h_R\|_{C^{2,\alpha}(\mathbb{S}^2, \mathrm{Sym}(T^*\mathbb{S}^2 \otimes T^*\mathbb{S}^2))} \leq \frac{C}{R}.$$

Here, C is also a positive constant independent of R. Thus, based on the above several inequalities, the constants in the proof of the openness, Theorem 1 are uniformly bounded. Thus, we conclude that there is a positive large constant R_0 such that for $R > R_0$, the constants appearing in Theorem 1 do not depend on R. Therefore, for a sufficiently large R, metric g is in a ϵ neighborhood of g_R. By the openness, $(\mathbb{S}^2, g)$ can be isometrically embedded into the (κ, m)-AdS–Sch space. q.e.d.

For the sake of the completeness, we roughly review existence results obtained by Guan and Lu [16] and Lu [31]. The following Heinz type C^2 a priori estimate has been proved by Lu [31]:

Theorem 16. *Suppose $(\mathbb{S}^2, g)$ is isometrically embedded into some ambient Riemannian manifold $(N^3, \bar{g})$ such that it is null homologous. Let X be some isometric embedding map. Assume that*

$$R(x) - \bar{R}(X(x)) + 2\inf\{Ric_{X(x)}(\mu, \mu) : \mu \in T_{X(x)}U, |\mu| = 1\} \geq C_0 > 0,$$

for any $x \in \mathbb{S}^2$ where C_0 is a positive constant. Further assume that $\bar{R} \geq -6\kappa^2$ and $R > -2\kappa^2$ for some constant κ. Then, the mean curvature H of the embedded surface is bounded from above,

$$H \leq C,$$

where C is a positive constant depending on C_0, $\inf(R + 2\kappa^2)$, $\|g\|_{C^3}$ and $\|\bar{g}\|_{C^3}$, but not depending on the position of the embedded surface.

We clarify the condition appearing in Theorem 2. Assume that there is a universal constant K_0 satisfying

(4.21)

$$\text{(a)} \limsup_{r\to\infty} \frac{1 - f^2}{r^2} \leq K_0, \quad \text{(b)} \limsup_{r\to\infty} -\frac{1 - f^2 + 2rff'}{r^2} \leq K_0.$$

The proof of Theorem 2 can be found in [16], [31].

5. Non-rigidity of slice spheres

In this section, we construct examples to show the non-rigidity of the isometric embedding problem in any dimensional ambient space. More precisely, we will find some convex hypersurface in an n-dimensional warped product space that is isometric to the unit slice sphere, but their second fundamental forms are not the same.

Reconsider the linearized system in the warped product space N,

$$2d\vec{r}\cdot D\tau = q, \tag{5.1}$$

where $\vec{r}$ is the position vector of a given surface M, τ is a variation field, and q is a symmetric $(0,2)$ tensor on the sphere. Similar to Section 3, in a local coordinate $\{x^1, x^2\}$ of $\mathbb{S}^2$, this system can be rewritten as

$$u_{i,j} + u_{j,i} = 2\phi h_{ij} + q_{ij}, \tag{5.2}$$

where $\xi = \sum_i u_i dx^i, u_i = \tau\cdot\frac{\partial}{\partial x^i}$ is a 1-form on the sphere, $\phi = \tau\cdot\nu$ is a function on the sphere, and h_{ij} is the second fundamental form of M. The comma again indicates the covariant derivative. Henceforth, we always use "$*$" to present the Euclidean inner product. By (2.12), for any r slice sphere, we have

$$\phi h_{ij} = \frac{f^2(r)}{r}\tau\cdot\frac{\partial}{\partial r}g_{ij} = \frac{1}{r}\tau * \frac{\partial}{\partial r}g_{ij}.$$

Let $\phi_E = \tau * \nu_E$ and h^E_{ij} be the second fundamental form of M in the Euclidean space, where ν_E is the exterior normal of M in the Euclidean space. Then, for the r slice sphere, we have

$$u_{i,j} + u_{j,i} = 2\phi_E h^E_{ij} + q_{ij},$$

which means that, if the given surface M is a slice sphere, then the system (5.1) can be viewed as a linearized isometric embedding system in the Euclidean space. Thus, we rewrite the above system in a polar coordinate.

Let $(u^1, u^2, \cdots, u^{n-1})$ be a spherical coordinate and r be the radius. Then, we can present the radius r slice sphere in the warped product space by the map

$$\begin{aligned}
&\vec{r}\left(u^1,\cdots,u^{n-1}\right)\\
=\ & r\left(\cos u^1\cos u^2\cdots\cos u^{n-1}, \cos u^1\cos u^2\cdots\cos u^{n-2}\sin u^{n-1},\cdots,\right.\\
&\left.\sin u^1\right)\\
=\ & \left(r^1, r^2, \cdots, r^n\right),
\end{aligned}$$

where r is a positive constant, and r^α is the scalar component of the position vector $\vec{r}$ with respect to $\frac{\partial}{\partial z^\alpha}$. By (5.1), we have

$$\frac{\partial}{\partial u^i} * \frac{\partial\tau}{\partial u^j} + \frac{\partial}{\partial u^j} * \frac{\partial\tau}{\partial u^i} = q_{ij}. \tag{5.3}$$

We define a symmetric $(0,2)$ tensor for two given vector fields $\vec{r}, \vec{y}$,

$$\begin{aligned}
&q_{ij}(\vec{r},\vec{y})\\
=\ & -\sigma_{\alpha\beta}(\vec{r})y^\alpha_i y^\beta_j - F_{\alpha\beta\gamma\lambda}(\vec{r},\vec{y})r^\alpha_i r^\beta_j y^\gamma y^\lambda\\
& -G_{\alpha\beta\gamma}(\vec{r},\vec{y})y^\gamma\left(r^\alpha_i y^\beta_j + y^\alpha_i r^\beta_j + y^\alpha_i y^\beta_j\right),
\end{aligned} \tag{5.4}$$

where the notations $\sigma_{\alpha\beta}$, r^α, y^γ, F, G have been defined in section 4. For convenience, we denote $r_i^\alpha = \frac{\partial r^\alpha}{\partial u^i}$, $y_j^\gamma = \frac{\partial y^\gamma}{\partial u^j}$. Let $\{z^1, \cdots, z^n\}$ be the standard coordinate of the underlying Euclidean space $\mathbb{R}^n$. Thus, we have the following result for the system (5.3),

Lemma 17. *Suppose θ is a smooth function of one variable. Let's define two vector fields*

$$\vec{y} = \theta\left(\sin u^1\right)\frac{\partial}{\partial z^n} \text{ and } \vec{r} = r\frac{\partial}{\partial r} \tag{5.5}$$

on the radius r slice sphere for a given positive constant r. Thus, the solution to the system (5.3) with q_{ij} given by (5.4) and $\vec{r}, \vec{y}$ defined by (5.5) can be chosen by

$$\tau = \tilde{\theta}\left(\sin u^1\right)\frac{\partial}{\partial z^n}, \tag{5.6}$$

where $\tilde{\theta}$ is another smooth function of one variable depending on θ.

Proof. We calculate q_{ij} in the Euclidean coordinate $\{z^1, \cdots, z^n\}$. The metric (1.3) can be rewritten as

$$\begin{aligned} ds^2 &= \frac{1}{f^2}dr^2 + r^2 dS_{n-1} \\ &= \left(\frac{1}{f^2} - 1\right)dr^2 + dr^2 + r^2 dS_{n-1} \\ &= \psi(r) z^\alpha z^\beta dz^\alpha dz^\beta + \delta_{\alpha\beta} dz^\alpha dz^\beta, \end{aligned} \tag{5.7}$$

where $\delta_{\alpha\beta}$ is Kronecker's symbol, and we let

$$\psi(r) = \frac{1}{r^2}\left(\frac{1}{f^2(r)} - 1\right), \text{ and } r^2 = \sum_\alpha \left(z^\alpha\right)^2. \tag{5.8}$$

We divide q into three terms

$$q = -I - II - III.$$

In the following, we calculate q_{ij} term by term. We first note that $y^n = \theta$, and $y^i = 0$ if $i < n$. For the first term, we have

$$I_{ij} = \sigma_{nn}(\vec{r})\theta_i\theta_j,$$

where θ_i indicates the derivative of θ with respect to $\frac{\partial}{\partial u^i}$. As $\theta = \theta(\sin u^1)$, except $i = j = 1$, we have $I_{ij} = 0$. For I_{11}, we have

$$\begin{aligned} I_{11} &= \left(\psi(r)(r^n)^2 + 1\right)\theta_1^2 \\ &= \left(r^2\psi(r)\sin^2 u^1 + 1\right)(\theta')^2\cos^2 u^1, \end{aligned}$$

where θ' indicates the first order derivative of θ. The second term is

$$II_{ij} = \int_0^1 (1-t)\partial_{nn}\sigma_{\alpha\beta}(\vec{r} + t\vec{y})r_i^\alpha r_j^\beta \theta^2 dt,$$

where r_i^α indicates the derivative of r^α with respect to $\frac{\partial}{\partial u^i}$. Let $\tilde{r} = |\vec{r} + t\vec{y}|$, then we have

$$\begin{aligned}(5.9)\quad \partial_n \sigma_{\alpha\beta}(\vec{r}+t\vec{y}) = &\frac{\psi'(\tilde{r})}{\tilde{r}}\left(r^n+ty^n\right)\left(r^\alpha+ty^\alpha\right)\left(r^\beta+ty^\beta\right)\\ &+\psi(\tilde{r})\left(\delta_{\alpha n}\left(r^\beta+ty^\beta\right)+\delta_{\beta n}\left(r^\alpha+ty^\beta\right)\right),\end{aligned}$$

where ψ' indicates the derivative of ψ with respect to the variable r. We also have

$$\begin{aligned}(5.10)\qquad &\partial_{nn}\sigma_{\alpha\beta}(\vec{r}+t\vec{y})\\ = &\left[\frac{\psi'(\tilde{r})}{\tilde{r}}\right]'\left(r^n+ty^n\right)^2\left(r^\alpha+ty^\alpha\right)\left(r^\beta+ty^\beta\right)\\ &+\frac{\psi'(\tilde{r})}{\tilde{r}}\left(r^\alpha+ty^\alpha\right)\left(r^\beta+ty^\beta\right)\\ &+2\frac{\psi'(\tilde{r})}{\tilde{r}}\left(r^n+ty^n\right)\left(\delta_{\alpha n}\left(r^\beta+ty^\beta\right)+\delta_{\beta n}\left(r^\alpha+ty^\alpha\right)\right)\\ &+2\psi(\tilde{r})\delta_{\alpha n}\delta_{\beta n}.\end{aligned}$$

Obviously,

$$\tilde{r}^2 = r^2 + t^2\left(\theta\left(\sin u^1\right)\right)^2 + 2tr\sin u^1\theta\left(\sin u^1\right),$$

depending only on $\sin u^1$ and t. As $\sum_\alpha (r^\alpha)^2 = r^2$, where r is constant, for any $i = 1, \cdots, n-1$, we have

$$\sum_\alpha r^\alpha r_i^\alpha = 0.$$

Thus, we have

$$\begin{aligned}II_{ij} = &\int_0^1 (1-t)\theta^2\left\{\left[\frac{\psi'(\tilde{r})}{\tilde{r}}\right]'\left(r^n+ty^n\right)^2+\frac{\psi'(\tilde{r})}{\tilde{r}}\right\}t^2(y^n)^2 r_i^n r_j^n dt\\ &+4\int_0^1(1-t)\theta^2\frac{\psi'(\tilde{r})}{\tilde{r}}\left(r^n+ty^n\right)t\vec{y}^n r_i^n r_j^n dt\\ &+2\int_0^1(1-t)\theta^2\psi(\tilde{r})r_i^n r_j^n dt.\end{aligned}$$

Therefore, II is zero except $i=j=1$ because $r^n = r\sin u^1$. For II_{11}, we have

$$\begin{aligned}&II_{11}\\ = &\int_0^1(1-t)t^2\theta^4\left\{\left[\frac{\psi'(\tilde{r})}{\tilde{r}}\right]'\left(r\sin u^1+t\theta\right)^2+\frac{\psi'(\tilde{r})}{\tilde{r}}\right\}r^2\cos^2 u^1 dt\\ &+4\int_0^1(1-t)t\theta^3\frac{\psi'(\tilde{r})}{\tilde{r}}\left(r\sin u^1+t\theta\right)r^2\cos^2 u^1 dt\\ &+2\int_0^1(1-t)\theta^2\psi(\tilde{r})r^2\cos^2 u^1 dt.\end{aligned}$$

For III, by (5.9), we have

(5.11)

$$\begin{aligned}
&III_{ij}\\
&= \int_0^1 \theta\partial_n\sigma_{\alpha\beta}(\vec{r}+t\vec{y})\left(r_i^\alpha y_j^\beta + y_i^\alpha r_j^\beta + y_i^\alpha y_j^\beta\right)dt\\
&= \int_0^1 \theta\frac{\psi'(\tilde{r})}{\tilde{r}}(r^n+t\theta)\big[ty^n(r_i^n y_j^n + r_j^n y_i^n)(r^n+ty^n)\\
&\quad + (r^n+ty^n)^2 y_i^n y_j^n\big]dt\\
&\quad + \int_0^1 \theta\psi(\tilde{r})\left[(r_i^n y_j^n + z_j^n y_i^n)(z_n+ty^n+ty^n) + 2y_i^n y_j^n(r^n+ty^n)\right]dt.
\end{aligned}$$

Therefore, except $i=j=1$, III is zero. For III_{11}, we have

(5.12)

$$\begin{aligned}
&III_{11}\\
&= \int_0^1 \theta\frac{\psi'(\tilde{r})}{\tilde{r}}(r\sin u^1+t\theta)^2\big[2tr\theta\theta'\cos^2 u^1\\
&\quad + \left(r\sin u^1+t\theta\right)(\theta')^2\cos^2 u^1\big]dt\\
&\quad + \int_0^1 \theta\psi(\tilde{r})\big[2r\theta'\cos^2 u^1\left(r\sin u^1+2t\theta\right)\\
&\quad + 2(\theta')^2\cos^2 u^1\left(r\sin u^1+t\theta\right)\big]dt.
\end{aligned}$$

If we let

$$\begin{aligned}
&W\left(r,\sin u^1,\theta\right)\\
&= \left(r^2\psi(r)\sin^2 u^1+1\right)(\theta')^2\\
&\quad + \int_0^1 (1-t)t^2\theta^4\left\{\left[\frac{\psi'(\tilde{r})}{\tilde{r}}\right]'\left(r\sin u^1+t\theta\right)^2 + \frac{\psi'(\tilde{r})}{\tilde{r}}\right\}r^2dt\\
&\quad + 4\int_0^1 (1-t)t\theta^3\frac{\psi'(\tilde{r})}{\tilde{r}}\left(r\sin u^1+t\theta\right)r^2dt + 2\int_0^1(1-t)\theta^2\psi(\tilde{r})r^2dt.\\
&\quad + \int_0^1 \theta\frac{\psi'(\tilde{r})}{\tilde{r}}\left(r\sin u^1+t\theta\right)^2\left[2tr\theta\theta' + \left(r\sin u^1+t\theta\right)(\theta')^2\right]dt\\
&\quad + \int_0^1 \theta\psi(\tilde{r})\left[2r\theta'\left(r\sin u^1+2t\theta\right) + 2(\theta')^2\left(r\sin u^1+t\theta\right)\right]dt,
\end{aligned}$$

then we have

(5.13)

$$q_{ij} = \begin{cases} -W\left(r,\sin u^1,\theta\right)\cos^2 u^1, & \text{if } i=j=1\\ 0, & \text{otherwise}\end{cases}.$$

Inserting (5.6) into the system (5.3), we have

$$2r\cos u^1\frac{d\tilde{\theta}}{du^1} = -W\cos^2 u^1,$$

more explicitly,

$$2rd\tilde{\theta} = -Wd\sin u^1.$$

Integrating the above ODE, we find

$$\tilde{\theta}(t) = -\frac{1}{2r}\int_0^t W(r,s,\theta(s))ds, \tag{5.14}$$

which implies that $\tilde{\theta}\left(\sin u^1\right)$ is required. q.e.d.

In the following, we apply Banach contraction mapping theorem to obtain the solution of (5.3). For any small positive constant $\epsilon \leq \epsilon_0$, we define a map

$$\begin{aligned} T : C^k([-1,1]) &\to C^k([-1,1]) \\ \theta &\mapsto -\frac{1}{2r\epsilon^2}\int_0^t W\left(r,s,\epsilon+\epsilon^2\theta(s)\right)ds, \end{aligned} \tag{5.15}$$

which has created a well defined map. We denote

$$m(s) = 2\int_0^1 (1-t)\psi(\tilde{r}(s,t))r^2dt, \text{ and } C_0 = \|m\|_{C^k([-1,1])},$$

where $\tilde{r} = |\vec{r}+t\vec{y}|$. If $\|\theta\|_{C^k([-1,1])} \leq 2C_0$, we estimate $T\theta$,

$$\|T\theta\|_{C^k([-1,1])} \leq C\left\|\frac{W\left(r,s,\epsilon+\epsilon^2\theta(s)\right)}{\epsilon^2}\right\|_{C^{k-1}([-1,1])} \leq 2C_0, \tag{5.16}$$

for any sufficiently small constant $\epsilon \leq \epsilon_0$. It remains to prove that T is a contraction map. For θ_1, θ_2 satisfying $\|\theta_1\|_{C^k([-1,1])}, \|\theta_2\|_{C^k([-1,1])} \leq 2C_0$, we have

$$\begin{aligned} &\|T\theta_1 - T\theta_2\|_{C^k([-1,1])} \\ \leq\ & \frac{C}{\epsilon^2}\left\|W\left(r,s,\epsilon+\epsilon^2\theta_1(s)\right) - W\left(r,s,\epsilon+\epsilon^2\theta_2(s)\right)\right\|_{C^{k-1}([-1,1])} \\ \leq\ & C(C_0)\epsilon\|\theta_1-\theta_2\|_{C^k([-1,1])}. \end{aligned} \tag{5.17}$$

Once we choose a sufficiently small positive constant ϵ_0, the map T is a contraction map for any $\epsilon \leq \epsilon_0$. Thus, T has a fixed point, which is denoted by $\theta^*(s)$. A C^k hypersurface Y^ϵ is defined by

$$Y^\epsilon = \vec{r} + \left(\epsilon+\epsilon^2\theta^*\left(\sin u^1\right)\right)\frac{\partial}{\partial z^n} \text{ and } \vec{y} = \left(\epsilon+\epsilon^2\theta^*\left(\sin u^1\right)\right)\frac{\partial}{\partial z^n}, \tag{5.18}$$

for $\epsilon \leq \epsilon_0$. By Lemma 17, it is obvious that

$$\frac{\partial}{\partial u^i} * \frac{\partial \vec{y}}{\partial u^j} + \frac{\partial}{\partial u^j} * \frac{\partial \vec{y}}{\partial u^i} = q_{ij}(\vec{r},\vec{y}).$$

We conclude, as we have shown in section 4, Y^ϵ is isometric to the r slice sphere.

Remark 18. If M is a 2-dimensional strictly convex surface, then the system (5.1) always is solvable in view of the argument in section 3. However, if M is a high dimensional hypersurface, the solvability of the system (5.1) is an open question in general.

Now, we can prove Theorem 3, i.e., the non-rigidity of slice spheres.

Proof of Theorem 3. In the preceding calculation, we have proved that every C^k hypersurface Y^ϵ defined by (5.18) is isometric to the r slice sphere. Furthermore, we can show that it is smooth. Using the spherical coordinate $\{u^1, \cdots, u^{n-1}\}$ of the sphere, by (2.6), the Gauss equations says that

$$\sigma_2(h_{ij}) = \frac{(n-1)(n-2)}{2r^{n-1}} - \frac{\bar{R}}{2} + \bar{Ric}(\nu,\nu), \tag{5.19}$$

where h_{ij} is the second fundamental form of Y^ϵ, $\bar{R}$ and $\bar{Ric}$ are the scalar and Ricci curvature of the ambient space N, and ν is the unit exterior normal vector field of Y^ϵ. As in section 2, using the squared distance function ρ, the second fundamental form can be rewritten as follows:

$$h_{ij} = \frac{1}{f\varphi}\left(-\rho_{i,j} + \frac{f_\rho}{f}\rho_i\rho_j + f^2 g_{ij}\right), \tag{5.20}$$

where φ is the support function and $2\rho = r^2$. We can assume that the hypersurface is C^3 at least. In view of section 2, the right hand side of (5.19) can also be expressed by ρ. Combining (5.19) with (5.20), and using the standard regularity theory of linear elliptic partial differential equations, we have the smoothness of ρ, which implies the smoothness of h_{ij}. By the Gauss formula, we have

$$Y^\epsilon_{i,j} = \Gamma^k_{ij} Y^\epsilon_k + h_{ij}\nu,$$

thus, we obtain the smoothness of Y^ϵ, where the comma indicates the covariant derivative using D and Γ^k_{ij} is the Christoffel symbol with respect to the connection ∇.

Now we can calculate the second fundamental form of Y^ϵ to compare with the slice sphere's. Note that

$$\frac{\partial}{\partial z^n} = \sin u^1 \frac{\partial}{\partial r} + \frac{\cos u^1}{r}\frac{\partial}{\partial u^1}, \vec{r} = r\frac{\partial}{\partial r}.$$

Thus, the squared distance function $\rho(Y^\epsilon)$ of Y^ϵ is a function of the variable ϵ. If $\epsilon = 0$, then Y^0 is exactly the r slice sphere. The Taylor expansion of $\rho(Y^\epsilon)$ in the ϵ neighborhood of the r slice sphere is

$$2\rho(Y^\epsilon) \quad = \quad r^2 + 2r\epsilon \sin u^1 + O\left(\epsilon^2\right),$$

which implies

$$|Y^\epsilon| = \sqrt{2\rho(Y^\epsilon)} = \sqrt{r^2 + 2r\epsilon \sin u^1 + O\left(\epsilon^2\right)} = r + \epsilon \sin u^1 + O\left(\epsilon^2\right).$$

As the derivatives of h^* are independent of ϵ, we can require ϵ to be sufficiently small such that the first order term of ϵ is dominant. As we know, the second order covariant derivative of ρ is defined by

$$\rho_{i,j} = \frac{\partial^2}{\partial u^i \partial u^j}\rho - \left(\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j}\right)\rho.$$

Thus, we have

$$\begin{aligned} &\rho_{1,1} = -r\epsilon \sin u^1 + O\left(\epsilon^2\right); \quad \rho_{i,j} = O\left(\epsilon^2\right) \text{ for } i \neq j; \\ &\rho_{i,i} = -r\epsilon \sin u^1 \Pi_{j=1}^{i-1} \cos^2 u^j + O\left(\epsilon^2\right) \text{ for } i \neq 1. \end{aligned} \tag{5.21}$$

A straightforward computation shows

$$\begin{aligned} &\rho_i \rho_j = O\left(\epsilon^2\right); |\nabla\rho|^2 = O\left(\epsilon^2\right); \\ &\varphi = \sqrt{2\rho - \frac{|\nabla\rho|^2}{f^2\left(|Y^\epsilon|\right)}} = r + \epsilon \sin u^1 + O\left(\epsilon^2\right). \end{aligned}$$

Using Taylor expansion, we have

$$\begin{aligned} &f(|Y^\epsilon|) = f(r) + f'(r)\epsilon \sin u^1 + O\left(\epsilon^2\right), \\ &f^2(|Y^\epsilon|) = f^2(r) + 2f(r)f'(r)\epsilon \sin u^1 + O\left(\epsilon^2\right), \\ &uf(|Y^\epsilon|) = rf(r) + (rf'(r) + f(r))\epsilon \sin u^1 + O\left(\epsilon^2\right). \end{aligned}$$

For $i \neq j$, we have $h_{ij} = O\left(\epsilon^2\right)$. Thus, the diagonal terms are

$$\begin{aligned} h_i^i &= g^{ii}h_{ii} = \frac{f^2(r) + \left(2f(r)f'(r) + \frac{1}{r}\right)\epsilon \sin u^1 + O\left(\epsilon^2\right)}{rf(r) + (rf'(r) + f(r))\epsilon \sin u^1 + O\left(\epsilon^2\right)} \\ &= \frac{f(r)}{r} + \left(\frac{f(r)f'(r)}{r} + \frac{1 - f^2(r)}{r^2}\right)\frac{\epsilon \sin u^1}{f(r)} + O\left(\epsilon^2\right), \end{aligned}$$

where the index i does not take summation here. Thus, h_i^i is the i-th diagonal term of the matrix $g^{-1}h$, which means it is the i-th principal curvature of Y^ϵ. Thus, if the positive constant ϵ is sufficiently small, then each principal curvature of Y^ϵ is not the same as the slice sphere's. Thus, we present our counterexamples. If every sphere is rigid, then the coefficient of the first order term vanishes for every r in the expression of h_i^i. Thus, the warped function should satisfy

$$\frac{f(r)f'(r)}{r} + \frac{1 - f^2(r)}{r^2} = 0, \tag{5.22}$$

for any r. It is well known that the solutions to the above equation are only the warping functions of space forms. We complete our proof. q.e.d.

An immediate corollary of the above theorem is that A. Ros's type theorems [42] are not always true in general warped product spaces, namely,

Theorem 19. *Constant scalar curvature hypersurfaces are not always round spheres in general warped product spaces. More precisely, if* (5.22) *does not hold for some r, then there exists a convex hypersurface which is a perturbation of the radius r slice sphere with the same constant scalar curvatures but different second fundamental forms.*

Proof. The examples in Theorem 3 are isometric to slice spheres, hence, their scalar curvatures are the same. q.e.d.

Remark 20. Using Brendle's theorem [**5**], in some cases, we have another proof that the second fundamental form of Y^ϵ is not the same as the slice sphere's. In fact, if the second fundamental form of Y^ϵ is the same as a slice sphere's, then its mean curvature should be constant, which implies that it must be a slice sphere [**5**]. However, the construction of Y^ϵ shows that it should not be a slice sphere, which is a contradiction.

The uniqueness of the solution to the Weyl problem in space forms is true, [**18**, **10**]. If the dimension of these space forms are large that 3, as we have more Gauss equations for a strictly convex embedded hypersurface M, these algebraic equations will determine the principal curvature of M. However, in general warped product spaces, Theorem 3 says that these more algebraic relations are not useful for the uniqueness of the solution to the isometric embedding problem.

The lack of rigidity of any strictly convex surface also appears in any 3-dimensional warped product space. An immediate corollary of Corollary 12 is the following non-infinitesimal rigidity of any strictly convex surface:

Corollary 21. *If the sectional curvature of a 3-dimensional warped product space is not a constant, namely*

$$\frac{f(r)f'(r)}{r} + \frac{1 - f^2(r)}{r^2} \neq 0,$$

for some r, then any embedded strictly convex surface is not infinitesimally rigid.

Proof. We first review some known facts about the isometry group of an n-dimensional Riemannian manifold N. The isometry group of N is a Lie group and its dimension is at most $n(n+1)/2$. If the dimension achieves $n(n+1)/2$, then the space should be a constant sectional curvature space. For further details, refer to [**26**] and [**12**]. In the present case, the 3-dimensional warped space is not a space form, the dimension of the isometry group is less than 6 which implies that the dimension of its Lie algebra is also less than 6. The argument in section 3 shows that for any element A in the Lie algebra, we can define a vector field τ using A satisfying (3.28). The details are the following: Suppose ϕ_t is a family of isometries generated by A and $\vec{r}$ is the position vector field of

an embedded surface M. Every $\phi_t M$ is isometric to M. Thus, we let τ be the variation of $\phi_t \vec{r}$, namely,

$$\tau = \left. \frac{d\,(\phi_t \vec{r})}{dt} \right|_{t=0}.$$

On the other hand, Corollary 12 states that the kernel of (3.28) is always 6-dimensional. Thus, there should exist a solution to (3.28) that does not come from the Lie algebra of the isometry group of N. Therefore, the embedded strictly convex surface is no longer infinitesimally rigid. q.e.d.

In view of the proof of openness, the previous corollary implies the non-rigidity of any strictly convex surface in a general 3-dimensional warped product space.

We point out that the meaning of the rigidity in the following theorem is different from the rigidity used in Theorem 3. In fact, we have two definitions of rigidity. One is that if two hypersurfaces have the same first fundamental forms, then they have the same second fundamental forms. The other is that if two hypersurfaces have the same first fundamental forms, then they can be isometrically translated to each other by an isometry of the ambient space. The second definition of rigidity is stronger than the first one. Let us see some examples. In (1.3), we choose the following warping function in $\mathbb{R}^3$, then

$$f(r) = \begin{cases} 1, & \text{if } 0 \le r \le r_0 \\ g(r), & \text{if } r > r_0 \end{cases}, \tag{5.23}$$

where r_0 is a given positive number and $g(r)$ is an arbitrary positive function. If there is a convex surface M contained in the ball B_{r_0} with radius r_0, then the translation of M in the ball does not change the shape of the surface. However, if the function g is not equal to 1, the warped product space defined by (5.23) is not a space form. Hence, M is rigid according to the first definition but not rigid according to the second one using the following theorem:

Theorem 22. *In any 3-dimensional warped product space N that is not a space form, any embedded strictly convex surface M is not rigid.*

Proof. Suppose $\vec{r}$ is the position vector of M. By the previous corollary, there exists a vector field T, which is the solution of (3.28), but not from the Lie algebra of the isometry group, i.e., there is no element A in the Lie algebra such that $T = A\vec{r}$. In the following, we generate a 1-parameter family of strictly convex surfaces M_ϵ, which is isometric to M using the vector field T. For any positive constant ϵ, we define the position vector field of M_ϵ as

$$\vec{r}_\epsilon = \vec{r} + \epsilon T + \epsilon^2 T_\epsilon,$$

where T_ϵ is a vector field to be determined later. As required, $\vec{r}_\epsilon$ satisfies the isometric embedding system

$$d\vec{r}_\epsilon \cdot d\vec{r}_\epsilon = d\vec{r} \cdot d\vec{r}.$$

According to a similar procedure to obtain (4.4) in section 4, the above system can be rewritten as

$$d\vec{r} \cdot D\left(\epsilon T + \epsilon^2 T_\epsilon\right) = q\left(\epsilon T + \epsilon^2 T_\epsilon, \nabla\left(\epsilon T + \epsilon^2 T_\epsilon\right)\right),$$

which implies that

$$d\vec{r} \cdot DT_\epsilon = \frac{1}{\epsilon^2} q\left(\epsilon T + \epsilon^2 T_\epsilon, \nabla\left(\epsilon T + \epsilon^2 T_\epsilon\right)\right).$$

We will reuse the argument in the proof of openness to find a solution T_ϵ that satisfies the above nonlinear system. Let TN again denote the tangent vector bundle of the ambient space N. For any given vector field T_ϵ of the pull-back bundle $\vec{r}^*TN$, we can define a map F whose image is a vector field T_ϵ^* of $\vec{r}^*TN$, namely,

$$\begin{aligned} F : C^{2,\alpha}\left(\mathbb{S}^2, \vec{r}^*TN\right) &\to C^{2,\alpha}\left(\mathbb{S}^2, \vec{r}^*TN\right), \\ \tilde{T}_\epsilon &\mapsto T_\epsilon^*, \end{aligned}$$

for $0 < \alpha < 1$, where T_ϵ^* solves the following system:

$$d\vec{r} \cdot DT_\epsilon^* = \frac{1}{\epsilon^2} q\left(\epsilon T + \epsilon^2 \tilde{T}_\epsilon, \nabla\left(\epsilon T + \epsilon^2 \tilde{T}_\epsilon\right)\right),$$

for the given $\tilde{T}_\epsilon$, and T_ϵ^* is perpendicular to its kernel, namely, the solutions to the system

$$d\vec{r} \cdot DT = 0.$$

Such T_ϵ^* exists uniquely as seen in section 4, so the map F is well defined. If we choose a sufficiently small ϵ, then the map F is a contraction map. By contraction mapping theorem, there exists a fixed point for the map F. We obtain a family of strictly convex surfaces M_ϵ if T_ϵ is exactly the fixed point. The detail is similar to what is presented in section 4.

We claim that there is some ϵ_0 such that there is no isometry of N by which we can translate M_{ϵ_0} to M. If the claim is not true, then we can find a 1-parameter family of isometries g_ϵ of the ambient space N such that

$$\vec{r}_\epsilon = g_\epsilon \vec{r}.$$

Now, we differentiate $\vec{r}_\epsilon$ with respect to ϵ, and then letting $\epsilon = 0$, we find that the vector field T is an infinitesimal vector field coming from an isometry of N, i.e., T comes from the Lie algebra of isometry, which contradicts our assumption. Thus, M_{ϵ_0} is isometric to the surface M but we cannot translate M_ϵ to M by any isometry of N. q.e.d.

6. A rigidity theorem for spheres

In the previous section, some counterexamples have been constructed to answer the rigidity problem for isometric embedding system negatively. In this section, we try to recover the rigidity of round spheres using condition (1.5), which restricts "translations" and exhibits the rotations. Thus, it may be compatible with the isometry group of the ambient space.

Suppose that Σ is an $n-1$ dimensional topological sphere and g is an Einstein metric with positive constant scalar curvature on Σ. The Einstein condition says

$$R_{ij} = (n-2)g_{ij} = (n-2)\delta_{ij}. \tag{6.1}$$

We need a Poincáre-type inequality. By Linchnerowicz's theorem [**36**], for Einstein metric defined by (6.1), the first eigenvalue λ_1 of the Laplacian with respect to the metric g defined on manifold Σ is not less than $n-1$. Thus, for any smooth function χ defined on Σ satisfying $\int_\Sigma \chi dV_g = 0$, where dV_g is the volume form of Σ, we have the following Poincáre type inequality:

$$(n-1)\int_\Sigma \chi^2 dV_g \le \lambda_1 \int_\Sigma \chi^2 dV_g \le \int_\Sigma |\nabla\chi|_g^2 dV_g.$$

Here, $|\cdot|_g$ is the norm with respect to the metric g on Σ.

Suppose that (Σ, g) can be isometrically embedded into an n-dimensional warped product space N. The embedded hypersurface is denoted by M. As the underlying manifold of N is the n-dimensional Euclidean space $\mathbb{R}^n$, let $\{z^1, \cdots, z^n\}$ be the standard coordinate of $\mathbb{R}^n$. We use $*$ to denote the standard inner product of the underlying Euclidean space. A vector field E is called a constant vector field, if every $E * \frac{\partial}{\partial z^i}$ is a constant for $i = 1, \cdots, n$. Thus, by the condition (1.5), we have

$$\int_M \vec{r} * E dV_g = 0,$$

where $\vec{r}$ is the position vector field of the hypesurface M. Again, suppose $\{e_1, e_2, \cdots, e_{n-1}\}$ is an orthonormal frame on M. If the metric on M is of Einstein, by the above Poincáre-type inequality, we have

$$\int_\Sigma |\vec{r} * E|^2 dV_g \le \frac{1}{n-1}\int_\Sigma \sum_i |e_i(\vec{r} * E)|^2 dV_g = \frac{1}{n-1}\int_\Sigma \sum_i |e_i * E|^2 dV_g.$$

Denote $r = |\vec{r}|$. Thus, we have

$$\int_\Sigma r^2 dV_g = \int_\Sigma \sum_j \left|\vec{r} * \frac{\partial}{\partial z^j}\right|^2 dV_g \le \frac{1}{n-1}\int_\Sigma \sum_i |e_i|_E^2 dV_g, \tag{6.2}$$

where $|\cdot|_E$ represents the Euclidean norm of some vector field. Now, we calculate the Euclidean norm of e_i using the polar coordinate. Let

$$\tilde{E}_1 = \frac{E_1}{f}, \tilde{E}_2 = E_2, \cdots \tilde{E}_n = E_n,$$

be another orthonormal frame with respect to the Euclidean metric, where E_i is defined in section 2. Suppose the scalar components of the vector e_i are a_i^α and $\tilde{a}_i^\alpha$ with respect to the two different frames

$$e_i = \sum_\alpha a_i^\alpha E_\alpha = \sum_\alpha \tilde{a}_i^\alpha \tilde{E}_\alpha,$$

then we have

$$\tilde{a}_i^1 = f a_i^1, \tilde{a}_i^\alpha = a_i^\alpha, \text{ for } \alpha = 2, \cdots, n.$$

Since every e_i is unit in N, we have $\sum_\alpha (a_i^\alpha)^2 = 1$. Thus, we obtain

$$|e_i|_E^2 = \sum_\alpha (\tilde{a}_i^\alpha)^2 = \left(\tilde{a}_i^1\right)^2 + 1 - \left(a_i^1\right)^2 = 1 + \left(f^2 - 1\right)\left(a_i^1\right)^2.$$

We also have

$$a_i^1 = e_i \cdot E_1 = \frac{e_i \cdot X}{r}, \text{ and } \rho_i = f e_i \cdot X = r f a_i^1.$$

Thus, we obtain

$$|e_i|_E^2 = 1 + \frac{f^2 - 1}{2\rho f^2}\rho_i^2.$$

Therefore, (6.2) becomes

$$\int_\Sigma 2\rho dV_g \leq \int_\Sigma dV_g + \int_\Sigma \frac{f^2 - 1}{2(n-1)\rho f^2}|\nabla \rho|^2 dV_g. \tag{6.3}$$

In the following, we try to find another integral equality using the Darboux equation. For the orthonormal frame $\{e_1, \cdots, e_{n-1}\}$, by (2.6), we have

$$\sigma_2(h) = \frac{(n-1)(n-2)}{2} + \sum_{i<j} \bar{R}_{ijij},$$

where the last term is the curvature of the ambient space defined by (2.7). Recall that $\varphi = X \cdot \nu$ is the support function of M, and then we have

$$\varphi^2 = 2\rho - \frac{|\nabla \rho|^2}{f^2}, \text{ and } (\nu^1)^2 = (\nu \cdot E_1)^2 = \frac{\varphi^2}{r^2} = 1 - \frac{|\nabla \rho|^2}{2\rho f^2}.$$

By (2.7), we have

$$\frac{1}{n-2}\sigma_2(h) = \frac{n-1}{2} + \frac{n-1}{2}\frac{f^2-1}{2\rho} + \frac{f_\rho}{2\rho f}|\nabla \rho|^2 + \frac{1 - f^2}{4\rho^2 f^2}|\nabla \rho|^2. \tag{6.4}$$

Denote

$$w_{ij} = -\rho_{i,j} + \frac{f_\rho}{f}\rho_i \rho_j + f^2 \delta_{ij}, \tag{6.5}$$

where the comma indicates covariant derivative with respect to ∇. By (2.11), we have $w_{ij} = f\varphi h_{ij}$. Then, by (6.4), we have

(6.6)
$$\begin{aligned}\frac{1}{n-2}\frac{\sigma_2(w)}{f^2} &= \varphi^2\left(\frac{n-1}{2}+\frac{1}{n-2}\sum_{i<j}\bar{R}_{ijij}\right)\\ &= (n-1)\rho+\frac{n-1}{2}(f^2-1)+\frac{f_\rho}{f}|\nabla\rho|^2+\frac{1-f^2}{2\rho f^2}|\nabla\rho|^2\\ &\quad-\left(\frac{n-1}{2}+\frac{1}{n-2}\sum_{i<j}\bar{R}_{ijij}\right)\frac{|\nabla\rho|^2}{f^2}.\end{aligned}$$

We calculate the left hand side. Obviously, we have

$$\begin{aligned}\sigma_2(w) &= \frac{1}{2}\sigma_2^{ij}\left(-\rho_{i,j}+\frac{f_\rho}{f}\rho_i\rho_j+f^2\delta_{ij}\right)\\ &= \frac{1}{2}\sigma_2^{ij}\left(-\rho_{i,j}\right)+\frac{f_\rho}{2f}\sigma_2^{ij}\rho_i\rho_j\\ &\quad+\frac{(n-2)f^2}{2}\left(-\Delta\rho+\frac{f_\rho}{f}|\nabla\rho|^2+(n-1)f^2\right).\end{aligned}$$

Thus, we obtain

(6.7)
$$\begin{aligned}&\int_\Sigma\frac{\sigma_2(w)}{f^2}dV_g\\ &=\int_\Sigma\left(\frac{\sigma_2^{ij}}{2f^2}\right)_{,j}\rho_i dV_g+\int_\Sigma\frac{f_\rho}{2f^3}\sigma_2^{ij}\rho_i\rho_j dV_g+\frac{n-2}{2}\int_\Sigma\frac{f_\rho}{f}|\nabla\rho|^2dV_g\\ &\quad+\frac{(n-1)(n-2)}{2}\int_\Sigma f^2dV_g\\ &=\int_\Sigma\frac{1}{2f^2}\left(\sigma_2^{ij}\right)_{,j}\rho_i dV_g-\int_\Sigma\frac{f_\rho}{2f^3}\sigma_2^{ij}\rho_i\rho_j dV_g+\frac{n-2}{2}\int_\Sigma\frac{f_\rho}{f}|\nabla\rho|^2dV_g\\ &\quad+\frac{(n-1)(n-2)}{2}\int_\Sigma f^2dV_g.\end{aligned}$$

We detail the first term of the above equality. We can rotate our frame to diagonalize the matrix w_{ij}, and then we have

$$\left(\sigma_2^{ij}\right)_{,j}\rho_i=\sigma_2^{ij,pq}w_{pq,j}=\sum_{j\neq i}\rho_i(w_{jji}-w_{ijj}). \tag{6.8}$$

It is obvious that

(6.9)
$$w_{bja}-w_{baj}=\rho_{baj}-\rho_{bja}+\frac{f_\rho}{f}(\rho_{ba}\rho_j-\rho_{bj}\rho_a)+2ff_\rho(\rho_a\delta_{bj}-\rho_j\delta_{ab}).$$

By applying Ricci identities, we have

$$\rho_{baj} - \rho_{bja} = \sum_c \rho_c R_{ajbc}. \tag{6.10}$$

By using (6.5), the definition of w_{ij}, we obtain

$$\begin{aligned}&\frac{f_\rho}{f}(\rho_{ba}\rho_j - \rho_{bj}\rho_a)\\ &= \frac{f_\rho}{f}\left[\rho_j\left(-w_{ba} + \frac{f_\rho}{f}\rho_b\rho_a + f^2\delta_{ba}\right) - \rho_a\left(-w_{bj} + \frac{f_\rho}{f}\rho_b\rho_j + f^2\delta_{bj}\right)\right].\end{aligned} \tag{6.11}$$

Combining (6.9) and (6.10) with (6.11), we obtain

$$w_{bja} - w_{baj} = \sum_c \rho_c R_{ajbc} + \frac{f_\rho}{f}(\rho_a w_{bj} - \rho_j w_{ba}) + f f_\rho(\rho_a\delta_{bj} - \rho_j\delta_{ab}). \tag{6.12}$$

Thus, combining (6.8) with the Einstein condition, we have

$$\begin{aligned}\left(\sigma_2^{ij}\right)_{,j}\rho_i &= \sum_{i,c}\sum_{j\neq i}\rho_i\rho_c R_{ijjc} + \frac{f_\rho}{f}\sigma_2^{ij}\rho_i\rho_j + (n-2)ff_\rho|\nabla\rho|^2\\ &= (n-2)|\nabla\rho|^2 + \frac{f_\rho}{f}\sigma_2^{ij}\rho_i\rho_j + (n-2)ff_\rho|\nabla\rho|^2.\end{aligned}$$

Thus, combining the above equality with (6.7), we have

$$\begin{aligned}&\int_\Sigma \frac{\sigma_2(w)}{f^2}dV_g\\ =\ &\frac{n-2}{2}\int_\Sigma \frac{|\nabla\rho|^2}{f^2}dV_g + (n-2)\int_\Sigma \frac{f_\rho}{f}|\nabla\rho|^2 dV_g\\ &+\frac{(n-1)(n-2)}{2}\int_\Sigma f^2 dV_g.\end{aligned}$$

Using (6.6) and the above equality, we have

$$\begin{aligned}\int_\Sigma 2\rho dV_g = &\int_\Sigma dV_g + \int_\Sigma \frac{|\nabla\rho|^2}{(n-1)f^2}dV_g + \int_\Sigma \frac{f^2-1}{(n-1)\rho f^2}|\nabla\rho|^2 dV_g\\ &+\int_\Sigma\left(1 + \frac{2}{(n-1)(n-2)}\sum_{i<j}\bar{R}_{ijij}\right)\frac{|\nabla\rho|^2}{f^2}dV_g.\end{aligned} \tag{6.13}$$

Combining (6.3) with (6.13), we obtain

$$0 \geq \int_\Sigma \frac{|\nabla\rho|^2}{(n-1)f^2}dV_g + \int_\Sigma \frac{f^2-1}{2(n-1)\rho f^2}|\nabla\rho|^2 dV_g \tag{6.14}$$

$$+\int_\Sigma \left(1+\frac{2}{(n-1)(n-2)}\sum_{i<j}\bar{R}_{ijij}\right)\frac{|\nabla\rho|^2}{f^2}dV_g.$$

Now, we will be able to prove Theorem 4.

Proof of Theorem 4. We define a function depending on the variable ρ

$$\phi(\rho)=2\rho+f^2-1.$$

As the function ρ is defined on M, ρ should range between its minimum and maximum values. Denote the two values by $\rho_{\min},\rho_{\max}$, then $\rho_{\min}\le\rho\le\rho_{\max}$. Obviously, we have

$$\phi_\rho=2(1+ff_\rho),$$

where ϕ_ρ indicates the derivative of the function ϕ with respect to ρ. By (6.4), we have

$$\frac{1}{n-2}\sigma_2(h)=\frac{n-1}{2}\frac{\phi(\rho)}{2\rho}+\frac{\phi_\rho(\rho)}{4\rho f^2}|\nabla\rho|^2-\frac{\phi(\rho)}{4\rho^2f^2}|\nabla\rho|^2. \tag{6.15}$$

By the assumption of the convexity, the right hand side of the above equality is always positive. Therefore, at the minimum point of ρ on M, we have $\nabla\rho=0$, which implies $\phi(\rho_{\min})>0$. We claim that ϕ is always positive between $\rho_{\min}$ and $\rho_{\max}$. If this is not true, let ρ_0 be the first zero of ϕ from $\rho_{\min}$, and then, at ρ_0, $\phi_\rho\le 0$. By (6.15), we obtain $\sigma_2(h)\le 0$ at $\rho=\rho_0$, which contradicts $\sigma_2(h)>0$. Thus, we always have $\phi>0$. By the positivity of $\phi(\rho)$, the assumption of the convexity, and (6.14), we obtain $\nabla\rho=0$. Therefore, ρ is a constant. Using (6.13), we have $2\rho=1$, which implies M is a unit slice sphere. We complete our proof. q.e.d.

In any 3-dimensional warped product space, Theorem 4 implies the following Corollary on the surfaces with constant Gauss curvature.

Corollary 23. *In any 3-dimensional warped product space, the only possible embedded strictly convex surface with constant scalar curvature is the slice sphere provided that the embedded surface satisfies condition (1.5).*

Proof. As the constant scalar curvature condition implies that the intrinsic metric of $\mathbb{S}^2$ is the standard metric of the sphere, applying Theorem 4 immediately yields the corollary. q.e.d.

If the warping function $f=1$, namely, that the ambient space is the Euclidean space, the above corollary is the classical rigidity of 2-dimensional strictly convex embedded surfaces.

7. Infinitesimal rigidity in space forms

In this section, we reprove the infinitesimal rigidity of strictly convex embedded surfaces in $\mathbb{R}^3$ using the maximum principle. Then, using Beltrami map, we can obtain the infinitesimal rigidity of strictly convex embedded surfaces in space form.

In the first two proofs on the infinitesimal rigidity in the Euclidean space, for simplification, we use the notation $\times$ to present the cross product induced by the Euclidean metric.

As in section 3, the infinitesimal rigidity problem of an isometric embedding system is to consider the linear system

$$d\vec{r}\cdot d\tau = 0 \tag{7.1}$$

for any given strictly convex embedded surface M, where $\vec{r}$ is its position vector field. Obviously, there is vector field A satisfying

$$d\tau = A\times d\vec{r}. \tag{7.2}$$

A is called the rotation vector. Differentiating the above equation (7.2), we have

$$d^2\tau = dA\times d\vec{r} = 0,$$

which implies that dA is parallel to the tangent space of M. Suppose $\{x^1, x^2\}$ is a local coordinate of M. Therefore, we can assume

$$d\vec{r} = \vec{r}_i dx^i, dA = A_i dx^i \text{ and } A_i = w_i^k \vec{r}_k,$$

where $\vec{r}_i, A_i$ are the derivatives of $\vec{r}$, A with respect to $\frac{\partial}{\partial x^i}$ and w_i^k is the corresponding scalar components. By $dA\times d\vec{r} = 0$, we have

$$w_i^k \vec{r}_k \times \vec{r}_j dx^i \wedge dx^j = \left(w_1^k \vec{r}_k \times \vec{r}_2 - w_2^k \vec{r}_k \times \vec{r}_1\right) dx^1 \wedge dx^2 = 0,$$

which implies

$$w_1^1 + w_2^2 = 0. \tag{7.3}$$

Now, we use a comma to indicate the covariant derivative. Using the Gauss formula, we have

$$A_{i,j} = w_{i,j}^k \vec{r}_k + w_i^k \vec{r}_{k,j} = w_{i,j}^k \vec{r}_k + w_i^k h_{kj}\nu, \tag{7.4}$$

where h_{ij} and ν are the second fundamental form and the unit exterior normal vector field of M. As $A_{i,j} = A_{j,i}$, by (7.4), we have the compatible equations

$$w_{i,j}^k = w_{j,i}^k \text{ and } w_i^k h_{kj} = w_j^k h_{ki}.$$

The second equation can be rewritten in detail as

$$-w_1^1 h_{12} - w_1^2 h_{22} + w_2^1 h_{11} + w_2^2 h_{21} = 0.$$

Multiplied by $\det(h_{ij})$, it becomes

$$w_1^1 h^{21} - w_1^2 h^{11} + w_2^1 h^{22} - w_2^2 h^{12} = 0. \tag{7.5}$$

We introduce a new tensor $a = a_{ij}dx^i dx^j$ defined by

$$\begin{aligned} a_{11} &= \sqrt{\det(g)}w_1^2, a_{12} = \sqrt{\det(g)}w_2^2, \\ a_{21} &= -\sqrt{\det(g)}w_1^1, a_{22} = -\sqrt{\det(g)}w_2^1. \end{aligned} \tag{7.6}$$

Then, by (7.3) (a_{ij}) is a symmetric matrix, and the above equations can be rewritten as

$$\begin{cases} h^{ij}a_{ij} = 0 \\ a_{ij,k} = a_{ik,j} \end{cases}, \tag{7.7}$$

which is exactly (3.7) in fact. For further detail, see [**34**] and [**21**]. Differentiating the first equation of (7.7), we have

$$-h^{ij}_{,2}a_{ij} = \sqrt{\det(g)}h^{ij}w^1_{i,j}, h^{ij}_{,1}a_{ij} = \sqrt{\det(g)}h^{ij}w^2_{i,j}.$$

Therefore, we obtain

(7.8)

$$h^{ij}A_{i,j} = h^{ij}w^k_{i,j}\vec{r}_k + w^i_i\vec{\nu} = -\frac{1}{\sqrt{\det(g)}}h^{ij}_{,2}a_{ij}\vec{r}_1 + \frac{1}{\sqrt{\det(g)}}h^{ij}_{,1}a_{ij}\vec{r}_2.$$

By Lemma 4 in [**19**], we have

$$a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 \leq -C\det(a_{ij}),$$

which means that

$$\left(w_1^1\right)^2 + \left(w_1^2\right)^2 + \left(w_2^1\right)^2 + \left(w_2^2\right)^2 \leq -C\det(w_{ij}).$$

Thus, by $A_i = w_i^k\vec{r}_k$, we conclude, in any non-degenerate point, namely, $\det w \neq 0$,

$$h^{ij}A_{i,j} = -h^{ij}_{,2}a_{ij}\frac{C_1^l A_l}{\sqrt{\det(g)}\det w} + h^{ij}_{,1}a_{ij}\frac{C_2^l A_l}{\sqrt{\det(g)}\det w},$$

where C_j^i is the cofactor of w_i^j with respect to matrix $\left(w_i^j\right)$.

Suppose the standard coordinate of $\mathbb{R}^3$ is $\{z^1, z^2, z^3\}$. A constant vector field E means that each $E \cdot \frac{\partial}{\partial z^i}$ is a constant for $i = 1, 2, 3$. Thus, for any constant vector E, taking the inner product on both sides of the previous equation with E, we obtain an elliptic partial differential equation of the function $A \cdot E$ with bounded coefficients for non-degenerate points. For degenerate points $\det w = 0$, using (7.8), we have

$$h^{ij}\left(A \cdot E\right)_{i,j} = 0,$$

which is an elliptic partial differential equation with bounded coefficients. Then, an application of the strong maximum principle yields that $A \cdot E$ is a constant that implies A is a constant vector field.

Thus, we have the following infinitesimal rigidity of any strictly convex embedded surface in the Euclidean space.

Proposition 24. *In the* 3*-dimensional Euclidean space, all solutions of* (7.1) *are*

$$A \times \vec{r} + B,$$

where A, B *are two constant vector fields and* $\vec{r}$ *is the position vector field of a given strictly convex embedded surface.*

Remark 25. We can provide another proof of the above proposition, though which we establish an elliptic equation for the vector field

$$B = \tau - A \times \vec{r},$$

where τ and A are defined by (7.1) and (7.2). Then, again using the strong maximum principle, we can obtain that B is a constant vector field.

Proof. Obviously, by (7.2), we have

$$dB = -dA \times \vec{r}. \tag{7.9}$$

Again, we suppose that $\{x^1, x^2\}$ is a local coordinate of $\mathbb{S}^2$. We write

$$B_k = -A_k \times \vec{r}, A_k = w_k^l \vec{r}_l,$$

where $B_i, \vec{r}_i, A_i$ are the derivatives of $B, \vec{r}$, A with respect to $\frac{\partial}{\partial x^i}$, and w_i^k is the corresponding scalar component. Then, we have

$$\begin{aligned} B_i &= -A_i \times \vec{r} = -w_i^k \vec{r}_k \times \vec{r}, \\ B_{i.j} &= -w_{i,j}^k \vec{r}_k \times \vec{r} - h_{kj} w_i^k \vec{n} \times \vec{r} - w_i^k \vec{r}_k \times \vec{r}_j. \end{aligned} \tag{7.10}$$

Multiplied by h^{ij}, where (h^{ij}) is the inverse matrix of (h_{ij}), the above equation becomes

$$\begin{aligned} h^{ij} B_{i.j} &= -h^{ij} w_{i,j}^k \vec{r}_k \times \vec{r} - w_k^k \vec{n} \times \vec{r} - h^{ij} w_i^k \vec{r}_k \times \vec{r}_j \\ &= -h^{ij} w_{i,j}^k \vec{r}_k \times \vec{r}, \end{aligned}$$

where we have used (7.3) and (7.5) in the last equality. Using the first equation of (7.10), we obtain an elliptic equation of $\vec{B}$ in any non degenerate point,

$$h^{ij} B_{i,j} = -h_{,2}^{ij} a_{ij} \frac{C_1^l A_l}{\sqrt{\det(g)} \det w} + h_{,1}^{ij} a_{ij} \frac{C_2^l B_l}{\sqrt{\det(g)} \det w},$$

where C_j^i is the cofactor of w_i^j with respect to matrix $\left(w_i^j\right)$ and a_{ij} is defined by (7.6). The strong maximum principle implies that B is a constant vector field. q.e.d.

Lin and Wang [**28**] discussed the infinitesimal rigidity of convex embedded surfaces in the 3-dimensional hyperbolic space. Here, using the Beltrami map, we turn the infinitesimal rigidity problem in space forms into the one in the Euclidean space. The idea used here may appear in

the literature, but we do not find an appropriate reference. For the sake of completeness, we include the detailed argument.

The Anti-de Sitter space time of signature $(3,1)$ is a hyperboloid $-\left(y^0\right)^2+\left(y^1\right)^2+\left(y^2\right)^2+\left(y^3\right)^2=-1$ in the Minkowski space $\mathbb{R}^{1,3}$, where y^0 is the time coordinate and $\{y^1,y^2,y^3\}$ is the space coordinate. It is known that the hyperboloid with the induced metric of the Minkowski space is isometric to the hyperbolic space. Thus, the infinitesimal rigidity problem in the hyperbolic space can be considered as the same problem in the hyperboloid.

Suppose that M is a strictly convex embedded surface in the hyperboloid and g is the induced metric on M. Let $\cdot$ denote the Minkowski inner product. For a 1-parameter family of surfaces M_t in the hyperboloid, let $\vec{r}_t$ be M_t's position vector field satisfying

$$\vec{r}_t\cdot\vec{r}_t=-1,\ \text{and}\ d\vec{r}_t\cdot d\vec{r}_t=g,$$

where $\vec{r}_0=\vec{r}$. Set $\tau=\left.\frac{\partial\vec{r}_t}{\partial t}\right|_{t=0}$, which is a variation of $\vec{r}$. Differentiating the above equations with respect to t and then letting $t=0$, we have

$$\tau\cdot\vec{r}=0,\ \text{and}\ d\vec{r}\cdot d\tau=0.$$

We can express $\tau,\vec{r}$ with respect to the coordinate of the Minkowski space

$$\tau=\left(\frac{1}{y^0}\sum_i A^iy^i,A^1,A^2,A^3\right),\quad \vec{r}=\left(y^0,\bar{r}\right),\ \text{and}\ \tau=\left(\frac{A*\bar{r}}{y^0},A\right),$$

where

$$\bar{r}=\left(y^1,y^2,y^3\right),A=\left(A^1,A^2,A^3\right),$$

and $*$ again denotes the standard inner product in the Euclidean space. We have

$$\begin{aligned}
& d\vec{r}\cdot d\tau\\
= & \left(dy^0,d\bar{r}\right)\cdot\left(\frac{dA*\bar{r}+A*d\bar{r}}{y^0}-A*\bar{r}\frac{dy^0}{\left(y^0\right)^2},dA\right)\\
= & \frac{1}{(y^0)^2}\left(\left(y^0\right)^2d\bar{r}*dA-y^0dA*\bar{r}dy^0-y^0A*d\bar{r}dy^0+A*\bar{r}\left(dy^0\right)^2\right)\\
= & \left(y^0\right)^2\frac{y^0d\bar{r}-\bar{r}dy^0}{\left(y^0\right)^2}*\frac{y^0dA-Ady^0}{\left(y^0\right)^2}\\
= & \left(y^0\right)^2d\frac{A}{y^0}*d\frac{\bar{r}}{y^0}=0,
\end{aligned}$$

which implies

$$d\left(\frac{\bar{r}}{y^0}\right)*d\left(\frac{A}{y^0}\right)=0. \tag{7.11}$$

The above equation indicates that the infinitesimal rigidity of the embedded surface M in the hyperboloid can be considered as a corresponding infinitesimal rigidity problem of the surface $\tilde{M}$ in the Euclidean space, where $\tilde{M}$ is defined by the position vector field $\tilde{r} = \bar{r}/y^0$. Thus, we need to study the convexity of $\tilde{M}$. We believe the following lemma is well known, but we do not find an appropriate reference. Thus, we provide a short proof here.

Lemma 26. *The second fundamental form of the surface $\tilde{M}$ in $\mathbb{R}^3$ is conformal to the second fundamental form of the surface M in the hyperboloid. The conformal function is always positive.*

Proof. Let $\{x^1, x^2\}$ be a local coordinate of M. The normal direction of M in the hyperboloid is

$$\frac{1}{\sqrt{\det(g)}}\begin{vmatrix} \frac{\partial}{\partial y^0} & \frac{\partial}{\partial y^1} & \frac{\partial}{\partial y^2} & \frac{\partial}{\partial y^3} \\ -\left(y^0\right)_1 & & \bar{r}_1 & \\ -\left(y^0\right)_2 & & \bar{r}_2 & \\ -y^0 & & \bar{r} & \end{vmatrix},$$

where $\left(y^0\right)_k = \frac{\partial y^0}{\partial x^k}$, $\bar{r}_k = \frac{\partial \bar{r}}{\partial x^k}$ for $k = 1, 2$ and g is the induced metric of M in the hyperboloid. Thus, the second fundamental form of M in the hyperboloid is

$$\frac{1}{\sqrt{\det(g)}}\begin{vmatrix} -\left(y^0\right)_{ij} & \bar{r}_{ij} \\ -\left(y^0\right)_1 & \bar{r}_1 \\ -\left(y^0\right)_2 & \bar{r}_2 \\ -y^0 & \bar{r} \end{vmatrix} = \frac{(-1)^{n+1} y^0}{\sqrt{\det(g)}}\begin{vmatrix} \bar{r}_{ij} - \frac{\left(y^0\right)_{ij}}{y^0}\bar{r} \\ \bar{r}_1 - \frac{\left(y^0\right)_1}{y^0}\bar{r} \\ \bar{r}_2 - \frac{\left(y^0\right)_2}{y^0}\bar{r} \end{vmatrix},$$

where $\left(y^0\right)_{ij} = \frac{\partial^2 y^0}{\partial x^i \partial x^j}$ and $\bar{r}_{ij} = \frac{\partial^2 \bar{r}}{\partial x^i \partial x^j}$. On the other hand, the normal vector field of the surface $\tilde{M}$ is

$$\frac{1}{\sqrt{\det(\tilde{g})}}\begin{vmatrix} \frac{\partial}{\partial y^1} & \frac{\partial}{\partial y^2} & \frac{\partial}{\partial y^3} \\ & \frac{\bar{r}_1}{y^0} - \frac{(y^0)_1}{(y^0)^2}\bar{r} & \\ & \frac{\bar{r}_2}{y^0} - \frac{(y^0)_2}{(y^0)^2}\bar{r} & \end{vmatrix},$$

where $\tilde{g}$ is the induced metric of $\tilde{M}$ in $\mathbb{R}^3$. The second derivative of $\tilde{r}$ is

$$\tilde{r}_{ij} = \frac{\bar{r}_{ij}}{y^0} - \frac{\left(y^0\right)_{ij}\bar{r}}{\left(y^0\right)^2} - \frac{\left(y^0\right)_j \tilde{r}_i + \left(y^0\right)_i \tilde{r}_j}{y^0}.$$

Since the second fundamental form of $\tilde{M}$ is $\tilde{h}_{ij} = \tilde{r}_{ij} * \tilde{n}$ where $*$ is the standard Euclidean metric, using the above three equalities, we obtain that the two fundamental forms are conformal and the conformal function is $\left(y^0\right)^4 \frac{\sqrt{\det(\tilde{g})}}{\sqrt{\det(g)}}$, which is positive. q.e.d.

Now, we can prove Theorem 5.

Proof of Theorem 5. To obtain the proof, we divided into three cases. For the Euclidean space, Proposition 24 states that any solution τ to (7.1) is generated by an element of the Lie algebra of the Lie group $O(3) \times \mathbb{R}^3$, which is the isometry group of the 3-dimensional Euclidean space.

For the hyperbolic space, we use the notations in the previous paragraphs. Lemma 26 and the argument before it show that we have two constant vector fields Y, Z in $\mathbb{R}^3$, such that

$$A = Y \times_E \bar{r} + y^0 Z,$$

where $\times_E$ is the cross product of the Euclidean space. Therefore, we have

$$\tau = \left(y^0 Z * \bar{r}, Y \times_E \bar{r} + y^0 Z\right).$$

For any point $p = \left(y^0, y^1, y^2, y^3\right) \in \mathbb{H}^3$, the Beltrami map β_p is defined as follows:

$$\begin{aligned} \beta_p : \mathbb{R}^3 &\to T_p\mathbb{R}^{3,1} \\ \frac{\partial}{\partial y^i} &\mapsto \frac{y^i}{y^0}\frac{\partial}{\partial y^0} + \frac{\partial}{\partial y^i}. \end{aligned} \tag{7.12}$$

If we let $X = \beta_p(\bar{r}), Y_0 = \beta_p(Y), Z_0 = \beta_p(Z)$, then it is easy to check that

$$\tau = y^0 Y_0 \times X + y^0 Z_0,$$

where $\times$ is the cross product of the hyperboloid. The above result was first proved by Lin and Wang [**28**]. Evidently, τ is generated by an element of the Lie algebra of the isometry group of the hyperboloid. See [**26**] for further details.

For the spherical space, any strictly convex surface in a 3-dimensional sphere can be included in a hemisphere by Bonnet-Myers' theorem [**33**]. Now for a positive constant κ, we let the 3-dimensional sphere with radius $1/\kappa$ be in $\mathbb{R}^4$ of which $\{y^0, y^1, y^2, y^3\}$ is the standard Euclidean coordinate. Without loss of generality, we assume that the surface lies in the upper hemisphere, namely, $y_0 > 0$. Using the Beltrami map, we project the hemisphere to its equator hyperplane. An argument similar to the proof in the hyperbolic space shows that solutions of the infinitesimal problem are

$$\tau = y^0 Y_0 \times X + y^0 Z_0,$$

where the meanings of $\times, X, Y_0, Z_0$ are similar to those in the hyperbolic case except y^0 replaced by $\sqrt{1 - (y^1)^2 - (y^2)^2 - (y^3)^2}$. These solutions are also generated by the Lie algebra of the Lie group $O(4)$, which is the isometry group of the 3-dimensional sphere. We complete the proof. q.e.d.

Remark 27. The previous argument can be generalized to hypersurfaces in high dimensional space forms. As we have more algebraic relations in view of Gauss equations, the condition of the convexity in Theorem 5 can be substituted by the condition that the rank of the second fundamental form is equal to or greater than three as first proved by Dajczer and Rodriguez [**11**]. Additional material on the infinitesimal rigidity of embedded hypersurfaces can be found in [**10**].

At the end of this section, we emphasize another fact that may be useful for the proof of openness in space forms.

Remark 28. In space forms, the linear system

$$d\vec{r}\cdot D\tau = q \tag{7.13}$$

can be rewritten as an inhomogeneous system

$$d\frac{\bar{r}}{y^0} * d\frac{A}{y^0} = \frac{q}{(y^0)^2}, \tag{7.14}$$

where the meanings of these notations are the same as in previous paragraphs. Nirenberg [**34**] proved (7.14) is always solvable in the Euclidean space, which implies the solvability of (7.13) in space forms.

8. A Shi-Tam type inequality and an example

In this section, we try to prove a Shi-Tam type inequality in the Schwarzschild manifold N_S, where the warping function is defined by

$$f(r) = \sqrt{1-\frac{m}{r}}. \tag{8.1}$$

We assume that Ω is a compact connected 3-dimensional Riemannian manifold bounded by the surface Σ and the scalar curvature of Ω is nonnegative. We denote the mean curvature of Σ in Ω by H. Furthermore, Σ can be isometrically embedded into the Schwarzschild manifold as a strictly convex surface M containing the black hole. We consider a geodesic flow

$$\begin{cases} \dfrac{d\Phi(t,\cdot)}{dt} &= \nu^t \quad , \text{ if } t\in(0,+\infty) \\ \Phi(0,\cdot) &= M \end{cases}, \tag{8.2}$$

where $\Phi(t,\cdot)$ is the position vector field and ν^t is the unit exterior normal vector field of the surface M_t defined by $\Phi(t,\cdot)$. Thus, we have a foliation of the space outside the M in N_S. Note that $M_0 = M$. The ambient space metric can be rewritten as $ds^2 = dt^2 + g_t$, where g_t is the metric on M_t. We try to find a conformal metric

$$d\tilde{s}^2 = u^2dt^2 + g_t$$

with the same scalar curvature as ds^2, where u is an unknown function to be determined in the following. We let H_0^t and H_1^t denote the mean

curvature of the surface M_t in the ambient space with the metric ds^2 and $d\tilde{s}^2$. Thus, we have an evolution equation of the function u [44],

$$\begin{cases} H_0^t \dfrac{\partial u}{\partial t} &= u^2 \Delta_t u + \dfrac{u-u^3}{2} R^t, \text{ if } t \in (0,+\infty) \\ u(0) &= H_0^t\big|_{t=0}/H \end{cases}, \tag{8.3}$$

where R^t is the scalar curvature of the surface M_t and Δ_t is the Laplacian operator on M_t.

The following explicit formula of sectional curvatures is useful in this section:

Proposition 29. *For any two vectors μ, ν in the warped product space $N = \left(\mathbb{R}^3, ds^2\right)$, the sectional curvature of N with respect to μ, ν is*

$$\bar{R}(\mu,\nu,\mu,\nu) = \frac{m}{2r^3}\left[|\mu\times\nu|^2 - 3\left(\mu\times\nu\cdot E_1\right)^2\right], \tag{8.4}$$

where the cross product $\times$ is induced by ds^2 and E_1 is defined in section 2.

Proof. We assume that

$$\mu = \sum_\alpha \mu^\alpha E_\alpha, \text{ and } \nu = \sum_\beta \nu^\beta E_\beta,$$

where E_α is defined by (2.2) and μ^α, ν^β are scalar components of μ, ν with respect to E_α. Thus, by (2.3), we have

$$\begin{aligned} \bar{R}(\mu,\nu,\mu,\nu) &= \sum_{\alpha,\beta,\gamma,\delta} \mu^\alpha\mu^\gamma\nu^\beta\nu^\delta \bar{R}_{\alpha\beta\gamma\delta} \\ &= \sum_{\alpha,\beta}\left((\mu^\alpha)^2\left(\nu^\beta\right)^2 - \mu^\alpha\mu^\beta\nu^\alpha\nu^\beta\right)\bar{R}_{\alpha\beta\alpha\beta} \\ &= \sum_{\alpha<\beta}\left(\mu^\alpha\nu^\beta - \mu^\beta\nu^\alpha\right)^2 \bar{R}_{\alpha\beta\alpha\beta}. \end{aligned}$$

Using (2.3), we have (8.4). q.e.d.

Using the above explicit formula, we can conclude that the geodesic flow preserves convexity.

Lemma 30. *If the metric on the strictly convex surface M is sufficiently close to a canonical metric of the round sphere, then for any $t > 0$, every leaf M_t is strictly convex as well.*

Proof. We suppose that X is the conformal Killing vector in N_S, and $\varphi^t = X\cdot\nu^t$ is the support function of M_t. We can consider the parameter r in the polar coordinate as a function depending on M_t and t. Thus, we have

$$r_t = \frac{dr}{dt} = \frac{f\varphi^t}{r}. \tag{8.5}$$

As $\Phi(t,\cdot)$ is a geodesic flow, we have

$$D_{\frac{\partial}{\partial t}}\nu = 0, \text{ and } \frac{d\varphi^t}{dt} = D_{\frac{\partial}{\partial t}}X\cdot\nu^t = f. \tag{8.6}$$

Combining (8.6) with (8.5), we have

$$\frac{d\left(r^2-\left(\varphi^t\right)^2\right)}{dt} = 0.$$

Solving the above equation, we have

$$\varphi = \sqrt{r^2-C},$$

where C is a positive constant only depending on $M = M_0$. On the other hand, we obtain

$$\left(\varphi^t\right)^2 = 2\rho - \frac{|\nabla\rho|^2}{f^2} = 2\rho - \frac{|\nabla\rho|^2}{1-m/r},$$

where $\rho = r^2/2$, which implies $C = \frac{|\nabla\rho|^2}{1-m/r}$. Note that, if M_0 is the slice sphere, we have $C = 0$. By continuity, we can require $3C < r^2$ at $t = 0$ because M_0 is sufficiently close to the slice sphere. Thus, for any unit tangential direction μ^t on M_t, by (8.4), we have

$$\begin{aligned}\bar{R}\left(\nu^t,\mu^t,\nu^t,\mu^t\right) &= \frac{m}{2r^3}\left[\left|\nu^t\times\mu^t\right|^2 - 3\left(\nu^t\times\mu^t\cdot E_1\right)^2\right] \\ &\geq \frac{m}{2r^3}\left(1-3\frac{C}{r^2}\right) > 0,\end{aligned} \tag{8.7}$$

where we have used the fact that ν^t and μ^t are unit vectors and perpendicular to each other.

By Riccati equation, any principal curvature λ^t of M_t satisfies

$$\frac{d}{dt}\lambda^t = -\left(\lambda^t\right)^2 + \bar{R}\left(\nu^t,\mu^t,\nu^t,\mu^t\right), \tag{8.8}$$

where μ^t is the corresponding unit principal direction. By (8.7), we have the convexity. q.e.d.

Now, we consider the following quantity:

Definition 31. Suppose the ambient space is a Schwarzschild manifold with mass m. For every leaf M_t, suppose H_0^t, H_1^t are the mean curvatures of M_t with respect to the metric $ds^2, d\tilde{s}^2$, respectively. Let

$$Q_t = \frac{1}{8\pi}\int_{M_t}\left(H_0^t - H_1^t\right) f dV_{g_t} + \frac{m}{2},$$

where dV_{g_t} is the volume form of M_t and f is defined by (8.1).

We observe that if the metric on M_0 is sufficiently close to the canonical metric on the round sphere, we have the monotonicity of Q_t.

Lemma 32. *The quantity Q_t monotonically decreases along the geodesic flow* (8.2).

Proof. The proof is modified from [45]. Taking the trace of the Riccati equation, we have

$$\frac{dH_0^t}{dt} = -\left|h^t\right|^2 - \bar{Ric}\left(\nu^t, \nu^t\right), \text{ and } \frac{dV_{g_t}}{dt} = H_0^t dV_{g_t},$$

where h^t is the second fundamental form of M_t with respect to ds^2 and $\left|h^t\right|^2$ is the square summation of the eigenvalues of h^t.

We calculate Q_t using the parabolic equation (8.3) and the formula (8.5),

$$\begin{aligned} & \frac{d}{dt}\int_{M_t} H_0^t\left(1-u^{-1}\right) f dV_{g_t} \\ = & -\int_{M_t} \frac{u^{-1}}{2}(u-1)^2 R^t f dV_{g_t} \\ & +\int_{M_t}\left[-\left(u^{-1}-1\right)\bar{Ric}\left(\nu^t,\nu^t\right) f + u\Delta_t f\right] dV_{g_t} \\ & +\int_{M_t} H_0^t\left(1-u^{-1}\right)\frac{ff'\varphi}{r} dV_{g_t}. \end{aligned}$$

By the static equation, we have

$$0 = \Delta f = \frac{d^2 f}{dt^2} + H_0^t \frac{df}{dt} + \Delta_t f,$$

where Δ is the Laplacian of the metric ds^2 in N_S and Δ_t is the Laplacian on M_t with respect to ds^2. Then, using (8.5), (8.6), the expression of $\bar{Ric}\left(\nu^t, \nu^t\right)$ in section 2 and the explicit formula of f, we obtain

$$\begin{aligned} & \int_{M_t}\left[-(1-u)\bar{Ric}\left(\nu^t,\nu^t\right) f + u\Delta_t f + H_0^t\varphi^t\left(1-u^{-1}\right)\frac{ff'}{r}\right] dV_{g_t} \\ = & -\int_{M_t} u^{-1}(1-u)^2 H_0^t\varphi^t\frac{m}{2r^3} dV_{g_t}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} & \frac{d}{dt}\int_{M_t} H_0^t\left(1-u^{-1}\right) f dV_{g_t} \\ = & -\int_{M_t}\frac{u^{-1}}{2}(u-1)^2\left(R^t + 2\bar{Ric}\left(\nu^t,\nu^t\right)\right) f dV_{g_t} \\ & -\int_{M_t} u^{-1}(1-u)^2 H_0^t\varphi^t\frac{m}{2r^3} dV_{g_t}. \end{aligned}$$

Lemma 30 states that the geodesic flow can preserve the convexity, therefore, the right hand side of the above formula is non-positive. Thus, we have the monotonicity. q.e.d.

In the following, we investigate the asymptotic behavior of the principal curvatures. By (8.5) and $\varphi^t = \sqrt{r^2 - C}$, we have

$$r_t = \sqrt{1 - \frac{m}{r}}\sqrt{1 - \frac{C}{r^2}} = \sqrt{1 - \frac{m}{r} - \frac{C}{r^2} + \frac{mC}{r^3}}.$$

The right hand side of the above equality is bounded from below. We can easily check that $r(t, \cdot)$ monotonically increases to infinity as t approaches infinity. Thus, for a sufficiently large T, if $t > T$, we have

$$r_t = 1 - \frac{m}{2r} + O\left(\frac{1}{r^2}\right), \text{ and } t - T = r - r(T) + \frac{m}{2}\log\frac{2r - m}{2r(T) - m} + O\left(\frac{1}{r}\right).$$

Since $r(t, \cdot)$ monotonically increases with respect to t, by (8.7), for $t \geq T$, we obtain

$$\frac{m}{4t^3} \leq \bar{R}\left(\nu^t, \mu^t, \nu^t, \mu^t\right) \leq \frac{m}{t^3},$$

where $-\bar{R}\left(\nu^t, \mu^t, \nu^t, \mu^t\right)$ is the sectional curvature of the plane spanned by μ^t, ν^t. Thus, we have

$$-\left(\lambda^t\right)^2 + \frac{m}{4t^3} \leq \frac{d\lambda^t}{dt} \leq -\left(\lambda^t\right)^2 + \frac{m}{t^3}. \tag{8.9}$$

Therefore, we obtain, for $t \geq T$,

$$\lambda^t = \frac{1}{t} + O\left(\frac{\log t}{t^2}\right).$$

By (8.4), the sectional curvature of the ambient space along the tangent vector fields is $O\left(\frac{1}{r^3}\right)$. Using Gauss equations, we have that the scalar curvature of M_t is

$$R^t = \frac{2}{t^2} + O\left(\frac{\log t}{t^3}\right).$$

We investigate the asymptotic behavior of the metric tensor on M_t. ν^t is the unit normal vector field of M_t with respect to ds^2. Evidently, we have

$$D\nu^t \cdot D\nu^t = d\frac{\nu^t}{|\nu^t|_E} \cdot_E d\frac{\nu^t}{|\nu^t|_E} + O\left(\frac{1}{t}\right) = g_{\mathbb{S}^2} + O\left(\frac{1}{t}\right),$$

where $\cdot_E$ is the standard Euclidean inner product, $|\cdot|_E$ is the corresponding norm and $g_{\mathbb{S}^2}$ is the standard metric on the round sphere. Thus, we have

$$g|_{M_t} = t^2 D\nu^t \cdot D\nu^t + O(t\log t) = t^2 g_{\mathbb{S}^2} + O(t\log t).$$

Then, the asymptotic behavior of the area form is

$$dV_{g_t} = t^2 dV_g + O(t\log t),$$

where dV_g is the standard area form of $\mathbb{S}^2$. Using a similar argument as in [44], we have the following lemma:

Lemma 33. *There exists a unique solution to the initial value problem* (8.3) *in* $[0,+\infty)$ *and the asymptotic behavior of the solution* u *is*

$$u(t) = 1 + \frac{m_0}{t} + O\left(\frac{\log t}{t^2}\right),$$

where m_0 *is a constant.*

Finally, we calculate the ADM mass of the metric ds^2. Let $\{z^1, \cdots, z^n\}$ be the standard coordinate of $\mathbb{R}^n$. The ADM mass of a metric g is defined by

$$ADM(g) = \frac{1}{2}\lim_{r\to+\infty}\int_{\mathbb{S}^2}\left(\frac{\partial g_{ij}}{\partial z^i} - \frac{\partial g_{ii}}{\partial z^j}\right) r z^j d\sigma.$$

It is well known that $ADM\left(ds_0^2\right) = m\omega$, where ω is the area of the round sphere. Let

$$b_{ij} = \left(u^2 - 1\right)\frac{\partial t}{\partial z^i}\frac{\partial t}{\partial z^j}.$$

Using Lemma 33, a straightforward calculation shows

$$\frac{\partial b_{ij}}{\partial z^i} - \frac{\partial b_{ii}}{\partial z^j} = \frac{2m_0}{t^2}\frac{\partial t}{\partial z^j} + O\left(\frac{\log t}{t^2}\right).$$

Thus, we have

$$ADM(b) = 2m_0 \lim_{t\to+\infty}\int_{\mathbb{S}^2}\sum_j \frac{z^j}{t}\frac{dt}{dr}\frac{\partial r}{\partial z^j} dV_g = 2m_0\omega.$$

By the choice of u in (8.3), the scalar curvature of $d\tilde{s}^2$ is the same as ds^2, thus is nonnegative. An application of the positive mass theorem [44] for Lipschitz metrics in asymptotic flat spaces with nonnegative scalar curvature immediately yields

$$ADM\left(ds^2\right) = ADM\left(ds_0^2\right) + ADM(b) = (2m_0 + m)\omega \geq 0.$$

Now, we can prove Theorem 6.

Proof of Theorem 6. By the monotonicity, it suffices to check that $2Q_t$ converges to $ADM(ds^2)$. We can find that

$$\begin{aligned} &\lim_{t\to+\infty} 2Q_t \\ =\ & 2\lim_{t\to+\infty}\frac{1}{8\pi}\int_{\mathbb{S}^2} H_0^t\left(1 - u^{-1}\right)\sqrt{1 - \frac{m}{t}}\, dV_{g_t} + m \\ =\ & 2m_0 + m \geq 0. \end{aligned}$$

Thus, we complete our proof. q.e.d.

Remark 34. We believe that this type of inequality holds in any AdS–Sch space.

We have proved an inequality (1.6) for convex surfaces of which the metric is closed to the metric of the round sphere. An interesting prob-

lem may be how to drop the constant m. At the end of this paper, we calculate the quantity defined in Definition 31 for the radius 1 slice sphere and its convex perturbation constructed in section 5 in the same AdS–Sch space. It seems that the positive constant m may be dropped, if the two surfaces lie in a same space.

We write down the second order approximation of the convex perturbation constructed in section 5 and also use the notations of that section with $n = 2$. In AdS–Sch spaces, the warping function f is defined by (1.4). Let $\vec{r}$ be the position vector field of the radius 1 slice sphere. We let (u^1, u^2) be a spherical coordinate of $\mathbb{S}^2$, which is

$$\vec{r}(u^1, u^2) = (\cos u^1 \cos u^2, \cos u^1 \sin u^2, \sin u^1).$$

We define a vector field

$$\vec{y} = \left(\epsilon + \epsilon^2 \tilde{\theta}(\sin u^1) + \epsilon^3 \theta(\sin u^1)\right) \frac{\partial}{\partial z^3},$$

where ϵ is a sufficiently small positive constant and $\tilde{\theta}, \theta$ are two smooth functions of one variable to be determined later. Then, by (5.13), we let

$$\tilde{\theta}(t) \quad = \quad -\frac{1}{2\epsilon^2} \int_0^t W(1, s, \epsilon) ds \Big|_{\epsilon=0} = -\frac{\psi(1)}{2} t,$$

where the function ψ is defined by (5.8). Using a similar argument of section 5, we can find a function θ such that the above vector field $\vec{y}$ defined by $\tilde{\theta}, \theta$ satisfies

$$d(\vec{r} + \vec{y}) \cdot d(\vec{r} + \vec{y}) = d\vec{r} \cdot d\vec{r}.$$

Thus, the convex surface $\vec{r} + \vec{y}$ is isometric to the radius 1 slice sphere. We denote

$$\alpha = -\frac{\psi(1)}{2} = \frac{f^2(1) - 1}{2f^2(1)} = \frac{\kappa - m}{2(1 - m + \kappa)}.$$

Then, we have the approximation of the square distance function ρ of the surface $\tilde{M}$ defined by the vector field $\vec{r} + \vec{y}$ with respect to ϵ,

(8.10)
$$\rho(\epsilon) = \frac{1}{2}|\vec{r} + \vec{y}|_E^2 = \frac{1}{2}\left(1 + 2\epsilon \sin u^1 + \epsilon^2(1 + 2\alpha \sin^2 u^1)\right) + O\left(\epsilon^3\right),$$

where $|\cdot|_E$ denotes the standard Euclidean norm. By (2.11), the second fundamental form and the support function are

$$h_{ij} = \frac{\rho_{i,j} - \frac{f_\rho}{f}\rho_i \rho_j - f^2 g_{ij}}{f\varphi}, \quad \varphi = \sqrt{2\rho - \frac{|\nabla \rho|^2}{f^2}}, \tag{8.11}$$

where the comma indicates covariant derivatives of ρ with respect to ∇. In our example, the mean curvature is defined by $H = -g^{ij} h_{ij}$, where g is the metric of the round sphere. By a straightforward computation,

for the surface $\tilde{M}$, we have the following expansion with respect to ϵ:

$$\frac{6}{4\pi}\int_{\mathbb{S}^2}\left[\frac{H}{2}f-\sqrt{1-m+\kappa}f\right]dV_g \quad = \quad \frac{3m}{4}\frac{m+2\kappa}{1-m+\kappa}\epsilon^2+O(\epsilon^3).$$

Since $2\sqrt{1-m+\kappa}$ is the mean curvature of the radius 1 slice sphere, the quantity calculated above is nothing but the mass defined by (1.6) without extra $m/2$. Thus, if ϵ is sufficiently small, we have

$$\int_{\mathbb{S}^2}\left[H-2\sqrt{1-m+\kappa}\right]fd\sigma>0.$$

Remark 35. Recently, Lu and Miao [32] considered the local version of the Riemannian Penrose inequality in the Schwarzschild manifold, thereby clarifying our previous example.

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