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AN OPTIMAL L^2 EXTENSION THEOREM ON WEAKLY PSEUDOCONVEX KÄHLER MANIFOLDS

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Abstract

In this paper, we prove an L^2 extension theorem for holomorphic sections of holomorphic line bundles equipped with singular metrics on weakly pseudoconvex Kähler manifolds. Furthermore, in our L^2 estimate, optimal constants corresponding to variable denominators are obtained. As applications, we prove an L^q extension theorem with an optimal estimate on weakly pseudoconvex Kähler manifolds and the log-plurisubharmonicity of the fiberwise Bergman kernel in the Kähler case.

1. Introduction and main results

L^2 extension theorems with uniform constant L^2 estimates on Stein manifolds are very useful in several complex variables and complex geometry (see [**25**], [**22**], [**23**], [**27**], [**28**], [**1**], [**11**], [**21**], etc). A recent progress is about the optimal L^2 extension and the applications. It turns out that L^2 extension theorems with optimal constant L^2 estimates are also quite interesting. For example, one may find some unexpected applications of the optimal L^2 extension in [**16**]. To optimize the uniform constants in the L^2 estimates is now an interesting aspect in studying L^2 extension theorems (see [**32**], [**14**], [**15**], [**18**], [**6**], [**7**], [**2**], [**24**], etc).

Another interesting aspect in studying L^2 extension theorems is to consider the singular metrics of holomorphic line bundles on weakly pseudoconvex Kähler manifolds. In this case, a difficulty arose, because, unlike the case of Stein or projective manifolds, there is a loss of positivity in the regularization process of singular weights on such manifolds.

In the present paper, we establish a method to obtain an optimal L^2 extension theorem for holomorphic sections of holomorphic line bundles with singular Hermitian metrics on weakly pseudoconvex Kähler manifolds.

Key words and phrases. Optimal L^2 extension, singular Hermitian metric, plurisubharmonic function, weakly pseudoconvex manifold, Kähler manifold.

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In order to overcome the difficulty in dealing with singular weights on such manifolds, not only the error term method of solving $\bar{\partial}$ equations (see Lemma 3.2) is needed, but also a limit problem about L^2 integrals with singular weights needs to be solved. In this paper, by replacing a fixed holomorphic function with a family of holomorphic functions, we solve the limit problem (see Proposition 5.2). Then by using Proposition 5.1, Proposition 5.2 and the strong openness conjecture (see Lemma 3.8) as the key tools, we construct a family of special smooth extensions (see Step 1 in Section 6) and overcome the difficulty in dealing with singular weights.

We began our work several years ago and partial results were announced at the Abel Symposium in 2013 by the first author (see [**31**], where the optimal constant L^2 estimate was obtained on weakly pseudoconvex Kähler manifolds for the smooth weights) and reported in several domestic conferences in China by the second author.

Our main theorem is stated below.

Let $\mathfrak{R}$ be the class of functions defined by

$$\Big\{R \in C^\infty(-\infty, 0] : R > 0,\ R' \leq 0,\ \int_{-\infty}^{0} \frac{1}{R(t)} dt < +\infty$$
$$\text{and } e^t R(t) \text{ is bounded above on } (-\infty, 0]\Big\}.$$

We will denote $\int_{-\infty}^{0} \frac{1}{R(t)} dt$ by C_R. The function $R(t)$ is equal to the function $\frac{1}{c_A(-t)e^t}$ defined just before the main theorems in [**16**] when $A = 0$.

Theorem 1.1. *(main theorem) Let (X, ω) be a weakly pseudoconvex complex n-dimensional manifold possessing a Kähler metric ω, ψ be a plurisubharmonic function on X, E be a holomorphic vector bundle of rank m over X equipped with a smooth Hermitian metric $(1 \leq m \leq n)$, and s be a global holomorphic section of E. Assume that s is transverse to the zero section, and let*

$$Y := \{x \in X : s(x) = 0\}.$$

Let L be a holomorphic line bundle over X equipped with a singular Hermitian metric h_L, which is written locally as $e^{-\varphi_L}$ for some function $\varphi_L \in L^1_{\mathrm{loc}}$ with respect to a local holomorphic frame of L. Assume that $\varphi_L + \psi$ is quasi-plurisubharmonic and φ_L is locally bounded above. Moreover, assume that

(i) $\quad \sqrt{-1}\Theta_L + \sqrt{-1}\partial\bar{\partial}\psi + m\sqrt{-1}\partial\bar{\partial}\log|s|^2_E \geq 0$ *holds on $X \setminus Y$,*

and that there is a continuous function $\alpha > 0$ on X such that the following two inequalities hold on $X \setminus Y$:

(ii) $\quad \sqrt{-1}\Theta_L + \sqrt{-1}\partial\bar{\partial}\psi + m\sqrt{-1}\partial\bar{\partial}\log|s|^2_E \geq \dfrac{\{\sqrt{-1}\Theta_E s, s\}_E}{\alpha|s|^2_E},$

(iii) $\quad \psi + m\log|s|^2_E \leq -2m\alpha.$

Then for every holomorphic section f on Y with values in the line bundle $K_X \otimes L$ (restricted to Y), such that

$$\int_Y \frac{|f|_L^2 e^{-\psi}}{|\wedge^m (ds)|_E^2} dV_Y < +\infty,$$

there exists a holomorphic section F on X with values in $K_X \otimes L$, such that $F = f$ on Y and

(1.1)
$$\int_X \frac{|F|_L^2}{e^{\psi+m\log|s|_E^2} R(\psi + m\log|s|_E^2)} dV_X \leq C_R \frac{(2\pi)^m}{m!} \int_Y \frac{|f|_L^2 e^{-\psi}}{|\wedge^m (ds)|_E^2} dV_Y.$$

Remark 1.1. We will explain some notations. Let $\{e_j\}_{j=1}^m$ be a local holomorphic frame of E which intersects with $\{s = 0\}$. Then s can be written locally as $\sum_{j=1}^m s^j e_j$, where s^j $(1 \leq j \leq m)$ are local holomorphic functions. $\wedge^m(ds)$ is defined locally by $(ds^1 \wedge \cdots \wedge ds^m) \otimes (e_1 \wedge \cdots \wedge e_m)$, which is a local section of $\wedge^m T_X^* \otimes \det E$ (however, ds is globally defined only on Y). The notation $\{\bullet, \bullet\}_E$ will be explained in Lemma 3.9. The norm $|\wedge^m (ds)|_E$ is computed here with respect to the metrics on $\wedge^m T_X^*$ and $\det E$ induced by the Kähler metric ω and the Hermitian metric on E. Similarly, the norms $|f|_L^2$ and $|F|_L^2$ are computed here with respect to the metrics on $K_X = \wedge^n T_X^*$ and L. The submanifold Y is equipped with the Kähler metric ω_Y induced from ω. $dV_X := \frac{\omega^n}{n!}$ and $dV_Y := \frac{\omega_Y^{n-m}}{(n-m)!}$ are the volume forms on X and Y respectively, where we regard ω and ω_Y as the associated Kähler forms. Then we have $|F|_L^2 dV_X = c_n\{F, F\}_L$ and $|f|_L^2 dV_Y = c_{n-m}\{f, f\}_L$, where $c_n := (\sqrt{-1})^{n^2}$ and $c_{n-m} := (\sqrt{-1})^{(n-m)^2}$.

Remark 1.2. By slight modifications of our proof, we can, in fact, replace the curvature assumption (ii) in Theorem 1.1 with the following weaker one: *assume that there exists a nonnegative number α_0 such that*

$$(ii)' \quad \sqrt{-1}\Theta_L + \sqrt{-1}\partial\bar{\partial}\psi + m\sqrt{-1}\partial\bar{\partial}\log|s|_E^2 \geq \frac{m\{\sqrt{-1}\Theta_E s, s\}_E}{\widetilde{\chi}(-2m\alpha)|s|_E^2}$$

holds on $X \setminus Y$, where

$$\widetilde{\chi}(t) := \alpha_0 + \frac{\alpha_0 \int_t^0 \left(\frac{1}{R(0)} - \frac{1}{R(t_1)}\right) dt_1 + \int_t^0 \left(\int_{t_2}^0 \frac{1}{R(t_1)} dt_1\right) dt_2}{\frac{\alpha_0}{R(0)} + \int_t^0 \frac{1}{R(t_1)} dt_1}.$$

Then the constant $C_R \frac{(2\pi)^m}{m!}$ in (1.1) should be replaced by $\left(\frac{\alpha_0}{R(0)} + C_R\right)\frac{(2\pi)^m}{m!}$.

It is not hard to see that $\widetilde{\chi}$ is a smooth strictly decreasing function from $(-\infty, 0)$ to $(\alpha_0, +\infty)$. It is also not hard to verify that $\widetilde{\chi}(t) \geq -\frac{t}{2}$ when $\alpha_0 = 0$. Furthermore, we can prove that $\left(\frac{\alpha_0}{R(0)} + C_R\right)\frac{(2\pi)^m}{m!}$ is the optimal constant corresponding to the assumption $(ii)'$.

REMARK 1.3. Theorem 1.1 is a generalization of the L^2 extension theorem on Stein manifolds with a negligible weight (see [**22**]) to the weakly pseudoconvex Kähler case with weaker curvature assumptions and an optimal estimate. In fact, if we take $R = e^{-t}$, then $C_R = 1$ and (1.1) becomes

$$\int_X |F|^2_L dV_X \leq \frac{(2\pi)^m}{m!} \int_Y \frac{|f|^2_L}{|\wedge^m (ds)|^2_E} e^{-\psi} dV_Y.$$

REMARK 1.4. Theorem 1.1 is also a generalization of Demailly's result in [**11**], where L is equipped with a smooth metric, $\alpha \geq 1$ and $\psi = 0$. In fact, if we take $R = (\frac{t}{2m})^2$ on $(-\infty, -2m]$, then (1.1) implies Demailly's L^2 estimate

$$\int_X \frac{|F|^2_L}{|s|^{2m}_E (-\log |s|_E)^2} dV_X \leq C \int_Y \frac{|f|^2_L}{|\wedge^m (ds)|^2_E} dV_Y,$$

where C is a positive constant depending only on m.

REMARK 1.5. The idea of considering variable denominators in (1.1) has been introduced in [**21**], where the optimal constant problem is not discussed. Theorem 1.1 gives an optimal version in some sense.

REMARK 1.6. In [**29**] and [**30**], Yi proved two L^2 extension theorems for holomorphic sections of holomorphic line bundles equipped with singular metrics on compact Kähler manifolds. Our result is stronger than hers since some strong additional hypotheses were assumed in her results.

REMARK 1.7. Recently, in [**8**] Cao also obtains a similar result as ours with different curvature assumptions and an additional assumption that there exists a sequence of analytic approximations of the singular metric. His proof seems also to be different from ours.

In [**4**] (see also [**5**]), a local $L^{\frac{2}{p}}$ extension theorem was obtained by using a L^2 extension theorem and an iterated method, where p is a positive integer. Using Theorem 1.1 and the similar method as in [**4**], we get the following L^q extension theorem ($0 < q \leq 2$) with optimal constants on weakly pseudoconvex Kähler manifolds. Write dV_X locally as $c_n e^{-\varphi_\omega} dz \wedge d\bar{z}$ with respect to local coordinates $z = (z^1, z^2, \cdots, z^n)$, where $c_n := (\sqrt{-1})^{n^2}$. Denote $\psi + m \log |s|^2_E$ by σ.

Theorem 1.2. *Let R, (X, ω), ψ, E, s, Y, L, h_L and φ_L be the same as in Theorem 1.1. Assume that $\frac{q}{2}\varphi_L + (1 - \frac{q}{2})\varphi_\omega + \psi$ is quasi-plurisubharmonic and φ_L is locally bounded above. Moreover, assume that*

(i) $\frac{q}{2}\sqrt{-1}\partial\bar{\partial}\varphi_L + (1 - \frac{q}{2})\sqrt{-1}\partial\bar{\partial}\varphi_\omega + \sqrt{-1}\partial\bar{\partial}\sigma \geq 0$ *holds on $X \setminus Y$,*

and that there is a continuous function $\alpha > 0$ on X such that the following two inequalities hold on $X \setminus Y$:

$$(ii) \quad \frac{q}{2}\sqrt{-1}\partial\bar{\partial}\varphi_L + (1-\frac{q}{2})\sqrt{-1}\partial\bar{\partial}\varphi_\omega + \sqrt{-1}\partial\bar{\partial}\sigma \geq \frac{\{\sqrt{-1}\Theta_E s, s\}_E}{\alpha|s|_E^2},$$

$$(iii) \quad \sigma \leq -2m\alpha.$$

Assume that f is a holomorphic section on Y with values in the line bundle $K_X \otimes L$ (restricted to Y), such that

$$C_f := \int_Y \frac{(|f|_L)^q e^{-\psi}}{|\wedge^m (ds)|_E^2} dV_Y < +\infty.$$

Furthermore, assume that there exists a holomorphic section F_1 on X with values in $K_X \otimes L$ such that $F_1 = f$ on Y and

$$C_{F_1} := \int_X \frac{(|F_1|_L)^q}{e^\sigma R(\sigma)} dV_X < +\infty.$$

Then there exists a holomorphic section F on X with values in $K_X \otimes L$, such that $F = f$ on Y and

$$\int_X \frac{(|F|_L)^q}{e^\sigma R(\sigma)} dV_X \leq C_R \frac{(2\pi)^m}{m!} C_f.$$

Let p be a positive integer. If we take $q = \frac{2}{p}$ and replace L by $K_X^{p-1} \otimes L$, which is equipped with the metric $e^{(p-1)\varphi_\omega - \varphi_L}$, then we can get from Theorem 1.2 the following corollary.

Corollary 1.1. *Assume that $\frac{\varphi_L}{p} + \psi$ is quasi-plurisubharmonic and φ_L is locally bounded above. Moreover, assume that*

$$(i) \quad \frac{\sqrt{-1}\Theta_L}{p} + \sqrt{-1}\partial\bar{\partial}\sigma \geq 0 \text{ holds on } X \setminus Y,$$

and that there is a continuous function $\alpha > 0$ on X such that the following two inequalities hold on $X \setminus Y$:

$$(ii) \quad \frac{\sqrt{-1}\Theta_L}{p} + \sqrt{-1}\partial\bar{\partial}\sigma \geq \frac{\{\sqrt{-1}\Theta_E s, s\}_E}{\alpha|s|_E^2},$$

$$(iii) \quad \sigma \leq -2m\alpha.$$

Assume that f is a holomorphic section on Y with values in the line bundle $K_X^p \otimes L$ (restricted to Y), such that

$$C_f := \int_Y \frac{(|f|_L)^{\frac{2}{p}} e^{-\psi}}{|\wedge^m (ds)|_E^2} dV_Y < +\infty.$$

Furthermore, assume that there exists a holomorphic section F_1 on X with values in $K_X^p \otimes L$ such that $F_1 = f$ on Y and

$$C_{F_1} := \int_X \frac{(|F_1|_L)^{\frac{2}{p}}}{e^\sigma R(\sigma)} dV_X < +\infty.$$

Then there exists a holomorphic section F on X with values in $K_X^p \otimes L$, such that $F = f$ on Y and

$$\int_X \frac{(|F|_L)^{\frac{2}{p}}}{e^{\sigma}R(\sigma)}dV_X \leq C_R \frac{(2\pi)^m}{m!}C_f.$$

The log-plurisubharmonicity of the fiberwise Bergman kernel was proved in [**3**] in the projective case (see also [**4**], [**5**], [**26**]). In [**16**], Guan and Zhou discovered its relation with the L^2 extension theorem with the optimal constant and gave another proof of it by their optimal L^2 extension theorem. Using Theorem 1.1 and a similar method as in [**16**], we can prove the log-plurisubharmonicity of the fiberwise Bergman kernel in the Kähler case as stated in the following theorem.

Theorem 1.3. *Let $\Pi : X \to Y$ be a surjective proper holomorphic map from a Kähler manifold X of dimension n to a complex manifold Y of dimension m. Denote by Y^0 the set of points which are not critical values of Π in Y. Set $X^0 = \Pi^{-1}(Y^0)$. Let L be a holomorphic line bundle on X equipped with a singular Hermitian metric h_L, such that*

(i) *the curvature current of (L, h_L) is semipositive on X,*
(ii) $H^0(X_{y_0}, K_{X_{y_0}} \otimes L|_{X_{y_0}} \otimes \mathcal{I}(h_L|_{X_{y_0}})) \neq 0$ *for some point $y_0 \in Y^0$.*

Then the logarithm of the fiberwise Bergman kernel of the line bundle $(K_{X/Y} \otimes L)|_{X^0}$ is plurisubharmonic on X^0. Hence, it defines a singular Hermitian metric on $(K_{X/Y} \otimes L)|_{X^0}$ with semipositive curvature current, which is called the fiberwise Bergman kernel metric. Furthermore, this metric extends across $X \backslash X^0$ to a metric with semipositive curvature current on all of X.

The rest sections of this paper are organized as follows. First, Section 2 is devoted to explain why the uniform constant in (1.1) is optimal. Next, some results are listed in Section 3, which will be used in the proof of Theorem 1.1. Then, we will prove a proposition in Section 4, which is a special case of Theorem 1.1. After that, two key propositions used to deal with singular metrics of holomorphic line bundles will be proved in Section 5. Then, we will prove Theorem 1.1 in Section 6 by using the results in Section 3, Section 4 and Section 5. Finally, we will prove Theorem 1.2 and Theorem 1.3 in Section 7 and Section 8 respectively.

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2. The optimal constant

In this section, we will prove the constant $C_R\frac{(2\pi)^m}{m!}$ in (1.1) is optimal by the following example.

Let $(\mathbb{B}^m, \omega_1)$ be the unit ball in $\mathbb{C}^m$ equipped with the Euclidean metric ω_1, and (Y, ω_2) be an $(n-m)$-dimensional compact Kähler manifold equipped with a Kähler metric ω_2 such that $H^0(Y, K_Y) \neq 0$. Then

$$X := \mathbb{B}^m \times Y$$

is a weakly pseudoconvex Kähler manifold equipped with the natural Kähler metric $\omega := \omega_1 + \omega_2$.

Assume that L and E are trivial Hermitian holomorphic bundles equipped with trivial metrics. Take $\psi = 0$ and take

$$s = z' := (z^1, \cdots, z^m),$$

with respect to a global orthonormal holomorphic frame of E, where $z^1, \cdots, z^m$ are coordinates of $\mathbb{B}^m$ and can be regarded as global functions on X. Let

$$\eta_1 := dz^1 \wedge \cdots \wedge dz^m,$$
$$\eta_2 := dz^{m+1} \wedge \cdots \wedge dz^n,$$

and

$$z'' := (z^{m+1}, \cdots, z^n),$$

where $z^{m+1}, \cdots, z^n$ are local coordinates of Y. We will write $\{0\} \times Y$ as Y for simplicity. Then K_Y is isomorphic to $K_X\big|_Y$ by the operator $\wedge\eta_1$.

It's obvious that the inequality (i) in Theorem 1.1 holds, and that there is a continuous function $\alpha > 0$ on X such that the inequalities (ii) and (iii) hold.

Write the factor

$$\frac{1}{e^{m\log|s|^2}R(m\log|s|^2)},$$

in (1.1) as $e^{-\Psi}$ and denote by $\mathrm{A}^2(X, \Psi)$ the weighted Bergman space

$$\left\{u : \ u \in H^0(X, K_X) \text{ and } \|u\|_\Psi := \left(\int_X |u|^2 e^{-\Psi} dV_X\right)^{\frac{1}{2}} < +\infty\right\},$$

where $|u|^2 dV_X = c_n u \wedge \bar{u} = c_n\{u, u\}$ and $c_n := (\sqrt{-1})^{n^2}$ as explained in Remark 1.1.

Let $B_{X,\Psi}(z', z'') \cdot c_n \eta_1 \wedge \eta_2 \wedge \overline{\eta_1} \wedge \overline{\eta_2}$ be the weighted Bergman kernel form on X with respect to the local coordinates (z', z'') of X. Similarly, we can define $\mathrm{A}^2(\mathbb{B}^m, \Psi)$, $\mathrm{A}^2(Y)$, $B_{\mathbb{B}^m,\Psi}(z') \cdot c_m \eta_1 \wedge \overline{\eta_1}$ and $B_Y(z'') \cdot c_{n-m}\eta_2 \wedge \overline{\eta_2}$. Then the product formula for the Bergman kernel form (see [**20**]) implies that

$$B_{X,\Psi}(z', z'') = B_{\mathbb{B}^m,\Psi}(z') \cdot B_Y(z'').$$

Let $z_0'' \in Y$ be a point such that $B_Y(z_0'') \neq 0$. By the extremal property of the Bergman kernel form, there exists a holomorphic $(n-m)$-form $f \in H^0(Y, K_Y)$ such that

$$\int_Y |f|^2 dV_Y = 1,$$

and

$$B_Y(z_0'') = |a(z_0'')|^2,$$

where $a(z'')$ is a local function defined by $f(z'') = a(z'')\eta_2$.

Let $\mathrm{S} := \{F \in \mathrm{A}^2(X, \Psi) : F = f \wedge \eta_1 \text{ on } Y\}$. Then any uniform constant C for the estimate (1.1) must satisfy

$$C \geq \left(\int_Y |f|^2 dV_Y \right)^{-1} \inf_{F \in \mathrm{S}} \|F\|_\Psi^2 = \inf_{F \in \mathrm{S}} \|F\|_\Psi^2 \geq \frac{|a(z_0'')|^2}{B_{X,\Psi}(0, z_0'')},$$

i.e.,

$$C \geq \frac{B_Y(z_0'')}{B_{\mathbb{B}^m,\Psi}(0) \cdot B_Y(z_0'')} = \frac{1}{B_{\mathbb{B}^m,\Psi}(0)}.$$

Since $e^{-\Psi}$ is a function of the variables $r_1, \cdots, r_m$, where $r_k = |z^k|$ $(1 \leq k \leq m)$, it is not hard to prove that

$$\left\{ \frac{(z^1)^{i_1} \cdots (z^m)^{i_m} \cdot \eta_1}{\|(z^1)^{i_1} \cdots (z^m)^{i_m} \cdot \eta_1\|_\Psi} \right\}_{(i_1, \cdots, i_m) \in \mathbb{N}^m},$$

form an orthonormal basis of $\mathrm{A}^2(\mathbb{B}^m, \Psi)$, where $\mathbb{N}$ denotes the set of nonnegative integers. Hence,

$$\begin{aligned} C \geq \frac{1}{B_{\mathbb{B}^m,\Psi}(0)} = \|\eta_1\|_\Psi^2 &= \int_{z' \in \mathbb{B}^m} \frac{2^m}{|z'|^{2m} R(\log |z'|^{2m})} dV_m \\ &= \int_{\mathbb{S}^{2m-1}} dS \int_0^1 \frac{2^m}{r R(\log r^{2m})} dr \\ &= \frac{2\pi^m}{(m-1)!} \int_{-\infty}^0 \frac{2^{m-1}}{m R(t)} dt \\ &= C_R \frac{(2\pi)^m}{m!}, \end{aligned}$$

where dV_m denotes the $2m$-dimensional Lebesgue measure on $\mathbb{C}^m$, $\mathbb{S}^{2m-1}$ is the unit sphere in $\mathbb{C}^m$ and dS is the surface measure on $\mathbb{S}^{2m-1}$.

Therefore, $C_R \frac{(2\pi)^m}{m!}$ is the optimal constant.

3. Some results used in the proof of Theorem 1.1

In this section, we give some results which will be used in the proof of Theorem 1.1.

Lemma 3.1. *Let Q be a Hermitian vector bundle on a Kähler manifold X of dimension n with a Kähler metric ω. Assume that $\tau, A > 0$ are smooth functions on X. Then for every form $v \in \mathcal{D}(X, \wedge^{n,q}T_X^* \otimes Q)$ with compact support we have*

$$\begin{aligned}&\int_X (\tau + A)|\mathrm{D}''^* v|_Q^2 dV_X + \int_X \tau |\mathrm{D}'' v|_Q^2 dV_X \\ \geq \ &\int_X \left\langle \left[\tau\sqrt{-1}\Theta_Q - \sqrt{-1}\partial\bar{\partial}\tau - \sqrt{-1}\frac{\partial\tau \wedge \bar{\partial}\tau}{A}, \Lambda\right] v, v \right\rangle_Q dV_X.\end{aligned}$$

Proof. The proof is almost the same as in [**11**], where the term

$$\int_X (\tau + A)|\mathrm{D}''^* v|_Q^2 dV_X,$$

in the above inequality is written as

$$\int_X (\sqrt{\tau} + \sqrt{A})^2 |\mathrm{D}''^* v|_Q^2 dV_X.$$

With slightly careful calculations, we can get this more precise inequality. q.e.d.

Lemma 3.2. *Let (X, ω) be a complete Kähler manifold equipped with a (non-necessarily complete) Kähler metric ω, and let Q be a Hermitian vector bundle over X. Assume that τ and A are smooth and bounded positive functions on X and let $\mathrm{B} := [\tau\sqrt{-1}\Theta_Q - \sqrt{-1}\partial\bar{\partial}\tau - \sqrt{-1}A^{-1}\partial\tau \wedge \bar{\partial}\tau, \Lambda]$. Assume that $\delta \geq 0$ is a number such that $\mathrm{B} + \delta\mathrm{I}$ is semi-positive definite everywhere on $\wedge^{n,q}T_X^* \otimes Q$ for some $q \geq 1$. Then given a form $g \in L^2(X, \wedge^{n,q}T_X^* \otimes Q)$ such that $\mathrm{D}''g = 0$ and $\int_X \langle (\mathrm{B} + \delta\mathrm{I})^{-1} g, g\rangle_Q dV_X < +\infty$, there exists an approximate solution $u \in L^2(X, \wedge^{n,q-1}T_X^* \otimes Q)$ and a correcting term $h \in L^2(X, \wedge^{n,q}T_X^* \otimes Q)$ such that $\mathrm{D}''u + \sqrt{\delta}h = g$ and*

$$\int_X \frac{|u|_Q^2}{\tau + A} dV_X + \int_X |h|_Q^2 dV_X \leq \int_X \langle (\mathrm{B} + \delta\mathrm{I})^{-1} g, g\rangle_Q dV_X.$$

Proof. By Lemma 3.1, Lemma 3.2 can be obtained by almost the same arguments as in [**11**], where the term $\int_X \langle (\mathrm{B} + \delta\mathrm{I})^{-1} g, g\rangle_Q dV_X$ in the above inequality is written as $2\int_X \langle (\mathrm{B} + \delta\mathrm{I})^{-1} g, g\rangle_Q dV_X$. q.e.d.

Lemma 3.3. *(see [**13**]) Let X be a Stein manifold and φ be a plurisubharmonic function on X. Then there exists a decreasing sequence of smooth strictly plurisubharmonic functions $\{\varphi_j\}_{j=1}^{+\infty}$ such that $\lim_{j\to+\infty} \varphi_j = \varphi$.*

Lemma 3.4. *Let (X, ω) be a complex manifold equipped with a Hermitian metric ω, and $\Omega \subset\subset X$ be an open subset. Assume that $T = \widetilde{T} + \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\varphi$ is a closed $(1,1)$-current on X, where $\widetilde{T}$ is a smooth real*

$(1,1)$-form and φ is a quasi-plurisubharmonic function. Let γ be a continuous real $(1,1)$-form such that $T \geq \gamma$. Suppose that the Chern curvature tensor of T_X satisfies

$$(\sqrt{-1}\Theta_{T_X} + \varpi \otimes \mathrm{Id}_{T_X})(\kappa_1 \otimes \kappa_2, \kappa_1 \otimes \kappa_2) \geq 0,$$

$$\forall \kappa_1, \kappa_2 \in T_X \text{ with } \langle \kappa_1, \kappa_2 \rangle = 0,$$

for some continuous nonnegative $(1,1)$-form ϖ on X. Then there is a family of closed $(1,1)$-currents $T_{\varsigma,\rho} = \widetilde{T} + \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\varphi_{\varsigma,\rho}$ on X ($\varsigma \in (0,+\infty)$ and $\rho \in (0,\rho_1)$ for some positive number ρ_1) independent of γ, such that

- (i) *$\varphi_{\varsigma,\rho}$ is quasi-plurisubharmonic on a neighborhood of $\overline{\Omega}$, smooth on $X \setminus E_\varsigma(T)$, increasing with respect to ς and ρ on Ω, and converges to φ on Ω as $\rho \to 0$,*
- (ii) *$T_{\varsigma,\rho} \geq \gamma - \varsigma\varpi - \delta_\rho\omega$ on Ω,*

where $E_\varsigma(T) := \{x \in X : \nu(T,x) \geq \varsigma\}$ $(\varsigma > 0)$ is the ς-upperlevel set of Lelong numbers, and $\{\delta_\rho\}$ is an increasing family of positive numbers such that $\lim_{\rho\to 0} \delta_\rho = 0$.

Proof. The reader is referred to Theorem 6.1 in [**10**], where Lemma 3.4 is stated in the case X is compact. Almost the same proof as in [**10**] shows that Lemma 3.4 holds in the noncompact case while uniform estimates are obtained only on the relatively compact subset Ω. One of the key points in our use is that the construction of $T_{\varsigma,\rho}$ is independent of γ. q.e.d.

Lemma 3.5. *(Theorem 1.5 in [**9**]) Let X be a Kähler manifold, and Z be an analytic subset of X. Assume that Ω is a relatively compact open subset of X possessing a complete Kähler metric. Then $\Omega \setminus Z$ carries a complete Kähler metric.*

Lemma 3.6. *(Theorem 4.4.2 in [**19**]) Let Ω be a pseudoconvex open set in $\mathbb{C}^n$, and φ be a plurisubharmonic function on Ω. For every $h \in L^2_{(p,q+1)}(\Omega,\varphi)$ with $\bar{\partial}h = 0$ there is a solution $v \in L^2_{(p,q)}(\Omega, \mathrm{loc})$ of the equation $\bar{\partial}v = h$ such that*

$$\int_\Omega \frac{|v|^2}{(1+|z|^2)^2} e^{-\varphi} dV \leq \int_\Omega |h|^2 e^{-\varphi} dV.$$

Lemma 3.7. *(Lemma 6.9 in [**9**]) Let Ω be an open subset of $\mathbb{C}^n$ and Z be a complex analytic subset of Ω. Assume that v is a $(p,q-1)$-form with L^2_{loc} coefficients and h is a (p,q)-form with L^1_{loc} coefficients such that $\bar{\partial}v = h$ on $\Omega \setminus Z$ (in the sense of distribution theory). Then $\bar{\partial}v = h$ on Ω.*

Lemma 3.8. *(strong openness conjecture, see [**17**]) Let φ be a negative plurisubharmonic function on the unit polydisk $\Delta^n \subset \mathbb{C}^n$. Assume* *that F is a holomorphic function on Δ^n satisfying*

$$\int_{\Delta^n} |F|^2 e^{-\varphi} dV_n < +\infty,$$

where dV_n is the $2n$-dimensional Lebesgue measure on $\mathbb{C}^n$. Then there exists $r \in (0,1)$ and $\beta \in (0,+\infty)$ such that

$$\int_{\Delta_r^n} |F|^2 e^{-(1+\beta)\varphi} dV_n < +\infty,$$

where $\Delta_r^n := \{(z^1, \cdots, z^n) \in \mathbb{C}^n : |z^k| < r, 1 \le k \le n\}$.

Lemma 3.9. *(Lagrange's inequality) Let X be a complex manifold, E be a Hermitian vector bundle over X of rank m, and $\{\bullet, \bullet\}_E : \wedge^{p_1,q_1} T_X^* \otimes E \times \wedge^{p_2,q_2} T_X^* \otimes E \longrightarrow \wedge^{p_1+q_2,q_1+p_2} T_X^*$ be the sesquilinear product which combines the wedge product $(u,v) \mapsto u \wedge \bar{v}$ on scalar valued forms with the Hermitian inner product on E. Then for any smooth section s of E over X and any smooth section w of $T_X^* \otimes E$ over X,*

$$\sqrt{-1}\{w,s\}_E \wedge \{s,w\}_E \le |s|_E^2 \sqrt{-1}\{w,w\}_E. \tag{3.1}$$

Proof. Since $\{\bullet, \bullet\}_E$ is a pointwise product, it's sufficient to prove (3.1) at every fixed point of X. Hence, we can regard T_X^* and E as vector spaces. Then s and w are regarded as elements in E and $T_X^* \otimes E$ respectively.

If $s = 0$, (3.1) is trivial. If $s \neq 0$, without loss of generality, we can assume that $|s|_E = 1$. Then we choose $e_2, \cdots, e_m \in E$ such that $s, e_2, \cdots, e_m$ form an orthonormal basis of E. Then w can be written as

$$w_1 \otimes s + \sum_{j=2}^{m} w_j \otimes e_j,$$

for some $w_j \in T_X^*$ $(1 \le j \le m)$. Then

$$\sqrt{-1}\{w,s\}_E \wedge \{s,w\}_E = \sqrt{-1} w_1 \wedge \bar{w}_1,$$

and

$$|s|_E^2 \sqrt{-1}\{w,w\}_E = \sqrt{-1} \sum_{j=1}^{m} w_j \wedge \bar{w}_j \ge \sqrt{-1} w_1 \wedge \bar{w}_1.$$

Hence, (3.1) holds. The lemma is, thus, proved. q.e.d.

4. Proof of a special case of Theorem 1.1

In order to prove Theorem 1.1, we prove the following proposition at first, which is a special case of Theorem 1.1 and will be used in Section 6. Although the following proposition can be implied by the main theorems in [**16**], we give its proof here for the self-contained purpose, which is also used in the proof of Theorem 1.1.

Proposition 4.1. *Let R be a function in $\mathfrak{R}$. Let (X,ω) be an n-dimensional Stein manifold possessing a Kähler metric ω, and $E = X\times\mathbb{C}^m$ be the trivial Hermitian holomorphic vector bundle of rank m equipped with the trivial metric ($1\le m\le n$). Assume that s is a global holomorphic section of E ($s=(s^1,\cdots,s^m)$ with respect to the standard orthonormal global holomorphic frame of E, where s^i ($1\le i\le m$) are global holomorphic functions on X). Assume that s is transverse to the zero section, and let*

$$Y := \{x\in X:\ s(x)=0\}.$$

Moreover, assume that $|s|\le 1$ on X. Let $L = X\times\mathbb{C}$ be the trivial holomorphic line bundle over X equipped with a singular Hermitian metric $h_L = e^{-\varphi}$, where φ is a plurisubharmonic function on X. Then for every holomorphic section f on Y with values in the line bundle $K_X\otimes L$ (restricted to Y) satisfying

$$\int_Y \frac{|f|^2 e^{-\varphi}}{|\wedge^m (ds)|^2} dV_Y < +\infty,$$

there exists a holomorphic section F on X with values in $K_X\otimes L$ satisfying $F=f$ on Y and

$$\int_X \frac{|F|^2 e^{-\varphi}}{e^{m\log|s|^2} R(m\log|s|^2)} dV_X \le C_R \frac{(2\pi)^m}{m!}\int_Y \frac{|f|^2 e^{-\varphi}}{|\wedge^m (ds)|^2} dV_Y.$$

Proof. Without loss of generality, we can suppose that $C_R = 1$. Otherwise, we replace R with $C_R R$ in the proof.

If $f=0$ on Y, then $F=0$ satisfies the conclusion of Proposition 4.1. In the following proof, we assume that f is not 0 identically.

Since X is Stein, there exists a smooth strictly plurisubharmonic exhaustion function P on X. Instead of working on X itself, we will work rather on the relatively compact subset $X_k\setminus Y$, where $X_k = \{P<k\}$ ($k=1,2,\cdots$, we choose P such that $X_1\ne\emptyset$). By Lemma 3.5, $X_k\setminus Y$ ($k=1,2,\cdots$) are complete Kähler.

We will discuss for fixed k until the end of the proof.

Since X is Stein, by Cartan's Theorem B, there exists a holomorphic section $\tilde f$ on X with values in $K_X\otimes L$ such that $\tilde f = f$ on Y.

Let $\zeta : (-\infty,0)\longrightarrow(0,+\infty)$ be a smooth strictly increasing function, and $\chi : (-\infty,0)\longrightarrow(0,+\infty)$ a smooth strictly decreasing function. Assume that $\chi(t)\ge -\frac{t}{2}$ for $t\in(-\infty,0)$. We will find more assumptions about ζ and χ in the proof, by which we will get explicit ζ and χ in the end of this section.

Let $a\in(0,1)$ and put $\sigma_\varepsilon = m\log(|s|^2+\varepsilon^2)-a$ and $\sigma = m\log|s|^2 - a$. Since $|s|\le 1$ on X, there exists a positive number $\varepsilon_a\in(0,1)$ such that $\sigma_\varepsilon\le -\frac{a}{2}$ on $\overline{X_k}$ for $\varepsilon\in(0,\varepsilon_a)$.

By Lemma 3.3, there exists a decreasing sequence of smooth plurisubharmonic functions $\{\varphi_j\}_{j=1}^{+\infty}$ on X such that $\lim_{j\to+\infty}\varphi_j=\varphi$. Let $L_{j,a,\varepsilon}$

denote the line bundle L on $X_k \setminus Y$ equipped with the new metric $h_{j,a,\varepsilon} := e^{-\varphi_j - \sigma - \zeta(\sigma_\varepsilon)}$.

Set $\tau_\varepsilon = \chi(\sigma_\varepsilon)$ and let A_ε be a smooth positive function on $\overline{X_k}$, which will be determined later. Set $\mathrm{B}_\varepsilon = [\Theta_\varepsilon, \Lambda]$ on $X_k \setminus Y$, where

$$\Theta_\varepsilon := \tau_\varepsilon \sqrt{-1}\Theta_{L_{j,a,\varepsilon}} - \sqrt{-1}\partial\bar{\partial}\tau_\varepsilon - \sqrt{-1}\frac{\partial\tau_\varepsilon \wedge \bar{\partial}\tau_\varepsilon}{A_\varepsilon}.$$

Set $\nu_\varepsilon = \frac{\sum_{i=1}^m \bar{s}^i ds^i}{|s|^2+\varepsilon^2}$. We want to find suitable ζ, χ and A_ε such that

$$\Theta_\varepsilon\big|_{X_k\setminus Y} \geq \frac{m\varepsilon^2}{|s|^2}\sqrt{-1}\nu_\varepsilon \wedge \bar{\nu}_\varepsilon. \tag{4.1}$$

Since $\chi(\sigma_\varepsilon) > 0$, $\sqrt{-1}\partial\bar{\partial}\varphi_j \geq 0$ and $\sqrt{-1}\partial\bar{\partial}\sigma \geq 0$, simple calculations yield

$$\begin{aligned}
& \Theta_\varepsilon\big|_{X_k\setminus Y} \\
= \ & \chi(\sigma_\varepsilon)(\sqrt{-1}\partial\bar{\partial}\varphi_j + \sqrt{-1}\partial\bar{\partial}\sigma) + \big(\chi(\sigma_\varepsilon)\zeta'(\sigma_\varepsilon) - \chi'(\sigma_\varepsilon)\big)\sqrt{-1}\partial\bar{\partial}\sigma_\varepsilon \\
& + \bigg(\chi(\sigma_\varepsilon)\zeta''(\sigma_\varepsilon) - \chi''(\sigma_\varepsilon) - \frac{\big(\chi'(\sigma_\varepsilon)\big)^2}{A_\varepsilon}\bigg)\sqrt{-1}\partial\sigma_\varepsilon \wedge \bar{\partial}\sigma_\varepsilon \\
\geq \ & \big(\chi(\sigma_\varepsilon)\zeta'(\sigma_\varepsilon) - \chi'(\sigma_\varepsilon)\big)\sqrt{-1}\partial\bar{\partial}\sigma_\varepsilon \\
& + \bigg(\chi(\sigma_\varepsilon)\zeta''(\sigma_\varepsilon) - \chi''(\sigma_\varepsilon) - \frac{\big(\chi'(\sigma_\varepsilon)\big)^2}{A_\varepsilon}\bigg)\sqrt{-1}\partial\sigma_\varepsilon \wedge \bar{\partial}\sigma_\varepsilon.
\end{aligned}$$

If the equalities

$$\chi(\sigma_\varepsilon)\zeta'(\sigma_\varepsilon) - \chi'(\sigma_\varepsilon) = 1, \tag{4.2}$$

and

$$\chi(\sigma_\varepsilon)\zeta''(\sigma_\varepsilon) - \chi''(\sigma_\varepsilon) - \frac{\big(\chi'(\sigma_\varepsilon)\big)^2}{A_\varepsilon} = 0 \tag{4.3}$$

hold, we obtain that

$$\Theta_\varepsilon\big|_{X_k\setminus Y} \geq \sqrt{-1}\partial\bar{\partial}\sigma_\varepsilon. \tag{4.4}$$

Furthermore, by (4.3) we can assume that $A_\varepsilon = \eta(\sigma_\varepsilon)$ for some smooth function $\eta : (-\infty, 0) \longrightarrow (0, +\infty)$ such that

$$\chi\zeta'' - \chi'' - \frac{(\chi')^2}{\eta} = 0. \tag{4.5}$$

Since it follows from Lemma 3.9 that

$$|s|^2\sqrt{-1}\sum_{i=1}^m ds^i \wedge d\bar{s}^i \geq \sqrt{-1}(\sum_{i=1}^m \bar{s}^i ds^i) \wedge (\sum_{i=1}^m s^i d\bar{s}^i), \tag{4.6}$$

we have

$$\begin{aligned}
&\sqrt{-1}\partial\bar{\partial}\sigma_\varepsilon\big|_{X_k\backslash Y}\\
= &\frac{m(|s|^2+\varepsilon^2)\sqrt{-1}\sum_{i=1}^m ds^i\wedge d\bar{s}^i - m\sqrt{-1}(\sum_{i=1}^m \bar{s}^i ds^i)\wedge(\sum_{i=1}^m s^i d\bar{s}^i)}{(|s|^2+\varepsilon^2)^2}\\
\geq &\frac{m\varepsilon^2\sqrt{-1}\sum_{i=1}^m ds^i\wedge d\bar{s}^i}{(|s|^2+\varepsilon^2)^2}\\
\geq &\frac{m\varepsilon^2\sqrt{-1}(\sum_{i=1}^m \bar{s}^i ds^i)\wedge(\sum_{i=1}^m s^i d\bar{s}^i)}{|s|^2(|s|^2+\varepsilon^2)^2}\\
= &\frac{m\varepsilon^2}{|s|^2}\sqrt{-1}\nu_\varepsilon\wedge\bar{\nu}_\varepsilon.
\end{aligned}$$

Then (4.1) follows from (4.4).

Hence,

$$\mathrm{B}_\varepsilon \geq \left[\frac{m\varepsilon^2}{|s|^2}\sqrt{-1}\nu_\varepsilon\wedge\bar{\nu}_\varepsilon,\, \Lambda\right] = \frac{m\varepsilon^2}{|s|^2}\mathrm{T}_{\bar{\nu}_\varepsilon}\mathrm{T}^*_{\bar{\nu}_\varepsilon} \tag{4.7}$$

on $X_k\backslash Y$ as an operator on $(n,1)$-forms, where $\mathrm{T}_{\bar{\nu}_\varepsilon}$ denotes the operator $\bar{\nu}_\varepsilon\wedge\bullet$ and $\mathrm{T}^*_{\bar{\nu}_\varepsilon}$ is its Hilbert adjoint operator.

Let $c\in(0,\frac{1}{2})$ be a positive number. It is easy to construct a smooth function $\theta:\mathbb{R}\longrightarrow[0,1]$ such that $\theta=0$ on $(-\infty,\frac{c}{2}]$, $\theta=1$ on $[1-\frac{c}{2},+\infty)$ and $|\theta'|\leq\frac{1+c}{1-c}$ on $\mathbb{R}$.

Define $g_\varepsilon=\bar{\partial}\big(\theta(\frac{\varepsilon^2}{|s|^2+\varepsilon^2})\tilde{f}\big)$, where $0<\varepsilon<\varepsilon_a$. Then $\bar{\partial}g_\varepsilon=0$ and

$$\begin{aligned}
g_\varepsilon &= -\theta'\big(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\big)\frac{\varepsilon^2\sum_{i=1}^m s^i d\bar{s}^i}{(|s|^2+\varepsilon^2)^2}\wedge\tilde{f}\\
&= -\bar{\nu}_\varepsilon\wedge\theta'(\frac{\varepsilon^2}{|s|^2+\varepsilon^2})\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\tilde{f}.
\end{aligned}$$

Then it follows from (4.7) that

$$\langle\mathrm{B}_\varepsilon^{-1}g_\varepsilon,g_\varepsilon\rangle_{L_{j,a,\varepsilon}}\big|_{X_k\backslash Y}\leq\frac{|s|^2}{m\varepsilon^2}\left|\theta'\big(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\big)\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\tilde{f}\right|^2_{L_{j,a,\varepsilon}}.$$

Hence,

$$\begin{aligned}
&\int_{X_k\backslash Y}\langle\mathrm{B}_\varepsilon^{-1}g_\varepsilon,g_\varepsilon\rangle_{L_{j,a,\varepsilon}}dV_X\\
\leq\ &\frac{e^a(1+c)^2}{m(1-c)^2}\int_{X_k\cap\{\sqrt{\frac{c}{2-c}}\varepsilon<|s|<\sqrt{\frac{2-c}{c}}\varepsilon\}}\frac{\varepsilon^2|\tilde{f}|^2e^{-\varphi_j}dV_X}{(|s|^2+\varepsilon^2)^2|s|^{2m-2}}.
\end{aligned}$$

Since s^i $(1 \leq i \leq m)$ can be viewed as transverse coordinates around Y, it is not hard to prove that

$$dV_X = dV_Y \cdot \frac{(\sqrt{-1})^{m^2} ds^1 \wedge \cdots \wedge ds^m \wedge d\bar{s}^1 \wedge \cdots \wedge d\bar{s}^m}{|ds^1 \wedge \cdots \wedge ds^m|^2},$$

at each point $x \in Y$ by a certain orthogonalization process on $T_X^*|_x$. Since $|\tilde{f}|^2 e^{-\varphi_j}$ and $| \wedge^m (ds)|^2$ are continuous around $\overline{X_k} \cap Y$, using a partition of unity $\{\xi_p\}_{p=1}^{p_0}$ around $\overline{X_k} \cap Y$ and the Fubini theorem, we get

$$\begin{aligned}
&\overline{\lim_{\varepsilon\to 0}} \int_{X_k\setminus Y} \langle \mathrm{B}_\varepsilon^{-1} g_\varepsilon, g_\varepsilon\rangle_{L_{j,a,\varepsilon}} dV_X \\
\leq\ & \frac{e^a(1+c)^2}{m(1-c)^2} \sum_{p=1}^{p_0} \overline{\lim_{\varepsilon\to 0}} \int_{X_k \cap \{\sqrt{\frac{c}{2-c}}\varepsilon<|s|<\sqrt{\frac{2-c}{c}}\varepsilon\}} \frac{\varepsilon^2 \xi_p |\tilde{f}|^2 e^{-\varphi_j} dV_X}{(|s|^2+\varepsilon^2)^2 |s|^{2m-2}} \\
\leq\ & \sum_{p=1}^{p_0} \Bigg(\overline{\lim_{\varepsilon\to 0}} \int_{\{z\in\mathbb{C}^m:\, \sqrt{\frac{c}{2-c}}\varepsilon<|z|<\sqrt{\frac{2-c}{c}}\varepsilon\}} \frac{\varepsilon^2 (\sqrt{-1})^{m^2} \wedge^m (dz) \wedge \wedge^m (d\bar{z})}{(|z|^2+\varepsilon^2)^2 |z|^{2m-2}} \\
& \times \frac{e^a(1+c)^2}{m(1-c)^2} \int_Y \frac{\xi_p |\tilde{f}|^2 e^{-\varphi_j}}{| \wedge^m (ds)|^2} dV_Y \Bigg) \\
\leq\ & \frac{e^a(1+c)^2}{m(1-c)^2} \int_Y \frac{|f|^2 e^{-\varphi_j} dV_Y}{| \wedge^m (ds)|^2} \overline{\lim_{\varepsilon\to 0}} \int_{z\in\mathbb{C}^m} \frac{\varepsilon^2 2^m dV_m}{(|z|^2+\varepsilon^2)^2 |z|^{2m-2}} \\
=\ & \frac{2^m e^a(1+c)^2}{m(1-c)^2} \int_Y \frac{|f|^2 e^{-\varphi_j} dV_Y}{| \wedge^m (ds)|^2} \int_{\mathbb{S}^{2m-1}} dS\, \overline{\lim_{\varepsilon\to 0}} \int_0^{+\infty} \frac{\varepsilon^2 r\, dr}{(r^2+\varepsilon^2)^2} \\
=\ & \frac{e^a(1+c)^2}{(1-c)^2} \frac{(2\pi)^m}{m!} \int_Y \frac{|f|^2 e^{-\varphi_j}}{| \wedge^m (ds)|^2} dV_Y,
\end{aligned}$$

where $z = (z^1, \cdots, z^m)$, $\wedge^m(dz) = dz^1 \wedge \cdots \wedge dz^m$, $\mathbb{S}^{2m-1}$ is the unit sphere in $\mathbb{C}^m$, dS is the surface measure on $\mathbb{S}^{2m-1}$ and $dV_m = 2^{-m}(\sqrt{-1})^{m^2} \wedge^m (dz) \wedge \wedge^m (d\bar{z})$. Then

$$\int_{X_k\setminus Y} \langle \mathrm{B}_\varepsilon^{-1} g_\varepsilon, g_\varepsilon\rangle_{L_{j,a,\varepsilon}} dV_X \leq \frac{e^a(1+c)^3}{(1-c)^2} \frac{(2\pi)^m}{m!} \int_Y \frac{|f|^2 e^{-\varphi_j}}{| \wedge^m (ds)|^2} dV_Y,$$

when ε is small enough. Then by Lemma 3.2 with $\delta = 0$, there exists $u_{k,j,a,c,\varepsilon} \in L^2(X_k \setminus Y, K_X \otimes L_{j,a,\varepsilon})$ such that

$$\bar{\partial} u_{k,j,a,c,\varepsilon} = g_\varepsilon \tag{4.8}$$

on $X_k \setminus Y$ and

$$\int_{X_k\setminus Y} \frac{|u_{k,j,a,c,\varepsilon}|^2 e^{-\varphi_j - \sigma - \zeta(\sigma_\varepsilon)}}{\tau_\varepsilon + A_\varepsilon} dV_X \leq \frac{e^a(1+c)^3}{(1-c)^2} \frac{(2\pi)^m}{m!} \int_Y \frac{|f|^2 e^{-\varphi_j} dV_Y}{| \wedge^m (ds)|^2}. \tag{4.9}$$

Since φ_j, σ, $\zeta(\sigma_\varepsilon)$, $\tau_\varepsilon + A_\varepsilon$ are all bounded above on $\overline{X_k}$ for each fixed ε, (4.9) implies that $u_{k,j,a,c,\varepsilon} \in L^2(X_k, K_X)$. Since g_ε is smooth, by Lemma 3.7, we get from (4.8) and (4.9) that

$$\bar\partial u_{k,j,a,c,\varepsilon} = g_\varepsilon = \bar\partial\Big(\theta(\frac{\varepsilon^2}{|s|^2+\varepsilon^2})\tilde f\Big) \tag{4.10}$$

holds on X_k and

$$\int_{X_k}\frac{|u_{k,j,a,c,\varepsilon}|^2e^{-\varphi_j-\sigma-\zeta(\sigma_\varepsilon)}}{\tau_\varepsilon+A_\varepsilon}dV_X \le \frac{e^a(1+c)^3}{(1-c)^2}\frac{(2\pi)^m}{m!}\int_Y\frac{|f|^2e^{-\varphi_j}dV_Y}{|\wedge^m(ds)|^2}. \tag{4.11}$$

Define $F_{k,j,a,c,\varepsilon} = -u_{k,j,a,c,\varepsilon} + \theta(\frac{\varepsilon^2}{|s|^2+\varepsilon^2})\tilde f$. Then (4.10) implies that $\bar\partial F_{k,j,a,c,\varepsilon} = 0$ on X_k. Hence, $F_{k,j,a,c,\varepsilon}$ is holomorphic on X_k. Thus, $u_{k,j,a,c,\varepsilon}$ is smooth on X_k. Since φ_j, $\zeta(\sigma_\varepsilon)$, $\tau_\varepsilon + A_\varepsilon$ are all bounded above on $\overline{X_k}$ for each fixed ε, it follows from (4.11) that $|u_{k,j,a,c,\varepsilon}|^2e^{-\sigma}$ is integrable on X_k. The non-integrability of $e^{-\sigma}$ along Y and the smoothness of $u_{k,j,a,c,\varepsilon}$ on X_k implies that $u_{k,j,a,c,\varepsilon} = 0$ on $X_k\cap Y$. Hence, $F_{k,j,a,c,\varepsilon} = f$ on $X_k\cap Y$.

Since

$$\langle\kappa_1+\kappa_2,\,\kappa_1+\kappa_2\rangle \le \langle\kappa_1,\kappa_1\rangle+\langle\kappa_2,\kappa_2\rangle+c\langle\kappa_1,\kappa_1\rangle+\frac{1}{c}\langle\kappa_2,\kappa_2\rangle, \tag{4.12}$$

for any inner product space $\big(\mathrm{H},\,\langle\bullet,\bullet\rangle\big)$, where $\kappa_1,\kappa_2\in\mathrm{H}$, it follows from $R(\sigma_\varepsilon)\le R(\sigma)$ and (4.11) that

$$\begin{aligned}
&\int_{X_k}\frac{|F_{k,j,a,c,\varepsilon}|^2e^{-\varphi_j}}{e^\sigma R(\sigma)}dV_X \\
\le\ & (1+c)\int_{X_k}\frac{1}{e^\sigma R(\sigma)}|u_{k,j,a,c,\varepsilon}|^2e^{-\varphi_j}dV_X \\
&+\frac{1+c}{c}\int_{X_k}\frac{1}{e^\sigma R(\sigma)}\Big|\theta(\frac{\varepsilon^2}{|s|^2+\varepsilon^2})\tilde f\Big|^2e^{-\varphi_j}dV_X \\
\le\ & (1+c)\Big(\sup_{X_k}\frac{(\tau_\varepsilon+A_\varepsilon)e^{\zeta(\sigma_\varepsilon)}}{R(\sigma_\varepsilon)}\Big)\int_{X_k}\frac{|u_{k,j,a,c,\varepsilon}|^2e^{-\varphi_j-\sigma-\zeta(\sigma_\varepsilon)}}{\tau_\varepsilon+A_\varepsilon}dV_X \\
&+\frac{1+c}{c}\int_{X_k\cap\{|s|<\sqrt{\frac{2-c}{c}}\varepsilon\}}\frac{1}{e^\sigma R(\sigma)}|\tilde f|^2e^{-\varphi_j}dV_X \\
\le\ & \Big(\sup_{X_k}\frac{(\tau_\varepsilon+A_\varepsilon)e^{\zeta(\sigma_\varepsilon)}}{R(\sigma_\varepsilon)}\Big)\frac{e^a(1+c)^4}{(1-c)^2}\frac{(2\pi)^m}{m!}\int_Y\frac{|f|^2e^{-\varphi_j}}{|\wedge^m(ds)|^2}dV_Y \\
&+C_1\int_{-\infty}^{2m\log\varepsilon+C_2}\frac{1}{R(t)}dt,
\end{aligned} \tag{4.13}$$

when ε is small enough, where C_1 and C_2 are two positive numbers independent of ε.

Since $\sup_{t\le 0}\big(e^tR(t)\big)<+\infty$, applying Montel's theorem and extracting weak limits of $\{F_{k,j,a,c,\varepsilon}\}_{\varepsilon>0}$ as $\varepsilon\to 0$, we get from (4.13) a holomorphic n-form $F_{k,j,a,c}$ such that $F_{k,j,a,c}=f$ on $X_k\cap Y$ and

$$\begin{aligned}(4.14)\qquad &\int_{X_k}\frac{|F_{k,j,a,c}|^2e^{-\varphi_j}}{e^{\sigma}R(\sigma)}dV_X\\ &\le\ \left(\sup_{X_k}\frac{(\tau_\varepsilon+A_\varepsilon)e^{\zeta(\sigma_\varepsilon)}}{R(\sigma_\varepsilon)}\right)\frac{e^a(1+c)^4}{(1-c)^2}\frac{(2\pi)^m}{m!}\int_Y\frac{|f|^2e^{-\varphi_j}}{|\wedge^m(ds)|^2}dV_Y.\end{aligned}$$

In order to get the optimal constant, it's natural to assume that

$$(4.15)\qquad \frac{(\tau_\varepsilon+A_\varepsilon)e^{\zeta(\sigma_\varepsilon)}}{R(\sigma_\varepsilon)}=1$$

on X_k. Then (4.14) and (4.15) imply that
(4.16)

$$\int_{X_k}\frac{|F_{k,j,a,c}|^2e^{-\varphi_j}dV_X}{e^{m\log|s|^2}R(m\log|s|^2-a)}\le\frac{(1+c)^4}{(1-c)^2}\frac{(2\pi)^m}{m!}\int_Y\frac{|f|^2e^{-\varphi_j}}{|\wedge^m(ds)|^2}dV_Y.$$

Since R is a continuous decreasing function on $(-\infty,0]$, $\sup_{t\le 0}\big(e^tR(t)\big)<+\infty$ and $\{\varphi_j\}_{j=1}^{+\infty}$ is a decreasing sequence such that $\lim_{j\to+\infty}\varphi_j=\varphi$, applying Montel's theorem and extracting weak limits of $\{F_{k,j,a,c}\}_{k,j,a,c}$, first as $c\to 0$, next as $a\to 0$, then as $j\to+\infty$, and, finally, as $k\to+\infty$, we get from (4.16) a holomorphic section F on X with values in $K_X\otimes L$ such that $F=f$ on Y and

$$\int_X\frac{|F|^2e^{-\varphi}}{e^{m\log|s|^2}R(m\log|s|^2)}dV_X\le\frac{(2\pi)^m}{m!}\int_Y\frac{|f|^2e^{-\varphi}}{|\wedge^m(ds)|^2}dV_Y.$$

Proposition 4.1 is, thus, proved.

Final step: solving ordinary differential equations.

We have already proved Proposition 4.1, provided that there exists suitable ζ, χ and η satisfying some assumptions. Then we will use those assumptions about ζ, χ and η to get their explicit expressions. Furthermore, we will check all the assumptions about ζ, χ and η.

(4.2), (4.5) and (4.15) amount to the following system of ordinary differential equations defined on $(-\infty,0)$:

$$(4.17)\qquad \chi(t)\zeta'(t)-\chi'(t)=1,$$

$$(4.18)\qquad \big(\chi(t)+\eta(t)\big)e^{\zeta(t)}=R(t),$$

$$(4.19)\qquad \frac{\big(\chi'(t)\big)^2}{\chi(t)\zeta''(t)-\chi''(t)}=\eta(t).$$

Moreover, we have assumed that ζ, χ and η are all smooth on $(-\infty,0)$, and that $\zeta>0$, $\chi>0$, $\eta>0$, $\zeta'>0$, $\chi'<0$ and $\chi(t)\ge-\frac{t}{2}$ on $(-\infty,0)$. In the proof of Proposition 4.1, we have assumed that $C_R=\int_{-\infty}^0\frac{1}{R(t)}dt=1$.

By (4.17), $(\chi\zeta' - \chi')' = 0$, i.e., $\chi\zeta'' - \chi'' = -\chi'\zeta'$. Then by (4.19), $\eta = -\frac{\chi'}{\zeta'}$. Hence, $\chi + \eta = \frac{\chi\zeta' - \chi'}{\zeta'} = \frac{1}{\zeta'}$ by (4.17). Then by (4.18), $e^{\zeta} = \zeta' R$, i.e., $(e^{-\zeta})' = -\frac{1}{R}$. Hence, $\zeta = -\log\big(b_1 - \int_{-\infty}^{t} \frac{1}{R(t_1)} dt_1\big)$ for some positive number $b_1 \geq 1$ and the assumption $\zeta' > 0$ on $(-\infty, 0)$ holds. The assumption $\zeta > 0$ on $(-\infty, 0)$ is equivalent to $b_1 - \int_{-\infty}^{t} \frac{1}{R(t_1)} dt_1 < 1$ on $(-\infty, 0)$. Hence, $b_1 \leq 1$ and then

$$\zeta = -\log\left(1 - \int_{-\infty}^{t} \frac{1}{R(t_1)} dt_1\right).$$

By (4.17), $e^{-\zeta}\chi\zeta' - e^{-\zeta}\chi' = e^{-\zeta}$, i.e., $(-e^{-\zeta}\chi)' = e^{-\zeta} = 1 - \int_{-\infty}^{t} \frac{1}{R(t_1)} dt_1$. Hence,

$$\chi = \frac{-t + \int_{-1}^{t} \big(\int_{-\infty}^{t_2} \frac{1}{R(t_1)} dt_1\big) dt_2 - b_2}{1 - \int_{-\infty}^{t} \frac{1}{R(t_1)} dt_1},$$

for some real number b_2. Define $\lambda_1 = -t + \int_{-1}^{t} \big(\int_{-\infty}^{t_2} \frac{1}{R(t_1)} dt_1\big) dt_2 - b_2$ on $(-\infty, 0]$. Then $\lambda_1 \in C^{\infty}(-\infty, 0]$, $\lambda_1' = -1 + \int_{-\infty}^{t} \frac{1}{R(t_1)} dt_1 < 0$ on $(-\infty, 0)$, $\lambda_1'' = \frac{1}{R} > 0$, $\lambda_1''' = -\frac{R'}{R^2} \geq 0$ and $\chi = -\frac{\lambda_1}{\lambda_1'}$ on $(-\infty, 0)$. The assumption $\chi > 0$ on $(-\infty, 0)$ is equivalent to $\lambda_1 > 0$ on $(-\infty, 0)$. Since $\lambda_1 \in C^{\infty}(-\infty, 0]$ and $\lambda_1' < 0$ on $(-\infty, 0)$, $\lambda_1 > 0$ on $(-\infty, 0)$ is equivalent to $\lambda_1(0) = \int_{-1}^{0} \big(\int_{-\infty}^{t_2} \frac{1}{R(t_1)} dt_1\big) dt_2 - b_2 \geq 0$, i.e., $b_2 \leq \int_{-1}^{0} \big(\int_{-\infty}^{t_2} \frac{1}{R(t_1)} dt_1\big) dt_2$. Since

$$\chi' = \frac{-(\lambda_1')^2 + \lambda_1\lambda_1''}{(\lambda_1')^2},$$

the assumption $\chi' < 0$ on $(-\infty, 0)$ is equivalent to $\lambda_2 := -(\lambda_1')^2 + \lambda_1\lambda_1'' < 0$ on $(-\infty, 0)$. Since $\lambda_2 \in C^{\infty}(-\infty, 0]$ and $\lambda_2' = -\lambda_1'\lambda_1'' + \lambda_1\lambda_1''' > 0$ on $(-\infty, 0)$, $\lambda_2 < 0$ on $(-\infty, 0)$ is equivalent to $\lambda_2(0) = R(0)^{-1}\big(\int_{-1}^{0} \big(\int_{-\infty}^{t_2} \frac{1}{R(t_1)} dt_1\big) dt_2 - b_2\big) \leq 0$, i.e., $b_2 \geq \int_{-1}^{0} \big(\int_{-\infty}^{t_2} \frac{1}{R(t_1)} dt_1\big) dt_2$. Hence, $b_2 = \int_{-1}^{0} \big(\int_{-\infty}^{t_2} \frac{1}{R(t_1)} dt_1\big) dt_2$ and

$$\chi = \frac{-t - \int_{t}^{0} \big(\int_{-\infty}^{t_2} \frac{1}{R(t_1)} dt_1\big) dt_2}{1 - \int_{-\infty}^{t} \frac{1}{R(t_1)} dt_1}.$$

Since $\eta = -\frac{\chi'}{\zeta'}$, $\chi' < 0$ and $\zeta' > 0$, we get $\eta > 0$ on $(-\infty, 0)$ and

$$\eta = \left(1 - \int_{-\infty}^{t} \frac{1}{R(t_1)} dt_1\right) R(t) + \frac{t + \int_{t}^{0} \big(\int_{-\infty}^{t_2} \frac{1}{R(t_1)} dt_1\big) dt_2}{1 - \int_{-\infty}^{t} \frac{1}{R(t_1)} dt_1}.$$

The smoothness on $(-\infty, 0)$ of ζ, χ and η is obvious. It's easy to check that the explicit expressions we have obtained are really solutions of the ordinary differential equations.

Define $\chi(0) = 0$. Then it's easy to check that χ is continuous on $(-\infty, 0]$. In order to check the assumption $\chi(t) \geq -\frac{t}{2}$ on $(-\infty, 0)$, it is sufficient to verify

$$\chi' + \frac{1}{2} = \frac{-\frac{1}{2}(\lambda_1')^2 + \lambda_1\lambda_1''}{(\lambda_1')^2} \leq 0$$

on $(-\infty, 0)$. Define $\lambda_3 = -\frac{1}{2}(\lambda_1')^2 + \lambda_1\lambda_1''$. Then $\lambda_3 \in C^\infty(-\infty, 0]$, $\lambda_3(0) = 0$ and $\lambda_3' = \lambda_1\lambda_1''' \geq 0$ on $(-\infty, 0)$. Hence, $\lambda_3 \leq 0$ on $(-\infty, 0)$ and $\chi' + \frac{1}{2} \leq 0$ on $(-\infty, 0)$. Thus, the assumption $\chi(t) \geq -\frac{t}{2}$ on $(-\infty, 0)$ holds.

In conclusion, all the previous assumptions about ζ, χ and η are suitable. q.e.d.

5. Two key propositions used to deal with singular metrics

At first, we define some notations in this section as follows.

$z := (z^1, \cdots, z^n)$,
$z' := (z^1, \cdots, z^m)$,
$z'' := (z^{m+1}, \cdots, z^n)$,
$\mathbb{B}_r^k :=$ the open ball in $\mathbb{C}^k$ centered at 0 with radius r,
$\mathbb{B}^k := \mathbb{B}_1^k$,
$\mu(\mathbb{B}_r^k) :=$ the $2k$-dimensional Lebesgue measure of $\mathbb{B}_r^k$,
$dV_k :=$ the $2k$-dimensional Lebesgue measure on $\mathbb{C}^k$,

where m and n are the same as in Theorem 1.1, k is a positive integer, and z'' will disappear if $m = n$.

The aim of this section is to prove two key propositions which will be used to deal with singular metrics of holomorphic line bundles in the proof of Theorem 1.1. One of them is a variant of a result in [**12**] concerning L^2 extensions for local holomorphic sections or functions, and the other one is a convergence property for integrals with plurisubharmonic weights.

Proposition 5.1. *Let R be a function in $\mathfrak{R}$, where $\mathfrak{R}$ is the same as in Theorem 1.1. Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain, ϕ be a plurisubharmonic function on Ω, and $w = (w^1, \cdots, w^m)$ be a family of holomorphic functions on Ω $(1 \leq m \leq n)$. Let*

$$Y := \{x \in \Omega : w(x) = 0\} \quad \textit{and} \quad U := \{x \in \Omega : |w(x)| < 1\}.$$

Assume that $\wedge^m(dw) := dw^1 \wedge \cdots \wedge dw^m$ is nonvanishing on Y. Then for every $\beta_1 \in (0, 1)$ and every holomorphic n-form f defined on a neighborhood of $\overline{U}$ in Ω satisfying

$$\int_U \frac{|f|^2 e^{-\phi}}{|w|^{2m} R(m \log |w|^2)} dV_n < +\infty,$$

there exists a holomorphic n-form F on Ω satisfying $F = f$ on Y,

(5.1)
$$\int_U \frac{|F|^2 e^{-\phi} dV_n}{|w|^{2m} R(m\log|w|^2)} \leq \left(2 + \frac{2(m+1)^2\beta_R}{R(0)\beta_1}\right) \int_U \frac{|f|^2 e^{-\phi} dV_n}{|w|^{2m} R(m\log|w|^2)},$$

and

(5.2)
$$\int_\Omega \frac{|F|^2 e^{-\phi} dV_n}{(1+|w|^2)^{m+\beta_1}} \leq \left(\beta_R + \frac{(m+1)^2\beta_R}{\beta_1 2^{\beta_1}}\right) \int_U \frac{|f|^2 e^{-\phi} dV_n}{|w|^{2m} R(m\log|w|^2)},$$

where $\beta_R := \sup_{t\leq 0}\left(e^t R(t)\right)$.

Proof. The proof is a slight modification of the one in [**12**], where $R(t)$ is equal to e^{-t}.

Since Ω is a pseudoconvex domain, there is a sequence of pseudoconvex subdomain $\Omega_k \subset\subset \Omega$ $(k = 1, 2, \cdots)$ such that $\cup_{k=1}^{+\infty}\Omega_k = \Omega$. Then for fixed k, by convolution we can get a decreasing family of smooth plurisubharmonic functions $\{\phi_j\}_{j=1}^{+\infty}$ defined on a neighborhood of $\overline{\Omega_k}$ such that $\lim_{j\to+\infty}\phi_j = \phi$.

Fix k and j. Let λ be the continuous n-form on Ω_k defined by

$$\begin{cases} (1-|w|^{m+1})f & \text{on } U\cap\Omega_k, \\ 0 & \text{on } \Omega_k\setminus U. \end{cases}$$

Then $\lambda = f$ on $Y\cap\Omega_k$ and it is easy to check that $g := \bar\partial\lambda$ is equal to

$$\begin{cases} -\frac{m+1}{2}|w|^{m-1}\sum\limits_{i=1}^{m} w^i d\bar w^i \wedge f & \text{on } U\cap\Omega_k, \\ 0 & \text{on } \Omega_k\setminus U, \end{cases}$$

in the sense of distribution theory. Then $g \in L^\infty(\Omega_k, \wedge^{n,1}T^*_\Omega)$.

Lemma 3.5 implies that $\Omega_k\setminus Y$ is a complete Kähler manifold. Let $\Omega_k\setminus Y$ be endowed with the Euclidean metric and let Q be the trivial line bundle on $\Omega_k\setminus Y$ equipped with the metric

$$e^{-\phi_j - m\log|w|^2 - \beta_1\log(1+|w|^2)}.$$

Then we want to solve a $\bar\partial$ equation on $\Omega_k\setminus Y$ by applying Lemma 3.2 to the case $\tau = 1$, $A = 0$ and $\delta = 0$ (in fact, the case $\tau = 1$ and $A = 0$ is the non-twisted version of Lemma 3.2). The key step in applying Lemma 3.2 is to estimate the term

$$\int_{\Omega_k\setminus Y} \langle \mathrm{B}^{-1}g, g\rangle_Q dV_n,$$

where $\mathrm{B} := [\sqrt{-1}\Theta_Q, \Lambda]$.

Set $\nu = \sum_{i=1}^m \bar w^i dw^i$. Then $g = -\frac{m+1}{2}|w|^{m-1}\bar\nu\wedge f$ on $U\cap\Omega_k$.

Since

$$\begin{aligned}
&\sqrt{-1}\Theta_Q\big|_{\Omega_k\setminus Y}\\
= \ & \sqrt{-1}\partial\bar\partial\phi_j + m\sqrt{-1}\partial\bar\partial\log|w|^2 + \beta_1\sqrt{-1}\partial\bar\partial\log(1+|w|^2)\\
\geq \ & \frac{\beta_1(1+|w|^2)\sqrt{-1}\sum_{i=1}^m dw^i\wedge d\bar w^i - \beta_1\sqrt{-1}(\sum_{i=1}^m \bar w^i dw^i)\wedge(\sum_{i=1}^m w^i d\bar w^i)}{(1+|w|^2)^2}\\
\geq \ & \frac{\beta_1\sqrt{-1}\nu\wedge\bar\nu}{|w|^2(1+|w|^2)^2},
\end{aligned}$$

by an inequality similar to (4.6), we get

$$\mathrm{B} \geq \frac{\beta_1}{|w|^2(1+|w|^2)^2}\mathrm{T}_{\bar\nu}\mathrm{T}_{\bar\nu}^*$$

on $\Omega_k \setminus Y$, where $T_{\bar\nu}$ is defined similarly as in (4.7). Then we get $\langle \mathrm{B}^{-1}g, g\rangle_Q\big|_{\Omega_k\setminus U} = 0$ and

$$\begin{aligned}
&\langle \mathrm{B}^{-1}g, g\rangle_Q\big|_{(U\cap\Omega_k)\setminus Y}\\
= \ & \Big\langle \mathrm{B}^{-1}\Big(-\frac{m+1}{2}|w|^{m-1}\bar\nu\wedge f\Big), -\frac{m+1}{2}|w|^{m-1}\bar\nu\wedge f\Big\rangle_Q\\
\leq \ & \frac{|w|^2(1+|w|^2)^2}{\beta_1}\left|\frac{m+1}{2}|w|^{m-1}f\right|^2 e^{-\phi_j - m\log|w|^2-\beta_1\log(1+|w|^2)}\\
= \ & \frac{(m+1)^2(1+|w|^2)^{2-\beta_1}}{4\beta_1}|f|^2e^{-\phi_j}\\
\leq \ & \frac{(m+1)^2}{\beta_1 2^{\beta_1}}|f|^2 e^{-\phi_j}.
\end{aligned}$$

Hence, it follows from Lemma 3.2 that there exists $u_{k,j} \in L^2(\Omega_k\setminus Y, K_\Omega\otimes Q)$ such that $\bar\partial u_{k,j} = g = \bar\partial\lambda$ on $\Omega_k\setminus Y$ and

$$\int_{\Omega_k\setminus Y}|u_{k,j}|_Q^2 dV_n \leq \int_{\Omega_k\setminus Y}\langle \mathrm{B}^{-1}g,g\rangle_Q dV_n.$$

Thus,

$$\begin{aligned}
&\int_{\Omega_k\setminus Y}\frac{|u_{k,j}|^2e^{-\phi_j}}{|w|^{2m}(1+|w|^2)^{\beta_1}}dV_n\\
\leq \ & \frac{(m+1)^2}{\beta_1 2^{\beta_1}}\int_{U\cap\Omega_k}|f|^2e^{-\phi_j}dV_n\\
\leq \ & \frac{(m+1)^2\beta_R}{\beta_1 2^{\beta_1}}\int_U \frac{|f|^2e^{-\phi}}{|w|^{2m}R(m\log|w|^2)}dV_n.
\end{aligned} \tag{5.3}$$

Hence, we have $u_{k,j} \in L^2(\Omega_k\setminus Y, K_\Omega)$. Since $g \in L^\infty(\Omega_k, \wedge^{n,1}T_\Omega^*)$, Lemma 3.7 implies that $\bar\partial u_{k,j} = g$ holds on Ω_k.

Let $F_{k,j} := \lambda - u_{k,j}$. Then $\bar\partial F_{k,j} = 0$ on Ω_k. Thus, $F_{k,j}$ is holomorphic on Ω_k. Hence, $u_{k,j}$ is continuous on Ω_k. Then the non-integrability of $|w|^{-2m}$ along Y implies that $u_{k,j} = 0$ on $Y \cap \Omega_k$. Therefore, $F_{k,j} = f$ on $Y \cap \Omega_k$.

It follows from (5.3) that

$$\int_{U\cap\Omega_k} \frac{|u_{k,j}|^2 e^{-\phi_j}}{|w|^{2m} R(m\log|w|^2)} dV_n \le \frac{2^{\beta_1}}{R(0)} \int_{U\cap\Omega_k} \frac{|u_{k,j}|^2 e^{-\phi_j}}{|w|^{2m}(1+|w|^2)^{\beta_1}} dV_n \le \frac{(m+1)^2\beta_R}{\beta_1 R(0)} \int_U \frac{|f|^2 e^{-\phi}}{|w|^{2m} R(m\log|w|^2)} dV_n.$$

Since

$$|F_{k,j}|^2\big|_{U\cap\Omega_k} \le 2|\lambda|^2 + 2|u_{k,j}|^2 \le 2|f|^2 + 2|u_{k,j}|^2,$$

we get

$$\begin{aligned} &\int_{U\cap\Omega_k} \frac{|F_{k,j}|^2 e^{-\phi_j}}{|w|^{2m} R(m\log|w|^2)} dV_n \\ \le\ & 2\int_{U\cap\Omega_k} \frac{(|f|^2 + |u_{k,j}|^2) e^{-\phi_j}}{|w|^{2m} R(m\log|w|^2)} dV_n \\ \le\ & \left(2 + \frac{2(m+1)^2\beta_R}{\beta_1 R(0)}\right) \int_U \frac{|f|^2 e^{-\phi}}{|w|^{2m} R(m\log|w|^2)} dV_n. \end{aligned} \tag{5.4}$$

Since $|F_{k,j}|^2\big|_{\Omega_k\setminus U} = |u_{k,j}|^2$ and

$$|F_{k,j}|^2\big|_{U\cap\Omega_k} \le (|f| + |u_{k,j}|)^2 \le (1+|w|^{2m})|f|^2 + (1 + \frac{1}{|w|^{2m}})|u_{k,j}|^2,$$

by (4.12), we get

$$\begin{aligned} \frac{|F_{k,j}|^2}{(1+|w|^2)^{m+\beta_1}}\bigg|_{U\cap\Omega_k} \ \le\ & \frac{|F_{k,j}|^2}{(1+|w|^{2m})(1+|w|^2)^{\beta_1}} \\ \le\ & |f|^2 + \frac{|u_{k,j}|^2}{|w|^{2m}(1+|w|^2)^{\beta_1}}, \end{aligned}$$

and

$$\frac{|F_{k,j}|^2}{(1+|w|^2)^{m+\beta_1}}\bigg|_{\Omega_k\setminus U} \le \frac{|u_{k,j}|^2}{|w|^{2m}(1+|w|^2)^{\beta_1}}.$$

Hence, it follows from (5.3) that

$$\begin{aligned} &\int_{\Omega_k} \frac{|F_{k,j}|^2 e^{-\phi_j}}{(1+|w|^2)^{m+\beta_1}} dV_n \\ \le\ & \int_U |f|^2 e^{-\phi} dV_n + \int_{\Omega_k} \frac{|u_{k,j}|^2 e^{-\phi_j}}{|w|^{2m}(1+|w|^2)^{\beta_1}} dV_n \\ \le\ & \left(\beta_R + \frac{(m+1)^2\beta_R}{\beta_1 2^{\beta_1}}\right) \int_U \frac{|f|^2 e^{-\phi}}{|w|^{2m} R(m\log|w|^2)} dV_n. \end{aligned} \tag{5.5}$$

The desired holomorphic n-form F on Ω and the L^2 estimates (5.1) and (5.2) can be obtained from (5.4) and (5.5) by applying Montel's theorem and extracting weak limits of $\{F_{k,j}\}_{k,j}$, first as $j \to +\infty$ and then as $k \to +\infty$. q.e.d.

Proposition 5.2. *Let $\phi(z', z'')$ be a plurisubharmonic function on $\mathbb{B}_r^m \times \mathbb{B}_r^{n-m}$ $(1 \leq m \leq n, r > 0)$ such that $\sup_{\mathbb{B}_r^m \times \mathbb{B}_r^{n-m}} \phi < +\infty$. Let $f(z'')$ be a holomorphic function on $\mathbb{B}_r^{n-m}$ and $h(z', z'')$ be a nonnegative continuous function on $\mathbb{B}_r^m \times \mathbb{B}_r^{n-m}$. Assume that C, β, β_1, c_1, c_2, r', r'' and $\varepsilon_{r'}$ are positive numbers such that $\beta_1 < 1$, $c_1 < c_2$, $r' < r'' < r$ and $\varepsilon_{r'} < \frac{r'}{4c_2}$. Suppose that $f_\varepsilon \in \mathcal{O}(\mathbb{B}_r^m \times \mathbb{B}_r^{n-m})$ $(\varepsilon \in (0, \varepsilon_{r'}))$ are a family of holomorphic functions satisfying $f_\varepsilon(0, z'') = f(z'')$ $(\forall z'' \in \mathbb{B}_r^{n-m},\ \forall \varepsilon \in (0, \varepsilon_{r'}))$,*

$$\sup_{\mathbb{B}_{r''}^m \times \mathbb{B}_{r''}^{n-m}} |f_\varepsilon| \leq C\varepsilon^{-\beta_1}, \ \forall \varepsilon \in (0, \varepsilon_{r'}), \tag{5.6}$$

and

$$\frac{1}{\varepsilon^{2m}} \int_{(z', z'') \in \mathbb{B}_{2c_2\varepsilon}^m \times \mathbb{B}_r^{n-m}} |f_\varepsilon(z', z'')|^2 e^{-(1+\beta)\phi(z', z'')} dV_n \leq C, \ \forall \varepsilon \in (0, \varepsilon_{r'}). \tag{5.7}$$

Then

$$\lim_{\varepsilon \to 0} \frac{1}{\mu(\mathbb{B}^m)} \int_{(\mathbb{B}_{c_2\varepsilon}^m \setminus \mathbb{B}_{c_1\varepsilon}^m) \times \mathbb{B}_{r'}^{n-m}} \frac{\varepsilon^2 h(z', z'') |f_\varepsilon(z', z'')|^2 e^{-\phi(z', z'')}}{m(|z'|^2 + \varepsilon^2)^2 |z'|^{2m-2}} dV_n$$
$$= \left(\frac{1}{c_1^2 + 1} - \frac{1}{c_2^2 + 1} \right) \int_{z'' \in \mathbb{B}_{r'}^{n-m}} h(0, z'') |f(z'')|^2 e^{-\phi(0, z'')} dV_{n-m}.$$

In order to prove Proposition 5.2, we prove the following lemma for plurisubharmonic functions at first.

Lemma 5.3. *Assume that c_2, r', $\varepsilon_{r'}$ and p are positive numbers such that $\varepsilon_{r'} < \frac{r'}{4c_2}$. Let m be a positive integer. Let $\widehat{\phi}(z')$ be a negative plurisubharmonic function on $\mathbb{B}_{r'}^m$ such that $\widehat{\phi}(0) > -\infty$. Put*

$$S_{p,\varepsilon} = \{w \in \mathbb{B}_{c_2}^m : \widehat{\phi}(\varepsilon w) < (1 + p)\widehat{\phi}(0)\}, \quad \varepsilon \in (0, \varepsilon_{r'}).$$

Then

$$\lim_{\varepsilon \to 0} \mu(S_{p,\varepsilon}) = 0,$$

where $\mu(S_{p,\varepsilon})$ denotes the $2m$-dimensional Lebesgue measure of $S_{p,\varepsilon}$.

Proof. Since $\widehat{\phi}(z')$ is a negative upper semicontinuous function on $\mathbb{B}_{r'}^m$ and $\widehat{\phi}(0) > -\infty$, we have that for every $q \in (0, 1)$, there exists $\varepsilon_q \in (0, \varepsilon_{r'})$ such that

$$\widehat{\phi}(\varepsilon w) \leq (1 - q)\widehat{\phi}(0),$$

for all $w \in \mathbb{B}_{c_2}^m$ whenever $\varepsilon \in (0, \varepsilon_q)$.

Since $\widehat{\phi}(\varepsilon w)$ is plurisubharmonic on $\mathbb{B}^m_{c_2}$ with respect to w for any $\varepsilon \in (0, \varepsilon_q)$, it follows from the mean value inequality that

$$\widehat{\phi}(0) \leq \frac{1}{\mu(\mathbb{B}^m_{c_2})} \int_{w \in \mathbb{B}^m_{c_2}} \widehat{\phi}(\varepsilon w) dV_m, \ \forall \varepsilon \in (0, \varepsilon_q).$$

Therefore, when $\varepsilon \in (0, \varepsilon_q)$, we have

$$\begin{aligned} \widehat{\phi}(0) &\leq \frac{1}{\mu(\mathbb{B}^m_{c_2})} \bigg(\int_{\mathbb{B}^m_{c_2} \setminus S_{p,\varepsilon}} \widehat{\phi}(\varepsilon w) dV_m + \int_{S_{p,\varepsilon}} \widehat{\phi}(\varepsilon w) dV_m \bigg) \\ &\leq \frac{(1-q)\widehat{\phi}(0)(\mu(\mathbb{B}^m_{c_2}) - \mu(S_{p,\varepsilon})) + (1+p)\widehat{\phi}(0)\mu(S_{p,\varepsilon})}{\mu(\mathbb{B}^m_{c_2})} \\ &= \widehat{\phi}(0) \bigg(1 - q + (p+q) \frac{\mu(S_{p,\varepsilon})}{\mu(\mathbb{B}^m_{c_2})} \bigg). \end{aligned}$$

Since $\widehat{\phi}(0) < 0$, we get

$$\mu(S_{p,\varepsilon}) \leq \frac{\mu(\mathbb{B}^m_{c_2})q}{p+q} \leq \frac{\mu(\mathbb{B}^m_{c_2})q}{p},$$

whenever $\varepsilon \in (0, \varepsilon_q)$. Hence, $\lim_{\varepsilon \to 0} \mu(S_{p,\varepsilon}) = 0$. q.e.d.

Now we begin to prove Proposition 5.2.

Proof. Let $\beta_h := \sup_{\mathbb{B}^m_{r'} \times \mathbb{B}^{n-m}_{r'}} h$.

Without loss of generality, we may suppose that ϕ is negative on $\mathbb{B}^m_r \times \mathbb{B}^{n-m}_r$. In fact, $\phi_1 := \phi - \sup_{\mathbb{B}^m_r \times \mathbb{B}^{n-m}_r} \phi - 1$ is a negative plurisubharmonic function on $\mathbb{B}^m_r \times \mathbb{B}^{n-m}_r$ and the conclusion of Proposition 5.2 will hold for ϕ if it holds for ϕ_1.

Now we want to estimate the supremum norms of f_ε and the partial derivatives of f_ε.

Since $|f_\varepsilon(z', z'')|^2$ is subharmonic with respect to z' and z'', applying the mean value inequality successively to z'' and z', we get from (5.7) that

$$\begin{aligned} (5.8) \qquad & \sup_{\mathbb{B}^m_{c_2\varepsilon} \times \mathbb{B}^{n-m}_{r'}} |f_\varepsilon|^2 \\ \leq \ & \frac{1}{\mu(\mathbb{B}^m_{c_2\varepsilon})\mu(\mathbb{B}^{n-m}_{r''-r'})} \int_{z' \in \mathbb{B}^m_{2c_2\varepsilon}} dV_m \int_{z'' \in \mathbb{B}^{n-m}_{r''}} |f_\varepsilon(z', z'')|^2 dV_{n-m} \\ \leq \ & \frac{C_1}{\varepsilon^{2m}} \int_{\mathbb{B}^m_{2c_2\varepsilon} \times \mathbb{B}^{n-m}_r} |f_\varepsilon|^2 e^{-(1+\beta)\phi} dV_n \\ \leq \ & C_1 C, \end{aligned}$$

where C_1 is a positive number independent of ε.

By (5.6) and Cauchy's estimate for holomorphic functions, we have

$$\sup_{\mathbb{B}^m_{2c_2\varepsilon}\times\mathbb{B}^{n-m}_{r'}}\left|\frac{\partial f_\varepsilon}{\partial z^j}\right|\le C_2\sup_{\mathbb{B}^m_{r''}\times\mathbb{B}^{n-m}_{r''}}|f_\varepsilon|\le C_2C\varepsilon^{-\beta_1},\tag{5.9}$$

for any $\varepsilon\in(0,\varepsilon_{r'})$ and any $j=1,\cdots,n$, where C_2 is a positive number independent of ε.

Let j be a positive integer. Then (5.7) implies that

$$\begin{aligned}
&\frac{1}{\varepsilon^{2m}}\int_{\{\phi\le -j\}\cap(\mathbb{B}^m_{c_2\varepsilon}\times\mathbb{B}^{n-m}_{r'})}|f_\varepsilon|^2e^{-\phi}dV_n\\
\le\ &\frac{1}{\varepsilon^{2m}}\int_{\{\phi\le -j\}\cap(\mathbb{B}^m_{c_2\varepsilon}\times\mathbb{B}^{n-m}_{r'})}|f_\varepsilon|^2e^{-(1+\beta)\phi-\beta j}dV_n\\
\le\ &Ce^{-\beta j},
\end{aligned}$$

for all $\varepsilon\in(0,\varepsilon_{r'})$.

Therefore, for every $b\in(0,1)$, there exists a positive integer j_b such that

$$\begin{aligned}
&\frac{1}{\mu(\mathbb{B}^m)}\int_{\{\phi\le -j_b\}\cap((\mathbb{B}^m_{c_2\varepsilon}\setminus\mathbb{B}^m_{c_1\varepsilon})\times\mathbb{B}^{n-m}_{r'})}\frac{\varepsilon^2h|f_\varepsilon|^2e^{-\phi}}{m(|z'|^2+\varepsilon^2)^2|z'|^{2m-2}}dV_n\\
\le\ &\frac{1}{m(c_1^2+1)^2c_1^{2m-2}\mu(\mathbb{B}^m)\varepsilon^{2m}}\int_{\{\phi\le -j_b\}\cap(\mathbb{B}^m_{c_2\varepsilon}\times\mathbb{B}^{n-m}_{r'})}h|f_\varepsilon|^2e^{-\phi}dV_n\\
\le\ &\frac{\beta_hCe^{-\beta j_b}}{m(c_1^2+1)^2c_1^{2m-2}\mu(\mathbb{B}^m)}\\
<\ &\frac{b}{2},
\end{aligned}\tag{5.10}$$

for all $\varepsilon\in(0,\varepsilon_{r'})$.

Set $\phi_b=\max\{\phi,-j_b\}$. Let

$$\Phi(z''):=\left(\frac{1}{c_1^2+1}-\frac{1}{c_2^2+1}\right)h(0,z'')|f(z'')|^2e^{-\phi_b(0,z'')},$$

and

$$\Phi_\varepsilon(z''):=\frac{1}{\mu(\mathbb{B}^m)}\int_{z'\in\mathbb{B}^m_{c_2\varepsilon}\setminus\mathbb{B}^m_{c_1\varepsilon}}\frac{\varepsilon^2h(z',z'')|f_\varepsilon(z',z'')|^2e^{-\phi_b(z',z'')}}{m(|z'|^2+\varepsilon^2)^2|z'|^{2m-2}}dV_m,$$

where $z''\in\mathbb{B}^{n-m}_{r'}$ and $\varepsilon\in(0,\varepsilon_{r'})$.

We claim that

$$\overline{\lim_{\varepsilon\to 0}}\Phi_\varepsilon(z'')\le\Phi(z''),\ \forall z''\in\mathbb{B}^{n-m}_{r'}.\tag{5.11}$$

It suffices to prove that (5.11) holds for every fixed $z_0''\in\mathbb{B}^{n-m}_{r'}$.

Set $\widehat{\phi}(z')=\phi_b(z',z_0'')$. Let $p>0$ be a positive number. Put

$$S_{p,\varepsilon}=\{w\in\mathbb{B}^m_{c_2}:\widehat{\phi}(\varepsilon w)<(1+p)\widehat{\phi}(0)\},\ \varepsilon\in(0,\varepsilon_{r'}).$$

Then by the change of variables $z' = \varepsilon w$, we get

$$
\begin{aligned}
&\Phi_\varepsilon(z_0'') \\
= \;& \frac{1}{\mu(\mathbb{B}^m)} \int_{w\in \mathbb{B}^m_{c_2}\setminus \mathbb{B}^m_{c_1}} \frac{h(\varepsilon w, z_0'')|f_\varepsilon(\varepsilon w, z_0'')|^2 e^{-\widehat{\phi}(\varepsilon w)}}{m(|w|^2+1)^2|w|^{2m-2}} dV_m \\
= \;& \frac{1}{\mu(\mathbb{B}^m)} \int_{w\in S_{p,\varepsilon}\setminus \mathbb{B}^m_{c_1}} \frac{h(\varepsilon w, z_0'')|f_\varepsilon(\varepsilon w, z_0'')|^2 e^{-\widehat{\phi}(\varepsilon w)}}{m(|w|^2+1)^2|w|^{2m-2}} dV_m \\
& + \frac{1}{\mu(\mathbb{B}^m)} \int_{w\in \mathbb{B}^m_{c_2}\setminus (\mathbb{B}^m_{c_1}\cup S_{p,\varepsilon})} \frac{h(\varepsilon w, z_0'')|f_\varepsilon(\varepsilon w, z_0'')|^2 e^{-\widehat{\phi}(\varepsilon w)}}{m(|w|^2+1)^2|w|^{2m-2}} dV_m \\
\leq \;& \frac{1}{\mu(\mathbb{B}^m)} \int_{S_{p,\varepsilon}\setminus \mathbb{B}^m_{c_1}} \frac{e^{j_b}\beta_h \sup\limits_{\mathbb{B}^m_{c_2\varepsilon}\times \mathbb{B}^{n-m}_{r'}} |f_\varepsilon|^2}{m(c_1^2+1)^2 c_1^{2m-2}} dV_m \\
& + \int_{\mathbb{B}^m_{c_2}\setminus (\mathbb{B}^m_{c_1}\cup S_{p,\varepsilon})} \frac{e^{-(1+p)\widehat{\phi}(0)} \sup\limits_{w\in \mathbb{B}^m_{c_2}} h(\varepsilon w, z_0'') \sup\limits_{w\in \mathbb{B}^m_{c_2}} |f_\varepsilon(\varepsilon w, z_0'')|^2}{m(|w|^2+1)^2|w|^{2m-2}\mu(\mathbb{B}^m)} dV_m \\
\leq \;& \frac{e^{j_b}\mu(S_{p,\varepsilon})C_1 C\beta_h}{m(c_1^2+1)^2 c_1^{2m-2}\mu(\mathbb{B}^m)} \\
& + \left(\frac{1}{c_1^2+1} - \frac{1}{c_2^2+1}\right) e^{-(1+p)\widehat{\phi}(0)} \sup_{w\in \mathbb{B}^m_{c_2}} h(\varepsilon w, z_0'') \sup_{w\in \mathbb{B}^m_{c_2}} |f_\varepsilon(\varepsilon w, z_0'')|^2,
\end{aligned}
$$

where we use the inequality (5.8) and the equality

$$
\frac{1}{\mu(\mathbb{B}^m)} \int_{w\in \mathbb{B}^m_{c_2}\setminus \mathbb{B}^m_{c_1}} \frac{1}{m(|w|^2+1)^2|w|^{2m-2}} dV_m = \frac{1}{c_1^2+1} - \frac{1}{c_2^2+1}, \tag{5.12}
$$

in the last inequality above. We will denote the two terms on the right-hand side of the last inequality above by $\gamma_{1,\varepsilon}$ and $\gamma_{2,\varepsilon}$ respectively.

Applying the mean value theorem to $f_\varepsilon(z', z'')$ on real lines, using the Cauchy–Schwarz inequality and then using the Cauchy–Riemann equation, we obtain from (5.9) that

$$
\begin{aligned}
& |f_\varepsilon(\varepsilon w, z'') - f(z'')|^2 \\
= \;& |f_\varepsilon(\varepsilon w, z'') - f_\varepsilon(0, z'')|^2 \\
\leq \;& |\varepsilon w|^2 \left(2 \sum_{j=1}^m \sup_{\mathbb{B}^m_{c_2\varepsilon}\times \mathbb{B}^{n-m}_{r'}} \left|\frac{\partial f_\varepsilon}{\partial z^j}\right|^2\right) \\
\leq \;& 2mc_2^2 C_2^2 C^2 \varepsilon^{2-2\beta_1},
\end{aligned} \tag{5.13}
$$

for any $(w, z'') \in \mathbb{B}^m_{c_2} \times \mathbb{B}^{n-m}_{r'}$. Then we have

$$
\lim_{\varepsilon\to 0} \sup_{(w,z'')\in \mathbb{B}^m_{c_2}\times \mathbb{B}^{n-m}_{r'}} |f_\varepsilon(\varepsilon w, z'') - f(z'')| = 0, \tag{5.14}
$$

since $\beta_1 \in (0,1)$. Hence,

$$\lim_{\varepsilon\to 0} \gamma_{2,\varepsilon} = \left(\frac{1}{c_1^2+1} - \frac{1}{c_2^2+1}\right) h(0,z_0'')|f(z_0'')|^2 e^{-(1+p)\widehat{\phi}(0)},$$

by the continuity of h. Since $\lim_{\varepsilon\to 0}\gamma_{1,\varepsilon} = 0$ by Lemma 5.3, we get

$$\overline{\lim_{\varepsilon\to 0}} \Phi_\varepsilon(z_0'') \le \left(\frac{1}{c_1^2+1} - \frac{1}{c_2^2+1}\right) h(0,z_0'')|f(z_0'')|^2 e^{-(1+p)\widehat{\phi}(0)}.$$

Then

$$\overline{\lim_{\varepsilon\to 0}} \Phi_\varepsilon(z_0'') \le \Phi(z_0''),$$

since p is an arbitrary positive number. Thus, we have proved (5.11).

Applying the change of variables $z' = \varepsilon w$ and (5.12) to the definition of $\Phi_\varepsilon(z'')$, we get that

$$\begin{aligned}
\Phi_\varepsilon(z'') &= \frac{1}{\mu(\mathbb{B}^m)} \int_{w\in \mathbb{B}^m_{c_2}\setminus \mathbb{B}^m_{c_1}} \frac{h(\varepsilon w, z'')|f_\varepsilon(\varepsilon w, z'')|^2 e^{-\phi_b(\varepsilon w, z'')}}{m(|w|^2+1)^2|w|^{2m-2}} dV_m \\
&\le e^{j_b}\beta_h \sup_{\mathbb{B}^m_{c_2\varepsilon}\times \mathbb{B}^{n-m}_{r'}} |f_\varepsilon|^2 \\
&\le C_1 C e^{j_b}\beta_h,
\end{aligned}$$

for all $z'' \in \mathbb{B}^{n-m}_{r'}$ by (5.8). Moreover, it is easy to see that

$$\Phi(z'') \le e^{j_b}\beta_h \sup_{\mathbb{B}^{n-m}_{r'}} |f|^2,$$

for all $z'' \in \mathbb{B}^{n-m}_{r'}$. Hence, by Fatou's lemma and (5.11), we obtain that

$$\begin{aligned}
&\overline{\lim_{\varepsilon\to 0}} \int_{\{\phi>-j_b\}\cap((\mathbb{B}^m_{c_2\varepsilon}\setminus \mathbb{B}^m_{c_1\varepsilon})\times \mathbb{B}^{n-m}_{r'})} \frac{\varepsilon^2 h(z',z'')|f_\varepsilon(z',z'')|^2 e^{-\phi(z',z'')}}{m(|z'|^2+\varepsilon^2)^2|z'|^{2m-2}} dV_n \\
\le\ & \overline{\lim_{\varepsilon\to 0}} \int_{(\mathbb{B}^m_{c_2\varepsilon}\setminus \mathbb{B}^m_{c_1\varepsilon})\times \mathbb{B}^{n-m}_{r'}} \frac{\varepsilon^2 h(z',z'')|f_\varepsilon(z',z'')|^2 e^{-\phi_b(z',z'')}}{m(|z'|^2+\varepsilon^2)^2|z'|^{2m-2}} dV_n \\
=\ & \mu(\mathbb{B}^m)\overline{\lim_{\varepsilon\to 0}} \int_{z''\in \mathbb{B}^{n-m}_{r'}} \Phi_\varepsilon(z'') dV_{n-m} \\
\le\ & \mu(\mathbb{B}^m) \int_{z''\in \mathbb{B}^{n-m}_{r'}} \overline{\lim_{\varepsilon\to 0}} \Phi_\varepsilon(z'') dV_{n-m} \\
\le\ & \mu(\mathbb{B}^m) \int_{z''\in \mathbb{B}^{n-m}_{r'}} \Phi(z'') dV_{n-m} \\
\le\ & \mu(\mathbb{B}^m)\left(\frac{1}{c_1^2+1} - \frac{1}{c_2^2+1}\right) \int_{z''\in \mathbb{B}^{n-m}_{r'}} h(0,z'')|f(z'')|^2 e^{-\phi(0,z'')} dV_{n-m}.
\end{aligned}$$

By combining (5.10) and the above inequality, we get that there exists $\varepsilon_b \in (0, \varepsilon_{r'})$ such that

$$\begin{aligned} & \frac{1}{\mu(\mathbb{B}^m)} \int_{(\mathbb{B}^m_{c_2\varepsilon} \setminus \mathbb{B}^m_{c_1\varepsilon}) \times \mathbb{B}^{n-m}_{r'}} \frac{\varepsilon^2 h(z', z'')|f_\varepsilon(z', z'')|^2 e^{-\phi(z', z'')}}{m(|z'|^2 + \varepsilon^2)^2 |z'|^{2m-2}} dV_n \\ < \ & \left(\frac{1}{c_1^2+1} - \frac{1}{c_2^2+1}\right) \int_{z'' \in \mathbb{B}^{n-m}_{r'}} h(0, z'')|f(z'')|^2 e^{-\phi(0, z'')} dV_{n-m} + b, \end{aligned}$$

for all $\varepsilon \in (0, \varepsilon_b)$. Hence,

$$\begin{aligned} & \overline{\lim_{\varepsilon \to 0}} \frac{1}{\mu(\mathbb{B}^m)} \int_{(\mathbb{B}^m_{c_2\varepsilon} \setminus \mathbb{B}^m_{c_1\varepsilon}) \times \mathbb{B}^{n-m}_{r'}} \frac{\varepsilon^2 h(z', z'')|f_\varepsilon(z', z'')|^2 e^{-\phi(z', z'')}}{m(|z'|^2 + \varepsilon^2)^2 |z'|^{2m-2}} dV_n \\ \leq \ & \left(\frac{1}{c_1^2+1} - \frac{1}{c_2^2+1}\right) \int_{z'' \in \mathbb{B}^{n-m}_{r'}} h(0, z'')|f(z'')|^2 e^{-\phi(0, z'')} dV_{n-m}. \end{aligned} \tag{5.15}$$

Since ϕ is plurisubharmonic,

$$\overline{\lim_{\varepsilon \to 0}} \phi(\varepsilon w, z'') = \phi(0, z''), \ \forall (w, z'') \in \mathbb{B}^m_{c_2} \times \mathbb{B}^{n-m}_{r'}.$$

Then using (5.12), (5.14), Fatou's lemma and the change of variables $z' = \varepsilon w$, we obtain that

$$\begin{aligned} & \mu(\mathbb{B}^m)\left(\frac{1}{c_1^2+1} - \frac{1}{c_2^2+1}\right) \int_{z'' \in \mathbb{B}^{n-m}_{r'}} h(0, z'')|f(z'')|^2 e^{-\phi(0, z'')} dV_{n-m} \\ = \ & \int_{(w, z'') \in (\mathbb{B}^m_{c_2} \setminus \mathbb{B}^m_{c_1}) \times \mathbb{B}^{n-m}_{r'}} \frac{h(0, z'')|f(z'')|^2 e^{-\phi(0, z'')}}{m(|w|^2+1)^2 |w|^{2m-2}} dV_n \\ = \ & \int_{(w, z'') \in (\mathbb{B}^m_{c_2} \setminus \mathbb{B}^m_{c_1}) \times \mathbb{B}^{n-m}_{r'}} \underline{\lim_{\varepsilon \to 0}} \frac{h(\varepsilon w, z'')|f_\varepsilon(\varepsilon w, z'')|^2 e^{-\phi(\varepsilon w, z'')}}{m(|w|^2+1)^2 |w|^{2m-2}} dV_n \\ \leq \ & \underline{\lim_{\varepsilon \to 0}} \int_{(w, z'') \in (\mathbb{B}^m_{c_2} \setminus \mathbb{B}^m_{c_1}) \times \mathbb{B}^{n-m}_{r'}} \frac{h(\varepsilon w, z'')|f_\varepsilon(\varepsilon w, z'')|^2 e^{-\phi(\varepsilon w, z'')}}{m(|w|^2+1)^2 |w|^{2m-2}} dV_n \\ = \ & \underline{\lim_{\varepsilon \to 0}} \int_{(z', z'') \in (\mathbb{B}^m_{c_2\varepsilon} \setminus \mathbb{B}^m_{c_1\varepsilon}) \times \mathbb{B}^{n-m}_{r'}} \frac{\varepsilon^2 h(z', z'')|f_\varepsilon(z', z'')|^2 e^{-\phi(z', z'')}}{m(|z'|^2+\varepsilon^2)^2 |z'|^{2m-2}} dV_n. \end{aligned}$$

Then the conclusion of Proposition 5.2 follows from (5.15) and the inequality above. q.e.d.

6. Proof of Theorem 1.1

Without loss of generality, we can suppose that $C_R = 1$. Otherwise, we replace R with $C_R R$ in the proof. If $f = 0$ on Y, then $F = 0$ satisfies the conclusion of Theorem 1.1. In the following proof, we assume that f is not 0 identically. Moreover, we will denote $|s|_E$ and $| \wedge^m (ds)|_E$ simply by $|s|$ and $| \wedge^m (ds)|$ respectively.

Let h_0 be any fixed smooth metric of L on X and let L_0 denote the line bundle L equipped with the metric h_0. Then $h_L = h_0 e^{-\varphi}$ for some global function φ on X. Let

$$\phi := \varphi + \psi.$$

Then ϕ is a quasi-plurisubharmonic function on X by the assumption in the theorem.

Since X is weakly pseudoconvex, there exists a smooth plurisubharmonic exhaustion function P on X. Let $X_k := \{P < k\}$ ($k = 1, 2, \cdots$, we choose P such that $X_1 \neq \emptyset$).

Our proof consists of several steps. We will discuss for fixed k until the end of Step 5.

Step 1: construction of a family of special smooth extensions $\tilde{f}_\varepsilon$ of f to a neighborhood of $\overline{X_k} \cap Y$ in X.

In order to deal with singular metrics of holomorphic line bundles on weakly pseudoconvex Kähler manifolds, we construct in this step a family of smooth extensions $\tilde{f}_\varepsilon$ of f satisfying some special estimates by using the results in Section 5.

Let $c \in (0, \frac{1}{2})$.

For the sake of clearness, we divide this step into four parts.

Part I: construction of local coordinate charts $\{V_i\}_{i=1}^N$, $\{U_i\}_{i=1}^N$ and a partition of unity $\{\xi_i\}_{i=1}^{N+1}$.

For any point $x \in Y$, we can find a local coordinate system

$$(V_x', z_x^1, \cdots, z_x^n),$$

in X centered at x and a local holomorphic frame $\{e_{x,j}\}_{j=1}^m$ of E on V_x', such that $s = \sum_{j=1}^m z_x^j e_{x,j}$ on V_x' and the frame $\{e_{x,j}\}_{j=1}^m$ is orthonormal at x.

Moreover, we assume that there exists a local holomorphic frame of L on V_x' and that the quasi-plurisubharmonic function ϕ can be written as a sum of a smooth function and a plurisubharmonic function on V_x'.

Let $\varepsilon_x \in (0, 1)$ be a fixed positive number such that

$$V_x := \{y \in V_x' : |z_x'(y)| < \varepsilon_x, |z_x''(y)| < \varepsilon_x\}$$

is relatively compact in V_x' and the inequalities

$$(1 - c)|z_x'|^2 \leq |s|^2 \leq (1 + c)|z_x'|^2, \tag{6.1}$$

and

$$1 - c \leq |e_{x,1} \wedge \cdots \wedge e_{x,m}|^2 \leq 1 + c$$

hold on a neighborhood of $\overline{V_x}$, where $z_x' := (z_x^1, \cdots, z_x^m)$ and $z_x'' := (z_x^{m+1}, \cdots, z_x^n)$ (z_x'' will disappear if $m = n$).

Since

$$| \wedge^m (ds)|^2 = |dz_i^1 \wedge \cdots \wedge dz_i^m|^2 |e_{i,1} \wedge \cdots \wedge e_{i,m}|^2,$$

where the norms are explained in Remark 1.1, we have

$$(1-c)|\wedge^m(dz'_x)|^2 \le |\wedge^m(ds)|^2 \le (1+c)|\wedge^m(dz'_x)|^2 \tag{6.2}$$

holds on a neighborhood of $\overline{V_x}$.

Let $\varepsilon'_x \in (0, \varepsilon_x)$ be a fixed positive number and set

$$U_x = \{y \in V'_x : |z'_x(y)| < \varepsilon'_x, |z''_x(y)| < \varepsilon'_x\}.$$

Since $\overline{X_k} \cap Y$ is compact, there exist points $x_1, x_2, \cdots, x_N \in \overline{X_k} \cap Y$ such that $\overline{X_k} \cap Y \subset \cup_{i=1}^N U_{x_i}$.

For simplicity, we will denote V_{x_i}, $z^j_{x_i}$ $(1 \le j \le n)$, $e_{x_i,j}$ $(1 \le j \le m)$, U_{x_i}, ε_{x_i}, ε'_{x_i}, z'_{x_i} and z''_{x_i} by V_i, z^j_i, $e_{i,j}$, U_i, ε_i, ε'_i, z'_i and z''_i respectively.

Choose an open set U_{N+1} in X such that $\overline{X_k} \cap Y \subset X \setminus \overline{U_{N+1}} \subset\subset \cup_{i=1}^N U_i$. Set $U = X \setminus \overline{U_{N+1}}$. For fixed k and c, we choose a fixed positive number $\varepsilon_0 \in (0, \sqrt{1-c}\min_{1\le i\le N}\varepsilon'_i)$ such that $\overline{X_k} \cap \big\{x \in X : |s(x)| \le \varepsilon_0\big\} \subset\subset U$.

Let $\{\xi_i\}_{i=1}^{N+1}$ be a partition of unity subordinate to the cover $\{U_i\}_{i=1}^{N+1}$ of X. Then $\operatorname{supp}\xi_i \subset\subset U_i$ for $i = 1, \cdots, N$ and $\sum_{i=1}^N \xi_i = 1$ on U.

Part II: construction of local holomorphic extensions $\widehat{f}_{i,\varepsilon}$ $(1 \le i \le N)$ of f to $V_i \cap \{|z'_i| < 3c_2\varepsilon\}$, where c_2 will be defined in this part.

Since we have assumed that ϕ can be written as a sum of a smooth function and a plurisubharmonic function on a neighborhood of $\overline{V_i}$ $(1 \le i \le N)$, by Lemma 3.8 and (6.2), there exists a positive number $\beta \in (0,1)$ such that

$$\int_{V_i\cap Y} \frac{|f|^2_{L_0} e^{-(1+\beta)\phi}}{|\wedge^m(dz'_i)|^2} dV_Y < +\infty \quad (1 \le i \le N). \tag{6.3}$$

Let $\widetilde{\lambda} : (-\infty, +\infty) \to [0, +\infty)$ be the function

$$\begin{cases} \big(\int_{-1}^1 e^{\frac{1}{t^2-1}} dt\big)^{-1} e^{\frac{1}{t^2-1}} & \text{if} \quad |t| < 1, \\ 0 & \text{if} \quad |t| \ge 1. \end{cases}$$

Then $\widetilde{\lambda}$ is smooth on $(-\infty, +\infty)$ with support contained in $[-1, 1]$ and $\int_{-\infty}^{+\infty} \widetilde{\lambda}(t)dt = 1$.

Set $R_0(t) = \frac{8m}{\beta} e^{-\frac{\beta t}{8m}}$, $t \in (-\infty, +\infty)$. Then $R_0 \in \mathfrak{R}$ and $C_{R_0} = 1$. Define $R(t) = R(0)$ on $(0, +\infty)$ and denote by R_1 the convolution $(\min\{R_0, R\}) * \widetilde{\lambda}$. Then it is easy to see that $R_1 \in \mathfrak{R}$ and

$$\min\{R_0(t+1), R(t+1)\} \le R_1(t) \le \min\{R_0(t-1), R(t-1)\}. \tag{6.4}$$

Let $c_1 := \sqrt{\frac{c}{(2-c)(1+c)}}$ and $c_2 := \sqrt{\frac{2-c}{c(1-c)}}$.

For each fixed $\varepsilon \in (0, \frac{\varepsilon_0}{4c_2})$, we apply Proposition 4.1 to the function R_1 and to the holomorphic section f on $V_i \cap Y$ with the L^2 condition (6.3), and then we obtain L^2 extensions of f from $V_i \cap Y$ to

$$V_i \cap \{|z_i'| < 3c_2\varepsilon\},$$

where we equip the line bundle L with the singular metric $h_0 e^{-(1+\beta)\phi}$ and take $s = \frac{z_i'}{3c_2\varepsilon}$ in Proposition 4.1. More precisely, there exists a uniform positive number $\widehat{C}$ (independent of ε) and holomorphic extensions $\widehat{f}_{i,\varepsilon}$ $(1 \le i \le N)$ of f from $V_i \cap Y$ to $V_i \cap \{|z_i'| < 3c_2\varepsilon\}$ such that

$$\begin{aligned}
&\int_{V_i \cap \{|z_i'| < 3c_2\varepsilon\}} \frac{|\widehat{f}_{i,\varepsilon}|^2_{L_0} e^{-(1+\beta)\phi}}{\left|\frac{z_i'}{3c_2\varepsilon}\right|^{2m} R_1\left(m \log\left|\frac{z_i'}{3c_2\varepsilon}\right|^2\right)} dV_X \\
\le\ & \widehat{C} \int_{V_i \cap Y} \frac{|f|^2_{L_0} e^{-(1+\beta)\phi}}{\left|\wedge^m d\left(\frac{z_i'}{3c_2\varepsilon}\right)\right|^2} dV_Y \\
=\ & \varepsilon^{2m} \widehat{C} (3c_2)^{2m} \int_{V_i \cap Y} \frac{|f|^2_{L_0} e^{-(1+\beta)\phi}}{|\wedge^m (dz_i')|^2} dV_Y.
\end{aligned}$$

Hence,

$$\int_{V_i \cap \{|z_i'| < 2c_2\varepsilon\}} \frac{|\widehat{f}_{i,\varepsilon}|^2_{L_0} e^{-(1+\beta)\phi}}{\left|\frac{z_i'}{2c_2\varepsilon}\right|^{2m} R_1\left(m \log\left|\frac{z_i'}{2c_2\varepsilon}\right|^2 + m \log \frac{4}{9}\right)} dV_X \le \widehat{C}_1 \varepsilon^{2m}, \tag{6.5}$$

for some positive number $\widehat{C}_1$ independent of ε.

Part III: construction of local holomorphic extensions $\tilde{f}_{i,\varepsilon}$ $(1 \le i \le N)$ of f to V_i.

For each fixed $\varepsilon \in (0, \frac{\varepsilon_0}{4c_2})$, applying Proposition 5.1 to the local extensions $\widehat{f}_{i,\varepsilon}$ $(1 \le i \le N)$ with the weight $(1+\beta)\phi$ and to the case $w = \frac{z_i'}{2c_2\varepsilon}$, $\Omega = V_i$ and $\beta_1 = \frac{1}{8}$, we obtain from (6.5) holomorphic sections $\tilde{f}_{i,\varepsilon}$ $(1 \le i \le N)$ on V_i satisfying $\tilde{f}_{i,\varepsilon} = \widehat{f}_{i,\varepsilon} = f$ on $V_i \cap Y$,

$$\int_{V_i \cap \{|z_i'| < 2c_2\varepsilon\}} \frac{|\tilde{f}_{i,\varepsilon}|^2_{L_0} e^{-(1+\beta)\phi}}{\left|\frac{z_i'}{2c_2\varepsilon}\right|^{2m} R_1\left(m \log\left|\frac{z_i'}{2c_2\varepsilon}\right|^2 + m \log \frac{4}{9}\right)} dV_X \le \widehat{C}_2 \varepsilon^{2m}, \tag{6.6}$$

and

$$\int_{V_i} \frac{|\tilde{f}_{i,\varepsilon}|^2_{L_0} e^{-(1+\beta)\phi}}{\left(1 + \left|\frac{z_i'}{2c_2\varepsilon}\right|^2\right)^{m+\frac{1}{8}}} dV_X \le \widehat{C}_3 \varepsilon^{2m}, \tag{6.7}$$

for some positive numbers $\widehat{C}_2$ and $\widehat{C}_3$ independent of ε.

Since $\sup_{t \le 0} \left(e^t R_1(t)\right) < +\infty$, it follows from (6.6) that

$$\int_{(z_i', z_i'') \in \mathbb{B}^m_{2c_2\varepsilon} \times \mathbb{B}^{n-m}_{\varepsilon_i}} |\tilde{f}_{i,\varepsilon}(z_i', z_i'')|^2_{L_0} e^{-(1+\beta)\phi(z_i', z_i'')} dV_X \le \widehat{C}_4 \varepsilon^{2m}, \tag{6.8}$$

for all $\varepsilon \in (0, \frac{\varepsilon_0}{4c_2})$, where $\widehat{C}_4$ is a positive number independent of ε.

Since $|z_i'| \le \varepsilon_i$ on V_i, it follows from (6.7) that

$$\int_{(z_i',z_i'')\in \mathbb{B}^m_{\varepsilon_i}\times\mathbb{B}^{n-m}_{\varepsilon_i}} |\tilde{f}_{i,\varepsilon}(z_i',z_i'')|^2_{L_0} e^{-(1+\beta)\phi(z_i',z_i'')} dV_X \le \widehat{C}_5 \varepsilon^{-\frac{1}{4}}, \tag{6.9}$$

for all $\varepsilon \in (0, \frac{\varepsilon_0}{4c_2})$, where $\widehat{C}_5$ is a positive number independent of ε.

Let $\varepsilon_i'' := \frac{\varepsilon_i'+\varepsilon_i}{2}$. Then by similar calculation as in (5.8), we get from (6.9) that

$$\sup_{\mathbb{B}^m_{\varepsilon_i''}\times\mathbb{B}^{n-m}_{\varepsilon_i''}} |\tilde{f}_{i,\varepsilon}(z_i',z_i'')|^2_{L_0} \le \widehat{C}_6 \varepsilon^{-\frac{1}{4}}, \tag{6.10}$$

for all $\varepsilon \in (0, \frac{\varepsilon_0}{4c_2})$, where $\widehat{C}_6$ is a positive number independent of ε.

(6.8) and (6.10) imply that the assumptions in Proposition 5.2 hold for $\tilde{f}_{i,\varepsilon}$. Since it is not hard to prove that

$$dV_X = dV_Y \cdot \frac{(\sqrt{-1})^{m^2} dz_i^1\wedge\cdots\wedge dz_i^m \wedge d\bar{z}_i^1\wedge\cdots\wedge d\bar{z}_i^m}{|dz_i^1\wedge\cdots\wedge dz_i^m|^2},$$

at each point $x \in U_i \cap Y$ by a certain orthogonalization process on $T_X^*|_x$, we apply Proposition 5.2 to $\tilde{f}_{i,\varepsilon}$ $(1 \le i \le N)$ and get

$$\begin{aligned} &\lim_{\varepsilon\to 0}\frac{1}{\mathrm{Vol}(\mathbb{B}^m)}\int_{U_i\cap\{c_1\varepsilon<|z_i'|<c_2\varepsilon\}} \frac{\varepsilon^2\xi_i|\tilde{f}_{i,\varepsilon}|^2_{L_0}e^{-\phi}}{m(|z_i'|^2+\varepsilon^2)^2|z_i'|^{2m-2}}dV_X \\ = \ & 2^m\left(\frac{1}{c_1^2+1}-\frac{1}{c_2^2+1}\right)\int_{U_i\cap Y}\frac{\xi_i|f|^2_{L_0}e^{-\phi}}{|\wedge^m dz_i'|^2}dV_Y, \end{aligned} \tag{6.11}$$

where $\mathrm{Vol}(\mathbb{B}^m)$ is the volume of the unit ball in $\mathbb{C}^m$ and the equality (6.11) will be used in Step 4.

Part IV: construction of a family of smooth extensions $\tilde{f}_\varepsilon$ of f to a neighborhood of $\overline{X_k} \cap Y$ in X.

Define $\tilde{f}_\varepsilon = \sum_{i=1}^N \xi_i \tilde{f}_{i,\varepsilon}$ for all $\varepsilon \in (0, \frac{\varepsilon_0}{4c_2})$.

Since for any $j = 1, \cdots, N$, $\tilde{f}_\varepsilon|_{U_j} = \sum_{i=1}^N \xi_i\tilde{f}_{j,\varepsilon} + \sum_{i=1}^N \xi_i(\tilde{f}_{i,\varepsilon}-\tilde{f}_{j,\varepsilon}) = \tilde{f}_{j,\varepsilon} + \sum_{i=1}^N \xi_i(\tilde{f}_{i,\varepsilon}-\tilde{f}_{j,\varepsilon})$, we have

$$|\mathrm{D}''\tilde{f}_\varepsilon|_{L_0}\big|_{U_j} = |\sum_{i=1}^N \bar\partial\xi_i\wedge(\tilde{f}_{i,\varepsilon}-\tilde{f}_{j,\varepsilon})|_{L_0}, \quad \forall\varepsilon\in(0,\frac{\varepsilon_0}{4c_2}). \tag{6.12}$$

For similar reasons as in (5.9) and (5.13), we get from (6.10) and (6.1) that

$$\begin{aligned} &|\bar\partial\xi_i\wedge(\tilde{f}_{i,\varepsilon}-\tilde{f}_{j,\varepsilon})|^2_{L_0}\big|_{U_i\cap U_j} \\ = \ & \left|\bar\partial\xi_i\wedge\left(\tilde{f}_{i,\varepsilon}(z_j',z_j'')-\tilde{f}_{i,\varepsilon}(0,z_j'')+\tilde{f}_{j,\varepsilon}(0,z_j'')-\tilde{f}_{j,\varepsilon}(z_j',z_j'')\right)\right|^2_{L_0} \end{aligned} \tag{6.13}$$

$$\leq \widehat{C}_7|z_j'|^2(\sup_{U_i}|\tilde{f}_{i,\varepsilon}|^2_{L_0}+\sup_{U_j}|\tilde{f}_{j,\varepsilon}|^2_{L_0})$$
$$\leq \widehat{C}_8|s|^2\varepsilon^{-\frac{1}{4}},$$

for all $\varepsilon\in(0,\frac{\varepsilon_0}{4c_2})$, where $\widehat{C}_7$ and $\widehat{C}_8$ are positive numbers independent of ε.

Step 2: the singularity attenuation process for the currents $\sqrt{-1}\partial\bar{\partial}\phi$ and $\sqrt{-1}\partial\bar{\partial}\psi$.

Part I: the process for the closed almost positive current $\sqrt{-1}\partial\bar{\partial}\phi$.

Since the singularities of $\sqrt{-1}\partial\bar{\partial}\log|s|^2$ and $\frac{\{\sqrt{-1}\Theta_E s,s\}}{|s|^2}$ obstruct the application of Lemma 3.4, we will work on the blow-up of X at first and then go back to X. The idea of using Lemma 3.4 and a blow-up to regularize curvature currents comes from [**30**].

Let $\widetilde{X}$ together with $\mu:\widetilde{X}\to X$ be the blow-up of X with center Y. Then μ is a proper holomorphic map and $\widetilde{X}$ is also weakly pseudoconvex. Let $\widetilde{X}_{k+1}:=\mu^{-1}(X_{k+1})$, $\widetilde{X}_k:=\mu^{-1}(X_k)$ and $\widetilde{Y}:=\mu^{-1}(Y)$. It is not hard to prove the following lemma and we won't give its proof.

Lemma 6.1. *There exists a positive number $\widetilde{n}_k$ such that*

$$\widetilde{\omega}_{k+1}:=\widetilde{n}_k\mu^*\omega+\sqrt{-1}\partial\bar{\partial}\log|s\circ\mu|^2-2\pi[\widetilde{Y}]$$

is a Kähler metric on $\widetilde{X}_{k+1}$.

It is not hard to see that

$$\sqrt{-1}\partial\bar{\partial}\log|s\circ\mu|^2-2\pi[\widetilde{Y}]$$

is a smooth real $(1,1)$-form on $\widetilde{X}$ and $\mu^*\big(\frac{\sqrt{-1}\{\Theta_E s,s\}}{|s|^2}\big|_{X\setminus Y}\big)$ is, in fact, smooth on $\widetilde{X}$ (not just smooth on $\widetilde{X}\setminus\widetilde{Y}$). Hence, there exists a smooth real $(1,1)$-form γ_1 on $\widetilde{X}$ such that

$$\gamma_1\big|_{\widetilde{X}\setminus\widetilde{Y}}=\mu^*\bigg(\frac{\sqrt{-1}\{\Theta_E s,s\}}{|s|^2}\bigg|_{X\setminus Y}\bigg).$$

Since $\mu:\widetilde{X}\setminus\widetilde{Y}\to X\setminus Y$ is biholomorphic and $[\widetilde{Y}]\big|_{\widetilde{X}\setminus\widetilde{Y}}=0$, the curvature inequalities (i) and (ii) in Theorem 1.1 implies that

$$\sqrt{-1}\partial\bar{\partial}(\phi\circ\mu)\big|_{\widetilde{X}\setminus\widetilde{Y}}+\gamma_2\big|_{\widetilde{X}\setminus\widetilde{Y}}\geq 0,$$

and

$$\sqrt{-1}\partial\bar{\partial}(\phi\circ\mu)\big|_{\widetilde{X}\setminus\widetilde{Y}}+\gamma_3\big|_{\widetilde{X}\setminus\widetilde{Y}}\geq 0$$

hold on $\widetilde{X}\setminus\widetilde{Y}$, where

$$\gamma_2:=\sqrt{-1}\mu^*\Theta_{L_0}+m\sqrt{-1}\partial\bar{\partial}\log|s\circ\mu|^2-2m\pi[\widetilde{Y}],\quad \gamma_3:=\gamma_2-\frac{\gamma_1}{\alpha\circ\mu}.$$

Since γ_2 and γ_3 are continuous real $(1,1)$-forms on $\widetilde{X}$, and $\phi\circ\mu$ is quasi-plurisubharmonic on $\widetilde{X}$, we get that

$$\sqrt{-1}\partial\bar{\partial}(\phi\circ\mu)+\gamma_2\geq 0, \tag{6.14}$$

and

$$\sqrt{-1}\partial\bar{\partial}(\phi\circ\mu)+\gamma_3\geq 0 \tag{6.15}$$

hold on $\widetilde{X}$. Since there must exist a continuous nonnegative $(1,1)$-form ϖ_{k+1} on the Kähler manifold $(\widetilde{X}_{k+1},\widetilde{\omega}_{k+1})$ such that

$$(\sqrt{-1}\Theta_{T_{\widetilde{X}_{k+1}}}+\varpi_{k+1}\otimes\mathrm{Id}_{T_{\widetilde{X}_{k+1}}})(\kappa_1\otimes\kappa_2,\kappa_1\otimes\kappa_2)\geq 0\quad(\forall\kappa_1,\kappa_2\in T_{\widetilde{X}_{k+1}})$$

holds on $\widetilde{X}_{k+1}$, by Lemma 3.4, we obtain from (6.14) and (6.15) a family of functions $\{\widetilde{\phi}_{\varsigma,\rho}\}_{\varsigma>0,\rho\in(0,\rho_1)}$ on $\widetilde{X}_{k+1}$ such that

(i) $\widetilde{\phi}_{\varsigma,\rho}$ is quasi-plurisubharmonic on a neighborhood of the closure of $\widetilde{X}_k$, smooth on $\widetilde{X}_{k+1}\setminus E_\varsigma(\phi\circ\mu)$, increasing with respect to ς and ρ on $\widetilde{X}_k$, and converges to $\phi\circ\mu$ on $\widetilde{X}_k$ as $\rho\to 0$,

(ii) $\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\widetilde{\phi}_{\varsigma,\rho}\geq-\frac{\gamma_2}{\pi}-\varsigma\varpi_{k+1}-\delta_\rho\widetilde{\omega}_{k+1}$ on $\widetilde{X}_k$,

(iii) $\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\widetilde{\phi}_{\varsigma,\rho}\geq-\frac{\gamma_3}{\pi}-\varsigma\varpi_{k+1}-\delta_\rho\widetilde{\omega}_{k+1}$ on $\widetilde{X}_k$,

where $E_\varsigma(\phi\circ\mu):=\{x\in\widetilde{X}:\nu(\phi\circ\mu,x)\geq\varsigma\}$ $(\varsigma>0)$ is the ς-upperlevel set of Lelong numbers of $\phi\circ\mu$, and $\{\delta_\rho\}$ is an increasing family of positive numbers such that $\lim_{\rho\to 0}\delta_\rho=0$.

Since $\widetilde{\omega}_{k+1}$ is a Kähler metric on $\widetilde{X}_{k+1}$ by Lemma 6.1 and $\widetilde{X}_k$ is relatively compact in $\widetilde{X}_{k+1}$, there exists a positive number $n_k>1$ such that $n_k\widetilde{\omega}_{k+1}\geq\varpi_{k+1}$ holds on $\widetilde{X}_k$. Take $\varsigma=\delta_\rho$ and denote $\widetilde{\phi}_{\delta_\rho,\rho}$ simply by $\widetilde{\phi}_\rho$. Then $\widetilde{\phi}_\rho$ is quasi-plurisubharmonic on a neighborhood of the closure of $\widetilde{X}_k$, smooth on $\widetilde{X}_{k+1}\setminus E_{\delta_\rho}(\phi\circ\mu)$, increasing with respect to ρ on $\widetilde{X}_k$, and converges to $\phi\circ\mu$ on $\widetilde{X}_k$ as $\rho\to 0$. Furthermore,

$$\sqrt{-1}\partial\bar{\partial}\widetilde{\phi}_\rho+\gamma_2+2\pi n_k\delta_\rho\widetilde{\omega}_{k+1}\geq 0,$$

and

$$\sqrt{-1}\partial\bar{\partial}\widetilde{\phi}_\rho+\gamma_3+2\pi n_k\delta_\rho\widetilde{\omega}_{k+1}\geq 0$$

hold on $\widetilde{X}_k$. Since $\mu:\widetilde{X}_k\setminus\widetilde{Y}\to X_k\setminus Y$ is biholomorphic, we get that

$$\sqrt{-1}\partial\bar{\partial}(\widetilde{\phi}_\rho\circ\mu^{-1})+(\mu^{-1})^*\gamma_2+2\pi n_k\delta_\rho(\mu^{-1})^*\widetilde{\omega}_{k+1}\geq 0,$$

and

$$\sqrt{-1}\partial\bar{\partial}(\widetilde{\phi}_\rho\circ\mu^{-1})+(\mu^{-1})^*\gamma_3+2\pi n_k\delta_\rho(\mu^{-1})^*\widetilde{\omega}_{k+1}\geq 0$$

hold on $X_k\setminus Y$. Then, replacing γ_2, γ_3 and $\widetilde{\omega}_{k+1}$ with their definitions, we obtain that

$$\begin{aligned}&\sqrt{-1}\partial\bar{\partial}(\widetilde{\phi}_\rho\circ\mu^{-1})+\sqrt{-1}\Theta_{L_0}+(m+2\pi n_k\delta_\rho)\sqrt{-1}\partial\bar{\partial}\log|s|^2\\ \geq\;&-2\pi n_k\widetilde{n}_k\delta_\rho\omega,\end{aligned} \tag{6.16}$$

and

$$\sqrt{-1}\partial\bar{\partial}(\widetilde{\phi}_\rho \circ \mu^{-1}) + \sqrt{-1}\Theta_{L_0} + (m + 2\pi n_k \delta_\rho)\sqrt{-1}\partial\bar{\partial}\log|s|^2 \geq \frac{\sqrt{-1}\{\Theta_E s, s\}}{\alpha|s|^2} - 2\pi n_k \widetilde{n}_k \delta_\rho \omega \tag{6.17}$$

hold on $X_k \setminus Y$.

Since $E_{\delta_\rho}(\phi \circ \mu)$ is an analytic set in $\widetilde{X}$, Remmert's proper mapping theorem implies that

$$\Sigma_\rho := \mu\big(E_{\delta_\rho}(\phi \circ \mu)\big)$$

is an analytic set in X. By Lemma 3.5, $X_k \setminus (Y \cup \Sigma_\rho)$ is a complete Kähler manifold.

It follows from the properties of $\widetilde{\phi}_\rho$ that $\widetilde{\phi}_\rho \circ \mu^{-1}$ is smooth on $X_{k+1} \setminus (Y \cup \Sigma_\rho)$, increasing with respect to ρ on $X_k \setminus Y$, uniformly bounded above on $X_k \setminus Y$ with respect to ρ, and converges to ϕ on $X_k \setminus Y$ as $\rho \to 0$.

In Step 3, we will use $\widetilde{\phi}_\rho \circ \mu^{-1}$ to construct a smooth metric of L on $X_k \setminus (Y \cup \Sigma_\rho)$.

Part II: the process for the closed positive current $\sqrt{-1}\partial\bar{\partial}\psi$.

Let l be a positive integer such that $l > 2m \sup_{X_{k+1}} \alpha + m \sup_{X_{k+1}} \log|s|^2$ and $-l < \sup_{X_{k+1}} \psi$. Set $\psi_l = \max\{\psi, -l\}$. Then ψ_l is plurisubharmonic on X and

$$\psi_l + m\log|s|^2 \leq -2m\alpha \tag{6.18}$$

on X_{k+1} by the inequality (iii) in Theorem 1.1. Since ψ_l is a locally bounded function on X, the Lelong numbers of ψ_l on X are all 0. Since there must exist a continuous nonnegative $(1,1)$-form ϖ on the Kähler manifold (X, ω) such that

$$(\sqrt{-1}\Theta_{T_X} + \varpi \otimes \mathrm{Id}_{T_X})(\kappa_1 \otimes \kappa_2, \kappa_1 \otimes \kappa_2) \geq 0 \quad (\forall \kappa_1, \kappa_2 \in T_X)$$

holds on X, Lemma 3.4 implies that there is a family of functions $\{\psi_{l,\varsigma,\rho}\}_{\varsigma>0, \rho\in(0,\rho_2)}$ on X such that

(i) $\psi_{l,\varsigma,\rho}$ is quasi-plurisubharmonic on X_{k+1}, smooth on X, increasing with respect to ς and ρ on X_{k+1}, and converges to ψ_l on X_{k+1} as $\rho \to 0$,

(ii) $\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\psi_{l,\varsigma,\rho} \geq -\varsigma\varpi - \delta'_\rho\omega$ on X_{k+1},

where $\{\delta'_\rho\}$ is an increasing family of positive numbers such that $\lim_{\rho\to 0} \delta'_\rho = 0$. We can assume that $\delta'_\rho = \delta_\rho$ since we can replace them by $\max\{\delta'_\rho, \delta_\rho\}$.

Since X_{k+1} is relatively compact in X, there exists a positive number $n'_k > 1$ such that $n'_k\omega \geq \varpi$ holds on X_{k+1}. Take $\varsigma = \delta_\rho$ and denote $\psi_{l,\delta_\rho,\rho}$ simply by $\psi_{l,\rho}$. Then $\psi_{l,\rho}$ is quasi-plurisubharmonic on X_{k+1}, smooth on X, increasing with respect to ρ on X_{k+1} and converges to ψ_l on X_{k+1} as $\rho \to 0$. Furthermore,

$$\sqrt{-1}\partial\bar{\partial}\psi_{l,\rho} \geq -2\pi n'_k \delta_\rho \omega \tag{6.19}$$

holds on X_{k+1}.

Step 3: construction of special weights and twist factors.

Let ζ, χ and η be the same functions as in Section 4, whose explicit expressions are given in the final step there.

Let $a \in (0,1]$ and put $\sigma_{\rho,\varepsilon} = \psi_{l,\rho} + m\log(|s|^2+\varepsilon^2) - a$. Then by the inequality (6.18), there exists a positive number $\varepsilon_a \in (0,\varepsilon_0)$ and a positive number $\rho_a \in (0, \min\{\rho_1,\rho_2\})$ such that $\sigma_{\rho,\varepsilon} \leq -2m\alpha - \frac{a}{2}$ on $\overline{X_k}$ for any $\varepsilon \in (0,\varepsilon_a)$ and any $\rho \in (0,\rho_a)$, where ε_0, ρ_1 and ρ_2 are the same as in Step 1 and Step 2 respectively.

Let $L_{\rho,\varepsilon}$ denote the line bundle L on $X_k \setminus (Y \cup \Sigma_\rho)$ equipped with the new metric

$$h_{\rho,\varepsilon} := h_0 e^{-\widetilde{\phi}_\rho \circ \mu^{-1} - (m+2\pi n_k\delta_\rho)\log|s|^2 - \zeta(\sigma_{\rho,\varepsilon})}.$$

Let $\tau_\varepsilon := \chi(\sigma_{\rho,\varepsilon})$ and $A_\varepsilon := \eta(\sigma_{\rho,\varepsilon})$. Set $\mathrm{B}_{\rho,\varepsilon} = [\Theta_{\rho,\varepsilon}, \Lambda]$ on $X_k \setminus (Y \cup \Sigma_\rho)$, where

$$\Theta_{\rho,\varepsilon} := \tau_\varepsilon \sqrt{-1}\Theta_{L_{\rho,\varepsilon}} - \sqrt{-1}\partial\bar{\partial}\tau_\varepsilon - \sqrt{-1}\frac{\partial\tau_\varepsilon \wedge \bar{\partial}\tau_\varepsilon}{A_\varepsilon}.$$

Set $\nu_\varepsilon = \frac{\{\mathrm{D}'s,s\}}{|s|^2+\varepsilon^2}$. We want to prove

$$\Theta_{\rho,\varepsilon}\big|_{X_k\setminus(Y\cup\Sigma_\rho)} \geq \frac{m\varepsilon^2}{|s|^2}\sqrt{-1}\nu_\varepsilon \wedge \bar{\nu}_\varepsilon - \big(2\pi n_k \widetilde{n}_k \chi(\sigma_{\rho,\varepsilon})\delta_\rho + 2\pi n'_k\delta_\rho\big)\omega. \tag{6.20}$$

It follows from (4.17) and (4.19) that

$$\begin{aligned} &\Theta_{\rho,\varepsilon}\big|_{X_k\setminus(Y\cup\Sigma_\rho)} \\ = \ & \chi(\sigma_{\rho,\varepsilon})\big(\sqrt{-1}\Theta_{L_0} + \sqrt{-1}\partial\bar{\partial}(\widetilde{\phi}_\rho \circ \mu^{-1}) \\ & +(m+2\pi n_k\delta_\rho)\sqrt{-1}\partial\bar{\partial}\log|s|^2\big) \\ & +\big(\chi(\sigma_{\rho,\varepsilon})\zeta'(\sigma_{\rho,\varepsilon}) - \chi'(\sigma_{\rho,\varepsilon})\big)\sqrt{-1}\partial\bar{\partial}\sigma_{\rho,\varepsilon} \\ & +\left(\chi(\sigma_{\rho,\varepsilon})\zeta''(\sigma_{\rho,\varepsilon}) - \chi''(\sigma_{\rho,\varepsilon}) - \frac{\big(\chi'(\sigma_{\rho,\varepsilon})\big)^2}{\eta(\sigma_{\rho,\varepsilon})}\right)\sqrt{-1}\partial\sigma_{\rho,\varepsilon} \wedge \bar{\partial}\sigma_{\rho,\varepsilon} \\ = \ & \chi(\sigma_{\rho,\varepsilon})\big(\sqrt{-1}\Theta_{L_0} + \sqrt{-1}\partial\bar{\partial}(\widetilde{\phi}_\rho \circ \mu^{-1}) \\ & +(m+2\pi n_k\delta_\rho)\sqrt{-1}\partial\bar{\partial}\log|s|^2\big) + \sqrt{-1}\partial\bar{\partial}\sigma_{\rho,\varepsilon}. \end{aligned} \tag{6.21}$$

By Lemma 3.9, we have

$$|s|^2\sqrt{-1}\{\mathrm{D}'s,\mathrm{D}'s\} \geq \sqrt{-1}\{\mathrm{D}'s,s\} \wedge \{s,\mathrm{D}'s\}.$$

Hence, (6.19) implies that

$$\begin{aligned}
&\sqrt{-1}\partial\bar\partial\sigma_{\rho,\varepsilon}\big|_{X_k\setminus(Y\cup\Sigma_\rho)}\\
= \;& \frac{m\sqrt{-1}\{\mathrm{D}'s,\mathrm{D}'s\}}{|s|^2+\varepsilon^2} - \frac{m\sqrt{-1}\{\mathrm{D}'s,s\}\wedge\{s,\mathrm{D}'s\}}{(|s|^2+\varepsilon^2)^2} - \frac{m\sqrt{-1}\{\Theta_E s,s\}}{|s|^2+\varepsilon^2}\\
&+\sqrt{-1}\partial\bar\partial\psi_{l,\rho}\\
\geq \;& \frac{m\varepsilon^2}{|s|^2}\frac{\sqrt{-1}\{\mathrm{D}'s,s\}\wedge\{s,\mathrm{D}'s\}}{(|s|^2+\varepsilon^2)^2} - \frac{m\sqrt{-1}\{\Theta_E s,s\}}{|s|^2+\varepsilon^2} - 2\pi n_k'\delta_\rho\omega\\
= \;& \frac{m\varepsilon^2}{|s|^2}\sqrt{-1}\nu_\varepsilon\wedge\bar\nu_\varepsilon - \frac{m\sqrt{-1}\{\Theta_E s,s\}}{|s|^2+\varepsilon^2} - 2\pi n_k'\delta_\rho\omega.
\end{aligned}$$

Then it follows from (6.21) that

$$\begin{aligned}
&\Theta_{\rho,\varepsilon}\big|_{X_k\setminus(Y\cup\Sigma_\rho)}\\
\geq \;& \chi(\sigma_{\rho,\varepsilon})\big(\sqrt{-1}\Theta_{L_0}+\sqrt{-1}\partial\bar\partial(\widetilde{\phi}_\rho\circ\mu^{-1})\\
&+(m+2\pi n_k\delta_\rho)\sqrt{-1}\partial\bar\partial\log|s|^2\big) - \frac{m\sqrt{-1}\{\Theta_E s,s\}}{|s|^2+\varepsilon^2}\\
&+\frac{m\varepsilon^2}{|s|^2}\sqrt{-1}\nu_\varepsilon\wedge\bar\nu_\varepsilon - 2\pi n_k'\delta_\rho\omega.
\end{aligned}$$

Since $\chi(\sigma_{\rho,\varepsilon}) \geq -\frac{\sigma_{\rho,\varepsilon}}{2} \geq m\alpha$ by the assumption $\chi(t) \geq -\frac{t}{2}$, it follows from (6.16) and (6.17) that

$$\begin{aligned}
&\chi(\sigma_{\rho,\varepsilon})\big(\sqrt{-1}\Theta_{L_0}+\sqrt{-1}\partial\bar\partial(\widetilde{\phi}_\rho\circ\mu^{-1})\\
&+(m+2\pi n_k\delta_\rho)\sqrt{-1}\partial\bar\partial\log|s|^2\big) - \frac{m\sqrt{-1}\{\Theta_E s,s\}}{|s|^2+\varepsilon^2}\\
= \;& \chi(\sigma_{\rho,\varepsilon})\big(\sqrt{-1}\Theta_{L_0}+\sqrt{-1}\partial\bar\partial(\widetilde{\phi}_\rho\circ\mu^{-1})\\
&+(m+2\pi n_k\delta_\rho)\sqrt{-1}\partial\bar\partial\log|s|^2+2\pi n_k\widetilde{n}_k\delta_\rho\omega\big)\\
&-2\pi n_k\widetilde{n}_k\chi(\sigma_{\rho,\varepsilon})\delta_\rho\omega - \frac{m\alpha|s|^2}{|s|^2+\varepsilon^2}\frac{\sqrt{-1}\{\Theta_E s,s\}}{\alpha|s|^2}\\
\geq \;& \frac{m\alpha|s|^2}{|s|^2+\varepsilon^2}\bigg(\sqrt{-1}\Theta_{L_0}+\sqrt{-1}\partial\bar\partial(\widetilde{\phi}_\rho\circ\mu^{-1})\\
&+(m+2\pi n_k\delta_\rho)\sqrt{-1}\partial\bar\partial\log|s|^2+2\pi n_k\widetilde{n}_k\delta_\rho\omega - \frac{\sqrt{-1}\{\Theta_E s,s\}}{\alpha|s|^2}\bigg)\\
&-2\pi n_k\widetilde{n}_k\chi(\sigma_{\rho,\varepsilon})\delta_\rho\omega\\
\geq \;& -2\pi n_k\widetilde{n}_k\chi(\sigma_{\rho,\varepsilon})\delta_\rho\omega
\end{aligned}$$

on $X_k\setminus(Y\cup\Sigma_\rho)$. Hence, we get (6.20) as desired.

Let β and c be as in Step 1. Define $\beta_0 = \frac{\beta}{2(1+\beta)}$. Inspired by an idea of Yi (see [**29**] or [**30**]), we choose an increasing family of positive numbers $\{\rho_\varepsilon\}_{\varepsilon\in(0,\varepsilon_a)}$ such that $\lim_{\varepsilon\to 0}\rho_\varepsilon = 0$, $\rho_\varepsilon < \rho_a$ $(\forall \varepsilon\in(0,\varepsilon_a))$,

$$2\pi n_k\widetilde{n}_k\chi(2m\log\varepsilon - l - 1)\delta_{\rho_\varepsilon} + 2\pi n_k'\delta_{\rho_\varepsilon} < \varepsilon^{\beta_0}, \quad \forall\varepsilon\in(0,\varepsilon_a), \tag{6.22}$$

$$4\pi n_k\delta_{\rho_\varepsilon} < \beta_0, \quad \forall\varepsilon\in(0,\varepsilon_a), \tag{6.23}$$

and

$$\left(\sqrt{\frac{c}{2-c}}\varepsilon\right)^{4\pi n_k\delta_{\rho_\varepsilon}} > \frac{1}{1+c}, \quad \forall\varepsilon\in(0,\varepsilon_a). \tag{6.24}$$

Since $\sigma_{\rho,\varepsilon} \geq 2m\log\varepsilon - l - 1$ on X_k and χ is decreasing, we have $\chi(\sigma_{\rho,\varepsilon}) \leq \chi(2m\log\varepsilon - l - 1)$ on X_k. Then it follows from (6.20) and (6.22) that

$$\Theta_{\rho_\varepsilon,\varepsilon}\big|_{X_k\setminus(Y\cup\Sigma_{\rho_\varepsilon})} \geq \frac{m\varepsilon^2}{|s|^2}\sqrt{-1}\nu_\varepsilon\wedge\bar{\nu}_\varepsilon - \varepsilon^{\beta_0}\omega.$$

Hence,

$$\mathrm{B}_{\rho_\varepsilon,\varepsilon} + \varepsilon^{\beta_0}\mathrm{I} \geq \left[\frac{m\varepsilon^2}{|s|^2}\sqrt{-1}\nu_\varepsilon\wedge\bar{\nu}_\varepsilon,\, \Lambda\right] = \frac{m\varepsilon^2}{|s|^2}\mathrm{T}_{\bar{\nu}_\varepsilon}\mathrm{T}^*_{\bar{\nu}_\varepsilon} \geq 0 \tag{6.25}$$

on $X_k\setminus(Y\cup\Sigma_{\rho_\varepsilon})$ as an operator on $(n,1)$-forms, where $\mathrm{T}_{\bar{\nu}_\varepsilon}$ denotes the operator $\bar{\nu}_\varepsilon\wedge\bullet$ and $\mathrm{T}^*_{\bar{\nu}_\varepsilon}$ is its Hilbert adjoint operator.

Step 4: construction of suitably truncated forms and solving $\bar{\partial}$ globally with L^2 estimates.

In this step and Step 5, we will denote $\mathrm{B}_{\rho_\varepsilon,\varepsilon}$, $L_{\rho_\varepsilon,\varepsilon}$ and $\sigma_{\rho_\varepsilon,\varepsilon}$ simply by B_ε, L_ε and σ_ε respectively.

Let c be as in Step 1. It is easy to construct a smooth function $\theta:\mathbb{R}\longrightarrow[0,1]$ such that $\theta = 0$ on $(-\infty,\frac{c}{2}]$, $\theta = 1$ on $[1-\frac{c}{2},+\infty)$ and $|\theta'| \leq \frac{1+c}{1-c}$ on $\mathbb{R}$.

Define $g_\varepsilon = \mathrm{D}''\big(\theta(\frac{\varepsilon^2}{|s|^2+\varepsilon^2})\tilde{f}_\varepsilon\big)$, where $\tilde{f}_\varepsilon$ is constructed in Step 1 and $0 < \varepsilon < \min\big\{\sqrt{\frac{c}{2-c}}\varepsilon_0,\, \varepsilon_a\big\}$ (ε_0 and ε_a are the same as in Step 1 and Step 3 respectively). Then $\mathrm{D}''g_\varepsilon = 0$ and

$$\begin{aligned} g_\varepsilon &= -\theta'\Big(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\Big)\frac{\varepsilon^2\{s,\mathrm{D}'s\}}{(|s|^2+\varepsilon^2)^2}\wedge\tilde{f}_\varepsilon + \theta\Big(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\Big)\mathrm{D}''\tilde{f}_\varepsilon \\ &= g_{1,\varepsilon} + g_{2,\varepsilon}, \end{aligned}$$

where $g_{1,\varepsilon}$ denotes $-\bar{\nu}_\varepsilon\wedge\theta'(\frac{\varepsilon^2}{|s|^2+\varepsilon^2})\frac{\varepsilon^2\tilde{f}_\varepsilon}{|s|^2+\varepsilon^2}$ and $g_{2,\varepsilon}$ denotes $\theta(\frac{\varepsilon^2}{|s|^2+\varepsilon^2})\mathrm{D}''\tilde{f}_\varepsilon$.

It follows from (4.12) and (6.25) that

$$\begin{aligned} &\langle(\mathrm{B}_\varepsilon + 2\varepsilon^{\beta_0}\mathrm{I})^{-1}g_\varepsilon, g_\varepsilon\rangle_{L_\varepsilon}\big|_{X_k\setminus(Y\cup\Sigma_{\rho_\varepsilon})} \\ \leq\;& (1+c)\langle(\mathrm{B}_\varepsilon + 2\varepsilon^{\beta_0}\mathrm{I})^{-1}g_{1,\varepsilon}, g_{1,\varepsilon}\rangle_{L_\varepsilon} \\ &+\frac{1+c}{c}\langle(\mathrm{B}_\varepsilon + 2\varepsilon^{\beta_0}\mathrm{I})^{-1}g_{2,\varepsilon}, g_{2,\varepsilon}\rangle_{L_\varepsilon} \\ \leq\;& (1+c)\langle(\mathrm{B}_\varepsilon + \varepsilon^{\beta_0}\mathrm{I})^{-1}g_{1,\varepsilon}, g_{1,\varepsilon}\rangle_{L_\varepsilon} + \frac{1+c}{c}\langle\frac{1}{\varepsilon^{\beta_0}}g_{2,\varepsilon}, g_{2,\varepsilon}\rangle_{L_\varepsilon}. \end{aligned} \tag{6.26}$$

By (6.25), we have

$$\begin{aligned}&\langle(\mathrm{B}_\varepsilon+\varepsilon^{\beta_0}\mathrm{I})^{-1}g_{1,\varepsilon},g_{1,\varepsilon}\rangle_{L_\varepsilon}\big|_{X_k\setminus(Y\cup\Sigma_{\rho_\varepsilon})}\\ \le\ &\frac{|s|^2}{m\varepsilon^2}\Big|\theta'\big(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\big)\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\tilde{f}_\varepsilon\Big|^2_{L_\varepsilon}.\end{aligned}$$

Then $\zeta>0$ implies that

$$\begin{aligned}&I_{1,\varepsilon}\\ :=\ &\int_{X_k\setminus(Y\cup\Sigma_{\rho_\varepsilon})}\langle(\mathrm{B}_\varepsilon+\varepsilon^{\beta_0}\mathrm{I})^{-1}g_{1,\varepsilon},g_{1,\varepsilon}\rangle_{L_\varepsilon}dV_X\\ \le\ &\frac{(1+c)^2}{(1-c)^2}\int_{X_k\cap\{\sqrt{\frac{c}{2-c}}\varepsilon<|s|<\sqrt{\frac{2-c}{c}}\varepsilon\}}\frac{\varepsilon^2|\tilde{f}_\varepsilon|^2_{L_0}e^{-\widetilde{\phi}_{\rho_\varepsilon}\circ\mu^{-1}}dV_X}{m(|s|^2+\varepsilon^2)^2|s|^{2m-2+4\pi n_k\delta_{\rho_\varepsilon}}}.\end{aligned}$$

Since $\widetilde{\phi}_{\rho_\varepsilon}\circ\mu^{-1}\ge\phi$ on $X_k\setminus Y$, it follows from (6.24) that

$$\begin{aligned}&I_{1,\varepsilon}\\ \le\ &\frac{(1+c)^2}{(1-c)^2}\int_{X_k\cap\{\sqrt{\frac{c}{2-c}}\varepsilon<|s|<\sqrt{\frac{2-c}{c}}\varepsilon\}}\frac{\big(\sqrt{\frac{c}{2-c}}\varepsilon\big)^{-4\pi n_k\delta_{\rho_\varepsilon}}\varepsilon^2|\tilde{f}_\varepsilon|^2_{L_0}e^{-\phi}dV_X}{m(|s|^2+\varepsilon^2)^2|s|^{2m-2}}\\ \le\ &\frac{(1+c)^3}{(1-c)^2}\int_{X_k\cap\{\sqrt{\frac{c}{2-c}}\varepsilon<|s|<\sqrt{\frac{2-c}{c}}\varepsilon\}}\frac{\varepsilon^2|\tilde{f}_\varepsilon|^2_{L_0}e^{-\phi}dV_X}{m(|s|^2+\varepsilon^2)^2|s|^{2m-2}}.\end{aligned}$$

Since

$$|\tilde{f}_\varepsilon|^2_{L_0}\big|_U=|\sum_{i=1}^N\sqrt{\xi_i}\cdot\sqrt{\xi_i}\tilde{f}_{i,\varepsilon}|^2_{L_0}\le(\sum_{i=1}^N\xi_i)(\sum_{i=1}^N\xi_i|\tilde{f}_{i,\varepsilon}|^2_{L_0})=\sum_{i=1}^N\xi_i|\tilde{f}_{i,\varepsilon}|^2_{L_0},$$

by the Cauchy–Schwarz inequality, we have

$$I_{1,\varepsilon}\le\frac{(1+c)^3}{(1-c)^2}\sum_{i=1}^N\int_{X_k\cap\{\sqrt{\frac{c}{2-c}}\varepsilon<|s|<\sqrt{\frac{2-c}{c}}\varepsilon\}}\frac{\varepsilon^2\xi_i|\tilde{f}_{i,\varepsilon}|^2_{L_0}e^{-\phi}dV_X}{m(|s|^2+\varepsilon^2)^2|s|^{2m-2}}.$$

Then it follows from (6.1), (6.11) and (6.2) that

$$\begin{aligned}&\overline{\lim_{\varepsilon\to0}}I_{1,\varepsilon}\\ \le\ &\sum_{i=1}^N\overline{\lim_{\varepsilon\to0}}\Big(\frac{(1+c)^3}{(1-c)^2}\int_{X_k\cap\{\sqrt{\frac{c}{2-c}}\varepsilon<|s|<\sqrt{\frac{2-c}{c}}\varepsilon\}}\frac{\varepsilon^2\xi_i|\tilde{f}_{i,\varepsilon}|^2_{L_0}e^{-\phi}dV_X}{m(|s|^2+\varepsilon^2)^2|s|^{2m-2}}\Big)\\ \le\ &\sum_{i=1}^N\overline{\lim_{\varepsilon\to0}}\int_{U_i\cap\{c_1\varepsilon\le|z_i'|\le c_2\varepsilon\}}\frac{\frac{(1+c)^3}{(1-c)^2}\cdot\varepsilon^2\xi_i|\tilde{f}_{i,\varepsilon}|^2_{L_0}e^{-\phi}dV_X}{m\big((1-c)|z_i'|^2+\varepsilon^2\big)^2\big((1-c)|z_i'|^2\big)^{m-1}}\\ \le\ &\sum_{i=1}^N\overline{\lim_{\varepsilon\to0}}\int_{U_i\cap\{c_1\varepsilon\le|z_i'|\le c_2\varepsilon\}}\frac{\frac{(1+c)^3}{(1-c)^{m+3}}\cdot\varepsilon^2\xi_i|\tilde{f}_{i,\varepsilon}|^2_{L_0}e^{-\phi}dV_X}{m(|z_i'|^2+\varepsilon^2)^2|z_i'|^{2m-2}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{2^m(1+c)^3\mathrm{Vol}(\mathbb{B}^m)}{(1-c)^{m+3}}\left(\frac{1}{c_1^2+1}-\frac{1}{c_2^2+1}\right)\sum_{i=1}^N\int_{U_i\cap Y}\frac{\xi_i|f|^2_{L_0}e^{-\phi}dV_Y}{|\wedge^m(dz_i')|^2}\\
&\le \frac{(1+c)^4}{(1-c)^{m+3}}\frac{(2\pi)^m}{m!}\int_Y\frac{|f|^2_{L_0}e^{-\phi}}{|\wedge^m(ds)|^2}dV_Y,
\end{aligned}
$$

where c_1 and c_2 are defined as in Step 1. Then

$$
I_{1,\varepsilon}\le\frac{(1+c)^5}{(1-c)^{m+3}}\frac{(2\pi)^m}{m!}\int_Y\frac{|f|^2_{L_0}e^{-\phi}}{|\wedge^m(ds)|^2}dV_Y, \tag{6.27}
$$

when ε is small enough.

Since $\zeta(\sigma_\varepsilon)>0$ and $\widetilde{\phi}_{\rho_\varepsilon}\circ\mu^{-1}\ge\phi$ on $X_k\setminus Y$, by (6.23), we have

$$
\begin{aligned}
I_{2,\varepsilon} &:= \int_{X_k\setminus(Y\cup\Sigma_{\rho_\varepsilon})}\langle\frac{1}{\varepsilon^{\beta_0}}g_{2,\varepsilon},g_{2,\varepsilon}\rangle_{L_\varepsilon}dV_X\\
&\le \frac{1}{\varepsilon^{\beta_0}}\int_{X_k\cap\{|s|<\sqrt{\frac{2-c}{c}}\varepsilon\}}\frac{|\mathrm{D}''\tilde{f}_\varepsilon|^2_{L_0}e^{-\widetilde{\phi}_{\rho_\varepsilon}\circ\mu^{-1}}}{|s|^{2m+4\pi n_k\delta_{\rho_\varepsilon}}}dV_X\\
&\le \frac{1}{\varepsilon^{\beta_0}}\int_{X_k\cap\{|s|<\sqrt{\frac{2-c}{c}}\varepsilon\}}\frac{|\mathrm{D}''\tilde{f}_\varepsilon|^2_{L_0}e^{-\phi}}{|s|^{2m+\beta_0}}dV_X.
\end{aligned}
$$

Then it follows from (6.12) that $I_{2,\varepsilon}$ is bounded by the sum of the terms

$$
\frac{N}{\varepsilon^{\beta_0}}\int_{U_{i,j,\varepsilon}}\frac{|\bar\partial\xi_i\wedge(\tilde{f}_{i,\varepsilon}-\tilde{f}_{j,\varepsilon})|^2_{L_0}e^{-\phi}}{|s|^{2m+\beta_0}}dV_X\quad(1\le i,j\le N),
$$

by the Cauchy–Schwarz inequality, where $U_{i,j,\varepsilon}:=U_i\cap U_j\cap\{|s|<\sqrt{\frac{2-c}{c}}\varepsilon\}$.

Since R_1 is a positive decreasing function, (6.1) and (6.6) imply that for $i=1,\cdots,N$,

$$
\begin{aligned}
&\int_{V_i\cap\{|s|<\sqrt{\frac{2-c}{c}}\varepsilon\}}\frac{|\tilde{f}_{i,\varepsilon}|^2_{L_0}e^{-(1+\beta)\phi}}{|s|^{2m}R_1(m\log|s|^2)}dV_X \qquad (6.28)\\
&\le \int_{V_i\cap\{|z_i'|<2c_2\varepsilon\}}\frac{|\tilde{f}_{i,\varepsilon}|^2_{L_0}e^{-(1+\beta)\phi}}{(1-c)^m|z_i'|^{2m}R_1(m\log|z_i'|^2+m\log(1+c))}dV_X\\
&\le \widehat{C}_9\widehat{C}_2,
\end{aligned}
$$

for some positive number $\widehat{C}_9$ independent of ε when ε is small enough.

By the Hölder inequality, (6.28), (6.13) and (6.4), we get (note that $\beta_0 := \frac{\beta}{2(1+\beta)}$)

$$\begin{aligned}
&\int_{U_{i,j,\varepsilon}} \frac{|\bar\partial\xi_i \wedge (\tilde f_{i,\varepsilon} - \tilde f_{j,\varepsilon})|^2_{L_0} e^{-\phi}}{|s|^{2m+\beta_0}} dV_X \\
\le\ & \Big(\int_{U_{i,j,\varepsilon}} \frac{|\bar\partial\xi_i \wedge (\tilde f_{i,\varepsilon} - \tilde f_{j,\varepsilon})|^2_{L_0} e^{-(1+\beta)\phi}}{|s|^{2m} R_1(m\log|s|^2)} dV_X\Big)^{\frac{1}{1+\beta}} \\
& \times\Big(\int_{U_{i,j,\varepsilon}} \frac{|\bar\partial\xi_i \wedge (\tilde f_{i,\varepsilon} - \tilde f_{j,\varepsilon})|^2_{L_0} \big(R_1(m\log|s|^2)\big)^{\frac{1}{\beta}}}{|s|^{2m+\beta_0\cdot\frac{1+\beta}{\beta}}} dV_X\Big)^{\frac{\beta}{1+\beta}} \\
\le\ & \widehat{C}'_{ij}\Big(\int_{U_{i,j,\varepsilon}} \frac{\varepsilon^{-\frac14}\big(R_0(m\log|s|^2-1)\big)^{\frac{1}{\beta}}}{|s|^{2m-\frac32}} dV_X\Big)^{2\beta_0} \\
\le\ & \widehat{C}''_{ij}\Big(\int_{U_{i,j,\varepsilon}} \frac{\varepsilon^{-\frac14}}{|s|^{2m-\frac54}} dV_X\Big)^{2\beta_0} \\
\le\ & \widehat{C}'''_{ij}\varepsilon^{2\beta_0},
\end{aligned}$$

when ε is small enough, where $\widehat{C}'_{ij}$, $\widehat{C}''_{ij}$ and $\widehat{C}'''_{ij}$ are positive numbers independent of ε. Hence,

$$I_{2,\varepsilon} \le C_1\varepsilon^{\beta_0}, \tag{6.29}$$

where C_1 is a positive number independent of ε.

Therefore, it follows from (6.26), (6.27) and (6.29) that

$$\begin{aligned}
&\int_{X_k\setminus(Y\cup\Sigma_{\rho_\varepsilon})} \langle(\mathrm{B}_\varepsilon + 2\varepsilon^{\beta_0}\mathrm{I})^{-1} g_\varepsilon, g_\varepsilon\rangle_{L_\varepsilon} dV_X \\
\le\ & (1+c)I_{1,\varepsilon} + \frac{1+c}{c} I_{2,\varepsilon} \\
\le\ & \frac{(1+c)^6}{(1-c)^{m+3}} \frac{(2\pi)^m}{m!} \int_Y \frac{|f|^2_{L_0} e^{-\phi}}{|\wedge^m (ds)|^2} dV_Y + \frac{1+c}{c} C_1\varepsilon^{\beta_0}.
\end{aligned}$$

Then by Lemma 3.2, there exists $u_{k,a,c,l,\varepsilon} \in L^2(X_k\setminus(Y\cup\Sigma_{\rho_\varepsilon}), K_X\otimes L_\varepsilon)$ and $h_{k,a,c,l,\varepsilon} \in L^2(X_k \setminus (Y\cup\Sigma_{\rho_\varepsilon}), \wedge^{n,1}T^*_X\otimes L_\varepsilon)$ such that

$$\mathrm{D}'' u_{k,a,c,l,\varepsilon} + \sqrt{2\varepsilon^{\beta_0}} h_{k,a,c,l,\varepsilon} = g_\varepsilon \tag{6.30}$$

on $X_k \setminus (Y\cup\Sigma_{\rho_\varepsilon})$ and

$$\begin{aligned}
&\int_{X_k\setminus(Y\cup\Sigma_{\rho_\varepsilon})} \frac{|u_{k,a,c,l,\varepsilon}|^2_{L_0} e^{-\widetilde\phi_{\rho_\varepsilon}\circ\mu^{-1} - (m+2\pi n_k\delta_{\rho_\varepsilon})\log|s|^2 - \zeta(\sigma_\varepsilon)}}{\tau_\varepsilon + A_\varepsilon} dV_X \\
&+ \int_{X_k\setminus(Y\cup\Sigma_{\rho_\varepsilon})} |h_{k,a,c,l,\varepsilon}|^2_{L_0} e^{-\widetilde\phi_{\rho_\varepsilon}\circ\mu^{-1} - (m+2\pi n_k\delta_{\rho_\varepsilon})\log|s|^2 - \zeta(\sigma_\varepsilon)} dV_X \\
\le\ & C(\varepsilon),
\end{aligned} \tag{6.31}$$

where

$$C(\varepsilon) := \frac{(1+c)^6}{(1-c)^{m+3}} \frac{(2\pi)^m}{m!} \int_Y \frac{|f|^2_{L_0} e^{-\phi}}{|\wedge^m (ds)|^2} dV_Y + \frac{1+c}{c} C_1 \varepsilon^{\beta_0}.$$

Since $\{\widetilde{\phi}_{\rho_\varepsilon} \circ \mu^{-1}\}_{\varepsilon \in (0,\varepsilon_a)}$ are uniformly bounded above on $X_k \setminus Y$ with respect to ε as obtained in Step 2, we have

$$e^{-\widetilde{\phi}_{\rho_\varepsilon} \circ \mu^{-1}} \geq C_2 \tag{6.32}$$

on $X_k \setminus Y$ for any $\varepsilon \in (0, \varepsilon_a)$, where C_2 is a positive number independent of ε. Since $2m \log \varepsilon - l - a \leq \sigma_\varepsilon \leq -\frac{a}{2}$ on $\overline{X_k}$ and $\log |s|^2$ is upper semicontinuous on X, we have that $\log |s|^2$, $\zeta(\sigma_\varepsilon)$ and $\tau_\varepsilon + A_\varepsilon$ are all bounded above on $\overline{X_k}$ for each fixed ε. Then it follows from (6.31) that $u_{k,a,c,l,\varepsilon} \in L^2(X_k, K_X \otimes L_0)$ and $h_{k,a,c,l,\varepsilon} \in L^2(X_k, \wedge^{n,1} T^*_X \otimes L_0)$. Hence, it follows from (6.30) and Lemma 3.7 that

$$\mathrm{D}'' u_{k,a,c,l,\varepsilon} + \sqrt{2\varepsilon^{\beta_0}} h_{k,a,c,l,\varepsilon} = \mathrm{D}'' \left(\theta\left(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\right) \tilde{f}_\varepsilon \right) \tag{6.33}$$

holds on X_k. Furthermore, (6.31) and (4.18) imply that

$$\begin{aligned} & \int_{X_k} \frac{|u_{k,a,c,l,\varepsilon}|^2_{L_0} e^{-\widetilde{\phi}_{\rho_\varepsilon} \circ \mu^{-1} - 2\pi n_k \delta_{\rho_\varepsilon} \lambda_s}}{|s|^{2m} R(\sigma_\varepsilon)} dV_X \\ & + \int_{X_k} |h_{k,a,c,l,\varepsilon}|^2_{L_0} e^{-\widetilde{\phi}_{\rho_\varepsilon} \circ \mu^{-1} - m \log |s|^2 - 2\pi n_k \delta_{\rho_\varepsilon} \lambda_s - \zeta(\sigma_\varepsilon)} dV_X \\ \leq\ & C(\varepsilon), \end{aligned} \tag{6.34}$$

where $\lambda_s := \sup_{X_k} \log |s|^2$.

Define $F_{k,a,c,l,\varepsilon} = -u_{k,a,c,l,\varepsilon} + \theta(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}) \tilde{f}_\varepsilon$. Then (6.33) implies that $\mathrm{D}'' F_{k,a,c,l,\varepsilon} = \sqrt{2\varepsilon^{\beta_0}} h_{k,a,c,l,\varepsilon}$ on X_k. Since $R(\sigma_\varepsilon) \leq R(\psi_l + m \log |s|^2 - a)$ and $\widetilde{\phi}_{\rho_\varepsilon} \circ \mu^{-1} \geq \phi$ on $X_k \setminus Y$, it follows from (4.12) and (6.34) that

$$\begin{aligned} & \int_{X_k} \frac{|F_{k,a,c,l,\varepsilon}|^2_{L_0} e^{-\widetilde{\phi}_{\rho_\varepsilon} \circ \mu^{-1}}}{|s|^{2m} R(\psi_l + m \log |s|^2 - a)} dV_X \\ \leq\ & (1+c) \int_{X_k} \frac{|u_{k,a,c,l,\varepsilon}|^2_{L_0} e^{-\widetilde{\phi}_{\rho_\varepsilon} \circ \mu^{-1}}}{|s|^{2m} R(\sigma_\varepsilon)} dV_X \\ & + \frac{1+c}{c} \int_{X_k} \frac{\left|\theta\left(\frac{\varepsilon^2}{|s|^2+\varepsilon^2}\right) \tilde{f}_\varepsilon\right|^2_{L_0} e^{-\widetilde{\phi}_{\rho_\varepsilon} \circ \mu^{-1}}}{|s|^{2m} R(\psi_l + m \log |s|^2 - a)} dV_X \\ \leq\ & (1+c) e^{2\pi n_k \delta_{\rho_\varepsilon} \lambda_s} C(\varepsilon) + \widetilde{C}(\varepsilon), \end{aligned} \tag{6.35}$$

when ε is small enough, where

$$\widetilde{C}(\varepsilon) := \frac{1+c}{c} \int_{X_k \cap \{|s| < \sqrt{\frac{2-c}{c}} \varepsilon\}} \frac{|\tilde{f}_\varepsilon|^2_{L_0} e^{-\phi}}{|s|^{2m} R(\psi_l + m \log |s|^2 - a)} dV_X.$$

Now we want to prove

$$\lim_{\varepsilon\to 0}\widetilde{C}(\varepsilon)=0.$$

As in (6.28), we have

$$\int_{V_i\cap\{|s|<\sqrt{\frac{2-c}{c}}\varepsilon\}}\frac{|\tilde{f}_{i,\varepsilon}|^2_{L_0}e^{-(1+\beta)\phi}}{|s|^{2m}R_1(m\log|s|^2-a+1+\lambda_\psi)}dV_X\leq\widehat{C}_{10}, \tag{6.36}$$

for some positive number $\widehat{C}_{10}$ independent of ε when ε is small enough, where $\lambda_\psi:=\sup_{X_{k+1}}\psi$.

For similar reasons as in (5.8), we get from (6.8) that

$$\sup_{U_i\cap\{|s|<\sqrt{\frac{2-c}{c}}\varepsilon\}}|\tilde{f}_{i,\varepsilon}|^2_{L_0}\leq\widehat{C}_{11}, \tag{6.37}$$

for some positive number $\widehat{C}_{11}$ independent of ε when ε is small enough.

By (6.4), (6.36), (6.37) and the Hölder inequality, for $i=1,\cdots,N$, we have that

$$\begin{aligned}
&\int_{U_i\cap X_k\cap\{|s|<\sqrt{\frac{2-c}{c}}\varepsilon\}}\frac{|\tilde{f}_{i,\varepsilon}|^2_{L_0}e^{-\phi}}{|s|^{2m}R(\psi_l+m\log|s|^2-a)}dV_X \\
\leq\ &\int_{U_i\cap\{|s|<\sqrt{\frac{2-c}{c}}\varepsilon\}}\frac{|\tilde{f}_{i,\varepsilon}|^2_{L_0}e^{-\phi}}{|s|^{2m}R_1(\lambda_\psi+m\log|s|^2-a+1)}dV_X \\
\leq\ &\left(\int_{U_i\cap\{|s|<\sqrt{\frac{2-c}{c}}\varepsilon\}}\frac{|\tilde{f}_{i,\varepsilon}|^2_{L_0}e^{-(1+\beta)\phi}}{|s|^{2m}R_1(m\log|s|^2-a+1+\lambda_\psi)}dV_X\right)^{\frac{1}{1+\beta}} \\
&\times\left(\int_{U_i\cap\{|s|<\sqrt{\frac{2-c}{c}}\varepsilon\}}\frac{|\tilde{f}_{i,\varepsilon}|^2_{L_0}}{|s|^{2m}R_1(m\log|s|^2-a+1+\lambda_\psi)}dV_X\right)^{\frac{\beta}{1+\beta}} \\
\leq\ &\widehat{C}_{12}\left(\int_{U_i\cap\{|s|<\sqrt{\frac{2-c}{c}}\varepsilon\}}\frac{1}{|s|^{2m}R_1(m\log|s|^2-a+1+\lambda_\psi)}dV_X\right)^{\frac{\beta}{1+\beta}} \\
\leq\ &\widehat{C}_{13}\left(\int_{-\infty}^{2m\log\varepsilon+\widehat{C}_{14}}\frac{1}{R_1(t)}dt\right)^{\frac{\beta}{1+\beta}},
\end{aligned} \tag{6.38}$$

when ε is small enough, where $\widehat{C}_{12}$, $\widehat{C}_{13}$ and $\widehat{C}_{14}$ are positive numbers independent of ε.

Since $\tilde{f}_\varepsilon:=\sum_{i=1}^N\xi_i\tilde{f}_{i,\varepsilon}$ and $\operatorname{supp}\xi_i\subset\subset U_i$, we get from (6.38) that

$$\lim_{\varepsilon\to 0}\widetilde{C}(\varepsilon)=0.$$

Since

$$\begin{aligned}
|s|^{2m}R(\psi_l+m\log|s|^2-a)\ &\leq\ e^{m\log|s|^2}R(m\log|s|^2-l-a) \\
&\leq\ e^{l+a}\sup_{t\leq 0}\left(e^tR(t)\right),
\end{aligned}$$

it follows from (6.32) and (6.35) that

$$\int_{X_k} |F_{k,a,c,l,\varepsilon}|^2_{L_0} dV_X \leq \widehat{C}_{15}, \tag{6.39}$$

for some positive number $\widehat{C}_{15}$ independent of ε when ε is small enough.

Since $\widetilde{\phi}_{\rho_\varepsilon} \circ \mu^{-1}$ is increasing with respect to ε and converges to ϕ on $X_k \setminus Y$ as $\varepsilon \to 0$, by extracting weak limits of $\{F_{k,a,c,l,\varepsilon}\}_{\varepsilon>0}$ as $\varepsilon \to 0$, we get from (6.39) and (6.35) a sequence $\{\varepsilon_j\}_{j=1}^{+\infty}$ and $F_{k,a,c,l} \in L^2(X_k, K_X \otimes L_0)$ such that $\lim_{j\to+\infty} \varepsilon_j = 0$, $F_{k,a,c,l,\varepsilon_j} \rightharpoonup F_{k,a,c,l}$ weakly in $L^2(X_k, K_X \otimes L_0)$ as $j \to +\infty$ and

$$\int_{X_k} \frac{|F_{k,a,c,l}|^2_{L_0} e^{-\phi} dV_X}{|s|^{2m} R(\psi_l + m\log|s|^2 - a)} \leq \frac{(1+c)^7}{(1-c)^{m+3}} \frac{(2\pi)^m}{m!} \int_Y \frac{|f|^2_{L_0} e^{-\phi} dV_Y}{|\wedge^m (ds)|^2}.$$

Since $\psi_l \geq \psi$ and R is decreasing, we get

$$\begin{aligned} &\int_{X_k} \frac{|F_{k,a,c,l}|^2_{L_0} e^{-\varphi}}{e^{\psi+m\log|s|^2} R(\psi + m\log|s|^2 - a)} dV_X \\ &\leq \frac{(1+c)^7}{(1-c)^{m+3}} \frac{(2\pi)^m}{m!} \int_Y \frac{|f|^2_{L_0} e^{-\varphi-\psi}}{|\wedge^m (ds)|^2} dV_Y. \end{aligned} \tag{6.40}$$

Since $\sigma_\varepsilon \leq -\frac{a}{2}$ on X_k and ζ is increasing, we get

$$e^{-\zeta(\sigma_\varepsilon)} \geq e^{-\zeta(-\frac{a}{2})} \tag{6.41}$$

on X_k. Then (6.34), (6.32) and (6.41) imply that

$$\int_{X_k} |h_{k,a,c,l,\varepsilon}|^2_{L_0} dV_X \leq e^{\zeta(-\frac{a}{2}) + (m+2\pi n_k \delta_{\rho_\varepsilon})\lambda_s} C_2^{-1} C(\varepsilon).$$

Hence, $\sqrt{2\varepsilon_j^{\beta_0}} h_{k,a,c,l,\varepsilon_j} \to 0$ in $L^2(X_k, \wedge^{n,1} T_X^* \otimes L_0)$ as $j \to +\infty$. Since $\mathrm{D}'' F_{k,a,c,l,\varepsilon} = \sqrt{2\varepsilon^{\beta_0}} h_{k,a,c,l,\varepsilon}$ on X_k, we get $\mathrm{D}'' F_{k,a,c,l} = 0$ on X_k. Then $F_{k,a,c,l}$ is a holomorphic section of $K_X \otimes L$ on X_k. In Step 5, we will prove that $F_{k,a,c,l} = f$ on $Y \cap X_k$ by solving $\bar\partial$ locally.

Step 5: solving $\bar\partial$ locally with L^2 estimates and the end of the proof.

For any $x \in Y \cap X_k$, let $(V_x, z_x^1, \cdots, z_x^n)$, z'_x, z''_x and $\varepsilon_x \in (0,1)$ be as in Step 1. Assume that $\widetilde{\varepsilon}_x \in (0, \varepsilon_x)$ is a positive number such that

$$W_x := \{y \in V_x : |z'_x(y)| < \widetilde{\varepsilon}_x, |z''_x(y)| < \widetilde{\varepsilon}_x\} \subset\subset X_k.$$

Since the bundle L is trivial on V_x, $u_{k,a,c,l,\varepsilon}$ and $h_{k,a,c,l,\varepsilon}$ can be regarded as forms on V_x with values in $\mathbb{C}$ and the metric of L_0 on V_x can be regarded as a positive smooth function. Obviously, the Kähler metric ω on W_x is bounded below and above by $C_3^{-1}\omega'$ and $C_3\omega'$ respectively, where ω' is the Euclidean metric on W_x and $C_3 > 1$ is some positive number independent of ε. In the following, we will denote the $2n$-dimensional Lebesgue measure on W_x by dV_n.

It is easy to see that $C(\varepsilon) \le C_0$ for some positive number C_0 independent of ε when ε is small enough. Then it follows from (6.34), (6.1), (6.41) and (6.32) that

$$\int_{W_x} |h_{k,a,c,l,\varepsilon}|^2 e^{-m\log|z'_x|^2} dV_n \le C_4 C_0, \tag{6.42}$$

for some positive number C_4 independent of ε when ε is small enough.

Since $\bar\partial h_{k,a,c,l,\varepsilon} = 0$ on W_x by (6.33), applying Lemma 3.6 to the $(n,1)$-form

$$\sqrt{2\varepsilon^{\beta_0}} h_{k,a,c,l,\varepsilon} \in L^2_{(n,1)}(W_x,\, m\log|z'_x|^2),$$

we get an $(n,0)$-form $v_{k,a,c,l,\varepsilon} \in L^2_{(n,0)}(W_x,\, m\log|z'_x|^2)$ such that

$$\bar\partial v_{k,a,c,l,\varepsilon} = \sqrt{2\varepsilon^{\beta_0}} h_{k,a,c,l,\varepsilon}$$

on W_x and

$$\int_{W_x} \frac{|v_{k,a,c,l,\varepsilon}|^2 e^{-m\log|z'_x|^2}}{(1+|z'_x|^2+|z''_x|^2)^2} dV_n \le \int_{W_x} |\sqrt{2\varepsilon^{\beta_0}} h_{k,a,c,l,\varepsilon}|^2 e^{-m\log|z'_x|^2} dV_n.$$

Since $|z'_x|^2+|z''_x|^2 < 2$ on W_x, by (6.42), we get

$$\int_{W_x} |v_{k,a,c,l,\varepsilon}|^2 e^{-m\log|z'_x|^2} dV_n \le 18 C_4 C_0 \varepsilon^{\beta_0}. \tag{6.43}$$

Since $e^{-m\log|z'_x|^2} > 1$ on W_x, (6.43) implies that

$$\int_{W_x} |v_{k,a,c,l,\varepsilon}|^2 dV_n \le 18 C_4 C_0 \varepsilon^{\beta_0}. \tag{6.44}$$

Now define $G_{k,a,c,l,\varepsilon} = -u_{k,a,c,l,\varepsilon} - v_{k,a,c,l,\varepsilon} + \theta(\frac{\varepsilon^2}{|s|^2+\varepsilon^2})\tilde f_\varepsilon$ on W_x. Then $G_{k,a,c,l,\varepsilon} = F_{k,a,c,l,\varepsilon} - v_{k,a,c,l,\varepsilon}$ and $\bar\partial G_{k,a,c,l,\varepsilon} = 0$. Hence, $G_{k,a,c,l,\varepsilon}$ is holomorphic in W_x. Therefore, $u_{k,a,c,l,\varepsilon} + v_{k,a,c,l,\varepsilon}$ is smooth in W_x. Furthermore, we get from (6.39) and (6.44) that

(6.45)
$$\int_{W_x} |G_{k,a,c,l,\varepsilon}|^2 dV_n \le 2\int_{W_x} |F_{k,a,c,l,\varepsilon}|^2 dV_n + 2\int_{W_x} |v_{k,a,c,l,\varepsilon}|^2 dV_n \le C_5,$$

for some positive number C_5 independent of ε when ε is small enough.

By (6.1) and (6.32), we get from (6.34) that

$$\int_{W_x} \frac{|u_{k,a,c,l,\varepsilon}|^2 e^{-m\log|z'_x|^2}}{R(\sigma_\varepsilon)} dV_n \le C_6 C(\varepsilon) \le C_6 C_0,$$

for some positive number C_6 independent of ε when ε is small enough. Since $R(\sigma_\varepsilon) \le R(2m\log\varepsilon - l - a)$ on W_x, we have

$$\int_{W_x} |u_{k,a,c,l,\varepsilon}|^2 e^{-m\log|z'_x|^2} dV_n \le C_6 C_0 R(2m\log\varepsilon - l - a).$$

Therefore, combining the last inequality and (6.43), we obtain that

$$\int_{W_x} \frac{|u_{k,a,c,l,\varepsilon} + v_{k,a,c,l,\varepsilon}|^2}{|z'_x|^{2m}} dV_n \le 2C_6 C_0 R(2m\log\varepsilon - l - a) + 36 C_4 C_0 \varepsilon^{\beta_0}.$$

Then the non-integrability of $|z'_x|^{-2m}$ along $W_x \cap Y$ and the smoothness of $u_{k,a,c,l,\varepsilon} + v_{k,a,c,l,\varepsilon}$ in W_x show that $u_{k,a,c,l,\varepsilon} + v_{k,a,c,l,\varepsilon} = 0$ on $W_x \cap Y$. Hence, $G_{k,a,c,l,\varepsilon} = f$ on $W_x \cap Y$.

Since $v_{k,a,c,l,\varepsilon_j} \to 0$ in $L^2_{(n,0)}(W_x)$ by (6.44) and $F_{k,a,c,l,\varepsilon_j} \rightharpoonup F_{k,a,c,l}$ weakly in $L^2_{(n,0)}(W_x)$ as $j \to +\infty$, we get $G_{k,a,c,l,\varepsilon_j} \rightharpoonup F_{k,a,c,l}$ weakly in $L^2_{(n,0)}(W_x)$ as $j \to +\infty$. Hence, it follows from (6.45) and routine arguments with applying Montel's theorem that a subsequence of $\{G_{k,a,c,l,\varepsilon_j}\}_{j=1}^{+\infty}$ converges to $F_{k,a,c,l}$ uniformly on compact subsets of W_x. Then $F_{k,a,c,l} = f$ on $W_x \cap Y$ and thereby on $Y \cap X_k$.

Since R is a continuous decreasing function on $(-\infty, 0]$, φ is locally bounded above and $\sup_{t\leq 0}\left(e^t R(t)\right) < +\infty$, applying Montel's theorem and extracting weak limits of $\{F_{k,a,c,l}\}_{k,a,c,l}$, first as $l \to +\infty$, next as $c \to 0$, then as $a \to 0$, and, finally, as $k \to +\infty$, we get from (6.40) a holomorphic section F on X with values in $K_X \otimes L$ such that $F = f$ on Y and

$$\int_X \frac{|F|_L^2}{e^{\psi+m\log|s|^2}R(\psi + m\log|s|^2)}dV_X \leq \frac{(2\pi)^m}{m!}\int_Y \frac{|f|_L^2 e^{-\psi}}{|\wedge^m(ds)|^2}dV_Y.$$

Theorem 1.1 is, thus, proved.

7. Proof of Theorem 1.2

K_X is naturally equipped with the smooth metric e^{φ_ω} with respect to the dual frame of dz. Let L' be the line bundle L equipped with the new metric $e^{-\varphi_{L'}}$, where $\varphi_{L'} := (2-q)\log|F_1|_L + \varphi_L$. Then the assumptions in the theorem imply that

(i) $\sqrt{-1}\Theta_{L'} + \sqrt{-1}\partial\bar{\partial}\sigma \geq 0,$

(ii) $\sqrt{-1}\Theta_{L'} + \sqrt{-1}\partial\bar{\partial}\sigma \geq \dfrac{\{\sqrt{-1}\Theta_E s, s\}_E}{\alpha|s|_E^2}.$

Since the holomorphic section $f \in H^0(Y, K_X|_Y \otimes L|_Y)$ satisfies

$$\int_Y \frac{|f|_{L'}^2 e^{-\psi}}{|\wedge^m(ds)|_E^2}dV_Y = C_f < +\infty,$$

by Theorem 1.1, there exists a holomorphic section F_2 on X with values in $K_X \otimes L$, such that $F_2 = f$ on Y and

$$\begin{aligned}
\int_X \frac{|F_2|_L^2}{(|F_1|_L)^{2-q}e^\sigma R(\sigma)}dV_X &= \int_X \frac{|F_2|_{L'}^2}{e^\sigma R(\sigma)}dV_X \\
&\leq C_R\frac{(2\pi)^m}{m!}\int_Y \frac{|f|_{L'}^2 e^{-\psi}}{|\wedge^m(ds)|_E^2}dV_Y \\
&= C_R\frac{(2\pi)^m}{m!}C_f.
\end{aligned}$$

Then Hölder's inequality gives that

$$\begin{aligned} C_{F_2} &:= \int_X \frac{(|F_2|_L)^q}{e^\sigma R(\sigma)} dV_X \\ &\leq \left(\int_X \frac{|F_2|_L^2}{(|F_1|_L)^{2-q} e^\sigma R(\sigma)} dV_X \right)^{\frac{q}{2}} \left(\int_X \frac{(|F_1|_L)^q}{e^\sigma R(\sigma)} dV_X \right)^{1-\frac{q}{2}} \\ &\leq \left(C_R \frac{(2\pi)^m}{m!} C_f \right)^{\frac{q}{2}} (C_{F_1})^{1-\frac{q}{2}}. \end{aligned}$$

We can then repeat the same argument with F_1 replaced by F_2, etc, and get a sequence of holomorphic extensions $\{F_k\}_{k=1}^{+\infty}$ of f and a sequence $\{C_{F_k}\}_{k=1}^{+\infty}$ such that

$$C_{F_{k+1}} \leq \left(C_R \frac{(2\pi)^m}{m!} C_f \right)^{\frac{q}{2}} (C_{F_k})^{1-\frac{q}{2}}, \qquad k = 1, 2, \cdots. \tag{7.1}$$

If $C_{F_k} \leq C_R \frac{(2\pi)^m}{m!} C_f$ for some C_{F_k}, then we finish the proof since F_k can be regarded as the desired holomorphic extension F in the conclusion.

If $C_{F_k} > C_R \frac{(2\pi)^m}{m!} C_f$ for any k, then $C_{F_{k+1}} < C_{F_k}$ for any k. Since φ_L is locally bounded above and $e^\sigma R(\sigma)$ is bounded above, applying Montel's theorem and extracting weak limits of $\{F_k\}_{k=1}^{+\infty}$, we can get from (7.1) a holomorphic section F on X with values in $K_X \otimes L$, such that $F = f$ on Y and

$$\int_X \frac{(|F|_L)^q}{e^\sigma R(\sigma)} dV_X \leq C_R \frac{(2\pi)^m}{m!} C_f.$$

Theorem 1.2 is, thus, proved.

8. Proof of Theorem 1.3

The fiberwise Bergman kernel $B(x)$ of $(K_{X/Y} \otimes L)|_{X^0}$ at a point $x \in X^0$ is defined by

$$B(x) = \sum_{u_y} u_y(x) \otimes \overline{u_y(x)},$$

for any choice of orthonormal basis $\{u_y\}$ of the Hilbert space

$$H^0(X_y, K_{X_y} \otimes L|_{X_y} \otimes \mathcal{I}(h_L|_{X_y})),$$

with the norm

$$\|u_y\|_{X_y} := \left(\int_{X_y} c_{n-m} \{u_y, u_y\}_L \right)^{\frac{1}{2}},$$

where $y := \Pi(x)$, $X_y := \Pi^{-1}(y)$, $(K_{X/Y} \otimes L)|_{X_y} \simeq K_{X_y} \otimes L|_{X_y}$, $c_{n-m} := (\sqrt{-1})^{(n-m)^2}$ and $\{\bullet, \bullet\}_L$ is defined as in Lemma 3.9.

The assumption (ii) in Theorem 1.3 implies that B is not equal to zero identically on X^0. If $\log B$ is proved to be plurisubharmonic on X^0, then $\log B \in L^1_{\text{loc}}(X^0)$ and the fiberwise Bergman kernel metric of $(K_{X/Y} \otimes L)|_{X^0}$ is defined to be B^{-1}.

We will divide the proof into two parts.

Part I. We will prove that $\log B$ is plurisubharmonic on X^0. Then the fiberwise Bergman kernel metric of the bundle $(K_{X/Y} \otimes L)|_{X^0}$ has semipositive curvature current on X^0.

Since it is not hard to prove the upper semicontinuity of $\log B$ by using global regularization results of the singular metric of L, we will only prove that for a coordinate chart $U \subset\subset X^0$ which is small enough, $\log B$ satisfies the mean value inequality on every complex line contained in U.

Without loss of generality, we can assume that $m = 1$, $U \simeq \mathbb{B}^{n-1} \times \Delta$ and $\Pi|_U$ is the projection from $\mathbb{B}^{n-1} \times \Delta$ to Δ, where Δ is the unit disc in $\mathbb{C}$. For any $t \in \Delta$, denote the compact fiber $\Pi^{-1}(t)$ by X_t. Let η be a local frame of L on U and let (z, t) be the coordinates on $U \simeq \mathbb{B}^{n-1} \times \Delta$. We will write the Bergman kernel of $K_{X_t} \otimes L|_{X_t} (\simeq K_{X/Y}|_{X_t} \otimes L|_{X_t})$ as $B_t(z) dz \otimes \eta \otimes d\bar{z} \otimes \bar{\eta}$ on $X_t \cap U$.

Since $\log B_t(z)$ is always plurisubharmonic with respect to z, we need only check that $\log B_t(z)$ satisfies the mean value inequality with respect to t for fixed z.

Fix $z = z_0$. For any given $t_0 \in \Delta$, if $B_{t_0}(z_0) = 0$, then $\log B_t(z_0)$ satisfies the mean value inequality at t_0. If $B_{t_0}(z_0) \neq 0$, by the extremal property of the Bergman kernel, there exists a holomorphic section $v_{t_0} \in H^0(X_{t_0}, K_{X_{t_0}} \otimes L|_{X_{t_0}} \otimes \mathcal{I}(h_L|_{X_{t_0}}))$ such that

$$B_{t_0}(z_0) = \frac{|v'_{t_0}(z_0)|^2}{\int_{X_{t_0}} c_{n-1}\{v_{t_0}, v_{t_0}\}_L}, \tag{8.1}$$

where $v_{t_0}|_U = v'_{t_0}(z) dz \otimes \eta$.

Applying Theorem 1.1 to the holomorphic section v_{t_0} in the case $R(t_1) = e^{-t_1}$, $m = 1$, $s = t - t_0$ and $\psi = -\log r^2$, we can obtain a holomorphic section $\widetilde{v} \in H^0(\Pi^{-1}(\Delta_r(t_0)), K_X \otimes L \otimes \mathcal{I}(h_L))$ such that $\widetilde{v}|_{X_{t_0}} = v_{t_0} \wedge dt$ and

$$\int_{\Pi^{-1}(\Delta_r(t_0))} c_n\{\widetilde{v}, \widetilde{v}\}_L \leq 2\pi r^2 \int_{X_{t_0}} c_{n-1}\{v_{t_0}, v_{t_0}\}_L, \tag{8.2}$$

where $\Delta_r(t_0) := \{t \in \mathbb{C} : |t - t_0| < r\}$ and r is an arbitrary positive number which is small enough.

Since $\int_{X_t} c_{n-1}\{\frac{\widetilde{v}}{dt}\big|_{X_t}, \frac{\widetilde{v}}{dt}\big|_{X_t}\}_L \neq 0$ for a.e. $t \in \Delta_r(t_0)$, the extremal property of the Bergman kernel implies that

$$B_t(z_0) \geq \frac{|\widetilde{v}'(t, z_0)|^2}{\int_{X_t} c_{n-1}\{\frac{\widetilde{v}}{dt}\big|_{X_t}, \frac{\widetilde{v}}{dt}\big|_{X_t}\}_L},$$

for a.e. $t \in \Delta_r(t_0)$, where $\widetilde{v}|_U = \widetilde{v}'(t,z)dz \wedge dt \otimes \eta$. Then we have

$$
\begin{aligned}
(8.3)\qquad & \frac{1}{\pi r^2}\int_{\Delta_r(t_0)} \log B_t(z_0)\frac{\sqrt{-1}dt\wedge d\bar{t}}{2} \\
\geq\; & \frac{1}{\pi r^2}\int_{\Delta_r(t_0)} \log|\widetilde{v}'(t,z_0)|^2\frac{\sqrt{-1}dt\wedge d\bar{t}}{2} \\
& -\frac{1}{\pi r^2}\int_{\Delta_r(t_0)}\left(\log\int_{X_t} c_{n-1}\left\{\frac{\widetilde{v}}{dt}\bigg|_{X_t}, \frac{\widetilde{v}}{dt}\bigg|_{X_t}\right\}_L\right)\frac{\sqrt{-1}dt\wedge d\bar{t}}{2}.
\end{aligned}
$$

Since the mean value inequality for subharmonic functions and Jensen's inequality for the convex function $-\log$ imply that

$$
\frac{1}{\pi r^2}\int_{\Delta_r(t_0)} \log|\widetilde{v}'(t,z_0)|^2\frac{\sqrt{-1}dt\wedge d\bar{t}}{2} \geq \log|\widetilde{v}'(t_0,z_0)|^2 = \log|v'_{t_0}(z_0)|^2,
$$

and

$$
\begin{aligned}
& -\frac{1}{\pi r^2}\int_{\Delta_r(t_0)}\left(\log\int_{X_t} c_{n-1}\left\{\frac{\widetilde{v}}{dt}\bigg|_{X_t}, \frac{\widetilde{v}}{dt}\bigg|_{X_t}\right\}_L\right)\frac{\sqrt{-1}dt\wedge d\bar{t}}{2} \\
\geq\; & -\log\left(\frac{1}{\pi r^2}\int_{\Delta_r(t_0)}\int_{X_t} c_{n-1}\left\{\frac{\widetilde{v}}{dt}\bigg|_{X_t}, \frac{\widetilde{v}}{dt}\bigg|_{X_t}\right\}_L\frac{\sqrt{-1}dt\wedge d\bar{t}}{2}\right) \\
=\; & -\log\left(\frac{1}{\pi r^2}\int_{\Pi^{-1}(\Delta_r(t_0))}\frac{c_n}{2}\{\widetilde{v},\widetilde{v}\}_L\right) \\
\geq\; & -\log\int_{X_{t_0}} c_{n-1}\{v_{t_0}, v_{t_0}\}_L,
\end{aligned}
$$

by Fubini's theorem and (8.2), we obtain from (8.3) and (8.1) that

$$
\begin{aligned}
& \frac{1}{\pi r^2}\int_{\Delta_r(t_0)} \log B_t(z_0)\frac{\sqrt{-1}dt\wedge d\bar{t}}{2} \\
\geq\; & \log|v'_{t_0}(z_0)|^2 - \log\int_{X_{t_0}} c_{n-1}\{v_{t_0}, v_{t_0}\}_L \\
=\; & \log B_{t_0}(z_0).
\end{aligned}
$$

Hence, $\log B_t(z_0)$ satisfies the mean value inequality with respect to t. Thus, we finish the proof of Part I.

Part II. Let $\Omega_1 \subset\subset \Omega_2$ be two small coordinate balls in X such that $\Pi(\Omega_2)$ is contained in a coordinate ball in Y, whose coordinates will be still denoted by $t = (t^1, t^2, \cdots, t^m)$. Let $\Sigma := X\backslash X_0$. We will prove that the fiberwise Bergman kernel is uniformly bounded on $\Omega_1\backslash\Sigma$. Then the fiberwise Bergman kernel metric on X^0 extends across $X\backslash X^0$ to a metric with semipositive curvature current on all of X. We will use similar arguments as in [**5**] in this part.

Let x be a point in $\Omega_1\backslash\Sigma$ and $y := \Pi(x)$. Denote $\Pi^{-1}(y)$ by X_y. Let $u \in H^0(X_y, K_{X_y} \otimes L|_{X_y} \otimes \mathcal{I}(h_L|_{X_y}))$. Then $u \wedge dt \in H^0(X_y, K_X|_{X_y} \otimes$

$L|_{X_y}\otimes\mathcal{I}(h_L|_{X_y}))$. Denote the coordinates on Ω_2 by $w=(w^1,w^2,\cdots,w^n)$ and denote the local holomorphic frame of L on Ω_2 by η. Then we can write $u\wedge dt$ as $u'dw\otimes\eta$ on $\Omega_2\cap X_y$. Hence, with respect to the local coordinates w and the local frame η, the fiberwise Bergman kernel at x is given by the supremum of $|u'(x)|^2$ when u is normalized by the condition

$$\int_{X_y} c_{n-m}\{u,u\}_L\leq 1.$$

By Theorem 1.1, we can obtain a holomorphic section

$$\widetilde{u}\in H^0(\Omega_2,K_X\otimes L\otimes\mathcal{I}(h_L)),$$

such that $\widetilde{u}|_{\Omega_2\cap X_y}=u\wedge dt=u'dw\otimes\eta$ and

$$\int_{\Omega_2} c_n\{\widetilde{u},\widetilde{u}\}_L\leq C_1\int_{\Omega_2\cap X_y} c_{n-m}\{u,u\}_L\leq C_1, \tag{8.4}$$

where C_1 is a positive number depending only on m and the diameter of Ω_2.

The mean value inequality applied to (8.4) shows that

$$|u'(x)|^2\leq C_2,$$

where C_2 is a positive number depending only on n, m, the diameter of Ω_1, the diameter of Ω_2 and the upper bound on Ω_2 of the local weight of h_L.

Since x is an arbitrary point in $\Omega_1\backslash\Sigma$, the fiberwise Bergman kernel is uniformly bounded on $\Omega_1\backslash\Sigma$. Therefore, we finish the proof of Part II.

Theorem 1.3 is, thus, proved.

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