J. DIFFERENTIAL GEOMETRY
109 (2018) 431-466

A DISCRETE UNIFORMIZATION THEOREM FOR POLYHEDRAL SURFACES II

Xianfeng Gu, Ren Guo, Feng Luo, Jian Sun & Tianqi Wu

Abstract

A notion of discrete conformality for hyperbolic polyhedral surfaces is introduced in this paper. This discrete conformality is shown to be computable. It is proved that each hyperbolic polyhedral metric on a closed surface is discrete conformal to a unique hyperbolic polyhedral metric with a given discrete curvature satisfying Gauss–Bonnet formula. Furthermore, the hyperbolic polyhedral metric with given curvature can be obtained using a discrete Yamabe flow with surgery. In particular, each hyperbolic polyhedral metric on a closed surface with negative Euler characteristic is discrete conformal to a unique hyperbolic metric.

1. Introduction

1.1. Statement of results. This is a continuation of [10] in which a discrete uniformization theorem for Euclidean polyhedral metrics on closed surfaces is established. The purpose of this paper is to prove the counterpart of discrete uniformization for hyperbolic polyhedral metrics. In particular, we will introduce a discrete conformality for hyperbolic polyhedral metrics on surfaces and show the discrete conformality is algorithmic.

Recall that a *surface with marked points* (S, V) is a pair of a closed connected surface S together with a finite non-empty subset of points V. A *triangulation* of a surface with marked points (S, V) is a triangulation of S so that its vertex set is V. A *hyperbolic polyhedral metric* d on a surface with marked points (S, V) is obtained as the isometric gluing of hyperbolic triangles along pairs of edges so that its cone points are in V. It is the same as a hyperbolic cone metric on S with cone points in V. We use the terminology *polyhedral metrics* to emphasize that these metrics are determined by finite sets of data (i.e., the finite set of lengths of edges). Every hyperbolic polyhedral metric has an associated *Delaunay triangulation* which has the property that if two adjacent triangles in the

Mathematics Subject Classification. 52C26, 58E30, 53C44.

Key words and phrases. Hyperbolic metrics, discrete uniformization, discrete conformality, discrete Yamabe flow, variational principle, and Delaunay triangulation.

Received April 16, 2014.

triangulation are isometrically embedded into the hyperoblic plane, the interior of the circumcircle of each triangle contains no other vertices.

Definition 1. (discrete conformality) Two hyperbolic polyhedral metrics d, d' on a closed surface with marked points (S, V) are *discrete conformal* if there exist a sequence of hyperbolic polyhedral metrics $d_1 = d, d_2, ..., d_m = d'$ on (S, V) and triangulations $\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_m$ of (S, V) satisfying

(a) each $\mathcal{T}_i$ is Delaunay in d_i,
(b) if $\mathcal{T}_i = \mathcal{T}_{i+1}$, there exists a function $u : V \to \mathbb{R}$, called a *conformal factor*, so that if e is an edge in $\mathcal{T}_i$ with end points v and v', then the lengths $x_{d_i}(e)$ and $x_{d_{i+1}}(e)$ of e in metrics d_i and d_{i+1} are related by
$$\sinh \frac{x_{d_{i+1}}(e)}{2} = e^{u(v)+u(v')} \sinh \frac{x_{d_i}(e)}{2},$$
(c) if $\mathcal{T}_i \neq \mathcal{T}_{i+1}$, then (S, d_i) is isometric to (S, d_{i+1}) by an isometry homotopic to the identity in (S, V).

This definition is the hyperbolic counterpart of discrete conformality introduced in [**10**]. The condition (b) first appeared in the work of Bobenko–Pinkall–Springborn [**4**]. In [**4**], Bobenko et al. took condition (b) as their definition of discrete conformality (4.1.4 Definition in [**4**]). Their definition depends on the triangulation $\mathcal{T}$ and does not involve Delaunay condition (a). It is different from definition 1.

Theorem 2. *Suppose d and d' are two hyperbolic (or Euclidean) polyhedral metrics given as isometric gluing of geometric triangles whose edge lengths are algebraic numbers on a closed surface with marked points (S, V). There exists an algorithm to decide if d and d' are discrete conformal.*

The above theorem shows that discrete conformality is computable. This is in contrasts to the conformality in Riemannian geometry. Indeed, it is highly unlikely that there exist algorithms to decide if two hyperbolic (or Euclidean) polyhedral metrics on (S, V) are conformal in the Riemannian sense.

The *discrete curvature* K of a polyhedral metric d is the function defined on V sending $v \in V$ to 2π less cone angle at v. It is well known that the discrete curvature satisfies the Gauss–Bonnet identity $\sum_{v \in V} K(v) = 2\pi\chi(S) + Area(d)$ where $Area(d)$ is the area of the metric d.

Theorem 3. *Suppose (S, V) is a closed connected surface with marked points and d is a hyperbolic polyhedral metric on (S, V). Then for any $K^* : V \to (-\infty, 2\pi)$ with $\sum_{v \in V} K^*(v) > 2\pi\chi(S)$, there exists a unique hyperbolic polyhedral metric d' on (S, V) so that d' is discrete conformal* *to d and the discrete curvature of d' is K^*. Furthermore, the discrete Yamabe flow with surgery associated to curvature K^* having initial value d converges to d' linearly fast.*

This theorem could be viewed as a discrete version of Troyanov's theorem on flat cone metrics in conformal classes [**25**].

For a closed connected surface S with $\chi(S) < 0$, by choosing $K^* = 0$, we obtain,

Corollary 4. *(discrete uniformization) Let S be a closed connected surface of negative Euler characteristic and $V \subset S$ be a finite non-empty subset. Then each hyperbolic polyhedral metric d on (S, V) is discrete conformal to a unique hyperbolic metric d^* on the surface (S, V) so that all cone angles of d^* are 2π, i.e., d^* is a hyperbolic metric on S. Furthermore, there exists a C^1-smooth flow on the Teichmuller space of hyperbolic polyhedral metrics on (S, V) which preserves discrete conformal classes and flows each polyhedral metric d to d^* as time goes to infinity.*

We thank the referee for informing us that the existence and uniqueness part of corollary 4 is equivalent to the following theorem of F. Fillastre.

Theorem 5 (Fillastre [**7**])**.** *Suppose S is a closed surface of negative Euler characteristic and V is a non-empty finite subset in S. Given any finite area complete hyperbolic metric d on $S-V$, there exists a Fuchsian group Γ acting on $\mathbb{H}^3$ and an isometric embedding $\phi : (S - V, d) \to \mathbb{H}^3/\Gamma$ so that (i) $\phi(S - V)$ is the boundary of a convex ideal hyperbolic polyhedron and (ii) $\mathbb{H}^3/\Gamma$ is homotopy equivalent to S. Furthermore, the group Γ is unique up to conjugation and ϕ is unique up to isometries of $\mathbb{H}^3$.*

The main steps in showing the equivalence of these two results involve the connection between discrete conformality and hyperbolic polyhedra explained in [**4**], the fact that the Delaunay triangulations in different geometries determine the same combinatorial type and length-cross-ratio can be computed using distances in Euclidean, or hyperbolic, or spherical geometries. It shows a close relationship between Alexandrov–Pogorelov convex embedding program and discrete conformal geometry.

The counterpart of corollary 4 holds for the spherical background geometry using Rivin's isometric embedding theorem. This was established in [**23**].

Theorem 6 (Sun–Wu–Zhu [**23**])**.** *Let S be the 2-sphere and $V \subset S$ be a finite subset with $|V| \geq 3$. Then each spherical cone metric d on (S, V) is discrete conformal to a spherical metric d^* on the surface (S, V) so that all cone angles of d^* are 2π. The metric d^* is unique up to Möbius transformations.*

1.2. Basic idea of the proof. The basic idea of the proof is similar to that of [**10**]. We first introduce the Teichmüller space $T_{hp}(S, V)$ of hyperbolic polyhedral metrics on (S, V). It is shown to be a real analytic manifold which admits a cell decomposition by the work of [**15**] and [**13**]. Using the work of Kubota [**14**] on hyperbolic Ptolemy identity, the work of Penner [**21**] and the work of Bobenko–Pinkall–Springborn [**4**], we show that $T_{hp}(S, V)$ is C^1 diffeomorphic to the decorated Teichmüller space so that two hyperbolic polyhedral metrics are discrete conformal if and only if their corresponding decorated metrics have the same underlying hyperbolic structure. Using this correspondence, we show Theorem 3 using a variational principle established in [**4**].

Many arguments in this paper are similar to that of [**10**]. The major difference between Euclidean and hyperbolic polyhedral metrics comes from the circumcircles of triangles. Namely, the circumcircle of a hyperbolic triangle may be non-compact, i.e., a horocycle or a curve of constant distance to a geodesic. This creates many difficulties when one uses the inner angle characterization of Delaunay triangulations. To overcome this, we prove (Theorem 14) that every triangle in a Delaunay triangulation of a hyperbolic polyhedral metric on a closed surface has a compact circumcircle.

1.3. Organization of the paper. Section 2 deals with the Teichmüller space of hyperbolic polyhedral metrics, its analytic cell decomposition and Delaunay triangulations. In section 3, we show that there is a C^1 diffeomorphism between the Teichmüller space of hyperbolic polyhedral metrics and the decorated Teichmüller space. Section 4 is devoted to the proof of Theorem 3. Section 5 proves theorem 2. Section 6 establishes the equivalence between the existence and uniqueness part of corollary 4 and Fillastre's work. In the appendix, some technical results are proved.

Acknowledgment. We thank the referee for his/her useful comments. In particular, for his/her observation that corollary 4 is a consequence of Fillastre's theorem. The work is supported in part by the NSF of USA and the NSF of China.

2. Teichmüller space of polyhedral metrics

2.1. Triangulations and some conventions. Take a finite disjoint union X of triangles and identify edges in pairs by homeomorphisms. The quotient space S is a compact surface together with a *triangulation* $\mathcal{T}$ whose simplices are the quotients of the simplices in the disjoint union X. Let $V = V(\mathcal{T})$ and $E = E(\mathcal{T})$ be the sets of vertices and edges in $\mathcal{T}$. We call $\mathcal{T}$ a *triangulation* of the surface with marked points (S, V). If each triangle in the disjoint union X is hyperbolic and the identification maps are isometries, then the quotient metric d on the quotient space (S, V) is a called *hyperbolic polyhedral metric.* The set of cone points of d is contained in V. Given a hyperbolic polyhedral metric d and a triangulation $\mathcal{T}$ on (S, V), if the interior of each triangle in $\mathcal{T}$ (in metric d) is locally isometric to a hyperbolic triangle, we say $\mathcal{T}$ is *geodesic* in d. If $\mathcal{T}$ is a triangulation of (S, V) isotopic to a geodesic triangulation $\mathcal{T}'$ in a hyperbolic polyhedral metric d, then the *length* of an edge $e \in E(\mathcal{T})$ (or *angle* of a triangle at a vertex in $\mathcal{T}$) is defined to be the length (respectively angle) of the corresponding geodesic edge $e' \in E(\mathcal{T}')$ (triangle at the vertex) measured in metric d.

Suppose e is an edge in $\mathcal{T}$ adjacent to two distinct triangles t, t'. Then the *diagonal switch* on $\mathcal{T}$ is a new triangulation $\mathcal{T}'$ obtained from $\mathcal{T}$ by replaces e by the other diagonal in the quadrilateral $t \cup_e t'$.

For simplicity, the terms metrics and triangulations in many places will mean isotopy classes of metrics and isotopy classes of triangulations. They can be understood from the context without causing confusion.

If X is a finite set, $|X|$ denotes its cardinality and $\mathbb{R}^X$ denotes the vector space $\{f : X \to \mathbb{R}\}$. For a finite set $W = \{w_1, ..., w_m\}$, we identify $\mathbb{R}^W$ with $\mathbb{R}^m$ by sending $x \in \mathbb{R}^W$ to $(x(w_1), ..., x(w_m))$.

All surfaces are assumed to be compact and connected in the rest of the paper.

2.2. The Teichmüller space and the length coordinates. Two hyperbolic polyhedral metrics d, d' on (S, V) are called *Teichmüller equivalent* if there is an isometry $h : (S, V, d) \to (S, V, d')$ so that h is isotopic to the identity map on (S, V). The *Teichmüller space* of all hyperbolic polyhedral metrics on (S, V), denoted by $T_{hp}(S, V)$, is the set of all Teichmüller equivalence classes of hyperbolic polyhedral metrics on (S, V).

Proposition 7. *$T_{hp}(S, V)$ is a real analytic manifold.*

Proof. Suppose $\mathcal{T}$ is a triangulation of (S, V) with the set of edges $E = E(\mathcal{T})$. Let $\mathbb{R}^{E(\mathcal{T})}_\Delta$ be the convex polytope in $\mathbb{R}^E$ defined by

$$\{x \in \mathbb{R}^E_{>0} | \forall \text{ triangle } t \text{ in } \mathcal{T} \text{ with edges } e_i, e_j, e_k, \quad x(e_i) + x(e_j) > x(e_k)\}.$$

For each $x \in \mathbb{R}^{E(\mathcal{T})}_\Delta$, one constructs a hyperbolic polyhedral metric d_x on (S, V) by replacing each triangle t of edges e_i, e_j, e_k by a hyperbolic triangle of edge lengths $x(e_i), x(e_j), x(e_k)$ and gluing them by isometries along the corresponding edges. This construction produces an injective map (the length coordinate associated to $\mathcal{T}$)

$$\Phi_{\mathcal{T}} : \mathbb{R}^{E(\mathcal{T})}_\Delta \to T_{hp}(S, V)$$

sending x to $[d_x]$. The image $P(\mathcal{T}) := \Phi_{\mathcal{T}}(\mathbb{R}^{E(\mathcal{T})}_\Delta)$ is the space of all hyperbolic polyhedral metrics $[d]$ on (S, V) for which $\mathcal{T}$ is isotopic to a geodesic triangulation in d. We call x the *length coordinate* of d_x and $[d_x] = \Phi_{\mathcal{T}}(x)$ (with respect to $\mathcal{T}$). In general $P(\mathcal{T}) \neq T_{hp}(S, V)$ (see §2.1 in [**10**]).

Since each hyperbolic polyhedral metric on (S, V) admits a geodesic triangulation (for instance, its Delaunay triangulation), we see that $T_{hp}(S, V) = \cup_{\mathcal{T}} P(\mathcal{T})$ where the union is over all triangulations of (S, V). The space $T_{hp}(S, V)$ is a real analytic manifold with real analytic coordinate charts $\{(P(\mathcal{T}), \Phi_{\mathcal{T}}^{-1}) | \mathcal{T}$ triangulations of $(S, V)\}$. To see transition functions $\Phi_{\mathcal{T}}^{-1}\Phi_{\mathcal{T}'}$ are real analytic, note that any two triangulations of (S, V) are related by a sequence of (topological) diagonal switches. On the other hand, we have the following stronger result to be proved in the appendix B,

Proposition 8. *Suppose* (S, V, d) *is a closed hyperbolic polyhedral metric surface. If* $\mathcal{T}$ *and* $\mathcal{T}'$ *are two geodesic triangulations of* (S, V, d)*, then there exists a sequence of geodesic triangulations* $\mathcal{T}_0 = \mathcal{T}, \mathcal{T}_1, ...,$ $\mathcal{T}_n = \mathcal{T}'$ *of* (S, V, d) *so that* $\mathcal{T}_{i+1}$ *and* $\mathcal{T}_i$ *are related by a diagonal switch.*

Therefore, it suffices to show the $\Phi_{\mathcal{T}}^{-1}\Phi_{\mathcal{T}'}$ is real analytic where $\mathcal{T}$ and $\mathcal{T}'$ are related by a diagonal switch along an edge e. In this case, the transition function $\Phi_{\mathcal{T}}^{-1}\Phi_{\mathcal{T}'}$ sends $(x_0, x_1, ..., x_m)$ to $(f(x_0, ..., x_m),$ $x_1, ..., x_m)$ where x_0 is the length of e and f is the length of the diagonal switched edge. Let t, t' be the triangles adjacent to e so that the lengths of edges of t, t' are $\{x_0, x_1, x_2\}$ and $\{x_0, x_3, x_4\}$. Using the cosine law, we see that f is a real analytic function of $x_0, ..., x_4$. q.e.d.

2.3. Delaunay triangulations and marked quadrilaterals. Each hyperbolic triangle t in $\mathbf{H}^2$ has a *circumcircle* which is the curve of constant geodesic curvature containing the three vertices of t. When the circumcircle is compact, it is a hyperbolic circle. When it is not compact, it is either a horocycle or a curve of constant distance to a geodesic. In the upper-half-space or the unit disk model X of the hyperbolic plane, a circumcircle is the same as the intersection of a Euclidean circle or line with X. We call the convex region bounded by the circumcircle the *circum-ball* of the triangle t. A *marked quadrilateral* Q is a hyperbolic quadrilateral together with a diagonal e inside Q. It is the same as a union of two hyperbolic triangles t, t' along a common edge e, i.e., $Q = t \cup_e t'$. A hyperbolic polygon is called *cyclic* if its vertices lie in a curve of constant geodesic curvature in the hyperbolic plane. A marked quadrilateral $t \cup_e t'$ is cyclic if and only if the two circumcircles for t and t' coincide.

A geodesic triangulation $\mathcal{T}$ of a hyperbolic polyhedral surface (S, V, d) is said to be *Delaunay* if for each edge e adjacent to two hyperbolic triangles t and t', the interior of the circumball of t does not contain the vertices of t' when the quadrilateral $t \cup_e t'$ is lifted to $\mathbf{H}^2$. The last condition is sometimes called the *empty ball condition.* We will call the marked quadrilateral $t \cup_e t'$ the *quadrilateral associated to the edge* e. G. Leibon [**15**] gave a very nice algebraic description of empty-ball condition in terms of the inner angles. The significance of Leibon's result is that the local Delaunay condition on edges implies the empty circle condition.

Lemma 9 (Leibon)**.** *A geodesic triangulation $\mathcal{T}$ is Delaunay if and only if*

$$\alpha + \alpha' \leq \beta + \beta' + \gamma + \gamma', \tag{1}$$

for each edge e, where $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are angles of the two triangles in $\mathcal{T}$ having e as the common edge so that α and α' are opposite to e. Furthermore, the equality holds for e if and only if the marked quadrilateral associated to e is cyclic.

The inequality (1) can be expressed in terms of the edge lengths as follows.

Proposition 10. *A geodesic triangulation $\mathcal{T}$ is Delaunay if and only if*

$$\frac{\sinh^2(x_1/2) + \sinh^2(x_2/2) - \sinh^2(x_0/2)}{\sinh(x_1/2)\sinh(x_2/2)} + \frac{\sinh^2(x_3/2) + \sinh^2(x_4/2) - \sinh^2(x_0/2)}{\sinh(x_3/2)\sinh(x_4/2)} \geq 0, \tag{2}$$

for each edge e adjacent two triangles t, t' of edge lengths x_0, x_1, x_2 and x_0, x_3, x_4 respectively. Furthermore, the equality holds for an edge e if and only if $t \cup_e t'$ is cyclic.

To prove Proposition 10, we need the following lemma.

Lemma 11. *Let x_1, x_2, x_3 be side lengths of a hyperbolic triangle and a_1, a_2, a_3 be the opposite angles so that a_i is facing the edge of length x_i. Then*

$$2 \sin \frac{a_2 + a_3 - a_1}{2} \cdot \cosh \frac{x_1}{2} = \frac{\sinh^2(x_2/2) + \sinh^2(x_3/2) - \sinh^2(x_1/2)}{\sinh(x_2/2)\sinh(x_3/2)}.$$

Proof. By the cosine law expressing x_i in terms of a_1, a_2, a_3, we have

$$\begin{aligned}
&\sinh^2(x_2/2) + \sinh^2(x_3/2) - \sinh^2(x_1/2)\\
&\quad = \frac{1}{2}(\cosh(x_2) + \cosh(x_3) - \cosh(x_1) - 1)\\
&\quad = \frac{1}{2}\Big[\frac{\cos a_2 + \cos a_1 \cos a_3}{\sin a_1 \sin a_3} + \frac{\cos a_3 + \cos a_1 \cos a_2}{\sin a_1 \sin a_2}\\
&\qquad - \frac{\cos a_1 + \cos a_2 \cos a_3}{\sin a_2 \sin a_3} - 1\Big]\\
&\quad = \frac{1}{2 \sin a_1 \sin a_2 \sin a_3}(\sin(a_2 + a_3) - \sin a_1)(\cos a_1 + \cos(a_2 - a_3))\\
&\quad = \frac{2 \sin \frac{a_2 + a_3 - a_1}{2} \cos \frac{a_1 + a_2 + a_3}{2} \cos \frac{a_1 + a_2 - a_3}{2} \cos \frac{a_1 - a_2 + a_3}{2}}{\sin a_1 \sin a_2 \sin a_3}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\sinh^2(x_i/2) &= \frac{1}{2}(\cosh x_i - 1)\\ &= \frac{1}{2}(\frac{\cos a_i + \cos a_j \cos a_k}{\sin a_j \sin a_k} - 1)\\ &= \frac{1}{2}\frac{\cos a_i + \cos(a_j + a_k)}{\sin a_j \sin a_k}\\ &= \frac{\cos\frac{a_i+a_j+a_k}{2}\cos\frac{a_i-a_j-a_k}{2}}{\sin a_j \sin a_k}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{\sinh^2(x_2/2)+\sinh^2(x_3/2)-\sinh^2(x_1/2)}{\sinh(x_2/2)\sinh(x_3/2)}\\ &= \frac{2\sin\frac{a_2+a_3-a_1}{2}\cos\frac{a_1+a_2+a_3}{2}\cos\frac{a_1+a_2-a_3}{2}\cos\frac{a_1-a_2+a_3}{2}}{\sin a_1 \sin a_2 \sin a_3\sqrt{\frac{\cos\frac{a_1+a_2+a_3}{2}\cos\frac{a_2-a_1-a_3}{2}}{\sin a_1 \sin a_3}}\sqrt{\frac{\cos\frac{a_1+a_2+a_3}{2}\cos\frac{a_3-a_1-a_2}{2}}{\sin a_1 \sin a_2}}}\\ &= 2\sin\frac{a_2+a_3-a_1}{2}\cdot\sqrt{\frac{\cos\frac{a_1+a_2-a_3}{2}\cos\frac{a_1-a_2+a_3}{2}}{\sin a_2 \sin a_3}}\\ &= 2\sin\frac{a_2+a_3-a_1}{2}\cdot\cosh\frac{x_1}{2}.\end{aligned}$$

In the last step above, we have used

$$\begin{aligned}(\cosh\frac{x_1}{2})^2 &= \frac{1}{2}(\cosh x_1 + 1)\\ &= \frac{1}{2}(\frac{\cos a_1 + \cos a_2 \cos a_3}{\sin a_2 \sin a_3} + 1)\\ &= \frac{1}{2}\frac{\cos a_1 + \cos(a_2 - a_3)}{\sin a_2 \sin a_3}\\ &= \frac{\cos\frac{a_1+a_2-a_3}{2}\cos\frac{a_1-a_2+a_3}{2}}{\sin a_2 \sin a_3}.\end{aligned}$$

q.e.d.

Proof of Proposition 10. Now (1) is equivalent to

$$\sin\frac{\beta+\gamma-\alpha}{2} + \sin\frac{\beta'+\gamma'-\alpha'}{2} \geq 0.$$

By Lemma 11 applied to triangles of lengths $\{x_0, x_1, x_2\}$ and $\{x_0, x_3, x_4\}$, we see that Delaunay is equivalent to (2). q.e.d.

2.4. Delaunay triangulations of compact hyperbolic polyhedral surfaces.

First, let's recall two results.

Lemma 12 (Penner [**21**] Lemma 5.2)**.** *Suppose $y : E(\mathcal{T}) \to \mathbb{R}_{>0}$ is a function satisfying for each edge e_0 adjacent to two triangles t, t' of* *edges* e_0, e_1, e_2 *and* e_0, e_3, e_4,

$$\frac{y_1^2 + y_2^2 - y_0^2}{y_1 y_2} + \frac{y_3^2 + y_4^2 - y_0^2}{y_3 y_4} \geq 0,$$

where $y_i = y(e_i)$. *Then* $y(e_i) + y(e_j) > y(e_k)$ *whenever* e_i, e_j, e_k *form edges of a triangle in* $\mathcal{T}$.

Proposition 13 (Fenchel [8] page 118)**.** *Let* C *be the circumcircle of a hyperbolic triangle of edge lengths* x_1, x_2, x_3. *Then* C *is a (compact) hyperbolic circle if and only if* $\sinh(\frac{x_i}{2}) + \sinh(\frac{x_j}{2}) > \sinh(\frac{x_k}{2})$ *for* $\{i, j, k\} = \{1, 2, 3\}$.

From these, we obtain,

Theorem 14. *If* $\mathcal{T}$ *is a Delaunay triangulation of a closed hyperbolic polyhedral surface* (S, V, d), *then each triangle has a compact circumcircle.*

Proof. By Proposition 10 for Delaunay triangulations inequality (2) holds. Taking $y(e) = \sinh(\frac{x(e)}{2})$ in Lemma 12 and using (2), we obtain

$$\sinh(\frac{x(e_i)}{2}) + \sinh(\frac{x(e_j)}{2}) > \sinh(\frac{x(e_k)}{2}). \tag{3}$$

Now Theorem 14 follows from (3) and Proposition 13. q.e.d.

Corollary 15. *Suppose* $x : E(\mathcal{T}) \to \mathbb{R}_{>0}$ *is a function so that* (2) holds at each edge. Then x is the edge length function (in $\mathcal{T}$) of a hyperbolic polyhedral metric on (S, V).

Proof. Since $\sinh(a + b) > \sinh(a) + \sinh(b)$ for $a, b > 0$, by (3), we obtain

$$x(e_i) + x(e_j) > x(e_k), \tag{4}$$

whenever e_i, e_j, e_k form edges of a triangle. q.e.d.

It is highly likely that Theorem 14 still holds for hyperbolic cone metrics on high dimensional compact manifolds, i.e., empty-ball condition implies compact circumsphere. The work of [6] shows that it holds for decorated finite volume hyperbolic metrics of any dimension.

We begin with a recall of the classical definition of *Delaunay triangulation* of a closed hyperbolic polyhedral metric (S, V, d). It is essentially the dual of the *Voronoi decomposition* of (S, V, d). See, for instance, [3] or [19] for more details. The Voronoi decomposition is a CW decomposition of the surface S whose open 2-cells are the path components of the set of points which have unique length-minimizing paths to V, whose open 1-cells are the path components of the set of points which have exactly two length minimizing paths to V and the 0-cells are points which have three or more length-minimizing paths to V. See page 470 of [19]. It can be shown that open 2-cells are of the form $R(v) = \{x \in S | d(x, v) < d(x, v') \text{ for all } v' \in V\}$, one for each $v \in V$.

The dual of the Voronoi decomposition is called a *Delaunay tessellation* $\mathcal{C}(d)$ of (S, V, d). It is a cell decomposition of (S, V) with vertices V and two vertices v, v' jointed by an edge if and only if $R(v) \cap R(v')$ is 1-dimensional. By definition, each open 2-cell in the Delaunay tessellation is isometric to an open convex polygon whose closure in $\mathbf{H}^2$ is inscribed to a compact circle in $\mathbf{H}^2$. The center of the compact circle corresponds to a 0-cell of the Voronoi decomposition. By further triangulating all non-triangular 2-dimensional cells (without introducing extra vertices) in $\mathcal{C}(d)$, one obtains a Delaunay triangulation of (S, V, d). This Delaunay triangulation has the property that the circumcircles of triangles are hyperbolic circles (i.e., compact). Indeed, the centers of the circumcircles are the vertices in the Voronoi cell decomposition. Conversely, if $\mathcal{T}$ is a Delaunay triangulation with compact circumcircles for all triangles, then it is a triangulation of the Delaunay tessellation.

Combining Theorem 14, we obtain part (a) of the following,

Proposition 16. *(a) Suppose $\mathcal{T}$ is a geodesic triangulation of a compact hyperbolic polyhedral surface (S, V, d). Then $\mathcal{T}$ satisfies the empty-ball condition if and only if it is a geodesic triangulation of the Delaunay tessellation.*

(b) If $\mathcal{T}$ and $\mathcal{T}'$ are Delaunay triangulations of a hyperbolic polyhedral metric d on a closed surface with marked points (S, V), then there exists a sequence of Delaunay triangulations $\mathcal{T}_1 = \mathcal{T}, \mathcal{T}_2, ..., \mathcal{T}_k = \mathcal{T}'$ of d so that $\mathcal{T}_{i+1}$ is obtained from $\mathcal{T}_i$ by a diagonal switch.

(c) Suppose $\mathcal{T}$ is a Delaunay triangulation of a compact hyperbolic polyhedral surface (S, V, d) whose diameter is D. Then the length of each edge e in $\mathcal{T}$ is at most $2D$. In particular, there exists an algorithm to find all Delaunay triangulations of a hyperbolic polyhedral surface.

Proof. Part (b) of the proposition follows from part (a) and the well known fact that any two geodesic triangulations of the Delaunay tessellation are related by a sequence of diagonal switches. Indeed, any two geodesic triangulations of a convex cyclic polygon are related by a sequence of (geodesic) diagonal switches. See, for instance, [3] for a proof.

To see part (c), if e is an edge dual to two Voronoi cells $R(v)$ and $R(v')$, then the length of e is at most the sum of the diameters of $R(v)$ and $R(v')$. However, the diameters of $R(v)$ and $R(v')$ are bounded by the diameter of the surface S. Thus, the length of e is at most $2D$.

An *edge path* of a triangulation is a path formed by edges in the triangulation. For any constant C, there exists an algorithm to list all the edge paths in (S, V, d) of length at most C joining V to V. On the other hand, the length of a geodesic path joining V to V is less than or equal to the length of a certain edge path joining V to V. Thus, there exists an algorithm to list all geodesic paths in (S, V, d) of lengths at most C joining V to V. Therefore, we can list algorithmically all Delaunay triangulations of a given polyhedral metric on (S, V). q.e.d.

Note that if we remove the compactness of the space S, then there are examples of geodesic triangulations with empty-ball condition which does not come from dual of Voronoi cell. See [6].

For a triangulation $\mathcal{T}$ of (S, V), the associated Delaunay cell in $T_{hp}(S, V)$ is defined to be

$$D_c(\mathcal{T}) = \{[d] \in T_{hp}(S, V) | \mathcal{T} \text{ is isotopic to a Delaunay triangulation of } d\}.$$

Proposition 10 and corollary 15 show that $D_c(\mathcal{T})$ is defined by a finite set of real analytic inequalities (i.e., (2)). On the other hand, Leibon showed in [15] that $D_c(\mathcal{T})$ is a cell. Putting these together, one obtains

Theorem 17 (Hazel [13], Leibon [15])**.** *There is a real analytic cell decomposition*

$$T_{hp}(S, V) = \cup_{[\mathcal{T}]} D_c(\mathcal{T}),$$

invariant under the action of the mapping class group where the union is over all isotopy classes $[\mathcal{T}]$ *of triangulations of* (S, V).

3. Diffeomorphism between two Teichmüller spaces

One of the main tools used in our proof is the decorated Teichmüller space theory developed by R. Penner [21]. See also [2], [11] and [10] for a discussion of Delaunay triangulations of decorated metrics.

Recall that S is a closed connected surface and $V = \{v_1, ..., v_n\} \subset S$ and let $\Sigma = S - V$. We assume $n \geq 1$ and the Euler characteristic $\chi(\Sigma) < 0$. A *decorated hyperbolic metric* is a complete hyperbolic metric d of finite area on Σ together with a horoball H_i at the i-th cusp for each v_i. The decorated metric will be written as a pair (d, w) where $w = (w_1, ..., w_n) \in \mathbb{R}^n_{>0}$ so that w_i is the length of the horocycle ∂H_i. The decorated Teichmüller space, denoted by $T_D(\Sigma)$, is the space of all decorated metrics on Σ modulo isometries homotopic to the identity and preserving decorations. For a given triangulation $\mathcal{T}$ of (S, V), let $\Psi_{\mathcal{T}} : \mathbb{R}^E_{>0} \to T_D(\Sigma)$ be the λ-length coordinate (see [21]) and let $D(\mathcal{T})$ be the set of all decorated hyperbolic metrics (d, w) in $T_D(\Sigma)$ so that $\mathcal{T}$ is isotopic to a Delaunay triangulation of (d, w). See [21] or [10] for details.

Fix a triangulation $\mathcal{T}$ of (S, V), we have two coordinate maps $\Phi^{-1}_{\mathcal{T}} : P(\mathcal{T}) \to \mathbb{R}^{E(\mathcal{T})}$ and $\Psi_{\mathcal{T}} : \mathbb{R}^{E(\mathcal{T})} \to T_D(S, V)$. Consider the smooth embedding $A_{\mathcal{T}} : P(\mathcal{T}) \to T_D(\Sigma)$ defined by $\Psi_{\mathcal{T}} \circ \Theta \circ \Phi^{-1}_{\mathcal{T}}$, where $\Theta : \mathbb{R}^{E(\mathcal{T})} \to \mathbb{R}^{E(\mathcal{T})}$ sends $(x_0, x_1, x_2, ...)$ to $(\sinh(x_0/2), \sinh(x_1/2), \sinh(x_2/2), ...)$, i.e., $\Theta(x)(e) = \sinh(x(e)/2)$.

Theorem 18. *For each triangulation $\mathcal{T}$ of (S,V), $A_{\mathcal{T}}|_{D_c(\mathcal{T})}$ is a real analytic diffeomorphism from $D_c(\mathcal{T})$ onto $D(\mathcal{T})$.*

Proof. To see that $A_{\mathcal{T}}$ maps $D_c(\mathcal{T})$ bijectively onto $D(\mathcal{T})$, it suffices to show that $\Theta\circ\Phi_{\mathcal{T}}^{-1}(D_c(\mathcal{T}))=\Psi_{\mathcal{T}}^{-1}(D(\mathcal{T}))$.

The space $\Psi_{\mathcal{T}}^{-1}(D(\mathcal{T}))$ can be characterized as follows. For each edge e in $(S,\mathcal{T})$ with a decorated hyperbolic metric (d,w), let a,a' be the two angles facing e and b,b',c,c' be the angles adjacent to the edge e. Then $\mathcal{T}$ is Delaunay in the metric (d,w) if and only if for each edge $e\in E(\mathcal{T})$ (see [**21**], or [**11**]),

$$a+a'\le b+b'+c+c'. \tag{5}$$

Let t and t' be the triangle adjacent to e and e,e_1,e_2 be edges of t and e,e_3,e_4 be the edges of t'. Let the λ-length of e be λ_0 and the λ-length of e_i be λ_i. Recall the cosine law for decorated ideal triangles [**21**] states that $\alpha=\frac{x}{yz}$ where α is the angle (i.e., the length of the horocyclic arc) and x,y,z are the λ-lengths so that x faces α. Using it, one sees that (5) is equivalent to

$$\frac{\lambda_0}{\lambda_1\lambda_2}+\frac{\lambda_0}{\lambda_3\lambda_4}\le\frac{\lambda_1}{\lambda_0\lambda_2}+\frac{\lambda_2}{\lambda_0\lambda_1}+\frac{\lambda_3}{\lambda_0\lambda_4}+\frac{\lambda_4}{\lambda_0\lambda_3}, \tag{6}$$

for each $e\in E(\mathcal{T})$.

Rearranging terms, we see (6) is equivalent to

$$0\le\frac{\lambda_1^2+\lambda_2^2-\lambda_0^2}{\lambda_1\lambda_2}+\frac{\lambda_3^2+\lambda_4^2-\lambda_0^2}{\lambda_3\lambda_4}, \tag{7}$$

for each $e\in E(\mathcal{T})$.

Therefore,

$$\Psi_{\mathcal{T}}^{-1}(D(\mathcal{T}))=\{(\lambda_0,\lambda_1,\dots,\lambda_{|E|})\in\mathbb{R}^E_{>0}|\ (7)\text{ holds at each edge } e\in E(\mathcal{T})\}.$$

By Theorem 14 and proposition 10, the characterization of a hyperbolic polyhedral metric d which is Delaunay in $\mathcal{T}$ in terms of the length coordinate $x=\Phi_{\mathcal{T}}^{-1}(d)$ is as follows. Take an edge $e\in E(\mathcal{T})$ and let t and t' be the triangles adjacent to e so that e,e_1,e_2 are edges of t and e,e_3,e_4 are the edge of t'. Suppose the length of e (in d) is x_0 and the length of e_i is x_i, $i=1,\dots,4$. Then, by Proposition 10,

$$0\le\frac{\sinh^2(x_1/2)+\sinh^2(x_2/2)-\sinh^2(x_0/2)}{\sinh(x_1/2)\sinh(x_2/2)}+\frac{\sinh^2(x_3/2)+\sinh^2(x_4/2)-\sinh^2(x_0/2)}{\sinh(x_3/2)\sinh(x_4/2)} \tag{8}$$

holds for each edge $e\in E(\mathcal{T})$.

This shows that

$$\Phi_{\mathcal{T}}^{-1}(D_c(\mathcal{T})) = \{x \in \mathbb{R}^E_{>0} | \text{ (8) holds for } e \in E, \text{ and (9) holds for each triangle}\},$$

where

(9) $\quad x(e_i) + x(e_j) > x(e_k), \quad e_i, e_j, e_k$ form edges of a triangle in $\mathcal{T}$.

Now inequality (7) is the same as (8) by taking λ_i to be $\sinh(x_i/2)$ for each i. This shows $\Theta \circ \Phi_{\mathcal{T}}^{-1}(D_c(\mathcal{T})) \subset \Psi_{\mathcal{T}}^{-1}(D(\mathcal{T}))$. On the other hand, corollary 15 implies that for each $\lambda \in \Psi_{\mathcal{T}}^{-1}(D(\mathcal{T}))$ and a triangle of edges e_i, e_j, e_k, we have $x(e_i) + x(e_j) > x(e_k)$ where $x(e) = 2\sinh^{-1}(\lambda(e))$, i.e., condition (9) is a consequence of (8). Therefore, $\Theta \circ \Phi_{\mathcal{T}}^{-1}(D_c(\mathcal{T})) = \Psi_{\mathcal{T}}^{-1}(D(\mathcal{T}))$.

Finally, since $\Phi_{\mathcal{T}}$, $\Psi_{\mathcal{T}}$ and Θ are real analytic diffeomorphisms and $A_{\mathcal{T}} = \Psi_{\mathcal{T}} \circ \Theta \circ \Phi_{\mathcal{T}}^{-1}$ and $A_{\mathcal{T}}^{-1} = \Phi_{\mathcal{T}} \circ \Theta^{-1} \circ \Psi_{\mathcal{T}}^{-1}$, we see that $A_{\mathcal{T}}$ is a real analytic diffeomorphism. q.e.d.

3.1. The Ptolemy identity and diagonal switch. Let Q be a convex quadrilateral Q in the Euclidean plane $\mathbf{E}^2$, or the hyperbolic plane $\mathbf{H}^2$ or the 2-sphere $\mathbf{S}^2$ so that its edges are a, b, a', b' counted cyclically and its diagonals are c, c'. We say Q is *cyclic* if it is circumscribed to a circle in $\mathbf{E}^2$, or $\mathbf{S}^2$, or a curve of constant geodesic curvature in $\mathbf{H}^2$. Let $l(e)$ to be the length of an edge e.

The classical Ptolemy theorem states that a Euclidean quadrilateral Q is cyclic if and only if the following holds

$$l(a)l(a') + l(b)l(b') = l(c)l(c').$$

In the 19-th century, Jean Darboux and Ferdinand Frobenius proved that a spherical quadrilateral Q is cyclic if and only if

$$\sin(\frac{l(a)}{2})\sin(\frac{l(a')}{2}) + \sin(\frac{l(b)}{2})\sin(\frac{l(b')}{2}) = \sin(\frac{l(c)}{2})\sin(\frac{l(c')}{2}).$$

The hyperbolic case was established by T. Kubota in 1912 [**14**]. He proved,

Proposition 19 (Kubota)**.** *A hyperbolic quadrilateral Q is inscribed to a curve of constant geodesic curvature in $\mathbf{H}^2$ if and only if*

(10)
$$\sinh(\frac{l(a)}{2})\sinh(\frac{l(a')}{2}) + \sinh(\frac{l(b)}{2})\sinh(\frac{l(b')}{2}) = \sinh(\frac{l(c)}{2})\sinh(\frac{l(c')}{2}).$$

Penner's Ptolemy identity [**21**] also takes the same form. Namely, if Q is a decorated ideal quadrilateral in $\mathbf{H}^2$ so that the λ-lengths of the its edges are A, B, A', B' counted cyclically and its diagonal are C, C', then

(11)
$$AA' + BB' = CC'.$$

The most remarkable feature of these theorems is that all equations take the same form as $xx' + yy' = zz'$ which we will call the Ptolemy identity. The Ptolemy identity also plays the key role for cluster algebras associated to surfaces [**9**].

The relationship between the Ptolemy identity and the diagonal switch operation on Delaunay triangulations is the following. If $\mathcal{T}$ and $\mathcal{T}'$ are two Delaunay triangulations of a Euclidean (or hyperbolic or spherical) polyhedral surface (S, V, d) so that they are related by a diagonal switch from edge e to edge e', then the change of the lengths from $l(e)$ and $l(e')$ is governed by one of the Ptolemy identities listed above.

3.2. A globally defined diffeomorphism.

Theorem 20. *Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two triangulations of (S, V) so that $D_c(\mathcal{T}) \cap D_c(\mathcal{T}') \neq \emptyset$. Then*

$$A_{\mathcal{T}}|_{D_c(\mathcal{T})\cap D_c(\mathcal{T}')} = A_{\mathcal{T}'}|_{D_c(\mathcal{T})\cap D_c(\mathcal{T}')}. \tag{12}$$

In particular, the gluing of these $A_{\mathcal{T}}|_{D_c(\mathcal{T})}$ mappings produces a homeomorphism $A = \cup_{\mathcal{T}} A_{\mathcal{T}}|_{D_c(\mathcal{T})} : T_{hp}(S, V) \to T_D(\Sigma)$ such that $A(d)$ and $A(d')$ have the same underlying hyperbolic structure if and only if d and d' are discrete conformal.

Proof. Suppose $d \in D_c(\mathcal{T}) \cap D_c(\mathcal{T}')$, i.e., $\mathcal{T}$ and $\mathcal{T}'$ are both Delaunay in the hyperbolic polyhedral metric d. Then by proposition 16 there exists a sequence of triangulations $\mathcal{T}_1 = \mathcal{T}, \mathcal{T}_2, ..., T_k = \mathcal{T}'$ on (S, V) so that each $\mathcal{T}_i$ is Delaunay in d and $\mathcal{T}_{i+1}$ is obtained from $\mathcal{T}_i$ by a diagonal switch. In particular, $A_{\mathcal{T}}(d) = A_{\mathcal{T}'}(d)$ follows from $A_{\mathcal{T}_i}(d) = A_{\mathcal{T}_{i+1}}(d)$ for $i = 1, 2, ..., k-1$. Thus, it suffices to show $A_{\mathcal{T}}(d) = A_{\mathcal{T}'}(d)$ when $\mathcal{T}'$ is obtained from $\mathcal{T}$ by a diagonal switch along an edge e. This is the same as showing $\Psi_{\mathcal{T}}^{-1}\Psi_{\mathcal{T}'} = \Theta\Phi_{\mathcal{T}}^{-1}\Phi_{\mathcal{T}'}\Theta^{-1}$ at the point $x = \Psi_{\mathcal{T}'}^{-1}(d)$. On the other hand, $\Psi_{\mathcal{T}}^{-1}\Psi_{\mathcal{T}'}(x)$ and $\Theta\Phi_{\mathcal{T}}^{-1}\Phi_{\mathcal{T}'}\Theta^{-1}(x)$ have the same coordinate except at the e edge of diagonal switch. For the edge e, the two coordinates are the same due to the Penner's Ptolemy identity (11) (for $\Psi_{\mathcal{T}}^{-1}\Psi_{\mathcal{T}'}$) and Kubota's Ptolemy identity (10) (for $\Phi_{\mathcal{T}}^{-1}\Phi_{\mathcal{T}'}$). These two identities differ by a change of variable $t \to \sinh(\frac{t}{2})$ which corresponds to Θ. Therefore, $A_{\mathcal{T}}(d) = A_{\mathcal{T}'}(d)$.

Taking the inverse, we obtain

$$A_{\mathcal{T}}^{-1}|_{D(\mathcal{T})\cap D(\mathcal{T}')} = A_{\mathcal{T}'}^{-1}|_{D(\mathcal{T})\cap D(\mathcal{T}')}. \tag{13}$$

Lemma 21. (a) *$D_c(\mathcal{T}) \cap D_c(\mathcal{T}') \neq \emptyset$ if and only if $D(\mathcal{T}) \cap D(\mathcal{T}') \neq \emptyset$.*

(b) *The gluing map $A = \cup_{\mathcal{T}} A_{\mathcal{T}}|_{D_c(\mathcal{T})} : T_c \to T_D$ is a homeomorphism invariant under the action of the mapping class group.*

Proof. By (12) and (13), the maps $A = \cup_{\mathcal{T}} A_{\mathcal{T}}|_{D_c(\mathcal{T})} : T_c \to T_D$ and $B = \cup_{\mathcal{T}} A_{\mathcal{T}}^{-1}|_{D(\mathcal{T})} : T_D \to T_c$ are well defined and continuous.

Since $A(D_c(\mathcal{T}) \cap D_c(\mathcal{T}')) \subset D(\mathcal{T}) \cap D(\mathcal{T}')$ and $B(D(\mathcal{T}) \cap D(\mathcal{T}')) \subset D_c(\mathcal{T}) \cap D_c(\mathcal{T}')$, part (a) follows. To see part (b), by Penner's result [**21**] that $T_D = \cup_{\mathcal{T}} D(\mathcal{T})$, the map A is onto. To see A is injective, suppose $x_1 \in D_c(\mathcal{T}_1), x_2 \in D_c(\mathcal{T}_2)$ so that $A(x_1) = A(x_2) \in D(\mathcal{T}_1) \cap D(\mathcal{T}_2)$. Apply (13) to $A^{-1}_{\mathcal{T}_1}|, A^{-1}_{\mathcal{T}_2}|$ on the set $D(\mathcal{T}_1) \cap D(\mathcal{T}_2)$ at the point $A(x_1)$, we conclude that $x_1 = x_2$. This shows that A is a bijection with inverse B. Since both A and B are continuous, A is a homeomorphism. q.e.d.

Now if d and d' are two discrete conformally equivalent hyperbolic polyhedral metrics, then $A(d)$ and $A(d')$ are of the form (p, w) and (p, w') due to the definitions. Indeed, if d and d' are related by condition (b) in definition 1, then the discrete conformality translates to the change of decoration without changing the hyperbolic metric. (This is the same proof as in [**10**], lemma 3.1). If d and d' are related by condition (c) in definition 1, then the two triangulations $\mathcal{T}_i$ and $\mathcal{T}_{i+1}$ are both Delaunay in $[d]$. Therefore, in this case, $A(d) = A(d')$.

On the other hand, if two hyperbolic cone metrics d, d' satisfy that $A(d)$ and $A(d')$ are of the form (p, w) and (p, w'), consider a generic smooth path $\gamma(t) = (p, w(t)), t \in [0, 1]$, in $T_D(\Sigma)$ from (p, w) to (p, w') so that $\gamma(t)$ intersects the cells $D(\mathcal{T})$'s transversely. This implies that γ passes through a finite set of cells $D(\mathcal{T}_i)$ and $\mathcal{T}_j$ and $\mathcal{T}_{j+1}$ are related by a diagonal switch. Let $t_0 = 0 < ... < t_m = 1$ be a partition of $[0, 1]$ so that $\gamma([t_i, t_{i+1}]) \subset D(\mathcal{T}_i)$. Say d_i is the hyperbolic polyhedral metric so that $A(d_i) = \gamma(t_i) \in D(\mathcal{T}_i) \cap D(\mathcal{T}_{i+1})$, $d_1 = d$ and $d_m = d'$. Then by definition, the sequences $\{d_1, ..., d_m\}$ and the associated Delaunay triangulations $\{\mathcal{T}_1, ..., T_m\}$ satisfy the definition of discrete conformality for d, d'. q.e.d.

Theorem 22. *The homeomorphism $A : T_{hp}(S, V) \to T_D(\Sigma)$ is a C^1 diffeomorphism.*

Proof. It suffices to show that for a point $d \in D_c(\mathcal{T}) \cap D_c(\mathcal{T}')$, the derivatives $DA_{\mathcal{T}}(d)$ and $DA_{\mathcal{T}'}(d)$ are the same. Since both $\mathcal{T}$ and $\mathcal{T}'$ are Delaunay in d and are related by a sequence of Delaunay triangulations (in d) $\mathcal{T}_1 = \mathcal{T}, \mathcal{T}_2, ..., \mathcal{T}_k = \mathcal{T}'$, $DA_{\mathcal{T}}(d) = DA_{\mathcal{T}'}(d)$ follows from $DA_{\mathcal{T}_i}(d) = DA_{\mathcal{T}_{i+1}}(d)$ for $i = 1, 2, ..., k-1$. Therefore, it suffices to show $DA_{\mathcal{T}}(d) = DA_{\mathcal{T}'}(d)$ when $\mathcal{T}$ and $\mathcal{T}'$ are related by a diagonal switch at an edge e. In the coordinates $\Phi_{\mathcal{T}}$ and $\Psi_{\mathcal{T}}$, the fact that $DA_{\mathcal{T}}(d) = DA_{\mathcal{T}'}(d)$ is equivalent to the following smoothness question on the diagonal lengths.

Lemma 23. *Suppose Q is a convex hyperbolic quadrilateral whose four edges are of lengths x, y, z, w (counted cyclically) and the length of a diagonal is a. Suppose $A(x, y, z, w, a)$ is the length of the other diagonal and $B(x, y, z, w, a) = s^{-1}(\frac{s(x)s(z)+s(y)s(w)}{s(a)})$ where $s(t) = \sinh(\frac{t}{2})$. If a point (x, y, z, w, a) satisfies $A(x, y, z, w, a) = B(x, y, z, w, a)$, i.e., Q is* *inscribed in a curve of constant geodesic curvature, then* $DA(x, y, z, w, a) = DB(x, y, z, w, a)$ *where* DA *is the derivative of* A.

Due to the lengthy proof of this lemma, we defer it to the appendix. q.e.d.

Corollary 24. *For a given hyperbolic polyhedral metric* d *on* (S, V), *the set of all Teichmüller equivalence classes of hyperbolic metrics on* (S, V) *which are discrete conformal to* d *is* C^1*-diffeomorphic to* $\mathbb{R}^{|V|}$.

4. Discrete uniformization for hyperbolic polyhedral metrics

This section proves theorem 3 which is the main result of this paper.

By Corollary 24, Theorem 3 is equivalent to a statement about the composition map of the discrete curvature map K and $(A|)^{-1}$ defined on $\{p\} \times \mathbb{R}^n_{>0} \subset T_D(\Sigma)$ for any complete hyperbolic metric p of finite area. Here $K : T_{hp}(S, V) \to (-\infty, 2\pi)^n$ is the map sending a metric d to its discrete curvature K_d. Let us make a change of variables from $w = (w_1, ..., w_n) \in \mathbb{R}^n_{>0}$ to $u = (u_1, ..., u_n) \in \mathbb{R}^n$ where $u_i = \ln(w_i)$. We write $w = w(u)$. For a given $p \in T(\Sigma)$, define F to be the composition of K and $(A|)^{-1}$ from $\mathbb{R}^n$ to $(-\infty, 2\pi)^n$ by

$$F(u) = K_{A^{-1}(p,w(u))}. \tag{14}$$

By the Gauss–Bonnet theorem, the image $F(u)$ lies in the open subset $\mathbf{P} = \{x \in (-\infty, 2\pi)^n | \sum_{i=1}^n x_i > 2\pi\chi(S)\}$ of $\mathbb{R}^n$. Theorem 3 is equivalent to that $F : \mathbb{R}^n \to \mathbf{P}$ is a bijection. We will show a stronger statement that F is a homeomorphism.

For simplicity, we use $s(t)$ to denote the function $\sinh(\frac{t}{2})$.

4.1. Injectivity of F.

Since A is a C^1 diffeomorphism and the discrete curvature $K : \mathcal{T}_{hp}(S, V) \to \mathbb{R}^V$ is real analytic, hence, the map F is C^1 smooth.

On the other hand, we have,

Theorem 25 (Akiyoshi [1]). *For any finite area complete hyperbolic metric* p *on* Σ, *there are only finitely many isotopy classes of triangulations* $\mathcal{T}$ *so that* $([p] \times \mathbb{R}^n_{>0}) \cap D(\mathcal{T}) \neq \emptyset$.

Let $\mathcal{T}_i$, $i = 1, ..., k$, be the set of all triangulations so that $(\{p\} \times \mathbb{R}^n) \cap D(\mathcal{T}_i) \neq \emptyset$ and $\{p\} \times \mathbb{R}^n \subset \cup_{i=1}^k D(\mathcal{T}_i)$.

Lemma 26. *Let* $\phi : \mathbb{R}^n \to \{p\} \times \mathbb{R}^n$ *be* $\phi(x) = (p, x)$ *and* $U_i = \phi^{-1}((\{p\} \times \mathbb{R}^n) \cap D(\mathcal{T}_i)) \subset \mathbb{R}^n$ *and* $J = \{i | \, int(U_i) \neq \emptyset\}$. *Then* $\mathbb{R}^n = \cup_{i \in J} U_i$ *and* U_i *is real analytic diffeomorphic to a convex polytope in* $\mathbb{R}^n$.

Proof. By definition, both $\{p\} \times \mathbb{R}^n$ and $D(\mathcal{T}_i)$ are closed and semi algebraic in $T_D(\Sigma)$. Therefore, U_i is closed in $\mathbb{R}^n$ and is diffeomorphic under $w = w(u)$ to a semi-algebraic set. Now by definition, $Y := \cup_{i \in J} U_i$ is a closed subset of $\mathbb{R}^n$ since U_i is closed. If $Y \neq \mathbb{R}^n$, then the complement $\mathbb{R}^n - Y$ is a non-empty open set which is diffeomorphic under $w = w(u)$ to a finite union of real algebraic sets of dimension less than n. This is impossible.

Finally, we will show that for any triangulation $\mathcal{T}$ of (S, V) and $p \in T(\Sigma)$, the intersection $U = \phi^{-1}((\{p\} \times R^n) \cap D(\mathcal{T}))$ is real analytically diffeomorphic to a convex polytope in a Euclidean space. In fact, $\Psi_{\mathcal{T}}^{-1}(U) \subset \mathbb{R}^{E(\mathcal{T})}$ is real analytically diffeomorphic to a convex polytope. To this end, let $b = \Psi_{\mathcal{T}}(p, (1, 1, ..., 1))$. By definition, $\Psi_{\mathcal{T}}^{-1}(U)$ is give by

$$\{x \in \mathbb{R}_{>0}^{E(\mathcal{T})} | \exists \lambda \in \mathbb{R}_{>0}^{V}, \sinh(x(e)/2) = b(e)\lambda(v_1)\lambda(v_2), \partial e = \{v_1, v_2\},$$
$$\text{Delaunay condition (2) holds for } x\}.$$

We claim that the Delaunay condition (2) consists of linear inequalities in the variable $\delta : V \to \mathbb{R}_{>0}$ where $\delta(v) = \lambda(v)^{-2}$. Indeed, suppose the two triangles adjacent to the edge $e = (v_1, v_2)$ have vertices v_1, v_2, v_3 and v_1, v_2, v_4. Let x_{ij} (respectively b_{ij}) be the value of x (respectively b) at the edge joining v_i, v_j, and $\lambda_i = \lambda(v_i)$ and let $s(t)$ be the function $\sinh(\frac{t}{2})$. By definition, $s(x_{ij}) = b_{ij}\lambda_i\lambda_j$. The Delaunay condition (2) at the edge $e = (v_1 v_2)$ says that

$$\frac{s(x_{12})^2}{s(x_{31})s(x_{32})} + \frac{s(x_{12})^2}{s(x_{41})s(x_{42})} \leq \frac{s(x_{31})}{s(x_{32})} + \frac{s(x_{32})}{s(x_{31})} + \frac{s(x_{41})}{s(x_{42})} + \frac{s(x_{42})}{s(x_{41})}.$$

It is the same as, using $s(x_{ij}) = b_{ij}\lambda_i\lambda_j$,

$$c_3\frac{\lambda_1\lambda_2}{\lambda_3^2} + c_4\frac{\lambda_1\lambda_2}{\lambda_4^2} \leq c_1\frac{\lambda_2}{\lambda_1} + c_2\frac{\lambda_1}{\lambda_2},$$

where c_i is some constant depending only on b_{jk}'s. Dividing above inequality by $\lambda_1\lambda_2$ and using $\delta_i = \lambda_i^{-2}$, we obtain

$$c_3\delta_3 + c_4\delta_4 \leq c_1\delta_1 + c_2\delta_2, \tag{15}$$

at each edge $e \in E(\mathcal{T})$. This shows for b fixed, the set of all possible values of δ form a convex polytope $\mathbf{Q}$ defined by (15) at all edges and $\delta(v) > 0$ at all $v \in V$. On the other hand, by definition, the map from $\mathbf{Q}$ to $\Psi_{\mathcal{T}}^{-1}(U)$ sending δ to $x = x(\delta)$ given by $x(vv') = 2\sinh^{-1}(\frac{b(vv')}{\sqrt{\delta(v)\delta(v')}})$ is a real analytic diffeomorphism. Thus, the result follows. q.e.d.

Write $F = (F_1, ..., F_n)$ which is C^1 smooth. The work of Bobenko–Pinkall–Springborn ([4], proposition 5.1.5) shows that

(a) $F_j|_{U_h}$ is real analytic so that $\frac{\partial F_i}{\partial u_j} = \frac{\partial F_j}{\partial u_i}$ in U_h for all $h \in J$,
(b) the Hessian matrix $[\frac{\partial F_i}{\partial u_j}]$ is positive definite on each U_h.

Therefore, the 1-form $\eta = \sum_i F_i(u)du_i$ is a C^1 smooth 1-form on $\mathbb{R}^n$ so that $d\eta = 0$ on each $U_h, h \in J$. This implies that $d\eta = 0$ in $\mathbb{R}^n$.

Hence, the integral

$$W(u) = \int_0^u \eta \tag{16}$$

is a well defined C^2 smooth function on $\mathbb{R}^n$ so that its Hessian matrix is positive definite. Therefore, W is convex in $\mathbb{R}^n$ so that its gradient $\nabla W = F$. Now F is injective due to the following well known lemma,

Lemma 27. *If $W : \Omega \to \mathbb{R}$ is a C^1-smooth strictly convex function on an open convex set $\Omega \subset \mathbb{R}^m$, then its gradient $\nabla W : \Omega \to \mathbb{R}^m$ is an embedding.*

4.2. The map F is onto. Since both $\mathbb{R}^n$ and

$$\mathbf{P} = \{x \in (-\infty, 2\pi)^n | \sum_{i=1}^n x_i > 2\pi\chi(S)\}$$

are connected manifolds of dimension n and F is injective and continuous, it follows that $F(\mathbb{R}^n)$ is open in $\mathbf{P}$. To show that F is onto, it suffices to prove that $F(\mathbb{R}^n)$ is closed in $\mathbf{P}$.

To this end, take a sequence $\{u^{(m)}\}$ in $\mathbb{R}^n$ which leaves every compact set in $\mathbb{R}^n$. We will show that $\{F(u^{(m)})\}$ leaves each compact set in $\mathbf{P}$. By taking subsequences, we may assume that for each index $i = 1, 2, ..., n$, the limit $\lim_m u_i^{(m)} = t_i$ exists in $[-\infty, \infty]$. Furthermore, by Akiyoshi's theorem that the space $p \times \mathbb{R}^n$ is in the union of a finite number of Delaunay cells $D(\mathcal{T})$, we may assume, after taking another subsequence, that the corresponding hyperbolic polyhedral metrics $d_m = A^{-1}(p, w(u^{(m)}))$ are in $D(\mathcal{T})$ for one triangulation $\mathcal{T}$. We will calculate in the length coordinate $\Phi_{\mathcal{T}}$ below.

Since $u^{(m)}$ does not converge to any vector in $\mathbb{R}^n$, there exists $t_i = \infty$ or $-\infty$. Let us label vertices $v \in V$ by *black* and *white* as follows. The vertex v_i is black if and only if $t_i = -\infty$ and all other vertices are white.

Lemma 28. *(a) There does not exist a triangle $\tau \in \mathcal{T}$ with exactly two white vertices.*

(b) If $\Delta v_1 v_2 v_3$ is a triangle with exactly one white vertex at v_1, then the inner angle of the triangle at v_1 converges to 0 as $m \to \infty$ in the metrics d_m.

Proof. To see (a), suppose otherwise, using the $\Phi_{\mathcal{T}}$ length coordinate, we see the given assumption is equivalent to following. There exists a hyperbolic triangle of lengths $l_1^{(m)}, l_2^{(m)}, l_3^{(m)}$ such that $s(l_i^{(m)}) = s(a_i)e^{u_j^{(m)} + u_k^{(m)}}$, $\{i, j, k\} = \{1, 2, 3\}$, where $\lim_m u_i^{(m)} > -\infty$ for $i = 2, 3$ and $\lim_m u_1^{(m)} = -\infty$. By applying $\sinh(t/2)$ to the triangle inequality $l_2^{(m)} + l_3^{(m)} > l_1^{(m)}$ and using angle sum formula for sinh, we obtain

$$s(l_2^{(m)})\sqrt{1 + s(l_3^{(m)})^2} + s(l_3^{(m)})\sqrt{1 + s(l_2^{(m)})^2} > s(l_1^{(m)}).$$

Thus,

$$s(a_2)e^{u_1^{(m)}+u_3^{(m)}}\sqrt{1+s(a_3)^2e^{2u_1^{(m)}+2u_2^{(m)}}} + s(a_3)e^{u_1^{(m)}+u_2^{(m)}}\sqrt{1+s(a_2)^2e^{2u_1^{(m)}+2u_3^{(m)}}} > s(a_1)e^{u_2^{(m)}+u_3^{(m)}}.$$

This is the same as

$$s(a_2)\sqrt{e^{-2u_2^{(m)}}+s(a_3)^2e^{2u_1^{(m)}}} + s(a_3)\sqrt{e^{-2u_3^{(m)}}+s(a_3)^2e^{2u_1^{(m)}}} > s(a_1)e^{-u_1^{(m)}}.$$

However, by the assumption, the right-hand-side tends to ∞ and the left-hand-side is bounded. The contradiction shows that (a) holds.

To see (b), we use the same notation as in the proof of (a). Let $\alpha_1^{(m)}$ be the inner angle at v_1 of the triangle $\Delta v_1v_2v_3$ in d_m metric. Our goal is to show $\lim_m \alpha_1^{(m)} = 0$.

Since the sequence of hyperbolic polyhedral metrics $\{d_m\}$ are Delaunay in the same triangulation $\mathcal{T}$, by proposition 13, the three numbers $s(l_1^{(m)}), s(l_2^{(m)}), s(l_3^{(m)})$ satisfy the triangle inequality. Therefore, for each m, there is a Euclidean triangle whose sides have lengths $s(l_1^{(m)}), s(l_2^{(m)}), s(l_3^{(m)})$. Since $s(l_i^{(m)}) = s(a_i)e^{u_j^{(m)}+u_k^{(m)}}$, this triangle is similar to the Euclidean triangle Δ whose sides have lengths $s(a_1)e^{-u_1^{(m)}}$, $s(a_1)e^{-u_2^{(m)}}$ and $s(a_1)e^{-u_3^{(m)}}$. By the assumption that $\lim_m u_1^{(m)} > -\infty$ and $\lim_m u_2^{(m)} = -\infty$ and $\lim_m u_3^{(m)} = -\infty$, the three edge lengths $s(a_1)e^{-u_1^{(m)}}, s(a_1)e^{-u_2^{(m)}}, s(a_1)e^{-u_3^{(m)}}$ tend to $t \in \mathbb{R}$, ∞ and ∞ respectively. Therefore, the angle in the Euclidean triangle Δ opposite to the edge of length $s(a_1)e^{-u_1^{(m)}}$ approaches 0. By the cosine law for Euclidean triangle, we obtain

$$\lim_m \frac{s(l_2^{(m)})^2 + s(l_3^{(m)})^2 - s(l_1^{(m)})^2}{2s(l_2^{(m)})s(l_3^{(m)})} = 1.$$

On the other hand, from Lemma 11, we have

$$\sin\frac{\alpha_2^{(m)}+\alpha_3^{(m)}-\alpha_1^{(m)}}{2}\cdot\cosh\frac{l_1^{(m)}}{2} = \frac{s(l_2^{(m)})^2 + s(l_3^{(m)})^2 - s(l_1^{(m)})^2}{2s(l_2^{(m)})s(l_3^{(m)})}.$$

Also we have $\lim_m l_1^{(m)} = 0$ due to $\lim_m u_2^{(m)} = -\infty$ and $\lim_m u_3^{(m)} = -\infty$. Hence,

$$\lim_m \sin\frac{\alpha_2^{(m)}+\alpha_3^{(m)}-\alpha_1^{(m)}}{2} = 1.$$

It is equivalent to

$$\lim_m(\alpha_2^{(m)} + \alpha_3^{(m)} - \alpha_1^{(m)}) = \pi \geq \lim_m(\alpha_2^{(m)} + \alpha_3^{(m)} + \alpha_1^{(m)}).$$

Thus,

$$\lim_m \alpha_1^{(m)} \leq 0.$$

Hence,

$$\lim_m \alpha_1^{(m)} = 0. \qquad \text{q.e.d.}$$

We now finish the proof of $F(\mathbb{R}^n) = \mathbf{P}$ as follows.

Case 1. All vertices are white. There exists $t_i = \infty$. Let $\triangle v_i v_j v_k$ be a triangle at vertex v_i. There exists a hyperbolic triangle of lengths $l_i^{(m)}, l_j^{(m)}, l_k^{(m)}$ such that $s(l_i^{(m)}) = s(a_i)e^{u_j^{(m)}+u_k^{(m)}}$ (similar formulas hold for $l_j^{(m)}$ and $l_k^{(m)}$). Then $\lim_m l_j^{(m)} = \lim_m l_k^{(m)} = \infty$. Let $\alpha_i^{(m)}$ be the inner angle at v_i. By the cosine rule,

$$\begin{aligned}
&\lim_m \cos\alpha_i^{(m)} \\
&\quad= \lim_m \frac{-\cosh l_i^{(m)} + \cosh l_j^{(m)} \cosh l_k^{(m)}}{\sinh l_j^{(m)} \sinh l_k^{(m)}} \\
&\quad= \lim_m \frac{-\cosh l_i^{(m)} + \cosh l_j^{(m)} \cosh l_k^{(m)}}{\cosh l_j^{(m)} \cosh l_k^{(m)}} \\
&\qquad\cdot \lim_m \frac{\cosh l_j^{(m)} \cosh l_k^{(m)}}{\sinh l_j^{(m)} \sinh l_k^{(m)}} \\
&\quad= \lim_m \frac{-\cosh l_i^{(m)} + \cosh l_j^{(m)} \cosh l_k^{(m)}}{\cosh l_j^{(m)} \cosh l_k^{(m)}} \\
&\quad= -\lim_m \frac{\cosh l_i^{(m)}}{\cosh l_j^{(m)} \cosh l_k^{(m)}} + 1 \\
&\quad= -\lim_m \frac{2s(l_i^{(m)})^2 + 1}{(2s(l_j^{(m)})^2 + 1)(2s(l_k^{(m)})^2 + 1)} + 1 \\
&\quad= -\lim_m \frac{2s(l_i^{(m)})^2}{(2s(l_j^{(m)})^2 + 1)(2s(l_k^{(m)})^2 + 1)} + 1 \\
&\quad= -\lim_m \frac{2s(a_i)^2 e^{2u_j^{(m)}+2u_k^{(m)}}}{(2s(a_j)^2 e^{2u_i^{(m)}+2u_k^{(m)}} + 1)(2s(a_k)^2 e^{2u_i^{(m)}+2u_j^{(m)}} + 1)} + 1
\end{aligned}$$

$$= -\lim_m \frac{2s(a_i)^2}{(2s(a_j)^2 e^{2u_i^{(m)}} + e^{-2u_k^{(m)}})(2s(a_k)^2 e^{2u_i^{(m)}} + e^{-2u_j^{(m)}})} + 1$$
$$= 1.$$

Therefore, each inner angle at v_i approaches 0. The curvature of d_m at v_i approaches 2π. This shows that $F(u^{(m)})$ tends to infinity of $\mathbf{P}$.
Case 2. All vertices are black. Then the length of each edge approaches 0. Each hyperbolic triangle approaches a Euclidean triangle. The sum of the curvatures at all vertices approaches $2\pi\chi(S)$. This shows that $F(u^{(m)})$ tends to infinity of $\mathbf{P}$.
Case 3. There exist both white and black vertices. Since the surface S is connected, there exists an edge e whose end points v, v_1 have different colors. Assume v is white and v_1 is black. Let $v_1, ..., v_k$ be the set of all vertices adjacent to v so that v, v_i, v_{i+1} form vertices of a triangle and let $v_{k+1} = v_1$. Now applying part (a) of Lemma 28 to triangle Δvv_1v_2 with v white and v_1 black, we conclude that v_2 must be black. Repeating this to Δvv_2v_3 with v white and v_2 black, we conclude v_3 is black. Inductively, we conclude that all v_i's, for $i = 1, 2, ..., k$, are black. By part (b) of Lemma 28, we conclude that the curvature of d_m at v tends to 2π. This shows that $F(u^{(m)})$ tends to infinity of $\mathbf{P}$.

Cases 1,2,3 show that $F(\mathbb{R}^n)$ is closed in $\mathbf{P}$. Therefore, $F(\mathbb{R}^n) = \mathbf{P}$.

4.3. Discrete Yamabe flow. Given $K^* \in (-\infty, 2\pi)^V$ so that

$$\sum_{v\in V} K^*(v) > 2\pi\chi(S),$$

by the proof above, there exists $u^* \in \mathbb{R}^n$ so that $F(u^*) = K^*$. Furthermore, the function F is the gradient $\bigtriangledown W$ of a strictly convex function $W(u)$ defined on (16) on $\mathbb{R}^n$.

The discrete Yamabe flow with surgery is defined to be the gradient flow of the strictly convex function $W^*(u) = W(u) - \sum_{i=1} K_i^* u_i$. This flow is a generalization of the discrete Yamabe flow introduced in [**16**]. Since $F(u^*) = K^*$, we see $\bigtriangledown W^*(u^*) = 0$, i.e., W^* has a unique minimal point u^* in $\mathbb{R}^n$. It follows that the gradient flow of W^* converges to the minimal point u^* as time approaches infinity.

In the formal notation, the flow takes the form $\frac{du_i(t)}{dt} = K_i - K_i^*$ and $u(0) = 0$. The linear convergence of the flow can be established using exactly the same method used for Theorem 1.4 of [**16**].

5. Algorithmic aspect of discrete conformality

We will prove theorem 2 in this section.

Suppose α and α' are two hyperbolic (or Euclidean) polyhedral metrics on (S, V) given in terms of edge lengths in two geodesic triangulations $\mathcal{T}$ and $\mathcal{T}'$, i.e., $l = \Phi_{\mathcal{T}}^{-1}(\alpha)$ and $l' = \Phi_{\mathcal{T}'}^{-1}(\alpha')$ are two vectors in

$\mathbb{R}^{E(\mathcal{T})}$ and $\mathbb{R}^{E(\mathcal{T}')}$. Suppose entries of l and l' are algebraic numbers. We will produce an algorithm to decide if d and d' are discrete conformal using the data $(\mathcal{T}, l)$ and $(\mathcal{T}', l')$.

There are two steps involved in the algorithm.

In the first step, using proposition 16(c), we may assume that both $\mathcal{T}$ and $\mathcal{T}'$ are Delaunay in metrics α and α' respectively. (The same also holds for Euclidean polyhedral metrics. This is a well known fact from computational geometry. See, for instance, [**3**]). Next, consider two decorated hyperbolic metrics $(d, w) = A_{\mathcal{T}}(\alpha)$ and $(d', w') = A_{\mathcal{T}'}(\alpha')$ with their respective Penner's λ-coordinates $y = \Psi_{\mathcal{T}}^{-1}(d, w)$ and $y' = \Psi_{\mathcal{T}'}^{-1}(d', w')$. By theorem 20, we see Theorem 2 follows from,

Proposition 29. *Suppose two decorated hyperbolic metrics (d, w) and (d', w') in $T_D(\Sigma)$ are given in terms of λ-lengths in two triangulations. There exists an algorithm to decide if $d = d'$.*

Proof. By the construction $y = \Psi_{\mathcal{T}}^{-1}(d, w)$ and $y' = \Psi_{\mathcal{T}'}^{-1}(d', w')$ are the two λ-lengths. Our goal is to use y and y' to decide if $d = d'$. There are two cases according to $\mathcal{T}$ and $\mathcal{T}'$ are isotopic or not.

In the first case, $\mathcal{T}$ and $\mathcal{T}'$ are isotopic. Then it is known by the work of Penner [**21**] that $d = d'$ if and only if the associated Thurston's shear coordinates of y and y' are the same. Here the shear coordinate z of y is defined to be $z(e) = \frac{y(e_1)y(e_3)}{y(e_2)y(e_4)}$ with e_1, e_2, e_3, e_4 being a (fixed) cyclically ordered edges of the quadrilateral associated to e. Thus, one can check algorithmically if $d = d'$ using y and y'.

In the second case that $\mathcal{T}$ and $\mathcal{T}'$ are not isotopic, we can algorithmically produce $y'' = \Psi_{\mathcal{T}}^{-1}(d', w')$ from y' and $\mathcal{T}'$. Indeed, a well known theorem of L. Mosher [**20**] says that there exists an algorithm to produce a finite set of triangulations $\mathcal{T}_1 = \mathcal{T}', \mathcal{T}_2, ..., \mathcal{T}_k = \mathcal{T}$ so that $\mathcal{T}_{i+1}$ is obtained from $\mathcal{T}_i$ by a diagonal switch. Penner's Ptolemy identity shows that one can compute algorithmically $\Psi_{\mathcal{T}_{i+1}}^{-1}(d', w')$ from $\Psi_{\mathcal{T}_i}^{-1}(d', w')$. Thus, we can algorithmically compute the new λ-length coordinate $y'' = \Psi_{\mathcal{T}}^{-1}(d', w')$ from $y' = \Psi_{\mathcal{T}'}^{-1}(d', w')$. This reduces the problem to the first case. q.e.d.

6. Equivalence between the existence and uniqueness part of corollary 4 and Fillastre's work

We begin with the following important theorem of Fillastre.

Theorem 30 (Fillastre [**7**])**.** *Suppose S is a closed surface of negative Euler characteristic and V is a non-empty finite subset in S. Given any finite area complete hyperbolic metric d on $S - V$, there exists a Fuchsian group Γ acting on $\mathbb{H}^3$ and an isometric embedding $\phi : (S - V, d) \to \mathbb{H}^3/\Gamma$ so that (i) $\phi(S - V)$ is the boundary of a convex ideal hyperbolic* *polyhedral surface disjoint from the compact core of* $\mathbb{H}^3/\Gamma$ *and (ii)* $\mathbb{H}^3/\Gamma$ *is homotopy equivalent to* S*. Furthermore, the group* Γ *is unique up to conjugation and* ϕ *is unique up to isometries of* $\mathbb{H}^3$*.*

Our goal in this section is to show that theorem 30 is equivalent to the following consequence of Theorem 3,

Corollary 31. *Let* S *be a closed connected surface of negative Euler characteristic and* $V \subset S$ *be a finite non-empty subset. Then each hyperbolic polyhedral metric* d_1 *on* (S, V) *is discrete conformal to a unique hyperbolic metric* d^* *on the surface* (S, V) *so that all cone angles are* 2π*, i.e.,* d^* *is a hyperbolic metric on* S*.*

Proof. We begin by showing that Theorem 30 implies Corollary 31. Let $(S, \mathcal{T}, d_1)$ be a Delaunay hyperbolic polyhedral metric on (S, V). Let d be the underlying complete hyperbolic metric on $S - V$ of $A(S, V, d_1)$ constructed from theorem 20. Namely, the metric d is obtained by first replacing each triangle in $\mathcal{T}$ by a decorated ideal triangle whose λ-length at an edge e is $\sinh(\frac{l_{d_1}(e)}{2})$, then gluing these triangles isometrically along edges preserving decorations and, finally, removing the decoration. For simplicity, we call $(S - V, d)$ the *underlying hyperbolic structure* of (S, V, d_1) in this section.

By theorem 30, we find a co-compact Fuchsian group Γ and an isometric embedding $\phi : (S - V, d) \to \mathbb{H}^3/\Gamma$ whose image is a convex ideal polyhedral surface. Let $\mathbb{H}^2 \subset \mathbb{H}^3$ be the totally geodesic plane invariant under Γ and L be the circle at infinite of $\mathbb{H}^2 \subset S^2$ where S^2 is the sphere at infinity of $\mathbb{H}^3$. It is well known that L is the limit set of Γ and Γ acts cocompactly on $\mathbb{H}^3 \cup (S^2 - L)$. The quotient space $(\mathbb{H}^3 \cup (S^2 - L))/\Gamma$ is the conformal compactification of the complete hyperbolic 3-manifold $\mathbb{H}^3/\Gamma$ so that $(\mathbb{H}^3 \cup (S^2 - L))/\Gamma = (\mathbb{H}^3/\Gamma) \cup S_+ \cup S_-$. Here $S_\pm = S^2_\pm/\Gamma$ where S^2_+ and S^2_- are the two open hemisphere components of $S^2 - L$ and $S_\pm$ is homeomorphic to S.

It is known [7] that the isometric embedding $\phi : (S - V, d) \to \mathbb{H}^3/\Gamma$ extends continuously to $\Phi : S \to (\mathbb{H}^3/\Gamma) \cup S_+ \cup S_-$ and $\phi(S - V) \cap (\mathbb{H}^2/\Gamma) = \emptyset$. In fact, $\phi(S - V)$ is the boundary of the convex hull of $\Phi(V)$ in $\mathbb{H}^3/\Gamma$. Therefore, $\Phi(V)$ is a subset of either S_+ or S_-, say $V_+ = \Phi(V) \subset S_+$. Equip S^2_+ the canonical hyperbolic metric d'. Then Γ acts isometrically on (S^2_+, d') and the quotient surface S_+ has the induced hyperbolic metric d_2. We claim that (S_+, V_+, d_2) is the hyperbolic metric discrete conformal to $(S, \mathcal{T}, d_1)$. Since the underlying complete hyperbolic metric of (S, V, d_1) is $(S-V, d)$ and ϕ is an isometry, it suffices to show that the ideal convex polyhedral surface $\phi(S - V)$ is the underlying complete hyperbolic metric of (S_+, V_+, d_2).

There are two steps involved in the proof. In the first step, let $\mathcal{T}_+$ be a Delaunay triangulation for (S_+, V_+, d_2). We will show that there is a geodesic ideal triangulation $\mathcal{T}'$ of the ideal convex polyhedral surface $\phi(S-V)$ so that (i) $\mathcal{T}'$ and $\mathcal{T}_+$ have the same vertex set V_+, (ii) $v, v' \in V_+$ are joint by an edge in $\mathcal{T}_+$ if and only if they are joint by an edge in $\mathcal{T}'$ and, (iii) all natural edges of the polyhedral surface $\phi(S-V)$ are in $\mathcal{T}'$. In the second step, we show that the shear coordinate at an edge vv' of $(\phi(S-V), \mathcal{T}')$ is equal to the length-cross-ratio of $(S_+, \mathcal{T}_+, d_2)$ at the edge vv'. The combination of these two steps shows that (S_+, V_+, d_2) is discrete conformal to (S, V, d_1).

Recall that the Euclidean convex hull $C_{E^n}(X)$ of a set X in the Euclidean space E^n is the intersection of all convex sets containing X. If X is a subset (containing at least two points) in the 2-sphere S^2, the (hyperbolic) convex hull $C(X)$ of X is defined to be $C_{E^n}(X) \cap B^3$ where B^3 is the open unit ball considered as the Klein model of $\mathbb{H}^3$. Given a discrete set W in the hyperbolic plane $\mathbb{H}^2$, the associated Voronoi diagram $\mathcal{V}(W)$ has 2-cells given by $K(w) = \{z \in H^2 | d(z,w) \leq d(z,w'), \forall w' \in W\}$, $w \in W$, 1-cells of the form $K(w_1) \cap K(w_2)$ where $|K(w_1) \cap K(w_2)| \geq 2$ and 0-cells v^*'s correspond to hyperbolic disks $B_r(v^*)$ centered at v^* of radius r such that $int(B_r(v^*)) \cap W = \emptyset$ and $|B_r(v^*) \cap W| \geq 3$. We will call $B_r(v^*)$ *maximum disks missing* W. The dual to the Voronoi diagram is the Delaunay tessellation $\mathcal{D}(W)$ whose vertices (0-cells) are W, 1-cells are given by geodesic segments $[w_1, w_2]$ where $w_1, w_2 \in W$ so that $|K(w_1) \cap K(w_2)| \geq 2$ and 2-cells are in 1-1 correspondence with maximum disks missing W. See [**3**] for more details. Let $cl(X)$ denote the closure of a set X.

Lemma 32. *Let W be a discrete subset of the hemisphere S^2_+ so that the limit points of W is in the boundary of the closed hemisphere $cl(S^2_+)$ and each Voronoi cell $K(w) = \{z \in S^2_+ | d'(z,w) \leq d'(z,w'), \forall w' \in W\}$ is compact and has finite sides for each $v \in W$. Then there exists an incidence preserving bijection between cells in the Delaunay tessellation associated to W in (S^2_+, d') and cells in the natural cell structure on the convex polyhedral surface $\partial C(W)$.*

Proof. By the definition of convex hull $C(W)$, $C(W) = (\cap_B C(B)) \cap C(S^2_+)$ where $S^2 - int(B)$ are the maximum disks missing W. Therefore, the 2-cells of $\partial C(W)$ are $C(W) \cap C(B)$ for each maximum disk B missing W and 1-cells (i.e., geodesic edges in $\partial C(W)$) are $\partial C(B_{r_1}(w_1)) \cap \partial C(B_{r_2}(w_2))$ where $|K(w_1) \cap K(w_2)| \geq 2$.

Let $r : S^2_+ \to \partial C(W)$ be the nearest point retraction (see [**18**]). It is defined as follows. For $z \in S^2_+$, consider the family of horospheres in $\mathbb{H}^3$ at z consisting of the family of Euclidean spheres in the open unit ball B^3 tangent to S^2_+ at z. There is a smallest horosphere that meets $\partial C(W)$ at a single point $r(z)$. It is known (see, for instance, [**24**], [**18**]) that $r(z)$ is continuous, $r|_W = id$, r sends 1-cells in $\mathcal{D}(W)$ to 1-cells in $\partial C(W)$ and r establishes a bijection between the 2-cells.

This gives the incidence preserving bijection between cells in the Delaunay tessellation $\mathcal{D}(W)$ and the cells in the natural cellar structure on the convex polyhedral surface $\partial C(W)$.

Now any geodesic triangulation of $\mathcal{D}(W)$ without extra vertices corresponds, under the nearest point projection r, to an ideal geodesic triangulation of the 2-cells of $\partial C(W)$. Thus, these two triangulations are isomorphic. q.e.d.

By taking W to be the preimage of V_+ in the universal cover S^2_+ of the compact hyperbolic surface (S_+, V_+, d_2) and using the fact that $\phi(S-V)$ is isometric to $\partial C(W)/\Gamma$, we conclude that the nearest projection map induces a bijection between the Delaunay triangulation $\mathcal{T}_+$ of (S_+, V_+, d_2) and a geometric triangulation $\mathcal{T}'$ of the natural cellular structure of $\phi(S-V)$.

The proof that $\phi(S-V)$ is the underlying hyperbolic structure of (S_+, V_+, d_2) now follows from a known lemma below by checking the length-cross-ratios based on $\mathcal{T}_+$ and $\mathcal{T}'$. Here the notion of length-cross-ratio was introduced in [**4**].

Lemma 33. *Suppose A_1, A_2, A_3, A_4 are four distinct points in S^2. Then the length-cross-ratio $|(A_1, A_2, A_3, A_4)| = \frac{l_{13}l_{24}}{l_{14}l_{23}}$ where l_{ij} is the Euclidean distance in $\mathbb{R}^3$ between A_i, A_j can be calculated as*

(a) $|(A_1, A_2, A_3, A_4)| = \frac{h_{13}h_{24}}{h_{14}h_{23}}$ where $h_{ij} = \sinh(\frac{d_B(A_i,A_j)}{2})$ and B is an open spherical ball in S^2 containing A_1, A_2, A_3, A_4 with the natural hyperbolic metric d_B,

(b) $|(A_1, A_2, A_3, A_4)| = \frac{s_{13}s_{24}}{s_{14}s_{23}}$ where $s_{ij} = \sin(\frac{d_{S^2}(A_i,A_j)}{2})$ and d_{S^2} is the spherical metric on S^2,

(c) $|(A_1, A_2, A_3, A_4)| = \frac{|B_1-B_3||B_2-B_4|}{|B_1-B_4||B_2-B_3|}$ where $B_i = \psi(A_i)$ is the image of A_i under a stereographic projection ψ from S^2 onto the extended complex plane $\mathbb{C} \cup \{\infty\}$,

*(d) ([**4**], [**21**]) $\ln(|(A_1, A_2, A_3, A_4)|)$ is the shear coordinate of the ideal hyperbolic tetrahedron with vertices A_1, A_2, A_3, A_4 at the edge A_1A_2.*

Proof. Part (b) follows from the definition since the Euclidean distance between two points in the 2-sphere of spherical distance d is $2\sin(\frac{d}{2})$. Therefore, we have $l_{ij} = 2\sin(\frac{d_{S^2}(A_i,A_j)}{2})$ and (b) holds. Part (c) follows from the fact that length-cross-ratio is invariant under Möbius transformation (see proposition 2.5.1 in [**4**]). Part (d) follows from part (c) by taking the Möbius transformation of the complex plane sending B_1, B_2, B_4 to $0, \infty, 1$ and a direct calculation. See proposition 5.3.2 in [**4**]. To see part (a), by part (d) and using a stereographic projection, it suffices to show that if B_1, B_2, B_3, B_4 are four distinct points in the open unit disk $\mathbb{D}$ with the Poincaré metric $d_{\mathbb{D}}$, then $\frac{|B_1-B_3||B_2-B_4|}{|B_1-B_4||B_2-B_3|} = \frac{h_{13}h_{24}}{h_{14}h_{23}}$ where $h_{ij} = \sinh(\frac{d_{\mathbb{D}}(B_i,B_j)}{2})$. This follows from the well known formula that the hyperbolic distance $d_{\mathbb{D}}(z, w)$ between two points $z, w \in \mathbb{D}$ is given by

$$\sinh(\frac{d_{\mathbb{D}}(z,w)}{2}) = \frac{|z-w|}{\sqrt{(1-|z|^2)(1-|w|^2)}}. \qquad \text{q.e.d.}$$

The uniqueness part of the Corollary 31 follows easily from the corresponding uniqueness part of theorem 30. We omit the details.

The proof that Corollary 31 implies Theorem 30 is as follows. We use the same notations introduced above. Given a complete finite area hyperbolic metric d on $S - V$, let d_1 be a hyperbolic polyhedral metric on (S, V) whose underlying metric is d. Now by Corollary 31, we produce a smooth hyperbolic metric d^* on S such that (S, V, d^*) is discrete conformal to (S, V, d_1). Let Γ be a Fuchsian group acting on (S^2_+, d') so that (S, d^*) is isometric to S^2_+/Γ and let V_+ be the subset of S^2_+/Γ corresponding to V under the isometry. Let $W \subset S^2_+$ be the preimage of V_+. Then the convex ideal polyhedral surface $\partial C(W)/\Gamma$ is isometric to $(S - V, d)$. Indeed, by construction and the argument above, both $(S - V, d)$ and $\partial C(W)/\Gamma$ are the underlying hyperbolic structures for the hyperbolic polyhedral surface (S, V, d_1).

Furthermore, this isometric embedding is unique due to the uniqueness part of discrete conformal metric. q.e.d.

7. Appendix

7.1. Appendix A: A proof of lemma 23.

Let $s(x) = \sinh \frac{x}{2}$.

Lemma 34 (Fenchel [8] page 118)**.** *Given a hyperbolic triangle with side lengths a, b, c, then*

$$\frac{(s(a)s(b)s(c))^2}{(s(a)+s(b)+s(c))(s(a)+s(b)-s(c))(s(b)+s(c)-s(a))(s(c)+s(b)-s(a))}$$

equals

- $\frac{1}{4}\sinh^2 r$ *if the triangle has a compact circumcircle of radius r,*
- ∞ *if the circumcircle is a horocycle,*
- $-\frac{1}{4}\cosh^2 D$ *if the circumcircle is of constant distance D to a geodesic.*

As a corollary we have,

Lemma 35. *Denote by α, β, γ the angles opposite to the sides with lengths a, b, c. Then*

$$\frac{\sinh a}{\sin \alpha} = 2\zeta \cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2},$$

where ζ equals

- $\tanh r$ *if the triangle has a compact circumcircle of radius r,*
- 1 *if the circumcircle is a horocycle,*
- $\coth D$ *if the circumcircle is of constant distance D to a geodesic.*

Proof. Assume that the triangle has a circumscribed circle of radius r. By using the cosine rule and Lemma 34,

$$\begin{aligned}
\sin\alpha &= (1-\cos^2\alpha)^{\frac{1}{2}} \\
&= \frac{(-\cosh^2 a - \cosh^2 b - \cosh^2 c + 1 + 2\cosh a \cosh b \cosh c)^{\frac{1}{2}}}{\sinh b \sinh c} \\
&= \frac{2}{\sinh b \sinh c} \cdot \{4s(a)^2 s(b)^2 s(c)^2 + \\
&\quad 2s(a)^2 s(b)^2 + 2s(b)^2 s(c)^2 + 2s(c)^2 s(a)^2 - s(a)^4 - s(b)^4 - s(c)^4\}^{\frac{1}{2}} \\
&= \frac{2}{\sinh b \sinh c} \cdot \{4s(a)^2 s(b)^2 s(c)^2 + \\
&\quad (s(a)+s(b)+s(c))(s(a)+s(b)-s(c)) \\
&\quad (s(b)+s(c)-s(a))(s(c)+s(b)-s(a))\}^{\frac{1}{2}} \\
&= \frac{2}{\sinh b \sinh c} \cdot \{4s(a)^2 s(b)^2 s(c)^2 + \frac{4s(a)^2 s(b)^2 s(c)^2}{\sinh^2 r}\}^{\frac{1}{2}} \\
&= \frac{4}{\sinh b \sinh c} \cdot s(a)s(b)s(c)\frac{\cosh r}{\sinh r}.
\end{aligned}$$

By taking limit with $r \to \infty$, we can prove the lemma for the case that the triangle has a horocyclic circumcircle.

Similar calculation can be used to prove the lemma for the case that the triangle has a circumscribed equidistant curve. q.e.d.

Lemma 36. *Let a, b, c, d be the side lengths of a hyperbolic quadrilateral and e, f the diagonal lengths so that a, b, c, d are cyclically ordered edge lengths and edges of lengths a, b, e form a triangle. The following three statements are equivalent.*

(i) *The vertices of this quadrilateral lie on a curve of constant geodesic curvature.*

(ii) *Ptolemy's formula holds:*

$$s(e)s(f) = s(a)s(c) + s(b)s(d).$$

(iii)

$$s(e)^2 = (s(a)s(c) + s(b)s(d))\frac{s(a)s(d) + s(b)s(c)}{s(a)s(b) + s(c)s(d)}, \tag{17}$$

and

$$s(f)^2 = (s(a)s(c) + s(b)s(d))\frac{s(a)s(b) + s(c)s(d)}{s(a)s(d) + s(b)s(c)}.$$

Proof. (i)$\Longrightarrow$(ii). It was proved by T. Kubota [**14**].
(ii)$\Longrightarrow$(i). It was proved by Joseph E. Valentine [**26**], Theorem 3.4.
(iii)$\Longrightarrow$(ii). The product of the two equations in (iii) produces the equation in (ii).
(i)$\Longrightarrow$(iii).

Case 1. When the vertices lie on a circle, it was proved in [**12**] (theorem 1, page 4).

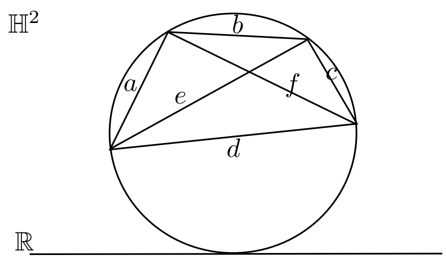

Figure 1.

Case 2. When the vertices lie on a horocycle, for example as in Figure 1, we have

$$s(e) = s(a) + s(b),$$
$$s(f) = s(b) + s(c),$$
$$s(d) = s(a) + s(b) + s(c).$$

Then the equations in (iii) hold.
Case 3. When the vertices lie on a geodesic, without loss of generality, we may assume

$$e = a + b,$$
$$f = b + c,$$
$$d = a + b + c.$$

Direct calculation shows that

$$s(a)s(c) + s(b)s(d) = s(a)s(c) + s(b)s(a + b + c) = s(a + b)s(c + b).$$

Similarly,

$$s(a)s(d) + s(b)s(c) = s(a + b)s(a + c),$$
$$s(a)s(b) + s(c)s(d) = s(c + a)s(c + b).$$

Therefore, the right hand side of (17) equals

$$s(a + b)s(c + b)\frac{s(a + b)s(a + c)}{s(c + a)s(c + b)} = s(a + b)^2 = s(e)^2.$$

Similar argument proves the equation involving $s(f)$.
Case 4. When the vertices lie on an equidistant curve with distance D to its geodesic axis, project the vertices to the geodesic axis. The corresponding distance between those projection of vertices are denoted by $\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}$.

By Case 3, we have

$$s(\overline{e})^2 = (s(\overline{a})s(\overline{c}) + s(\overline{b})s(\overline{d}))\frac{s(\overline{a})s(\overline{d}) + s(\overline{b})s(\overline{c})}{s(\overline{a})s(\overline{b}) + s(\overline{c})s(\overline{d})}.$$

Since

$$s(x) = s(\overline{x}) \cosh D,$$

for $x = a, b, c, d, e, f$, we have

$$s(e)^2 = (s(a)s(c) + s(b)s(d)) \frac{s(a)s(d) + s(b)s(c)}{s(a)s(b) + s(c)s(d)}. \qquad \text{q.e.d.}$$

First, we verify that

$$\frac{\partial A}{\partial x}|_{A=B} = \frac{\partial B}{\partial x}.$$

The role of x, y, z, w are the same with respect to a. It is enough to verify the case of variable x.

Now let $\alpha, \alpha', \beta, \beta'$ be the angles formed by the pairs of edges $\{a, y\}$, $\{a, x\}$, $\{a, z\}$, $\{a, w\}$ as Figure 2.

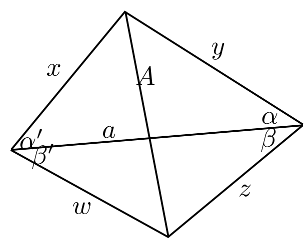

Figure 2.

In the triangle of lengths y, z, A, by the cosine rule,

$$\cosh A = \cosh y \cosh z - \sinh y \sinh z \cos(\alpha + \beta).$$

Taking derivative of both sides with respect to x, we have

$$\frac{\partial A}{\partial x} = \frac{\sinh y \sinh z \sin(\alpha + \beta)}{\sinh A} \cdot \frac{\partial \alpha}{\partial x}.$$

In the triangle of lengths x, y, a, by the derivative of cosine rule [**17**], we have

$$\frac{\partial \alpha}{\partial x} = \frac{\sinh x}{\sinh y \sinh a \sin \alpha}.$$

Therefore,

$$\frac{\partial A}{\partial x} = \frac{\sinh z}{\sinh a} \cdot \frac{\sin(\alpha + \beta)}{\sinh A} \cdot \frac{\sinh x}{\sin \alpha}.$$

In the triangle of lengths y, z, A, Lemma 35 implies that

$$\frac{\sinh A}{\sin(\alpha + \beta)} = 2\zeta_1 \cosh \frac{A}{2} \cosh \frac{y}{2} \cosh \frac{z}{2}. \tag{18}$$

In the triangle of lengths x, y, a, Lemma 35 implies that

$$\frac{\sinh x}{\sin \alpha} = 2\zeta_2 \cosh \frac{x}{2} \cosh \frac{y}{2} \cosh \frac{a}{2}. \tag{19}$$

Therefore,

$$\begin{aligned}\frac{\partial A}{\partial x} =& \frac{\sinh z}{\sinh a} \cdot \frac{2\zeta_2 \cosh\frac{x}{2}\cosh\frac{y}{2}\cosh\frac{a}{2}}{2\zeta_1 \cosh\frac{A}{2}\cosh\frac{y}{2}\cosh\frac{z}{2}}\\ =& \frac{\sinh\frac{z}{2}\cosh\frac{x}{2}\zeta_2}{\sinh\frac{a}{2}\cosh\frac{A}{2}\zeta_1}.\end{aligned}$$

When $A = B$, by Lemma 36, the vertices of the hyperbolic quadrilateral lie on a circle, a horocycle or an equidistant curve. Thus, $\zeta_1 = \zeta_2$.

Therefore,

$$\frac{\partial A}{\partial x}|_{A=B} = \frac{\sinh\frac{z}{2}\cosh\frac{x}{2}}{\sinh\frac{a}{2}\cosh\frac{B}{2}} = \frac{\partial B}{\partial x}.$$

Second, we verify that

$$\frac{\partial A}{\partial a}|_{A=B} = \frac{\partial B}{\partial a}.$$

In the triangle of lengths y, z, A, by the cosine rule,

$$\cosh A = \cosh y \cosh z - \sinh y \sinh z \cos(\alpha + \beta).$$

Taking derivative of both sides with respect to a, we have

$$\frac{\partial A}{\partial a} = \frac{\sinh y \sinh z \sin(\alpha+\beta)}{\sinh A} \cdot (\frac{\partial \alpha}{\partial a} + \frac{\partial \beta}{\partial a}).$$

In the triangle of length x, y, a, by the derivative of cosine rule [17], we have

$$\frac{\partial \alpha}{\partial a} = -\frac{\sinh x}{\sinh y \sinh a \sin\alpha}\cos\alpha'.$$

In the triangle of length w, z, a, by the derivative of cosine rule [17], we have

$$\frac{\partial \beta}{\partial a} = -\frac{\sinh w}{\sinh z \sinh a \sin\beta}\cos\beta'.$$

Therefore,

$$\frac{\partial A}{\partial a} = -\frac{\sin(\alpha+\beta)}{\sinh A \sinh a}(\frac{\sinh z \sinh x \cos\alpha'}{\sin\alpha} + \frac{\sinh y \sinh w \cos\beta'}{\sin\beta}).$$

By the equations (18) and (19), we have

$$\frac{\sin(\alpha+\beta)}{\sin\alpha} = \frac{\zeta_2 \cosh\frac{a}{2}\sinh\frac{A}{2}}{\zeta_1 \cosh\frac{z}{2}\sinh\frac{x}{2}}.$$

By the similar calculation, we have

$$\frac{\sin(\alpha+\beta)}{\sin\beta} = \frac{\zeta_3 \cosh\frac{a}{2}\sinh\frac{A}{2}}{\zeta_1 \cosh\frac{y}{2}\sinh\frac{w}{2}},$$

there ζ_3 is the corresponding quantity of the triangle of lengths w, z, a.

Therefore,

$$\frac{\partial A}{\partial a} = -\frac{1}{\cosh\frac{A}{2}\sinh\frac{a}{2}}(\frac{\zeta_2}{\zeta_1}\sinh\frac{z}{2}\cosh\frac{x}{2}\cos\alpha' + \frac{\zeta_3}{\zeta_1}\sinh\frac{y}{2}\cosh\frac{w}{2}\cos\beta').$$

When $A = B$, by Lemma 36, the vertices of the hyperbolic quadrilateral lie on a circle, a horocycle or an equidistant curve. Thus, $\zeta_1 = \zeta_2 = \zeta_3$.

Therefore,

$$\frac{\partial A}{\partial a}|_{A=B} = -\frac{1}{\cosh\frac{B}{2}\sinh\frac{a}{2}}(\sinh\frac{z}{2}\cosh\frac{x}{2}\cos\alpha' + \sinh\frac{y}{2}\cosh\frac{w}{2}\cos\beta').$$

On the other hand,

$$\frac{\partial B}{\partial a} = -\frac{\sinh\frac{B}{2}\cosh\frac{a}{2}}{\cosh\frac{B}{2}\sinh\frac{a}{2}}.$$

To prove $\frac{\partial A}{\partial a}|_{A=B} = \frac{\partial B}{\partial a}$, it remains to show that

$$\sinh\frac{z}{2}\cosh\frac{x}{2}\cos\alpha' + \sinh\frac{y}{2}\cosh\frac{w}{2}\cos\beta' = \sinh\frac{B}{2}\cosh\frac{a}{2}. \tag{20}$$

In the triangle of length x, y, a, by the cosine rule,

$$\cos\alpha' = \frac{-\cosh y + \cosh x\cosh a}{\sinh x\sinh a}.$$

In the triangle of length w, z, a, by the cosine rule,

$$\cos\beta' = \frac{-\cosh z + \cosh w\cosh a}{\sinh w\sinh a}.$$

Therefore, the equation (20) is equivalent to

(21)

$$\frac{\sinh\frac{z}{2}}{2\sinh\frac{x}{2}}(-\cosh y + \cosh x\cosh a) + \frac{\sinh\frac{y}{2}}{2\sinh\frac{w}{2}}(-\cosh z + \cosh w\cosh a)$$
$$= \sinh\frac{B}{2}\cosh\frac{a}{2}\sinh a.$$

Using the notation $s(t) = \sinh\frac{t}{2}$, we have $\cosh t = 2s(t)^2 + 1$. Therefore, the equation (21) is equivalent to

$$\begin{aligned}
&\frac{s(z)}{s(x)}(2s(a)^2s(x)^2 + s(a)^2 + s(x)^2 - s(y)^2) \\
&+ \frac{s(y)}{s(w)}(2s(a)^2s(w)^2 + s(a)^2 + s(w)^2 - s(z)^2) \\
&= 2s(B)s(a)(s(a)^2 + 1) \\
&= 2(s(x)s(z) + s(y)s(w))(s(a)^2 + 1),
\end{aligned}$$

the second equality is due to Ptolemy's formula.

After simplify we obtain

$$s(a)^2 = (s(x)s(z) + s(y)s(w))\frac{s(x)s(w) + s(y)s(z)}{s(x)s(y) + s(z)s(w)}.$$

This is exactly the result of Lemma 36.

7.2. Appendix B: A proof of proposition 8. We begin with a result which implies proposition 8.

Theorem 37. *Suppose (S, V, d) is a closed hyperbolic polyhedral metric surface. If $\mathcal{T}$ is a geodesic triangulation of (S, V, d), then there exists a sequence of geodesic triangulations $\mathcal{T}_0 = \mathcal{T}, \mathcal{T}_1, ..., \mathcal{T}_n$ of (S, V, d) so that $\mathcal{T}_{i+1}$ and $\mathcal{T}_i$ are related by a diagonal switch and $\mathcal{T}_n$ is Delaunay.*

Note that the Euclidean counterpart of the result was well known. See, for instance, Bobenko–Springborn [3] for a proof.

Proof. For any geodesic triangulation $\mathcal{T}$ of (S, V, d), we define the complexity of $\mathcal{T}$ (in the metric d) to be

$$||\mathcal{T}|| = \sum_{triangle\ t} \sinh \frac{x_t}{2} \sinh \frac{y_t}{2} \sinh \frac{z_t}{2},$$

where x_t, y_t, z_t are the three edge lengths of t.

We call an edge e of a geodesic triangulation $\mathcal{T}$ *local Delaunay* if

$$\alpha + \alpha' \leq \beta + \beta' + \gamma + \gamma', \tag{22}$$

where $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are angles of the two triangles in $\mathcal{T}$ having e as the common edge so that α and α' are opposite to e.

Lemma 38. *If $\mathcal{T}$ and $\mathcal{T}'$ are two geodesic triangulations of (S, V, d) so that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by a diagonal switch at a non local Delaunay edge e, then $||\mathcal{T}'|| < ||\mathcal{T}||$.*

This follows from,

Lemma 39. *Assume x_0, x_1, x_2 and x_0, x_3, x_4 are edge lengths of two adjacent hyperbolic triangles t, t' respectively, and x_0' is the edge length of the other diagonal of the quadrilateral $Q = t \cup_e t'$ where e is the edge of length x_0. If the quadrilateral Q is not local Delaunay at e, then*

$$\sinh \frac{x_2}{2} \sinh \frac{x_3}{2} \sinh \frac{x_0'}{2} + \sinh \frac{x_4}{2} \sinh \frac{x_1}{2} \sinh \frac{x_0'}{2} < \sinh \frac{x_1}{2} \sinh \frac{x_2}{2} \sinh \frac{x_0}{2} + \sinh \frac{x_3}{2} \sinh \frac{x_4}{2} \sinh \frac{x_0}{2}.$$

Proof. By proposition 10 and that e is not local Delaunay in Q, we have,

$$\frac{\sinh^2(x_1/2) + \sinh^2(x_2/2) - \sinh^2(x_0/2)}{\sinh(x_1/2)\sinh(x_2/2)} + \frac{\sinh^2(x_3/2) + \sinh^2(x_4/2) - \sinh^2(x_0/2)}{\sinh(x_3/2)\sinh(x_4/2)} < 0,$$

which is equivalent to

$$(\sinh\frac{x_1}{2}\sinh\frac{x_3}{2}+\sinh\frac{x_2}{2}\sinh\frac{x_4}{2})(\sinh\frac{x_1}{2}\sinh\frac{x_4}{2}+\sinh\frac{x_2}{2}\sinh\frac{x_3}{2}) < \sinh^2\frac{x_0}{2}(\sinh\frac{x_1}{2}\sinh\frac{x_2}{2}+\sinh\frac{x_3}{2}\sinh\frac{x_4}{2}).$$

By a result of Valentine ([26], the corollary on page 820),

$$\sinh\frac{x_1}{2}\sinh\frac{x_3}{2}+\sinh\frac{x_2}{2}\sinh\frac{x_4}{2}\geq\sinh\frac{x_0'}{2}\sinh\frac{x_0}{2}.$$

Combining the above two inequalities, we have

$$\sinh\frac{x_0'}{2}\sinh\frac{x_0}{2}(\sinh\frac{x_1}{2}\sinh\frac{x_4}{2}+\sinh\frac{x_2}{2}\sinh\frac{x_3}{2}) < \sinh^2\frac{x_0}{2}(\sinh\frac{x_1}{2}\sinh\frac{x_2}{2}+\sinh\frac{x_3}{2}\sinh\frac{x_4}{2}),$$

i.e.

$$\sinh\frac{x_2}{2}\sinh\frac{x_3}{2}\sinh\frac{x_0'}{2}+\sinh\frac{x_4}{2}\sinh\frac{x_1}{2}\sinh\frac{x_0'}{2} < \sinh\frac{x_1}{2}\sinh\frac{x_2}{2}\sinh\frac{x_0}{2}+\sinh\frac{x_3}{2}\sinh\frac{x_4}{2}\sinh\frac{x_0}{2}.$$

q.e.d.

Now for triangulation $\mathcal{T}$, if all edges are local Delaunay, then $\mathcal{T}$ is Delaunay by Leibon's theorem. If one of the edge e is not local Delaunay, we do the flip at e to produce a geodesic triangulation $\mathcal{T}_1$. In this way, we produce a sequence of geodesic triangulations $\mathcal{T}, \mathcal{T}_1$,....., so that $\mathcal{T}_{i+1}$ is obtained from $\mathcal{T}_i$ by a diagonal switch (flip) at a none local Delaunay edge in $\mathcal{T}_i$. We claim that one of $\mathcal{T}_n$ is Delaunay. For otherwise, by the lemma above, $||\mathcal{T}_i|| < ||\mathcal{T}||$. On the other hand, there exists $\delta > 0$ so that $l(e) \geq \delta$ for all geodesic (non-constant) paths e joining V to V in (S, V, d). This shows the length $l(e)$ of each edge e in $\mathcal{T}_i$ is bounded due to $\sinh(l(e)/2) \leq \frac{||\mathcal{T}||}{\sinh^2(\delta/2)}$ which follows from $\sinh(l(e)/2)\sinh^2(\delta/2) \leq ||\mathcal{T}_i|| \leq ||\mathcal{T}||$. On the other hand, it is well known that given any number $M > 0$, there are only finitely many geodesic path in (S, V, d) of lengths at most M joining V to V. Thus, we conclude that there are only finitely many geodesic triangulations $\mathcal{T}_i$'s one can produce and the last one in the sequence must be Delaunay. q.e.d.

Since any two Delaunay triangulations of (S, V, d) are related by sequence of geometric flips (see, for instance, [3] for a proof), we obtain the proposition 8.

References

[1] Hirotaka Akiyoshi, *Finiteness of polyhedral decompositions of cusped hyperbolic manifolds obtained by the Epstein–Penner's method*. Proc. Amer. Math. Soc. 129 (2001), no. 8, 2431–2439, MR1823928, Zbl 0974.51023.

[2] B. H. Bowditch, D. B. A. Epstein, *Natural triangulations associated to a surface*. Topology 27 (1988), no. 1, 91–117, MR0935529, Zbl 0649.32017.

[3] Alexander I. Bobenko, Boris A. Springborn, *A discrete Laplace–Beltrami operator for simplicial surfaces*. Discrete Comput. Geom. 38 (2007), no. 4, 740–756, MR2365833, Zbl 1144.65011.

[4] Alexander Bobenko, Ulrich Pinkall, Boris Springborn, *Discrete conformal maps and ideal hyperbolic polyhedra*. Geom. Topol. 19 (2015), no. 4, 2155–2215, MR3375525, Zbl 1327.52040.

[5] Mark de Berg, Otfried Cheong, Marc van Kreveld, Mark Overmars, Computational geometry. Algorithms and applications. Third edition. Springer-Verlag, Berlin, 2008, MR2723879, Zbl 1140.68069.

[6] Jason DeBlois, *The Delaunay tessellation in hyperbolic space*. Math. Proc. Cambridge Philos. Soc. Published online: 27 September 2016.

[7] Fillastre, François, *Polyhedral hyperbolic metrics on surfaces*. Geom. Dedicata 134 (2008), 177–196, MR2399657, Zbl 1153.57012.

[8] Werner Fenchel, Elementary geometry in hyperbolic space. With an editorial by Heinz Bauer. de Gruyter Studies in Mathematics, 11. Walter de Gruyter & Co., Berlin, 1989, MR1004006, Zbl 0674.51001.

[9] Sergey Fomin, Michael Shapiro, Dylan Thurston, *Cluster algebras and triangulated surfaces. I. Cluster complexes*. Acta Math. 201 (2008), no. 1, 83–146, MR2448067, Zbl 1263.13023.

[10] Xianfeng Gu, Feng Luo, Jian Sun, Tianqi Wu, *A discrete uniformization theorem for polyhedral surfaces*. arXiv:1309.4175, to appear in J. Differential Geom.

[11] Ren Guo, Feng Luo, *Rigidity of polyhedral surfaces II*. Geom. Topol. 13 (2009), no. 3, 1265–1312, MR2496046, Zbl 1160.52012.

[12] Ren Guo, Nilgün Sönmez, *Cyclic polygons in classical geometry*. Comptes rendus de l'Académie bulgare des Sciences 64 (2011), no. 2, 185–194, MR2865551, Zbl 1289.51013.

[13] Graham P. Hazel, *Triangulating Teichmüller space using the Ricci flow*. PhD thesis, University of California San Diego, 2004. available at www.math.ucsd.edu/~thesis/thesis/ghazel/ghazel.pdf.

[14] T. Kubota, *On the extended Ptolemy's theorem in hyperbolic geometry*. Science reports of the Tohoku University. Series 1; Physics, chemistry, astronomy. Vol. 2 (1912), 131–156.

[15] Gregory Leibon, *Characterizing the Delaunay decompositions of compact hyperbolic surface*. Geom. Topol. 6 (2002), 361–391, MR1914573, Zbl 1028.52014.

[16] Feng Luo, *Combinatorial Yamabe flow on surfaces*. Commun. Contemp. Math. 6 (2004), no. 5, 765–780, MR2100762, Zbl 1075.53063.

[17] Feng Luo, *Rigidity of polyhedral surfaces I*. J. Differential Geom. 96 (2014), no. 2, 241–302, MR3178441, Zbl 06287981.

[18] Marden, Albert, The view from above. Pure Appl. Math. Q. 7 (2011), no. 2, Special Issue: In honor of Frederick W. Gehring, Part 2, 383–394, MR2815381, Zbl 1291.30244.

[19] Howard Masur, John Smillie, *Hausdorff dimension of sets of nonergodic measured foliations*. Ann. of Math. (2) 134 (1991), no. 3, 455–543, MR1135877, Zbl 0774.58024.

[20] Lee Mosher, *Tiling the projective foliation space of a punctured surface.* Trans. Amer. Math. Soc. 306 (1988), no. 1, 1–70, MR0927683, Zbl 0647.57005.

[21] Robert C. Penner, *The decorated Teichmüller space of punctured surfaces*. Comm. Math. Phys. 113 (1987), no. 2, 299–339, MR0919235, Zbl 0642.32012.

[22] M. Roček, R. M. Williams, *The quantization of Regge calculus*. Z. Phys. C 21 (1984), no. 4, 371–381, MR0734698.

[23] Jian Sun, Tianqi Wu, Xiang Zhu, *Discrete uniformization for spherical polyhedral surfaces*, preprint, 2015.

[24] Sullivan, Dennis, Travaux de Thurston sur les groupes quasi-fuchsiens et les variétés hyperboliques de dimension 3 fibrèes sur S1. (French) Bourbaki Seminar, Vol. 1979/80, pp. 196–214, Lecture Notes in Math., 842, Springer, Berlin-New York, 1981, MR0636524, Zbl 0459.57006.

[25] Marc Troyanov, *Les surfaces euclidiennes à singularités coniques*. (French) [Euclidean surfaces with conical singularities] Enseign. Math. (2) 32 (1986), nos 1–2, 79–94, MR0850552, Zbl 0611.53035.

[26] Joseph E. Valentine, *An analogue of Ptolemy's theorem and its converse in hyperbolic geometry*. Pacific J. Math. 34 (1970) 817–825, MR0270261, Zbl 0198.25601.

Department of Computer Science
Stony Brook University
Stony Brook, NY, 11794
USA

E-mail address: gu@cs.stonybrook.edu

Department of Mathematics
Oregon State University
Corvallis, OR, 97330
USA

E-mail address: guoren@oregonstate.edu

Department of Mathematics
Rutgers University
Piscataway, NJ, 08854
USA

E-mail address: fluo@math.rutgers.edu

Yau Mathematical Science Center
Tsinghua University
Beijing, 100084
China

E-mail address: jsun@math.tsinghua.edu.cn

Courant Institute of Mathematics
New York University
New York, NY, 10012
USA
E-mail address: tianqi@cims.nyu.edu