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BARYCENTER TECHNIQUE AND THE RIEMANN MAPPING PROBLEM OF CHERRIER–ESCOBAR

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Abstract

We solve in the affirmative the remaining cases of the Riemann mapping problem of Cherrier[**35**]–Escobar[**38**] first raised by Cherrier[**35**] in 1984. Indeed, performing a suitable scheme of the barycenter technique of Bahri–Coron[**14**] via the Almaraz[**3**]–Chen[**34**]'s bubbles, we completely solve all the cases left open after the work of Chen[**34**]. Hence, combining our work with the ones of Almaraz[**2**], Chen[**34**], Cherrier[**35**], Escobar[**38**],[**40**] and Marques[**55**],[**56**], we have that every compact Riemannian manifold with boundary, of dimension greater or equal than three, and with finite Sobolev quotient, carries a conformal scalar flat metric with constant mean curvature on the boundary.

1. Introduction and statement of the results

In their attempt to suitably generalize the celebrated Riemann mapping theorem of Complex Analysis which asserts that every simply connected proper domain of the plane is conformally diffeomorphic to a disk, Cherrier[**35**]–Escobar[**38**] raised the question of whether every n-dimensional compact Riemannian manifold with boundary and $n \geq 3$ carries a conformal scalar flat Riemannian metric with constant mean curvature on the boundary. In [**35**], Cherrier gives a positive answer provided the curvatures are not *too big* in a reasonable geometric sense. In [**38**] and [**40**], Escobar provides a positive answer when $n = 3$, $n = 4$ or when $n = 5$ and the boundary is umbilic and when $n \geq 6$ with the boundary being non-umbilic or the Riemannian manifold being locally conformally flat and the boundary being umbilic. Later, Marques[**55**],[**56**] gives a positive answer to some remaining cases, precisely when $n = 4$ or 5 and the boundary is not umbilic, when $n \geq 8$ and the boundary is umbilic and not locally conformally flat with respect to the induced Riemannian metric, and when $n \geq 9$ with the boundary being umbilic and the Weyl tensor does not vanish identi-

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cally on the boundary. In [**2**], Almaraz gives a positive answer when $n = 6, 7, 8$, the boundary is umbilic and the Weyl tensor does not vanish identically on the boundary. Recently, Chen[**34**] solves the problem for many situations of the cases remaining after the above cited works and reduces the other ones to the positivity of the ADM mass of some class of asymptotically flat Riemannian manifolds, like she did in a joint work with S. Brendle for the boundary Yamabe problem in [**23**]. However, as in [**23**], the latter positivity is *not known to hold* for non-spin Riemannian manifolds of dimension greater or equal than eight.

Our main goal in this work is to use the algebraic topological argument of Bahri–Coron[**14**] to solve the cases left open by Almaraz[**2**], Chen[**34**], Cherrier[**35**], Escobar[**38**],[**40**] and Marques[**55**],[**56**], as we did in [**57**] for the boundary Yamabe problem to settle the cases remaining after the works of Escobar[**37**] and Brendle–Chen[**23**]. Indeed, performing a suitable scheme of the barycenter technique of Bahri–Coron[**14**] via the Almaraz[**3**]–Chen[**34**]'s bubbles, we prove a result for the Riemann mapping problem of Cherrier[**35**]–Escobar[**38**] which covers all the cases left open after the above cited works. We mention that there has been a lot of works about Yamabe and scalar curvature type problems for compact Riemannian manifolds with and without boundary, and their fully nonlinear and high-dimensional extensions, see, for example, [**1**], [**4**], [**5**], [**6**], [**7**], [**8**], [**10**], [**11**], [**13**], [**14**], [**15**], [**16**], [**18**], [**19**], [**20**], [**21**], [**22**], [**23**], [**24**], [**25**], [**26**], [**27**], [**28**], [**29**], [**30**], [**31**], [**32**], [**33**], [**34**], [**36**], [**37**], [**38**], [**39**], [**41**], [**42**], [**43**], [**47**], [**48**], [**49**], [**51**], [**52**], [**53**], [**54**], [**55**], [**56**], [**57**], [**58**], [**59**], [**60**], [**61**], [**62**], [**63**], [**64**], [**67**], [**68**] and the references therein.

In order to state clearly our theorem, we first fix some notation. Given $(\overline{M}, g)$ a n-dimensional compact Riemannian manifold with boundary ∂M, interior M and $n \geq 3$, we denote by $L_g = -4\frac{n-1}{n-2}\Delta_g + R_g$ the conformal Laplacian of $(\overline{M}, g)$ and $B_g = \frac{4(n-1)}{n-2}\frac{\partial}{\partial n_g} + 2(n-1)H_g$ the conformal Neumann operator of (M, g), with R_g denoting the scalar curvature of $(\overline{M}, g)$, Δ_g denoting the Laplace–Beltrami operator with respect to g, H_g is the mean curvature of ∂M in $(\overline{M}, g)$, namely $H_g := \frac{1}{n-1}tr_{\hat{g}}A_g$ where A_g is the second fundamental form of ∂M in $(\overline{M}, g)$ with respect to the inner normal direction, $\hat{g} := g|_{\partial M}$, $\frac{\partial}{\partial n_g}$ is the outer Neumann operator on ∂M with respect to g. We remark that our (L_g, B_g) is $\frac{4(n-1)}{n-2}$ times the one used in the work of Escobar[**37**] (see page 5). Furthermore, we define the following Escobar functional:

$$\mathcal{E}_g(u) := \frac{\langle L_g u, u\rangle + \langle B_g u, u\rangle}{(\oint_{\partial M} u^{\frac{2(n-1)}{n-2}} dS_g)^{\frac{n-2}{n-1}}}, \qquad u \in W^{1,2}_+(\overline{M}), \tag{1}$$

where

$$\langle L_g u, u\rangle := \langle L_g u, u\rangle_{L^2(M)}, \quad \langle B_g u, u\rangle := \langle B_g u, u\rangle_{L^2(\partial M)},$$

dS_g is the volume form with respect to the Riemannian metric induced by g on ∂M and

$$W_+^{1,2}(\overline{M}) := \{u \in W^{1,2}(\overline{M}): \quad u > 0\},$$

with $W^{1,2}(\overline{M})$ denoting the usual Sobolev space of functions which are L^2-integrable with their first derivatives (for more information, see [**9**] and [**46**]). Moreover, we recall that the Sobolev quotient of $(M, \partial M, g)$ is defined as

$$\mathcal{Q}(M, \partial M, g) := \inf_{u \in W_+^{1,2}(\overline{M})} \mathcal{E}_g(u).$$

As in [**57**], here also we remind that the following conformally invariant set plays an important role in the existence part in [**34**]

$$\mathcal{Z}_g := \{x \in \partial M: \quad \limsup_{d_g(x,y)\longrightarrow 0} (d_g(x,y))^{2-d}|W_g(y)| = 0\},$$

where d_g and W_g denote, respectively, the Riemannian distance and the Weyl tensor of $(\overline{M}, g)$ and $d := [\frac{n-2}{2}]$ with $[\cdot]$ denoting the integer part function.

Now, having fixed the needed notation, we are ready to state our theorem which reads as follows:

Theorem 1.1. *Assuming that $(\overline{M}, g)$ is a n-dimensional compact Riemannian manifold with boundary ∂M and interior M such that ∂M is umbilic in $(\overline{M}, g)$, $n \geq 6$, $\mathcal{Q}(M, \partial M, g) > 0$, and $\mathcal{Z}_g = \partial M$, then $(\overline{M}, g)$ carries a conformal scalar flat Riemannian metric with respect to which ∂M has constant mean curvature.*

Hence, since the only *possible* open cases for the Riemann mapping problem of Cherrier[**35**]–Escobar[**38**] are when the dimension of the manifold is greater or equal than 6 with umbilic boundary, positive Sobolev quotient and $\mathcal{Z}_g = \partial M$, then clearly Theorem 1.1 and the works of Almaraz[**2**], Chen[**34**], Cherrier[**35**], Escobar[**38**],[**40**] and Marques[**55**],[**56**] imply the following positive answer to the high-dimensional generalization by Cherrier[**35**]–Escobar[**38**] of the celebrated Riemann mapping problem of Complex Analysis.

Theorem 1.2. *Every n-dimensional compact Riemannian manifold with boundary, $n \geq 3$, and finite Sobolev quotient, carries a conformal scalar flat Riemannian metric with constant mean curvature on the boundary.*

To give a positive answer to the high-dimensional generalization by Cherrier[**35**]–Escobar[**38**] of the celebrated Riemann mapping problem of Riemann surface theory is equivalent to solving a second order elliptic boundary value problem with critical Sobolev nonlinearity on the boundary. Indeed, under the assumptions of Theorem 1.1, the Riemann mapping problem of Cherrier[**35**]–Escobar[**38**] is equivalent to finding a smooth and positive solution of the following semilinear elliptic boundary value problem:

$$\begin{cases} L_g u = 0 \ \text{ in } \ M, \\ B_g u = 2(n-1)u^{\frac{n}{n-2}} \qquad \text{on } \ \partial M. \end{cases} \tag{2}$$

The boundary value problem (2) has a variational structure. Indeed, thanks to the work of Cherrier[**35**], smooth solutions of (2) can be found by looking at critical points of the Escobar functional $\mathcal{E}_g$, and as in [**57**], we will pursue such an approach here. Precisely, we will perform a suitable application of the barycenter technique of Bahri–Coron[**14**] via the Almaraz[**3**]–Chen[**34**]'s bubbles. However, since the application of the barycenter technique that we are going to present in this work is similar to the one we perform in [**57**], and since the algebraic topological argument of Bahri–Coron[**14**] (see [**44**], [**45**], [**62**], [**65**] for some applications of it) seems to be less known ("mastered") in the Geometric Analysis community in comparison to his companion argument, namely the Aubin[**8**]–Schoen[**66**]'s minimization technique, then we feel more convenient to describe it more in details here, for those who are not familiar with. We are going to describe the barycenter technique of Bahri–Coron[**14**] focusing on $\mathcal{E}_g$. To begin, the algebraic topological argument of Bahri–Coron[**14**] belongs the class of indirect methods. Precisely, it is an argument by contradiction. Thus, assuming that the Euler–Lagrange functional $\mathcal{E}_g$ has no critical points, one looks for a contradiction by using the quantization and strong interaction phenomenon that $\mathcal{E}_g$ verifies and the *cone* structure of the space of barycenters of ∂M. To describe how the combination of the latter works to give the desired contradiction, we think it is for the sake of understanding of the reader more useful to do it with pictures and some *talking* mathematical formulas rather than only exact mathematical ones, of course at the price of precision, but with the right intuition of what is going on with the barycenter technique of Bahri–Coron[**14**]. First of all, and recalling the assumption $\mathcal{E}_g$ has no critical points (to keep in mind), one has that the quantization phenomenon that $\mathcal{E}_g$ enjoys implies Fig. 1, which traduces the fact that by bubbling ∂M survives topologically between the first and second critical levels of $\mathcal{E}_g$ that we denote by (W_1, W_0) (see (7) for its precise definition). Next, realizing $B_2(\partial M)$ (for its definition see (19)) as a cone over $B_1(\partial M) = \partial M$ with top ∂M, one has that the quantization phenomenon that $\mathcal{E}_g$ satisfies implies again Fig. 2 which shows the fact that by bubbling $B_2(\partial M)$ as a cone over $B_1(\partial M)$ survives as a nontrivial cone between the second and third critical levels of $\mathcal{E}_g$ that we denote by (W_2, W_1). Similarly, realizing $B_3(\partial M)$ as a cone

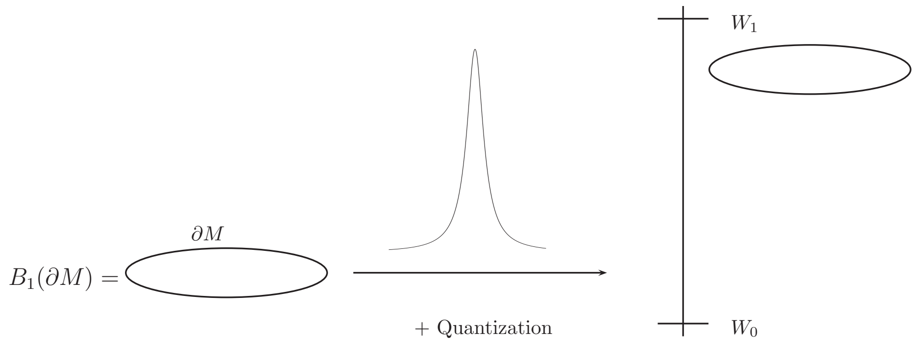

Figure 1

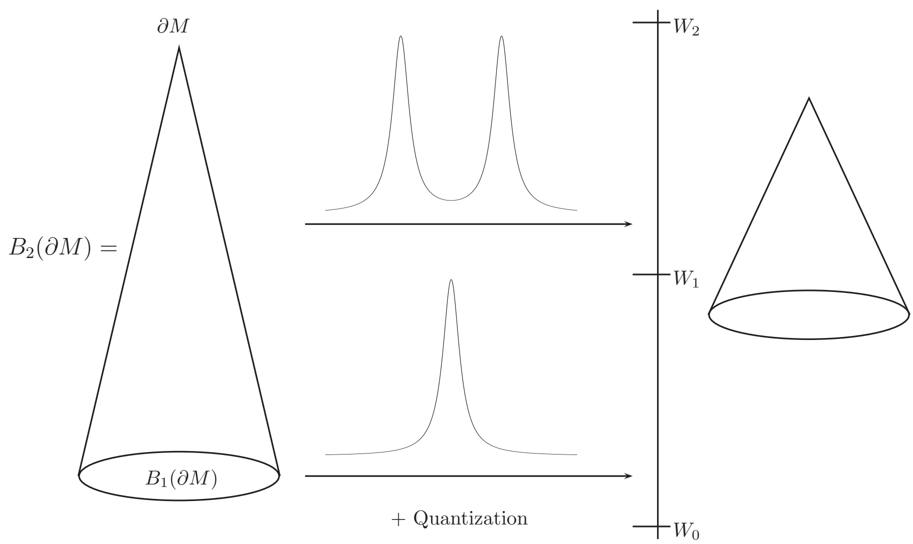

Figure 2

over $B_2(\partial M)$ with top ∂M, one has that the quantization phenomenon that $\mathcal{E}_g$ verifies implies again Fig. 3 which traduces the fact that by bubbling $B_3(\partial M)$ as a cone over $B_2(\partial M)$ survives as a nontrivial cone between the third and fourth critical levels of $\mathcal{E}_g$ that we denote by (W_3, W_2). Hence, recursively for $p \in \mathbb{N}^*$, realizing $B_{p+1}(\partial M)$ as a cone over $B_p(\partial M)$ with top ∂M, we have that the quantization phenomenon that $\mathcal{E}_g$ enjoys implies again Fig. 4 which shows the fact that by bubbling $B_{p+1}(\partial M)$ as a cone over $B_p(\partial M)$ survives as a nontrivial cone between the $(p+1)$ and $(p+2)$ critical levels of $\mathcal{E}_g$ that we denote by (W_{p+1}, W_p). On the other hand, the latter recursion leads to a contradiction because of the strong interaction phenomenon that $\mathcal{E}_g$ enjoys. To see this, one first observes that, as standard bubbles, the Almaraz[**3**]–Chen[**34**]'s bubbles $\varphi_{a,\lambda}$ verify the *every where* selfaction

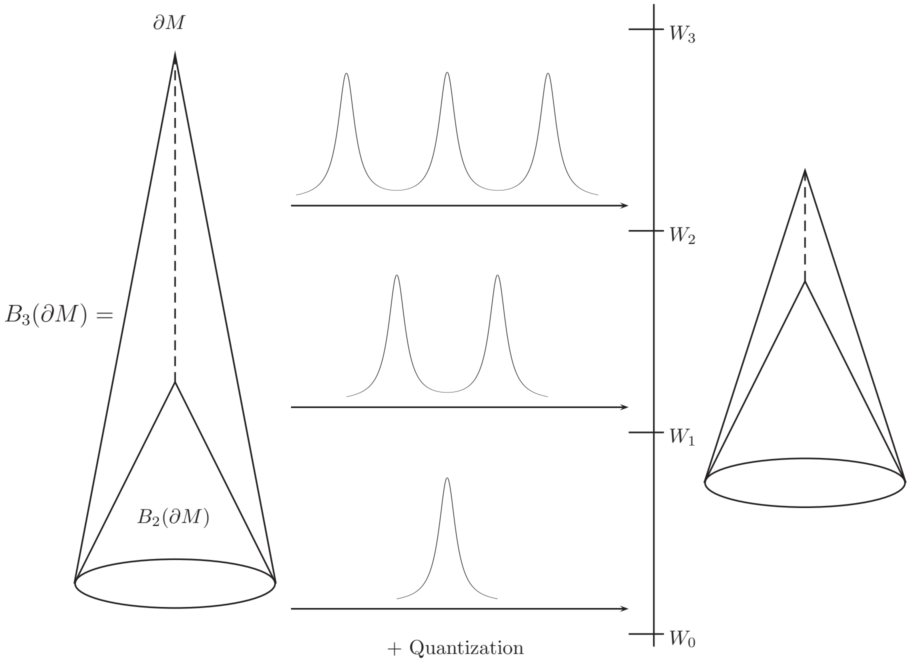

Figure 3

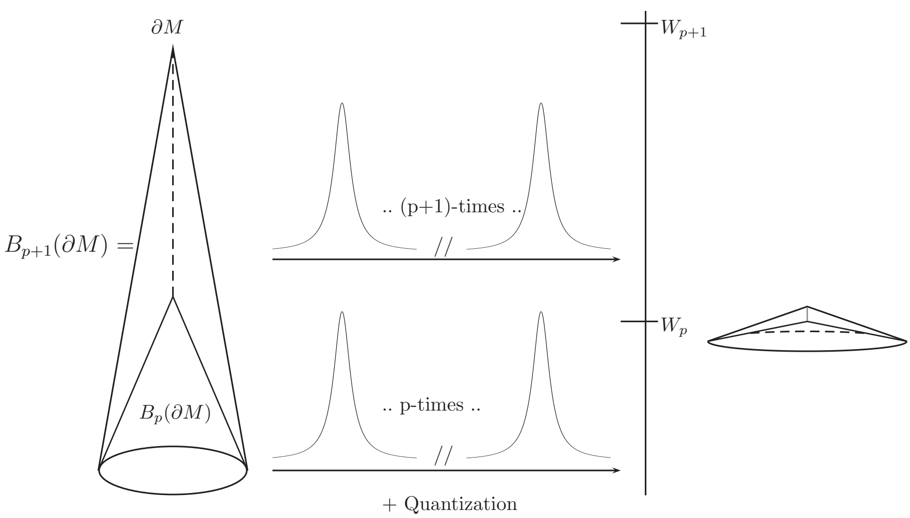

Figure 4

and interaction identities (see Lemma 2.8)

$$\begin{aligned}(3)\quad &\langle L_g\varphi_{a_i,\lambda},\varphi_{a_j,\lambda}\rangle+\langle B_g\varphi_{a_i,\lambda},\varphi_{a_j,\lambda}\rangle\\ &=(1+o(1))(n-2)\oint_{\partial M}\varphi_{a_i,\lambda}^{\frac{n}{n-2}}\varphi_{a_j,\lambda}dS_g,\quad a_i,a_j\in\partial M.\end{aligned}$$

Moreover, one observes also that, as standard bubbles, the Almaraz[**3**]–Chen[**34**]'s bubbles $\varphi_{a,\lambda}$ verify the following interaction identity at *infinity* (see formula (18) and Lemma 2.8)

$$(4)\qquad \oint_{\partial M}\varphi_{a_i,\lambda}^{\frac{n}{n-2}}\varphi_{a_j,\lambda}dS_g=\frac{c_n+o(1)}{(1+\lambda^2G_g^{\frac{2}{2-n}}(a_i,a_i))^{\frac{n-2}{2}}},\quad a_i\neq a_j\in\partial M,$$

where c_n is a positive constant pedending only on n, G_g is the Green's function of (L_g,B_g) suitably normalized, and $o(1)$ in the above formula means a quantity tending to 0 uniformly in $A:=(a_i,a_j)$ as λ tends to $+\infty$. We would like to emphasize that, formula (3) for different particles $a_i\neq a_j$ is saying nothing else than, the linear interaction given by the right hand side of (3) is almost (up to the multiplication by $(n-2)$) the same as the nonlinear interaction given by the right hand side of (4), and that the value of the nonlinear interaction is almost $\frac{G_g(a_i,a_j)}{\lambda^{n-2}}$ (up to the multiplication by c_n). We would like to add also that the latter discussion reflects also nothing else that the *restoration* of symmetry, which is part of what defines infinity, because of the conformal invariance of the problem, and is consistent with invariance principles. Furthermore, for $p\in\mathbb{N}^*$, (3), (4) and the scale invariance of $\mathcal{I}_g$ imply that for optimal configurations $\sum_{i=1}^p\varphi_{a_i,\lambda}$ there holds

$$(5)\ \mathcal{I}_g(\sum_{i=1}^p\varphi_{a_i,\lambda})\leq\sum_{i=1}^p\left(\mathcal{E}_g(\varphi_{a_i,\lambda})-(1+o(1))c^n\sum_{j\neq i}^p Int_g(a_i,a_j,\lambda)\right),$$

where c^n is a positive constant depending only on n, $Int_g(a_i,a_j,\lambda)$ denotes the interaction between $\varphi_{a_i,\lambda}$ and $\varphi_{a_j,\lambda}$ measured with respect to the left hand side of (3) or (4), and of course c^n depends on which way one sees $Int_g(a_i,a_j,\lambda)$. We would like to comment on the $-$ sign in formula (5) and say some words about optimal configurations. The latter sign does not come by miracle, but is part of what defines "true" infinity for the associated variational problem, and analytically, it can be seen just by Taylor expansion (which is possible because of being at infinity), and the fact that the scale invariance of $\mathcal{E}_g$ and (3) imply that, the denominator in the definition of $\mathcal{E}_g$ brings (roughly speaking) a contribution of -2 times the linear interaction in the calculation of $\mathcal{I}_g(\sum_{i=1}^p\varphi_{a_i,\lambda})$, and the numerator brings a contribution of $+1$ times the linear interaction in the latter calculation. We have chosen to focus on optimal configurations to show how the strong interaction pushes the energy down for a large number of particles, this is due to the fact that those configurations are the most "variationally" critical ones, and to have a picture in mind, one can just think of the top of a mountain, or the down of a black hole. Now, an important piece in the argument is the fact that, the concentration points live in a

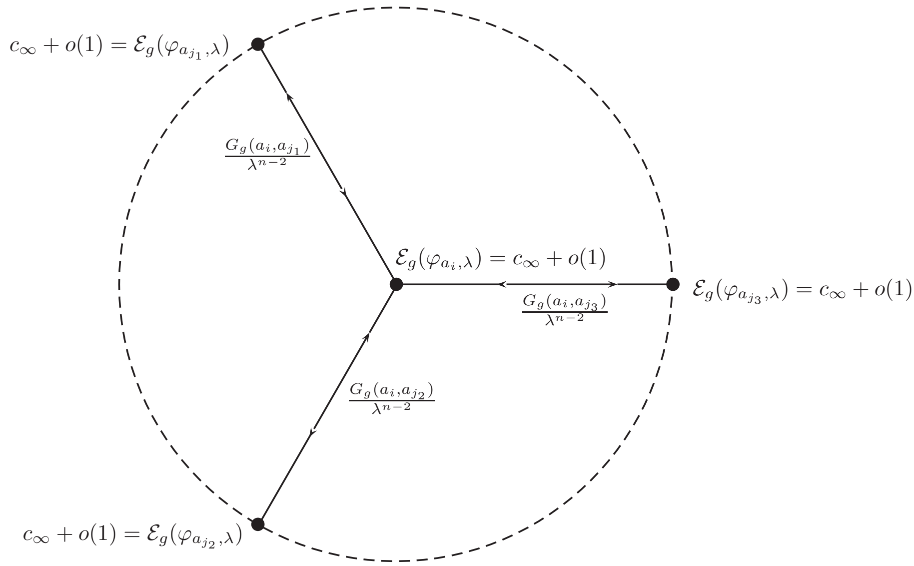

Figure 5

world with quantization and strong interaction phenomenon as shown by Fig. 5 where c_∞ is the first critical value and that they verify the sharp mass energy estimate

$$\mathcal{E}_g(\varphi_{a,\lambda}) \leq c_\infty - \frac{MD_g(a)}{\lambda^{n-2}} + o(\frac{1}{\lambda^{n-2}}),$$

where $MD_g(a)$ denotes the mass distribution of the particle a, $o(1)$ means a quantity tending to 0 uniformly in a as λ tends to $+\infty$, and we recall that the interaction terms in the picture (5) are nothing else than a "representation" of $Int_g(a_i, a_j, \lambda)$, namely

$$Int_g(a_i, a_j, \lambda) = \bar{c}_n(1 + o(1))\frac{G_g(a_i, a_j)}{\lambda^{n-2}}, \tag{6}$$

where $\bar{c}_n$ is a positive constant depending only on n, and still as above $\bar{c}_n$ depends on how one sees $Int_g(a_i, a_j, \lambda)$ according to the left hand side of (3) and (4). Moreover, MD_g is an L^∞-function and $\min_{\overline{M}^2} G_g > 0$ which is what we mean by $\mathcal{E}_g$ verifies a strong interaction phenomenon. We would like to remark the fact that in the pictures (1)–(4), we choose to emphasize that more the number of particles is large, less is the height of the cone, and this is to traduce the fact that (roughly speaking) for a very large number of particles in consideration, more is the number of particles, less is the "topological" space that we have to put the corresponding cone, as explained analytically by the formulas (5)–(6). Thus, clearly (5)–(6) imply Fig. 6 which shows the fact that for p_0 large, $B_{p_0}(\partial M)$ as a cone over

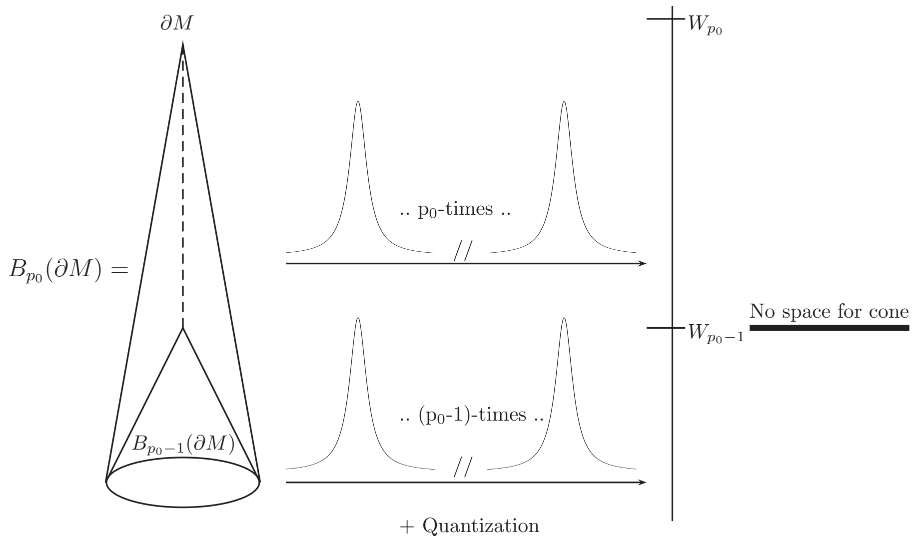

Figure 6

$B_{p_0-1}(\partial M)$ cannot be embedded by bubbling and still be a nontrivial cone between the p_0 and p_0+1 critical levels of $\mathcal{E}_g$. Hence, clearly the figures (4) and (6) lead to a contradiction.

The structure of this paper is as follows. In Section 2, we fix some notation and give some preliminaries, like the set of formal barycenters of ∂M and present some useful topological properties of them. Furthermore, we recall the Almaraz[**3**]–Chen[**23**]'s bubbles and the fact that they can be used to replace the standard bubbles in the analysis of diverging Palais–Smale (PS) sequences of the Euler–Lagrange functional $\mathcal{E}_g$. Moreover, using a result of Almaraz[**3**] and another one of Chen[**34**] (or of Almaraz[**3**]), we derive selfaction and interaction estimates for the Almaraz[**3**]–Chen[**34**]'s bubbles. In Section 3, we use the latter estimates to map the space of barycenters of ∂M of any order into suitable sublevels of $\mathcal{E}_g$ via the Almaraz[**3**]–Chen[**34**]'s bubbles. Finally, in Section 4, we define the neighborhood of potential critical points at infinity of $\mathcal{E}_g$ and use the results of Section 3 to carry our scheme of the barycenter technique of Bahri–Coron[**14**] to prove Theorem 1.1.

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2. Notation and preliminaries

In this section, we fix some notations and give some preliminaries. First of all, since the problem under study is conformally invariant and we are dealing with the umbilic case, then from now until the end of the paper $(\overline{M}, g)$ will be the given underlying compact n-dimensional Riemannian manifold with boundary ∂M and interior M, ∂M is totally geodesic in $(\overline{M}, g)$, $n \geq 6$, $\mathcal{Q}(M, \partial M, g) > 0$, and $\mathcal{Z}_g = \partial M$.

In the following, for any Riemannian metric $\bar{g}$ on $\overline{M}$ and $p \in \partial M$, we will use the notation $B^{\bar{g}}_p(r)$ to denote the coordinate ball with respect to $\bar{g}$ of radius r and center p. For $p \in \partial M$, we use the notation $\hat{B}^{\bar{g}}_p(r)$ to denote the geodesic ball in ∂M with respect to the Riemannian metric $\hat{\bar{g}}$ induced by $\bar{g}$ on ∂M of radius r and center p. We also denote, respectively, by $d_{\bar{g}}(x, y)$ and $\bar{d}_{\bar{g}}(x, y)$, the geodesic distance with respect to $\bar{g}$ between two points x and y of $\overline{M}$, and the coordinate distance between $x \in \partial M$ and $y \in \overline{M}$. Similarly, we denote by $d_{\hat{\bar{g}}}(x, y)$, the geodesic distance with respect to $\hat{\bar{g}}$ between two points x and y of ∂M. $inj_{\bar{g}}(\overline{M})$, $inj_{\hat{\bar{g}}}(\partial M)$ stand for the injectivity radius of $(\overline{M}, \bar{g})$, $(\partial M, \hat{\bar{g}})$. $dV_{\bar{g}}$ denotes the Riemannian measure associated to the metric $\bar{g}$ and $dS_{\bar{g}}$ the volume form on ∂M with respect to $\hat{\bar{g}}$ on ∂M. For simplicity, given $p \in \partial M$, we will use, respectively, $B_p(r)$ and $\hat{B}_p(r)$ to denote $B^g_p(r)$ and $\hat{B}^g_p(r)$. For $a \in \overline{M}$, we use the notation $exp^{\bar{g}}_a$ to denote the exponential map with respect to $\bar{g}$ and set for simplicity $\exp_a := \exp^g_a$. For $a \in \partial M$, we denote by $\hat{\exp}^{\bar{g}}_a$ the exponential map with respect to $\hat{\bar{g}}$ and set $\hat{\exp}_a := \hat{\exp}^g_a$.

$\mathbb{N}$ denotes the set of nonnegative integers, $\mathbb{N}^*$ stands for the set of positive integers and for $k \in \mathbb{N}^*$, $\mathbb{R}^k$ stands for the standard k-dimensional Euclidean space, $\mathbb{R}^k_+$ the open positive half-space of $\mathbb{R}^k$ and $\bar{\mathbb{R}}^k_+$ its closure in $\mathbb{R}^k$. For simplicity, we will use the notation $\mathbb{R}_+ := \mathbb{R}^1_+$ and $\bar{\mathbb{R}}_+ := \bar{\mathbb{R}}^1_+$. For $r > 0$, $B^{\mathbb{R}^k}_0(r)$ denotes the open ball of $\mathbb{R}^k$ of center 0 and radius r and set for simplicity $B^k := B^{\mathbb{R}^k}_0(1)$. We use g_{B^k} to denote the Euclidean metric on B^k. For $p \in \mathbb{N}^*$, σ_p denotes the permutation group of p elements, $(\partial M)^p$ denotes the Cartesian product of p copies of ∂M. We define $((\partial M)^2)^* := (\partial M)^2 \setminus Diag((\partial M)^2)$ where $Diag((\partial M)^2)$ is the diagonal of $(\partial M)^2$, namely $Diag((\partial M)^2) := \{(a, a) : \ a \in \partial M\}$. For $p \in \mathbb{N}^*$, Δ_{p-1} the following simplex $\Delta_{p-1} := \{(\alpha_1, \cdots, \alpha_p) : \alpha_i \geq 0, i = 1, \cdots, p, \sum_{i=1}^p \alpha_i = 1\}$.

For $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, $\beta \in]0, 1[$, $L^p(M)$ and $L^p(\partial M)$, $W^{k,p}(M)$, $C^k(\overline{M})$ and $C^{k,\beta}(\overline{M})$ stand, respectively, for the standard p-Lebesgue space on M and ∂M, (k, p)-Sobolev space, k-continuously differentiable space and k-continuously differential space of Hölder exponent β, all with respect to g (if the definition needs a metric structure) and for precise definitions and properties, see, for example, [**9**] or [**46**].

For $a \in \partial M$, $O_a(1)$ stands for quantities bounded uniformly in a. For ϵ positive and small and $a \in \partial M$, $O_{a,\epsilon}(1)$ stands for quantities uniformly bounded in a and ϵ. For ϵ positive and small, $o_\epsilon(1)$ means quantities which tend to 0 as ϵ tends to 0. For λ large and $a \in \partial M$, $O_{a,\lambda}(1)$ stands for quantities uniformly bounded in a and λ. For $a \in \partial M$, ϵ and δ positive and small and λ large, $O_{a,\epsilon,\delta}(1)$ and $O_{a,\lambda}(1)$ stand, respectively, for quantities which are bounded uniformly in a, δ and ϵ and in a and λ. For $a \in \partial M$, ϵ positive and small and λ large, $o_{a,\epsilon}(1)$ and $o_{a,\lambda}(1)$ stand, respectively, for quantities which tend to 0 uniformly in a as ϵ tends to 0 and as λ tends to $+\infty$. For $A \in (\partial M)^2$ and λ large, $O_{A,\lambda}(1)$ and $o_{A,\lambda}(1)$ stand, respectively, for quantities which are bounded uniformly in A and λ, and which tend to 0 uniformly in A as λ tends to $+\infty$. For $p \in \mathbb{N}^*$, $A \in (\partial M)^p$, $\bar{\alpha} \in \Delta_{p-1}$ and λ large, $O_{A,\bar{\alpha},\lambda}(1)$ and $o_{A,\bar{\alpha},\lambda}(1)$ stand, respectively, for quantities which are uniformly bounded in p, A, $\bar{\alpha}$ and λ and for quantities which tend to 0 uniformly in p, A and $\bar{\alpha}$ as λ tends to $+\infty$. For $x \in \mathbb{R}$, we will use the notation $O(x)$ and $o(x)$ to mean, respectively, $|x|O(1)$ and $|x|o(1)$ where $O(1)$ and $o(1)$ will be specified in all the contexts where they are used. Large positive constants are usually denoted by C and the value of C is allowed to vary from formula to formula and also within the same line. Similarly small positive constants are denoted by c and their values may vary from formula to formula and also within the same line. The symbol $\sum_{i\neq j}$ always means a double sum over the associated index set under the assumption $i \neq j$.

For X a topological space, $H_*(X)$ will denote the singular homology of X with $\mathbb{Z}_2$ coefficients and $H^*(X)$ for the cohomology. For Y a subspace of X, $H_*(X, Y)$ will stand for the relative homology. The symbol $\frown$ will denote the cap product between cohomology and homology. For a map $f : X \to Y$, with X and Y topological spaces, f_* stands for the induced map in homology and f^* for the induced map in cohomology. For $p \in \mathbb{N}$, we set

$$W_p := \{u \in W^{1,2}_+(\overline{M}) : \quad \mathcal{E}_g(u) \le (p+1)^{\frac{1}{n-1}} \mathcal{Q}(B^n)\}, \tag{7}$$

where

$$\mathcal{Q}(B^n) := \mathcal{Q}(B^n, \partial B^n, g_{B^n}). \tag{8}$$

For a Riemannian metric $\bar{g}$ defined on $\overline{M}$, we denote by $G_{\bar{g}}$ the Green's function of $(L_{\bar{g}}, B_{\bar{g}})$ satisfying the normalization

$$\lim_{\bar{d}_{\bar{g}}(a,x)\longrightarrow 0} (\bar{d}_{\bar{g}}(a,x))^{n-2} G_{\bar{g}}(a,x) = 1, \tag{9}$$

and set

$$G := G_g.$$

Using the existence of conformal normal Fermi coordinates (see [**55**]) and recalling that ∂M is totally geodesic in $(\overline{M}, g)$, we have that for every large positive integer m and for every $a \in \partial M$, there exists a positive function $u_a \in C^\infty(\overline{M})$ such that the metric

$$g_a := u_a^{\frac{4}{n-2}} g$$

verifies

$$detg_a(x) = 1 + O_{a,x}((\bar{d}_{g_a}(x,a))^m) \quad \text{for} \quad x \in B_a^{g_a}(\varrho_a), \tag{10}$$

with $O_{a,x}(1)$ meaning bounded by a constant independent of a and x, $0 < \varrho_a < \min\{\frac{inj_{g_a}(\overline{M})}{10}, \frac{inj_{\hat{g}_a}(\partial M)}{10}\}$. Moreover, we can take the family of functions u_a, g_a and ϱ_a such that

$$\text{the maps } \ a \longrightarrow u_a, \ g_a \ \text{ are } C^0 \ \text{ and } \ \frac{1}{4} \geq \varrho_a \geq \varrho_0 > 0,$$

for some small positive ϱ_0 satisfying $\varrho_0 < \min\{\frac{inj_g(\overline{M})}{10}, \frac{inj_{\hat{g}}(\partial M)}{10}\}$, and

$$\begin{aligned} &||u_a||_{C^2(\overline{M})} = O_a(1), \ \ \frac{1}{\overline{C}^2} g \leq g_a \leq \overline{C}^2 g, \quad a \in \overline{M}, \\ &u_a(a) = 1 \text{ and } \ H_a := H_{g_a} = 0, \\ &u_a(x) = 1 + O_{a,x}(\bar{d}^2_{g_a}(a,x)) = 1 + O_{a,x}(\bar{d}^2_g(a,x)), \end{aligned} \tag{11}$$

for $x \in B_a^{g_a}(\varrho_0) \supset B_a(\frac{\varrho_0}{2\overline{C}})$ and some large positive constant $\overline{C}$ independent of a and $O_{a,x}(1)$ is as in (10). We remark that in general, namely for g arbitrary, we know from [**55**] that

$$H_a = O_{a,x}(d_{\hat{g}_a}^{m-1}(a,x)) = 1 + O_{a,x}(d_{\hat{g}}^{m-1}(a,x)),$$

for $x \in \hat{B}_a^{g_a}(\varrho_0) \supset \hat{B}_a(\frac{\varrho_0}{2\overline{C}})$, where $\hat{g}_a := g_a|_{\partial M}$, and $O_{a,x}$ is as in (11). We point out that the latter suffices for our purpose, since we will be working with the Almaraz[**3**]–Chen[**34**]'s bubbles. However, we quickly provide some indications on how one can use Marques[**55**]'s work to achieve $H_a = 0$ in the totally geodesic case (even under the only assumption $H_g = 0$ on ∂M). From Marques[**55**], one has a function $\tilde{u}_a > 0$ and smooth such that $\tilde{g}_a := \tilde{u}_a^{\frac{4}{n-2}} g$ verifies (10)–(11) with some $\tilde{C}$ and some $\tilde{\varrho}_a > 0$ except the global vanishing of the associated mean curvature, but one always has its local vanishing, namely $\frac{\partial \tilde{u}_a}{\partial n_g} = 0$ in $\hat{B}_a^{\tilde{g}_a}(\tilde{\varrho}_a)$ (realling that we are in the totally geodesic case), see (1) of Proposition 3.2 in [**55**], however, as remark in the proof of Proposition in [**55**], the local vanishing of the mean curvature always holds, namely for g arbitrary. Thus, setting $u_a := \tilde{\chi}_a u_a + (1 - \tilde{\chi}_a)$ with $\tilde{\chi}_a = \chi(\frac{\tilde{C}\bar{d}_g(a,\cdot)}{\tilde{\varrho}_a})$, for some large $\overline{C}$ such that $g_a := u_a^{\frac{4}{n-2}} g$ verifies the comparability $\frac{1}{\overline{C}^2}g \leq g_a \leq \overline{C}^2$, and χ is a cut-off function defined on $\bar{\mathbb{R}}_+$ satisfying χ is non-negative, $\chi(t)=1$ if $t \leq \frac{1}{2}$ and $\chi(t)=0$ if $t \geq 1$, then one has that $g_a = \tilde{g}_a$ in $B^{g_a}_a(\varrho_a)$ with $\varrho_a := \frac{\tilde{\varrho}_a}{2\overline{C}\tilde{C}}$ and $\frac{\partial u_a}{\partial n_g} = 0$ on ∂M. Hence, recalling again that we are in the totally geodesic case (however, just $H_g = 0$ on ∂M suffices), one has that the mean curvature H_a of g_a vanishes identically on ∂M, thus clearly g_a satisfies fully (10)–(11) as desired. Now, having finished to give indications on how one can achieve $H_a = 0$ on ∂M, we continue to make some definitions. For $a \in \partial M$ and ϵ positive, we recall that the standard bubbles of the geometric problem under study are defined as follows:

$$\delta_{a,\epsilon}(x) := \left(\frac{\epsilon}{(\epsilon + x_n)^2 + |x'|^2}\right)^{\frac{n-2}{2}}, \qquad x \in B^{g_a}_a(\varrho_0), \tag{12}$$

where (x', x_n) is the Fermi normal coordinate of x with respect to g_a at a. For $a \in \partial M$ and $0 < r < \varrho_0$, we set also

$$\begin{aligned} &G_a := G_{g_a}, \quad \exp^a_a = \exp^{g_a}_a, \quad \hat{\exp}^a_a = \exp^{\hat{g}_a}_a, \\ &B^a_a(r) := B^{g_a}_a(r) \quad \text{and} \quad \hat{B}^a_a(r) := \hat{B}^{g_a}_a(r). \end{aligned} \tag{13}$$

On the other hand, the conformal invariance properties of the couple conformal Laplacian and conformal Neumann operator imply

$$\begin{aligned} &\mathcal{E}_g(u) = \mathcal{E}_{g_a}(u_a^{-1}u), \quad \oint_{\partial M} u^{\frac{2(n-1)}{n-2}} dS_g = \oint_{\partial M} (u_a^{-1}u)^{\frac{2(n-1)}{n-2}} dS_{g_a}, \\ &G_g(x,y) = G_a(x,y)u_a(x)u_a(y), \end{aligned} \tag{14}$$

for $(x,y) \in \overline{M}^2$, $a \in \partial M$ and $u \in W^{1,2}_+(\overline{M})$. We also define

$$\begin{aligned} &c_0 := (n-2), \quad c_1 := \int_{\mathbb{R}^{n-1}} \left(\frac{1}{1+|x|^2}\right)^{n-1} dx, \text{ and} \\ &c_2 := 4\frac{n-1}{n-2} \int_{\mathbb{R}^n_+} \left|\nabla\left[\left(\frac{1}{(1+x_n)^2 + |x'|^2}\right)^{\frac{n-2}{2}}\right]\right|^2 dx. \end{aligned} \tag{15}$$

Furthermore, we set

$$c_3 := \int_{\mathbb{R}^{n-1}} \left(\frac{1}{1+|x|^2}\right)^{\frac{n}{2}} dx,$$

and define the following quantity which depends only on $(\overline{M}, g)$

$$c_g = \frac{c_3}{4c_1} \min_{((\partial M)^2)^*} G, \tag{16}$$

and see above for the definition of $((\partial M)^2)^*$, G, c_1 and c_3. We recall that the numbers c_i $(i = 0,1,2)$ and $\mathcal{Q}(B^n)$ verify the following relation:

$$c_2 = c_0 c_1 \quad \text{and} \quad \mathcal{Q}(B^n) = \frac{c_2}{c_1^{\frac{n-2}{n-1}}}. \tag{17}$$

Moreover, for $p \in \mathbb{N}^*$, $A := (a_1, \cdots, a_p) \in (\partial M)^p$, $\bar{\lambda} := (\lambda_1, \cdots, \lambda_p) \in (\mathbb{R}_+)^p$, we associate the following quantities (which appear in the analysis of diverging PS sequences of the Euler–Lagrange functional $\mathcal{E}_g$)

$$\varepsilon_{i,j} := \varepsilon_{i,j}(A, \bar{\lambda}) := \frac{c_3}{\left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j G^{\frac{2}{2-n}}(a_i, a_i)\right)^{\frac{n-2}{2}}}, \tag{18}$$

where $i, j = 1, \cdots, p$, $i \neq j$.

Now, we are going to present some topological properties of the space of formal barycenter of ∂M that we will need for our algebraic topological argument for existence. To do that, for $p \in \mathbb{N}^*$, we recall that the set of formal barycenters of ∂M of order p is defined as follows:

$$B_p(\partial M) := \{\sum_{i=1}^{p} \alpha_i \delta_{a_i} \; : \; a_i \in \partial M, \; \alpha_i \geq 0 \, , \sum_{i=1}^{p} \alpha_i = 1\}, \tag{19}$$

and set

$$B_0(\partial M) = \emptyset.$$

Furthermore, we have the existence of $\mathbb{Z}_2$ orientation classes $w_p \in H_{np-1}(B_p(\partial M), B_{p-1}(\partial M))$ and that the cap product acts as follows (see, for example, [**14**], page 263, line 5):

$$\begin{aligned} &H^l((\partial M)^p/\sigma_p) \times H_k(B_p(\partial M), B_{p-1}(\partial M)) \\ &\qquad \stackrel{\frown}{\longrightarrow} H_{k-l}(B_p(\partial M), B_{p-1}(\partial M)). \end{aligned} \tag{20}$$

On the other hand, since ∂M is a closed $(n-1)$-dimensional manifold, then we have

$$\text{an orientation class } 0 \neq O^*_{\partial M} \in H^{n-1}(\partial M). \tag{21}$$

Moreover, there is a natural way to see (precisely to transfer) $O^*_{\partial M} \in H^{n-1}(\partial M)$ as a nontrivial element of $H^{n-1}((\partial M)^p/_{\sigma_p})$ (see [**12**] formula (6.159) or [**14**] page 265 and [**45**] page 161 and Definition 23), namely

$$O^*_{\partial M} \simeq O^*_p \text{ with } 0 \neq O^*_p \in H^{n-1}((\partial M)^p/\sigma_p). \tag{22}$$

We recall briefly the construction of O^*_p , and for more details see [**12**] or [**14**]. The main ingredient in the definition of O^*_p is a transfer map (see [**17**])

$$tr : H^{n-1}((\partial M)^p/\sigma_1 \times \sigma_{p-1}) \longrightarrow H^{n-1}((\partial M)^p/\sigma_p), \tag{23}$$

where $\sigma_1 \times \sigma_{p-1}$ is the subgroup of σ_p consisting of elements of σ_p which map 1 to 1. We now describe the main map and facts that are used in the construction of the map tr, and for more details see [**12**] or [**45**]. The first fact is the following equivalence:

$$(\partial M)^p/\sigma_p \simeq ((\partial M)^p/\sigma_1 \times \sigma_{p-1}) \,/\, (\sigma_p/\sigma_1 \times \sigma_{p-1}),$$

and the associated projection

$$q : (\partial M)^p/\sigma_1 \times \sigma_{p-1} \longrightarrow (\partial M)^p/\sigma_p.$$

The next fact and the main map used in the construction of tr is Poincaré duality and q_*. With the transfer map tr at hand, the next ingredient in the definition of O_p^* is the projection

$$\pi : (\partial M)^p/\sigma_1 \times \sigma_{p-1} \longrightarrow \partial M. \tag{24}$$

Indeed, (21), (23) and (24) defines O_p^* via the following formula:

$$O_p^* := tr \circ \pi^*(O_{\partial M}^*).$$

Now, identifying $O_{\partial M}^*$ and O_p^* via (22) and using (20) we have the following well-know formula, see [**12**] formula (25), or [**14**] and [**50**] for related issues.

Lemma 2.1. *There holds*

$$\begin{aligned} H^{n-1}((\partial M)^p/_{\sigma_p}) \times H_{np-1}(B_p(\partial M), B_{p-1}(\partial M)) \\ \xrightarrow{\frown} H_{np-n}(B_p(\partial M), B_{p-1}(\partial M)) \\ \xrightarrow{\partial} H_{np-n-1}(B_{p-1}(\partial M), B_{p-2}(\partial M)), \end{aligned}$$

and

$$\omega_{p-1} = \partial(O_{\partial M}^* \frown w_p).$$

Next, we are going to discuss some important properties of the Almaraz[**3**]–Chen[**34**]'s bubbles. Using the techniques of Brendle[**21**], Almaraz[**3**]–Chen[**34**] have introduced a family of bubbles which verify the same properties as the Brendle[**21**]'s bubbles and the Brendle–Chen[**23**]'s bubbles. Indeed, for δ small, they define a family of bubbles $v_{a,\epsilon,\delta}$ (see page 2643 in [**3**] and page 16 in [**34**]), $a \in \partial M$ and ϵ positive and small such that they can replace the standard bubbles in the analysis of diverging PS sequences of $\mathcal{E}_g$ and more importantly verify a sharp energy estimate. Precisely, $v_{a,\epsilon,\delta}$ is defined as a suitable perturbation of the standard bubbles glued with an appropriate scale of the Green's function G_a centered at a as follows:

$$v_{a,\epsilon,\delta}(\cdot) = \chi_\delta(\cdot)(\delta_{a,\epsilon}(\cdot) + w_{a,\epsilon}(\cdot)) + (1 - \chi_\delta(\cdot))\epsilon^{\frac{n-2}{2}} G_a(a, \cdot), \tag{25}$$

where

$$\chi_\delta(x) := \chi\left(\frac{\bar{d}_{g_a}(a, x)}{\delta}\right),$$

and χ is a cut-off function defined on $\bar{\mathbb{R}}_+$ satisfying χ is non-negative, $\chi(t) = 1$ if $t \le 1$ and $\chi(t) = 0$ if $t \ge 2$, $\delta_{a,\epsilon}$ is defined as in (12),

$G_a(a,\cdot)$ is defined as in (13), and in Fermi normal coordinates around a with respect to g_a we have that $w_{a,\epsilon}$ satisfies the pointwise estimate

$$|\partial^\beta w_{a,\epsilon}(x)| \leq C_n(|\beta|)\frac{\epsilon^{\frac{n-2}{2}}}{(\epsilon^2+r^2)^{\frac{n-4+\beta}{2}}}, \tag{26}$$

where $r=\bar{d}_{g_a}(a,x)$, $x\in B^a_a(\varrho_0)$, ϱ_0 is as in (11) and $C_n(|\beta|)$ is a large positive constant which depends only on n and $|\beta|$. Furthermore, recalling the assumption $\mathcal{Z}_g=\partial M$, we have that $v_{a,\epsilon,\delta}$ verifies the following energy estimate which is a weak form (but sufficient for the purpose of this paper) of a combination either of Proposition 3.11 in [**3**] and Proposition 3.9 in [**3**] or of Proposition 3.11 in [**3**] and Proposition 9 in [**34**] (for a reader who feels comfortable with the property $H_a=0$).

Lemma 2.2. *There exists* $0<\delta_0\leq\varrho_0$ *small such that for every* $0<2\epsilon\leq\delta\leq\delta_0$ *and for every* $a\in\partial M$*, there holds*

$$\begin{aligned}\langle L_{g_a}v_{a,\epsilon,\delta},v_{a,\epsilon,\delta}\rangle+\langle B_{g_a}v_{a,\epsilon,\delta},v_{a,\epsilon,\delta}\rangle&\\ \leq \mathcal{Q}(B^n)\left(\oint_{\partial M}v_{a,\epsilon,\delta}^{\frac{2(n-1)}{n-2}}dS_{g_a}\right)^{\frac{n-2}{n-1}}&-\epsilon^{n-2}\mathcal{I}(a,\delta)\\ +O_{a,\epsilon,\delta}(\delta\epsilon^{n-2}+\epsilon^{n-1}\delta^{-n+1}),&\end{aligned}$$

where $\mathcal{Q}(B^n)$ *is defined by* (8)*,* $\mathcal{I}(a,\delta)$ *is a flux integral verifying* $\mathcal{I}(a,\delta)=\mathcal{I}(a)+o_{a,\delta}(1)$*,* $\mathcal{I}(a)=O_a(1)$*, and for the meaning of* $o_{a,\delta}(1)$ $O_a(1)$ *and* $O_{a,\epsilon,\delta}(1)$*, see Section* 2*.*

On the other hand, using the work of Almaraz[**3**] (Lemma 3.15 in [**3**]), we have that the $v_{a,\epsilon,\delta}$'s verify the following interaction estimates.

Lemma 2.3. *There exists a large constant* $C_1>0$ *such that for every* $2\epsilon_1\leq 2\epsilon_2\leq\delta^2\leq\delta_0$ *and every* $a_1,a_2\in\partial M$*, there holds*

$$\begin{aligned}&\int_M v_{a_1,\epsilon_1,\delta}\left|L_{g_{a_2}}v_{a_2,\epsilon_2,\delta}\right|dV_{g_{a_2}}\\ &\quad+\oint_{\partial M}v_{a_1,\epsilon_1,\delta}\left|B_{g_{a_2}}v_{a_2,\epsilon_2,\delta}-c_0v_{a_2,\epsilon_2,\delta}^{\frac{n}{n-2}}\right|dS_{g_{a_2}}\\ &\qquad\leq C_1\left(\delta+\frac{\epsilon_2}{\delta}\right)\left(\frac{\epsilon_2^2+d^2_{g_{a_2}}(a_1,a_2)}{\epsilon_1\epsilon_2}\right)^{\frac{2-n}{2}},\end{aligned}$$

where c_0 *is defined by* (15)*.*

Furthermore, using (25), it is easy to see that the following estimate holds:

Lemma 2.4. *Assuming that* $0<\epsilon\leq\delta_0^n$ *and* $a\in\partial M$*, then we have*

$$\oint_{\partial M}v_{a,\epsilon,\epsilon^{\frac{1}{n}}}^{\frac{2(n-1)}{n-2}}dS_{g_a}=c_1+o_{a,\epsilon}(1),$$

where c_1 *is as in* (15)*, and for the meaning of* $o_{a,\epsilon}(1)$*, see Section* 2*.*

Thus, setting

$$v_a^\lambda := v_{a,\frac{1}{\lambda},(\frac{1}{\lambda})^{\frac{1}{n}}}, \quad a \in \partial M, \quad \lambda \geq \frac{1}{\delta_0^n}, \tag{27}$$

and

$$\varphi_{a,\lambda} := u_a v_a^\lambda, \quad a \in \partial M, \quad \lambda \geq \frac{1}{\delta_0^n}, \tag{28}$$

where u_a is as in (10) and δ_0 is still given by Lemma 2.2, we have clearly that Lemma 2.2, Lemma 2.3 and Lemma 2.4 combined with (14) imply the following lemmata which will play an important role in our application of the barycenter technique of Bahri–Coron[**14**]:

Lemma 2.5. *Assuming that* $a \in \partial M$ *and* $\lambda \geq \frac{1}{\delta_0^n}$, *then the following estimate holds:*

$$\mathcal{E}_g(\varphi_{a,\lambda}) \leq \mathcal{Q}(B^n)\left(1 + O_{a,\lambda}\left(\frac{1}{\lambda^{n-2}}\right)\right),$$

where $\mathcal{Q}(B^n)$ *is as in* (8) *and for the meaning of* $O_{a,\lambda}(1)$, *see Section* 2.

Lemma 2.6. *There exists a large constant* $C_2 > 0$ *such that for every* $a_1, a_2 \in \partial M$ *and for every* $\lambda \geq \frac{1}{\delta_0^n}$, *we have*

$$\begin{aligned}\int_M \varphi_{a_1,\lambda}\left|L_{g_{a_2}}\varphi_{a_2,\lambda}\right| dV_{g_{a_2}} + \oint_{\partial M} \varphi_{a_1,\lambda}\left|B_{g_{a_2}}\varphi_{a_2,\lambda} - c_0\varphi_{a_2,\lambda}^{\frac{n}{n-2}}\right| dS_{g_{a_2}} \\ \leq C_1\left(\frac{1}{\lambda}\right)^{\frac{1}{n}}\left(1 + \lambda^2 d_{g_{a_2}}^2(a_1,a_2)\right)^{\frac{2-n}{2}},\end{aligned}$$

where c_0 *is as in* (15).

Lemma 2.7. *Assuming that* $a \in \partial M$ *and* $\lambda \geq \frac{1}{\delta_0^n}$, *then there holds*

$$\oint_{\partial M} \varphi_{a,\lambda}^{\frac{2(n-1)}{n-2}} dS_{g_a} = c_1 + o_{a,\lambda}(1),$$

where c_1 *is as in* (15) *and for the meaning of* $o_{a,\lambda}(1)$, *see Section* 2.

On the other hand, using (25)–(26) and (27), we have that v_a^λ decomposes as follows:

$$v_a^\lambda(\cdot) = \chi^\lambda(\cdot)\left(\delta_a^\lambda(\cdot) + w_a^\lambda(\cdot)\right) + \left(1 - \chi^\lambda(\cdot)\right)\frac{G_a(a,\cdot)}{\lambda^{\frac{n-2}{2}}}, \tag{29}$$

where

$$w_a^\lambda := w_{a,\frac{1}{\lambda}}, \quad \delta_a^\lambda := \delta_{a,\frac{1}{\lambda}}, \quad \text{and} \quad \chi^\lambda = \chi_{(\frac{1}{\lambda})^{\frac{1}{n}}}, \tag{30}$$

and w_a^λ satisfies the following pointwise estimate:

$$|\partial^\beta w_a^\lambda(x)| \leq C_n(|\beta|)\frac{\lambda^{\frac{n-6}{2}+\beta}}{(1+\lambda^2 r^2)^{\frac{n-4+\beta}{2}}}, \tag{31}$$

where $r = \bar{d}_{g_a}(a,x)$ and $x \in B^a_a(\varrho_0)$. Now, for $p \in \mathbb{N}^*$, and $A := (a_1, \cdots, a_p) \in (\partial M)^p$ and $\lambda \geq \frac{1}{\delta_0^n}$, we associate the following quantities:

$$\epsilon_{i,j} := \epsilon_{i,j}(A,\lambda) := \oint_{\partial M} \varphi_{a_i,\lambda}^{\frac{n}{n-2}} \varphi_{a_j,\lambda} dS_g, \quad i,j = 1,\cdots,p, \;\; i \neq j. \tag{32}$$

Using (28), (29)–(32), we have the following lemma which provides self- and interaction estimates and a relation between $\epsilon_{i,j}(A,\lambda)$ and $\varepsilon_{i,j}(A,\bar{\lambda})$ with $\bar{\lambda} := (\lambda, \cdots, \lambda)$; for the meaning of $\varepsilon_{i,j}(A,\bar{\lambda})$ see (18).

Lemma 2.8. *Assuming that $p \in \mathbb{N}^*$, $A := (a_1, \cdots, a_p) \in (\partial M)^p$ and $\lambda \geq \frac{1}{\delta_0^n}$, then*
1) For every $i,j = 1, \cdots, p$ with $i \neq j$, we have
i)

$$\epsilon_{i,j} \longrightarrow 0 \Longleftrightarrow \varepsilon_{i,j} \longrightarrow 0,$$

where $\varepsilon_{i,j} := \varepsilon_{i,j}(A,\bar{\lambda})$ with $\bar{\lambda} := (\lambda,\cdots,\lambda)$ and $\epsilon_{i,j} := \epsilon_{i,j}(A,\lambda)$, and for their definitions see, respectively, (18) *and* (32).
ii) There exists $0 < C_3 < \infty$ independent of p, A and λ such that the following estimate holds:

$$C_3^{-1} < \frac{\epsilon_{i,j}}{\varepsilon_{i,j}} < C_3.$$

iii) If $\varepsilon_{i,j} \longrightarrow 0$, then

$$\epsilon_{i,j} = (1 + o_{\varepsilon_{i,j}}(1))\varepsilon_{i,j},$$

and for the meaning of $o_{\varepsilon_{i,j}}(1)$, see Section 2.
2) For every $i = 1, \cdots, p$, there holds

$$\langle L_g \varphi_{a_i,\lambda}, \varphi_{a_i,\lambda}\rangle + \langle B_g \varphi_{a_i,\lambda}, \varphi_{a_i,\lambda}\rangle = c_0(1 + o_{a_i,\lambda}(1)) \oint_{\partial M} \varphi_{a_i,\lambda}^{\frac{2(n-1)}{n-2}} dS_g,$$

where c_0 is given by (15) *and for the meaning of $o_{a_i,\lambda}(1)$, see Section* 2.
3) For every $i,j = 1,\cdots,p$ with $i \neq j$, there holds

$$\langle L_g \varphi_{a_i}, \varphi_{a_j}\rangle + \langle B_g \varphi_{a_i,\lambda}, \varphi_{a_j,\lambda}\rangle = (1 + o_{A_{i,j},\lambda}(1)) c_0 \epsilon_{i,j}, \quad \text{and}$$
$$\epsilon_{j,i} = (1 + o_{A_{i,j},\lambda}(1))\epsilon_{i,j},$$

where $A_{i,j} := (a_i, a_j)$ and for the meaning of $o_{A_{i,j},\lambda}(1)$, see Section 2.

Proof. To prove Lemma 2.8, we use the same strategy as in [**57**]. First of all, to simplify notation, for every $i = 1, \cdots, p$, we set

$$\begin{aligned} &\varphi_i := \varphi_{a_i,\lambda}, \;\; v_i := v^\lambda_{a_i}, \;\; \delta_i := \delta^\lambda_{a_i}, \;\; w_i := w^\lambda_{a_i}, \;\; G_i(\cdot) := G_{a_i}(a_i, \cdot), \\ &B_i := \hat{B}^{a_i}_{a_i}(\delta_0), \quad \exp_i := \hat{\exp}^{a_i}_{a_i} \;\text{ and }\; u_i := u_{a_i}, \end{aligned} \tag{33}$$

and for the meaning of $\hat{B}^{a_i}_{a_i}(\delta_0)$ and $\hat{\exp}^{a_i}_{a_i}$, see Section 2. Next, using (31) and (33), we have that v_i verifies the following pointwise estimate:

$$v_i = (1 + o_{a_i,\lambda}(1)) \left(\chi^\lambda \delta_i + (1 - \chi^\lambda) \frac{G_i}{\lambda^{\frac{n-2}{2}}}\right).$$

Moreover, using (9), (12), (30) and (33), we obtain (on ∂M)

$$\chi^\lambda \delta_i = \chi^\lambda \left(\frac{\lambda}{1+\lambda^2 r^2}\right)^{\frac{n-2}{2}} = \chi^\lambda \left(\frac{\lambda}{1+\lambda^2 G_i^{\frac{2}{2-n}} \frac{r^2}{G_i^{\frac{2}{2-n}}}}\right)^{\frac{n-2}{2}} \tag{34}$$
$$= (1+o_{a_i,\lambda}(1))\, \chi^\lambda \left(\frac{\lambda}{1+\lambda^2 G_i^{\frac{2}{2-n}}}\right)^{\frac{n-2}{2}},$$

where r is as in (26) with a replaced by a_i, and

(35)

$$\left(1-\chi^\lambda\right)\left(\frac{\lambda}{1+\lambda^2 G_i^{\frac{2}{2-n}}}\right)^{\frac{n-2}{2}} = \left(1-\chi^\lambda\right)\frac{G_i}{\lambda^{\frac{n-2}{2}}}\left(\frac{1}{1+O_{a_i,\lambda}(|\frac{1}{\lambda}|^{\frac{2(n-1)}{n}})}\right)$$
$$= (1+o_{a_i,\lambda}(1))\left(1-\chi^\lambda\right)\frac{G_i}{\lambda^{\frac{n-2}{2}}}.$$

Hence, combining (34) and (35), we obtain

$$v_i = (1+o_{a_i,\lambda}(1))\left(\frac{\lambda}{1+\lambda^2 G_i^{\frac{2}{2-n}}}\right)^{\frac{n-2}{2}}. \tag{36}$$

Now, using (11), (28), (32), (33) and (36), we derive the following estimate for $\epsilon_{i,j}$ $(i,j=1,\cdots,p$ and $i\neq j)$:

$$\epsilon_{i,j} = \oint_{\partial M} v_i^{\frac{n}{n-2}} v_j \frac{u_j}{u_i} dS_{g_{a_i}} = \int_{B_i} v_i^{\frac{n}{n-2}} v_j \frac{u_j}{u_i} dS_{g_{a_i}} + O_{A_{i,j},\lambda}(\lambda^{1-n})$$
$$= \left(1+o_{A_{i,j},\lambda}(1)\right) u_j(a_i) \tag{37}$$
$$\int_{B^\lambda(0)} \left(\frac{1}{1+|x|^2}\right)^{\frac{n}{2}} \left[\frac{1}{1+\lambda^2 G_j^{\frac{2}{2-n}}(\exp_i(\frac{x}{\lambda}))}\right]^{\frac{n-2}{2}} dx$$
$$+ O_{A_{i,j},\lambda}(\lambda^{1-n}),$$

where

$$B^\lambda(0) := B_0^{\mathbb{R}^{n-1}}(\lambda\delta_0),$$

and for the meaning of $B_0^{\mathbb{R}^{n-1}}(\lambda\delta_0)$, see Section 2. From (37) it follows that:

$$\epsilon_{i,j} \longrightarrow 0 \Longleftrightarrow \lambda^2 G_j^{\frac{2}{2-n}}(a_i) \longrightarrow +\infty \Longleftrightarrow \varepsilon_{i,j} \longrightarrow 0. \tag{38}$$

Thus, we have that the proof of i) of point 1) is complete. Now, since $\epsilon_{i,j}$ and $\varepsilon_{i,j}$ are bounded by definition, then thanks to (38), to prove

ii) of point 1), we can assume without loss of generality that

$$\lambda^2 G_j^{\frac{2}{2-n}}(a_j) \gg 1. \tag{39}$$

Thus under the latter assumption, setting

$$\mathcal{A} = B_0^{\mathbb{R}^{n-1}}(\gamma\lambda\sqrt{G_j^{\frac{2}{2-n}}(a_i)}),$$

for $\gamma > 0$ small and using Taylor expansion, we obtain that the following estimate holds on $\mathcal{A}$:

(40)

$$\begin{aligned}
&\left(\frac{1}{1+\lambda^2 G_j^{\frac{2}{2-n}}(\exp_i(\frac{x}{\lambda}))}\right)^{\frac{n-2}{2}} \\
&= \left(\frac{1}{1+\lambda^2 G_j^{\frac{2}{2-n}}(a_i)}\right)^{\frac{n-2}{2}} \left(\frac{1}{1+\frac{\lambda^2\left[G_j^{\frac{2}{2-n}}(\exp_i(\frac{x}{\lambda}))-G_j^{\frac{2}{2-n}}(a_i)\right]}{1+\lambda^2 G_j^{\frac{2}{2-n}}(a_i)}}\right)^{\frac{n-2}{2}} \\
&= \left(\frac{1}{1+\lambda^2 G_j^{\frac{2}{2-n}}(a_i)}\right)^{\frac{n-2}{2}} \left(\frac{1}{1+\frac{\lambda\nabla G_j^{\frac{2}{2-n}}(a_i)x+O(|x|^2)}{1+\lambda^2 G_j^{\frac{2}{2-n}}(a_i)}}\right)^{\frac{n-2}{2}} \\
&= \left(\frac{1}{1+\lambda^2 G_j^{\frac{2}{2-n}}(a_i)}\right)^{\frac{n-2}{2}} - \frac{n-2}{2}\lambda\left(\frac{1}{1+\lambda^2 G_j^{\frac{2}{2-n}}(a_i)}\right)^{\frac{n}{2}} \nabla G_j^{\frac{2}{2-n}}(a_i)x \\
&\quad + O\left(\frac{|x|^2}{(1+\lambda^2 G_j^{\frac{2}{2-n}}(a_i))^{\frac{n}{2}}}\right).
\end{aligned}$$

Now, combining (37) and (40), we obtain

$$\begin{aligned}
\epsilon_{i,j} =&(1+o_{A_{i,j},\lambda}(1))\frac{u_j(a_i)c_3}{\left(1+\lambda^2 G_j^{\frac{2}{2-n}}(a_i)\right)^{\frac{n-2}{2}}} + o_{\varepsilon_{i,j}}(\varepsilon_{i,j}) \\
&+ \int_{\mathcal{A}^c\cap B^\lambda} \left(\frac{1}{1+|x|^2}\right)^{\frac{n}{2}} \left[\left(\frac{1}{1+\lambda^2 G_j^{\frac{2}{2-n}}}\right)^{\frac{n-2}{2}}\right] \circ \exp_i(\frac{x}{\lambda})\, dx.
\end{aligned} \tag{41}$$

Next, using (14), (39) and Taylor expansion, we derive that

(42)
$$\begin{aligned}\frac{u_j(a_i)c_3}{\left(1+\lambda^2 G_j^{\frac{2}{2-n}}(a_i)\right)^{\frac{n-2}{2}}} =& c_3\left(1+o_{\varepsilon_{i,j}}(1)\right)u_j(a_i)\frac{G_j(a_i)}{\lambda^{n-2}}\\ =& c_3\left(1+o_{\varepsilon_{i,j}}(1)\right)\frac{G(a_i,a_j)}{\lambda^{n-2}} = \left(1+o_{\varepsilon_{i,j}}(1)\right)\varepsilon_{i,j}.\end{aligned}$$

Thus, combining (38), (39), (41) and (42), we obtain

$$\epsilon_{i,j} = \left(1+o_{\varepsilon_{i,j}}(1)\right)\varepsilon_{i,j} + I_{\mathcal{A}^c},$$

where

$$I_{\mathcal{A}^c} = \int_{\mathcal{A}^c\cap B^\lambda(0)}\left(\frac{1}{1+|x|^2}\right)^{\frac{n}{2}}\left[\left(\frac{1}{1+\lambda^2 G_j^{\frac{2}{2-n}}}\right)^{\frac{n-2}{2}}\right]\circ \exp_i\left(\frac{x}{\lambda}\right)dx,$$

and $\mathcal{A}^c = \mathbb{R}^{n-1}\setminus\mathcal{A}$. Hence, to end the proof of ii) of point 1) and to prove iii) of point 1), we are going to show that $I_{\mathcal{A}^c}$ satisfies

$$I_{\mathcal{A}^c} = o_{\varepsilon_{i,j}}(\varepsilon_{i,j}). \tag{43}$$

In order to do that, we first decompose $\mathcal{A}^c$ into

$$\mathcal{B} = \{x\in\mathbb{R}^{n-1}:\ \gamma\lambda\sqrt{G_j^{\frac{2}{2-n}}(a_i)} \le |x| \le \gamma^{-1}\lambda\sqrt{G_j^{\frac{2}{2-n}}(a_i)}\}, \tag{44}$$

and

$$\mathcal{C} = \{x\in\mathbb{R}^{n-1}:\ |x| > \gamma^{-1}\lambda\sqrt{G_j^{\frac{2}{2-n}}(a_i)}\},$$

and have

$$I_{\mathcal{A}^c} = I_{\mathcal{B}} + I_{\mathcal{C}},$$

where

$$I_{\mathcal{B}} := \int_{\mathcal{B}\cap B^\lambda(0)}\left(\frac{1}{1+|x|^2}\right)^{\frac{n}{2}}\left[\left(\frac{1}{1+\lambda^2 G_j^{\frac{2}{2-n}}}\right)^{\frac{n-2}{2}}\right]\circ \exp_i\left(\frac{x}{\lambda}\right)dx, \tag{45}$$

and

$$I_{\mathcal{C}} := \int_{\mathcal{C}\cap B^\lambda(0)}\left(\frac{1}{1+|x|^2}\right)^{\frac{n}{2}}\left[\left(\frac{1}{1+\lambda^2 G_j^{\frac{2}{2-n}}}\right)^{\frac{n-2}{2}}\right]\circ \exp_i\left(\frac{x}{\lambda}\right)dx. \tag{46}$$

To prove (43), we are going to estimate separately $I_{\mathcal{B}}$ and $I_{\mathcal{C}}$. We start with $I_{\mathcal{B}}$. Using (44) and (45), we have clearly that $I_{\mathcal{B}}$ verifies the following estimate:

(47)

$$I_{\mathcal{B}} \leq \frac{C_\gamma}{\left(1+\lambda^2 G_j^{\frac{2}{2-n}}(a_i)\right)^{\frac{n}{2}}} \times \int_{\mathcal{B}\cap B^\lambda} \left[\left(\frac{1}{1+\lambda^2 G_j^{\frac{2}{2-n}}}\right)^{\frac{n-2}{2}}\right] \circ \exp_i(\frac{x}{\lambda})dx,$$

for some large positive constant C_γ depending only on γ. Thus, rescaling and changing coordinates via $\exp_j \circ \exp_i^{-1}$ (if necessary), we have that (47) implies

$$I_{\mathcal{B}} \leq \hat{C}_\gamma \varepsilon_{i,j}^{\frac{n}{n-2}} \int_{[|x|\leq \lambda \tilde{C}_\gamma d_{\hat{g}}(a_i,a_j)]} \left(\frac{1}{1+|x|^2}\right)^{\frac{n-2}{2}} dx \leq \bar{C}_\gamma \varepsilon_{i,j}^{\frac{n-1}{n-2}}, \tag{48}$$

for some large positive constants $\hat{C}_\gamma$, $\tilde{C}_\gamma$ and $\bar{C}_\gamma$ which are depending only on γ. Finally, we estimate $I_{\mathcal{C}}$. To do that, we fix $\gamma > 0$ sufficiently small and use (38), (39) and (46) to obtain

$$I_{\mathcal{C}} \leq \frac{C_\gamma}{\left(1+\lambda^2 G_j^{\frac{2}{2-n}}(a_i)\right)^{\frac{n-2}{2}}} \int_{\mathcal{C}} \left(\frac{1}{1+|x|^2}\right)^{\frac{n}{2}} dx = o_{\varepsilon_{i,j}}(\varepsilon_{i,j}), \tag{49}$$

for some large constant C_γ depending only on γ. Hence (48) and (49) imply (43), thereby ending the proof of point 1). On the other hand, we have clearly that point 2) follows from Lemma 2.6 and Lemma 2.7. Furthermore, the first equation of point 3) follows from Lemma 2.6 and ii) of point 1), while the second equation follows from the first equation and from the self-adjointness of (L_g, B_g). q.e.d.

Now, using (12), (28), (29)–(31), we have the following interaction type estimate:

Lemma 2.9. *Assuming that* $p \in \mathbb{N}^*$, $A := (a_1, \cdots, a_p) \in (\partial M)^p$ *and* $\lambda \geq \frac{1}{\delta_0^n}$, *then for every* $i, j = 1, \cdots, p$ *with* $i \neq j$, *there holds*

$$\oint_{\partial M} \varphi_{a_i,\lambda}^{\frac{n-1}{n-2}} \varphi_{a_j,\lambda}^{\frac{n-1}{n-2}} dS_g = O_{A_{i,j},\lambda}(\varepsilon_{i,j}^{\frac{n-1}{n-2}} \log \varepsilon_{i,j}), \tag{50}$$

where $A_{i,j} = (a_i, a_j)$, $\varepsilon_{i,j} := \varepsilon_{i,j}(A, \bar{\lambda})$ *with* $\bar{\lambda} := (\lambda, \cdots, \lambda)$, *and for the meaning of* $O_{A_{i,j},\lambda}(1)$ *and* $\epsilon_{i,j}(A, \bar{\lambda})$, *see, respectively, Section* 2 *and* (18).

Proof. Using (12), (28), (29)–(31) and setting $\varphi_i = \varphi_{a_i,\lambda}$ for $i = 1, \cdots, p$, we have that for every $i = 1, \cdots, p$, the following estimate holds:

$$\varphi_i \leq C\left(\frac{\lambda}{1+\lambda^2 d^2_{\hat{g}}(a_i,\cdot)}\right)^{\frac{n-2}{2}} \quad \text{on } \partial M, \tag{51}$$

for some large positive constant independent of a_i and λ with $\hat{g}$ denoting the Riemannian metric induced by g on ∂M. Hence, using (51), we have for $c > 0$ and small that the following estimate holds:

(52)

$$\begin{aligned}
&\oint_{\partial M} \varphi_i^{\frac{n-1}{n-2}} \varphi_j^{\frac{n-1}{n-2}} dS_g \\
&\leq C \int_{B_0^{\mathbb{R}^{n-1}}(c)} \left(\frac{\lambda}{1+\lambda^2|x|^2}\right)^{\frac{n-1}{2}} \left(\frac{\lambda}{1+\lambda^2 d^2_{\hat{g}}(a_j, \exp_i(x))}\right)^{\frac{n-1}{2}} dx \\
&\quad + C\frac{1}{\lambda^{\frac{n-1}{2}}} \int_{B_0^{\mathbb{R}^{n-1}}(c)} \left(\frac{\lambda}{1+\lambda^2|x|^2}\right)^{\frac{n-1}{2}} dx + O_{A_{i,j},\lambda}\left(\frac{1}{\lambda^{n-1}}\right) \\
&= C \int_{B_0^{\mathbb{R}^{n-1}}(c\lambda)} \left(\frac{1}{1+r^2}\right)^{\frac{n-1}{2}} \left(\frac{1}{1+\lambda^2 d^2_{\hat{g}}(a_j, \exp_i(\frac{x}{\lambda}))}\right)^{\frac{n-1}{2}} dx \\
&\quad + C\frac{1}{\lambda^{n-1}} \int_{B_0^{\mathbb{R}^{n-1}}(c\lambda)} \left(\frac{1}{1+|x|^2}\right)^{\frac{n-1}{2}} dx + O_{A_{i,j},\lambda}\left(\frac{1}{\lambda^{n-1}}\right),
\end{aligned}$$

for some large positive constant C independent of $A_{i,j}$ and λ with $r = |x|$ and $\exp_i := \hat{\exp}_{a_i}$ (for its meaning see Section 2). Thus appealing to (52), we infer that

(53)

$$\begin{aligned}
&\oint_{\partial M} \varphi_i^{\frac{n-1}{n-2}} \varphi_j^{\frac{n-1}{n-2}} dS_g \\
&\leq C \int_{B_0^{\mathbb{R}^{n-1}}(c\lambda)} \left(\frac{1}{1+|x|^2}\right)^{\frac{n-1}{2}} \left(\frac{1}{1+\lambda^2 d^2_{\hat{g}}(a_j, \exp_i(\frac{x}{\lambda}))}\right)^{\frac{n-1}{2}} dx \\
&\quad + O_{A_{i,j},\lambda}\left(\frac{\log\lambda}{\lambda^{n-1}}\right).
\end{aligned}$$

Thus (50) follows from (53) if

$$d_{\hat{g}}(a_i, a_j) \geq 3c.$$

Hence, to complete the proof of the lemma it remains to treat the case $d_{\hat{g}}(a_i, a_j) < 3c$. To do that, we set

$$\mathcal{B} = \{x \in \bar{\mathbb{R}}^{n-1} : \quad \frac{1}{2} d_{\hat{g}}(a_i, a_j) \leq |\frac{x}{\lambda}| \leq 2 d_{\hat{g}}(a_i, a_j)\},$$

and use (53) and the triangle inequality to get for $c > 0$ sufficiently small that the following estimate holds:

$$\begin{aligned}
&\oint_{\partial M} \varphi_i^{\frac{n-1}{n-2}} \varphi_j^{\frac{n-1}{n-2}} dS_g \\
&\leq C \int_{\mathcal{B}} \left(\frac{1}{1+|x|^2}\right)^{\frac{n-1}{2}} \left(\frac{1}{1+\lambda^2 d_{\hat{g}}^2(a_j, \exp_i(\frac{x}{\lambda}))}\right)^{\frac{n-1}{2}} dx \\
&\quad + O_{A_{i,j},\lambda}(\varepsilon_{i,j}^{\frac{n-1}{n-2}} \log \varepsilon_{i,j}) \\
&\leq C \left(\frac{1}{1+|\lambda d_{\hat{g}}(a_i, a_j)|^2}\right)^{\frac{n-1}{2}} \int_{\{|\frac{x}{\lambda}| \leq 4 d_{\hat{g}}(a_i, a_j)\}} \left(\frac{1}{1+|x|^2}\right)^{\frac{n-1}{2}} dx \\
&\quad + O_{A_{i,j},\lambda}(\varepsilon_{i,j}^{\frac{n-1}{n-2}} \log \varepsilon_{i,j}) \\
&= O_{A_{i,j},\lambda}(\varepsilon_{i,j}^{\frac{n-1}{n-2}} \log \varepsilon_{i,j}),
\end{aligned}$$

where C is a large positive constant independent of $A_{i,j}$ and λ, thereby completing the proof of the lemma. q.e.d.

3. Energy estimates for the barycenter technique

In this section, we map $B_p(\partial M)$ into some appropriate sublevels of the Euler–Lagrange functional $\mathcal{E}_g$ via the Almaraz[**3**]–Chen[**34**]'s bubbles. Precisely, we are going to derive sharp energy estimates for convex combinations of the bubbles $\varphi_{a,\lambda}$ given by (28) so that we can use them in the next section to run a suitable scheme of the barycenter technique of Bahri–Coron[**14**]. In order to do that, we first make the following definition. For $p \in \mathbb{N}^*$, $\sigma := \sum_{i=1}^p \alpha_i \delta_{a_i} \in B_p(\partial M)$ and $\lambda \geq \frac{1}{\delta_0^n}$, where δ_0 is given by Lemma 2.2, we define $f_p(\lambda) : B_p(\partial M) \longrightarrow W_+^{1,2}(\overline{M})$ as follows:

$$f_p(\lambda)(\sigma) := \sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda}. \tag{54}$$

We start this section with the following proposition which provides the first step to apply our scheme of the algebraic topological argument of Bahri–Coron[**14**].

Proposition 3.1. *There exists a large constant* $C_0 > 0$, $\nu_0 > 1$ *and* $0 < \varepsilon_0 \leq \delta_0$ *such that for every* $p \in \mathbb{N}^*$, $p \geq 2$ *and every* $0 < \varepsilon \leq \varepsilon_0$, *there exists* $\lambda_p := \lambda_p(\varepsilon) := \lambda_p(\nu_0, \varepsilon) \geq \frac{1}{\delta_0^n}$ *such that for every* $\lambda \geq \lambda_p$

and for every $\sigma = \sum_{i=1}^{p} \alpha_i \delta_{a_i} \in B_p(\partial M)$*, we have*
1) If there exist $i_0 \neq j_0$ *such that* $\frac{\alpha_{i_0}}{\alpha_{j_0}} > \nu_0$ *or if* $\sum_{i \neq j} \varepsilon_{i,j} > \varepsilon$*, then*

$$\mathcal{E}_g(f_p(\lambda)(\sigma)) \leq p^{\frac{1}{n-1}} \mathcal{Q}(B^n),$$

where $\mathcal{Q}(B^n)$ *is defined by* (8)*,* $\varepsilon_{i,j} := \varepsilon_{i,j}(A, \bar{\lambda})$ *with* $\bar{\lambda} := (\lambda, \cdots, \lambda)$ *and for the definition of* $\varepsilon(A, \bar{\lambda})$*, see* (18)*.*
2) If for every $i \neq j$ *we have* $\frac{\alpha_i}{\alpha_j} \leq \nu_0$ *and if* $\sum_{i \neq j} \varepsilon_{i,j} \leq \varepsilon$*, then*

$$\mathcal{E}_g(f_p(\lambda)(\sigma)) \leq p^{\frac{1}{n-1}} \mathcal{Q}(B^n) \left(1 + \frac{C_0}{\lambda^{n-2}} - c_g \frac{(p-1)}{\lambda^{n-2}}\right),$$

where c_g *is defined by* (16)*.*

As in [**57**], Proposition 3.1 will be derived from the following technical lemma:

Lemma 3.2. *We have that the following holds:*
1) For every $\epsilon > 0$ *and small and for every* $p \in \mathbb{N}^*$ *with* $p \geq 2$*, there exists* $\lambda_p := \lambda_p(\epsilon) \geq \frac{1}{\delta_0^n}$ *such that for every* $\lambda \geq \lambda_p$ *and for every* $\sigma := \sum_{i=1}^{p} \alpha_i \delta_{a_i} \in B_p(\partial M)$*, we have that*

$$\sum_{i \neq j} \epsilon_{i,j} > \epsilon$$

implies

$$\mathcal{E}_g(f_p(\lambda)(\sigma)) < p^{\frac{1}{n-1}} \mathcal{Q}(B^n),$$

where $\epsilon_{i,j} := \epsilon_{i,j}(A, \lambda)$ *is defined by* (32)*.*
2) For every $\nu > 1$*, for every* $\epsilon > 0$ *and small and for every* $p \in \mathbb{N}^*$ *with* $p \geq 2$*, there exists* $\lambda_p := \lambda_p(\epsilon, \nu) \geq \frac{1}{\delta_0^n}$ *such that for every* $\lambda \geq \lambda_p$ *and for every* $\sigma := \sum_{i}^{p} \alpha_i \delta_{a_i} \in B_p(\partial M)$*, we have*

$$\exists\, i_0 \neq j_0 \ \textit{such that} \ \frac{\alpha_{i_0}}{\alpha_{j_0}} > \nu \quad \textit{and} \quad \sum_{i \neq j} \epsilon_{i,j} \leq \epsilon$$

imply

$$\mathcal{E}_g(f_p(\lambda)(\sigma)) < p^{\frac{1}{n-1}} \mathcal{Q}(B^n).$$

3) There exists $C_0 > 0$*,* $\nu_0 > 1$*,* $\lambda_0 \geq \frac{1}{\delta_0^n}$ *and* $0 < \epsilon_0 \leq \delta_0$ *such that for every* $1 < \nu \leq \nu_0$*, for every* $0 < \epsilon \leq \epsilon_0$*, for every* $p \in \mathbb{N}^*$ *with* $p \geq 2$*, for every* $\lambda \geq \lambda_0$ *and for every* $\sigma := \sum_{i=1}^{p} \alpha_i \delta_{a_i} \in B_p(\partial M)$*, we have*

$$\frac{\alpha_i}{\alpha_j} \leq \nu \quad \forall i, j, \quad \textit{and} \quad \sum_{i \neq j} \epsilon_{i,j} \leq \epsilon$$

imply

$$\mathcal{E}_g(f_p(\lambda)(\sigma)) \leq p^{\frac{1}{n-1}} \mathcal{Q}(B) \left(1 + \frac{C_0}{\lambda^{n-2}} - c_g \frac{(p-1)}{\lambda^{n-2}}\right).$$

Proof. The strategy of the proof is the same as the one of Lemma 3.2 in [**57**]. For the sake of completeness we will provide full details. First of all, we set

$$\mathcal{N}_g(u) := \langle L_g u, u\rangle + \langle B_g u, u\rangle, \;\; \mathcal{D}_g(u) := \left(\oint_{\partial M} u^{\frac{2(n-1)}{n-2}} dS_g\right)^{\frac{n-2}{n-1}}, \tag{55}$$

for $u \in W^{1,2}_+(\overline{M})$ and use (1) to have

$$\mathcal{E}_g(u) = \frac{\mathcal{N}_g(u)}{\mathcal{D}_g(u)}, \quad u \in W^{1,2}_+(\overline{M}). \tag{56}$$

Furthermore, for $p \in \mathbb{N}^*$, $\sigma := \sum_{i=1}^p \alpha_i \delta_{a_i} \in B_p(\partial M)$ and $\lambda \geq \frac{1}{\delta_0^n}$, we set (as in the proof of Lemma 2.8)

$$\varphi_i = \varphi_{a_i,\lambda}, \;\; i = 1, \cdots, p. \tag{57}$$

Now, we start with the proof of point 1). To do so, we first use Lemma 2.8, (54), (55), (57) and Hölder's inequality to estimate $\mathcal{N}_g(f_p(\lambda)(\sigma))$ as follows:

(58)

$$\begin{aligned}\mathcal{N}_g(f_p(\lambda)(\sigma)) = c_0(1+o_{A,\bar{\alpha},\lambda}(1)) \oint_{\partial M} \left(\sum_{i=1}^p \alpha_i \varphi_i^{\frac{n}{n-2}}\right)\left(\sum_{j=1}^p \alpha_j\varphi_j\right) dS_g \\ = c_0(1+o_{A,\bar{\alpha},\lambda}(1)) \oint_{\partial M} \left(\frac{\sum_{i=1}^p \alpha_i \varphi_i^{\frac{n}{n-2}}}{\sum_{j=1}^p \alpha_j \varphi_j}\right)\left(\sum_{j=1}^p \alpha_j\varphi_j\right)^2 dS_g \\ \leq c_0(1+o_{A,\bar{\alpha},\lambda}(1)) \mathcal{D}_g(f_p(\lambda)(\sigma)) \left\|\frac{\sum_{i=1}^p \alpha_i \varphi_i^{\frac{n}{n-2}}}{\sum_{j=1}^p \alpha_j \varphi_j}\right\|_{L^{n-1}(\partial M)},\end{aligned}$$

where $A := (a_1, \cdots, a_p)$, $\bar{\alpha} := (\alpha_1, \cdots, \alpha_p)$ and for the meaning of $o_{A,\bar{\alpha},\lambda}(1)$, see Section 2. Thus, using the convexity of the map $x \longrightarrow x^\beta$ with $\beta > 1$, we derive that (58) implies

$$\begin{aligned}\mathcal{N}_g(f_p(\lambda)(\sigma)) \leq & c_0(1+o_{A,\bar{\alpha},\lambda}(1))\mathcal{D}_g(f_p(\lambda)(\sigma)) \\ & \left(\oint_M \left(\sum_{i=1}^p \frac{\alpha_i\varphi_i}{\sum_{j=1}^p \alpha_j\varphi_j}\varphi_i^{\frac{2}{n-2}}\right)^{n-1} dS_g\right)^{\frac{1}{n-1}} \\ \leq & c_0(1+o_{A,\bar{\alpha},\lambda}(1))\mathcal{D}_g(f_p(\lambda)(\sigma)) \\ & \left(\sum_{i=1}^p \oint_{\partial M} \frac{\alpha_i\varphi_i}{\sum_{j=1}^p \alpha_j\varphi_j}\varphi_i^{\frac{2(n-1)}{n-2}} dS_g\right)^{\frac{1}{n-1}}.\end{aligned} \tag{59}$$

Hence, clearly Lemma 2.7, (56) and (59) imply for any pair $i \neq j$ $(i, j = 1, \cdots, p)$

$$
\begin{aligned}
\mathcal{E}_g(f_p(\lambda)(\sigma)) \leq & c_0(1 + o_{A,\bar{\alpha},\lambda}(1)) \\
& \left(c_1(p-1) + \oint_{\partial M} \frac{\alpha_i \varphi_i}{\alpha_i \varphi_i + \alpha_j \varphi_j} \varphi_i^{\frac{2(n-1)}{n-2}} dS_g \right)^{\frac{1}{n-1}} \\
\leq & c_0(1 + o_{A,\bar{\alpha},\lambda}(1)) \\
& \left(c_1 p - \oint_{\partial M} \frac{\alpha_j \varphi_j}{\alpha_i \varphi_i + \alpha_j \varphi_j} \varphi_i^{\frac{2(n-1)}{n-2}} dS_g \right)^{\frac{1}{n-1}},
\end{aligned} \tag{60}
$$

and we may assume $\alpha_i \leq \alpha_j$ by symmetry. Now, we are going to estimate from below the quantity $\oint_{\partial M} \frac{\alpha_j \varphi_j}{\alpha_i \varphi_i + \alpha_j \varphi_j} \varphi_i^{\frac{2(n-1)}{n-2}} dS_g$. In order to do that, for $\gamma > 0$, we set

$$
\mathcal{A}_{i,j} = \{x \in \partial M : \ \varphi_i(x) \geq \gamma(\frac{\alpha_i}{\alpha_j}\varphi_i(x) + \varphi_j(x))\}, \tag{61}
$$

and use (61) to have

$$
\begin{aligned}
& \oint_{\partial M} \frac{\alpha_j \varphi_j \varphi_i^{\frac{2(n-1)}{n-2}}}{\alpha_i \varphi_i + \alpha_j \varphi_j} dS_g \geq \oint_{\mathcal{A}_{i,j}} \frac{\varphi_j}{\frac{\alpha_i}{\alpha_j}\varphi_i + \varphi_j} \varphi_i^{\frac{2(n-1)}{n-2}} dS_g \\
\geq & \gamma \oint_{\mathcal{A}_{i,j}} \varphi_i^{\frac{n}{n-2}} \varphi_j dS_g \\
= & \gamma \left(\oint_{\partial M} \varphi_i^{\frac{n}{n-2}} \varphi_j dS_g - \oint_{\mathcal{A}^c_{i,j}} \varphi_i^{\frac{n}{n-2}} \varphi_j dS_g \right) \\
\geq & \gamma \left(\oint_{\partial M} \varphi_i^{\frac{n}{n-2}} \varphi_j dS_g - \gamma^{\frac{2}{n-2}} \oint_{\mathcal{A}^c_{i,j}} \left(\frac{\alpha_i}{\alpha_j}\varphi_i + \varphi_j \right)^{\frac{2}{n-2}} \varphi_i \varphi_j dS_g \right),
\end{aligned} \tag{62}
$$

where $\mathcal{A}^c_{i,j} := \partial M \setminus \mathcal{A}_{i,j}$. Next, since $\frac{\alpha_i}{\alpha_j} \leq 1$, then appealing to (62), we infer that the following estimate holds:

$$
\begin{aligned}
& \oint_{\partial M} \frac{\alpha_j \varphi_j \varphi_i^{\frac{2(n-1)}{n-2}}}{\alpha_i \varphi_i + \alpha_j \varphi_j} dS_g \\
& \quad \geq \gamma \left(\oint_{\partial M} \varphi_i^{\frac{n}{n-2}} \varphi_j dS_g - C\gamma^{\frac{2}{n-2}} \oint_{\partial M} (\varphi_i^{\frac{2}{n-2}} + \varphi_j^{\frac{2}{n-2}}) \varphi_i \varphi_j dS_g \right),
\end{aligned} \tag{63}
$$

for some large positive constant C independent of A, λ and γ. Thus, ii) of point 1) and point 3) of Lemma 2.8 together with (63) imply that for $\gamma > 0$ sufficiently small, there holds

$$
\oint_{\partial M} \frac{\alpha_j \varphi_j}{\alpha_i \varphi_i + \alpha_j \varphi_j} \varphi_i^{\frac{2(n-1)}{n-2}} dS_g \geq \frac{\gamma}{2} \int_{\partial M} \varphi_i^{\frac{n}{n-2}} \varphi_j dS_g. \tag{64}
$$

Hence, combining (17), (60) and (64), we conclude that for any pair $i \neq j$, the following estimate holds:

$$\begin{aligned} &\mathcal{E}_g(f_p(\lambda)(\sigma)) \\ &\leq (1+o_{A,\bar{\alpha},\lambda}(1))\, \mathcal{Q}(B^n) \left(p - \frac{\gamma}{2c_1} \oint_{\partial M} \varphi_i^{\frac{n}{n-2}} \varphi_j dS_g\right)^{\frac{1}{n-1}}. \end{aligned} \tag{65}$$

Clearly (65) implies, that we always have

$$\mathcal{E}_g(f_p(\lambda)(\sigma)) \leq (1+o_{A,\bar{\alpha},\lambda}(1))\, p^{\frac{1}{n-1}}\, \mathcal{Q}(B^n),$$

and in case $\sum_{i\neq j} \epsilon_{i,j} > \epsilon$

$$\mathcal{E}_g(f_p(\lambda)(\sigma)) \leq (1+o_{A,\bar{\alpha},\lambda}(1))\, p^{\frac{1}{n-1}}\, \mathcal{Q}(B^n) \left(1 - \frac{\gamma\epsilon}{2pc_1}\right)^{\frac{1}{n-1}},$$

thereby ending the proof of point 1). Now, we are going to treat the second case. Hence, we may assume

$$\sum_{i\neq j} \epsilon_{i,j} \ll 1,$$

and thus according to Lemma 2.8

$$\epsilon_{i,j} = (1+o_{\varepsilon_{i,j}}(1))\varepsilon_{i,j} \ \text{ and } \ \lambda d_{\hat{g}}(a_i,a_j) \gg 1, \tag{66}$$

and for the meaning of $o_{\epsilon_{i,j}}(1)$, see Section 2. We then use Lemma 2.8, (55) and (66) to have

$$\begin{aligned} \mathcal{N}_g(f_p(\lambda)(\sigma)) &= \sum_{i=1}^p \sum_{j=1}^p \alpha_i\alpha_j \left(\langle L_g\varphi_i, \varphi_j\rangle + \langle B_g\varphi_i, \varphi_j\rangle\right) \\ &= \sum_{i=1}^p \alpha_i^2 \left(\langle L_g\varphi_i, \varphi_i\rangle + \langle B_g\varphi_i, \varphi_i\rangle\right) \\ &\quad + \sum_{i\neq j} \alpha_i\alpha_j \left(\langle L_g\varphi_i, \varphi_j\rangle + \langle B_g\varphi_i, \varphi_j\rangle\right) \\ &= \sum_i \alpha_i^2 \mathcal{E}_g(\varphi_i)\mathcal{D}_g(\varphi_i) + c_0(1+o_{\sum_{i,\neq j}\varepsilon_{i,j}}(1)) \sum_{i\neq j} \alpha_i\alpha_j\varepsilon_{i,j}, \end{aligned} \tag{67}$$

and (for the meaning of $o_{\sum_{i\neq j}\epsilon_{i,j}}(1)$, see Section 2)

$$\begin{aligned} \mathcal{D}_g^{\frac{n-1}{n-2}}(f_p(\lambda)(\sigma)) &= \oint_{\partial M} \left(\sum_{i=1}^p \alpha_i\varphi_i\right)^{\frac{2(n-1)}{n-2}} dS_g \\ &= \sum_{i=1}^p \alpha_i \oint_{\partial M} \left(\sum_{j=1}^p \alpha_j\varphi_j\right)^{\frac{n}{n-2}} \varphi_i dS_g \end{aligned}$$

$$= \sum_{i=1}^{p} \alpha_i \oint_{\partial M} \left(\alpha_i \varphi_i + \sum_{j=1,\, j\neq i}^{p} \alpha_j \varphi_j \right)^{\frac{n}{n-2}} \varphi_i dS_g.$$

To proceed further, we set

$$\mathcal{A}_i = \{x \in \partial M : \alpha_i \varphi_i(x) > \sum_{j=1,\, j\neq i}^{p} \alpha_j \varphi_j(x)\},$$

and use Taylor expansion to obtain

$$\begin{aligned} \mathcal{D}_g^{\frac{n-1}{n-2}}(f_p(\lambda)(\sigma)) = & \sum_{i=1}^{p} \alpha_i^{\frac{2(n-1)}{n-2}} \oint_{\mathcal{A}_i} \varphi_i^{\frac{2(n-1)}{n-2}} dS_g \\ & + \frac{n}{n-2} \sum_{i\neq j} \alpha_i^{\frac{n}{n-2}} \alpha_j \oint_{\mathcal{A}_i} \varphi_i^{\frac{n}{n-2}} \varphi_j dS_g \\ & + \sum_{i=1}^{p} \alpha_i \oint_{\mathcal{A}_i^c} \left(\sum_{j=1,\, j\neq i}^{p} \alpha_j \varphi_j \right)^{\frac{n}{n-2}} \varphi_i dS_g \\ & + O_{A,\bar{\alpha},\lambda} \left(\sum_{i\neq j} \alpha_i^{\frac{n-1}{n-2}} \alpha_j^{\frac{n-1}{n-2}} \oint_{\partial M} \varphi_i^{\frac{n-1}{n-2}} \varphi_j^{\frac{n-1}{n-2}} dS_g \right), \end{aligned} \tag{68}$$

where $\mathcal{A}_i^c := \partial M \setminus \mathcal{A}_i$, $A := (a_1, \cdots, a_p)$, $\bar{\alpha} := (\alpha_1, \cdots, \alpha_p)$, $O_{A,\bar{\alpha},\lambda}(1)$ is defined as in Section 2, and we made use of $n \geq 3$ and the algebraic relation

$$(a+b)^{\frac{n-1}{n-2}} \leq C_n (a^{\frac{n-1}{n-2}} + b^{\frac{n-1}{n-2}}),$$

for $a, b \geq 0$ and C_n a positive constant depending only on n. Moreover, since

$$(a+b)^{\frac{n}{n-2}} \geq a^{\frac{n}{n-2}} + b^{\frac{n}{n-2}}, \tag{69}$$

then (68) and (69) implies

$$\begin{aligned} \mathcal{D}_g^{\frac{n-1}{n-2}}(f_p(\lambda)(\sigma)) \geq & \sum_{i=1}^{p} \alpha_i^{\frac{2(n-1)}{n-2}} \oint_{\partial M} \varphi_i^{\frac{2(n-1)}{n-2}} dS_g \\ & + \frac{n}{n-2} \sum_{i\neq j} \alpha_i^{\frac{n}{n-2}} \alpha_j \oint_{\partial M} \varphi_i^{\frac{n}{n-2}} \varphi_j dS_g \\ & + \sum_{i\neq j} \alpha_i \alpha_j^{\frac{n}{n-2}} \oint_{A_i^c} \varphi_j^{\frac{n}{n-2}} \varphi_i dS_g \\ & + O_{A,\lambda} \left(\sum_{i\neq j} \alpha_i^{\frac{n-1}{n-2}} \alpha_j^{\frac{n-1}{n-2}} \oint_{\partial M} \varphi_i^{\frac{n-1}{n-2}} \varphi_j^{\frac{n-1}{n-2}} dS_g \right) \end{aligned} \tag{70}$$

$$\geq \sum_{i=1}^{p} \alpha_i^{\frac{2(n-1)}{n-2}} \oint_{\partial M} \varphi_i^{\frac{2(n-1)}{n-2}} dS_g$$
$$+ 2\frac{n-1}{n-2} \sum_{i\neq j} \alpha_i^{\frac{n}{n-2}} \alpha_j \oint_{\partial M} \varphi_i^{\frac{n}{n-2}} \varphi_j dS_g$$
$$+ O_{A,\bar{\alpha},\lambda} \left(\sum_{i\neq j} \alpha_i^{\frac{n-1}{n-2}} \alpha_j^{\frac{n-1}{n-2}} \oint_{\partial M} \varphi_i^{\frac{n-1}{n-2}} \varphi_j^{\frac{n-1}{n-2}} dS_g \right).$$

So, using Lemma 2.9 and (66), we have that (70) implies

$$(71) \quad \begin{aligned} \mathcal{D}_g^{\frac{n-1}{n-2}}(f_p(\lambda)(\sigma)) \geq & \sum_{i=1}^{p} \alpha_i^{\frac{2(n-1)}{n-2}} \oint_{\partial M} \varphi_i^{\frac{2(n-1)}{n-2}} dS_g \\ & + \frac{2(n-1)}{n-2} \left(1 + o_{\sum_{i\neq j}\varepsilon_{i,j}}(1)\right) \sum_{i\neq j} \alpha_i^{\frac{n}{n-2}} \alpha_j \varepsilon_{i,j} \\ & + o_{\sum_{i\neq j}\varepsilon_{i,j}} \left(\sum_{i\neq j} \alpha_i^{\frac{n-1}{n-2}} \alpha_j^{\frac{n-1}{n-2}} \varepsilon_{i,j} \right). \end{aligned}$$

Thus, using Young's inequality and the symmetry of $\varepsilon_{i,j}$, we infer from (71) that the following estimate holds:

$$(72) \quad \begin{aligned} \mathcal{D}_g^{\frac{n-1}{n-2}}(f_p(\lambda)(\sigma)) \geq & \sum_{i=1}^{p} \alpha_i^{\frac{2(n-1)}{n-2}} \oint_{\partial M} \varphi_i^{\frac{2(n-1)}{n-2}} dS_g \\ & + \frac{2(n-1)}{n-2} (1 + o_{\sum_{i\neq j}\varepsilon_{i,j}}(1)) \sum_{i\neq j} \alpha_i^{\frac{n}{n-2}} \alpha_j \varepsilon_{i,j}. \end{aligned}$$

Hence, using again Young's inequality, Taylor expansion and Lemma 2.7, we have that (72) gives

$$\mathcal{D}_g(f_p(\lambda)(\sigma))$$
$$\geq \left(\sum_{i=1}^{p} \alpha_i^{\frac{2(n-1)}{n-2}} \oint_{\partial M} \varphi_i^{\frac{2(n-1)}{n-2}} dS_g \right)^{\frac{n-2}{n-1}}$$
$$\left(1 + \frac{\frac{2(n-1)}{n-2} \left(1 + o_{\sum_{i\neq j}\varepsilon_{i,j}}(1)\right) \sum_{i\neq j} \alpha_i^{\frac{n}{n-2}} \alpha_j \varepsilon_{i,j}}{\sum_{i=1}^{p} \alpha_i^{\frac{2(n-1)}{n-2}} \oint_{\partial M} \varphi_i^{\frac{2(n-1)}{n-2}} dS_g} \right)^{\frac{n-2}{n-1}}$$
$$(73) \quad = \left(\sum_{i=1}^{p} \alpha_i^{\frac{2(n-1)}{n-2}} \oint_{\partial M} \varphi_i^{\frac{2(n-1)}{n-2}} dS_g \right)^{\frac{n-2}{n-1}}$$

$$+2\frac{\left(1+o_{\sum_{i\neq j}\varepsilon_{i,j}}(1)\right)\sum_{i\neq j}\alpha_i^{\frac{n}{n-2}}\alpha_j\varepsilon_{i,j}}{\left(\sum_{i=1}^p\alpha_i^{\frac{2(n-1)}{n-2}}\oint_{\partial M}\varphi_i^{\frac{2(n-1)}{n-2}}dS_g\right)^{\frac{1}{n-1}}}$$

$$=\left(\sum_{i=1}^p\alpha_i^{\frac{2(n-1)}{n-2}}\mathcal{D}_g^{\frac{n-1}{n-2}}(\varphi_i)\right)^{\frac{n-2}{n-1}}$$

$$+2c_1^{-\frac{1}{n-1}}\frac{\left(1+o_{\sum_{i\neq j}\varepsilon_{i,j}}(1)\right)\sum_{i\neq j}\alpha_i^{\frac{n}{n-2}}\alpha_j\varepsilon_{i,j}}{\left(\sum_{i=1}^p\alpha_i^{\frac{2(n-1)}{n-2}}\right)^{\frac{1}{n-1}}}.$$

Now letting

$$\mathcal{G}=\sum_{i=1}^p\alpha_i^2\mathcal{E}_g(\varphi_i)\mathcal{D}_g(\varphi_i)+c_0\left(1+o_{\sum_{i\neq j}\varepsilon_{i,j}}(1)\right)\sum_{i\neq j}\alpha_i\alpha_j\varepsilon_{i,j},$$

and

$$\mathcal{H}=\left(\sum_{i=1}^p\alpha_i^{\frac{2(n-1)}{n-2}}\mathcal{D}_g^{\frac{n-1}{n-2}}(\varphi_i)\right)^{\frac{n-2}{n-1}}$$

$$+2c_1^{-\frac{1}{n-1}}\frac{\left(1+o_{\sum_{i\neq j}\varepsilon_{i,j}}(1)\right)\sum_{i\neq j}\alpha_i^{\frac{n}{n-2}}\alpha_j\varepsilon_{i,j}}{\left(\sum_{i=1}^p\alpha_i^{\frac{2(n-1)}{n-2}}\right)^{\frac{1}{n-1}}},$$

we find from combining (67) and (73) via Taylor expansion

$$\mathcal{E}_g(f_p(\lambda)(\sigma))\leq\frac{\mathcal{H}}{\mathcal{G}}$$

$$=\frac{\sum_{i=1}^p\mathcal{E}_g(\varphi_i)\alpha_i^2\mathcal{D}_g(\varphi_i)}{\left(\sum_{i=1}^p\alpha_i^{\frac{2(n-1)}{n-2}}\mathcal{D}_g^{\frac{n-1}{n-2}}(\varphi_i)\right)^{\frac{n-2}{n-1}}}$$

$$+\frac{c_0}{c_1^{\frac{n-2}{n-1}}}\frac{\left(1+o_{\sum_{i\neq j}\varepsilon_{i,j}}(1)\right)\sum_{i\neq j}\alpha_i\alpha_j\varepsilon_{i,j}}{\left(\sum_{i=1}^p\alpha_i^{\frac{2(n-1)}{n-2}}\right)^{\frac{n-2}{n-1}}} \tag{74}$$

$$-\frac{2c_0}{c_1^{\frac{n-2}{n-1}}}\frac{\left(1+o_{\sum_{i\neq j}\varepsilon_{i,j}}(1)\right)\left(\sum_{i=1}^p\alpha_i^2\right)\left(\sum_{i\neq j}\alpha_i^{\frac{n}{n-2}}\alpha_j\varepsilon_{i,j}\right)}{\left(\sum_{i=1}^p\alpha_i^{\frac{2(n-1)}{n-2}}\right)^{\frac{2n-3}{n-1}}}.$$

Hence, using (17) and rearranging the terms in (74), we get

$$\mathcal{E}_g(f_p(\lambda)(\sigma)) \leq \max_{i=1,\cdots,p} \mathcal{E}_g(\varphi_i) \frac{\sum_{i=1}^p \alpha_i^2 \mathcal{D}_g(\varphi_i)}{\left(\sum_{i=1}^p \left(\alpha_i^2 \mathcal{D}_g(\varphi_i)\right)^{\frac{n-1}{n-2}}\right)^{\frac{n-2}{n-1}}}$$

$$+ \frac{\left(1 + o_{\sum_{i\neq j}\varepsilon_{i,j}}(1)\right) \mathcal{Q}(B^n)}{\left(\sum_{i=1}^p \alpha_i^{\frac{2(n-1)}{n-2}}\right)^{\frac{n-2}{n-1}}} \tag{75}$$

$$\left(\sum_{i\neq j}\left[1 - 2\frac{\alpha_i^{\frac{2}{n-2}}\left(\sum_{k=1}^p \alpha_k^2\right)}{\left(\sum_{k=1}^p \alpha_k^{\frac{2(n-1)}{n-2}}\right)}\right]\alpha_i\alpha_j\frac{\varepsilon_{i,j}}{c_1}\right).$$

This inequality has the following impact. First note for the first summand of (75), that the function

$$\Gamma : \{\gamma \in [0,1]^p : \sum_{i=1}^p \gamma_i = 1\} \longrightarrow \mathbb{R}_+,$$

$$\gamma \longrightarrow \frac{\sum_{i=1}^p \gamma_i^2}{(\sum_{i=1}^p \gamma_i^{\frac{2(n-1)}{n-2}})^{\frac{n-2}{n-1}}}, \tag{76}$$

has the strict global maximum

$$\gamma_{\max} = \left(\frac{1}{p}, \ldots, \frac{1}{p}\right), \tag{77}$$

with $\Gamma(\gamma_{\max}) = p^{\frac{1}{n-1}}$. Thus, using Lemma 2.7, Lemma 2.5, (55), (66) and (75), we infer that for any $\nu > 0$, for every $\epsilon > 0$ and small and for every $p \in \mathbb{N}^*$, there exists $\lambda_p := \lambda_p(\nu, \epsilon) \geq \frac{1}{\delta_0^n}$ such for every $\lambda \geq \lambda_p$ and for every $\sigma := \sum_{i=1}^p \alpha_i \delta_{a_i} \in B_p(\partial M)$, there holds

$$\mathcal{E}_g(f_p(\lambda)(\sigma)) < p^{\frac{1}{n-1}} \mathcal{Q}(B^n),$$

whenever

$$\exists i_o \neq j_0 \quad \text{such that} \quad \frac{\alpha_{i_0}}{\alpha_{j_0}} > \nu \quad \text{and} \quad \sum_{i\neq j}\varepsilon_{i,j} \leq \epsilon,$$

thereby ending the proof of point 2). Now, we are going to treat point 3) and end the proof of the Lemma. Thus, we may assume

$$\forall\ i,j \ \ \frac{\alpha_i}{\alpha_j} \leq 1 + o_\mu^+(1) \quad \text{and} \quad \sum_{i\neq j}\epsilon_{i,j} \ll 1, \tag{78}$$

where $o_\mu^+(1)$ is a positive quantity depending only μ with μ small and verifying the property that it tends to 0 as μ tends to 0. So, using (75), (78) and the properties of Γ (see (76) and (77)), we infer that the following estimate holds:

$$\begin{aligned}\mathcal{E}_g(f_p(\lambda)(\sigma)) & \\ \leq & \max_{i=1,\cdots,p} \mathcal{E}_g(\varphi_i)p^{\frac{1}{n-1}} \\ & - (1 + o_{\sum_{i\neq j}\varepsilon_{i,j}}(1) + o_\mu(1))\frac{\mathcal{Q}(B^n)\sum_{i\neq j}\alpha_i\alpha_j\frac{\varepsilon_{i,j}}{c_1}}{(\sum_{i=1}^p \alpha_i^{\frac{2(n-1)}{n-2}})^{\frac{n-2}{n-1}}} \\ = & \max_{i=1,\cdots,p} \mathcal{E}_g(\varphi_i)p^{\frac{1}{n-1}} \\ & - (1 + o_{\sum_{i\neq j}\varepsilon_{i,j}}(1) + o_\mu(1))\mathcal{Q}(B^n)p^{\frac{2-n}{n-1}}\sum_{i\neq j}\frac{\varepsilon_{i,j}}{c_1}.\end{aligned} \tag{79}$$

Now, using Lemma 2.5, (66), (78) and (79), we have that there exists $C_0 > 0$, $\nu_0 > 1$, $\lambda_0 \geq \frac{1}{\delta_0^n}$ and $0 < \epsilon_0 \leq \delta_0$ such that for every $1 < \nu \leq \nu_0$, for every $0 < \epsilon \leq \epsilon_0$, for every $p \in \mathbb{N}^*$, for every $\lambda \geq \lambda_0$ and for every $\sigma := \sum_{i=1}^p \alpha_i\delta_{a_i} \in B_p(\partial M)$, we have if $\frac{\alpha_i}{\alpha_j} \leq \nu \quad \forall i, j$ and $\sum_{i\neq j}\epsilon_{i,j} \leq \epsilon$, then there holds

$$\mathcal{E}_g(f_p(\lambda)(\sigma)) \leq p^{\frac{1}{n-1}}\mathcal{Q}(B^n)\left(1 + \frac{C_0}{\lambda^{n-2}} - \frac{1}{2c_1p}\sum_{i\neq j}\varepsilon_{i,j}\right). \tag{80}$$

Thus, recalling that (see (42))

$$\varepsilon_{i,j} = (1 + o_{\varepsilon_{i,j}}(1))c_3\frac{G(a_i, a_j)}{\lambda^{n-2}},$$

and using again (66), we infer from (80) that up to taking ϵ_0 smaller, for every $1 < \nu \leq \nu_0$, for every $0 < \epsilon \leq \epsilon_0$, for every $p \in \mathbb{N}^*$, for every $\lambda \geq \lambda_0$ and for every $\sigma := \sum_{i=1}^p \alpha_i\delta_{a_i} \in B_p(\partial M)$, there holds

$$\frac{\alpha_i}{\alpha_j} \leq \nu \quad \forall i, j \quad \text{and} \quad \sum_{i\neq j}\epsilon_{i,j} \leq \epsilon$$

imply

$$\begin{aligned}\mathcal{E}_g(f_p(\lambda)(\sigma)) \leq & p^{\frac{1}{n-1}}\mathcal{Q}(B^n)\left(1 + \frac{C_0}{\lambda^{n-2}} - \frac{c_3}{4c_1p\lambda^{n-2}}\sum_{i\neq j}G(a_i, a_j)\right) \\ \leq & p^{\frac{1}{n-1}}\mathcal{Q}(B^n)\left(1 + \frac{C_0}{\lambda^{n-2}} - c_g\frac{(p-1)}{\lambda^{n-2}}\right),\end{aligned}$$

thereby ending the proof of point 3), hence of the Lemma. q.e.d.

Proof of Proposition 3.1. It follows from Lemma 3.2 by taking C_0 and ν_0 to be the ones given by Lemma 3.2, while $\varepsilon_0 := \frac{\epsilon_0}{2}$ and

$$\lambda_p := \lambda_p(\varepsilon, \nu_0) := \max\{\lambda_p(\frac{\varepsilon}{2}), \lambda_p(2\varepsilon, \nu_0), \lambda_0\},$$

where ϵ_0, $\lambda_p(\frac{\varepsilon}{2})$, $\lambda_p(2\varepsilon, \nu_0)$ and λ_0 are as in Lemma 3.2. q.e.d.

Now, using Proposition 3.1, we have the following corollary which will be used together with Proposition 3.1 in the next section to carry a suitable algebraic topological argument of Bahri–Coron[**14**].

Corollary 3.3. *There exists* $p_0 \in \mathbb{N}^*$ *large enough such that for every* $0 < \varepsilon \leq \varepsilon_0$ *and for every for every* $\lambda \geq \lambda_{p_0}$ *(where* ε_0 *and* λ_{p_0} *are given by Proposition* 3.1*), there holds*

$$f_{p_0}(\lambda)(B_{p_0}(\partial M)) \subset W_{p_0-1}.$$

Proof. It follows directly from Proposition 3.1 and the definition of W_{p_0-1} (see (7) with p replaced by $p_0 - 1$). q.e.d.

4. Application of Bahri–Coron's barycenter technique

In this section, we are going to use directly the results of the previous one to run a suitable scheme of the barycenter technique of Bahri–Coron[**14**]. To do so, we first introduce the notion of *neighborhood of potential critical points at infinity* of the Euler–Lagrange functional $\mathcal{E}_g$. Precisely, for $p \in \mathbb{N}^*$, $0 < \varepsilon \leq \varepsilon_0$ (where ε_0 is given by Proposition 3.1), we define $V(p, \varepsilon)$ the (p, ε)-neighborhood of potential critical points at infinity of $\mathcal{E}_g$, namely

$$V(p,\varepsilon) := \{u \in W^{1,2}(\overline{M}) \mid \exists a_1, \cdots, a_p \in \partial M,\ \alpha_1, \cdots, \alpha_p > 0,$$
$$\lambda_1, \cdots, \lambda_p \geq \frac{1}{\varepsilon} \ :\ ||u - \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i}|| \leq \varepsilon, \frac{\alpha_i}{\alpha_j} \leq \nu_0$$
$$\text{and for } i \neq j \ :\ \varepsilon_{i,j} \leq \varepsilon\},$$

where $|| \cdot ||$ denotes the standard $W^{1,2}$-norm, $\varepsilon_{i,j} := \varepsilon_{i,j}(A, \bar{\lambda})$ with $A := (a_1, \cdots, a_p)$, $\bar{\lambda} := (\lambda_1, \cdots, \lambda_p)$, $(\varepsilon_{i,j}(A, \bar{\lambda}))$'s are defined by (18), $\varphi_{a_i,\lambda}$ is given by (28) for $i = 1, \cdots, p$ and ν_0 is given by Proposition 3.1.

Concerning the sets $V(p, \varepsilon)$, for every $p \in \mathbb{N}^*$, we have that there exists $0 < \varepsilon_p \leq \varepsilon_0$ such that for every $0 < \varepsilon \leq \varepsilon_p$, we have that

(81) $\forall u \in V(p, \varepsilon)$ the minimization problem $\displaystyle\min_{B_{C_p\varepsilon}} ||u - \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i}||$

has a unique solution, up to permutations, where $B_{C_p\varepsilon}$ is defined as

$$B_{C_p\varepsilon} := \{(\bar{\alpha}, A, \bar{\lambda}) \in \mathbb{R}_+^p \times (\partial M)^p \times (0, +\infty)^p \mid,$$
$$\lambda_i \geq \frac{C_p}{\epsilon},\ \frac{\alpha_i}{\alpha_j} \leq \nu_0 \ \text{ and for } i \neq j \ :\ \varepsilon_{i,j} \leq C_p \varepsilon p\},$$

and $C_p > 1$. Furthermore, we define the selection map

$$s_p : V(p, \varepsilon) \longrightarrow (\partial M)^p / \sigma_p,$$

as follows:

(82) $s_p(u) := A, \ \ u \in V(p, \varepsilon),$ and A is given by (81).

Now having introduced the neighborhoods of potential critical points at infinity of the Euler–Lagrange functional $\mathcal{E}_g$, we are ready to present our algebraic topological argument for existence. In order to do that, we start by the following classical deformation lemma which follows from the same arguments as for its counterparts in classical application of the algebraic topological argument of Bahri–Coron[**14**] (see, for example, Proposition 6 in [**14**]) and the fact that the $\varphi_{a,\lambda}$ can replace the standard bubbles in the analysis of diverging PS sequences of the Euler–Lagrange functional $\mathcal{E}_g$.

Lemma 4.1. *Assuming that $\mathcal{E}_g$ has no critical points, then for every $p \in \mathbb{N}^*$, up to taking ε_p smaller (where ε_p is given by* (81)*), we have that for every $0 < \varepsilon \leq \varepsilon_p$, there holds $(W_p,\, W_{p-1})$ retracts by deformation onto $(W_{p-1} \cup \mathcal{A}_p,\, W_{p-1})$ with $V(p,\, \tilde{\varepsilon}) \subset \mathcal{A}_p \subset V(p,\, \varepsilon)$ where $0 < \tilde{\varepsilon} < \frac{\varepsilon}{4}$ is a very small positive real number and depends on ε.*

Using Proposition 3.1 and Lemma 4.1, we are going to show that if $\mathcal{E}_g$ has no critical points, then for λ large enough, the map $(f_1(\lambda))_*$ is well defined and maps ∂M (in top homology) in a nontrivial way in $(W_1,\, W_0)$. Precisely, we show:

Lemma 4.2. *Assuming that $\mathcal{E}_g$ has no critical points and $0 < \varepsilon \leq \varepsilon_1$ (where ε_1 is given by* (81)*), then up to taking ε_1 smaller and λ_1 larger (where λ_1 is given by Proposition* 3.1*), we have that for every $\lambda \geq \lambda_1$*

$$f_1(\lambda) : \ (B_1(\partial M),\, B_0(\partial M)) \longrightarrow (W_1,\, W_0)$$

is well defined and satisfies

$$(f_1(\lambda))_*(w_1) \neq 0 \quad in \quad H_{n-1}(W_1,\, W_0).$$

Proof. It follows from the selection map s_1 given by (82), Lemma 2.5 and the same arguments as in Lemma 26 in [**45**]. q.e.d.

Next, as in [**57**], using Lemma 2.1, Proposition 3.1, Lemma 4.1 and the algebraic topological argument of Bahri–Coron[**14**], we are going to show that if for λ large $B_p(\partial M)$ (in top homology) survives "topologically" the embedding into $(W_p,\, W_{p-1})$ via $f_p(\lambda)$, then for λ large $B_{p+1}(\partial M)$ (in top homology and as a cone with base $B_{p-1}(\partial M)$ and top ∂M) survives "topologically" the embedding into $(W_{p+1},\, W_p)$ via $f_{p+1}(\lambda)$. Precisely, we prove the following proposition:

Proposition 4.3. *Assuming that $\mathcal{E}_g$ has no critical points and $0 < \varepsilon \leq \varepsilon_{p+1}$ (where ε_{p+1} is given by* (81)*), then up to taking ε_{p+1}*

smaller and λ_p *and* λ_{p+1} *larger (where* λ_p *and* λ_{p+1} *are given by Proposition* 3.1*), we have that for every* $\lambda \geq \max\{\lambda_p, \lambda_{p+1}\}$*, there holds*

$$f_{p+1}(\lambda) : (B_{p+1}(\partial M),\ B_p(\partial M)) \longrightarrow (W_{p+1},\ W_p),$$

and

$$f_p(\lambda) : (B_p(\partial M),\ B_{p-1}(\partial M)) \longrightarrow (W_p,\ W_{p-1})$$

are well defined and satisfy

$$(f_p(\lambda))_*(w_p) \neq 0 \quad in \quad H_{np-1}(W_p,\ W_{p-1})$$

implies

$$(f_{p+1}(\lambda))_*(w_{p+1}) \neq 0 \quad in \quad H_{n(p+1)-1}(W_{p+1},\ W_p).$$

Proof. First of all, we let $p \in \mathbb{N}^*$ and $0 < \varepsilon_{p+1}$, where ε_{p+1} is given by (81). Next, recalling that we have assumed that $\mathcal{E}_g$ has no critical points, and using Lemma 4.1, then up to taking ε_{p+1} smaller, we infer that the following holds:

$$(W_{p+1},\ W_p) \simeq (W_p \cup \mathcal{A}_{p+1},\ W_p), \tag{83}$$

with

$$V(p+1,\ \tilde{\varepsilon}) \subset \mathcal{A}_{p+1} \subset V(p+1,\ \varepsilon), \ \ 0 < 4\tilde{\varepsilon} < \varepsilon. \tag{84}$$

Now, using Lemma 2.8 and Proposition 3.1, we have that for every $\lambda \geq \max\{\lambda_p, \lambda_{p+1}\}$ (where λ_p and λ_{p+1} are given by Proposition 3.1), there holds

$$f_{p+1}(\lambda) : (B_{p+1}(\partial M),\ B_p(\partial M)) \longrightarrow (W_{p+1},\ W_p), \tag{85}$$

and

$$f_p(\lambda) : (B_p(\partial M),\ B_{p-1}(\partial M)) \longrightarrow (W_p,\ W_{p-1}) \tag{86}$$

are well defined and hence have that the first point is proven. Next, using Proposition 3.1, (85) and (86), we have that up to taking λ_{p+1} and λ_p larger (for example, larger than $4\max\{\lambda_{p+1}(\varepsilon),\ \lambda_p(\varepsilon),\ \lambda_p(2\tilde{\varepsilon}), \lambda_p(\frac{\tilde{\varepsilon}}{2}), \frac{1}{\tilde{\varepsilon}}\}$, where $\lambda_p(\varepsilon),\ \lambda_{p+1}(\varepsilon), \lambda_p(2\tilde{\varepsilon})$, and $\lambda_p(\frac{\tilde{\varepsilon}}{2})$ are given by Proposition 3.1 and $\tilde{\varepsilon}$ is given by (84)) the following diagram:

$$\begin{array}{ccc} (B_{p+1}(\partial M),\ \mathcal{O}(B_p(\partial M))) & \xrightarrow{f_{p+1}(\lambda)} & (W_{p+1},\ W_p) \\ \uparrow & & \uparrow \\ (\mathcal{O}(B_p(\partial M)),\ B_{p-1}(\partial M)) & \xrightarrow{f_p(\lambda)} & (W_p,\ W_{p-1}) \end{array} \tag{87}$$

is well defined and commutes, where

$$\mathcal{O}(B_p(\partial M)) := \{\sigma = \sum_{i=1}^{p+1} \alpha_i \delta_{a_i} \in B_{p+1}(\partial M) \mid$$
$$\exists\, i_0 \neq j_0 : \ \frac{\alpha_{i_0}}{\alpha_{j_0}} > \nu_0 \ \text{ or } \ \sum_{i \neq j} \varepsilon_{i,j} > \tilde{\varepsilon}\},$$

with ν_0 given by Proposition 3.1. On the other hand, we have

$$\mathcal{O}(B_p(\partial M)) \simeq B_p(\partial M), \tag{88}$$

and

$$B_{p+1}(\partial M) \setminus \mathcal{O}(B_p(\partial M)) \simeq B_{p+1}(\partial M) \setminus B_p(\partial M). \tag{89}$$

Now, using Lemma 2.1, we derive

$$\begin{aligned} &H^{n-1}((\partial M)^{p+1}/\sigma_{p+1}) \times H_{n(p+1)-1}(B_{p+1}(\partial M),\ B_p(\partial M)) \\ &\quad \stackrel{\frown}{\longrightarrow} H_{n(p+1)-n}(B_{p+1}(\partial M),\ B_p(\partial M)) \\ &\quad\quad \stackrel{\partial}{\longrightarrow} H_{n(p+1)-n-1}(B_p(\partial M),\ B_{p-1}(\partial M)). \end{aligned} \tag{90}$$

Furthermore, using (83), we infer that

$$\begin{aligned} &H^{n-1}(\mathcal{A}_{p+1}) \times H_{n(p+1)-1}(W_{p+1},\ W_p) \\ &\quad \stackrel{\frown}{\longrightarrow} H_{n(p+1)-n}(W_{p+1},\ W_p) \stackrel{\partial}{\longrightarrow} H_{n(p+1)-n-1}(W_p,\ W_{p-1}). \end{aligned} \tag{91}$$

Moreover, passing to homologies in (87) and using (89), we derive

$$\begin{aligned} &H_{n(p+1)-1}(B_{p+1}(\partial M),\ B_p(\partial M)) \\ &\quad \xrightarrow{(f_{p+1}(\lambda))_*} H_{n(p+1)-1}(W_{p+1},\ W_p), \end{aligned} \tag{92}$$

and

$$H_{np-1}(B_p(\partial M),\ B_{p-1}(\partial M)) \xrightarrow{(f_p(\lambda))_*} H_{np-1}(W_p,\ W_{p-1})$$

are well defined and the following diagram commutes:
(93)

$$\begin{array}{ccc} H_{n(p+1)-n}(B_{p+1}(\partial M),\ B_p(\partial M)) & \xrightarrow{(f_{p+1}(\lambda))_*} & H_{n(p+1)-n}(W_{p+1},\ W_p) \\ \partial\big\downarrow & & \partial\big\downarrow \\ H_{np-1}(B_p(\partial M),\ B_{p-1}(\partial M)) & \xrightarrow{(f_p(\lambda))_*} & H_{np-1}(W_p,\ W_{p-1}). \end{array}$$

Next, recalling that we have taken λ_{p+1} and λ_p larger than

$$4\max\{\lambda_{p+1}(\varepsilon),\ \lambda_p(\varepsilon),\ \lambda_p(2\tilde{\varepsilon}), \lambda_p(\frac{\tilde{\varepsilon}}{2}), \frac{1}{\tilde{\varepsilon}}\},$$

we derive that

$$\begin{aligned} &f_{p+1}(\lambda)\,(B_{p+1}(\partial M) \setminus \mathcal{O}(B_p(\partial M))) \\ &\quad \subset V(p+1,\ \tilde{\varepsilon}) \subset \mathcal{A}_{p+1} \subset V(p+1, \varepsilon). \end{aligned} \tag{94}$$

Thus, using (22), (82), (88), (89) and (94), we infer as in formula (6.3) in [**45**] that (see also [**12**] or [**14**])

$$\begin{aligned} &(f_{p+1}(\lambda))^*(s^*_{p+1}(O^*_{\partial M})) = O^*_{\partial M}, \quad \text{with} \\ &\quad s^*_{p+1}(O^*_{\partial M}) \neq 0 \quad \text{in} \quad H^{n-1}(\mathcal{A}_{p+1}). \end{aligned} \tag{95}$$

On the other hand, using (93), we derive that

$$\partial(f_{p+1}(\lambda))_* = (f_p(\lambda))_*\partial \quad \text{in} \quad H_{n(p+1)-n}(B_{p+1}(\partial M),\ B_p(\partial M)). \tag{96}$$

Now, combining Lemma 2.1, (88), (90), (91), (92), (95) and (96), we obtain

$$\begin{aligned}(f_p(\lambda))_*(\omega_p) &= (f_p(\lambda))_* \left(\partial(O^*_{\partial M} \frown \omega_{p+1})\right)\\ &= (f_p(\lambda))_* \left(\partial(((f_{p+1}(\lambda))^*(s^*_{p+1}(O^*_{\partial M}))) \frown \omega_{p+1})\right)\\ &= \partial\left((f_{p+1}(\lambda))_*((f_{p+1}(\lambda))^*(s^*_{p+1}(O^*_{\partial M})) \frown \omega_{p+1})\right)\\ &= \partial(s^*_{p+1}(O^*_{\partial M}) \frown ((f_{p+1}(\lambda))_*(\omega_{p+1}))),\end{aligned} \tag{97}$$

with all the equalities holding in $H_{np-1}(W_p,\ W_{p-1})$. Hence, clearly (97) and the assumption

$$(f_p(\lambda))_*(w_p) \neq 0 \quad \text{in} \quad H_{np-1}(W_p,\ W_{p-1})$$

imply

$$(f_{p+1}(\lambda))_*(w_{p+1}) \neq 0 \quad \text{in} \quad H_{n(p+1)-1}(W_{p+1},\ W_p),$$

as desired, thereby completing the proof of Proposition 4.3. q.e.d.

Now, we are ready to present the proof of Theorem 1.1.

Proof of Theorem 1.1. As in [**57**], it follows by a contradiction argument from Corollary 3.3, Lemma 4.2 and Proposition 4.3. q.e.d.

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