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G-INVARIANT HOLOMORPHIC MORSE INEQUALITIES

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Abstract

Consider an action of a connected compact Lie group on a compact complex manifold M, and two equivariant vector bundles L and E on M, with L of rank 1. The purpose of this paper is to establish holomorphic Morse inequalities à la Demailly for the invariant part of the Dolbeault cohomology of tensor powers of L twisted by E. To do so, we define a moment map μ by the Kostant formula and we define the reduction of M under a natural hypothesis on $\mu^{-1}(0)$. Our inequalities are given in term of the curvature of the bundle induced by L on this reduction.

0. Introduction

Morse Theory investigates the topological information carried by Morse functions on a manifold and in particular their critical points. Let f be a Morse function on a compact manifold of real dimension n. We suppose that f has isolated critical points. Let m_j $(0 \leq j \leq n)$ be the number of critical points of f of Morse index j, and let b_j be the Betti numbers of the manifold. Then the strong Morse inequalities states that for $0 \leq q \leq n$,

$$\sum_{j=0}^{q}(-1)^{q-j}b_j \leq \sum_{j=0}^{q}(-1)^{q-j}m_j, \tag{0.1}$$

with equality if $q = n$. From (0.1), we get the weak Morse inequalities:

$$b_j \leq m_j \qquad \text{for} \quad 0 \leq j \leq n. \tag{0.2}$$

In his seminal paper [**23**], Witten gave an analytic proof of the Morse inequalities by analyzing the spectrum of the Schrödinger operator $\Delta_t = \Delta + t^2|df|^2 + tV$, where $t > 0$ is a real parameter and V an operator of order 0. For $t \to +\infty$, Witten shows that the spectrum of Δ_t approaches in some sense the spectrum of a sum of harmonic oscillators attached to the critical point of f.

In [**7**], Demailly established analogous asymptotic Morse inequalities for the Dolbeault cohomology associated with high tensor powers $L^p :=$

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$L^{\otimes p}$ of a holomorphic Hermitian line bundle (L, h^L) over a compact complex manifold (M, J). The inequalities of Demailly give asymptotic bounds on the Morse sums of the Betti numbers of $\bar{\partial}$ on L^p in terms of certain integrals of the Chern curvature R^L of (L, h^L). More precisely, we define $\dot{R}^L \in \operatorname{End}(T^{(1,0)}M)$ by $g^{TM}(\dot{R}^L u, \overline{v}) = R^L(u, \overline{v})$ for $u, v \in T^{(1,0)}M$, where g^{TM} is a J-invariant Riemannian metric on TM. We denote by $M(\leq q)$ the set of points where $\dot{R}^L$ is non-degenerate and have at most q negative eigenvalues, and we set $n = \dim_{\mathbb{C}} M$. Then we have for $0 \leq q \leq n$

(0.3)
$$\sum_{j=0}^{q} (-1)^{q-j} \dim H^j(M, L^p) \leq \frac{p^n}{n!} \int_{M(\leq q)} (-1)^q \left(\frac{\sqrt{-1}}{2\pi} R^L \right)^n + o(p^n),$$

with equality if $q = n$. Here $H^j(M, L^p)$ denotes the Dolbeault cohomology in bidegree $(0, j)$, which is also the j-th group of cohomology of the sheaf of holomorphic sections of L^p.

These inequalities have found numerous applications. In particular, Demailly used them in [**7**] to find new geometric characterizations of Moishezon spaces, which improve Siu's solution in [**18**, **19**] of the Grauert–Riemenschneider conjecture [**11**]. Another notable application of the holomorphic Morse inequalities is the proof of the effective Matsusaka theorem by Siu [**20**, **9**]. Recently, Demailly used these inequalities in [**10**] to prove a significant step of a generalized version of the Green–Griffiths–Lang conjecture.

To prove these inequalities, the key remark of Demailly was that in the formula for the Kodaira Laplacian $\square_p$ associated with L^p, the metric of L plays formally the role of the Morse function in the paper by Witten [**23**], and that the parameter p plays the role of the parameter t. Then the Hessian of the Morse function becomes the curvature of the bundle. The proof of Demailly was based on the study of the semi-classical behavior as $p \to +\infty$ of the spectral counting functions of $\square_p$. Subsequently, Bismut gave an other proof of the holomorphic Morse inequalities in [**2**] by adapting his heat kernel proof of the Morse inequality [**1**]. The key point is that we can compare the left hand side of (0.3) with the alternate trace of the heat kernel acting on forms of degree $\leq q$, i.e.,

(0.4)
$$\sum_{j=0}^{q} (-1)^{q-j} \dim H^j(M, L^p) \leq \sum_{j=0}^{q} (-1)^{q-j} \operatorname{Tr}^{\Omega^{0,j}(M, L^p)} \left[\exp\Big(-\frac{u}{p} \square_p \Big) \right],$$

with equality if $q = n$. Then, Bismut obtained the holomorphic Morse inequalities by showing the convergence of the heat kernel thanks to probability theory. Demailly [**8**] and Bouche [**5**] gave an analytic approach of this result. In [**15**], Ma and Marinescu gave a new proof of this convergence, replacing the probabilistic arguments of Bismut [**2**] by arguments inspired by the analytic localization techniques of Bismut–Lebeau [**4**, Chap. 11].

When the bundle L is positive, (0.3) is a consequence of the Hirzebruch–Riemann–Roch theorem and of the Kodaira vanishing theorem, and reduces to

$$\dim H^0(M, L^p) = \frac{p^n}{n!}\int_M \Big(\frac{\sqrt{-1}}{2\pi}R^L\Big)^n + o(p^n). \tag{0.5}$$

In this case, a local estimate can be obtained by the study of the asymptotic of the Bergman kernel (the kernel of the orthogonal projection from $\mathscr{C}^\infty(M, L^p)$ onto $H^0(M, L^p)$) when $p \to +\infty$. We refer to [**15**] and the reference therein for the study of the Bergman kernel.

In the equivariant case, a connected compact Lie group G acts on M and its action lifts on L. When L is positive, Ma and Zhang [**16**] have studied the invariant Bergman kernel, i.e., the kernel of the projection from $\mathscr{C}^\infty(M, L^p)$ onto the G-invariant part of $H^0(M, L^p)$. Let μ be the moment map associated with the G-action on M (see (0.7)). Ma and Zhang [**16**] established that the invariant Bergman kernel concentrate to any neighborhood U of $\mu^{-1}(0)$, and that near $\mu^{-1}(0)$, we have a full off-diagonal asymptotic development. They also obtain a fast decay of the invariant Bergman kernel in the normal directions to $\mu^{-1}(0)$, which does not appear in the classical case.

In this paper, we establish G-invariant holomorphic Morse inequalities under certain natural condition, in the context of Ma–Zhang [**16**] but without the assumption that L is positive.

More precisely, we consider an action of a connected compact Lie group G on a compact complex manifold M and two G-equivariant vector bundles L and E on M, with L of rank 1, and we establish asymptotic holomorphic Morse inequalities similar to (0.3) for the G-invariant part of the Dolbeault cohomology of $L^p \otimes E$ (see Theorems 0.3 and 0.5). To do so, we define a "moment map" $\mu\colon M \to \mathrm{Lie}(G)$ by the Kostant formula and we define the reduction of M under natural hypothesis on $\mu^{-1}(0)$ (see Assumption 0.1). Our inequalities are then given in term of the curvature of the bundle induced by L on this reduction, and the integral in (0.3) will be over subsets of the reduction.

A new feature in our setting when compared to Demailly's result is the localization near $\mu^{-1}(0)$. We use a heat kernel method inspired by [**2**] (see also [**15**, Sects. 1.6–1.7]), the key being that an analogue of (0.4) still holds (see Theorem 0.7) for the Kodaira Laplacian restricted to the space of invariant forms. We show that the heat kernel will concentrate in any neighborhood U of $\mu^{-1}(0)$, and we study the asymptotic of the heat kernel near $\mu^{-1}(0)$. For this last part, we work with the operator induced by the Kodaira Laplacian on the quotient of U. However, as we will have to integrate the heat kernel in the normal directions to $\mu^{-1}(0)$, we need a more precise convergence result that in [**15**, Sect. 1.6]. Indeed we also need to prove a uniform fast decay of the heat kernel in the normal directions, which is analogous to the decay encountered in [**16**, Thm. 0.2]. Our approach is largely inspired by [**16**].

Note that in the literature, there exists another type of equivariant holomorphic Morse inequalities [**24**, **17**, **25**], which relate the Dolbeault cohomology groups of the fixed point-set of a compact Kähler manifold M endowed with an action of a compact connected Lie group G to the Dolbeault cohomology groups of M itself.

We now give more details about our results. Let (M, J) be a connected compact complex manifold. Let $n = \dim_{\mathbb{C}} M$. Let (L, h^L) be a holomorphic Hermitian line bundle on M, and (E, h^E) a holomorphic Hermitian vector bundle on M. We denote the Chern (i.e., holomorphic and Hermitian) connections of L and E, respectively, by ∇^L and ∇^E, and their respective curvatures by $R^L = (\nabla^L)^2$ and $R^E = (\nabla^E)^2$. Let ω be the first Chern form of (L, h^L), i.e., the $(1,1)$-form defined by

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L. \tag{0.6}$$

We *do not assume* that ω is a positive $(1,1)$-form.

Let G be a connected compact Lie group with Lie algebra $\mathfrak{g}$. Let $d = \dim_{\mathbb{R}} G$. We assume that G acts holomorphically on (M, J), and that the action lifts to a holomorphic action on L and E. We assume that h^L and h^E are preserved by the G-action. Then R^L, R^E and ω are G-invariant forms.

In the sequel, if F is any G-representation, we denote by F^G the space of elements of F invariant under the action of G. The infinitesimal action of $K \in \mathfrak{g}$ on any F will be denoted by $\mathcal{L}^F_K$, or simply by $\mathcal{L}_K$ when it entails no confusion.

For $K \in \mathfrak{g}$, let K^M be the vector field on M induced by K (see (1.2)). We can define a map $\mu \colon M \to \mathfrak{g}^*$ by the Kostant formula

$$\mu(K) = \frac{1}{2i\pi} \left(\nabla^L_{K^M} - \mathcal{L}_K \right). \tag{0.7}$$

Then for any $K \in \mathfrak{g}$ (see Lemma 2.1),

$$d\mu(K) = i_{K^M} \omega. \tag{0.8}$$

Moreover, the set defined by

$$P = \mu^{-1}(0) \tag{0.9}$$

is stable by G.

We make the following assumption:

Assumption 0.1. 0 *is a regular value of* μ.

Under Assumption 0.1, P is a submanifold. Moreover, by Lemma 2.2, G acts locally freely on P, so that the quotient $M_G = P/G$ is an orbifold, which we call the *reduction* of M. For definition and basic properties of orbifolds, we refer to [**15**, Sect. 5.4], for instance. The projection $P \to M_G$ is denoted by π.

We denote by TY the tangent bundle of the G-orbits in P. As G acts locally freely on P, we know that $TY = \mathrm{Span}(K^M, K \in \mathfrak{g})$ and that it is a vector bundle on P.

The following analogue of the classical Kähler reduction (see [**12**]) holds.

Theorem 0.2. *The complex structure J on M induces a complex structure J_G on M_G, for which the orbifold bundles L_G, E_G induced by L, E on M_G are holomorphic. Moreover, the form ω descends to a form ω_G on M_G and if R^{L_G} is the Chern curvature of L_G for the metric h^{L_G} induced by h^L, then*

$$\omega_G = \frac{\sqrt{-1}}{2\pi} R^{L_G}. \tag{0.10}$$

Finally, for $x \in M_G$, π_* induces an isomorphism

$$(\ker \omega_G)_x \simeq (\ker \omega)|_{\pi^{-1}(x)}. \tag{0.11}$$

Let b^L be the bilinear form on TM

$$b^L(\cdot\,,\cdot) = \frac{\sqrt{-1}}{2\pi} R^L(\cdot\,, J\cdot) = \omega(\cdot\,, J\cdot). \tag{0.12}$$

Then we will show in Lemma 2.3 that when restricted to $TY \times TY$, the bilinear form b^L is non-degenerate on P. In particular, the signature of $b^L|_{TY\times TY}$ is constant on P. We denote by $(r, d-r)$ this signature, i.e., in any orthogonal (with respect to b^L) basis of $TY|_P$, the matrix of b^L will have r negative diagonal elements and $d-r$ positive diagonal elements.

We define $\dot{R}^{L_G} \in \mathrm{End}(T^{(1,0)}M_G)$ by $g(\dot{R}^{L_G}u, \overline{v}) = R^{L_G}(u, \overline{v})$ for $u, v \in T^{(1,0)}M_G$, where g is a J_G-invariant Riemannian metric on the orbifold tangent bundle TM_G. We denote by $M_G(q)$ the set of $x \in M_G$ such that $\dot{R}^{L_G}_x$ is invertible and has exactly q negative eigenvalues, with the convention that if $q \notin \{0, \dots, n-d\}$, then $M_G(q) = \emptyset$. Set $M_G(\leq q) = \cup_{i \leq q} M_G(i)$. Note that $M_G(q)$ does not depend on the metric g.

As G preserves every structure we are given, it acts naturally on the Dolbeault cohomology $H^\bullet(M, L^p \otimes E)$. The following theorem is the main result of our paper:

Theorem 0.3. *Assume that G acts effectively on M (i.e., the only element of G acting as Id_M is the identity). Then as $p \to +\infty$, the following strong Morse inequalities hold for $q \in \{1, \dots, n\}$*

$$\sum_{j=0}^{q} (-1)^{q-j} \dim H^j(M, L^p \otimes E)^G \tag{0.13}$$

$$\leq \mathrm{rk}(E)\frac{p^{n-d}}{(n-d)!}\int_{M_G(\leq q-r)}(-1)^{q-r}\omega_G^{n-d}+o(p^{n-d}),$$

with equality for $q=n$.

In particular, we get the weak Morse inequalities

$$\dim H^q(M,L^p\otimes E)^G \leq \mathrm{rk}(E)\frac{p^{n-d}}{(n-d)!}\int_{M_G(q-r)}(-1)^{q-r}\omega_G^{n-d}+o(p^{n-d}). \tag{0.14}$$

REMARK 0.4. We assume temporarily that G acts freely on P, so that M_G is a manifold.

If L is positive, then ω is a Kähler form and μ is a genuine moment map. Moreover, (M_G,ω_G) is the usual Kähler reduction of M (see [**12**]). Zhang [**26**, Theorem 1.1 and Proposition 1.2] proved that in this case quantization and reduction commute: for p large enough,

$$H^\bullet(M,L^p\otimes E)^G\simeq H^\bullet(M_G,L_G^p\otimes E_G). \tag{0.15}$$

We refer to Vergne's Bourbaki seminar [**22**] for a survey on the Guillemin–Sternberg geometric quantization conjecture.

In particular, as in the non-equivariant setting, Theorem 0.3 is, in this case, a consequence of (0.15) and of the Hirzebruch–Riemann–Roch theorem and of the Kodaira vanishing theorem, both applied on M_G.

We prove here that even if ω is degenerate or if G does not act freely on P, under Assumption 0.1, we have the same estimate for $\sum_{j=0}^q(-1)^{q-j}\dim H^j(M,L^p\otimes E)^G$ as the one given by the holomorphic Morse inequalities on M_G for $\sum_{j=0}^{q-r}(-1)^{q-j}\dim H^j(M_G,L_G^p\otimes E_G)$.

Theorem 0.3 is in fact a particular case of the more general Theorem 0.5 below.

Set

$$G^0=\{g\in G\,:\,g\cdot x=x \text{ for any } x\in M\}, \tag{0.16}$$

which is a finite normal subgroup of G. Note that we will see in (5.23) that we also have $G^0=\{g\in G\,:\,g\cdot x=x \text{ for any } x\in P\}$.

Observe that $\dim(L_v^p\otimes E_v)^{G^0}$ does not depend on $v\in M$. We will thus denote it simply by $\dim(L^p\otimes E)^{G^0}$.

Theorem 0.5. *As* $p\to+\infty$, *the following strong Morse inequalities hold for* $q\in\{1,\dots,n\}$

$$\sum_{j=0}^q(-1)^{q-j}\dim H^j(M,L^p\otimes E)^G \leq \dim(L^p\otimes E)^{G^0}\frac{p^{n-d}}{(n-d)!}\int_{M_G(\leq q-r)}(-1)^{q-r}\omega_G^{n-d}+o(p^{n-d}), \tag{0.17}$$

with equality for $q=n$.

In particular, we get the weak Morse inequalities

$$\dim H^q(M, L^p \otimes E)^G \leq \dim(L^p \otimes E)^{G^0} \frac{p^{n-d}}{(n-d)!} \int_{M_G(q-r)} (-1)^{q-r} \omega_G^{n-d} + o(p^{n-d}). \tag{0.18}$$

REMARK 0.6. The integer $\dim(L^p \otimes E)^{G^0}$ depends on p. However, as G^0 is finite and acts by rotations on L, there exists $k \in \mathbb{N}$ (a divisor of the cardinal of G^0) such that G^0 acts trivially on L^k. In particular, we have $\dim(L^{kp} \otimes E)^{G^0} = \dim E^{G^0}$.

We now explain what are the main steps of our proof.

Let g^{TM} be a J- and G-invariant metric on TM. Let dv_M be the corresponding Riemannian volume on M, and let ∇^{TM} be the Levi-Civita connection on (TM, g^{TM}). Let $\bar{\partial}^{L^p\otimes E}$ be the Dolbeault operator acting on $\Omega^{0,\bullet}(M, L^p \otimes E)$. Let $\bar{\partial}^{L^p\otimes E,*}$ be its dual with respect to the L^2 product induced by g^{TM}, h^L and h^E on $\Omega^{0,\bullet}(M, L^p \otimes E)$ (see (1.8)). We set

$$D_p = \sqrt{2}\left(\bar{\partial}^{L^p\otimes E} + \bar{\partial}^{L^p\otimes E,*}\right), \tag{0.19}$$

and we denote by $e^{-uD_p^2}$ the associated heat kernel.

We denote P_G the orthogonal projection from $\Omega^{0,\bullet}(M, L^p \otimes E)$ onto $\Omega^{0,\bullet}(M, L^p \otimes E)^G$. Let $(P_G e^{-\frac{u}{p}D_p^2} P_G)(v, v')$ be the Schwartz kernel of $P_G e^{-\frac{u}{p}D_p^2} P_G$ with respect to $dv_M(v')$.

Note that the operator D_p^2 acts on $\Omega^{0,\bullet}(M, L^p \otimes E)^G$ (i.e., commutes with P_G) and preserves the $\mathbb{Z}$-grading. We denote by $\mathrm{Tr}_q[P_G e^{-\frac{u}{p}D_p^2} P_G]$ the trace of $P_G e^{-\frac{u}{p}D_p^2} P_G$ acting on $\Omega^{0,q}(M, L^p \otimes E)$. We then have an analogue of (0.4):

Theorem 0.7. *For any $u > 0$, $p \in \mathbb{N}^*$ and $0 \leq q \leq n$, we have*

$$\sum_{j=0}^{q} (-1)^{q-j} \dim H^j(M, L^p \otimes E)^G \leq \sum_{j=0}^{q} (-1)^{q-j} \mathrm{Tr}_j \left[P_G e^{-\frac{u}{p}D_p^2} P_G\right], \tag{0.20}$$

with equality for $q = n$.

We now give the estimates on $P_G e^{-\frac{u}{p}D_p^2} P_G$ to treat the right-hand side of (0.20).

Let U be a small open G-invariant neighborhood of P, such that G acts locally freely on its closure $\overline{U}$.

First, we have away from P the following theorem. The analogous result of Ma–Zhang for the invariant Bergman kernel is [**16**, Thm. 0.1].

Theorem 0.8. *For any fixed $u > 0$ and $k, \ell \in \mathbb{N}$, there exists $C > 0$ such that for any $p \in \mathbb{N}^*$ and $v, v' \in M$ with $v, v' \in M \setminus U$,*

$$\left| P_G e^{-\frac{u}{p} D_p^2} P_G(v, v') \right|_{\mathscr{C}^\ell} \leq C p^{-k}, \tag{0.21}$$

where $|\cdot|_{\mathscr{C}^\ell}$ is the $\mathscr{C}^\ell$-norm induced by ∇^L, ∇^E, ∇^{TM}, h^L, h^E and g^{TM}.

We now turn to the "near P" asymptotic of the heat kernel. To explain simply this asymptotic, we assume now that G acts freely on P. We can thus also assume that G acts freely on $\overline{U}$. Let $B = U/G$. Then M_G and B are here genuine manifolds. We will explain in Section 5.2 how to adapt the proof of Theorems 0.3 and 0.5 to the case of a locally free action.

We again denote by $TY = \mathrm{Span}(K^M, K \in \mathfrak{g})$ the tangent bundle of the orbits in U. By Lemma 2.3, we have

$$TU = TY \oplus (TY)^{\perp_{b^L}}. \tag{0.22}$$

Then we can choose the horizontal bundles of the fibrations $U \to B$ and $P \to M_G$ to be, respectively,

$$T^H U = (TY)^{\perp_{b^L}} \quad \text{and} \quad T^H P = T^H U\big|_P \cap TP. \tag{0.23}$$

Indeed, using (0.22) and the fact that $TY \subset TP$, we see that

$$TP = TY \oplus T^H P. \tag{0.24}$$

Let $g^{T^H P}$ be a G-invariant and J-invariant metric on $T^H P$. Let g^{TY} be a G-invariant metric on TY and let g^{JTY} be the G-invariant metric on JTY induced by J and g^{TY}. Then by (2.19), we can chose the metric g^{TM} on TM so that on P:

$$g^{TM}|_P = g^{TY}|_P \oplus g^{JTY}|_P \oplus g^{T^H P}. \tag{0.25}$$

We will use this condition on g^{TM} in the rest of the introduction as well as in Sections 4.1–5.2.

Let g^{TB} be the metric on TB induced by g^{TM} and $T^H U$, and let dv_B be the associated volume form.

Suppose that U is small enough so that it can be identified with a ε-neighborhood, $\varepsilon > 0$, of the zero section of the normal bundle N of P in U via exponential map. We denote the corresponding coordinate by $v = (y, Z^\perp) \in U$ with $y \in P$ and $Z^\perp \in N_y$. Note that by (0.24) and (0.25) we can identify N_y and JTY_y.

Let $\mathbf{J} \in \mathrm{End}(TM|_P)$ be such that on P

$$\omega = g^{TM}(\mathbf{J}\cdot, \cdot). \tag{0.26}$$

By (4.3), the normal bundle N_G of M_G in B can be identified with the bundle $(JTY)_B$ induced on B by $JTY \simeq N$ (see Section 1). In particular, if $\pi(y) = x$, we keep the same notation for an element of N_y and the corresponding element in $N_{G,x}$.

We will see in Section 4.3 that $\mathbf{J}$ intertwines the bundles TY and JTY. In particular, $\mathbf{J^2}$ induces an endomorphism of N_G and if $\{a_1^\perp, \dots, a_d^\perp\} = -2\sqrt{-1}\pi \mathrm{Sp}(\mathbf{J}|_{(TY\oplus JTY)^{(1,0)}})$ ($a_j^\perp \in \mathbb{R}^*$), then

$$\mathrm{Sp}(\mathbf{J^2}|_{N_G}) = -\frac{1}{4\pi^2}\{a_1^{\perp,2}, \dots, a_d^{\perp,2}\}. \tag{0.27}$$

Let g^{N_G} be the induced metric on N_G and dv_{N_G} the corresponding volume form. For $x \in M_G$, let $\{e_i^\perp\}_{i=1}^d$ be an orthonormal basis of $N_{G,x}$ such that $\mathbf{J}_x^2 e_i^\perp = -\frac{1}{4\pi^2} a_i^{\perp,2}(x) e_i^\perp$. We can then identify $\mathbb{R}^d$ with $N_{G,x}$ via the map

$$(Z_1^\perp, \dots, Z_d^\perp) \in \mathbb{R}^d \mapsto Z^\perp = \sum_{i=1}^d Z_i^\perp e_i^\perp. \tag{0.28}$$

We now define the operator $\mathscr{L}_x^\perp$ acting on $N_{G,x} \simeq \mathbb{R}^d$ by

$$\mathscr{L}_x^\perp = -\sum_{i=1}^d \Big((\nabla_{e_i^\perp})^2 - |a_i^\perp Z_i^\perp|^2 \Big) - \sum_{j=1}^d a_j^\perp, \tag{0.29}$$

where ∇_U denotes the ordinary differentiation operator on $\mathbb{R}^d$ in the direction U. We denote by $e^{-u\mathscr{L}_x^\perp}(Z^\perp, Z'^\perp)$ the heat kernel of $\mathscr{L}_x^\perp$ with respect to $dv_{N_{G,x}}(Z'^\perp)$. Note that we have an explicit formula for $e^{-u\mathscr{L}_x^\perp}(Z^\perp, Z'^\perp)$ (see (5.12)), but we do not give it to have a simpler asymptotic formula for the heat kernel.

Let g^{TM_G} be the metric on M_G induced by g^{TM} and T^HP and dv_{M_G} the corresponding volume form. We denote by $\langle\cdot\,,\cdot\rangle_G$ the $\mathbb{C}$-bilinear extension of g^{TM_G} on $TM_G \otimes \mathbb{C}$. Then we can identify R^{L_G} with the Hermitian matrix $\dot{R}^{L_G} \in \mathrm{End}(T^{(1,0)}M_G)$ such that for $V, V' \in T^{(1,0)}M_G$,

$$R^{L_G}(V, V') = \langle \dot{R}^{L_G} V, \overline{V'}\rangle_G. \tag{0.30}$$

Let $\{w_j\}$ be a local orthonormal frame of $T^{(1,0)}M$ with dual frame $\{w^j\}$. Set

$$\omega_d = -\sum_{i,j} R^L(w_i, \overline{w}_j) \overline{w}^j \wedge i_{\overline{w}_i}. \tag{0.31}$$

Let h be the G-invariant smooth function on M given by (see Section 1)

$$h(x) = \sqrt{\mathrm{vol}(G.x)}, \tag{0.32}$$

and let $\kappa \in \mathscr{C}^\infty(TB|_{M_G})$ be the function defined by $\kappa|_{M_G} = 1$ and for $x \in M_G$, $Z \in T_xB$,

$$dv_B(x,Z) = \kappa(x,Z) dv_{T_xB}(Z) = \kappa(x,Z) dv_{M_G}(x) dv_{N_{G,x}}(Z). \tag{0.33}$$

The following result is a version of [**16**, Thm. 2.21] in our situation for the heat kernel.

Theorem 0.9. *Assume that G acts freely on P. For any fixed $u>0$ and $m\in\mathbb{N}$, we have the following convergence as $p\to+\infty$ for $|Z^\perp|<\varepsilon$:*

$$\begin{aligned}(0.34)\quad & h(y,Z^\perp)^2\big(P_G e^{-\frac{u}{p}D_p^2}P_G\big)\big((y,Z^\perp),(y,Z^\perp)\big) = \\ & \frac{\kappa^{-1}(x,Z^\perp)}{(2\pi)^{n-d}}\frac{\det(\dot{R}_x^{L_G})e^{2u\omega_d(x)}}{\det\big(1-\exp(-2u\dot{R}_x^{L_G})\big)}e^{-u\mathscr{L}_x^\perp}(\sqrt{p}Z^\perp,\sqrt{p}Z^\perp)\otimes \mathrm{Id}_E\, p^{n-d/2} \\ & \qquad + O\big(p^{n-d/2-1/2}(1+\sqrt{p}|Z^\perp|)^{-m}\big),\end{aligned}$$

where $x=\pi(y)\in M_G$ and the term $O(\cdot)$ is uniform. The convergence is in the $\mathscr{C}^\infty$-topology in $y\in P$. Here, we use the convention that if an eigenvalue of $\dot{R}_{x_0}^{L_G}$ is zero, then its contribution to $\frac{\det(\dot{R}_{x_0}^{L_G})}{\det(1-\exp(-u\dot{R}_{x_0}^{L_G}))}$ is $\frac{1}{2u}$.

From Theorems 0.7, 0.8 and 0.9, we get Theorem 0.3 in the case where G acts freely on P by integrating on M the trace of $(P_G e^{-\frac{u}{p}D_p^2}P_G)(m,m)$, then taking the limit $u\to+\infty$.

This paper is organized as follows. In Section 1, we recall some constructions associated with a principal bundle. In Section 2, we apply the constructions and results of Section 1 to our situation to define the reduction of M and to descend on it the different objects we are given, thus proving Theorem 0.2. In Section 3 we prove the localization of the heat kernel near P, i.e., Theorem 0.8. In Sections 4, we assume for simplicity that G acts freely on P and $\overline{U}$, and study the asymptotic of the heat kernel near P by localizing the problem and studying a rescaled Laplacian on B. We thus obtain Theorem 0.9. Finally, in Section 5, we prove the G-invariant holomorphic Morse inequalities (Theorems 0.3 and 0.5) and we also show how to use Theorem 0.5 to get estimates on the other isotypic components of the cohomology $H^\bullet(M,L^p\otimes E)$.

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1. Connections and Laplacians associated with a principal bundle

In this section, we review some results of [**16**, Chp. 1] for the convenience of the reader.

Let G be a connected compact Lie group of dimension d that acts smoothly and locally freely on the left on a smooth manifold M of dimension m. Then $\pi\colon M\to B=M/G$ is a G-principal bundle and B is an orbifold. We denote by TY the relative tangent bundle of this fibration.

Note that in [**16**, Chp. 1], Ma and Zhang assumed that G acts freely on M, but as they explain in the introduction of [**16**, Chp. 1] and in [**16**, Sect. 4.1], the results of [**16**, Chp. 1] extend to the case where G acts only locally freely, essentially because when we work on orbifold quotients, we in fact work with invariant sections on M.

Let g^{TM} be a G-invariant metric on TM, and ∇^{TM} the corresponding Levi-Civita connection on TM. We denote by T^HM the orthogonal complement of TY in TM. For $U \in TB$, we denote by U^H the horizontal lift of U in TM, that is $\pi_* U^H = U$ and $U^H \in T^HM$. Let $\theta\colon TM \to \mathfrak{g}$ be the connection form corresponding to T^HM, and let Θ be its curvature, i.e., the horizontal form such that

$$\Theta(U^H, V^H) = -P^{TY}[U^H, V^H], \tag{1.1}$$

where P^{TY} is the natural projection $TM = TY \oplus T^HM \to TY$.

The metric g^{TM} induces a metric g^{TY} (resp. g^{T^HM}) on TY (resp. T^HM). Let g^{TB} be the metric on TB induced by g^{T^HM}, and let ∇^{TB} be the corresponding Levi-Civita connection.

Let (F, h^F) be a G-equivariant Hermitian vector bundle with G-equivariant Hermitian connection ∇^F. Then G acts on $\mathscr{C}^\infty(M, F)$ by $(g.s)(x) = g.s(g^{-1}x)$.

Any $K \in \mathfrak{g}$ induces a vector field K^M on M given by

$$K^M_x = \frac{\partial}{\partial s}\bigg|_{s=0} e^{-sK}.x. \tag{1.2}$$

For $K \in \mathfrak{g}$, recall that $\mathcal{L}_K$ is the infinitesimal action of K on any G representation. Let $\mu^F \in \mathscr{C}^\infty(M, \mathfrak{g}^* \otimes \operatorname{End}(F))$ be defined by

$$\mu^F(K) = \nabla^F_{K^M} - \mathcal{L}_K. \tag{1.3}$$

Using the identification $TY \simeq M \times \mathfrak{g}$, we can identify μ^F with $\widetilde{\mu}^F \in \mathscr{C}^\infty(M, TY \otimes \operatorname{End}(F))^G$ such that

$$\langle \widetilde{\mu}^F, K^M \rangle = \mu^F(K). \tag{1.4}$$

Let F_B be the orbifold bundle on B induced by F, i.e., $F_{B,x} = \mathscr{C}^\infty(\pi^{-1}(x), F|_{\pi^{-1}(x)})^G$. Then there is a canonical isomorphism

$$\pi_G\colon \mathscr{C}^\infty(M, F)^G \xrightarrow{\sim} \mathscr{C}^\infty(B, F_B). \tag{1.5}$$

The invariant metric h^F induces a metric h^{F_B} on F_B. For $s \in \mathscr{C}^\infty(B, F_B)$ and $U \in TB$, we define

$$\nabla^{F_B}_U s := \nabla^F_{U^H} s. \tag{1.6}$$

Observe that ∇^{F_B} is the restriction of the connection $\nabla^F - \mu(\theta)$ to $\mathscr{C}^\infty(M, F)^G$. Let R^{F_B} be the curvature of ∇^{F_B}. Then by [**16**, (1.18)] we have for $V, V' \in TB$

$$R^{F_B}(V, V') = R^F(V^H, V'^H) - \mu^F(\Theta)(V, V'). \tag{1.7}$$

Let dv_M be the Riemannian volume on (M, g^{TM}). We endow $\mathscr{C}^\infty(M,F)$ with the L^2 product induced by g^{TM} and h^F:

$$\langle s, s' \rangle = \int_M \langle s, s' \rangle_{h^F}(x) dv_M(x). \tag{1.8}$$

In the same way, g^{TB} and h^{F_B} induce a L^2 product $\langle \cdot\,, \cdot \rangle$ on $\mathscr{C}^\infty(B, F_B)$.

For $x \in M$, we denote by $\mathrm{vol}(G.x)$ the volume of the orbit of $G.x$ endowed with the restriction of g^{TM}. Define the G-invariant function h on M by

$$h(x) = \sqrt{\mathrm{vol}(G.x)}. \tag{1.9}$$

Then h define a function on B, which is still denoted by h. Note that h is smooth only on the regular part of B. However, we can extend it continuously to get a smooth function $\widehat{h}$ on B. Then $\widehat{h}$ also define a smooth function on U.

The map

$$\Phi := \widehat{h}\pi_G \colon \big(\mathscr{C}^\infty(M,F)^G, \langle \cdot\,, \cdot \rangle\big) \to \big(\mathscr{C}^\infty(B, F_B), \langle \cdot\,, \cdot \rangle\big) \tag{1.10}$$

is then an isometry.

Let $\{u_i\}_{i=1}^m$ be an orthonormal frame of TM. For any Hermitian bundle with Hermitian connection (E, h^E, ∇^E) on M, the Bochner Laplacians Δ^E, Δ_M are given by

$$\Delta^E = -\sum_{i=1}^m \left((\nabla^E_{u_i})^2 - \nabla^E_{\nabla^{TM}_{u_i} u_i} \right), \quad \Delta_M = \Delta^{\mathbb{C}}. \tag{1.11}$$

Let $\{f_l\}_{l=1}^d$ be a G-invariant orthonormal frame of TY with dual frame $\{f^l\}_{l=1}^d$, and let $\{e_i\}_{i=1}^{m-d}$ be an orthonormal frame of TB. Then $\{e_i^H, f_l\}$ form an orthonormal frame of TM.

For $\sigma, \sigma' \in TY \otimes \mathrm{End}(F)$, let $\langle \sigma, \sigma' \rangle_{g^{TY}} \in \mathrm{End}(F)$ be the contraction of the part of $\sigma \otimes \sigma'$ in $TY \otimes TY$ with g^{TY}. Note that

$$\langle \widetilde{\mu}^F, \widetilde{\mu}^F \rangle_{g^{TY}} = \sum_{l=1}^d \langle \widetilde{\mu}^F, f_l \rangle^2_{g^{TY}} \in \mathrm{End}(F). \tag{1.12}$$

Theorem 1.1. *As an operator on $\mathscr{C}^\infty(B, F_B)$, $\Phi \Delta^F \Phi^{-1}$ is given by*

$$\Phi \Delta^F \Phi^{-1} = \Delta^{F_B} - \langle \widetilde{\mu}^F, \widetilde{\mu}^F \rangle_{g^{TY}} - \widehat{h}^{-1} \Delta_B \widehat{h}. \tag{1.13}$$

Proof. This is proved in [**16**, Thm. 1.3]. q.e.d.

2. The reduction of M and the Laplacian on B

This Section is organized as follows. In Section 2.1 we apply the constructions and results of Section 1 to our situation in order to define the reduction of M and to descend on it the different objects we are given. We prove, under Assumption 0.1, some properties of the reduction that are well-known in the case where ω is positive and get Theorem 0.2. In Section 2.2, we compute the operator induced on U/G by the Kodaira Laplacian.

We use here the notations of the introduction. In particular, let (M, J) be a connected compact complex manifold of dimension n, let (L, h^L) be a holomorphic Hermitian line bundle on M and (E, h^E) a Hermitian complex vector bundle on M. We denote the associated Chern curvatures by R^L and R^E. Let $\omega = \frac{\sqrt{-1}}{2\pi} R^L$ be the first Chern form of (L, h^L), which is not assumed to be positive. Let G be a connected compact Lie group with Lie algebra $\mathfrak{g}$. Let $d = \dim_{\mathbb{R}} G$. We assume that G acts holomorphically on (M, J), and that the action lifts in a holomorphic action on L and E. We assume that h^L and h^E are preserved by the G-action.

Recall that ∇^L denotes the Chern connection of (L, h^L) and that the moment map μ is defined by $2i\pi\mu(K) = \nabla^L_{K^M} - \mathcal{L}_K$ for $K \in \mathfrak{g}$. Let $P = \mu^{-1}(0)$ and let U be a small tubular neighborhood of P. Finally, we set $M_G = P/G$.

2.1. The reduction of M. We begin by proving the following result:

Lemma 2.1. *The map μ is smooth on M and is linear in K. Moreover, it is moment map of the G-action on M, i.e., μ is G-equivariant and for any $K \in \mathfrak{g}$,*

$$d\mu(K) = i_{K^M}\omega. \tag{2.1}$$

Proof. First, as both $\nabla^L_{K^M}$ and $\mathcal{L}_K$ satisfies the Leibniz rules and preserves h^L, we know that $\nabla^L_{K^M} - \mathcal{L}_K$ is $\mathscr{C}^\infty(M)$-linear, and, moreover, it is a skew-adjoint operator. Thus, under the canonical isomorphism $\operatorname{End}(L) = \mathbb{C}$,

$$\nabla^L_{K^M} - \mathcal{L}_K \in \mathscr{C}^\infty(M, i\mathbb{R}). \tag{2.2}$$

This proves the first part of Lemma 2.1.

As ∇^L is G-invariant, we have $g \cdot (\nabla^L_Y s) = \nabla^L_{g_* Y}(g \cdot s)$ for $Y \in \mathscr{C}^\infty(M, TM)$, $s \in \mathscr{C}^\infty(M, L)$ and $g \in G$. Thus, taking $g = e^{-tK}$ for $K \in \mathfrak{g}$ and differentiating at $t = 0$, we get

$$\mathcal{L}_K \nabla^L_Y s = \nabla^L_{[K^M, Y]} s + \nabla^L_Y \mathcal{L}_K s, \qquad \text{that is} \quad [\mathcal{L}_K, \nabla^L] = 0. \tag{2.3}$$

Using the definition of μ (0.7), (2.3) becomes

$$\big(2i\pi\mu(K) + \nabla^L_{K^M}\big)\nabla^L_Y s = \nabla^L_{[K^M, Y]} s + \nabla^L_Y \big(2i\pi\mu(K) + \nabla^L_{K^M}\big)s. \tag{2.4}$$

This, together with (0.6), yields to

$$Y(\mu(K)) = \omega(K^M, Y), \tag{2.5}$$

which is (2.1).

Finally, it is easy to prove that $g_* K^M = (\mathrm{Ad}_g K)^M$ and $g \cdot (\mathcal{L}_K s) = \mathcal{L}_{\mathrm{Ad}_g K}(g \cdot s)$, so

$$2i\pi\, g \cdot (\mu(K)s) = \big(\nabla^L_{(\mathrm{Ad}_g K)^M} - \mathcal{L}_{\mathrm{Ad}_g K}\big)(g \cdot s), \tag{2.6}$$

and thus

$$\mu(g^{-1}x) = \mathrm{Ad}^*_{g^{-1}}\mu. \tag{2.7}$$

The proof of Lemma 2.1 is complete. q.e.d.

Lemma 2.2. *The group G acts locally freely on P.*

Proof. By (2.1), we have for $x \in P$, $V \in T_xM$ and $K \in \mathfrak{g}$,

$$\omega(K^M, V)_x = \big(d_x\mu(V)\big)(K). \tag{2.8}$$

In particular, if $K^M = 0$, then $(d_x\mu(V))(K)$ vanishes for all $V \in T_xM$. However, by Assumption 0.1, the differential $d_x\mu\colon T_xM \to \mathfrak{g}^*$ is surjective, hence $K = 0$. q.e.d.

Lemma 2.3. *When restricted to $TY \times TY$, the bilinear form b^L is non-degenerate on P.*

Proof. First, observe that for $x \in P$, $V \in T_xM$ and $K \in \mathfrak{g}$, equations (0.12) and (2.1) yield

$$b^L(K^M, JV)_x = -\omega(K^M, V)_x = -\big(d_x\mu(V)\big)(K). \tag{2.9}$$

Let $x \in P$, $V \in T_xM$ and $K \in \mathfrak{g}$. Then by (2.9)

$$JV \in (TY)^{\perp_{b^L}}\Big|_P \iff d_x\mu(V) = 0 \iff V \in TP, \tag{2.10}$$

the last equivalence coming from the fact that $P = \mu^{-1}(0)$. In particular, $\dim(TY)^{\perp_{b^L}} = \dim TP = 2n - d$, the last identity coming from the fact that 0 is a regular value of μ. Moreover, $\dim TY = d$ (because G acts locally freely on U) and $TY + (TY)^{\perp_{b^L}} = TU$. This is possible only if this sum is direct, i.e., $TY \cap (TY)^{\perp_{b^L}} = \{0\}$. We have proved our lemma. q.e.d.

By Lemma 2.3, we have

$$TU = TY \oplus (TY)^{\perp_{b^L}}. \tag{2.11}$$

Then we can choose the horizontal bundles of the fibrations $U \to B$ and $P \to M_G$ to be

$$T^HU = (TY)^{\perp_{b^L}} \quad \text{and} \quad T^HP = T^HU\big|_P \cap TP. \tag{2.12}$$

Indeed, using (2.11) and the fact that $TY \subset TP$, we see that

$$TP = TY \oplus T^HP. \tag{2.13}$$

Let $(L_G, h^{L_G}, \nabla^{L_G})$ and $(E_G, h^{E_G}, \nabla^{E_G})$ be defined from (L, h^L, ∇^L), (E, h^E, ∇^E) and T^HP as indicated in Section 1. We also define ω_G by

$$\omega_G(V, V') = \omega(V^H, V'^H). \tag{2.14}$$

Note that (1.7) restricted to $P = \mu^{-1}(0)$ gives

$$R^{L_B}|_{M_G}(V, V') = R^L|_P(V^H, V'^H). \tag{2.15}$$

From (0.6), (2.14) and (2.15), we see that if R^{L_G} is the curvature of ∇^{L_G}, then

$$\omega_G = \frac{\sqrt{-1}}{2\pi} R^{L_G}. \tag{2.16}$$

Lemma 2.4. *We have*

$$\begin{aligned} T^H U\big|_P &= JTP, \\ TU|_P &= TP \oplus JTY. \end{aligned} \tag{2.17}$$

In the second line, the sum is orthogonal with respect to b^L.

Proof. Recall that $T^H U = (TY)^{\perp_{b^L}}$. Thus the first identity in (2.17) follows from (2.10).

Concerning the second, we have for $V \in TP$ and $K \in \mathfrak{g}$,

$$b^L(JK^M, V)_x = \omega(K^M, V)_x = \big(d_x\mu(V)\big)_x(K) = 0. \tag{2.18}$$

Using (2.18) and the facts that b^L is non-degenerate on JTY and that $\dim TU = \dim TP + \dim JTY$, we get the second identity in (2.17). q.e.d.

Using Lemma 2.4 and (2.13), we find firstly

$$TU|_P = T^H P \oplus TY \oplus JTY, \tag{2.19}$$

the decomposition being orthogonal for b^L, and secondly by Lemma 2.4 and (2.12),

$$T^H P = TP \cap JTP. \tag{2.20}$$

In particular, $T^H P$ is stable by J, so we can define an almost-complex structure on M_G in the following way. For $V \in TM_G$, we denote V^H its lift in $T^H P$, and we define the almost complex structure J_G on M_G by

$$(J_G V)^H = J(V^H). \tag{2.21}$$

Lemma 2.5. *The almost complex structure J_G is integrable, thus (M_G, J_G) is a complex manifold.*

Proof. Let $u, v \in \mathscr{C}^\infty(M_G, T^{1,0}M_G)$. Then there are $U, V \in \mathscr{C}^\infty(M_G, TM_G)$ such that

$$u = U - \sqrt{-1} J_G U\,, \quad v = V - \sqrt{-1} J_G V. \tag{2.22}$$

Using (2.21), we find

$$u^H = U^H - \sqrt{-1} JU^H\,, \quad v = V^H - \sqrt{-1} JV^H \in T^{1,0}M \cap T_{\mathbb{C}}P. \tag{2.23}$$

As both $T^{1,0}M$ and $T_{\mathbb{C}}P$ are integrable, we have $[u^H, v^H] \in T^{1,0}M \cap T_{\mathbb{C}}P$, i.e., there is $W \in \mathscr{C}^\infty(M, TM)$ such that

$$[u^H, v^H] = W - \sqrt{-1} JW, \tag{2.24}$$

and, moreover, $W, JW \in TP$. Thus, $W \in TP \cap JTP = T^HP$ and we can write $W = X^H$ for X a section of TM_G. Hence

(2.25)
$$[u, v] = \pi_*[u^H, v^H] = \pi_*(X^H - \sqrt{-1}JX^H) = X - \sqrt{-1}J_GX \in T^{1,0}M_G.$$

By the Newlander–Nirenberg theorem, (2.25) means that J_G is integrable. q.e.d.

Lemma 2.6. *The bundles L_G and E_G are holomorphic. Moreover, ∇^{L_G} and ∇^{E_G} are the respective Chern connections of (L_G, h^{L_G}) and (E_G, h^{E_G}).*

Proof. We first prove the result for L_G.

Observe that for $U, V \in TM_G$,

$$\omega_G(J_GU, J_GV) = \omega(JU^H, JV^H) = \omega(U^H, V^H) = \omega_G(U, V). \tag{2.26}$$

Hence, ω_G is a $(1,1)$-form, and so is R^{L_G} by (2.16). We decompose ∇^{L_G} into holomorphic part and anti-holomorphic part,

$$\nabla^{L_G} = (\nabla^{L_G})^{1,0} + (\nabla^{L_G})^{0,1}. \tag{2.27}$$

As R^{L_G} is $(1,1)$, we have

$$\left((\nabla^{L_G})^{0,1}\right)^2 = 0. \tag{2.28}$$

For $s \in \mathscr{C}^\infty(M_G, L_G)$, we define

$$\bar\partial^{L_G} s = (\nabla^{L_G})^{0,1}s. \tag{2.29}$$

Let s_0 be a local frame of L_G near $x_0 \in M_G$. Then we can write $(\nabla^{L_G})^{0,1}s_0 = \alpha s_0$ for some $(0,1)$-form α. By (2.28), we have

$$0 = \left((\nabla^{L_G})^{0,1}\right)^2 s_0 = (\bar\partial\alpha)s_0. \tag{2.30}$$

Thus, $\bar\partial\alpha = 0$. By the (local) $\bar\partial$-lemma, there is a function f defined near x_0 such that $\bar\partial f = -\alpha$. Thus,

$$\bar\partial^{L_G} s_0 + (\bar\partial f)s_0 = 0. \tag{2.31}$$

This shows that (2.29) defines a holomorphic structure on L_G, for which $e^f s_0$ is a local holomorphic frame near x_0.

Finally, ∇^{L_G} is clearly Hermitian with respect to h^{L_G}, and is holomorphic by the definition (2.29), so ∇^{L_G} is indeed the Chern connection on L_G.

We now turn to E_G. Here again, it is enough to prove that R^{E_G} is a $(1,1)$-form (see, for instance, [**13**, Prop. I.3.7]). As R^E is a $(1,1)$-form, (1.7) shows that it is equivalent to prove that $\Theta|_{T^HP\times T^HP}$ is a $(1,1)$-form.

Let $u = U - \sqrt{-1}JU$ and $v = V - \sqrt{-1}JV$ be in $(T^HP)^{1,0}$. As U, V, JU and JV are in $T^HP = TP \cap JTP$ and TP is integrable, we have $[u, v] \in T_{\mathbb{C}}P$. Moreover, as u and v are of type $(1,0)$ and J is integrable,

$[u,v]$ is also of type $(1,0)$, and thus $[u,v] = -iJ[u,v] \in JT_{\mathbb{C}}P$. In conclusion, $[u,v] \in T^H_{\mathbb{C}}P$ and by (1.1), $\Theta(u,v) = 0$. q.e.d.

Lemma 2.7. *We have* $\ker\omega|_P \subset T^H P$, *and for* $x \in M_G$, π_* *induces an isomorphism*

$$(\ker\omega_G)_x \simeq (\ker\omega)|_{\pi^{-1}(x)}. \tag{2.32}$$

Proof. Let $V \in TU|_P$ be such that $\omega(V,\cdot) = 0$. Then we also have $b^L(V,\cdot) = 0$. Thus V is in particular in $(TY)^{\perp_{b^L}} = T^H U$. Moreover, V is also orthogonal (for b^L) to JTY, so is in $T^H P$ by Lemma 2.4.

As $\omega_G(\pi_*\cdot,\pi_*\cdot) = \omega(\cdot,\cdot)$, we know that π_* maps $\ker\omega|_P$ in $\ker\omega_G$, and is injective as $\ker\omega|_P \subset T^H P$. Finally, if $V \in \ker\omega_G$, then $\omega(V^H, V') = 0$ for $V' \in T^H P$. In fact, as the decomposition in (2.19) is orthogonal for b^L, we have $\omega(V^H, V') = 0$ for $V' \in TU|_P$, and thus $V^H \in \ker\omega$. The proof of our lemma is complete. q.e.d.

By (2.16) and Lemmas 2.5, 2.6 and 2.7 we have proved Theorem 0.2.

2.2. The Kodaira Laplacian and the operator induced on B. We define the vector bundle $\mathcal{E}$, and $\mathbb{E}_p$ $(p \geq 1)$ over M by

$$\begin{aligned} \mathcal{E} &= \Lambda^{0,\bullet}(T^*M) \otimes E, \\ \mathbb{E}_p &= \Lambda^{0,\bullet}(T^*M) \otimes E \otimes L^p. \end{aligned} \tag{2.33}$$

Recall that g^{TM} is a J- and G-invariant metric on TM (we do not assume that (0.25) holds in this section). We endow $\mathscr{C}^\infty(M,\mathbb{E}_p)$ with the L^2 scalar product associated with g^{TM}, h^L and h^E as in (1.8). Then the Dolbeault–Dirac operator D_p defined in (0.19) is a formally self-adjoint operator acting on $\mathscr{C}^\infty(M,\mathbb{E}_p)$.

We now recall the Lichnerowicz formula for the Kodaira Laplacian D_p^2.

Let ∇^{TM} be the Levi-Civita connection on (M, g^{TM}). We denote by $P^{T^{(1,0)}M}$ the orthogonal projection form $TM \otimes_{\mathbb{R}} \mathbb{C}$ onto $T^{(1,0)}M$. Let $\nabla^{T^{(1,0)}M} = P^{T^{(1,0)}M}\nabla^{TM}P^{T^{(1,0)}M}$ be the induced connection on $T^{(1,0)}M$. We endow $\det(T^{1,0}M)$ with the metric induced by g^{TM}, and we denote by $\nabla^{\det}$ the Hermitian connection on $\det(T^{1,0}M)$ induced by $\nabla^{T^{(1,0)}M}$. Let $R^{\det}$ be the curvature of $\nabla^{\det}$.

Let $(w_1,\ldots,w_n)$ be an orthonormal frame of $(T^{(1,0)}M, g^{TM})$, and $(e_1,\ldots,e_{2n})$ be the orthonormal frame of (TM, g^{TM}) given by

$$e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j) \quad \text{and} \quad e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j). \tag{2.34}$$

Let $\{e^k\}$ be the dual basis of $\{e_k\}$. The Clifford action of $T_{\mathbb{C}}M$ on $\Lambda^{0,\bullet}(T^*M)$ is defined by linearity from

$$c(w_j) := \sqrt{2}\overline{w}^j \wedge \quad \text{and} \quad c(\overline{w}_j) := -\sqrt{2}i_{\overline{w}_j}. \tag{2.35}$$

We then define a map, still denoted by $c(\cdot)$, on $\Lambda(T^*_{\mathbb{C}}M)$ by setting for $j_1 < \cdots < j_k$:

$$c(e^{j_1} \wedge \cdots \wedge e^{j_k}) := c(e_{j_1}) \ldots c(e_{j_k}). \tag{2.36}$$

Let Γ^{TM} and $\Gamma^{\det}$ be the connection forms of ∇^{TM} and $\nabla^{\det}$ associated with the frames $\{e_i\}$ and $w_1 \wedge \cdots \wedge w_n$. Define the *Clifford connection* on $\Lambda^{0,\bullet}(T^*M)$ (see [**15**, (1.3.5)]) by the following local formula in the frame $\{\overline{w}^{i_1} \wedge \cdots \wedge \overline{w}^{i_k}\}$:

$$\nabla^{\mathrm{Cl}} = d + \frac{1}{4}\sum_{i,j}\langle \Gamma^{TM} e_i, e_j\rangle c(e_i)c(e_j) + \frac{1}{2}\Gamma^{\det}. \tag{2.37}$$

We also denote by ∇^{Cl} the connection on $\mathcal{E}$ induced by ∇^{Cl} and ∇^E.

Let Ω be the real $(1,1)$-form defined by

$$\Omega = g^{TM}(J\cdot\,,\cdot). \tag{2.38}$$

On $\Lambda^{0,\bullet}(T^*M)$, we define the *Bismut connection* ∇^{Bi} by

$$\nabla^{\mathrm{Bi}}_V = \nabla^{\mathrm{Cl}}_V + \frac{\sqrt{-1}}{4} c\big(i_V(\partial - \bar{\partial})\Omega\big). \tag{2.39}$$

This connection, along with ∇^E and ∇^L, induces connections $\nabla^{\mathcal{E}}$ and $\nabla^{\mathbb{E}_p}$ on $\mathcal{E}$ and $\mathbb{E}_p$. Moreover, we know that (see, e.g., [**15**, Thm. 1.4.5])

$$D_p = \sum_{i=1}^{2n} c(e_i)\nabla^{\mathbb{E}_p}_{e_i}. \tag{2.40}$$

Let $\Delta^{\mathbb{E}_p}$ be the Bochner Laplacian on $\mathbb{E}_p$ induced by $\nabla^{\mathbb{E}_p}$. It is given by the following formula: if (g^{ij}) is the inverse of the matrix $(g_{ij}) = (g^{TM}_Z(e_i, e_j))$, then

$$\Delta^{\mathbb{E}_p} = -g^{ij}\left(\nabla^{\mathbb{E}_p}_{e_i}\nabla^{\mathbb{E}_p}_{e_j} - \nabla^{\mathbb{E}_p}_{\nabla^{TM}_{e_i} e_j}\right). \tag{2.41}$$

Let r^M be the scalar curvature of (M, g^{TM}). Let $\Psi_{\mathcal{E}}$ be the smooth self-adjoint section of $\mathrm{End}(\mathcal{E})$ given by

$$\Psi_{\mathcal{E}} = \frac{r^M}{4} + c\big(R^E + \frac{1}{2}R^{\det}\big) + \frac{\sqrt{-1}}{2}c(\bar{\partial}\partial\Omega) - \frac{1}{8}\big|(\partial - \bar{\partial})\Omega\big|^2. \tag{2.42}$$

Set also

$$\begin{aligned} \omega_d &= -\sum_{i,j} R^L(w_i, \overline{w}_j)\overline{w}^j \wedge i_{\overline{w}_i}, \\ \tau &= \sum_i R^L(w_i, \overline{w}_i). \end{aligned} \tag{2.43}$$

The Lichnerowicz formula (see, for instance, [**15**, Thm. 1.4.7 and (1.5.17)]) reads

$$D_p^2 = \Delta^{\mathbb{E}_p} - p(2\omega_d + \tau) + \Psi_{\mathcal{E}}. \tag{2.44}$$

Let μ^E, μ^{Bi} and $\mu^{\mathbb{E}_p}$ be the moment maps induced by ∇^E, ∇^{Bi} and $\nabla^{\mathbb{E}_p}$ as in (1.3). Recall that μ is defined in (0.7). Then we have

$$\begin{cases} \mu^L = 2i\pi\mu, \\ \mu^{\mathbb{E}_p} = 2i\pi p\mu + \mu^E + \mu^{\mathrm{Bi}}. \end{cases} \tag{2.45}$$

Assume now that G acts freely on P, and recall that we then choose the G-invariant neighborhood U of P so that G acts freely on its closure $\overline{U}$. Using the procedure of Section 1 for $U \to U/G = B$ and $g^{TM}|_U$, we can define the operator $\Phi D_p^2 \Phi^{-1}$ induced by D_p^2 on B. Thanks to Theorem 1.1 and (2.44), we find that in the case of a free G-action on P,

$$\Phi D_p^2 \Phi^{-1} = \Delta^{\mathbb{E}_{p,B}} - p(2\omega_d + \tau) + \Psi_{\mathcal{E}} - \langle \widetilde{\mu}^{\mathbb{E}_p}, \widetilde{\mu}^{\mathbb{E}_p} \rangle_{g^{TY}} - \widehat{h}^{-1}\Delta_B \widehat{h}. \tag{2.46}$$

Here, we have kept the same notation for an element in $\mathscr{C}^\infty(U, \mathrm{End}(\mathbb{E}_p))^G$ and the induced element in $\mathscr{C}^\infty(B, \mathrm{End}(\mathbb{E}_{p,B}))$, and we will always do this in the sequel.

3. Localization near P

The goal of this section is to prove the localization of $P_G e^{-\frac{u}{p}D_p^2} P_G$ near P, i.e., we prove Theorem 0.8.

Let inj^M be the injectivity radius of (M, g^{TM}), and $\varepsilon \in]0, \mathrm{inj}^M[$.

For $x_0 \in M$, we denote by $B^M(x_0, \varepsilon)$ and $B^{T_{x_0}M}(0, \varepsilon)$ the open balls in M and $T_{x_0}M$ with center x_0 and 0 and radius ε, respectively. If $\exp^M_{x_0}$ is the exponential map of M, then $Z \in B^{T_{x_0}M}(0, \varepsilon) \mapsto \exp^M_{x_0}(Z) \in B^M(x_0, \varepsilon)$ is a diffeomorphism, which gives local coordinates by identifying $T_{x_0}M$ with $\mathbb{R}^{2n}$ via an orthonormal basis $\{e_i\}$ of $T_{x_0}M$:

$$(Z_1, \dots, Z_{2n}) \in \mathbb{R}^{2n} \mapsto \sum_i Z_i e_i \in T_x M. \tag{3.1}$$

From now on, we will always identify $B^{T_{x_0}M}(0, \varepsilon)$ and $B^M(x_0, \varepsilon)$.

Let $x_1, \dots x_N$ be points of M such that $\{U_k = B^M(x_k, \varepsilon)\}_{k=1}^N$ is an open covering of M. On each U_k we identify E_Z, L_Z and $\Lambda^{0,\bullet}(T_Z^*M)$ to E_{x_k}, L_{x_k} and $\Lambda^{0,\bullet}(T_{x_k}^*M)$ by parallel transport with respect to ∇^E, ∇^L and ∇^{Bi} along the geodesic ray $t \in [0,1] \mapsto tZ$. We fix for each $k = 1, \dots, N$ an orthonormal basis $\{e_i\}_i$ of $T_{x_k}M$ (without mentioning the dependence on k).

We denote by ∇_V the ordinary differentiation operator in the direction V on $T_{x_k}M$.

Let $\{\varphi_k\}_k$ be a partition of unity subordinate to $\{U_k\}_k$. For $\ell \in \mathbb{N}$, we define a Sobolev norm $||\cdot||_{\boldsymbol{H}^\ell(p)}$ on the ℓ-th Sobolev space $\boldsymbol{H}^\ell(M, \mathbb{E}_p)$ by

$$||s||^2_{\boldsymbol{H}^\ell(p)} = \sum_k \sum_{j=0}^{\ell} \sum_{i_1,\dots,i_j=1}^{d} ||\nabla_{e_{i_1}} \dots \nabla_{e_{i_j}}(\varphi_k s)||^2_{L^2}. \tag{3.2}$$

Lemma 3.1. *For any $m \in \mathbb{N}$, there exists $C_m > 0$ such that for any $p \in \mathbb{N}^*$ and any $s \in \boldsymbol{H}^{2m+2}(M, \mathbb{E}_p)$,*

$$||s||^2_{\boldsymbol{H}^{2m+2}(p)} \leq C_m p^{4m+4} \sum_{j=0}^{m+1} p^{-4j} ||D_p^{2j} s||_{L^2}. \tag{3.3}$$

Proof. This is proved in [**15**, Lem. 1.6.2]. q.e.d.

Let $f \colon \mathbb{R} \to [0,1]$ be a smooth even function such that

$$f(t) = \begin{cases} 1 \text{ for } |t| < \varepsilon/2, \\ 0 \text{ for } |t| > \varepsilon. \end{cases} \tag{3.4}$$

For $u > 0$, $\varsigma \geq 1$ and $a \in \mathbb{C}$, set

$$\begin{aligned} \mathrm{F}_u(a) &= \int_{\mathbb{R}} e^{iv\sqrt{2}a} \exp(-v^2/2) f(v\sqrt{u}) \frac{dv}{\sqrt{2\pi}}, \\ \mathrm{G}_u(a) &= \int_{\mathbb{R}} e^{iv\sqrt{2}a} \exp(-v^2/2)(1 - f(v\sqrt{u})) \frac{dv}{\sqrt{2\pi}}. \end{aligned} \tag{3.5}$$

These functions are even holomorphic functions. Moreover, the restrictions of F_u and G_u to $\mathbb{R}$ lie in the Schwartz space $\mathcal{S}(\mathbb{R})$, and

$$\mathrm{F}_u(vD_p) + \mathrm{G}_u(vD_p) = \exp\left(-v^2 D_p^2\right) \text{ for } v > 0. \tag{3.6}$$

Let $\mathrm{G}_u(vL_p)(x,x')$ be the smooth kernel of $\mathrm{G}_u(vL_p)$ with respect to $dv_M(x')$.

Proposition 3.2. *For any $m \in \mathbb{N}$, $u_0 > 0$, $\varepsilon > 0$, there exist $C > 0$ and $N \in \mathbb{N}$ such that for any $u > u_0$ and any $p \in \mathbb{N}^*$,*

$$\left| \mathrm{G}_{\frac{u}{p}} \left(\sqrt{u/p} D_p \right) (\cdot\,,\cdot) \right|_{\mathscr{C}^m(M\times M)} \leq C p^N \exp\left(-\frac{\varepsilon^2 p}{16u} \right). \tag{3.7}$$

Here, the $\mathscr{C}^m$-norm is induced by ∇^L, ∇^E, ∇^{Bi}, h^L, h^E and g^{TM}.

Proof. This is proved in [**15**, Prop. 1.6.4]. q.e.d.

Proof of Theorem 0.8. As 0 is a regular value of μ, there is ϵ_0 such that

$$\mu \colon M_{2\epsilon_0} := \mu^{-1}(B^{\mathfrak{g}^*}(0, 2\epsilon_0)) \to B^{\mathfrak{g}^*}(0, 2\epsilon_0) \tag{3.8}$$

is a submersion. Note that $M_{2\epsilon_0}$ is an open G-invariant subset of M.

Fix ε, ϵ_0 small enough so that $M_{2\epsilon_0} \subset U$ and $d^M(x,y) > 4\varepsilon$ if $x \in M_{\epsilon_0}$ and $y \in M \setminus U$. We set $V_{\epsilon_0} = M \setminus M_{\epsilon_0}$, which is a smooth G-manifold with boundary ∂V_{ϵ_0}. Then $M \setminus U \subset V_{\epsilon_0}$.

We denote by $D_{p,D}$ the operator D_p acting on V_{ϵ_0} with the Dirichlet boundary condition. Then $D_{p,D}$ is self-adjoint.

By [**21**, Sects. 2.6, 2.8] and [**15**, Append. D.2], we know that the wave operator $\cos(uD_{p,D})$ is well defined and its Schwartz kernel

$\cos(uD_{p,D})(x,x')$ only depends on the restriction of D_p to $G\cdot B^M(x,u)\cap V_{\epsilon_0}$ and vanish if $d^M(x,x')\geq u$. Thus, by (3.5),

$$\forall\, x,x'\in M\backslash U,\quad \mathrm{F}_{\frac{u}{p}}\left(\sqrt{u/p}D_p\right)(x,x')=\mathrm{F}_{\frac{u}{p}}\left(\sqrt{u/p}D_{p,D}\right)(x,x'). \tag{3.9}$$

Let $s\in\mathscr{C}^\infty(M,\mathbb{E}_p)^G$ with $\mathrm{supp}(s)\subset \overset{\circ}{V}_{\epsilon_0}$. Since D_p commutes with the G-action, we know that $D_ps\in\Omega^{0,\bullet}(M,L^p\otimes E)^G$. Moreover, from the Lichnerowicz formula (2.44), we get

$$\langle D_p^2s,s\rangle=\|\nabla^{\mathbb{E}_p}s\|^2_{L^2}-p\langle(\omega_d+\tau)s,s\rangle+\langle\Psi_{\mathcal{E}}s,s\rangle. \tag{3.10}$$

Observer that, as $s\in\Omega^{0,\bullet}(M,L^p\otimes E)^G$, (1.3) gives

$$\nabla^{\mathbb{E}_p}_{K^M}s=(\mathcal{L}_K+\mu^{\mathbb{E}_p}(K))s=\mu^{\mathbb{E}_p}(K)s, \tag{3.11}$$

and thus by (2.45) and the fact that $\mathrm{supp}(s)\subset \overset{\circ}{V}_{\epsilon_0}$,

$$\begin{aligned}\|\nabla^{\mathbb{E}_p}s\|^2_{L^2}\geq & C\sum_i\|\nabla^{\mathbb{E}_p}_{K_i^M}s\|^2_{L^2}=C\sum_i\|\mu^{\mathbb{E}_p}(K_i)s\|^2_{L^2}\\ &\geq Cp^2\|\,|\mu|s\|^2_{L^2}-C'\|s\|^2_{L^2}\\ &\geq C\epsilon_0^2p^2\|s\|^2_{L^2}-C'\|s\|^2_{L^2}.\end{aligned} \tag{3.12}$$

Thanks to (3.10) and (3.12), we have

$$\langle D_p^2s,s\rangle\geq Cp^2\|s\|^2_{L^2}. \tag{3.13}$$

In particular, as P_G preserve the Dirichlet boundary condition, there are $C,C'>0$ such that for $p\geq 1$,

$$\mathrm{Sp}(P_GD^2_{p,D}P_G)\subset[Cp^2,+\infty[\,. \tag{3.14}$$

By the elliptic estimate for the Laplacian with Dirichlet boundary condition [**21**, Thm. 5.1.3], we can see that the proof of Lemma 3.1 (see [**15**, Lem. 1.6.2]) still works if we replace therein D_p by $D_{p,D}$ and take $s\in\boldsymbol{H}^{2m+2}(M,\mathbb{E}_p)\cap\boldsymbol{H}^1_0(M,\mathbb{E}_p)$. Using this modification of Lemma 3.1, (3.14) and

$$\sup_{a\geq Cp}|a^m\mathrm{F}_{\frac{u}{p}}(a\sqrt{u/p})|\leq C_{m,k,u}p^{-k}, \tag{3.15}$$

we find that for any Q,Q differential operators of order $2m,2m'$ with scalar principal symbol and with support in U_i,U_j and for any $k\in\mathbb{N}$

$$\left\|Q\mathrm{F}_{\frac{u}{p}}\left(\sqrt{u/p}D_{p,D}\right)Q's\right\|_{L^2}\leq C_{m,m',u}p^{-k}\|s\|_{L^2}. \tag{3.16}$$

Thus, using Sobolev inequalities with (3.16), and (3.6), (3.7) and (3.9), we get Theorem 0.8. q.e.d.

4. Asymptotic of the heat kernel near P for a free action

We assume in this Section that G acts freely on P and $\overline{U}$.

In this section, we prove Theorem 0.9. In Section 4.1, we work near P and replace our geometric setting by a model setting, in which M is replaced by $G\times\mathbb{R}^{2n-d}$, P by $G\times\mathbb{R}^{2n-2d}\times\{0\}$ and the different bundles are trivial. We can then define a rescaled version of $\frac{1}{p}D_p^2$, and in Section 4.2, we prove the convergence of the heat kernel of the rescaled operator. In Section 4.3 we compute the limiting heat kernel to finish the proof of Theorem 0.9.

4.1. Rescaling the operator $\Phi D_p^2\Phi^{-1}$. This section is analogous to [**16**, Sect. 2.6], with the necessary changes made.

We fix $x_0 \in M_G$ and $\varepsilon \in]0, \mathrm{inj}^M/4[$.

Recall that we have the following diagram:

$$\begin{array}{ccc} P=\mu^{-1}(0) & \hookrightarrow & U \\ \downarrow G & & \downarrow G \\ M_G & \hookrightarrow & B \end{array}$$

Recall also that g^{T^HP} is a G-invariant and J-invariant metric on T^HP, g^{TY} is a G-invariant metric on TY and g^{JTY} is the G-invariant metric on JTY induced by J and g^{TY}. Then by (2.19), we can chose be a G-invariant metric g^{TM} on M such that on P:

$$g^{TM}|_P = g^{TY}|_P \oplus g^{JTY}|_P \oplus g^{T^HP}. \tag{4.1}$$

Let g^{T^HU} be the restriction of g^{TM} on T^HU. Let g^{TB} (resp. g^{TM_G}) be the metric on TB (resp. TM_G) induced by g^{T^HU} (resp. g^{T^HP}).

By (0.24) and Lemma 2.4, we know that

$$T^HU|_P = JTY|_P \oplus JT^HP = JTY|_P \oplus T^HP. \tag{4.2}$$

As a consequence, if N_G denotes the normal bundle of M_G in B, then N_G can be identified as

$$N_G \simeq (TM_G)^{\perp_{g^{TB}}} = (JTY)_B|_{M_G}, \tag{4.3}$$

where $(JTY)_B$ denotes the bundle over B induced by JTY.

Let ∇^{TB} be the Levi-Civita connection on (TB, g^{TB}). Let P^{N_G} and P^{TM_G} be the orthogonal projections from $TB|_{M_G}$ to N_G and TM_G, respectively. Set

$$\begin{aligned} &\nabla^{N_G} = P^{N_G}(\nabla^{TB}|_{M_G})P^{N_G}, \quad && \nabla^{TM_G} = P^{TM_G}(\nabla^{TB}|_{M_G})P^{TM_G}, \\ &{}^0\nabla^{TB} = \nabla^{N_G} \oplus \nabla^{TM_G}, && A = \nabla^{TB}|_{M_G} - {}^0\nabla^{TB}. \end{aligned} \tag{4.4}$$

For $W \in T_{x_0}M_G$, let $u \in \mathbb{R} \mapsto x_u = \exp_{x_0}^{M_G}(uW) \in M_G$ be the geodesic in M_G starting at x_0 with speed W. If $|W| \le 4\varepsilon$ and $V \in N_{G,x_0}$, let $\tau_W V$ be the parallel transport of V with respect to ∇^{N_G} along to curve $u \in [0,1] \mapsto x_u = \exp_{x_0}^{M_G}(uW)$.

If $Z \in T_{x_0}B$, we decompose Z as $Z = Z^0 + Z^\perp$ with $Z^0 \in T_{x_0}M_G$ and $Z^\perp \in N_{G,x_0}$, and if $|Z^0|, |Z^\perp| \leq \varepsilon$ we identify Z with $\exp^B_{\exp_{x_0}^{M_G}(Z^0)}(\tau_{Z^0}Z^\perp)$. This gives a diffeomorphism

$$\mathcal{F}\colon B^{T_{x_0}M_G}(0,4\varepsilon) \times B^{N_{G,x_0}}(0,4\varepsilon) \xrightarrow{\sim} \mathscr{U}(x_0) \subset B, \tag{4.5}$$

where $\mathscr{U}(x_0)$ is an open neighborhood of x_0 in B. Note that under this diffeomorphism, $\mathscr{U}(x_0) \cap M_G$ is identified with $(B^{T_{x_0}M_G}(0,4\varepsilon) \times \{0\})$.

In the sequel, we will indifferently write $B^{T_{x_0}M_G}(0,4\varepsilon) \times B^{N_{G,x_0}}(0,4\varepsilon)$ or $\mathscr{U}(x_0)$, x_0 or 0, etc.

We identify $(L_B)_Z$, $(E_B)_Z$ and $(\mathbb{E}_p)_{B,Z}$ with $(L_B)_{x_0}$, $(E_B)_{x_0}$ and $(\mathbb{E}_p)_{B,x_0}$ by using parallel transport with respect to ∇^{L_B}, ∇^{E_B} and $\nabla^{(\mathbb{E}_p)_B}$ along the curve $u \in [0,1] \mapsto \gamma_u = uZ$.

Fix $y_0 \in \pi^{-1}(x_0)$. We define $\widetilde{\gamma}\colon [0,1] \to M$ to be the curve lifting γ such that $\frac{\partial \widetilde{\gamma}_u}{\partial u} \in T^H_{\widetilde{\gamma}_u}U$. As above, on $\pi^{-1}(B^{T_{x_0}B}(0,4\varepsilon))$, we can trivialize L, E and $\mathbb{E}_p$ using the parallel transport along $\widetilde{\gamma}$ with respect to the corresponding connections. By (1.6), the previous trivialization are naturally induced by this one.

This also gives a diffeomorphism

$$\pi^{-1}(B^{T_{x_0}B}(0,4\varepsilon)) \simeq G \times B^{T_{x_0}B}(0,4\varepsilon), \tag{4.6}$$

and the induced G-action on $G \times B^{T_{x_0}B}(0,\varepsilon)$ is then

$$g.(g', Z) = (gg', Z). \tag{4.7}$$

Let $\{e_i^0\}$ and $\{e_i^\perp\}$ be orthonormal basis of $T_{x_0}M_G$ and N_{G,x_0}, respectively. Then $\{e_i\} = \{e_i^0, e_i^\perp\}$ is an orthonormal basis of $T_{x_0}B$. Let $\{e^i\}$ be its dual basis. We will also denote $\mathcal{F}_*(e_i^0)$, $\mathcal{F}_*(e_i^\perp)$ by $\{e_i^0\}$, $\{e_i^\perp\}$, so that in our coordinates,

$$\frac{\partial}{\partial Z_i^0} = e_i^0\,, \quad \frac{\partial}{\partial Z_i^\perp} = e_i^\perp. \tag{4.8}$$

In what follows, we will extend the geometric object from $B^{T_{x_0}B}(0,4\varepsilon)$ to $\mathbb{R}^{2n-d} \simeq T_{x_0}B$ (here the identification is similar to (3.1)) to get analogue geometric structures on $G \times \mathbb{R}^{2n-d}$ as on M, an thus work on

$$M_0 := G \times \mathbb{R}^{2n-d}, \tag{4.9}$$

instead of M.

Let L_0 be the trivial bundle $L|_{G.y_0}$ lifted on M_0. We still denote by ∇^L, h^L the connection and metric on L_0 over $\pi^{-1}(B^{T_{x_0}B}(0,4\varepsilon))$ induced by the above identification. Then h^L is identified with the constant metric $h^{L_0} = h^{L_{y_0}}$. We use similar notations for the bundle E.

Let $\varphi\colon \mathbb{R} \to [0,1]$ be a smooth even function such that

$$\varphi(v) = \begin{cases} 1 \text{ for } |v| < 2, \\ 0 \text{ for } |v| > 4. \end{cases} \tag{4.10}$$

Let $\varphi_\varepsilon\colon M_0 \to M_0$ be defined by

$$\varphi_\varepsilon(g,Z) = (g, \varphi(|Z|/\varepsilon)Z). \tag{4.11}$$

Let $\nabla^{E_0} = \varphi_\varepsilon^* \nabla^E$. Then ∇^{E_0} is an extension of ∇^E outside $\pi^{-1}(B^{T_{x_0}B}(0,4\varepsilon))$.

Let P^{TY} be the orthogonal projection from TM onto TY. For $W \in TB$, let $W^H \in T^H U$ be the horizontal lift of W. Then we can define the Hermitian connection ∇^{L_0} on $(L_0, h^{L_0}) \to G \times \mathbb{R}^{2n-d}$ by

$$\nabla^{L_0} = \varphi_\varepsilon^* \nabla^L + \big(1 - \varphi(|Z|/\varepsilon)\big) R^L_{y_0}(Z^H, P^{TY}_{y_0}\cdot). \tag{4.12}$$

We can compute directly the curvature R^{L_0} of ∇^{L_0}: if we denote $(1,Z)$ just by Z, then

$$\begin{aligned} R^{L_0}_Z =& R^L_{\varphi_\varepsilon(Z)}(P^{TY}_{y_0}\cdot, P^{TY}_{y_0}\cdot) + R^L_{y_0}(P^{T^HU}_{y_0}\cdot, P^{TY}_{y_0}\cdot) \\ &+ \varphi^2(|Z|/\varepsilon) R^L_{\varphi_\varepsilon(Z)}(P^{T^HU}_{y_0}\cdot, P^{T^HU}_{y_0}\cdot) \\ &+ \varphi(|Z|/\varepsilon)\big[R^L_{\varphi_\varepsilon(Z)} - R^L_{y_0}\big](P^{T^HU}_{y_0}\cdot, P^{TY}_{y_0}\cdot) \\ &+ \varphi'(|Z|/\varepsilon)\frac{Z^*}{\varepsilon|Z|} \wedge \big[R^L_{\varphi_\varepsilon(Z)} - R^L_{y_0}\big](Z^H, P^{TY}_{y_0}\cdot) \\ &+ (\varphi\varphi')(|Z|/\varepsilon)\frac{Z^*}{\varepsilon|Z|} \wedge R^L_{\varphi_\varepsilon(Z)}(Z^H, P^{T^HU}_{y_0}\cdot), \end{aligned} \tag{4.13}$$

where $Z^* \in T^*_{x_0}B$ is the dual of $Z \in T_{x_0}B$ with respect to the metric $g^{TB}_{x_0}$.

The group G acts naturally on M_0 by (4.7) and under our identifications, the action of G on L, E on $G \times^{T_{x_0}B} (0,\varepsilon)$ is exactly the G-action on $L|_{G.y_0}$, $E|_{G.y_0}$.

We define a G-action on L_0, E_0 by the action of G on $G.y_0$. Then it extends the G-action on L, E on $G \times^{T_{x_0}B} (0,\varepsilon)$ to M_0.

By Lemma 2.4, we know that

$$R^L_{(1,Z^0)}(Z^H, K^M) = R^L_{(1,Z^0)}\big((Z^\perp)^H, K^M\big). \tag{4.14}$$

For $(1,Z) \in G \times \mathbb{R}^{2n-d}$, (4.7) gives $\varphi_{\varepsilon *} K^{M_0}_{(1,Z)} = K^M_{y_0}$ for $K \in \mathfrak{g}$. Thus, by (0.7), (4.12) and (4.14), the moment map $\mu_0\colon M_0 \to \mathfrak{g}^*$ of the G-action on (M_0, L_0) is given by

$$\mu_0(K)_{(1,Z)} = \mu(K)_{\varphi_\varepsilon(1,Z)} + \frac{1}{2i\pi}\big(1 - \varphi(|Z|/\varepsilon)\big) R^L_{y_0}((Z^\perp)^H, K^M_{y_0}). \tag{4.15}$$

Now, from the construction of our coordinate, we have $\mu_0 = 0$ on $G \times \mathbb{R}^{2n-2d} \times \{0\}$. Moreover,

(4.16)
$$\mu(K)_{\varphi_\varepsilon(1,Z)} = \frac{1}{2i\pi} R^L_{(1,Z)}\big(\varphi(|Z|/\varepsilon)(Z^\perp)^H, K^M\big) + O\big(\varphi(|Z|/\varepsilon)|Z||Z^\perp|\big).$$

Thus, from our construction, Lemma 2.3 and (4.3), (4.15) and (4.16), we know that

$$\mu_0^{-1}(0) = G \times \mathbb{R}^{2n-2d} \times \{0\}. \tag{4.17}$$

Let

$$g^{TM_0}(g,Z) = g^{TM}(\varphi_\varepsilon(g,Z)) \quad \text{and} \quad J_0(g,Z) = J(\varphi_\varepsilon(g,Z)) \tag{4.18}$$

be the metric and almost-complex structure on M_0. Let $T^{*(0,1)}M_0$ be the anti-holomorphic cotangent bundle of (M_0, J_0). Since $J_0(g,Z) = J(\varphi_\varepsilon(g,Z))$, $T^{*(0,1)}_{(g,Z),J_0}M_0$ is naturally identified with $T^{*(0,1)}_{\varphi_\varepsilon(g,Z),J}M_0$.

We can now construct all the objects corresponding to those of Section 2.2 in this new setting and denotes them by adding subscripts 0, e.g., $\mathbb{E}_{0,p}$, $\nabla^{\det_0}$, ∇^{Cl_0}, ∇^{Bi_0}, $\nabla^{\mathbb{E}_{0,p}}, \dots$. Then, we can define the Dirac operator $D_p^{M_0}$ on M_0, which satisfies

$$D_p^{M_0,2} = \Delta^{\mathbb{E}_{0,p}} - p(2\omega_{d,0} + \tau_0) + \Psi_{\mathcal{E}_0}. \tag{4.19}$$

By (2.44) and the above constructions, we know that D_p^2 and $D_p^{M_0,2}$ coincide on $\pi^{-1}(B^{T_{x_0}B}(0,2\varepsilon))$.

We can identify $\Lambda^{0,\bullet}(T^*_{(g,Z)}M_0)$ with $\Lambda^{0,\bullet}(T^*_{gy_0}M)$ by identifying first $\Lambda^{0,\bullet}(T^*_{(g,Z)}M_0)$ with $\Lambda^{0,\bullet}(T^*_{\varphi_\varepsilon(g,Z),J}M)$ and then identifying $\Lambda^{0,\bullet}(T^*_{\varphi_\varepsilon(g,Z),J}M)$ with $\Lambda^{0,\bullet}(T^*_{gy_0}M)$ by parallel transport with respect to ∇^{Bi_0} (see (2.39)) along $u \in [0,1] \mapsto (g, u\varphi(|Z|/\varepsilon)Z)$. We also trivialize $\det(T^{(1,0)}M_0)$ in this way using $\nabla^{\det_0}$.

Let g^{TB_0} be the metric on $B_0 := \mathbb{R}^{2n-d}$ induced by g^{TM_0}, and let dv_{B_0} be the corresponding Riemannian volume. We denote by TY_0 the relative tangent bundle of the fibration $M_0 \to B_0$, and by g^{TY_0} the metric on TY_0 induced by g^{TM_0}.

The operator $\Phi D_p^{M_0,2}\Phi^{-1}$ is also well-defined on $T_{x_0}B \simeq \mathbb{R}^{2n-d}$. More precisely, it is an operator on the bundle $(\mathbb{E}_{0,p})_{B_0}$ over B_0 induced by $\mathbb{E}_{0,p}$, and by (2.46), it is given by

$$\Phi D_p^{M_0,2}\Phi^{-1} = \Delta^{(\mathbb{E}_{0,p})_{B_0}} - p(2\omega_{0,d} + \tau_0) + \Psi_{\mathcal{E}_0} - \langle \widetilde{\mu}^{\mathbb{E}_{0,p}}, \widetilde{\mu}^{\mathbb{E}_{0,p}} \rangle_{g^{TY_0}} - \frac{1}{h_0}\Delta_{B_0} h_0. \tag{4.20}$$

Let $\exp(-uD_p^{M_0,2})(Z,Z')$ be the smooth heat kernel of $D_p^{M_0,2}$ with respect to $dv_{M_0}(Z')$, the volume form associated with g^{TM_0}.

Lemma 4.1. *Under notation of Proposition* 3.2 *and the above trivializations, the following estimate holds uniformly on* $v = (g, Z), v' = (g', Z') \in G \times B^{T_{x_0}B}(0, \varepsilon)$*:*

$$\left| e^{-\frac{u}{p}D_p^2}(v, v') - e^{-\frac{u}{p}D_p^{M_0,2}}\Big((g,Z),(g',Z')\Big)\right| \leq Cp^N \exp\big(-\frac{\varepsilon^2 p}{16u}\big). \tag{4.21}$$

Proof. By (4.19), $D_p^{M_0,2}$ has the same structure as D_p^2. Thus Lemma 3.1 and Proposition 3.2 are still true if we replace D_p^2 therein by $D_p^{M_0,2}$. Moreover, as $D_p^{M_0,2}$ and D_p^2 coincide for $|Z|$ small, by the finite propagation speed of the wave equation (see, e.g., [**15**, Thm. D.2.1]), we know that

$$\mathrm{F}_{\frac{u}{p}}\left(\sqrt{u/p}D_p\right)(v,\cdot) = \mathrm{F}_{\frac{u}{p}}\left(\sqrt{u/p}D_p^{M_0}\right)\big((g,Z),\cdot\big), \tag{4.22}$$

if $v = (g, Z)$ under the above trivializations. Thus, we get our Lemma by (3.6). q.e.d.

We still denote P_G the orthogonal projection from $\Omega^{0,\bullet}(U, L^p \otimes E)$ onto $\Omega^{0,\bullet}(U, L^p \otimes E)^G$. Let dg be the Haar measure on G. Then we have

$$\big(P_G e^{-\frac{u}{p}D_p^2} P_G\big)(v, v') = \int_{G\times G} (g, g'^{-1}) \cdot e^{-\frac{u}{p}D_p}(g^{-1}v, g'v')dgdg'. \tag{4.23}$$

If we again denote by P_G the orthogonal projection from $\Omega^{0,\bullet}(M_0, L_0^p \otimes E_0)$ onto $\Omega^{0,\bullet}(M_0, L_0^p \otimes E_0)^G$, then we have a similar formula as (4.23) for $(P_G e^{-\frac{u}{p}D_p^{M_0,2}} P_G)$. Thus, as G preserves every metric and connection, Lemma 4.1 implies

Corollary 4.2. *Under notation of Proposition* 3.2*, the following estimate holds uniformly on* $v = (g, Z), v' = (g', Z') \in G \times B^{T_{x_0}B}(0, \varepsilon)$*:*

$$\begin{aligned}\left|\big(P_G e^{-\frac{u}{p}D_p^2} P_G\big)(v, v) - \big(P_G e^{-\frac{u}{p}D_p^{M_0,2}} P_G\big)\big((g,Z),(g',Z')\big)\right| \\ \leq Cp^N \exp\big(-\frac{\varepsilon^2 p}{16u}\big).\end{aligned} \tag{4.24}$$

Let S_L be a G-invariant unit section of $L|_{Gy_0}$. Let pr be the projection $G \times \mathbb{R}^{2n-d} \to G$. Using S_L and the above discussion, we get two isometries

$$\begin{aligned}&\mathbb{E}_{0,p} = \Lambda^{0,\bullet}(T^*M_0) \otimes E_0 \otimes L_0^p \simeq \mathrm{pr}^*(\mathcal{E}|_{Gy_0}) \\ &(\mathbb{E}_{0,p})_{B_0} \simeq \mathcal{E}_{B,x_0}.\end{aligned} \tag{4.25}$$

Thus, $\Phi D_p^{M_0,2}\Phi^{-1}$ can be seen as an operator on $\mathcal{E}_{B,x_0}$. Note that our formulas will not depend on the choice of S_L as the isomorphism $\mathrm{End}((\mathbb{E}_{0,p})_{B_0}) \simeq \mathrm{End}(\mathcal{E}_{B,x_0})$ is canonical.

Let dv_{TB} be the Riemannian volume of $(T_{x_0}B, g^{T_{x_0}B})$. Let κ be the smooth positive function defined by

$$dv_{B_0}(Z) = \kappa(Z) dv_{TB}(Z) = \kappa(Z) dv_{M_G}(x_0) dv_{N_{G,x_0}}, \tag{4.26}$$

with $\kappa(0) = 1$.

As in (1.7), we denote by R^{L_B}, R^{E_B} and R^{Bi_B} the curvature on L_B, E_B and $(\Lambda^{0,\bullet}(T^*M))_B$ induced by ∇^L, ∇^E and ∇^{Bi} on M.

As in (1.4), $\widetilde{\mu} \in TY$, $\widetilde{\mu}^E \in TY \otimes \mathrm{End}(E)$ and $\widetilde{\mu}^{\mathrm{Bi}} \in TY \otimes \mathrm{End}(\Lambda^{0,\bullet}(T^*M))$ are the sections induced by μ, μ^E and μ^{Bi} in (1.3) and (2.45).

We denote by ∇_V the ordinary differentiation operator on $T_{x_0}B = B_0$ in the direction V.

We will now make the change of parameter $t = \frac{1}{\sqrt{p}} \in]0,1]$.

Definition 4.3. For $s \in \mathscr{C}^\infty(\mathbb{R}^{2n-d}, \mathcal{E}_{B,x_0})$ and $Z \in \mathbb{R}^{2n-d}$ set

$$\begin{aligned}
&(S_t s)(Z) = s(Z/t),\\
&\nabla_t = t S_t^{-1} \kappa^{1/2} \nabla^{(\mathbb{E}_{0,p})_{B_0}} \kappa^{-1/2} S_t,\\
&\nabla_0 = \nabla + \frac{1}{2} R^{L_B}_{x_0}(Z, \cdot),\\
&\mathscr{L}_t = t^2 S_t^{-1} \kappa^{1/2} \Phi D_p^{M_0,2} \Phi^{-1} \kappa^{-1/2} S_t,\\
&\mathscr{L}_0 = -\frac{1}{2} \sum_{i=1}^{2n-d} (\nabla_{0,e_i})^2 - 2\omega_{d,x_0} - \tau_{x_0} + 4\pi^2 |P^{TY} \mathbf{J}_{x_0} Z|^2.
\end{aligned} \tag{4.27}$$

Proposition 4.4. *When $t \to 0$, we have*

$$\nabla_{t,e_i} = \nabla_{0,e_i} + O(t) \text{ and } \quad \mathscr{L}_t = \mathscr{L}_0 + O(t). \tag{4.28}$$

Proof. Let Γ^{L_B}, Γ^{E_B} and Γ^{Bi_B} be the connection form of ∇^{L_B}, ∇^{E_B} and ∇^{Bi_B} with respect to fixed frame of L_B, E_B and $(\Lambda^{0,\bullet}(T^*M))_B$ which are parallel along the curve $u \in [0,1] \mapsto uZ$ under our trivialization on $B^{T_{x_0}B}(0, 4\varepsilon)$.

By (4.27), we have for $|Z| \le \varepsilon/t$

(4.29)

$$\begin{aligned}
&\nabla_{t,e_i}(Z)\\
&\quad = \kappa^{1/2}(tZ) \Big\{ \nabla_{e_i} + \Big(t^{-1} \Gamma^{L_B}_{tZ}(e_i) + t \Gamma^{E_B}_{tZ}(e_i) + t \Gamma^{\mathrm{Bi}_B}_{tZ}(e_i) \Big) \Big\} \kappa^{-1/2}(tZ).
\end{aligned}$$

It is a well known fact (see, for instance, [**15**, Lemma 1.2.4]) that for if $\Gamma = \Gamma^{L_B}$ (resp. Γ^{E_B}, Γ^{Bi_B}) and $R = R^L$ (resp. R^{E_B}, R^{Bi_B}), then

$$\Gamma_Z(e_i) = \frac{1}{2} R_{x_0}(Z, e_i) + O(|Z|^2). \tag{4.30}$$

Thus,

$$t\Gamma^{E_B}_{tZ}(e_i) + t\Gamma^{\mathrm{Bi}_B}_{tZ}(e_i) = O\left(t^2\right),$$
$$t^{-1}\Gamma^{L}_{tZ}(e_i) = \frac{1}{2}R^L_{x_0}(Z,e_i) + O\left(t\right). \tag{4.31}$$

The first asymptotic development in Proposition 4.4 follows from $\varphi(0) = \kappa(0) = 1$, (4.29), (4.30) and (4.31).

Let $(g^{ij}(Z))$ be the inverse of the matrix $(g_{ij}(Z)) := (g^{T_{x_0}B}_Z(e_i,e_j))$. By (2.41), (4.20) and (4.27) we have

$$\begin{aligned}\mathscr{L}_t(Z) = -g^{ij}(tZ)\Big(\nabla_{t,e_i}\nabla_{t,e_j} - t\nabla_{t,\nabla^{TB_0}_{e_i}e_j}\Big) \\ - \langle t\widetilde{\mu}^{\mathbb{E}_{0,p}}, t\widetilde{\mu}^{\mathbb{E}_{0,p}}\rangle_{g^{TY}}(tZ) - (2\omega_{0,d} + \tau_0)(tZ) \\ + t^2\Big(\Psi_{\mathcal{E}_0} + \frac{1}{h_0}\Delta_{B_0}h_0\Big)(tZ).\end{aligned} \tag{4.32}$$

With the asymptotic of ∇_t above, (2.41) and the fact that $g^{ij}(0) = \delta_{ij}$ we find

$$-g^{ij}(tZ)\Big(\nabla_{t,e_i}\nabla_{t,e_j} - t\nabla_{t,\nabla^{TB_0}_{e_i}e_j}\Big) = \sum_i \left(\nabla_{0,e_i}\right)^2 + O(t). \tag{4.33}$$

Moreover,

$$\begin{aligned}-(2\omega_{0,d} + \tau_0)(tZ) + t^2\Big(\Psi_{\mathcal{E}_0} + \frac{1}{h_0}\Delta_{B_0}h_0\Big)(tZ) \\ = -2\omega_{d,x_0} - \tau_{x_0} + O(t).\end{aligned} \tag{4.34}$$

Now, by (2.1), (0.26) and the fact that $\widetilde{\mu}_{y_0} = 0$ for $y_0 \in P$, $\pi(y_0) = x_0$, we get for $K \in \mathfrak{g}$:

$$\begin{aligned}-\langle \mathbf{J}e^H_i, K^M\rangle_{y_0} &= \omega(K^M, e^H_i)_{y_0} \\ &= \nabla_{e^H_i}(\mu(K))(y_0) = \langle \nabla^{TY}_{e^H_i}\widetilde{\mu}, K^M\rangle_{y_0}.\end{aligned} \tag{4.35}$$

Thus,

$$|\widetilde{\mu}|^2_{g^{TY}}(Z) = |\nabla^{TY}_Z\widetilde{\mu}|^2_{g^{TY}} + O(|Z|^3) = |P^{TY}\mathbf{J}_{x_0}Z|^2_{g^{TY}} + O(|Z|^3). \tag{4.36}$$

Note that

$$\begin{aligned}\langle t\widetilde{\mu}^{\mathbb{E}_p}, t\widetilde{\mu}^{\mathbb{E}_p}\rangle_{g^{TY}} = \\ -4\pi^2\frac{1}{t^2}|\widetilde{\mu}|^2_{g^{TY}} + \left\langle 4i\pi\widetilde{\mu} + t^2(\widetilde{\mu}^E + \widetilde{\mu}^{\mathrm{Bi}}), \widetilde{\mu}^E + \widetilde{\mu}^{\mathrm{Bi}}\right\rangle_{g^{TY}}.\end{aligned} \tag{4.37}$$

Thus, we get the second asymptotic development in Proposition 4.4 by using (4.32), (4.33), (4.34), (4.36) and (4.37). q.e.d.

4.2. Convergence of the heat kernel. In this section, we prove the convergence of the heat kernel of the rescaled operator. Note that here we must have a more precise result than in [**15**, Sect. 1.6] because in the proof of Theorem 0.3 (see Section 5.1) we will have to integrate along the normal directions, and thus we need a result of decay in these directions. To obtain it, we draw our inspiration form [**16**].

Recall that $\mathcal{E}_0 = \Lambda^{0,\bullet}(T^*M_0) \otimes E_0$ and that we have trivialized the Hermitian bundle $(\mathcal{E}_{0,B_0}, h^{\mathcal{E}_{0,B_0}})$ on $B_0 = T_{x_0}B$ by identifying it to $(\mathcal{E}_{B,x_0}, h^{\mathcal{E}_{B,x_0}})$. Recall also that $\mu_0 \colon M_0 \to \mathfrak{g}^*$ is the moment map of the G-action on M_0.

Let $\|\cdot\|_{L^2}$ be the L^2-norm on $\mathscr{C}^\infty(B_0, \mathcal{E}_{B,x_0})$ induced by $g^{T_{x_0}B}$ and $h^{\mathcal{E}_{B,x_0}}$ as in (1.8).

Let $\{f_l\}$ be a G-invariant orthonormal frame of TY on $\pi^{-1}(B^B(x_0, 4\varepsilon))$, then $\{f_{0,l}(Z) = f_l(\varphi_\varepsilon(g, Z))\}$ is a G-invariant orthonormal frame of TY_0 on M_0.

Definition 4.5. Set

$$\mathcal{D}_t = \Big\{\nabla_{t,e_i}\,,\ 1 \le i \le 2n-d\,;\ \frac{1}{t}\langle\widetilde{\mu}_0, f_{0,l}\rangle(tZ)\,,\ 1 \le l \le d\Big\}, \tag{4.38}$$

and for $k \in \mathbb{N}^*$, let $\mathcal{D}_t^m$ be the family of operators Q acting on $\mathscr{C}^\infty(T_{x_0}B, \mathcal{E}_{B,x_0})$ which can be written in the form $Q = Q_1 \dots Q_m$ with $Q_i \in \mathcal{D}_t$.

For $s \in \mathscr{C}^\infty(B_0, \mathcal{E}_{B,x_0})$ and $k \in \mathbb{N}^*$, set

$$\begin{aligned} ||s||_{t,0}^2 &= ||s||_{L^2}^2, \\ ||s||_{t,m}^2 &= ||s||_{t,0}^2 + \sum_{\ell=1}^{m} \sum_{Q \in \mathcal{D}_t^\ell} ||Qs||_{t,0}^2. \end{aligned} \tag{4.39}$$

We denote by $\boldsymbol{H}_t^m$ the Sobolev space $\boldsymbol{H}^m(B_0, \mathcal{E}_{B,x_0})$ endowed with the norm $||\cdot||_{t,m}$, and by $\boldsymbol{H}_t^{-1}$ the Sobolev space of order -1 endowed with the norm

$$||s||_{t,-1} = \sup_{s' \in \boldsymbol{H}_p^1 \setminus \{0\}} \frac{\langle s, s'\rangle_{t,0}}{||s'||_{t,0}}\,. \tag{4.40}$$

Finally, if $A \in \mathscr{L}(\boldsymbol{H}_t^k, \boldsymbol{H}_t^m)$, we denote by $||A||_t^{k,m}$ the operator norm of A associated with $||\cdot||_{t,k}$ and $||\cdot||_{t,m}$.

Then $\mathscr{L}_t$ is a formally self-adjoint elliptic operator with respect to $\|\cdot\|_{t,0}$ and is a smooth family of operators with respect to the parameter $x_0 \in M_G$.

We denote by $\mathscr{C}_c^\infty(B_0, \mathcal{E}_{B,x_0})$ the set of smooth section of $\mathcal{E}_{B,x_0}$ over B_0 with compact support.

Proposition 4.6. *There exist constants $C_1, C_2, C_3 > 0$ such that for any $t \in]0,1]$ and any $s, s' \in \mathscr{C}_c^\infty(B_0, \mathcal{E}_{B,x_0})$,*

$$\begin{aligned} \langle \mathscr{L}_t s, s\rangle_{t,0} &\ge C_1||s||_{t,1}^2 - C_2||s||_{t,0}^2, \\ \big|\langle \mathscr{L}_t s, s'\rangle_{t,0}\big| &\le C_3||s||_{t,1}||s'||_{t,1}. \end{aligned} \tag{4.41}$$

Proof. From (4.32) and (4.39), we have

$$(4.42)\quad \langle \mathscr{L}_t s, s\rangle_{t,0} = \|\nabla_t s\|^2_{t,0} - t^2\Big\langle \langle \widetilde{\mu}^{\mathbb{E}_{0,p}}, \widetilde{\mu}^{\mathbb{E}_{0,p}}\rangle_{g^{TY}}(tZ)s, s\Big\rangle_{t,0} + \left\langle S_t^{-1}\Big(-(2\omega_{0,d}+\tau_0) + t^2\Big(\Psi_{\mathcal{E}_0} + \frac{1}{h_0}\Delta_{B_0} h_0\Big)\Big)s, s\right\rangle_{t,0}.$$

By (4.15) and our constructions, we know that for $Z \in T_{\mathbb{R},x_0}B$ with $|Z| > 4\varepsilon$,

$$(4.43)\quad \mu^{\mathbb{E}_{0,p}}(K)_{(1,Z)} = 2i\pi p\,\mu_0(K)_{(1,Z)} = pR^L_{y_0}\big((Z^\perp)^H, K^X_{y_0}\big).$$

Thus, from (1.12), (4.15), (4.37) and (4.43), we get

$$(4.44)\quad -t^2\Big\langle \langle \widetilde{\mu}^{\mathbb{E}_{0,p}}, \widetilde{\mu}^{\mathbb{E}_{0,p}}\rangle_{g^{TY}}(tZ)s, s\Big\rangle_{t,0} \geq 2\pi^2 \sum_{l=1}^{d} \Big\| \frac{1}{t}\langle \widetilde{\mu}_0, f_{0,l}\rangle(tZ)s\Big\|^2_{t,0} - Ct\|s\|^2_{t,0}.$$

Now, (4.41) follows from (4.42) and (4.44). q.e.d.

Let Γ be the contour in $\mathbb{C}$ defined in Figure 1.

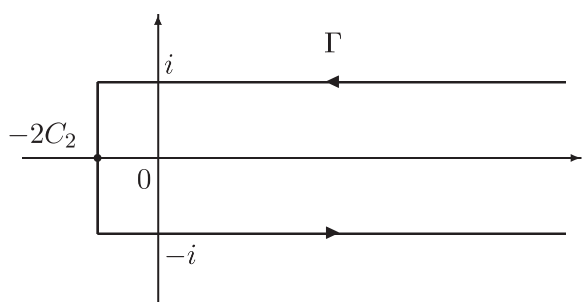

Figure 1

Proposition 4.7. *There exist $t_0 > 0$ and $C > 0$, $a, b \in \mathbb{N}$ such that for any $t \in]0, t_0]$ and any $\lambda \in \Gamma$, the resolvent $(\lambda - \mathscr{L}_t)^{-1}$ exists and*

$$(4.45)\quad \begin{aligned} \Big\|(\lambda - \mathscr{L}_t)^{-1}\Big\|^{0,0}_t &\leq C, \\ \Big\|(\lambda - \mathscr{L}_t)^{-1}\Big\|^{-1,1}_t &\leq C(1+|\lambda|^2). \end{aligned}$$

Proof. Note that $\mathscr{L}_t$ is self-adjoint operator, thus (4.41) implies that $(\lambda - \mathscr{L}_t)^{-1}$ exists for $\lambda \in \Gamma$ and there is a constant $C > 0$ (independent of λ) such that

$$(4.46)\quad \Big\|\big(\lambda - \mathscr{L}_t\big)^{-1}\Big\|^{0,0}_t \leq C.$$

On the other hand, if $\lambda_0 \in]-\infty, -2C_2]$, then (4.41) also implies that

$$\left\|\left(\lambda_0 - \mathscr{L}_t\right)^{-1}\right\|_t^{-1,1} \leq \frac{1}{C_1}. \tag{4.47}$$

Then, using the fact that

$$\left(\lambda - \mathscr{L}_t\right)^{-1} = \left(\lambda_0 - \mathscr{L}_t\right)^{-1} - (\lambda - \lambda_0)\left(\lambda - \mathscr{L}_t\right)^{-1}\left(\lambda_0 - \mathscr{L}_t\right)^{-1}, \tag{4.48}$$

we find that

$$\left\|\left(\lambda - \mathscr{L}_t\right)^{-1}\right\|_t^{-1,0} \leq \frac{1}{C_1}\left(1 + C|\lambda - \lambda_0|\right). \tag{4.49}$$

Finally, exchanging the last two factors in (4.48) and applying (4.49), we get

$$\begin{aligned}\left\|\left(\lambda - \mathscr{L}_t\right)^{-1}\right\|_t^{-1,1} &\leq \frac{1}{C_1} + \frac{|\lambda - \lambda_0|}{C_1^2}\left(1 + C|\lambda - \lambda_0|\right) \\ &\leq C(1 + |\lambda|^2).\end{aligned} \tag{4.50}$$

The proof of our Proposition is complete. q.e.d.

Proposition 4.8. *Take $m \in \mathbb{N}^*$. Then there exists a contant $C_m > 0$ such that for any $t \in]0,1]$, $Q_1, \ldots, Q_m \in \mathcal{D}_t \cup \{Z_i\}_{i=1}^{2n-d}$ and $s, s' \in \mathscr{C}_c^\infty(B_0, \mathcal{E}_{B,x_0})$,*

$$\left|\left\langle [Q_1, [Q_2, \ldots [Q_m, \mathscr{L}_t] \ldots]]s, s'\right\rangle_{t,0}\right| \leq C_m ||s||_{t,1} ||s'||_{t,1}. \tag{4.51}$$

Proof. First, note that $[\nabla_{t,e_i}, Z_j] = \delta_{ij}$. Thus by (4.32), we know that $[Z_j, \mathscr{L}_t]$ satisfies (4.51).

Using (4.15) and (4.43), we see that $(\nabla_{e_i}\langle\widetilde{\mu}_0, f_{0,l}\rangle)(tZ)$ is uniformly bounded with its derivatives for $t \in [0,1]$, and for $|Z| \geq 4\varepsilon$,

$$\left(\nabla_{e_i}\langle\widetilde{\mu}_0, f_{0,l}\rangle\right)(Z) = \left(e_i\langle\widetilde{\mu}_0, f_{0,l}\rangle\right)_{x_0} = \omega_{x_0}(f_{0,l}, e_i). \tag{4.52}$$

Thus, $[\frac{1}{t}\langle\widetilde{\mu}_0, f_{0,l}\rangle(tZ), \mathscr{L}_t]$ also satisfies (4.51).

Let $R^{(L_0)_{B_0}}$ and $R^{(\mathcal{E}_0)_{B_0}}$ be the curvatures of the connections on $(L_0)_{B_0}$ and $(\mathcal{E}_0)_{B_0}$ induced by ∇^{L_0}, ∇^{E_0} and ∇^{Bi_0}. Then by (4.27), we have

$$\left[\nabla_{t,e_i}, \nabla_{t,e_j}\right] = \left(R^{L_{0,B_0}} + t^2 R^{\mathcal{E}_{0,B_0}}\right)_{tZ}(e_i, e_j). \tag{4.53}$$

By (4.32), (4.52) and (4.53), we find that $[\nabla_{t,e_i}, \mathscr{L}_t]$ has the same structure as $\mathscr{L}_t$ for $t \in]0,1]$, by which we mean that it is of the form

$$\begin{aligned}&\sum_{i,j} a_{ij}(t, tZ)\nabla_{t,e_i}\nabla_{t,e_i} + \sum_i b_i(t, tZ)\nabla_{t,e_i} + c(t, tZ) \\ &\qquad + \sum_l \left[d_l(t, tZ)\frac{1}{t}\langle\widetilde{\mu}_0, f_{0,l}\rangle(tZ) + d'\left|\frac{1}{t}\widetilde{\mu}_0\right|^2_{g^{TY}}\right],\end{aligned} \tag{4.54}$$

where $d' \in \mathbb{C}$, and a_{ij}, b_i, c and d_l are polynomials in the first variable, and have all their derivatives in the second variable uniformly bounded for $Z \in \mathbb{R}^{2n-d}$ and $t \in [0,1]$. Note that in fact, for $[\nabla_{t,e_i}, \mathscr{L}_t]$, $d' = 0$ in (4.54).

The adjoint connection $(\nabla_t)^*$ of ∇_t with respect to $\langle \cdot, \cdot \rangle_{t,0}$ is given by

$$(\nabla_t)^* = -\nabla_t - t\big(\kappa^{-1}\nabla\kappa\big)(tZ). \tag{4.55}$$

Note that the last term of (4.55) and all its derivative in Z are uniformly bounded for $Z \in \mathbb{R}^{2n-d}$ and $t \in [0,1]$. Thus, by (4.54) and (4.55), we find that (4.51) holds when $m = 1$.

Finally, we can prove by induction that $[Q_1, [Q_2, \dots [Q_m, \mathscr{L}_t] \dots]]$ has also the same structure as in (4.54), and thus satisfies (4.51) thanks to (4.55). q.e.d.

Proposition 4.9. *For any $t \in]0, t_0]$, $\lambda \in \Gamma$ and $m \in \mathbb{N}$,*

$$(\lambda - \mathscr{L}_t)^{-1}\big(\boldsymbol{H}_t^m\big) \subset \boldsymbol{H}_t^{m+1}. \tag{4.56}$$

Moreover, for any $\alpha \in \mathbb{N}^{2n-d}$, there exist $K \in \mathbb{N}$ and $C_{\alpha,m} > 0$ such that for any $t \in]0,1]$, $\lambda \in \Gamma$ and $s \in \mathscr{C}_c^\infty(B_0, \mathcal{E}_{B,x_0})$,

$$\big\|Z^\alpha(\lambda - \mathscr{L}_t)^{-1}s\big\|_{t,m+1} \le C_{\alpha,m}(1+|\lambda|^2)^K \sum_{\alpha' \le \alpha} ||Z^{\alpha'}s||_{t,m}. \tag{4.57}$$

Proof. Let $Q_1, \dots, Q_m \in \mathcal{D}_t$ and $Q_{m+1}, \dots, Q_{m+|\alpha|} \in \{Z_i\}_{i=1}^{2n}$. Then we can express the operator $Q_1 \dots Q_{m+|\alpha|}(\lambda - \mathscr{L}_t)^{-1}$ as a linear combination of operators of the type

$$[Q_1, [Q_2, \dots [Q_\ell, (\lambda - \mathscr{L}_t)^{-1}] \dots]]Q_{\ell+1} \dots Q_{m+|\alpha|} \quad \text{with} \quad \ell \le m + |\alpha|. \tag{4.58}$$

We denote by $\mathscr{F}_t$ the family of operator $\mathscr{F}_t = \{[Q_{j_1}, [Q_{j_2}, \dots [Q_{j_k}, \mathscr{L}_t] \dots]]\}$. Then any commutator $[Q_1, [Q_2, \dots [Q_\ell, (\lambda - \mathscr{L}_t)^{-1}] \dots]]$ can be expressed as a linear combination operators of the form

$$(\lambda - \mathscr{L}_t)^{-1}F_1(\lambda - \mathscr{L}_t)^{-1}F_2 \dots F_\ell(\lambda - \mathscr{L}_t)^{-1} \quad \text{with} \quad F_j \in \mathscr{F}_t. \tag{4.59}$$

Moreover, by Proposition 4.8, the norm $\| \cdot \|_t^{1,-1}$ of any element of $\mathscr{F}_t$ is uniformly bounded by C. As a consequence, using Proposition 4.7 we see that there is $C > 0$ and $N \in \mathbb{N}$ such that the $\| \cdot \|_t^{0,1}$-norm of operators in (4.59) is bounded by $C(1+|\lambda|^2)^N$. Thus, Proposition 4.9 holds. q.e.d.

Let $e^{-\mathscr{L}_t}(Z, Z')$ be the smooth kernel of the operator $e^{-\mathscr{L}_t}$ with respect to $dv_{TB}(Z')$. Let $\pi_{M_G} : TB \times_{M_G} TB \to M_G$ be the projection from the fiberwise product $TB \times_{M_G} TB$ onto M_G (here we should rather write $TB|_{M_G}$ but we drop the subscript to simplify the notations). As $\mathscr{L}_t$ depends on the parameter $x_0 \in M_G$, then $e^{-\mathscr{L}_t}(\cdot, \cdot)$ can be viewed as a section of $\pi_{M_G}^*(\mathrm{End}(\mathcal{E}_B))$ over $TB \times_{M_G} TB$.

Let $\nabla^{\pi^*_{M_G}\mathrm{End}(\mathcal{E}_B)}$ be the connection on $\pi^*_{M_G}\mathrm{End}(\mathcal{E}_B)$ induced by $\nabla^{\mathcal{E}_B}$. Then $\nabla^{\pi^*_{M_G}\mathrm{End}(\mathcal{E}_B)}$, h^E and g^{TM} induce naturally a $\mathscr{C}^m$-norm for the parameter $x_0 \in M_G$ on sections of $\pi^*_{M_G}(\mathrm{End}(\mathcal{E}_B))$.

As above, we will decompose any $Z \in T_{x_0}B$ as $Z = Z^0 + Z^\perp$, with $Z_0 \in T_{x_0}M_G$ and $Z^\perp \in N_{G,x_0}$.

Theorem 4.10. *There exists $C' > 0$ such that for any $m, m', m'', r \in \mathbb{N}$ and $u_0 > 0$, there is $C > 0$ such that for any $t \in]0, t_0]$, $u \geq u_0$ and $Z, Z' \in T_{x_0}B = B_0$*

(4.60)
$$\sup_{|\alpha|,|\alpha'|\leq m} \big(1 + |Z^\perp| + |Z'^\perp|\big)^{m''} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \frac{\partial^r}{\partial t^r} e^{-u\mathscr{L}_t}(Z, Z') \right|_{\mathscr{C}^{m'}(M_G)}$$
$$\leq C\big(1 + |Z^0| + |Z'^0|\big)^{2(n+r+m'+1)+m} \exp\left(4C_2 u - \frac{C'}{u}|Z - Z'|^2\right),$$

where $|\cdot|_{\mathscr{C}^{m'}(M)}$ denotes the $\mathscr{C}^m$-norm for the parameter $x_0 \in M_G$.

Proof. By (4.45), we know that for $k \in \mathbb{N}^*$,

$$e^{-u\mathscr{L}_t} = \frac{(-1)^{k-1}(k-1)!}{2i\pi u^{k-1}} \int_\Gamma e^{-u\lambda}(\lambda - \mathscr{L}_t)^{-k} d\lambda. \tag{4.61}$$

Then for $m \in \mathbb{N}$, we know from Proposition 4.9 that for $Q \in \cup_{\ell=1}^m \mathcal{D}_t^\ell$, there are $C_m > 0$ and $M \in \mathbb{N}$ such that for $\lambda \in \Gamma$,

$$\|Q(\lambda - \mathscr{L}_t)^{-m}\|_t^{0,0} \leq C_m(1 + |\lambda|^2)^M. \tag{4.62}$$

Moreover, taking the adjoint of (4.62), we deduce

$$\|(\lambda - \mathscr{L}_t)^{-m}Q\|_t^{0,0} \leq C_m(1 + |\lambda|^2)^M. \tag{4.63}$$

From (4.61), (4.62) and (4.63), we have for $Q, Q' \in \cup_{\ell=1}^m \mathcal{D}_t^\ell$:

$$\left\| Q e^{-u\mathscr{L}_t} Q' \right\|_t^{0,0} \leq C_m e^{2C_2 u}. \tag{4.64}$$

Let $\|\cdot\|_m$ be the usual Sobolev norm on $\mathscr{C}^\infty(T_{x_0}B, \mathcal{E}_{x_0})$ induced by $h^{\mathcal{E}_{x_0}}$ and the volume form $dv_{TX}(Z)$:

$$||s||_m^2 = \sum_{\ell\leq m} \sum_{i_1,\ldots,i_\ell} ||\nabla_{e_{i_1}} \cdots \nabla_{e_{i_\ell}} s||_0^2. \tag{4.65}$$

Then by (4.29) and (4.39), for any $m \in \mathbb{N}$ there exists $C'_m > 0$ such that for $s \in \mathscr{C}^\infty(T_{x_0}B, \mathcal{E}_{x_0})$ with support in $B^{T_{x_0}B}(0, q)$ and $t \in [0, 1]$,

$$\frac{1}{C'_m(1+q)^m}\|s\|_{t,m} \leq \|s\|_m \leq C'_m(1+q)^m\|s\|_{t,m}. \tag{4.66}$$

From (4.64), (4.66) and Sobolev inequalities (for $\|\cdot\|_m$) we find that if $Q, Q' \in \cup_{\ell=1}^m \mathcal{D}_t^\ell$, then

$$\sup_{|Z|,|Z'|\leq q} \left| Q_Z Q'_{Z'} e^{-u\mathscr{L}_t}(Z,Z') \right| \leq C(1+q)^{2n+2} e^{2C_2 u}. \tag{4.67}$$

Moreover, by Lemma 2.3 and (4.15), (4.16) and (4.43), we have

$$\sum_{l=1}^d \left| \frac{1}{t} \langle \widetilde{\mu}_0, f_{0,l}\rangle (tZ) \right|^2 = \left| \frac{1}{t}\widetilde{\mu}_0 \right|^2_{g^{TY}} (tZ) \geq C|Z^\perp|^2. \tag{4.68}$$

Thus, (4.29), (4.67) and (4.68) imply (4.60) with the exponential e^{2C_2u} for the case where $r = m' = 0$ and $C' = 0$, i.e., for any $m, m'' \in \mathbb{N}$, there is $C > 0$ such that for any $t \in]0, t_0]$, $Z, Z' \in T_{x_0}B = B_0$

$$\sup_{|\alpha|,|\alpha'|\leq m} \left(1 + |Z^\perp| + |Z'^\perp|\right)^{m''} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} e^{-u\mathscr{L}_t}(Z,Z') \right| \\ \leq C\left(1 + |Z^0| + |Z'^0|\right)^{2n+2+m} \exp(2C_2 u). \tag{4.69}$$

To obtain the right exponential factor in the right hand side of (4.60), we proceed as in the proof of [**3**, Thm. 11.14] (see also [**15**, Thm. 4.2.5]).

Recall that the function f is defined in (3.4). For $\varsigma > 1$ and $a \in \mathbb{C}$, set

$$K_{u,\varsigma}(a) = \int_{\mathbb{R}} e^{iv\sqrt{2u}a} \exp(-v^2/2) \left(1 - f\left(\sqrt{2u}v/\varsigma\right)\right) \frac{dv}{\sqrt{2\pi}}. \tag{4.70}$$

Then there are $C'', C_1 > 0$ such that for any $c > 0$ and $m, m' \in \mathbb{N}$, there is $C > 0$ such that for $u \geq u_0$, $\varsigma > 1$ and $a \in \mathbb{C}$ with $|\mathrm{Im}(a)| \leq c$, we have

$$|a|^m |K_{u,\varsigma}^{(m')}(a)| \leq C \exp\left(C'' c^2 u - \frac{C_1}{u}\varsigma^2\right). \tag{4.71}$$

For $c > 0$, let V_c be the image of $\{a \in \mathbb{C} \,:\, |\mathrm{Im}(a)| \leq c\}$ by the map $a \mapsto a^2$, that is

$$V_c = \left\{ \lambda \in \mathbb{C} \,:\, \mathrm{Re}(\lambda) \geq \frac{1}{4c^2}\mathrm{Im}(\lambda) - c^2 \right\}. \tag{4.72}$$

Then the contour Γ of Figure 1 satisfies $\Gamma \subset V_c$ for c large enough.

As $K_{u,\varsigma}$ is even, there exist a unique holomorphic function $\widetilde{K}_{u,\varsigma}$ such that $\widetilde{K}_{u,\varsigma}(a^2) = K_{u,\varsigma}(a)$. By (4.71), we have for $\lambda \in V_c$

$$|\lambda|^m |\widetilde{K}_{u,\varsigma}^{(m')}(\lambda)| \leq C \exp\left(C'' c^2 u - \frac{C_1}{u}\varsigma^2\right). \tag{4.73}$$

Using the finite propagation speed of the wave equation and (4.70), we know that there exists $c' > 0$ such that for any $\varsigma > 1$

$$\widetilde{K}_{u,\varsigma}(\mathscr{L}_t)(Z,Z') = e^{-u\mathscr{L}_t}(Z,Z') \qquad \text{if} \quad |Z - Z'| \geq c'\varsigma. \tag{4.74}$$

From (4.73), we see that for $k \in \mathbb{N}$, there is a unique holomorphic function $\widetilde{K}_{u,\varsigma,k}$ defined on a neighborhood of V_c which satisfies the same estimates as $\widetilde{K}_{u,\varsigma}$ in (4.73) and

$$\frac{\widetilde{K}^{(k-1)}_{u,\varsigma,k}(\lambda)}{(k-1)!} = \widetilde{K}_{u,\varsigma}(\lambda). \tag{4.75}$$

In particular, as in (4.61), we have

$$\widetilde{K}_{u,\varsigma}(\mathscr{L}_t) = \frac{1}{2i\pi}\int_\Gamma \widetilde{K}_{u,\varsigma,k}(\lambda-\mathscr{L}_t)^{-k}d\lambda. \tag{4.76}$$

Using (4.64) and proceeding as in (4.66)–(4.69), we find

$$\sup_{|\alpha|,|\alpha'|\le m}\big(1+|Z^\perp|+|Z'^\perp|\big)^{m''}\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\widetilde{K}_{u,\varsigma}(Z,Z')\right| \le C\big(1+|Z^0|+|Z'^0|\big)^{2n+2+m}\exp\Big(C''c^2u-\frac{C_1}{u}\varsigma^2\Big). \tag{4.77}$$

For $Z\neq Z'$, we set $\varsigma\in\mathbb{N}^*$ such that $|\varsigma-\frac{1}{c'}|Z-Z'||<1$ in the previous estimate and get

$$\sup_{|\alpha|,|\alpha'|\le m}\big(1+|Z^\perp|+|Z'^\perp|\big)^{m''}\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\widetilde{K}_{u,\varsigma}(Z,Z')\right| \le C\big(1+|Z^0|+|Z'^0|\big)^{2n+2+m}\exp\Big(C''c^2u-\frac{C_1}{c'^2u}|Z-Z'|^2\Big). \tag{4.78}$$

Now, take $\delta_1=\frac{C''c^2+2C_2}{C''c^2+4C_2}$, then from $(4.69)^{\delta_1}\times(4.78)^{1-\delta_1}$ and (4.74) (and from (4.69) if $Z=Z'$), we get (4.60) for $r=m'=0$, i.e., for all $Z,Z'\in T_{x_0}B$

$$\sup_{|\alpha|,|\alpha'|\le m}\big(1+|Z^\perp|+|Z'^\perp|\big)^{m''}\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}e^{-u\mathscr{L}_t}(Z,Z')\right| \le C\big(1+|Z^0|+|Z'^0|\big)^{2n+2+m}\exp\Big(4C_2u-\frac{C'}{u}|Z-Z'|^2\Big). \tag{4.79}$$

We now turn to the case $r\ge 1$. By (4.61), we have

$$\frac{\partial^r}{\partial t^r}e^{-u\mathscr{L}_t} = \frac{(-1)^{k-1}(k-1)!}{2i\pi u^{k-1}}\int_\Gamma e^{-\lambda}\frac{\partial^r}{\partial t^r}(\lambda-\mathscr{L}_t)^{-1}d\lambda. \tag{4.80}$$

For $k,q\in\mathbb{N}^*$, set

$$I_{k,r} = \Big\{(\mathbf{k},\mathbf{r})=(k_i,r_i)\in(\mathbb{N}^*)^{j+1}\times(\mathbb{N}^*)^j \,:\, \sum_{i=0}^j k_i=k+j\,,\ \sum_{i=1}^j r_i=r\Big\}. \tag{4.81}$$

For $(\mathbf{k},\mathbf{r}) \in I_{k,r}$, $\lambda \in \Gamma$, $t > 0$ set

$$(4.82)\quad A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t) = (\lambda - \mathscr{L}_t)^{-k_0}\frac{\partial^{r_1}\mathscr{L}_t}{\partial t^{r_1}}(\lambda - \mathscr{L}_t)^{-k_1}\cdots\frac{\partial^{r_j}\mathscr{L}_t}{\partial t^{r_j}}(\lambda - \mathscr{L}_t)^{-k_j}.$$

Then there exist $a^{\mathbf{k}}_{\mathbf{r}} \in \mathbb{R}$ such that

$$(4.83)\qquad \frac{\partial^r}{\partial v^r}(\lambda - \mathscr{L}_t)^{-k} = \sum_{(\mathbf{k},\mathbf{r})\in I_{k,q}} a^{\mathbf{k}}_{\mathbf{r}} A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t).$$

We claim that for any $m \in \mathbb{N}$, $k > 2(m+r+1)$ and $Q, Q' \in \cup_{\ell=1}^m \mathcal{D}^\ell_t$, there exist $C > 0$, $N \in \mathbb{N}$ such that for $\lambda \in \Gamma$

$$(4.84)\qquad \left\|QA^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)Q's\right\|_0 \le C(1+|\lambda|)^N \sum_{|\beta|\le 2r}\|Z^\beta s\|_0.$$

Indeed, we know by (4.32) that $\frac{\partial^r}{\partial t^r}\mathscr{L}_t$ is a combination of

$$(4.85)\qquad \begin{array}{ll} \dfrac{\partial^{r_1}}{\partial v^{r_1}} g^{ij}(tZ), & \dfrac{\partial^{r_2}}{\partial t^{r_2}}\nabla_{t,e_i}, \\ \dfrac{\partial^{r_1}}{\partial t^{r_1}}\theta(tZ), & \dfrac{\partial^{r_1}}{\partial t^{r_1}} t\langle\tilde{\mu}^{\mathbb{E}_{0,p}}, f_{0,l}(tZ)\rangle, \end{array}$$

where θ runs over the functions r^X, etc., appearing in (4.32).

Now, if $f = g^{ij}$ or $f = \theta$ in (4.85) (resp. $f = \nabla_{t,e_i}$ or $f = t\langle\tilde{\mu}^{\mathbb{E}_{0,p}}, f_{0,l}(tZ)\rangle$), then for $r_1 \ge 1$, $\frac{\partial^{r_1}}{\partial v^{r_1}} f(tZ)$ is a function of the type $g(tZ)Z^\beta$ where $|\beta| \le r_1$ (resp. r_1+1) and $g(Z)$ and its derivatives in Z are uniformly bounded for $Z \in \mathbb{R}^{2n}$.

Let $\mathscr{F}'_t$ be the family of operators of the form

$$(4.86)\qquad \mathscr{F}'_t = \left\{[f_{j_1}Q_{j_1}, [f_{j_2}Q_{j_2}, \dots [f_{j_m}Q_{j_m}, \mathscr{L}_t]\dots]]\right\},$$

where f_{j_i} is smooth and bounded (with its derivatives) and $Q_{j_i} \in \mathcal{D}_t \cup \{Z_l\}_{l=1}^{2n-d}$.

We will now deal with the operator $A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)Q'$. First, we move all the terms Z^β in the terms $g(tZ)Z^\beta$ (defined above) to the right-hand side of this operator. To do so, we use the same commutator trick as in the proof of Theorem 4.9, that is we perform the commutations once at a time with each Z_i (and not directly with Z^β, $|\beta| > 1$). Then we obtain that $A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)Q'$ is of the form $\sum_{|\beta|\le 2r} L_{t,\beta}Q''_\beta Z^\beta$ where Q''_β is obtained from Q' and its commutation with Z^β. Next, we move all the terms ∇_{t,e_i} and $\langle\frac{1}{t}\tilde{\mu}^{\mathbb{E}_{0,p}}, f_{0,l}(tZ)\rangle$ in $\frac{\partial^r}{\partial t^r}\mathscr{L}_t$ to the right-hand side of the operators $L_{t,\beta}$. Then as in the proof of Theorem 4.9, we finally get that $QA^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)Q'$ is of the form $\sum_{|\beta|\le 2r}\mathscr{L}_{t,\beta}Z^\beta$, where $\mathscr{L}_{t,\beta}$ is a linear combination of operators of the type

$$(4.87)\qquad Q(\lambda - \mathscr{L}_t)^{-k'_0}R_1(\lambda - \mathscr{L}_t)^{-k'_1}R_2\cdots R_{l'}(\lambda - \mathscr{L}_t)^{-k'_{l'}}Q'''Q'',$$

where $\sum_j k'_j = k + l'$, $R_j \in \mathscr{F}'_t$, $Q''' \in \cup_{\ell=1}^{2r}\mathcal{D}^\ell_t$ and $Q'' \in \cup_{\ell=1}^m \mathcal{D}^\ell_t$ is obtained from Q' and its commutation with Z^β. Since $k > 2(m+r+1)$, we can use Proposition 4.9 and the arguments leading to (4.62) and (4.63) in order to split the operator in (4.87) into two parts:

$$Q(\lambda-\mathscr{L}_t)^{-k'_0}R_1(\lambda-\mathscr{L}_t)^{-k'_1}R_2\cdots R_i(\lambda-\mathscr{L}_t)^{-k''_i}\times (\lambda-\mathscr{L}_t)^{-(k'_i-k''_i)}R_{i+1}\cdots R_{l'}(\lambda-\mathscr{L}_t)^{-k'_{l'}}Q'''Q'', \tag{4.88}$$

such that the $\|\cdot\|_t^{0,0}$-norm each part is bounded by $C(1+|\lambda|^2)^N$. This conclude the proof of (4.84).

By (4.80), (4.83) and (4.84), we get (4.60) for $m'=0$ using a similar reasoning that for (4.79).

For $m'=1$, observe that if $U\in TM_G$, then

$$\nabla_U^{\pi_{M_G}^*\mathrm{End}(\mathcal{E})}e^{-u\mathscr{L}_p}= \frac{(-1)^{k-1}(k-1)!}{2i\pi u^{k-1}}\int_\Gamma e^{-\lambda}\nabla_U^{\pi_{M_G}^*\mathrm{End}(\mathcal{E})}(\lambda-\mathscr{L}_t)^{-k}d\lambda. \tag{4.89}$$

Moreover, $\nabla_U^{\pi_{M_G}^*\mathrm{End}(\mathcal{E})}(\lambda-\mathscr{L}_t)^{-k}$ is a linear combination operators of the form

$$(\lambda-\mathscr{L}_t)^{-i_1}\big(\nabla_U^{\pi_{M_G}^*\mathrm{End}(\mathcal{E})}\mathscr{L}_t\big)(\lambda-\mathscr{L}_t)^{-i_2}\times \big(\nabla_U^{\pi_{M_G}^*\mathrm{End}(\mathcal{E})}\mathscr{L}_t\big)(\lambda-\mathscr{L}_t)^{-i_\ell}, \tag{4.90}$$

and $\nabla_U^{\pi_{M_G}^*\mathrm{End}(\mathcal{E})}\mathscr{L}_t$ is a differential operator with the same structure as $\mathscr{L}_t$. In particular, $\nabla_U^{\pi_{M_G}^*\mathrm{End}(\mathcal{E})}\mathscr{L}_t$ satisfies an estimates analogous to (4.51). Thus, above arguments can be repeated to prove (4.60) for $m'=1$. The case $m'\geq 2$ is similar. q.e.d.

Remark 4.11. In the sequel, we will in fact only use Theorem 4.10 with $r=0,1$, but we prefer to state it in the general case.

Proposition 4.12. *There are constants $C>0$ and $M\in\mathbb{N}^*$ such that for $t\in[0,t_0]$ and $\lambda\in\Gamma$,*

$$\big\|\big((\lambda-\mathscr{L}_t)^{-1}-(\lambda-\mathscr{L}_0)^{-1}\big)s\big\|_{0,0} \leq Ct(1+|\lambda|^2)^M\sum_{|\alpha|\leq 3}||Z^\alpha s||_{0,0}. \tag{4.91}$$

Proof. From (4.29) and (4.39), for $t\in[0,1]$ and $m\in\mathbb{N}^*$ we find

$$\|s\|_{t,m}\leq C\sum_{|\alpha|\leq m}\|Z^\alpha s\|_{0,m}. \tag{4.92}$$

Moreover, for s,s' with compact support, a Taylor expansion of (4.32) gives

$$\Big|\big\langle(\mathscr{L}_t-\mathscr{L}_0)s,s'\big\rangle_{t,0}\Big|\leq Ct\|s'\|_{t,1}\sum_{|\alpha|\leq 3}\|Z^\alpha s\|_{0,1}. \tag{4.93}$$

Thus,

$$\|(\mathscr{L}_t - \mathscr{L}_0)s\|_{t,-1} \leq Ct \sum_{|\alpha|\leq 3} \|Z^\alpha s\|_{0,1}. \tag{4.94}$$

Note that

$$(\lambda - \mathscr{L}_t)^{-1} - (\lambda - \mathscr{L}_0)^{-1} = (\lambda - \mathscr{L}_t)^{-1}(\mathscr{L}_t - \mathscr{L}_0)(\lambda - \mathscr{L}_0)^{-1}. \tag{4.95}$$

Moreover, Propositions 4.7, 4.8 and 4.9 still holds for $t = 0$. Thus, Proposition 4.9, (4.94) and (4.95) yields to (4.91). q.e.d.

Theorem 4.13. *There exists $C' > 0$ such that for any $m, m', m'' \in \mathbb{N}$ and $u_0 > 0$, there is $C > 0$ such that for any $t \in]0, t_0]$, $u \geq u_0$ and $Z, Z' \in B_0$*

$$\begin{aligned} &\sup_{|\alpha|,|\alpha'|\leq m} \big(1 + |Z^\perp| + |Z'^\perp|\big)^{m''} \\ &\qquad \times \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left(e^{-u\mathscr{L}_t} - e^{-u\mathscr{L}_0}\right)(Z, Z') \right|_{\mathscr{C}^{m'}(M_G)} \\ &\quad \leq Ct\big(1 + |Z^0| + |Z'^0|\big)^{2(n+m'+1)+m} \exp\left(4C_2 u - \frac{C'}{u}|Z - Z'|^2\right). \end{aligned} \tag{4.96}$$

Proof. Let $\mathcal{B}_q = B^{T_{x_0}B}(0, q)$. Let $\|s\|^2_{\mathcal{B}_q} = \int_{|Z|\leq q} |s|^2_{h^{\mathcal{E}_{x_0}}} dv_{TX}(Z)$, and let $J_{q,x_0} = L^2(\mathcal{B}_q, \mathcal{E}_{B,x_0})$. If A is a bounded operator on J_{q,x_0}, we denote its operator norm by $\|A\|_{\mathcal{B}_q}$. By (4.61) and (4.91), we know that there is $C' > 0$ and $N, M \in \mathbb{N}$ such that for $t \in]0, 1]$,

$$\begin{aligned} &\left\| e^{-u\mathscr{L}_t} - e^{-u\mathscr{L}_0} \right\|_{\mathcal{B}_q} \\ &\qquad \leq \frac{1}{2\pi} \int_\Gamma |e^{-u\lambda}| \left\| (\lambda - \mathscr{L}_t)^{-1} - (\lambda - \mathscr{L}_0)^{-1} \right\|_{\mathcal{B}_q} d\lambda \\ &\qquad \leq Ct \int_\Gamma e^{-u\mathrm{Re}(\lambda)} (1 + |\lambda|^2)^M (1 + q)^N d\lambda \leq C't(1 + q)^N. \end{aligned} \tag{4.97}$$

Let $\phi \colon T_{x_0}B \to [0, 1]$ be a smooth function with compact support, equal to 1 near 0 and such that $\int_{T_{x_0}B} \phi(Z) dv_{TX}(Z) = 1$. Let $\nu \in]0, 1]$. By the proof of Theorem 4.10, we see that $e^{-u\mathscr{L}_0}$ satisfies an inequality similar to (4.60). By Theorem 4.10, there exists $C > 0$ such that for $|Z|, |Z'| \leq q$ and $U, U' \in \mathcal{E}_{x_0}$,

$$\begin{aligned} &\Big| \Big\langle \big(e^{-u\mathscr{L}_t} - e^{-u\mathscr{L}_0}\big)(Z, Z')U, U' \Big\rangle \\ &\qquad - \int_{T_{x_0}B \times T_{x_0}B} \Big\langle \big(e^{-u\mathscr{L}_t} - e^{-u\mathscr{L}_0}\big)(Z - W, Z' - W')U, U' \Big\rangle \\ &\quad \times \frac{1}{\nu^{4n-2d}} \phi(W/\nu)\phi(W'/\nu) dv_{TX}(W) dv_{TX}(W') \Big| \leq C\nu(1 + q)^N |U||U'|. \end{aligned} \tag{4.98}$$

Moreover, by (4.97), we have

$$\begin{aligned}(4.99)\quad &\Big|\int_{T_{x_0}B\times T_{x_0}B}\Big\langle\big(e^{-u\mathscr{L}_t}-e^{-u\mathscr{L}_0}\big)(Z-W,Z'-W')U,U'\Big\rangle\\ &\qquad\times\frac{1}{\nu^{4n-2d}}\phi(W/\nu)\phi(W'/\nu)dv_{TX}(W)dv_{TX}(W')\Big|\\ &\qquad\qquad\leq\frac{Ct}{\nu^{2n-d}}(1+q)^N|U||U'|.\end{aligned}$$

Hence, taking $\nu=t^{1/(2n-d+1)}$ we find that there is $C>0$ and $K\in\mathbb{N}$ such that for any $t\in\,]0,t_0]$, $Z,Z'\in B^{B_0}(0,q)$,

$$(4.100)\qquad \Big|\big(e^{-u\mathscr{L}_t}-e^{-u\mathscr{L}_0}\big)(Z,Z')\Big|\leq Ct^{1/(2n-d+1)}(1+q)^K.$$

In particular, we have

$$(4.101)\qquad e^{-u\mathscr{L}_t}\big|_{t=0}=e^{-u\mathscr{L}_0}.$$

From Theorem 4.10, (4.101) and the formula

$$(4.102)\qquad G(t)-G(0)=\int_0^t G'(s)ds,$$

we get (4.96). q.e.d.

Remark 4.14. As we have estimates on every derivatives of $e^{-u\mathscr{L}_t}(Z,Z')$, we can in fact use the same method as in Theorem 4.13 to get an asymptotic expansion at every order of $e^{-u\mathscr{L}_t}(Z,Z')$.

4.3. Computation of the limiting heat kernel. In this section, we will evaluate the limiting heat kernel $e^{-u\mathscr{L}_0}((0,Z^\perp),(0,Z^\perp))$ for $(0,Z^\perp)\in T_{x_0}B$ and thus obtain Theorem 0.9.

Recall that we have the following splitting of vector bundle over P, which is orthogonal for both b^L and g^{TM} (see (0.25) and (2.19)):

$$(4.103)\qquad TU|_P=T^HP\oplus TY\oplus JTY.$$

Note also that by (0.12) and (0.26), we have

$$(4.104)\qquad b^L(\cdot\,,\cdot)=\langle(-J\mathbf{J})\cdot\,,\cdot\rangle,$$

and thus $-J\mathbf{J}$ preserves both TY and JTY on P. In particular, on P, $\mathbf{J}$ intertwines TY and JTY, and is invertible on $TY\oplus JTY$ because g^{TM} and b^L are definite positive on this bundle. Thus,

$$(4.105)\qquad \mathbf{J}^2TY=TY\,,\quad \mathbf{J}TY=JTY\,,\quad \mathbf{J}T^HP=JT^HP=T^HP.$$

Thus, $\mathbf{J}$ induces naturally $\mathbf{J}_G\in\operatorname{End}(TM_G)$, and we see with (4.3) that $(\mathbf{J}TY)_B|_{M_G}$ is the orthogonal complement of TM_G in TB. We will identify the normal bundle N_G of M_G in B with $(\mathbf{J}TY)_B|_{M_G}$. From this fact and (4.105), we know that for $U,V\in T_{x_0}B$,

$$(4.106)\qquad \omega(U^H,V^H)=\omega_G(P^{TM_G}U,P^{TM_G}V).$$

From the above discussion, we can diagonalize $\mathbf{J}$ on $(T^HP)^{(1,0)}$ and $(TY \oplus JTY)^{(1,0)}$, and we thus can get orthonormal basis $\{w_j^0\}_{j=1}^{n-d}$ and $\{e_i^\perp\}_{i=1}^d$ of $T_{x_0}^{(1,0)}M_G$ and $N_{G,x_0} = (\mathbf{J}TY)_{B,x_0} \subset TB$, respectively, such that in these basis

$$\begin{cases} \mathbf{J}|_{T_{x_0}^{(1,0)}M_G} = \dfrac{\sqrt{-1}}{2\pi}\operatorname{diag}(a_1^0, \dots, a_{n-d}^0), \\ \mathbf{J}^2|_{N_{G,x_0}} = -\dfrac{1}{4\pi^2}\operatorname{diag}(a_1^{\perp,2}, \dots, a_d^{\perp,2}), \end{cases} \tag{4.107}$$

where $a_j^0 \in \mathbb{R}$ and $a_j^\perp \in \mathbb{R}^*$ are the respective eigenvalues of $-2\sqrt{-1}\pi\mathbf{J}|_{(T^HP)^{(1,0)}}$ and $-2\sqrt{-1}\pi\mathbf{J}|_{(TY\oplus JTY)^{(1,0)}}$. Let $\{w^{0,j}\}_{j=1}^{n-d}$ and $\{e^{\perp,i}\}_{i=1}^d$ be their dual basis. We also set

$$e_{2j-1}^0 = \frac{1}{\sqrt{2}}(w_j^0 + \overline{w}_j^0) \quad \text{and} \quad e_{2j}^0 = \frac{\sqrt{-1}}{\sqrt{2}}(w_j^0 - \overline{w}_j^0). \tag{4.108}$$

Then $\{e_i^0\}_{i=1}^{2n-2d}$ is an orthonormal basis of $T_{x_0}M_G$.

From now on, we will use the coordinates in Section 4.1 induced by the above basis as in (3.1).

We denote by $Z^0 = (Z_1^0, \dots, Z_{2n-2d}^0)$ and $Z^\perp = (Z_1^\perp, \dots, Z_d^\perp)$ the elements in $T_{x_0}M_G$ and N_{G,x_0}. Then $Z \in T_{x_0}B$ can be decomposed as $Z = (Z^0, Z^\perp)$. We will also use the complex coordinates $z^0 = (z_1^0, \dots, z_{n-d}^0)$, so that

$$\begin{aligned} &Z^0 = z^0 + \overline{z}^0, \\ &w_j^0 = \sqrt{2}\frac{\partial}{\partial z_j^0}\,, \quad \overline{w}_j^0 = \sqrt{2}\frac{\partial}{\partial \overline{z}_j^0}, \\ &e_{2j-1}^0 = \frac{\partial}{\partial z_j^0} + \frac{\partial}{\partial \overline{z}_j^0}\,, \quad e_{2j}^0 = \sqrt{-1}(\frac{\partial}{\partial z_j^0} - \frac{\partial}{\partial \overline{z}_j^0}). \end{aligned} \tag{4.109}$$

When we consider z^0 or $\overline{z}^0$ as vector fields, we identify them with $\sum_j z_j^0 \frac{\partial}{\partial z_j^0}$ and $\sum_j \overline{z}_j^0 \frac{\partial}{\partial \overline{z}_j^0}$. Note that

$$\Big|\frac{\partial}{\partial z_j^0}\Big|^2 = \Big|\frac{\partial}{\partial \overline{z}_j^0}\Big|^2 = \frac{1}{2} \quad \text{and} \quad |z^0|^2 = |\overline{z}^0|^2 = \frac{1}{2}|Z^0|^2. \tag{4.110}$$

Set

$$\mathscr{L} = -\sum_{i=1}^{2n-2d} (\nabla_{0,e_i^0})^2 - \sum_{j=1}^{n-d} a_j^0\,, \tag{4.111}$$

and recall that

$$\mathscr{L}^\perp = -\sum_{i=1}^{d} \Big((\nabla_{e_i^\perp})^2 - |a_i^\perp Z_i^\perp|^2 \Big) - \sum_{j=1}^{d} a_j^\perp. \tag{4.112}$$

As in [**16**, (3.11) and (3.13)], we can show using (2.15), (4.27) and (4.106), that

$$\begin{aligned} R^{L_B}_{x_0}(U,V) &= -2\pi\sqrt{-1}\langle \mathbf{J}P^{TM_G}U, P^{TM_G}V\rangle, \\ \mathscr{L}_0 &= \mathscr{L} + \mathscr{L}^{\perp} - 2\omega_d(x_0). \end{aligned} \tag{4.113}$$

Thus,

$$e^{-u\mathscr{L}_0}(Z,Z') = e^{-u\mathscr{L}}(Z^0,Z'^0)e^{-u\mathscr{L}^{\perp}}(Z^{\perp},Z'^{\perp})e^{2u\omega_d(x_0)}. \tag{4.114}$$

Moreover, using (4.107), (4.111), (4.113) and the formula for the heat kernel of a harmonic oscillator (see [**15**, (E.2.4), (E.2.5)], for instance), we find (with the convention of Theorem 0.9):

$$e^{-u\mathscr{L}}(0,0) = \frac{1}{(2\pi)^{n-d}}\frac{\det(\dot{R}^{L_G}_x)}{\det\big(1-\exp(-2u\dot{R}^{L_G}_x)\big)}. \tag{4.115}$$

We can now prove Theorem 0.9. We fix $u>0$.

Let $s\in\mathscr{C}^{\infty}_c(B_0,\mathcal{E}_{x_0})$. Then by (4.26) and (4.27)

$$\begin{aligned} &e^{-u\mathscr{L}_t}s(Z) \\ &= S_t^{-1}\kappa^{1/2}e^{-\frac{u}{p}\Phi D_p^{M_0,2}\Phi^{-1}}\kappa^{-1/2}S_t s(Z) \\ &= \kappa^{1/2}(tZ)\int_{\mathbb{R}^{2n-d}} e^{-\frac{u}{p}\Phi D_p^{M_0,2}\Phi^{-1}}(tZ,Z')(S_t s)(Z')\kappa^{1/2}(Z')dv_{TB}(Z') \\ &= p^{-n+d/2}\kappa^{1/2}(tZ) \\ &\qquad\times\int_{\mathbb{R}^{2n-d}} e^{-\frac{u}{p}\Phi D_p^{M_0,2}\Phi^{-1}}(tZ,tZ'')s(Z'')\kappa^{1/2}(tZ'')dv_{TB}(Z''), \end{aligned} \tag{4.116}$$

which yields to

$$e^{-u\mathscr{L}_t}(Z,Z') = p^{-n+d/2}e^{-\frac{u}{p}\Phi D_p^{M_0,2}\Phi^{-1}}(tZ,tZ')\kappa^{1/2}(tZ)\kappa^{1/2}(tZ'). \tag{4.117}$$

On the other hand, for $s\in\mathscr{C}^{\infty}_c(B_0,(\mathbb{E}_{0,p})_{B_0})$ and $v\in M_0$,

$$\begin{aligned} &\Big(e^{-\frac{u}{p}\Phi D_p^{M_0,2}\Phi^{-1}}s\Big)(\pi(v)) \\ &= \left(\Phi e^{-\frac{u}{p}D_p^{M_0,2}}\Phi^{-1}s\right)(\pi(v)) \\ &= h(v)\int_{M_0} e^{-\frac{u}{p}D_p^{M_0,2}}(v,v')h^{-1}(v')s(v')dv_{M_0}(v') \\ &= h(v)\int_{B_0} e^{-\frac{u}{p}D_p^{M_0,2}}(v,y')h(y')s(y')dv_{B_0}(y'), \end{aligned} \tag{4.118}$$

thus we find

$$h(v)h(v')\big(P_G e^{-\frac{u}{p}D_p^{M_0,2}}P_G\big)(v,v') = e^{-\frac{u}{p}\Phi D_p^{M_0,2}\Phi^{-1}}(\pi(v),\pi(v')). \tag{4.119}$$

Let $v = (g, Z) \in U \simeq G \times B^{T_{x_0}B}(0, \varepsilon)$. We suppose that in the decomposition $Z = Z^0 + Z^{\perp}$, we have $Z^0 = 0$. Then from Corollary 4.2, Theorem 4.13, (4.117), and (4.119), we find that for any $m, m' \in \mathbb{N}$, there exists $C > 0$ (independent of $Z^{\perp}$) such

$$\begin{aligned}(4.120)\quad \Big| p^{-n+d/2} h(v)h(v)\big(P_G e^{-\frac{u}{p}D_p^2} P_G\big)(v,v) \\ - \kappa^{-1}(Z^{\perp}) e^{-u\mathscr{L}_0}(\sqrt{p}Z^{\perp}, \sqrt{p}Z^{\perp})\Big|_{\mathscr{C}^{m'}(M_G)} \\ \leq Cp^{-1/2}\big(1 + \sqrt{p}|Z^{\perp}|\big)^{-m}.\end{aligned}$$

Now, for $v \in U$, we write as in the Introduction of this paper $v = (y, Z^{\perp})$ with $y \in P$ and $Z^{\perp} \in N_{P/U,y}$. Let $x = \pi(y) \in M_G$. Then we do the procedure of Sections 4.1 and 4.3 with $x_0 = x$ and $y_0 = y$. Then Theorem 0.9 follows from (4.114), (4.115) and (4.120) applied to $Z = (0, Z^{\perp}) \in T_{x_0}B = T_{x_0}M_G \oplus N_{G,x_0}$.

5. Proof of the inequalities

In this Section, we prove our main results: Theorems 0.3 and 0.5. In Section 5.1 we prove Theorem 0.7 and, as a consequence, we obtain the G-invariant holomorphic Morse inequalities in the case of a free G-action on P. Then, we explain in Section 5.2 how to modify the arguments in Sections 4 and 5.1 to get our inequalities under Assumption 0.1 in full generality. Finally, in Section 5.3, we apply Theorem 0.5 to get estimates on the other isotypic components of the cohomology $H^{\bullet}(M, L^p \otimes E)$.

5.1. Proof of Theorem 0.3 when G acts freely on P. We assume in this Section that G acts freely on P and $\overline{U}$. We keep here the notations of Sections 4.

In this section, we will first prove Theorem 0.7, and then show how to use it in conjunction with the convergence of the heat kernel of the rescaled operator to get Theorem 0.3. The method is inspired by [**2**] (see also [**15**, Sect. 1.7]).

For $0 \leq q \leq n$, set

$$b_q^{p,G} = \dim H^q(M, L^p \otimes E)^G. \tag{5.1}$$

By Hodge theory, there is a G-equivariant isomorphism $H^{\bullet}(M, L^p \otimes E) \simeq \ker D_p^2$, and in particular we get for the invariant part:

$$H^{\bullet}(M, L^p \otimes E)^G \simeq (\ker D_p^2)^G \quad \text{and} \quad b_q^{p,G} = \dim(\ker D_p^2)^G. \tag{5.2}$$

We begin by proving Theorem 0.7.

Proof of Theorem 0.7. If λ is an eigenvalue of D_p^2 acting on $\Omega^{0,\bullet}(M, L^p \otimes E)^G$, we denote by $F_j^{\lambda} \subset \Omega^{0,j}(M, L^p \otimes E)$ the corresponding finite-dimensional eigenspace in degree j. As $\bar{\partial}^{L^p\otimes E}$ and $\bar{\partial}^{L^p\otimes E,*}$ act on

$\Omega^{0,\bullet}(M, L^p \otimes E)^G$ and commute with D_p^2, we deduce that

$$\bar{\partial}^{L^p\otimes E}(F_j^\lambda) \subset F_{j+1}^\lambda \qquad \text{and} \qquad \bar{\partial}^{L^p\otimes E,*}(F_j^\lambda) \subset F_{j-1}^\lambda. \tag{5.3}$$

As a consequence, we have a complex

$$0 \longrightarrow F_0^\lambda \stackrel{\bar{\partial}^{L^p\otimes E}}{\longrightarrow} F_1^\lambda \stackrel{\bar{\partial}^{L^p\otimes E}}{\longrightarrow} \cdots \stackrel{\bar{\partial}^{L^p\otimes E}}{\longrightarrow} F_n^\lambda \longrightarrow 0. \tag{5.4}$$

If $\lambda = 0$, we have $F_j^0 \simeq H^j(M, L^p \otimes E)^G$ by (5.2). If $\lambda > 0$, then the complex (5.4) is exact. Indeed, if $\bar{\partial}^{L^p\otimes E}s = 0$ and $s \in F_j^\lambda$, then

$$s = \lambda^{-1}D_p^2 s = \lambda^{-1}\bar{\partial}^{L^p\otimes E}\bar{\partial}^{L^p\otimes E,*}s \in \mathrm{Im}(\bar{\partial}^{L^p\otimes E}). \tag{5.5}$$

In particular, we get for $\lambda > 0$

$$\sum_{j=0}^{q}(-1)^{q-j}\dim F_j^\lambda = \dim\big(\bar{\partial}^{L^p\otimes E}(F_q^\lambda)\big) \geq 0, \tag{5.6}$$

with equality if $q = n$.

Now,

$$\mathrm{Tr}_j[P_G e^{-\frac{u}{p}D_p^2}P_G] = b_j^{p,G} + \sum_{\lambda>0} e^{-\frac{u}{p}\lambda}\dim F_j^\lambda. \tag{5.7}$$

Thus, (5.6) and (5.7) entail (0.20).

Note that this proof does not depend on the metric we chose on TM, so we get (0.20) in general. q.e.d.

We denote by $\mathrm{Tr}_{\Lambda^{0,q}}$ the trace on $\Lambda^{0,q}(T^*M) \otimes L^p \otimes E$ or $\Lambda^{0,q}(T^*M)$. We know that

$$\mathrm{Tr}_q[P_G e^{-\frac{u}{p}D_p^2}P_G] = \int_M \mathrm{Tr}_{\Lambda^{0,q}}\Big[\big(P_G e^{-\frac{u}{p}D_p^2}P_G\big)(v,v)\Big]\,dv_M(v). \tag{5.8}$$

With Theorem 0.8 and (4.23), we in fact have

$$\mathrm{Tr}_q[P_G e^{-\frac{u}{p}D_p^2}P_G] = \int_U \mathrm{Tr}_{\Lambda^{0,q}}\Big[\big(P_G e^{-\frac{u}{p}D_p^2}P_G\big)(v,v)\Big]\,dv_M(v) + O(p^{-\infty}). \tag{5.9}$$

By Theorems 0.7 and 0.9, (5.9), and using the change of variable $Z^\perp \leftrightarrow \sqrt{p}Z^\perp$, we deduce that for every $u > 0$,

$$p^{-n+d}\sum_{j=0}^{q}(-1)^{q-j}b_j^{p,G} \leq \frac{\mathrm{rk}(E)}{(2\pi)^{n-d}}\Big(\int_{x\in M_G,\,|Z^\perp|\leq\sqrt{p}\varepsilon} \frac{\det(\dot{R}_x^{L_G})\sum_{j=0}^{q}(-1)^{q-j}\,\mathrm{Tr}_{\Lambda^{0,j}}[e^{2u\omega_d(x)}]}{\det\big(1-\exp(-2u\dot{R}_x^{L_G})\big)} \times e^{-u\mathscr{L}_x^\perp}(Z^\perp, Z^\perp)dv_{TB}(x, Z^\perp)\Big) + o(1). \tag{5.10}$$

For $u>0$, set

$$f(u)=\frac{1}{\tanh(2u)}-\frac{1}{\sinh(2u)}. \tag{5.11}$$

Then there is $c>0$ such that for $u>1$, $f(u)>c$, and $f(u)\xrightarrow[u\to\pm\infty]{}\pm 1$. By (0.29) and Mehler's formula (see [**15**, Thm. E.1.4], for instance), we know that

$$e^{-u\mathscr{L}_x^{\perp}}(Z^{\perp},Z^{\perp})=\prod_{i=1}^{d}\sqrt{\frac{a_i^{\perp}}{\pi(1-e^{-4ua_i^{\perp}})}}\exp\left\{-a_i^{\perp}f(ua_i^{\perp})Z_i^{\perp,2}\right\}. \tag{5.12}$$

Thus, as $a_i^{\perp}f(ua_i^{\perp})>\text{cste}>0$ for $u>1$,

$$\begin{aligned}\int_{|Z^{\perp}|\leq\sqrt{p}\varepsilon}&e^{-u\mathscr{L}_x^{\perp}}(Z^{\perp},Z^{\perp})dv_{N_{G,x}}(Z^{\perp})\\ &=\int_{\mathbb{R}^d}e^{-u\mathscr{L}_x^{\perp}}(Z^{\perp},Z^{\perp})dv_{N_{G,x}}(Z^{\perp})+O(p^{-\infty})\\ &=\prod_{i=1}^{d}\sqrt{\frac{1}{f(ua_i^{\perp})(1-e^{-4ua_i^{\perp}})}}+O(p^{-\infty}).\end{aligned} \tag{5.13}$$

Let $\{w_j^0\}$ be a local orthonormal frame of $T^{(1,0)}M_G$ such that $\dot{R}^{L_G}w_j^0=a_j^0\,w_j^0$ (see (4.107)). Its dual frame is denoted by $\{w^{0,j}\}$. Then

$$\omega_{G,d}=-\sum_{j=1}^{n-d}a_j^0\,\overline{w}^{0,j}\wedge i_{\overline{w}_j^0}. \tag{5.14}$$

We again denote by w_j^0 the horizontal lift of w_j^0 in T^HP. In the same way, let $\{w_j^{\perp}\}$ be a local orthonormal frame of $(TY\oplus JTY)^{(1,0)}$ such that $\dot{R}^Lw_j^{\perp}=a_j^{\perp}\,w_j^{\perp}$ (see Section 4.3). Its dual frame is denoted by $\{w^{\perp,j}\}$. Then

$$\omega_d=-\sum_{j=1}^{n-d}a_j^0\,\overline{w}^{0,j}\wedge i_{\overline{w}_j^0}-\sum_{j=1}^{d}a_j^{\perp}\,\overline{w}^{\perp,j}\wedge i_{\overline{w}_j^{\perp}}. \tag{5.15}$$

Thus, writing $\{w_j\}=\{w_j^0,w_j^{\perp}\}$ and $\{a_j\}=\{a_j^0,a_j^{\perp}\}$, we get

$$e^{2u\omega_d}=1+\sum_j(e^{-2ua_j}-1)\overline{w}^j\wedge i_{\overline{w}_j}, \tag{5.16}$$

and

$$\operatorname{Tr}_{\Lambda^{0,q}}[e^{2u\omega_d}]=\sum_{j_1<\cdots<j_q}\exp\Big(-2u\sum_{k=1}^{q}a_{j_k}\Big). \tag{5.17}$$

In particular, there exist $C > 0$ such that for $x \in M_G$, $u > 1$ and $0 \le q \le n$,

$$\left| \frac{\det(\dot{R}^{L_G}_x) \operatorname{Tr}_{\Lambda^{0,q}}[e^{2u\omega_d(x)}]}{\det\left(1 - \exp(-2u\dot{R}^{L_G}_x)\right)} \prod_{i=1}^{d} \sqrt{\frac{1}{f(ua_i^\perp)(1 - e^{-4ua_i^\perp})}} \right| \le C. \tag{5.18}$$

On the other hand the signature of b^L on JTY is the same as on TY (i.e., $(r, d-r)$), so by Lemma 2.3 and (2.15), (2.19) and (2.32) we have for $0 \le q \le n$

$$\pi(P \cap M(q)) = M_G(q - r), \tag{5.19}$$

where $M(q)$ is define in an analogue way as $M_G(q)$ in the introduction. Thus, by (5.17) and (5.19),

$$\begin{aligned} \lim_{u \to +\infty} \frac{\det(\dot{R}^{L_G}_x) \operatorname{Tr}_{\Lambda^{0,q}}[e^{2u\omega_d(x)}]}{\det\left(1 - \exp(-2u\dot{R}^{L_G}_x)\right)} \prod_{i=1}^{d} \sqrt{\frac{1}{f(ua_i^\perp)(1 - e^{-4ua_i^\perp})}} \\ = \mathbf{1}_{M_G(q-r)}(x)(-1)^{q-r} \det(\dot{R}^{L_G}), \end{aligned} \tag{5.20}$$

where the function $\mathbf{1}_S$ takes the value 1 on S and 0 elsewhere.

Using (5.10), (5.13), (5.18), (5.20) and dominated convergence as $u \to +\infty$, we find

$$\begin{aligned} &\limsup_{p \to +\infty} p^{-n+d} \sum_{j=0}^{q} (-1)^{q-j} b_j^{p,G} \\ &\le \frac{\operatorname{rk}(E)}{(2\pi)^{n-d}} \prod_{i=1}^{d} \sqrt{\frac{1}{f(ua_i^\perp)(1 - e^{-4ua_i^\perp})}} \\ &\qquad \times \int_{M_G} \frac{\det(\dot{R}^{L_G}_x) \sum_{j=0}^{q} (-1)^{q-j} \operatorname{Tr}_{\Lambda^{0,j}}[e^{2u\omega_{G,d}(x)}]}{\det\left(1 - \exp(-2u\dot{R}^{L_G}_x)\right)} dv_{M_G}(x) \\ &\le (-1)^{q-r} \int_{M_G(\le q-r)} \det\left(\frac{\dot{R}^{L_G}_x}{2\pi}\right) dv_{M_G}(x). \end{aligned} \tag{5.21}$$

Finally, note that by (2.16),

$$\det\left(\frac{\dot{R}^{L_G}_x}{2\pi}\right) dv_{M_G}(x) = \left(\frac{\sqrt{-1}}{2\pi} R^{L_G}\right)^{n-d} / (n-d)! = \frac{\omega_G^{n-d}}{(n-d)!}. \tag{5.22}$$

Then (5.21) and (5.22) entail Theorem 0.3.

5.2. The case of a locally free action. In this section, we prove Theorem 0.3 under Assumption 0.1. In particular, the action of G on P and $\overline{U}$ is only locally free, and thus M_G and B are orbifolds. The proof relies on a similar method as the case of a free G-action, but the main difference is that we need to work off-diagonal to get uniform estimates near the orbifold singularities. We explain below how to adapt the arguments in Sections 4 and 5.1 to get the general result.

Recall that $G^0 = \{g \in G \,:\, g \cdot x = x \text{ for any } x \in M\}$. Then G^0 is a finite normal subgroup of G and the quotient G/G^0 acts effectively on M.

It is a well-known fact that if $\phi\colon (M, g^{TM}) \to (M, g^{TM})$ is an isometry and $x \in M$ is a point such that $\phi(x) = x$ and $d\phi_x = \mathrm{Id}_{T_xM}$ then $\phi = \mathrm{Id}_M$. In particular, suppose that $g \in G$ satisfies $g|_P = \mathrm{Id}_P$. Then we have for $x \in P$: $gx = x$, $dg_x|_{T_xP} = \mathrm{Id}_{T_xP}$ and g preserves J so $dg_x|_{JT_xP} = \mathrm{Id}_{JT_xP}$. As $TP + JTP = TM$, we deduce that g acts as the identity on M. Thus,

$$G^0 = \{g \in G \,:\, g \cdot x = x \text{ for any } x \in P\}. \tag{5.23}$$

Recall that the function h defined in (1.9) is smooth only on the regular part of B and we have denoted by $\widehat{h}$ its smooth extension from the regular part of B to B.

First, we need to modify Section 4.1 as follows:

Recall that TM is endowed with a metric g^{TM} satisfying (0.25). We identify the normal bundle N of P in U to the orthogonal complement of TP. By (0.24) and (0.25), this means that N is identified with JTY. By (0.24) and (2.17), we have in particular $T^HU = T^HP \oplus N$.

Let g^{TY}, g^{T^HU} be the restriction of g^{TM} on TY, T^HU. Let g^{TB} (resp. g^{TM_G}) be the metric on TB (resp. TM_G) induced by g^{T^HU} (resp. g^{T^HP}).

Here, unlike in Section 4, we will not work on the quotient B but directly on M. Let ∇^{TB} be the Levi-Civita connection on (TB, g^{TB}). Let P^N and P^{T^HP} be the orthogonal projections from $T^HU|_P$ to N and T^HP, respectively. Set

$$\begin{aligned} \nabla^{T^HU} &= \pi^*\nabla^{TB}, & \nabla^N &= P^N(\nabla^{T^HU}|_P)P^N, \\ \nabla^{T^HP} &= P^{T^HP}(\nabla^{T^HU}|_P)P^{T^HP}, & {}^0\nabla^{T^HU} &= \nabla^N \oplus \nabla^{T^HP}. \end{aligned} \tag{5.24}$$

Fix $y_0 \in P$. For $V \in T^HU$ (resp. T^HP), we define $t \mapsto x_t = \exp^{T^HU}_{y_0}(tV) \in U$ (resp. $\exp^{T^HP}_{y_0}(tV) \in P$) the curve such that $x_0 = y_0$, $\dot{x}_0 = V$, $\dot{x} \in T^HU$ and $\nabla^{T^HU}_{\dot{x}}\dot{x} = 0$ (resp. $\dot{x} \in T^HP$ and $\nabla^{T^HP}_{\dot{x}}\dot{x} = 0$). For $W \in T^HP$ small and $V \in N_{y_0}$, let $\tau_W V$ be the parallel transport of V with respect to ∇^N along to curve $t \in [0,1] \mapsto \exp^{T^HP}_{y_0}(tW)$.

As in Section 4.1, we identify $B^{T^H_{y_0}U}(0,\varepsilon)$ to a subset of U as follows: for $Z \in B^{T^H_{y_0}U}(0,\varepsilon)$, we decompose Z as $Z = Z^0 + Z^\perp$ with $Z^0 \in T^H_{y_0}P$ and $Z^\perp \in N_{y_0}$, and then we identify Z with $\exp^{T^HU}_{\exp^{T^HP}_{y_0}(Z^0)}(\tau_{Z^0}Z^\perp)$.

Moreover, if $G_{y_0} = \{g \in G \,:\, gy_0 = y_0\}$ is the stabilizer of y_0 and $g \in G_{y_0}$, we can decompose $T^H_{y_0}P$ as

$$T^H_{y_0}P = (T^H_{y_0}P)^g \oplus \mathcal{N}_{y_0,g}, \tag{5.25}$$

where $(T^H_{y_0}P)^g$ is the fixed point-set of g in $T^H_{y_0}P = T_{y_0}P \cap JT_{y_0}P$ and $\mathcal{N}_{y_0,g}$ is its orthogonal complement. Hence we get, for each $g \in G_{y_0}$, a decomposition of the coordinate Z^0 as $Z^0 = Z^0_{1,g} + Z^0_{2,g}$ with $Z^0_{1,g} \in (T^H_{y_0}P)^g$ and $Z^0_{2,g} \in \mathcal{N}_{y_0,g}$. Note that $\mathrm{rk}(\mathcal{N}_{y_0,g}) = 0$ if and only if $g \in G^0$.

Observe that $U \simeq G \cdot B^{T^H_{y_0}U}(0,\varepsilon) = G \times_{G_{y_0}} B^{T^H_{y_0}U}(0,\varepsilon)$ is a G-neighborhood of the orbit $G \cdot y_0$ and $(B^{T^H_{y_0}U}(0,\varepsilon), G_{y_0})$ gives local chart on B.

As the constructions in Section 4.1 are G_{y_0}-equivariant, we can extend in the same way the geometric objects from $G \times_{G_{y_0}} B^{T^H_{y_0}U}(0,\varepsilon)$ to

$$M_0 := G \times_{G_{y_0}} \mathbb{R}^{2n-d}, \tag{5.26}$$

where $\mathbb{R}^{2n-d} \simeq T^H_{y_0}U$. Note that Lemma 4.1 and Corollary 4.2 still hold, because do not work on the quotient to get them: we only use finite propagation speed of the wave equation on M.

Set

$$\begin{aligned} &B_0 = M_0/G = \mathbb{R}^{2n-d}/G_{y_0}, \\ &\widehat{M_0} = G \times \mathbb{R}^{2n-d}, \qquad \widehat{B}_0 = \widehat{M_0}/G = \mathbb{R}^{2n-d}. \end{aligned} \tag{5.27}$$

Then we have a covering $\widehat{M_0} \to M_0$ (resp. $\widehat{B}_0 \to B_0$) which gives a (global) orbifold chart on M_0 (resp. B_0). We can then extend the geometric objects from M_0 to $\widehat{M_0}$. We will add a hat to denote the corresponding objects on $\widehat{B}_0$ or $\widehat{M_0}$. In particular, we have a Dirac operator $D^{\widehat{M_0}}_p$ on $\widehat{M_0}$ corresponding to $D^{M_0}_p$ in Section 4.1.

Let $\widehat{\pi}_G \colon G \times \mathbb{R}^{2n-d} \to \mathbb{R}^{2n-d}$ be the projection on the second factor. As in (1.10), we define

$$\widehat{\Phi} = \widehat{h}\widehat{\pi}_G \colon \mathscr{C}^\infty(G \times \mathbb{R}^{2n-d}, \mathbb{E}_{0,p})^G \to \mathscr{C}^\infty(\mathbb{R}^{2n-d}, (\mathbb{E}_{0,p})_{\widehat{B}_0}). \tag{5.28}$$

We also denote by $\widehat{\Phi}$ the map induced from $\mathscr{C}^\infty(M_0, \mathbb{E}_{0,p})^G$ to $\mathscr{C}^\infty(B_0, (\mathbb{E}_{0,p})_{B_0})$.

Let g^{TM_0} be defined as in (4.18) and let $g^{T^HM_0}$ be the metric on $\mathbb{R}^{2n-d}$ induced by g^{TM_0}, with corresponding Riemannian volume on $(\mathbb{R}^{2n-d}, g^{T^HM_0})$ denoted by $dv_{T^HM_0}$.

Let $e^{-u\widehat{\Phi}D^{\widehat{M_0},2}_p\widehat{\Phi}}$ be the heat kernel of the operator $\widehat{\Phi}D^{\widehat{M_0},2}_p\widehat{\Phi}$ on $\widehat{B}_0$ and $e^{-u\widehat{\Phi}D^{\widehat{M_0},2}_p\widehat{\Phi}}(Z,Z')$ $(Z,Z' \in \widehat{B}_0)$ be its smooth kernel with respect to $dv_{T^HM_0}(Z')$. Concerning heat kernels on orbifolds, we refer the reader to [**14**, Sect. 2.1]. Then we have for $v = [g,Z]$ and $v' = [g',Z']$ in M_0,

$$\begin{aligned} &\widehat{h}(v)\widehat{h}(v')\big(P_G e^{-\frac{u}{p}D^{M_0,2}_p} P_G\big)(v,v') \\ &\qquad = e^{-\frac{u}{p}\widehat{\Phi}D^{M_0,2}_p\widehat{\Phi}^{-1}}(\pi(v),\pi(v')) \end{aligned} \tag{5.29}$$

$$= \frac{1}{|G^0|} \sum_{g \in G_{y_0}} (g,1) \cdot e^{-\frac{u}{p}\widehat{\Phi} D_p^{\widehat{M}_0,2}\widehat{\Phi}}(g^{-1}Z, Z'),$$

where $|G^0|$ is the cardinal of G^0. Indeed, the first equality in (5.29) is analogous to (4.119), and the second follows from a similar computation as in [**6**, (5.19)] or [**15**, (5.4.17)].

Note that our trivialization of the restriction of L (resp. E) on $B^{T^H_{y_0}U}(0,\varepsilon)$ is not G_{y_0}-invariant, except if G_{y_0} acts trivially on L_{y_0} (resp. E_{y_0}). More precisely, let $\widehat{M}_{G,0} = \mathbb{R}^{2n-2d} \times \{0\} \subset \widehat{B}_0$ and for $g \in G_{y_0}$, let $\widehat{M}^g_{G,0}$ be the fixed point-set of g in $\widehat{M}_{G,0}$. Then the action of g on $L|_{\widehat{M}^g_{G,0}}$ is the multiplication by $e^{i\theta_g}$ and θ_g is locally constant on $\widehat{M}^g_{G,0}$. Likewise, the action of g on $E|_{\widehat{M}^g_{G,0}}$ is given by $g_E \in \mathscr{C}^\infty(\widehat{M}^g_{G,0}, \mathrm{End}(E))$ which is parallel with respect to ∇^E.

Now, as we work on $\widehat{B}_0$ and $\widehat{M}_0$, we can apply the results of Sections 4.1–4.3 to the operator $\widehat{\Phi} D_p^{\widehat{M}_0,2}\widehat{\Phi}$. We will use the same notation as in these sections, and add a subscript to indicate the base-point (e.g., κ_x, $\mathscr{L}_{0,x}, \ldots$). By Theorem 4.13 and (4.117), we obtain for $g \in G_{y_0}$ and $u > 0$ fixed

$$\begin{aligned}(5.30)\quad &\Big| p^{-n+d/2} e^{-\frac{u}{p}\widehat{\Phi} D_p^{\widehat{M}_0,2}\widehat{\Phi}}(g^{-1}Z, Z) \\ &- \kappa^{-1}_{Z_{1,g}}(Z^\perp) e^{-u\mathscr{L}_{0,Z_{1,g}}}\big(\sqrt{p} g^{-1}(Z_{2,g}+Z^\perp), \sqrt{p}(Z_{2,g}+Z^\perp)\big)\Big|_{\mathscr{C}^{m'}(M_G)} \\ &\leq C p^{-1/2}\big(1+\sqrt{p}|Z_{2,g}|\big)^N \big(1+\sqrt{p}|Z^\perp|\big)^{-m} \exp\big(-Cp \inf_{h \in G_{y_0}} |h^{-1}Z - Z|^2\big).\end{aligned}$$

On the other hand, note that there is $\rho > 0$ such that for $g \in G_{y_0}$, $|g^{-1}Z - Z|^2 \geq \rho |Z_{2,g}|^2$, so

$$\begin{aligned}(5.31)\quad &\Big| p^{-n+d/2} e^{-\frac{u}{p}\widehat{\Phi} D_p^{\widehat{M}_0,2}\widehat{\Phi}}(g^{-1}Z, Z) \\ &- \kappa^{-1}_{Z_{1,g}}(Z^\perp) e^{-u\mathscr{L}_{0,Z_{1,g}}}\big(\sqrt{p} g^{-1}(Z_{2,g}+Z^\perp), \sqrt{p}(Z_{2,g}+Z^\perp)\big)\Big|_{\mathscr{C}^{m'}(M_G)} \\ &\qquad\qquad \leq C p^{-1/2}\big(1+\sqrt{p}|Z^\perp|\big)^{-m} \exp\big(-C'p|Z_{2,g}|^2\big).\end{aligned}$$

We can now prove Theorem 0.5. First, observe that Theorem 0.7 is still true here because we work on M to prove it in Section 5.1. Thus, we can use a similar approach to prove Theorem 0.5 as in Section 5.1.

Note that the estimate (5.9) still holds. Consider now a G-invariant function $\psi \in \mathscr{C}^\infty(M)$ such that the induced function (again denoted by ψ) on B is compactly supported in a small neighborhood (in B) of $x_0 \in M_G$.

Similarly to (4.26), we denote by $dv_{T^H U}$ the Riemannian volume of $(T^H_{y_0}U, g^{T^H_{y_0}U})$. Then, as in (5.10), (5.31) and dominated convergence imply that

$$(5.32)\quad p^{-n+d}\int_U \psi(v)\,\mathrm{Tr}_q\left[\left(P_G e^{-\frac{u}{p}D_p^{M_0,2}}P_G\right)(v,v)\right]dv_M(v) = \frac{1}{|G_{y_0}/G^0||G^0|}\sum_{g\in G_{y_0}}\frac{p^{-\mathrm{rk}(\mathcal{N}_{y_0,g})/2}}{(2\pi)^{n-d}}\times \int_{A(p,\varepsilon)}\psi\left(Z_{1,g}+\frac{Z_{2,g}}{\sqrt{p}}\right)\mathrm{Lim}(g,Z)dv_{T^HU}(Z)+o(1),$$

where $A(p,\varepsilon)=\{Z\in \widehat{B}_0 \,:\, |Z_{1,g}|\le\varepsilon, |Z_{2,g}|\le\varepsilon\sqrt{p}, |Z^\perp|\le\varepsilon\sqrt{p}\}$ and $\mathrm{Lim}(g,Z)$ is the limiting term given by

$$(5.33)\quad \mathrm{Lim}(g,Z)= \mathrm{Tr}_q\left[(g,1)\cdot\frac{\det(\dot{R}^{L_G}_{Z_{1,g}})e^{2u\omega_d(Z_{1,g})}}{\det\left(1-\exp(-2u\dot{R}^{L_G}_{Z_{1,g}})\right)} \times e^{-u\mathscr{L}^\perp_{Z_{1,g}}}\left(g^{-1}(Z_{2,g}+Z^\perp),Z_{2,g}+Z^\perp\right)\otimes \mathrm{Id}_E\right].$$

In particular, in (5.32), every term involving a g such that $\mathrm{rk}(\mathcal{N}_{y_0,g})>0$, i.e., $g\notin G^0$, disappears when we look at the leading term in p.

Thus, we now consider $g\in G^0$. The action of g on M and $\Lambda^{0,\bullet}(T^*M)$ is trivial, so we have

$$(5.34)\quad \mathrm{Tr}_q\left[(g,1)\cdot\frac{\det(\dot{R}^{L_G}_{Z^0})e^{2u\omega_d(Z^0)}}{\det\left(1-\exp(-2u\dot{R}^{L_G}_{Z^0})\right)} \times e^{-u\mathscr{L}^\perp_{Z^0}}(g^{-1}Z^\perp,Z^\perp)\otimes\mathrm{Id}_E\right] = e^{ip\theta_g}\frac{\det(\dot{R}^{L_G}_{Z^0})\,\mathrm{Tr}_{\Lambda^{0,q}}[e^{2u\omega_d(Z^0)}]}{\det\left(1-\exp(-2u\dot{R}^{L_G}_{Z^0})\right)}e^{-u\mathscr{L}^\perp_{Z^0}}(Z^\perp,Z^\perp)\otimes g_E(Z^0).$$

Using (5.32), (5.33) and (5.34), we get as in (5.13)–(5.21):

$$\limsup_{p\to+\infty} p^{-n+d}\int_U\psi(v)\,\mathrm{Tr}_q\left[\left(P_G e^{-\frac{u}{p}D_p^2}P_G\right)(v,v)\right]dv_M(v)$$
$$\le \frac{1}{(2\pi)^{n-d}}\frac{1}{|G^0|}\sum_{g\in G^0}\prod_{i=1}^d\sqrt{\frac{1}{f(ua_i^\perp)(1-e^{-4ua_i^\perp})}}\times \int_{M_G}\psi(x)\frac{\det(\dot{R}^{L_G}_x)\sum_{j=0}^q(-1)^{q-j}\,\mathrm{Tr}_{\Lambda^{0,j}}[e^{2u\omega_{G,d}(x)}]}{\det\left(1-\exp(-2u\dot{R}^{L_G}_x)\right)} \times e^{ip\theta_g}\,\mathrm{Tr}^E[g_E(x)]dv_{M_G}(x)$$

$$(5.35)\quad \leq (-1)^{q-r}\int_{M_G(\leq q-r)} \det\Big(\frac{\dot{R}^{L_G}_x}{2\pi}\Big)\frac{\psi(x)}{|G^0|}\Big(\sum_{g\in G^0} e^{ip\theta_g}\operatorname{Tr}^E[g_E(x)]\Big)dv_{M_G}(x)$$
$$= (-1)^{q-r}\dim(L^p\otimes E)^{G^0}\int_{M_G(\leq q-r)}\psi(x)\det\Big(\frac{\dot{R}^{L_G}_x}{2\pi}\Big)dv_{M_G}(x).$$

Finally, we take some functions ψ_k as ψ above and such that $\sum_k \psi_k = 1$ in a neighborhood of M_G in B and we apply (5.35) for those ψ_k. We get Theorem 0.5 by taking the sum over k of the obtained estimates and using Theorem 0.7 and (5.9).

5.3. The other isotypic components of the cohomology. In this subsection, we show how to use Theorem 0.5 to get estimates on the other isotypic components of the cohomology $H^\bullet(M, L^p\otimes E)$.

Let $\mathcal{V}_\gamma$ be the finite dimensional irreducible representation of G with highest weight γ.

For a representation F of G, we denote by F_γ its isotopic component associated with γ. Then we have

$$(5.36)\quad \begin{aligned} H^\bullet(M, L^p\otimes E)_\gamma &= \mathcal{V}_\gamma\otimes \operatorname{Hom}_G(\mathcal{V}_\gamma, H^\bullet(M, L^p\otimes E)) \\ &= \mathcal{V}_\gamma\otimes\big(H^\bullet(M, L^p\otimes E)\otimes\mathcal{V}_\gamma^*\big)^G \\ &= \mathcal{V}_\gamma\otimes H^\bullet(M, L^p\otimes E\otimes\mathcal{V}_\gamma^*)^G, \end{aligned}$$

where $\mathcal{V}_\gamma^*$ is viewed as a trivial bundle over M.

By Theorem 0.5 applied replacing E by $E\otimes\mathcal{V}_\gamma^*$ and (5.36) we have as $p\to+\infty$,

$$(5.37)\quad \sum_{j=0}^{q}(-1)^{q-j}\dim H^j(M, L^p\otimes E)_\gamma$$
$$\leq \dim\mathcal{V}_\gamma\dim(L^p\otimes E\otimes\mathcal{V}_\gamma^*)^{G^0}\frac{p^{n-d}}{(n-d)!}\int_{M_G(\leq q-r)}(-1)^{q-r}\omega_G^{n-d}+o(p^{n-d}),$$

with equality for $q=n$.

In particular, we get the weak inequalities

$$(5.38)\quad \dim H^q(M, L^p\otimes E)_\gamma$$
$$\leq \dim\mathcal{V}_\gamma\dim(L^p\otimes E\otimes\mathcal{V}_\gamma^*)^{G^0}\frac{p^{n-d}}{(n-d)!}\int_{M_G(q-r)}(-1)^{q-r}\omega_G^{n-d}+o(p^{n-d}).$$

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