

GENERALIZED DONALDSON–THOMAS INVARIANTS ON THE LOCAL PROJECTIVE PLANE

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Abstract

We show that the generating series of generalized Donaldson–Thomas invariants on the local projective plane with any positive rank is described in terms of modular forms and theta type series for indefinite lattices. In particular, it absolutely converges to give a holomorphic function on the upper half plane.

1. Introduction

1.1. Motivation. Let

$$\pi: X \rightarrow \mathbb{P}^2$$

be the total space of the canonical line bundle on \mathbb{P}^2 . The space X is a non-compact Calabi–Yau 3-fold, and the enumerative invariants (e.g., Gromov–Witten invariants, Donaldson–Thomas invariants) on X have drawn attention in connection with string theory. Among such invariants, we focus on the *generalized Donaldson–Thomas (DT) invariants* introduced by Thomas [Tho00], Joyce–Song [JS12] and Kontsevich–Soibelman [KS]. Given an element

$$(r, l, \Delta) \in \mathbb{Z}^{\oplus 3},$$

with $r > 0$, the generalized DT invariant

$$(1) \quad \text{DT}(r, l, \Delta) \in \mathbb{Q}$$

counts semistable¹ sheaves E on X supported on the zero section² of π , satisfying

$$(2) \quad \text{rank}(\pi_* E) = r, \quad c_1(\pi_* E) = l, \quad \Delta(\pi_* E) = \Delta.$$

Here $\Delta(\pi_* E)$ is the discriminant

$$\Delta(\pi_* E) = l^2 - 2r \text{ch}_2(\pi_* E).$$

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¹The invariant (1) is independent of a choice of a stability, i.e., slope stability or Gieseker stability. See Lemma 2.10.

²Indeed such sheaves are scheme theoretically supported on the zero section. See Lemma 2.3.

We are interested in the generating series:

$$(3) \quad \text{DT}(r, l) := \sum_{\Delta \in \mathbb{Z}_{\geq 0}} \text{DT}(r, l, \Delta) (-q^{\frac{1}{2r}})^{\Delta}.$$

If r and l are coprime, then the series (3) is the generating series of Euler numbers of moduli spaces of stable sheaves on \mathbb{P}^2 , which has been explicitly computed up to rank three in several literatures [G90], [Kly91], [Yos94], [Yos96], [G99], [Man11], [BM13], [Koo15], [Wei11]. If r and l are not coprime, then the definition of the invariant (1) involves the logarithm of the moduli stack in the Hall algebra, and its explicit computation is more subtle. Nevertheless, there exist works [Man13], [GS15] in which the series (3) is studied for non-coprime (r, l) up to rank three. In any case, the resulting closed formula is quite complicated even in the rank three case³, and it seems hopeless to obtain a neat closed formula for an arbitrary rank.

On the other hand, by Vafa–Witten’s S-duality conjecture [VW94], the series (3) is expected to have a certain modular invariance property. The computation of the series (3) in the rank two case [Kly91], [BM13] indicates that (3) is not a modular form in a strict sense, but may be so in a broad sense including *mock modular forms* [Zwe], [Zag09]. In order to approach the S-duality conjecture, we may not have to worry about the complexity of the explicit closed formula: it is enough to know that the series (3) is a finite linear combination of modular forms of the same weight in a broad sense. The purpose of this paper is to show that the series (3) for any $r \geq 1$ is always written in terms of modular forms and certain theta type series for indefinite lattices, which converge and hopefully have a modular invariance property in a broad sense.

1.2. Main result. We construct theta type series from data

$$(4) \quad \xi = (\Gamma, B, \bar{\nu}, c_1, c_2, \dots, c_b, c'_1, c'_2, \dots, c'_b, \alpha_1, \dots, \alpha_k).$$

Here $(\Gamma, B(-, -))$ is a non-degenerate lattice with index (a, b) , and $\bar{\nu}, c_i, c'_i, \alpha_i$ are elements of $\Gamma_{\mathbb{Q}}$, satisfying certain conditions described in Subsection 2.7. Given data (4), we construct the series

$$(5) \quad \Theta_{\xi}(q) := \sum_{\nu \in \bar{\nu} + \Gamma} \prod_{i=1}^b (\text{sgn}(B(c_i, \nu)) - \text{sgn}(B(c'_i, \nu))) \prod_{j=1}^k B(\alpha_j, \nu) \cdot q^{Q(\nu)}.$$

Here $Q(\nu) = B(\nu, \nu)/2$ and $\text{sgn}(x)$ is defined by

$$\text{sgn}(x) = \begin{cases} x/|x| & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

³For instance, a formula in the rank three case occupies 1.5 pages in [Koo15, Section 4.3].

The series (5) is a generalization of known theta type series, and our conditions in Subsection 2.7 allow us to show the convergence (cf. Lemma 2.16) of (5) after the substitution $q = e^{2\pi i\tau}$, where $\tau \in \mathcal{H} \subset \mathbb{C}$ and \mathcal{H} is the upper half plane. For example, if $b = k = 0$, then Q is a positive definite quadratic form on Γ , and the series (5) is nothing but the classical theta series. In this case, we call data (4) as *classical data*. If $b = 1$ and $k = 0$, then the series (5) is a mock theta series studied by Zwegers [Zwe]. Some more detail on the series (5) will be discussed in Subsection 2.7. The following is the main result in this paper:

Theorem 1.1. (Corollary 3.7.) *For any $r \in \mathbb{Z}_{\geq 1}$ and $l \in \mathbb{Z}$, there is a finite number of data ξ_1, \dots, ξ_n as in (4), classical data $\xi', a_1, \dots, a_n \in \mathbb{Q}$ and $N \in \mathbb{Z}_{\geq 1}$ such that the following holds:*

$$DT(r, l) = q^{\frac{r}{8}} \eta(q)^{-3r} \cdot \Theta_{\xi'}(q)^{-1} \cdot \left(\sum_{i=1}^n a_i \Theta_{\xi_i}(q^{\frac{1}{N}}) \right).$$

Here $\eta(q)$ is the Dedekind eta function

$$(6) \quad \eta(q) = q^{\frac{1}{24}} \prod_{m \geq 1} (1 - q^m).$$

Since the series (5) converges, we obtain the following corollary:

Corollary 1.2. *For any $r \in \mathbb{Z}_{\geq 1}$ and $l \in \mathbb{Z}$, the generating function*

$$\sum_{\Delta \in \mathbb{Z}_{\geq 0}} DT(r, l, \Delta) e^{2\pi i \Delta \tau}$$

converges absolutely on the upper half plane $\tau \in \mathcal{H} \subset \mathbb{C}$.

The strategy of the proof of Theorem 1.1 is described below. Although it follows from a traditional approach, the result of Theorem 1.1 is a new structure result for the series (3) with an arbitrary positive rank⁴.

1.3. Strategy of the proof of Theorem 1.1. So far there have been two kinds of approaches toward the study of the invariants (1): one is to use the localization with respect to the torus action [Kly91], [Koo15], [Wei11], [GS15], and the other one is to use the blow-up formula and the wall-crossing formula [Yos96], [Man11], [Man13], [BM13]. We follow the latter strategy. In fact, the latter one has been used to com-

⁴In [Koo15, Theorem 3.7], by the torus localization, Kool described the Euler numbers of moduli spaces of stable sheaves on \mathbb{P}^2 with any positive rank in terms of Euler numbers of certain explicit varieties given by GIT quotients. Since the computation of the latter numbers is not obvious, his result does not imply Theorem 1.1 even if r and l are coprime.

pute Betti numbers (rather than Euler numbers) of moduli spaces of stable sheaves on \mathbb{P}^2 . Let

$$f: \widehat{\mathbb{P}}^2 \rightarrow \mathbb{P}^2$$

be a blow-up at a point and C the exceptional divisor of f . The blow-up formula [Yos96], [LQ99], [G99] describes Betti numbers of the moduli spaces of stable sheaves on \mathbb{P}^2 in terms of those on $\widehat{\mathbb{P}}^2$ with respect to the f^*H -stability and classical theta series. Here H is the hyperplane class of \mathbb{P}^2 . Note that $\widehat{\mathbb{P}}^2$ admits a \mathbb{P}^1 -fibration $\widehat{\mathbb{P}}^2 \rightarrow \mathbb{P}^1$, and we denote by F a fiber class. Let us consider a one parameter family of \mathbb{R} -divisors on $\widehat{\mathbb{P}}^2$:

$$H_t = f^*H - tC, \quad t \in [0, 1).$$

The \mathbb{R} -divisor H_t is ample for $t \in (0, 1)$. It is well-known that, for t sufficiently close to 1, there is no H_t -semistable sheaf E on $\widehat{\mathbb{P}}^2$ with $\text{rank}(E) \geq 2$ and $c_1(E) \cdot F = 1$. This fact together with the wall-crossing from H_0 to H_t with $t \rightarrow 1 - 0$ enable us to describe Betti numbers of moduli spaces of H_0 -semistable sheaves on $\widehat{\mathbb{P}}^2$ in terms of those with lower rank. Here $t \rightarrow 1 - 0$ means $t = 1 - \varepsilon$ for $0 < \varepsilon \ll 1$. Combined with the blow-up formula, we can compute the desired Betti numbers on \mathbb{P}^2 by the induction of the rank. The above argument was considered by Yoshioka [Yos96] in the rank two case, by Manschot [Man11], [Man13] in the rank three case. As pointed out in [Man13], the argument in principal can be applied for an arbitrary rank. However, there are some issues to apply the above arguments to study the series (3):

- If r and l are not coprime, then a relationship between Betti numbers of the moduli spaces⁵ and the generalized DT invariants is not yet established.
- To obtain a result for the series (3) from the result of Betti numbers, one has to take a specialization, whose computation is not obvious.
- The wall-crossing formula is quite complicated, and it is hard to describe the result in a general rank.

In order to avoid the first and the second issues, we directly work with the generalized DT invariants, rather than Betti numbers. Instead of using the results of [Yos96], [LQ99], [G99], we use the result of [Tod15] in which a blow-up formula for the series (3) was obtained by interpreting a blow-up of a surface as a 3-fold flop. As for the third issue, we work with Joyce's wall-crossing coefficients [Joy08] with respect to the polarization change from H_0 to H_t with $t \rightarrow 1 - 0$ in detail, and extract the theta type series (5).

⁵In this case, the moduli space is an algebraic stack, and its Betti numbers are interpreted as a rational function given by the ratio of Poincaré polynomials.

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2. Preliminary

This section is devoted to a preliminary to the proof of Theorem 1.1. Throughout this paper, all the varieties or stacks are defined over \mathbb{C} .

2.1. Stability conditions on local surfaces. Let S be a smooth projective surface and

$$\pi: X = \omega_S \rightarrow S,$$

the total space of the canonical line bundle on S . Note that X is a non-compact Calabi–Yau 3-fold, i.e., $\omega_X \cong \mathcal{O}_X$. Let

$$\text{Coh}_c(X) \subset \text{Coh}(X)$$

be the abelian category of coherent sheaves on X whose supports are compact. We recall two kinds of stability conditions on $\text{Coh}_c(X)$ which depend on a choice of an ample \mathbb{R} -divisor H on S : slope stability condition and Gieseker stability condition.

The slope stability condition uses the following slope function for $0 \neq E \in \text{Coh}_c(X)$:

$$\mu_H(E) = \frac{c_1(\pi_*E) \cdot H}{\text{rank}(\pi_*E)} \in \mathbb{R} \cup \{\infty\}.$$

Here we set $\mu_H(E) = \infty$ if $\text{rank}(\pi_*E) = 0$.

Definition 2.1. A pure two dimensional sheaf $E \in \text{Coh}_c(X)$ is H -slope (semi)stable if for any short exact sequence $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ in $\text{Coh}_c(X)$ with $F, G \neq 0$, we have $\mu_H(F) < (\leq) \mu_H(G)$.

The Gieseker stability condition uses the reduced Hilbert polynomial for $E \in \text{Coh}_c(X)$ ⁶ :

$$(7) \quad \bar{\chi}_H(E) = \chi(\pi_*E \otimes \mathcal{O}_S(mH))/a_d,$$

where a_d is the leading coefficient of $\chi(\pi_*E \otimes \mathcal{O}_S(mH))$. We write $\bar{\chi}_H(F) \prec \bar{\chi}_H(E)$ if $\bar{\chi}_H(F) < \bar{\chi}_H(E)$ for $m \gg 0$.

Definition 2.2. A pure two dimensional sheaf $E \in \text{Coh}_c(X)$ is Gieseker (semi)stable if for any non-zero proper subsheaf $F \subset E$, we have $\bar{\chi}_H(F) \prec (\preceq) \bar{\chi}_H(E)$.

⁶By the Riemann–Roch theorem, we can formally define (7) for any \mathbb{R} -divisor H .

We have the obvious implications:

$$(8) \quad \begin{aligned} &\text{slope stable} \Rightarrow \text{Gieseker stable} \\ &\qquad \qquad \qquad \Rightarrow \text{Gieseker semistable} \Rightarrow \text{slope semistable.} \end{aligned}$$

We regard S as a closed subscheme of X by the zero section of π . In some situation, an object in $\text{Coh}_c(X)$ is supported on S :

Lemma 2.3. *Suppose that $K_S \cdot H < 0$. Then any H -slope semistable sheaf $E \in \text{Coh}_c(X)$ with $\text{rank}(\pi_* E) > 0$ is an \mathcal{O}_S -module. In particular, any pure two dimensional sheaf on X is supported on S .*

Proof. Applying $\otimes_{\mathcal{O}_X} E$ to the inclusion $\mathcal{O}_X(-S) \subset \mathcal{O}_X$, we obtain the map

$$E(-S) \rightarrow E.$$

The above map is zero since

$$\mu_H(E(-S)) = \mu_H(E) - K_S \cdot H > \mu_H(E)$$

and $E, E(-S)$ are H -slope semistable. This implies that E is an \mathcal{O}_S -module. (Also see [GS15, Lemma 2.1].) q.e.d.

2.2. Hall algebras. We recall the stack theoretic Hall algebras of $\text{Coh}_c(X)$ introduced by Joyce [Joy08]. Let \mathfrak{M} be the moduli stack of all the objects in $\text{Coh}_c(X)$. The stack theoretic Hall algebra $H(X)$ is \mathbb{Q} -spanned by the isomorphism classes of the symbols (cf. [Joy08])

$$(9) \quad [\rho: \mathcal{X} \rightarrow \mathfrak{M}],$$

where \mathcal{X} is an algebraic stack of finite type with affine geometric stabilizers and ρ is a 1-morphism. The relation is generated by

$$(10) \quad [\rho: \mathcal{X} \rightarrow \mathfrak{M}] \sim [\rho|_{\mathcal{Y}}: \mathcal{Y} \rightarrow \mathfrak{M}] + [\rho|_{\mathcal{U}}: \mathcal{U} \rightarrow \mathfrak{M}],$$

where $\mathcal{Y} \subset \mathcal{X}$ is a closed substack and $\mathcal{U} := \mathcal{X} \setminus \mathcal{Y}$. There is an associative $*$ -product on $H(X)$ based on the Ringel–Hall algebras. Let $\mathfrak{E}\mathfrak{r}$ be the stack of short exact sequences

$$0 \rightarrow E_1 \rightarrow E_3 \rightarrow E_2 \rightarrow 0,$$

in $\text{Coh}_c(X)$ and

$$p_i: \mathfrak{E}\mathfrak{r} \rightarrow \mathfrak{M},$$

the 1-morphism sending E_\bullet to E_i . The $*$ -product on $H(X)$ is given by

$$[\rho_1: \mathcal{X}_1 \rightarrow \mathfrak{M}] * [\rho_2: \mathcal{X}_2 \rightarrow \mathfrak{M}] = [\rho_3: \mathcal{X}_3 \rightarrow \mathfrak{M}],$$

where

$$(\mathcal{X}_3, \rho_3 = p_3 \circ (\rho'_1, \rho'_2))$$

is given by the following Cartesian diagram

$$\begin{array}{ccccc}
 \mathcal{X}_3 & \xrightarrow{(\rho'_1, \rho'_2)} & \mathfrak{E}\mathfrak{r} & \xrightarrow{p_3} & \mathfrak{M} \\
 \downarrow & & \square & & \downarrow (p_1, p_2) \\
 \mathcal{X}_1 \times \mathcal{X}_2 & \xrightarrow{(\rho_1, \rho_2)} & \mathfrak{M}^{\times 2} & &
 \end{array}$$

Let cl be the group homomorphism

$$(11) \quad \text{cl}: K(\text{Coh}_c(X)) \rightarrow H^*(S, \mathbb{Q})$$

defined in the following way:

$$(12) \quad \text{cl}(E) = (\text{rank}(\pi_* E), c_1(\pi_* E), \text{ch}_2(\pi_* E)).$$

We denote by $\Lambda \subset H^*(S, \mathbb{Q})$ the image of cl . We write an element $\gamma \in \Lambda$ as (r, l, s) as in the RHS of (12). For $\gamma \in \Lambda$, let

$$(13) \quad \mathfrak{M}(\gamma) \subset \mathfrak{M}$$

be the substack of $E \in \text{Coh}_c(X)$ with $\text{cl}(E) = \gamma$. The algebra $H(X)$ is Λ -graded

$$H(X) = \bigoplus_{\gamma \in \Lambda} H_\gamma(X),$$

where $H_\gamma(X)$ is \mathbb{Q} -spanned by the symbols (9) which factor through (13).

2.3. Integration map. Let χ be the pairing on Λ given by

$$(14) \quad \chi((r_1, l_1, s_1), (r_2, l_2, s_2)) = K_S(r_2 l_1 - r_1 l_2).$$

Since X is a non-compact Calabi–Yau 3-fold, the Serre duality and the Riemann–Roch theorem implies

$$\begin{aligned}
 \chi(\text{cl}(E_1), \text{cl}(E_2)) &= \dim \text{Hom}(E_1, E_2) - \dim \text{Ext}^1(E_1, E_2) \\
 &\quad + \dim \text{Ext}^1(E_2, E_1) - \dim \text{Hom}(E_2, E_1),
 \end{aligned}$$

for $E_1, E_2 \in \text{Coh}_c(X)$. Let $C(X)$ be the Lie algebra

$$C(X) = \bigoplus_{\gamma \in \Lambda} \mathbb{Q} \cdot c_\gamma,$$

with bracket given by

$$(15) \quad [c_{\gamma_1}, c_{\gamma_2}] = (-1)^{\chi(\gamma_1, \gamma_2)} \chi(\gamma_1, \gamma_2) c_{\gamma_1 + \gamma_2}.$$

There is a Lie subalgebra

$$H^{\text{Lie}}(X) \subset H(X)$$

consisting of virtual indecomposable objects (cf. [Joy07, Section 5.2]) and a linear homomorphism (cf. [JS12, Theorem 5.12])

$$(16) \quad \Pi: H^{\text{Lie}}(X) \rightarrow C(X),$$

such that if \mathcal{X} is a \mathbb{C}^* -gerb over an algebraic space \mathcal{X}' , we have

$$\Pi([\rho: \mathcal{X} \rightarrow \mathfrak{M}(\gamma)]) = - \left(\sum_{k \in \mathbb{Z}} k \cdot \chi(\nu^{-1}(k)) \right) c_\gamma.$$

Here ν is Behrend’s constructible function [Beh09] on \mathcal{X}' . Moreover, the map (16) preserves the brackets for the elements $[\rho_i: \mathcal{X}_i \rightarrow \mathfrak{M}]$ for $i = 1, 2$ if \mathfrak{M} is a smooth stack at $\rho_i(x)$ for any $x \in \mathcal{X}_i$, $i = 1, 2$. In the case we are interested in, this condition is satisfied:

Lemma 2.4. *Suppose that $K_S \cdot H < 0$. Then for any H -slope semistable $E \in \text{Coh}_c(X)$ with $\text{rank}(\pi_* E) > 0$, the stack \mathfrak{M} is smooth at $[E]$.*

Proof. By Lemma 2.3, we have $E \in \text{Coh}(S)$. Since the stability is an open condition, the obstruction space of the deformation theory of E lies in $\text{Ext}_S^2(E, E)$. By the Serre duality, we have

$$\text{Ext}_S^2(E, E) \cong \text{Hom}(E, E \otimes \omega_S)^\vee,$$

which vanishes since $E, E \otimes \omega_S$ are H -slope semistable and $\mu_H(E) > \mu_H(E \otimes \omega_S)$ by $K_S \cdot H < 0$. q.e.d.

Remark 2.5. By the argument of [JS12, Theorem 5.12], the map (16) is a Lie algebra homomorphism if we know that \mathfrak{M} is analytically locally written as a critical locus of a certain holomorphic function in the sense of [JS12, Theorem 5.3]. However, since our situation is a non-compact Calabi–Yau 3-fold, we are not able to use [JS12, Theorem 5.12] to conclude that (16) is a Lie algebra homomorphism.

2.4. Generalized DT invariants. For $\gamma \in \Lambda$, let

$$(17) \quad \mathfrak{M}_H^{s(ss)}(\gamma) \subset \mathfrak{M}(\gamma)$$

be the substack of H -slope (semi)stable sheaves $E \in \text{Coh}_c(X)$ satisfying $\text{cl}(E) = \gamma$. The stack (17) determines the element

$$(18) \quad \delta_H(\gamma) = [\mathfrak{M}_H^{ss}(\gamma) \subset \mathfrak{M}(\gamma)] \in H_\gamma(X).$$

The above element also defines the element of $H_\gamma(X)^7$:

$$(19) \quad \epsilon_H(\gamma) = \sum_{\substack{\gamma_1 + \dots + \gamma_m = \gamma \\ \mu_H(\gamma_i) = \mu_H(\gamma)}} \frac{(-1)^{m-1}}{m} \delta_H(\gamma_1) * \dots * \delta_H(\gamma_m).$$

Here the slope $\mu_H(\gamma)$ for non-zero $\gamma = (r, l, s) \in \Lambda$ is given by $l \cdot H/r$, i.e.,

$$\mu_H(\text{cl}(E)) = \mu_H(E)$$

holds for any non-zero $E \in \text{Coh}_c(X)$.

⁷It is straightforward to check that (19) is a finite sum.

Definition 2.6. The generalized DT invariant $\text{DT}_H(\gamma) \in \mathbb{Q}$ is defined by the formula:

$$(20) \quad \Pi(\epsilon_H(\gamma)) = -\text{DT}_H(\gamma) \cdot c_\gamma.$$

Remark 2.7. If $\mathfrak{M}_H^s(\gamma) = \mathfrak{M}_H^{ss}(\gamma)$, then they are \mathbb{C}^* -gerb over a quasi-projective scheme $M_H^s(\gamma)$. In this case, the invariant $\text{DT}_H(\gamma)$ is written as

$$\text{DT}_H(\gamma) = \int_{M_H^s(\gamma)} \nu \, d\chi,$$

where ν is the Behrend function [Beh09] on $M_H^s(\gamma)$.

By formally replacing the Behrend function by the constant function 1 in the construction of (16), and removing the minus sign in (20), we obtain another invariant (cf. [Joy08]):

$$(21) \quad \text{Eu}_H(\gamma) \in \mathbb{Q}.$$

In the situation of Remark 2.7, the above invariant is the usual Euler number:

$$\text{Eu}_H(\gamma) = \chi(M_H^s(\gamma)).$$

Also in the same situation of Lemma 2.3, the invariant (21) essentially coincides with the generalized DT invariant:

Lemma 2.8. *Suppose that $K_S \cdot H < 0$. Then for $\gamma = (r, l, s) \in \Lambda$ with $r > 0$, we have the equality:*

$$(22) \quad \text{DT}_H(\gamma) = (-1)^{r^2\chi(\mathcal{O}_S)+1+\Delta(\gamma)} \text{Eu}_H(\gamma).$$

Here the discriminant $\Delta(\gamma)$ is defined to be

$$\Delta(\gamma) = l^2 - 2rs.$$

Proof. For any closed point $[E] \in \mathfrak{M}_H^{ss}(\gamma)$, the stack \mathfrak{M} is smooth at $[E]$ by Lemma 2.4. Its dimension is

$$\dim \text{Ext}_S^1(E, E) - \dim \text{Hom}_S(E, E) = r^2\chi(\mathcal{O}_S) - \Delta(\gamma),$$

by the Riemann–Roch theorem. Hence the Behrend function of \mathfrak{M} at $[E]$ is given by $(-1)^{r^2\chi(\mathcal{O}_S)-\Delta(\gamma)}$. Taking the minus sign in (20) into account, we obtain the desired equality. q.e.d.

For $r \in \mathbb{Z}_{\geq 1}$ and $l \in \text{NS}(S)$, we set

$$(23) \quad \text{DT}_H(r, l) := \sum_s \text{DT}_H(r, l, s) (-q^{\frac{1}{2r}})^{l^2 - 2rs}.$$

Note that if $K_S \cdot H < 0$, the equality (22) implies

$$(24) \quad \text{DT}_H(r, l) = (-1)^{r^2\chi(\mathcal{O}_S)+1} \sum_s \text{Eu}_H(r, l, s) q^{\frac{l^2}{2r} - s}.$$

If, furthermore, $r = 1$, the moduli stack $\mathfrak{M}_H^{ss}(1, l, s)$ is isomorphic to the \mathbb{C}^* -gerb over the Hilbert scheme of points on S . Hence we have (cf. [G90]):

$$(25) \quad \text{DT}(1, l) = (-1)^{\chi(\mathcal{O}_S)+1} q^{\frac{\chi(S)}{24}} \eta(q)^{-\chi(S)}.$$

Here $\eta(q)$ is the Dedekind eta function (6). Also our definition of the generating series (23) implies that $\text{DT}_H(r, l)$ depends on l only on modulo r :

Lemma 2.9. *For any $l' \in \text{NS}(S)$, we have*

$$\text{DT}_H(r, l + rl') = \text{DT}_H(r, l).$$

Proof. Let us take $L' \in \text{Pic}(X)$ such that $c_1(L'|_S) = l'$. The lemma follows since $E \mapsto E \otimes L'$ preserves the μ_H -semistability and $\Delta(E \otimes L') = \Delta(E)$. q.e.d.

2.5. Generalized DT invariants for Gieseker semistable sheaves.

One may also be interested in generalized DT invariants counting Gieseker semistable sheaves. Indeed in the situation we are interested in (i.e., $S = \mathbb{P}^2$), they coincide with the invariants $\text{DT}_H(\gamma)$ in Definition 2.6. Let

$$\mathfrak{M}_{G,H}^{ss}(\gamma) \subset \mathfrak{M}$$

be the substack of Gieseker semistable sheaves $E \in \text{Coh}_c(X)$ satisfying $\text{cl}(E) = \gamma$. Similarly to (18), (19), we have the following elements of $H_\gamma(X)$:

$$(26) \quad \begin{aligned} \delta_{G,H}(\gamma) &= [\mathfrak{M}_{G,H}^{ss}(\gamma) \subset \mathfrak{M}(\gamma)], \\ \epsilon_{G,H}(\gamma) &= \sum_{\substack{\gamma_1 + \dots + \gamma_m = \gamma \\ \bar{\chi}_H(\gamma_i) = \bar{\chi}_H(\gamma)}} \frac{(-1)^{m-1}}{m} \delta_{G,H}(\gamma_1) * \dots * \delta_{G,H}(\gamma_m). \end{aligned}$$

Here $\bar{\chi}_H(\gamma)$ for $\gamma \in \Lambda$ is determined by the condition $\bar{\chi}_H(\text{cl}(E)) = \bar{\chi}_H(E)$ for any $E \in \text{Coh}_c(X)$. Similarly to Definition 2.6, we can define the invariant

$$\Pi(\epsilon_{G,H}(\gamma)) = -\text{DT}_{G,H}(\gamma) \cdot c_\gamma.$$

Lemma 2.10. *Suppose that $-K_S$ is ample and $H = -aK_S$ for $a \in \mathbb{R}_{>0}$. Then we have the equality*

$$(27) \quad \text{DT}_H(\gamma) = \text{DT}_{G,H}(\gamma).$$

Proof. By (8), the argument of [Joy08, Theorem 5.11] shows that

$$(28) \quad \delta_H(\gamma) = \sum_{\substack{\gamma_1 + \dots + \gamma_m = \gamma \\ \mu_H(\gamma_i) = \mu_H(\gamma) \\ \bar{\chi}_H(\gamma_1) \succ \dots \succ \bar{\chi}_H(\gamma_m)}} \delta_{G,H}(\gamma_1) * \dots * \delta_{G,H}(\gamma_m).$$

By substituting (28) into the RHS of (19), and using the inversion formula of (26) as in [Joy08, Equation (23)], we can describe $\epsilon_H(\gamma)$ in terms of $\epsilon_{G,H}(\gamma_i)$ with $\mu_H(\gamma_i) = \mu_H(\gamma)$. Using [Joy08, Theorem 5.4], the same argument of [Joy08, Theorem 5.2] shows that $\epsilon_H(\gamma)$ is written as

$$(29) \quad \epsilon_H(\gamma) = \epsilon_{G,H}(\gamma) + \left(\begin{array}{c} \text{multiple commutators of} \\ \epsilon_{G,H}(\gamma_i) \text{ with } \mu_H(\gamma_i) = \mu_H(\gamma) \end{array} \right).$$

By our assumption $H = -aK_S$, (14), (15), and Lemma 2.4, we have

$$\Pi[\epsilon_{G,H}(\gamma_1), \epsilon_{G,H}(\gamma_2)] = 0, \quad \text{if } \mu_H(\gamma_i) = \mu_H(\gamma).$$

Applying Π to (29), we obtain the desired equality (27). q.e.d.

2.6. Wall-crossing formula. The behavior of $DT_H(\gamma)$, $Eu_H(\gamma)$ under the change of H is described by the wall-crossing formula given in [Joy08], [JS12]. Here we recall its explicit formula for $Eu_H(\gamma)$. Let H_1, H_2 be \mathbb{R} -divisors on S . We recall some combinatorial numbers:

Definition 2.11. ([Joy08, Definition 4.2]) For non-zero $\gamma_1, \dots, \gamma_m \in \Lambda$, we define

$$S(\{\gamma_1, \dots, \gamma_m\}, H_1, H_2) \in \{0, \pm 1\}$$

as follows: if for each $i = 1, \dots, m - 1$, we have either (30) or (31)

$$(30) \quad \mu_{H_1}(\gamma_i) \leq \mu_{H_1}(\gamma_{i+1}) \text{ and } \mu_{H_2}(\gamma_1 + \dots + \gamma_i) > \mu_{H_2}(\gamma_{i+1} + \dots + \gamma_m),$$

$$(31) \quad \mu_{H_1}(\gamma_i) > \mu_{H_1}(\gamma_{i+1}) \text{ and } \mu_{H_2}(\gamma_1 + \dots + \gamma_i) \leq \mu_{H_2}(\gamma_{i+1} + \dots + \gamma_m),$$

then define

$$S(\{\gamma_1, \dots, \gamma_m\}, H_1, H_2) = (-1)^k,$$

where k is the number of $i = 1, \dots, m - 1$ satisfying (30). Otherwise we define $S(\{\gamma_1, \dots, \gamma_m\}, H_1, H_2) = 0$.

Another combinatorial number is defined as follows:

Definition 2.12. ([Joy08, Definition 4.4]) For non-zero $\gamma_1, \dots, \gamma_m \in \Lambda$, we define

$$(32) \quad U(\{\gamma_1, \dots, \gamma_m\}, H_1, H_2) = \sum_{1 \leq m'' \leq m' \leq m} \sum_{\substack{\psi: \{1, \dots, m\} \rightarrow \{1, \dots, m'\} \\ \psi': \{1, \dots, m'\} \rightarrow \{1, \dots, m''\}}} \prod_{a=1}^{m''} S(\{\Upsilon_i\}_{i \in \psi'^{-1}(a)}, H_1, H_2) \frac{(-1)^{m''-1}}{m''} \prod_{b=1}^{m'} \frac{1}{|\psi^{-1}(b)|!}.$$

Here ψ, ψ', Υ_i are as follows:

- ψ and ψ' are non-decreasing surjective maps.
- For $1 \leq i, j \leq m$ with $\psi(i) = \psi(j)$, we have $\mu_{H_1}(\gamma_i) = \mu_{H_1}(\gamma_j)$.

- For $1 \leq i, j \leq m''$, we have

$$(33) \quad \mu_{H_2} \left(\sum_{k \in \psi^{-1}\psi'^{-1}(i)} \gamma_k \right) = \mu_{H_2} \left(\sum_{k \in \psi^{-1}\psi'^{-1}(j)} \gamma_k \right).$$

- The elements $\Upsilon_i \in \Lambda$ for $1 \leq i \leq m'$ are defined to be

$$(34) \quad \Upsilon_i = \sum_{j \in \psi^{-1}(i)} \gamma_j.$$

For $m \in \mathbb{Z}_{\geq 1}$, let $G(m)$ be the set of connected, simply connected graphs with vertex $\{1, \dots, m\}$, such that $i \rightarrow j$ in G implies $i < j$. The wall-crossing formula for $\text{Eu}_H(\gamma)$ is described in the following way:

Theorem 2.13. ([Joy08, Theorem 6.28, Equation (130)]) *Suppose that H_1, H_2 are ample \mathbb{R} -divisors. We have the formula:*

$$(35) \quad \begin{aligned} \text{Eu}_{H_2}(\gamma) = & \sum_{\substack{m \geq 1, \gamma_1, \dots, \gamma_m \in \Lambda \\ \gamma_1 + \dots + \gamma_m = \gamma}} \sum_{G \in G(m)} \frac{1}{2^{m-1}} U(\{\gamma_1, \dots, \gamma_m\}, H_1, H_2) \\ & \prod_{i \rightarrow j \text{ in } G} \chi(\gamma_i, \gamma_j) \prod_{i=1}^m \text{Eu}_{H_1}(\gamma_i). \end{aligned}$$

Remark 2.14. If we know that the stack \mathfrak{M} satisfies the property as in Remark 2.5, we can apply (16) to show the wall-crossing formula for $\text{DT}_H(\gamma)$ similar to (35) as in [JS12, Theorem 6.28]. Alternatively, if $K_S \cdot H_i < 0$, we can substitute the equality (22) to (35) and obtain the wall-crossing formula for $\text{DT}_H(\gamma)$.

2.7. Theta type series for indefinite lattices. We introduce the theta type series from data

$$(36) \quad \xi = (\Gamma, B, \bar{\nu}, c_1, c_2, \dots, c_b, c'_1, c'_2, \dots, c'_b, \alpha_1, \dots, \alpha_k)$$

satisfying the following conditions:

- (i) Γ is a finitely generated free abelian group and

$$B: \Gamma \times \Gamma \rightarrow \mathbb{Z},$$

a non-degenerate symmetric bilinear pairing with index (a, b) for $a \geq b$.

- (ii) The elements $c_1, \dots, c_b \in \Gamma_{\mathbb{Q}}$ span a b -dimensional negative definite subspace in $\Gamma_{\mathbb{Q}}$.
- (iii) The elements $c'_1, \dots, c'_b \in \Gamma_{\mathbb{Q}}$ satisfy

$$\begin{aligned} B(c_i, c'_j) &= 0 \quad \text{for all } 1 \leq i, j \leq b, \quad i \neq j, \\ B(c'_i, c'_j) &= 0 \quad \text{for all } 1 \leq i, j \leq b, \\ B(c_i, c'_i) &< 0 \quad \text{for all } 1 \leq i \leq b. \end{aligned}$$

- (iv) The element $\bar{\nu} \in \Gamma_{\mathbb{Q}}$ satisfies that

$$B(c'_i, \nu) \neq 0 \text{ for all } 1 \leq i \leq b \text{ and } \nu \in \bar{\nu} + \Gamma.$$

- (v) $k \in \mathbb{Z}_{\geq 0}$ and $\alpha_1, \dots, \alpha_k$ are elements of $\Gamma_{\mathbb{Q}}$.

As in the introduction, we set $Q(\nu) = B(\nu, \nu)/2$ and consider the series

$$(37) \quad \Theta_{\xi}(q) := \sum_{\nu \in \bar{\nu} + \Gamma} \prod_{i=1}^b (\text{sgn}(B(c_i, \nu)) - \text{sgn}(B(c'_i, \nu))) \prod_{j=1}^k B(\alpha_j, \nu) \cdot q^{Q(\nu)}.$$

When $b = k = 0$, the series (37) becomes

$$(38) \quad \Theta_{\xi}(q) = \sum_{\nu \in \bar{\nu} + \Gamma} q^{Q(\nu)}.$$

The series (38) is a classical theta series with respect to the positive definite quadratic form Q on Γ , which is a modular form with weight $a/2$. If $b = 0$ and $k > 0$, then the series (37) is obtained as derivations of Jacobi theta series with respect to elliptic variables.

If $b = 1$ and $k = 0$, then the series (37) is not always a modular form. Instead Zwegers [Zwe] showed that the series

$$\sum_{\nu \in \bar{\nu} + \Gamma} \left(E \left(\frac{B(c_1, \nu)}{\sqrt{-Q(c_1)}} y^{\frac{1}{2}} \right) - \text{sgn}(B(c'_1, \nu)) \right) e^{2\pi i Q(\nu)\tau},$$

for $y = \text{Im } \tau$ gives a real analytic modular form of weight $(a + 1)/2$. Here $E(x)$ is defined by

$$E(x) = 2 \int_0^x e^{-\pi u^2} du.$$

A series which admits a modular completion as above is called a mock modular form [Zag09]. The case $b = 1$ and $k > 0$ is obtained by the derivations of mock Jacobi forms in [Zwe] with respect to the elliptic variables.

Suppose that $b \geq 2$ and $k = 0$. If we further assume that

$$(39) \quad B(c_i, c_j) = 0, \quad i \neq j,$$

then the argument of [Zwe] can be easily generalized to show that

$$(40) \quad \sum_{\nu \in \bar{\nu} + \Gamma} \prod_{i=1}^b \left(E \left(\frac{B(c_i, \nu)}{\sqrt{-Q(c_i)}} y^{\frac{1}{2}} \right) - \text{sgn}(B(c'_i, \nu)) \right) e^{2\pi i Q(\nu)\tau}$$

is a real analytic modular form. Indeed the series (37) in this case is a mixed mock modular form in the sense of [BM13].

Remark 2.15. Unfortunately the series (37) without the condition (39) is involved in Theorem 1.1. In that case, the proof of [Zwe] is not directly applied to show the modularity of (40). The study of the

modularity of (40) without (39), or other kind of modular completion of the series (37), would be required to understand the S-duality for an arbitrary rank.

2.8. Some properties of theta type series. Let us consider the series (37) determined by data (36) satisfying (i) to (v). We first show the convergence of (37):

Lemma 2.16. *For $\tau \in \mathcal{H}$, the series*

$$(41) \quad \sum_{\nu \in \bar{\nu} + \Gamma} \prod_{i=1}^b (\operatorname{sgn}(B(c_i, \nu)) - \operatorname{sgn}(B(c'_i, \nu))) \prod_{j=1}^k B(\alpha_j, \nu) \cdot e^{2\pi i Q(\nu)\tau}$$

converges absolutely.

Proof. The convergence for $(b, k) = (1, 0)$ follows from [Zwe, Proposition 2.4]. The conditions (i) to (v) allow us to apply a similar argument. Since the series (37) is unchanged by replacing c_i, c'_i by multiplications of positive integers, we may assume that $c_i, c'_i \in \Gamma$. By the condition (iii) of data (36), the element $\nu \in \bar{\nu} + \Gamma$ is uniquely written as

$$\nu = \mu + \sum_{i=1}^b m_i c'_i,$$

for some $\mu \in \bar{\nu} + \Gamma$, $m_i \in \mathbb{Z}$ satisfying

$$\frac{B(c_i, \mu)}{B(c_i, c'_i)} \in [0, 1) \quad \text{for all } 1 \leq i \leq b.$$

Therefore, the series (41) is written as

$$(42) \quad \sum_{\substack{\mu \in \bar{\nu} + \Gamma \\ \frac{B(c_i, \mu)}{B(c_i, c'_i)} \in [0, 1) \\ \text{for all } 1 \leq i \leq b}} e^{2\pi i Q(\mu)\tau} \sum_{m_1, \dots, m_b \in \mathbb{Z}} \prod_{i=1}^b \left(\operatorname{sgn} \left(\frac{B(c_i, \mu)}{B(c_i, c'_i)} + m_i \right) + \operatorname{sgn}(B(c'_i, \mu)) \right) \cdot (-1)^b \prod_{i=1}^k B \left(\alpha_i, \mu + \sum_{i=1}^b m_i c'_i \right) e^{\sum_{j=1}^b 2\pi i B(c'_j, \mu) m_j \tau}.$$

Here we have used $B(c_i, c'_i) < 0$ from the condition (ii). Since there is a finite number of possibilities for the value $B(c_i, \mu)/B(c_i, c'_i) \in [0, 1)$ in (42), there is a finite number of $\mu_1, \dots, \mu_p \in \Gamma$ such that any $\mu \in \bar{\nu} + \Gamma$ in (42) is written as

$$\mu = \bar{\nu} + \mu_e + \mu',$$

for some $1 \leq e \leq p$ and $\mu' \in \Gamma'$, where $\Gamma' \subset \Gamma$ is the orthogonal

complement of the $(0, b)$ -space spanned by c_1, \dots, c_b . Therefore, the series (42) is a finite linear combination of the series of the form

$$\sum_{\mu \in \bar{\nu} + \mu_e + \Gamma'} \prod_{s \in S} B(\alpha_s, \mu) e^{2\pi i Q(\mu)\tau} \cdot \prod_{j=1}^b \sum_{m_j \in \mathbb{Z}} m_j^{k_j} \left(\operatorname{sgn} \left(\frac{B(c_j, \mu)}{B(c_j, c'_j)} + m_j \right) + \operatorname{sgn}(B(c'_j, \mu)) \right) e^{2\pi i B(c'_j, \mu) m_j \tau},$$

for some fixed $e \in \{1, \dots, p\}$, a finite set $S \subset \{1, \dots, k\}$ and some $k_1, \dots, k_b \in \mathbb{Z}_{\geq 0}$. Let us also fix the elements

$$\mu \in \bar{\nu} + \mu_e + \Gamma', \quad j \in \{1, \dots, b\}.$$

Then we have

$$\begin{aligned} & \sum_{m_j \in \mathbb{Z}} m_j^{k_j} \left(\operatorname{sgn} \left(\frac{B(c_j, \mu)}{B(c_j, c'_j)} + m_j \right) + \operatorname{sgn}(B(c'_j, \mu)) \right) e^{2\pi i B(c'_j, \mu) m_j \tau} \\ &= 2 \operatorname{sgn}(B(c'_j, \mu)) \sum_{m_j \in \mathbb{Z}_{\geq 0}} m_j^{k_j} e^{2\pi i |B(c'_j, \mu)| m_j \tau} + C \\ &= 2 \operatorname{sgn}(B(c'_j, \mu)) (2\pi i |B(c'_j, \mu)|)^{-k_j} \left(\frac{d}{d\tau} \right)^{k_j} \frac{1}{1 - e^{2\pi i |B(c'_j, \mu)| \tau}} + C. \end{aligned}$$

Here $C \in \{0, \pm 1\}$, depending on the signs of $B(c'_j, \mu)$ and $B(c_j, \mu)/B(c_j, c'_j)$. Since we have

$$\inf\{|B(c'_j, \mu)| : \mu \in \bar{\nu} + \Gamma\} > 0,$$

due to the condition (iv), we have

$$\sup \left\{ \left| B(c'_j, \mu)^{-k_j} \left(\frac{d}{d\tau} \right)^{k_j} \frac{1}{1 - e^{2\pi i |B(c'_j, \mu)| \tau}} \right| : \mu \in \bar{\nu} + \Gamma \right\} < \infty.$$

We are reduced to showing the absolute convergence of the series

$$(43) \quad \sum_{\mu \in \bar{\nu} + \mu_e + \Gamma'} \prod_{s \in S} B(\alpha_s, \mu) e^{2\pi i Q(\mu)\tau}.$$

Since Γ' is positive definite by the condition (ii), the series (43) converges absolutely by the absolute convergence of the classical theta series. q.e.d.

By the proof of Lemma 2.16, the series $\Theta_\xi(q)$ in (37) makes sense, and determines the element

$$\Theta_\xi(q) \in \mathbb{Q}((q^{\frac{1}{N}})),$$

for some $N \in \mathbb{Z}_{\geq 1}$.

Definition 2.17. We define

$$(44) \quad \mathcal{M} \subset \varinjlim \mathbb{Q}(q^{\frac{1}{N}}),$$

to be the \mathbb{Q} -subalgebra generated by $\Theta_\xi(q^{\frac{1}{N}})$ for all the data (36) and $N \in \mathbb{Z}_{\geq 1}$.

We will use the following lemma:

Lemma 2.18. *Let ξ be data (36), $V \subset \Gamma_{\mathbb{Q}}$ a linear subspace which contains $c_1, \dots, c_b, c'_1, \dots, c'_b$, and $T \subset \{1, \dots, b\}$ a subset. Then the series*

$$(45) \quad \sum_{\substack{\nu \in (\bar{\nu} + \Gamma) \cap V \\ B(\nu, c_i) = 0 \text{ for all } i \in T}} \prod_{i \notin T} (\text{sgn}(B(c_i, \nu)) - \text{sgn}(B(c'_i, \nu))) \prod_{j=1}^k B(\alpha_j, \nu) q^{Q(\nu)}$$

is an element of \mathcal{M} .

Proof. If the series (45) is not zero, there is $\nu_0 \in (\bar{\nu} + \Gamma) \cap V$ with $B(\nu_0, c_i) = 0$ for all $i \in T$. Any element ν in the series (45) satisfies that

$$\nu - \nu_0 \in \Gamma' := \{v \in \Gamma \cap V : B(v, c_i) = 0 \text{ for all } i \in T\}.$$

Note that c'_i for $i \notin T$ and ν_0 are elements of $\Gamma'_{\mathbb{Q}}$. We consider the decomposition

$$\Gamma_{\mathbb{Q}} = \Gamma'_{\mathbb{Q}} \oplus \Gamma'^{\perp}_{\mathbb{Q}},$$

where $\Gamma'^{\perp}_{\mathbb{Q}}$ is the orthogonal complement of $\Gamma'_{\mathbb{Q}}$ in $\Gamma_{\mathbb{Q}}$ with respect to $B(-, -)$. For $\nu \in \Gamma_{\mathbb{Q}}$, we denote by ν^+ the $\Gamma'_{\mathbb{Q}}$ -component of ν with respect to the above decomposition. The series (45) is written as

$$(46) \quad \sum_{\nu \in \nu_0 + \Gamma'} \prod_{i \notin T} (\text{sgn}(B(c_i^+, \nu)) - \text{sgn}(B(c'_i, \nu))) \prod_{j=1}^k B(\alpha_j^+, \nu) \cdot q^{Q(\nu)}.$$

Since c_1, \dots, c_b span a $(0, b)$ -space, the elements c_i with $i \in T$ span a $(0, |T|)$ -space and c_i^+ with $i \notin T$ span a $(0, b - |T|)$ -space. Hence the data

$$(\Gamma', B|_{\Gamma'}, \nu_0, c_i^+, c'_i, i \notin T, \alpha_1^+, \dots, \alpha_k^+)$$

satisfies the conditions (i) to (v) in the previous subsection. Therefore, the series (46) is an element of \mathcal{M} . q.e.d.

Lemma 2.19. *Any element in \mathcal{M} is written as*

$$\sum_{i=1}^n a_i \Theta_{\xi_i}(q^{\frac{1}{N}}),$$

for a finite number of data ξ_1, \dots, ξ_n as in (36), $a_1, \dots, a_n \in \mathbb{Q}$ and $N \in \mathbb{Z}_{\geq 1}$.

Proof. If ξ_1, ξ_2 are data (36), then we have

$$\Theta_{\xi_1}(q) \cdot \Theta_{\xi_2}(q) = \Theta_{\xi_1 \oplus \xi_2}(q),$$

where $\xi_1 \oplus \xi_2$ is the direct product of data ξ_1, ξ_2 in an obvious sense. Moreover, $\Theta_{\xi}(q) = \Theta_{\xi'}(q^{1/N})$ where ξ' is data (36) with B replaced by NB . Therefore, the lemma holds. q.e.d.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Below, we denote by H the hyperplane class of \mathbb{P}^2 . We identify $\text{NS}(\mathbb{P}^2)$ with \mathbb{Z} by $lH \mapsto l$. For $r \in \mathbb{Z}_{\geq 1}$ and $l \in \mathbb{Z}$, we consider the generating series

$$\text{DT}(r, l) := \text{DT}_H(r, l)$$

defined by (23) for $S = \mathbb{P}^2$. By the Bogomolov inequality, the above series coincides with the series (3) in the introduction.

3.1. Blow-up formula. Let

$$f: \widehat{\mathbb{P}}^2 \rightarrow \mathbb{P}^2$$

be a blow-up at a point in \mathbb{P}^2 . Note that we have

$$\text{NS}(\widehat{\mathbb{P}}^2) = \mathbb{Z}[f^*H] \oplus \mathbb{Z}[C].$$

Below we write an element $lf^*H + aC$ of $\text{NS}(\widehat{\mathbb{P}}^2)$ as (l, a) . We have the following blow-up formula of the series (23):

Proposition 3.1. *For any $r \in \mathbb{Z}_{\geq 1}$, $l \in \mathbb{Z}$ and $a \in \mathbb{Z}$, we have the following formula:*

$$(47) \quad \text{DT}_{H_0}(r, (l, a)) = q^{\frac{r}{24}} \eta(q)^{-r} \cdot \vartheta_{r,a}(q) \cdot \text{DT}(r, l).$$

Here $H_0 = f^*H$ and $\vartheta_{r,a}(q)$ is defined by

$$\vartheta_{r,a}(q) := \sum_{(k_1, \dots, k_{r-1}) \in (a/r, \dots, a/r) + \mathbb{Z}^{r-1}} q^{\sum_{1 \leq i \leq j \leq r-1} k_i k_j}.$$

Proof. If r and l are coprime, the result essentially follows from [Yos96], [LQ99], [G99]. In a general case, we use the blow-up formula in [Tod15] for the invariants $\text{Eu}_H(\gamma)$ obtained as an application of the flop transformation formula of generalized DT type invariants. We note that, although H_0 is not ample, the LHS of (47) is well-defined due to the boundedness of μ_{H_0} -semistable sheaves on $\widehat{\mathbb{P}}^2$ (cf. [Tod15, Proposition 2.17]). By [Tod15, Theorem 4.3], we have

$$\begin{aligned}
 & \sum_{s,a} \text{Eu}_{H_0}(r, (l, a), -s) q^{\frac{r}{12} - \frac{a}{2} + s} t^{\frac{r}{2} - a} \\
 (48) \quad & = \sum_s \text{Eu}_H(r, l, -s) q^s \cdot \eta(q)^{-r} \vartheta_{1,0}(q, t)^r.
 \end{aligned}$$

Here $\eta(q)$ is given by (6) and $\vartheta_{1,0}(q, t)$ is given by

$$\vartheta_{1,0}(q, t) = \sum_{k \in \mathbb{Z}} q^{\frac{1}{2}(k+\frac{1}{2})^2} t^{k+\frac{1}{2}}.$$

The formulas (24) and (48) immediately imply

$$\begin{aligned}
 & \text{DT}_{H_0}(r, (l, a)) \\
 & = q^{\frac{r}{24}} \eta(q)^{-r} \cdot \left(\sum_{\substack{k_1, \dots, k_r \in \mathbb{Z} \\ k_1 + \dots + k_r = -a}} q^{\frac{1}{2}(k_1^2 + \dots + k_r^2) - \frac{a^2}{2r}} \right) \cdot \text{DT}(r, l).
 \end{aligned}$$

By the substitution $k_r = -a - k_1 - \dots - k_{r-1}$, it is straightforward to check that

$$\sum_{\substack{k_1, \dots, k_r \in \mathbb{Z} \\ k_1 + \dots + k_r = -a}} q^{\frac{1}{2}(k_1^2 + \dots + k_r^2) - \frac{a^2}{2r}} = \vartheta_{r,a}(q). \quad \text{q.e.d.}$$

3.2. Combinatorial generating series. In this subsection, we introduce some generating series defined by the combinatorial numbers in Subsection 2.6. For $t \in \mathbb{R}$, we set

$$H_t := f^*H - tC \in \text{NS}(\widehat{\mathbb{P}}^2)_{\mathbb{R}}.$$

Note that H_t is ample for $t \in (0, 1)$, $H_0 = f^*H$ is nef and big, and

$$F := H_1 = f^*H - C$$

is a fiber class of the \mathbb{P}^1 -fibration $\widehat{\mathbb{P}}^2 \rightarrow \mathbb{P}^1$. Also we denote by $\Lambda \subset H^*(\widehat{\mathbb{P}}^2, \mathbb{Q})$ the image of cl in (11) for $S = \widehat{\mathbb{P}}^2$. Let us take $m \in \mathbb{Z}_{\geq 1}$ and

$$r_1, \dots, r_m \in \mathbb{Z}_{\geq 1}, \beta_1, \dots, \beta_m \in \text{NS}(\widehat{\mathbb{P}}^2).$$

We set

$$S(\{(r_i, \beta_i)\}_{i=1}^m, H, F_+) := \lim_{t \rightarrow 1-0} S(\{(r_i, \beta_i, 0)\}_{i=1}^m, H_0, H_t),$$

$$U(\{(r_i, \beta_i)\}_{i=1}^m, H, F_+) := \lim_{t \rightarrow 1-0} U(\{(r_i, \beta_i, 0)\}_{i=1}^m, H_0, H_t).$$

Here we regard $(r_i, \beta_i, 0)$ as an element⁸ of Λ , and S, U are combinatorial numbers in Subsection 2.6. Obviously the limits of the RHS are well-defined.

⁸The choice 0 in the H^4 -component can be arbitrary, since the slope is independent of the second Chern character.

We introduce some more notation. For $r \geq 1$, we set

$$\begin{aligned} \text{NS}_{<r}(\widehat{\mathbb{P}^2}) \\ := \{xf^*H + yC : x, y \in \mathbb{Z}, 0 \leq x \leq r - 1, 0 \leq y \leq r - 1\}. \end{aligned}$$

Note that $\text{NS}_{<r}(\widehat{\mathbb{P}^2})$ is a finite subset of $\text{NS}(\widehat{\mathbb{P}^2})$. Also we denote by $G'(m)$ the set of oriented graphs G with vertex a subset in $\{1, \dots, m\}$, which may not be connected nor simply connected, and $i \rightarrow j$ implies $i \leq j$. Note that $G(m) \subset G'(m)$, where $G(m)$ is the set of graphs in Subsection 2.6.

Definition 3.2. Given data

$$(49) \quad \begin{aligned} l \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 1}, G \in G'(m), \\ r_1, \dots, r_m \in \mathbb{Z}_{\geq 1}, \bar{\beta}_i \in \text{NS}_{<r_i}(\widehat{\mathbb{P}^2}) \quad (1 \leq i \leq m), \end{aligned}$$

we define the following generating series

$$(50) \quad \begin{aligned} S_{(r_1, \bar{\beta}_1), \dots, (r_m, \bar{\beta}_m)}^{l, G}(q) := & \sum_{\substack{\beta_i \in \text{NS}(\widehat{\mathbb{P}^2}), \beta_i \equiv \bar{\beta}_i \pmod{r_i} \\ \beta_1 + \dots + \beta_m = (l, 1-l)}} S(\{(r_i, \beta_i)\}_{i=1}^m, H, F_+) \\ & \cdot \prod_{i \rightarrow j \text{ in } G} K_{\widehat{\mathbb{P}^2}}(r_j \beta_i - r_i \beta_j) q^{-\sum_{1 \leq i < j \leq m} \frac{(r_j \beta_i - r_i \beta_j)^2}{2r_i r_j}}, \end{aligned}$$

$$(51) \quad \begin{aligned} U_{(r_1, \bar{\beta}_1), \dots, (r_m, \bar{\beta}_m)}^{l, G}(q) := & \sum_{\substack{\beta_i \in \text{NS}(\widehat{\mathbb{P}^2}), \beta_i \equiv \bar{\beta}_i \pmod{r_i} \\ \beta_1 + \dots + \beta_m = (l, 1-l)}} U(\{(r_i, \beta_i)\}_{i=1}^m, H, F_+) \\ & \cdot \prod_{i \rightarrow j \text{ in } G} K_{\widehat{\mathbb{P}^2}}(r_j \beta_i - r_i \beta_j) q^{-\sum_{1 \leq i < j \leq m} \frac{(r_j \beta_i - r_i \beta_j)^2}{2r_i r_j}}. \end{aligned}$$

The generating series (50), (51) are well-defined. Indeed, we have the following proposition:

Proposition 3.3. *The series (50), (51) are elements of \mathcal{M} .*

Here \mathcal{M} is given in Definition 2.17. The proof of Proposition 3.3 will be given in Subsection 3.4 and Subsection 3.5.

3.3. Rank reduction formula. In this subsection, we apply Theorem 2.13 to describe $\text{DT}(r, l)$ for $r \geq 2$ in terms of the series (51) and the series $\text{DT}(r', l')$ with $r' < r$. We first collect some well-known lemmas:

Lemma 3.4. *For a fixed $\gamma = (r, \beta, s) \in \Lambda$, there exist*

$$0 = t_0 < t_1 < \dots < t_m = 1,$$

such that the stack $\mathfrak{M}_{H_t}^{ss}(\gamma)$ is constant on $t \in (t_i, t_{i+1})$.

Proof. It is enough to show the finiteness of the set of $t \in [0, 1)$ satisfying the following: there exist μ_{H_t} -semistable sheaves E_i with $\text{cl}(E_i) = \gamma_i = (r_i, \beta_i, s_i) \in \Lambda$ for $i = 1, 2$ such that

$$\mu_{H_t}(\gamma_i) = \mu_{H_t}(\gamma), \quad \gamma_1 + \gamma_2 = \gamma.$$

By the left equality and the Hodge index theorem, we have

$$(52) \quad (r\beta_1 - r_1\beta)^2 \leq 0.$$

On the other hand, we have $s_i \leq \beta_i^2/2r_i$ by Bogomolov inequality, hence $\beta_1^2/r_1 + \beta_2^2/r_2 \geq 2s$. By substituting $\beta_2 = \beta - \beta_1$, we obtain

$$(53) \quad -r_1r_2\Delta(\gamma) \leq (r\beta_1 - r_1\beta)^2.$$

Note that there is only a finite number of possible r_i . By (52), (53), the possible β_1 are also finite. Hence the possible $t \in [0, 1)$ is finite. q.e.d.

Lemma 3.5. *For $\gamma = (r, \beta, s) \in \Lambda$ with $r \geq 2$ and $F \cdot \beta = 1$, we have $\mathfrak{M}_{H_t}^{ss}(\gamma) = \emptyset$ for $t \rightarrow 1 - 0$.*

Proof. By Lemma 3.4, the moduli stack $\mathfrak{M}_{H_t}^{ss}(\gamma)$ for $t \rightarrow 1 - 0$ is well-defined. Also if there is $[E] \in \mathfrak{M}_{H_t}^{ss}(\gamma)$ for $t \rightarrow 1 - 0$, then it must be F -slope semistable. By [Moz, Lemma 4.3], any F -slope semistable sheaf on $\widehat{\mathbb{P}}^2$ is restricted to a semistable sheaf on a generic fiber of $\widehat{\mathbb{P}}^2 \rightarrow \mathbb{P}^1$. Since there is no semistable sheaf on \mathbb{P}^1 with rank bigger than or equal to two and degree one, we have $\mathfrak{M}_{H_t}^{ss}(\gamma) = \emptyset$ for $t \rightarrow 1 - 0$. q.e.d.

By combining Theorem 2.13, Proposition 3.1 and the above two lemmas, we show the following:

Proposition 3.6. *For any $r \in \mathbb{Z}_{\geq 2}$ and $l \in \mathbb{Z}$, we have the following formula:*

$$(54) \quad \begin{aligned} \text{DT}(r, l) = & \sum_{\substack{m \geq 2, r_1, \dots, r_m \in \mathbb{Z}_{\geq 1} \\ r_1 + \dots + r_m = r}} \sum_{\substack{\bar{\beta}_i = (l_i, a_i) \in \text{NS}_{< r_i}(\widehat{\mathbb{P}}^2) \\ 1 \leq i \leq m}} \sum_{G \in G(m)} \frac{(-1)^m}{2^{m-1}} \\ & \cdot U_{(r_1, \bar{\beta}_1), \dots, (r_m, \bar{\beta}_m)}^{l, G}(q) \cdot \vartheta_{r, 1-l}(q)^{-1} \prod_{i=1}^m \vartheta_{r_i, a_i}(q) \cdot \prod_{i=1}^m \text{DT}(r_i, l_i). \end{aligned}$$

Proof. We apply Theorem 2.13 for $S = \widehat{\mathbb{P}}^2$, $(H_1, H_2) = (H_0, H_t)$ with $t \in (0, 1)$, and $\gamma = (r, \beta, s)$ with $\beta = (l, 1 - l)$. Using (14) and (35), we obtain the identity⁹ :

$$(55) \quad \begin{aligned} \text{Eu}_{H_t}(r, \beta, s) = & \sum_{\substack{m \geq 1, \gamma_i = (r_i, \beta_i, s_i) \in \Lambda \\ \gamma_1 + \dots + \gamma_m = \gamma}} \sum_{G \in G(m)} \frac{1}{2^{m-1}} U(\{\gamma_i\}_{i=1}^m, H_0, H_t) \\ & \cdot \prod_{i \rightarrow j \text{ in } G} K_{\widehat{\mathbb{P}}^2}(r_j \beta_i - r_i \beta_j) \prod_{i=1}^m \text{Eu}_{H_0}(r_i, \beta_i, s_i). \end{aligned}$$

⁹It is straightforward to generalize the result of Theorem 2.13 for non-ample H_0 .

Since $F \cdot \beta = 1$, we have

$$\lim_{t \rightarrow 1-0} \text{Eu}_{H_t}(r, \beta, s) = 0,$$

by Lemma 3.5. Therefore, by taking $t \rightarrow 1 - 0$ of both sides of (55), and moving the $m = 1$ term to the LHS, we obtain the identity:

$$\begin{aligned} \sum_s \text{Eu}_{H_0}(r, \beta, s) q^{\frac{\beta^2}{2r} - s} &= - \sum_{\substack{m \geq 2, r_1, \dots, r_m \in \mathbb{Z}_{\geq 1} \\ r_1 + \dots + r_m = r}} \sum_{G \in G(m)} \frac{1}{2^{m-1}} \\ &\cdot \sum_{\substack{\beta_i \in \text{NS}(\widehat{\mathbb{P}}^2), 1 \leq i \leq m \\ \beta_1 + \dots + \beta_m = \beta}} U(\{(r_i, \beta_i)\}_{i=1}^m, H, F_+) \\ &\cdot \prod_{i \rightarrow j \text{ in } G} K_{\widehat{\mathbb{P}}^2}(r_j \beta_i - r_i \beta_j) q^{\frac{\beta^2}{2r} - \sum_{i=1}^m \frac{\beta_i^2}{2r_i}} \prod_{i=1}^m \left(\sum_{s_i} \text{Eu}_{H_0}(r_i, \beta_i, s_i) q^{\frac{\beta_i^2}{2r_i} - s_i} \right). \end{aligned}$$

For $\beta_i \in \text{NS}(\widehat{\mathbb{P}}^2)$, let $\bar{\beta}_i \in \text{NS}_{<r_i}(\widehat{\mathbb{P}}^2)$ be the unique element such that

$$\beta_i \equiv \bar{\beta}_i \pmod{r_i}.$$

Then by (24) and Lemma 2.9, we have

$$\sum_{s_i} \text{Eu}_{H_0}(r_i, \beta_i, s_i) q^{\frac{\beta_i^2}{2r_i} - s_i} = (-1)^{r_i^2+1} \text{DT}_{H_0}(r_i, \bar{\beta}_i).$$

Also noting that

$$(56) \quad \frac{\beta^2}{2r} - \sum_{i=1}^m \frac{\beta_i^2}{2r_i} = - \sum_{1 \leq i < j \leq m} \frac{(r_j \beta_i - r_i \beta_j)^2}{2r r_i r_j},$$

we obtain the following identity:

$$\begin{aligned} \text{DT}_{H_0}(r, \beta) &= \sum_{\substack{m \geq 2, r_1, \dots, r_m \in \mathbb{Z}_{\geq 1} \\ r_1 + \dots + r_m = r}} \sum_{\substack{\bar{\beta}_i \in \text{NS}_{<r_i}(\widehat{\mathbb{P}}^2), \\ 1 \leq i \leq m.}} \sum_{G \in G(m)} \frac{(-1)^m}{2^{m-1}} \\ &\cdot U_{(r_1, \bar{\beta}_1), \dots, (r_m, \bar{\beta}_m)}^{l, G}(q) \prod_{i=1}^m \text{DT}_{H_0}(r_i, \bar{\beta}_i). \end{aligned}$$

Applying (47) to both sides of the above identity, we obtain the desired formula (54). q.e.d.

We have the following corollary which, together with Lemma 2.19, prove Theorem 1.1.

Corollary 3.7. *For any $r \in \mathbb{Z}_{\geq 1}$ and $l \in \mathbb{Z}$, there exist classical data ξ^l such that*

$$q^{-\frac{r}{8}} \eta(q)^{3r} \cdot \Theta_{\xi^l}(q) \cdot \text{DT}(r, l) \in \mathcal{M}.$$

Proof. The case of $r = 1$ follows from (25). The case of $r \geq 2$ follows from the induction of r by Proposition 3.3 and Proposition 3.6, noting that $\vartheta_{r,a}(q)$ is a classical theta series. q.e.d.

3.4. Proof of Proposition 3.3 for (50). In this subsection, we show that

$$(57) \quad S_{(r_1, \bar{\beta}_1), \dots, (r_m, \bar{\beta}_m)}^{l, G}(q) \in \mathcal{M}.$$

We first prepare the following lemma:

Lemma 3.8. *For $l \in \mathbb{Z}$, there are no $r_1, r_2 \in \mathbb{Z}_{\geq 1}$, $\beta_1, \beta_2 \in \text{NS}(\widehat{\mathbb{P}^2})$ such that $\beta_1 + \beta_2 = (l, 1 - l)$ and*

$$(58) \quad \frac{\beta_1}{r_1} \cdot F = \frac{\beta_2}{r_2} \cdot F.$$

Proof. Suppose that there exist such (r_i, β_i) . By substituting $\beta_2 = (l, 1 - l) - \beta_1$ into (58), and noting $(l, 1 - l) \cdot F = 1$, we obtain $\beta_1 \cdot F = r_1 / (r_1 + r_2)$. Since the LHS is an integer and the RHS is not an integer, this is a contradiction. q.e.d.

We describe the series (50) in terms of the theta type series in Subsection 2.7. In the sum (50), we set

$$(59) \quad \beta_i = \bar{\beta}_i + r_i u_i,$$

for $u_i \in \text{NS}(\widehat{\mathbb{P}^2})$ and

$$(60) \quad \begin{aligned} \nu_i &:= \frac{\beta_i}{r_i} - \frac{\beta_{i+1}}{r_{i+1}} \\ &= \bar{\nu}_i + u_i - u_{i+1}, \end{aligned}$$

for $1 \leq i \leq m - 1$. Here we have set

$$\bar{\nu}_i = \frac{\bar{\beta}_i}{r_i} - \frac{\bar{\beta}_{i+1}}{r_{i+1}}.$$

Then we have

$$(61) \quad \nu_i \in \bar{\nu}_i + \text{NS}(\widehat{\mathbb{P}^2}).$$

Conversely given ν_i for $1 \leq i \leq m - 1$ as in (61), we can write u_i satisfying (60) as follows:

$$u_i = u_1 - (\nu_1 - \bar{\nu}_1) - \dots - (\nu_{i-1} - \bar{\nu}_{i-1}).$$

By substituting into (59) and $\beta_1 + \dots + \beta_m = \beta$, where $\beta = (l, 1 - l)$, we have

$$\beta - \bar{\beta} + \sum_{i=1}^{m-1} \sum_{j=i+1}^m r_j (\nu_i - \bar{\nu}_i) = r u_1.$$

Here we have set $\bar{\beta} = \bar{\beta}_1 + \dots + \bar{\beta}_m$. Therefore, the necessary and

sufficient condition for ν_i in (61) to have the solution $(u_1, \dots, u_m) \in \text{NS}(\widehat{\mathbb{P}^2})^{\times m}$ is

$$\beta - \bar{\beta} + \sum_{i=1}^{m-1} \sum_{j=i+1}^m r_j(\nu_i - \bar{\nu}_i) \in r \text{NS}(\widehat{\mathbb{P}^2}).$$

On the other hand, for each $1 \leq i \leq m - 1$, we have

$$\begin{aligned} (62) \quad & \text{sgn} \left(F \cdot \left(\frac{\beta_1 + \dots + \beta_i}{r_1 + \dots + r_i} - \frac{\beta_{i+1} + \dots + \beta_m}{r_{i+1} + \dots + r_m} \right) \right) \\ &= \text{sgn} \left(F \cdot \sum_{k \leq i < j} (\beta_k r_j - \beta_j r_k) \right) \\ &= \text{sgn} \left(F \cdot \sum_{k \leq i < j} r_j r_k (\nu_k + \nu_{k+1} + \dots + \nu_{j-1}) \right). \end{aligned}$$

By Lemma 3.8, the value (62) is non-zero for (r_i, β_i) in the series (50). Therefore, the series (50) is written as

$$\begin{aligned} (63) \quad & \prod_{i \rightarrow j \text{ in } G} r_i r_j \cdot \frac{1}{2^{m-1}} \sum_{\substack{\nu_i \in \bar{\nu}_i + \text{NS}(\widehat{\mathbb{P}^2}), 1 \leq i \leq m-1 \\ \beta - \bar{\beta} + \sum_{i=1}^{m-1} \sum_{j=i+1}^m r_j(\nu_i - \bar{\nu}_i) \in r \text{NS}(\widehat{\mathbb{P}^2})}} \\ & \prod_{i=1}^{m-1} \left(\text{sgn}(H_0 \cdot \nu_i) - \delta_{0, H_0 \cdot \nu_i} \right. \\ & \quad \left. - \text{sgn} \left(F \cdot \sum_{k \leq i < j} r_j r_k (\nu_k + \nu_{k+1} + \dots + \nu_{j-1}) \right) \right) \\ & \cdot \prod_{i \rightarrow j \text{ in } G} K_{\widehat{\mathbb{P}^2}}(\nu_i + \nu_{i+1} + \dots + \nu_{j-1}) \cdot q^{-\sum_{1 \leq i < j \leq m} \frac{r_i r_j (\nu_i + \nu_{i+1} + \dots + \nu_{j-1})^2}{2r}}. \end{aligned}$$

We set

$$\Gamma = \left\{ (\nu_1, \dots, \nu_{m-1}) \in \text{NS}(\widehat{\mathbb{P}^2})^{\times m-1} : \sum_{i=1}^{m-1} \sum_{j=i+1}^m r_j \nu_i \in r \text{NS}(\widehat{\mathbb{P}^2}) \right\}.$$

Let $\bar{\nu}'$ be one of $(\nu_1, \dots, \nu_{m-1})$ in the series (63). Since $\Gamma \subset \text{NS}(\widehat{\mathbb{P}^2})^{\times m-1}$ is of finite index, we have $\bar{\nu}' \in \Gamma_{\mathbb{Q}}$. By expanding, the series (63) is a linear combination of the series

$$\begin{aligned}
 & \sum_{\substack{\nu \in \mathcal{T}' + \Gamma \\ H_0 \cdot \nu_i = 0 \\ \text{for all } i \in T}} \prod_{i \notin T} (\text{sgn}(H_0 \cdot \nu_i) \\
 & - \text{sgn} \left(F \cdot \sum_{k \leq i < j} r_j r_k (\nu_k + \nu_{k+1} + \cdots + \nu_{j-1}) \right)) \\
 (64) \quad & \cdot \prod_{i \rightarrow j \text{ in } G} K_{\widehat{\mathbb{P}^2}}(\nu_i + \nu_{i+1} + \cdots + \nu_{j-1}) \cdot q^{-\sum_{1 \leq i < j \leq m} \frac{r_i r_j (\nu_i + \nu_{i+1} + \cdots + \nu_{j-1})^2}{2r}}.
 \end{aligned}$$

Here $T \subset \{1, \dots, m-1\}$ is a subset. By Lemma 2.18, it is enough to show that the series (64) with $T = \emptyset$ is written as $\Theta_\xi(q^{\frac{1}{r}})$ for data ξ as in (36).

Let $A = \{a_{ij}\}_{1 \leq i, j \leq m-1}$ be the $(m-1) \times (m-1)$ -matrix given by

$$a_{ij} = \begin{cases} -\sum_{k \leq i, j < l} r_l r_k & (i \leq j), \\ a_{ji} & (i > j). \end{cases}$$

We define the integer valued symmetric bilinear pairing $B(-, -)$ on Γ by

$$B(\nu, \nu') = \nu \cdot A \cdot {}^t \nu'.$$

Here we regard an element $\nu \in \Gamma$ as a row vector $(\nu_1, \dots, \nu_{m-1})$ in $\text{NS}(\widehat{\mathbb{P}^2})^{\times m-1}$. It is straightforward to check that

$$\begin{aligned}
 Q(\nu) & := \frac{B(\nu, \nu)}{2} \\
 (65) \quad & = - \sum_{1 \leq i < j \leq m} \frac{r_i r_j (\nu_i + \nu_{i+1} + \cdots + \nu_{j-1})^2}{2}.
 \end{aligned}$$

Since $\text{NS}(\widehat{\mathbb{P}^2})$ with its intersection form is a lattice with index $(1, 1)$, it follows that $(\Gamma, B(-, -))$ is a non-degenerate lattice with index $(m-1, m-1)$. In particular, we have $\det A \neq 0$.

We set $c_i, c'_i \in \Gamma_{\mathbb{Q}}$ for $1 \leq i \leq m-1$ as follows:

$$\begin{aligned}
 (66) \quad c_i & = (0, \dots, 0, \overset{i}{H_0}, 0, \dots, 0)A^{-1}, \\
 c'_i & = (0, \dots, 0, \overset{i}{-F}, 0, \dots, 0).
 \end{aligned}$$

Here $\overset{i}{*}$ means that $*$ is located on the i -th column. Let $E(G)$ be the set of arrows in G , and take $e = (i \rightarrow j) \in E(G)$. Since (Γ, B) is non-degenerate, there exists $\alpha_e \in \Gamma_{\mathbb{Q}}$ such that

$$B(\alpha_e, \nu) = K_{\widehat{\mathbb{P}^2}}(\nu_i + \nu_{i+1} + \cdots + \nu_{j-1}),$$

for any $\nu \in \Gamma_{\mathbb{Q}}$. By the above constructions, the series (64) with $T = \emptyset$ is written in the following way:

$$\sum_{\nu \in \overline{\nu}' + \Gamma} \prod_{i=1}^{m-1} (\text{sgn}(B(c_i, \nu)) - \text{sgn}(B(c'_i, \nu))) \prod_{e \in E(G)} B(\alpha_e, \nu) \cdot q^{\frac{Q(\nu)}{r}}.$$

Hence the following lemma shows that (57) holds:

Lemma 3.9. *The data*

$$(\Gamma, B, \overline{\nu}', c_1, \dots, c_{m-1}, c'_1, \dots, c'_{m-1}, \alpha_e, e \in E(G))$$

satisfies the conditions (i) to (v) in Subsection 2.7.

Proof. The condition (i) is already stated. Let $V \subset \Gamma_{\mathbb{Q}}$ be the sub \mathbb{Q} -vector space spanned by c_i for $1 \leq i \leq m - 1$. By (66), V is $m - 1$ -dimensional, and

$$V = \bigoplus_{i=1}^{m-1} \mathbb{Q} \cdot (0, \dots, 0, \overset{i}{H}_0, 0, \dots, 0).$$

By (65), it follows that Q is negative definite on V , hence the condition (ii) holds. The condition (iii) follows from

$$\begin{aligned} B(c_i, c'_j) &= (0, \dots, \overset{i}{H}_0, \dots, 0) \cdot {}^t(0, \dots, \overset{j}{-F}, \dots, 0) \\ &= -\delta_{ij}, \\ B(c'_i, c'_j) &= (0, \dots, \overset{i}{-F}, \dots, 0) \cdot {}^t(b_1 F, \dots, b_{m-1} F) \\ &= 0. \end{aligned}$$

Here b_1, \dots, b_{m-1} are some rational numbers, and the last equality follows from $F^2 = 0$. The condition (iv) follows from Lemma 3.8, and there is nothing to prove for (v). q.e.d.

3.5. Proof of Proposition 3.3 for (51). We finish the proof of Proposition 3.3 by proving that

$$(67) \quad U_{(r_1, \overline{\beta}_1), \dots, (r_m, \overline{\beta}_m)}^{l, G}(q) \in \mathcal{M}.$$

By Lemma 3.8 and (33), the rational number $U(\{(r_i, \beta_i)\}_{i=1}^m, H, F_+)$ in the RHS of (51) does not contain contributions of the terms in (32) with $m'' \geq 2$. For data (49) and a fixed non-decreasing surjection

$$\psi: \{1, \dots, m\} \twoheadrightarrow \{1, \dots, m'\},$$

we consider the series:

$$(68) \quad \begin{aligned} S_{(r_1, \overline{\beta}_1), \dots, (r_m, \overline{\beta}_m)}^{l, G, \psi} &:= \sum_{\substack{\beta_j \in \text{NS}(\widehat{\mathbb{P}}^2), \beta_j \equiv \overline{\beta}_j \pmod{r_j} \\ 1 \leq j \leq m, \beta_1 + \dots + \beta_m = (l, 1-l) \\ \beta_j \cdot H_0 / r = \beta_k \cdot H_0 / r \text{ if } \psi(j) = \psi(k)}} S(\{(R_i, \mathcal{B}_i)\}_{i=1}^{m'}, H, F_+) \\ &\cdot \prod_{i \rightarrow j \text{ in } G} K_{\widehat{\mathbb{P}}^2}(r_j \beta_i - r_i \beta_j) q^{-\sum_{1 \leq i < j \leq m} \frac{(r_j \beta_i - r_i \beta_j)^2}{2rr_i r_j}}. \end{aligned}$$

Here (R_i, \mathcal{B}_i) is given by

$$(69) \quad R_i = \sum_{j \in \psi^{-1}(i)} r_j, \quad \mathcal{B}_i = \sum_{j \in \psi^{-1}(i)} \beta_j.$$

Since (51) is a \mathbb{Q} -linear combination of the series (68), it is enough to show that (68) is an element of \mathcal{M} .

Let us consider $(\beta_1, \dots, \beta_m)$ in the RHS of (68). For each $1 \leq j \leq m$ with $\psi(j) = i$, we can write

$$(70) \quad \frac{\beta_j}{r_j} = \frac{\mathcal{B}_i}{R_i} + l_j C,$$

for some $l_j \in \mathbb{Q}$. For $1 \leq i \leq m'$, let $\overline{\mathcal{B}}_i \in \text{NS}_{<R_i}(\widehat{\mathbb{P}}^2)$ be the unique element such that

$$\mathcal{B}_i \equiv \overline{\mathcal{B}}_i \pmod{R_i}.$$

Then we have

$$(71) \quad l_j C \in \frac{\overline{\beta}_j}{r_j} - \frac{\overline{\mathcal{B}}_i}{R_i} + \text{NS}(\widehat{\mathbb{P}}^2).$$

By applying $\cdot H_0$ and $\cdot C$, the condition (71) is equivalent to the two conditions:

$$\frac{\overline{\mathcal{B}}_i}{R_i} \cdot H_0 \in \frac{\overline{\beta}_j}{r_j} \cdot H_0 + \mathbb{Z}, \quad l_j \in \frac{\overline{\mathcal{B}}_i}{R_i} \cdot C - \frac{\overline{\beta}_j}{r_j} \cdot C + \mathbb{Z}.$$

Also using (70), the condition (69) for \mathcal{B}_i is equivalent to

$$\sum_{j \in \psi^{-1}(i)} r_j l_j = 0.$$

Using (70) and noting $K_{\widehat{\mathbb{P}}^2} \cdot C = -1$, we have

$$\prod_{i \rightarrow j \text{ in } G} K_{\widehat{\mathbb{P}}^2}(r_j \beta_i - r_i \beta_j) = \prod_{i \rightarrow j \text{ in } G} r_i r_j \cdot \sum_{\substack{G', G'' \subset G \\ E(G') \amalg E(G'') = E(G)}} \prod_{i \rightarrow j \text{ in } G'} (l_j - l_i) \cdot \prod_{i \rightarrow j \text{ in } \psi(G'')} K_{\widehat{\mathbb{P}}^2} \left(\frac{\mathcal{B}_i}{R_i} - \frac{\mathcal{B}_j}{R_j} \right).$$

Here $G', G'' \subset G$ are oriented subgraphs, $E(G)$ is the set of arrows in G , and $\psi(G'') \in G'(m')$ is obtained by the image of G'' under ψ . Also

using (56) and setting $\beta = (l, 1 - l)$, the power of q in (68) is written as

$$\begin{aligned} & \frac{\beta^2}{2r} - \sum_{i=1}^m \frac{\beta_i^2}{2r_i} \\ &= \frac{\beta^2}{2r} - \sum_{i=1}^{m'} \frac{\mathcal{B}_i^2}{2R_i} + \sum_{i=1}^{m'} \left(\frac{\mathcal{B}_i^2}{2R_i} - \sum_{j \in \psi^{-1}(i)} \frac{\beta_j^2}{2r_j} \right) \\ &= - \sum_{1 \leq i < j \leq m'} \frac{(R_j \mathcal{B}_i - R_i \mathcal{B}_j)^2}{2r R_i R_j} + \sum_{i=1}^{m'} \sum_{j < k \text{ in } \psi^{-1}(i)} \frac{r_j r_k (l_k - l_j)^2}{2R_i}. \end{aligned}$$

Combing the above calculations, the series (68) is written as

$$\begin{aligned} & \prod_{i \rightarrow j \text{ in } G} r_i r_j \cdot \sum_{\substack{G', G'' \subset G \\ E(G') \amalg E(G'') = E(G)}} \sum_{\substack{\bar{\mathcal{B}}_i \in \text{NS}_{< R_i}(\widehat{\mathbb{P}}^2), 1 \leq i \leq m' \\ \bar{\mathcal{B}}_i \cdot H_0 / R_i \in \bar{\beta}_j \cdot H_0 / r_j + \mathbb{Z} \text{ for all } j \in \psi^{-1}(i)}} \\ & \cdot \sum_{\substack{l_j \in \bar{\mathcal{B}}_{\psi(j)} \cdot C / R_{\psi(j)} - \bar{\beta}_j \cdot C / r_j + \mathbb{Z}, 1 \leq j \leq m \\ \sum_{j \in \psi^{-1}(i)} r_j l_j = 0 \text{ for all } 1 \leq i \leq m'}} \prod_{i \rightarrow j \text{ in } G'} (l_j - l_i) \\ & \cdot q^{\sum_{i=1}^{m'} \sum_{j < k \text{ in } \psi^{-1}(i)} \frac{r_j r_k (l_k - l_j)^2}{2R_i}} \\ & \cdot \sum_{\substack{\mathcal{B}_i \in \text{NS}(\widehat{\mathbb{P}}^2), \mathcal{B}_i \equiv \bar{\mathcal{B}}_i \pmod{R_i} \\ \mathcal{B}_1 + \dots + \mathcal{B}_{m'} = (l, 1-l)}} S(\{(R_i, \mathcal{B}_i)\}_{i=1}^{m'}, H, F_+) \\ & \cdot \prod_{i \rightarrow j \text{ in } \psi(G'')} K_{\widehat{\mathbb{P}}^2} \left(\frac{\mathcal{B}_i}{R_i} - \frac{\mathcal{B}_j}{R_j} \right) q^{-\sum_{1 \leq i < j \leq m'} \frac{(R_j \mathcal{B}_i - R_i \mathcal{B}_j)^2}{2r R_i R_j}}. \end{aligned}$$

By Lemma 2.18, the series

$$\begin{aligned} & \sum_{\substack{l_j \in \bar{\mathcal{B}}_{\psi(j)} \cdot C / R_{\psi(j)} - \bar{\beta}_j \cdot C / r_j + \mathbb{Z}, 1 \leq j \leq m \\ \sum_{j \in \psi^{-1}(i)} r_j l_j = 0 \text{ for all } 1 \leq i \leq m'}} \prod_{i \rightarrow j \text{ in } G'} (l_j - l_i) \\ & \cdot q^{\sum_{i=1}^{m'} \sum_{j < k \text{ in } \psi^{-1}(i)} \frac{r_j r_k (l_k - l_j)^2}{2R_i}} \end{aligned}$$

is an element of \mathcal{M} . Combined with (57), we conclude that (68) is an element of \mathcal{M} .

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