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MINIMAL HYPERSURFACES OF LEAST AREA

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Abstract

In this paper, we study closed embedded minimal hypersurfaces in a Riemannian $(n+1)$-manifold $(2 \leq n \leq 6)$ that minimize area among such hypersurfaces. We show they exist and arise either by minimization techniques or by min–max methods: they have index at most 1. We apply this to obtain a lower area bound for such minimal surfaces in some hyperbolic 3-manifolds.

1. Introduction

A classical result in minimal hypersurfaces theory is that, in $\mathbb{S}^{n+1}$ with the round metric, the totally geodesic equatorial $\mathbb{S}^n$ has least area among minimal hypersurfaces in $\mathbb{S}^{n+1}$. Actually, it is a consequence of the monotonicity formula for minimal hypersurfaces in $\mathbb{R}^{n+2}$. Another consequence of the monotonicity formula in a general closed Riemannian manifold M is that any closed minimal hypersurface has area at least some positive constant depending on M (in the following all minimal hypersurfaces are assumed to be closed). So one can ask to precise this constant or to find a minimal hypersurface of least area among minimal hypersurfaces in M.

One way to understand this question is to look at how minimal hypersurfaces can be constructed as critical points of the area functional in a closed Riemannian $(n+1)$-manifold M. If S is some closed hypersurface in M non-vanishing in homology, geometric measure theory [**9**] tells us that the area can be minimized in the homology class of S to produce a closed embedded minimal hypersurface Σ in M which minimizes the area. Actually, Σ is a smooth hypersurface outside some singular subset of Hausdorff dimension less than or equal to $n-7$. This approach produces minimal hypersurfaces that are stable, *i.e.*, the Jacobi operator on Σ has index 0.

If the homology group $H_n(M)$ vanishes, for example $M = \mathbb{S}^{n+1}$, the above idea cannot be applied. Almgren and Pitts [**1**, **24**] then developed a min–max approach to construct minimal hypersurfaces in such a manifold M. They prove that the fundamental class $[M] \in$

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$H_{n+1}(M)$ is associated to a particular positive number W_M called the width of the manifold. Then this number is realized as the area of some particular minimal hypersurface (maybe with multiplicities); this minimal hypersurface is called a min–max hypersurface associated to the fundamental class $[M]$. Pitts proved the result when $2 \leq n \leq 5$, it was extended by Schoen and Simon [**27**] later to higher values of n. Here also, the minimal hypersurface may have a singular subset of Hausdorff dimension less than or equal to $n-7$. As a consequence, there always exists a smooth minimal hypersurface in M if $2 \leq n \leq 6$. This min–max approach works with one parameter families of hypersurfaces called sweep-outs, so the min–max hypersurface is expected to have index at most 1. For example, all such min–max hypersurfaces in the round $\mathbb{S}^{n+1}$ are the equatorial $\mathbb{S}^n$. Indeed, the monotonicity formula tells that any minimal hypersurface has area at least the one of the equatorial $\mathbb{S}^n$ but, looking at the sweep-out (see definitions in Section 3) $t \mapsto \{x = (x_1, \dots, x_{n+2}) \in \mathbb{S}^{n+1} | x_{n+2} = 2t - 1\}$, we see that $W_{\mathbb{S}^{n+1}}$ is at most the area of an equatorial $\mathbb{S}^n$ (see also pages 213–214 in [**18**]); hence equal to the area of an equatorial $\mathbb{S}^n$.

Coming back to the question of finding a minimal hypersurface of least area among minimal hypersurfaces, the main result of our paper mainly says that such a hypersurface exists and can be constructed by one of the above approaches. To be more precise, we take into account the possible non-orientability of hypersurfaces: let $\mathcal{O}$ be the collection of all smooth orientable connected closed embedded minimal hypersurfaces in M and $\mathcal{U}$ be the collection of the non-orientable ones. If $2 \leq n \leq 6$, we know that at least one of them is non-empty. We then define

$$\mathcal{A}_1(M) = \inf(\{|\Sigma|, \Sigma \in \mathcal{O}\} \cup \{2|\Sigma|, \Sigma \in \mathcal{U}\}),$$

where $|\cdot|$ denotes the area. The non-orientable hypersurfaces are chosen to be counted twice since, in several constructions, non-orientable minimal hypersurfaces appear with multiplicity 2. So our main theorem can be stated as follows:

Theorem A. *Let M be an oriented closed Riemannian $(n+1)$-manifold ($2 \leq n \leq 6$). Then $\mathcal{A}_1(M)$ is equal to one of the following possibilities:*

1) *$|\Sigma|$ where $\Sigma \in \mathcal{O}$ is a min–max hypersurface of M associated to the fundamental class of $H_{n+1}(M)$ and has index 1.*
2) *$|\Sigma|$ where $\Sigma \in \mathcal{O}$ is stable.*
3) *$2|\Sigma|$ where $\Sigma \in \mathcal{U}$ is stable and its orientable 2-sheeted cover has index 0 or 1; if the index is 1, $2|\Sigma| = W_M$.*

Moreover, if $\Sigma \in \mathcal{O}$ satisfies $|\Sigma| = \mathcal{A}_1(M)$, then Σ is of type 1 or 2 and if $\Sigma \in \mathcal{U}$ satisfies $2|\Sigma| = \mathcal{A}_1(M)$, then Σ is of type 3.

So the theorem says that $\mathcal{A}_1(M)$ is realized, moreover, it characterizes all minimal hypersurfaces that realize $\mathcal{A}_1(M)$. Let us first notice that the restriction on the dimension is the classical restriction about the regularity for minimal hypersurfaces in high dimensions. The main property of the hypersurface Σ is expressed in terms of the index of its Jacobi operator: it is 0 (stable case) or 1. One consequence of the above theorem is the following corollary:

Corollary 1. *Let M be an oriented closed Riemannian $(n+1)$-manifold ($2 \le n \le 6$). Then there exists a closed embedded minimal hypersurface of index 0 or 1. If there are no stable hypersurfaces in M then there is an index one embedded min–max minimal hypersurface that realizes W_M.*

If M has positive Ricci curvature, it is known that there is no stable orientable minimal hypersurface. So, in that case, the above theorem is similar to the main result obtained in [**32**] where Zhou characterizes the min–max hypersurface in the positive Ricci case. Actually, the estimate on the index of the double cover in the non-orientable case does not appear in the work of Zhou. For the rest, the proof of our result is based on similar ideas to the work of Zhou.

Of course, it would be interesting to say more about the hypersurface that appears in Theorem A, for example about its topology. In dimension 3 ($n = 2$), we are able to give some improvements to our main results. In fact, we prove that in the index 1 case for type 1 and 3 surfaces, the genus of the surface Σ cannot be too small and is controlled by the Heegaard genus of the ambient manifold M; this will be Theorem B. In [**16**], Marques and Neves look also for control on the genus of index 1 minimal surfaces. In fact, finding upper bounds for the genus of min–max surfaces was first present in the work of Smith [**30**] about the existence of minimal 2-spheres in Riemannian 3-spheres and has received major contributions by De Lellis and Pellandini [**6**] and Ketover [**13**].

Actually, sometimes, index and genus can be combined to estimate the area of a minimal surface (see [**16**] for an example). So one consequence of our improvement is that we can give a lower bound for the area of minimal surfaces in hyperbolic 3-manifolds.

Theorem C. *Let M be a closed orientable hyperbolic 3-manifold. If the Heegaard genus of M is at least 7 then $\mathcal{A}_1(M) \ge 2\pi$. In other words, any orientable minimal surface in M has area at least 2π and any non-orientable minimal surface has area at least π.*

Let us notice that, in the above result, if M does not have any non-orientable surface, we need only assume that the Heegard genus is at least 6. In fact, after this paper was made public, Ketover, Marques and Neves obtained results concerning the index and the multiplicity of some min–max hypersurface. The main point is that the arguments in the proof of Theorems 3.5 and 4.1 in [**14**] imply that, in the last possibility of Theorem A, the orientable 2-sheeted cover has index 0.

As a consequence, Theorem C is true if we only assume the Heegaard genus us at least 6.

To prove Theorem A, one idea would be to consider a minimizing sequence and use some compactness result for minimal hypersurfaces to get some limit hypersurface. The main default with this approach is that *a priori* the eventual limit need not be a smooth hypersurface. However, this minimization argument can be done among stable minimal hypersurfaces to produce a stable minimal hypersurface with least area. So we can construct a stable minimal hypersurface that realizes $\mathcal{A}_{\mathcal{S}}(M)$ where $\mathcal{A}_{\mathcal{S}}(M)$ is defined as $\mathcal{A}_1(M)$ but with an infimum computed only among stable minimal hypersurfaces. If $\mathcal{A}_1(M) = \mathcal{A}_{\mathcal{S}}(M)$, this almost gives the proof of the main theorem.

In fact, the proof of Theorem A mainly consists in proving that $\mathcal{A}_1(M) = \min(W_M, \mathcal{A}_{\mathcal{S}}(M))$. So we need to understand minimal hypersurfaces Σ with area less than $\mathcal{A}_{\mathcal{S}}(M)$. Actually, we prove that Σ can be seen as a leaf of maximal area of some sweep-out of the manifold M. As a consequence, this implies that the area of the min–max hypersurface constructed by Pitts is less than the area of Σ so this min–max hypersurface has to realize $\mathcal{A}_1(M)$. The proof of the existence of the above sweep-out uses another point of view about min–max theory for minimal hypersurfaces which is developed by Colding, De Lellis and Tasnady [**4**, **7**].

Actually, the questions we look at in this paper can be generalized. Consider the space $\mathcal{M} = \mathcal{O} \cup \mathcal{U}$ of closed embedded minimal hypersurfaces on a manifold M. Let $\mathcal{A} : \mathcal{M} \to \mathbb{R}^+$ be the area function. What are the properties of $\mathcal{A}$? Is the image of $\mathcal{A}$ infinite? More generally, is $\mathcal{A}$ always unbounded? An affirmative reply to either question would answer a question of Yau concerning the existence of an infinite number of minimal hypersurfaces. In this paper we discussed $\mathcal{A}_1(M)$, the minimum of $\mathcal{A}$. In general the values of $\mathcal{A}$ are difficult to understand. For example, when M is the standard 3-sphere, we know that $\mathcal{A}_1(M) = 4\pi$. The next value of $\mathcal{A}$ is $2\pi^2$, the area of the Clifford torus. This is very difficult to prove and is an important part of the solution of the Willmore conjecture by Marques and Neves [**17**]. Let us also notice that there is a gap in the values of $\mathcal{A}$ after $2\pi^2$. Indeed, if a sequence of minimal surfaces in $\mathbb{S}^3$ has area converging to $2\pi^2$ then the same argument as in Appendix A of [**17**] implies that, up to a subsequence, it converges smoothly to a minimal surface of area $2\pi^2$, a Clifford torus by Theorem B in [**17**]. This means the sequence is made of minimal tori after a certain rank so they are Clifford tori and have area $2\pi^2$ by the proof of the Lawson conjecture [**2**]. After the area $2\pi^2$, one has the Lawson examples that are genus g surfaces in $\mathcal{M}$ whose areas converge to 8π as $g \to \infty$. One can see these surfaces (as $g \to \infty$) as desingularizing two orthogonal geodesic 2-spheres along their intersection.

Actually, we do not believe 8π can be realized as the area of a minimal surface. Do the areas of closed embedded minimal surfaces of $\mathbb{S}^3$ form a discrete set of real numbers?

By desingularizing k geodesic 2-spheres meeting along a common geodesic at equal angles, one obtains surfaces in $\mathcal{M}$ whose areas converge to $4\pi k$.

For any M of dimension 3, one can consider the surfaces in $\mathcal{M}$ of genus at most g (there may not be any) and try to calculate the minimum $\mathcal{A}^g(M)$ of $\mathcal{A}$ on these surfaces. $\mathcal{A}^g(M)$ is realized (provided some genus g surface exists in M). The behavior of $\mathcal{A}^g(M)$ would be interesting to understand.

As an example, what are these quantities for the space $M = \mathbb{S}^2 \times \mathbb{S}^1$, $\mathbb{S}^2$ the unit 2-sphere and $\mathbb{S}^1$ the circle of length ℓ?

We also notice that Theorem A does not solve the following question: if $(\Sigma_n)_n$ is a sequence of minimal hypersurfaces whose areas converge to $\mathcal{A}_1(M)$, do we have convergence of $(\Sigma_n)_n$ to one of the smooth hypersurfaces of Theorem A?

This article is organized as follow. In Section 2, we recall some classical definitions about the index of minimal hypersurfaces. In Section 3, we give a quick presentation of the min–max theories of Colding, De Lellis and Tasnady (the continuous setting) and of Almgren and Pitts (the discrete setting).

Section 4 is devoted to the minimization among stable hypersurfaces, we define $\mathcal{A}_{\mathcal{S}}(M)$ and prove that it is realized. In Section 5, we construct the sweep-out associated to a minimal hypersurface with area less than $\mathcal{A}_{\mathcal{S}}(M)$. Finally, the proof of the main theorem is given in Section 6.

From Section 7, we look at the dimension 3 case; in Section 7, we improve Theorem A to obtain some control of the topology of the surface. This result is then applied in Section 8 to give a lower bound for the area of minimal surfaces in hyperbolic 3-manifolds.

Our work is strongly influenced by the paper of Marques and Neves [**16**]. Indeed, at a recent meeting, when we told Fernando C. Marques about our work, he returned the next day with the ideas we had used to prove $\mathcal{A}_1(M)$ is realized.

2. Minimal hypersurfaces

In this section, we give some definitions and recall some basic facts about minimal hypersurfaces.

In this paper, we look at hypersurfaces Σ in a certain Riemannian $(n+1)$-manifold M. All along the paper, M will be orientable. If it is not precised, all hypersurfaces are assumed to be embedded.

2.1. Minimal hypersurfaces. Minimal hypersurfaces in M are those with vanishing mean curvature vector, they appear as critical points of the area functional for hypersurfaces.

In the following, we will denote by $\mathcal{O}$ the collection of all orientable minimal hypersurfaces and by $\mathcal{U}$ the collection of all non-orientable ones.

As in the introduction, we define

$$\mathcal{A}_1(M) = \inf(\{|\Sigma|, \Sigma \in \mathcal{O}\} \cup \{2|\Sigma|, \Sigma \in \mathcal{U}\}).$$

2.2. The stability operator. Minimal hypersurfaces are critical points of the area functional on hypersurfaces. The study of the second derivative of the area functional on such a critical point is given by the stability operator.

Let Σ be a minimal hypersurface in an orientable Riemannian $(n+1)$-manifold M. The stability operator is a quadratic differential form acting on sections of the normal bundle $N\Sigma$ to Σ. If $\xi \in \Gamma(N\Sigma)$ is such a section, we have

$$Q_\Sigma(\xi, \xi) = \int_\Sigma \|\nabla^\perp \xi\|^2 - \mathrm{Ric}_M(\xi, \xi) - \|A\|^2 \|\xi\|^2 d\mathrm{vol}_\Sigma,$$

where $\nabla^\perp$ is the normal connection on $N\Sigma$ coming from the Levi-Civita connection on M, Ric_M is the Ricci curvature tensor on M and $\|A\|$ is the norm of the second fundamental form on Σ.

A minimal hypersurface is called stable if Q is non-negative. This means that Σ is a minimum of order 2 for the area functional. The index of Σ is the maximal dimension of linear subspaces E of $\Gamma(N\Sigma)$ such that Q is negative definite on E.

If Σ is 2-sided, *i.e.*, $N\Sigma$ is a trivial line bundle, there is a unit normal vector field ν along Σ so any section ξ can be written as $\xi = u\nu$ where u is a function. Thus, the stability operator becomes an operator on functions

$$Q_\Sigma(u, u) = \int_\Sigma \|\nabla u\|^2 - (\mathrm{Ric}_M(\nu, \nu) + \|A\|^2) u^2 d\mathrm{vol}_\Sigma = -\int_\Sigma u \mathcal{L}_\Sigma u \, d\mathrm{vol}_\Sigma,$$

where $\mathcal{L}_\Sigma u = \Delta u + (\mathrm{Ric}_M(\nu, \nu) + \|A\|^2) u$ is called the Jacobi operator on Σ.

If Σ is a closed minimal hypersurface, $-\mathcal{L}_\Sigma$ has a discrete spectrum $\lambda_1 < \lambda_2 \le \cdots$. The index of Σ is then the number of negative eigenvalues of $-\mathcal{L}_\Sigma$.

If Σ is orientable, Σ is 2-sided since M is orientable and the above description applies. If Σ is non-orientable, Σ is not 2-sided but we can consider $\pi : \widetilde{\Sigma} \to \Sigma$ the orientable double cover of Σ. The map π defines a minimal immersion of $\widetilde{\Sigma}$ in M which is 2-sided so the Jacobi operator $\mathcal{L}_{\widetilde{\Sigma}}$ is defined. The covering map π comes with a unique non-trivial deck transformation σ which is an involution. If ν is a unit normal vector field along $\widetilde{\Sigma}$ we have $\nu \circ \sigma = -\nu$. So sections of $N\Sigma$ correspond to σ-odd functions on $\widetilde{\Sigma}$ and σ-even functions on $\widetilde{\Sigma}$ correspond to functions on Σ. We also notice that, for a function u on $\widetilde{\Sigma}$, $\mathcal{L}_{\widetilde{\Sigma}}(u \circ \sigma) = (\mathcal{L}_{\widetilde{\Sigma}} u) \circ \sigma$. Thus the hypersurface Σ is stable if and only if $Q_{\widetilde{\Sigma}}$ is non-negative on

σ-odd functions. As another consequence, if Σ is stable and u is an eigenfunction of $-\mathcal{L}_{\widetilde{\Sigma}}$ with a negative eigenvalue, u is σ-even.

3. Preliminaries about min–max theory

In this paper, we will use several times the min–max approach to construct minimal hypersurfaces. There are two major settings for the min–max theory: the discrete setting which is due to Almgren and Pitts [**24**] and the continuous setting due to Colding, De Lellis [**4**] and De Lellis, Tasnady [**7**]. Both settings have their own interest, the continuous setting is easier to consider for some geometric considerations and the discrete setting is more linked to the topology of the ambient space.

Good introductions to both settings can be found in several papers (see [**4**, **7**, **17**, **32**]). So here, we only summarize facts that we will really use.

Let M be a compact Riemannian $(n+1)$-manifold with or without boundary. $\mathcal{H}^k$ will denote the k-dimensional Hausdorff measure and, when Σ is an n-dimensional submanifold, we use the following notation $|\Sigma| = \mathcal{H}^n(\Sigma)$ and we say that $|\Sigma|$ is the area of Σ even if it has dimension larger than 2. If Σ is an immersed hypersurface, we also use $|\Sigma|$ to compute its volume which could be different from the $\mathcal{H}^n$-measure of its image in M.

3.1. The continuous setting. Let us recall some definitions and results from the papers of De Lellis and Tasnady [**7**] and Zhou [**32**]. First let us define what kind of family of hypersurfaces we will consider.

Definition 2. A family $\{\Gamma_t\}_{t\in[a,b]}$ of closed subsets of M with finite $\mathcal{H}^n$-measure is called a *generalized smooth family* if

(s1) For each t there is a finite set $P_t \subset M$ such that $\Gamma_t \setminus P_t$ is either a smooth hypersurface in $M \setminus P_t$ or the empty set;
(s2) $\mathcal{H}^n(\Gamma_t)$ depends continuously on t and $t \mapsto \Gamma_t$ is continuous in the Hausdorff sense;
(s3) on any $U \subset\subset M \setminus P_{t_0}$, $\Gamma_t \xrightarrow{t\to t_0} \Gamma_{t_0}$ smoothly in U.

Now let us define the continuous sweep-outs for ambient manifolds with or without boundary.

Definition 3. Let M be a closed manifold. A generalized smooth family $\{\Gamma_t\}_{t\in[a,b]}$ is a *continuous sweep-out* of M if there exists a family $\{\Omega_t\}_{t\in[a,b]}$ of open subsets of M such that

(sw1) $(\Gamma_t \setminus \partial\Omega_t) \subset P_t$ for any t;
(sw2) $\mathcal{H}^{n+1}(\Omega_t \triangle \Omega_s) \to 0$ as $t \to s$ (where $\triangle$ denotes the symmetric difference of subsets);
(sw3) $\Omega_a = \emptyset$ and $\Omega_b = M$;

Definition 4. Let M be a compact manifold with non-empty boundary. A generalized smooth family $\{\Gamma_t\}_{t\in[a,b]}$ is a *continuous sweep-out* of M if there exists a family $\{\Omega_t\}_{t\in[a,b]}$ of open subsets of M satisfying *(sw2)* and

(sw0') $\partial M \subset \Omega_t$ for $t > a$;
(sw1') $(\Gamma_t \setminus \partial_*\Omega_t) \subset P_t$ for any $t > a$ where $\partial_*\Omega_t = \partial\Omega_t \setminus \partial M$;
(sw3') $\Omega_a = \emptyset$, $\Omega_b = M$ and there are $\varepsilon > 0$ and a smooth function $w : [0,\varepsilon]\times\partial M \to \mathbb{R}$ with $w(0,p) = 0$ and $\partial_t w(0,p) > 0$ such that

$$\Gamma_{a+t} = \{\exp_p(w(t,p)\nu(p)), p \in \partial M\},$$

for $t \in [0,\varepsilon]$ and ν the inward unit normal to ∂M.

For a continuous sweep-out as above $\{\Gamma_t\}_{t\in[a,b]}$, we define the quantity $\mathbf{L}(\Gamma_t) = \max_{t\in[a,b]} |\Gamma_t|$.

Two continuous sweep-outs $\{\Gamma_t^1\}_{t\in[a,b]}$ and $\{\Gamma_t^2\}_{t\in[a,b]}$ are said to be homotopic if, informally, they can be continuously deformed one to the other (the precise definitions are Definition 0.6 in [**7**] and Definition 2.5 in [**32**]). Then a family Λ of sweep-outs is called homotopically closed if it contains the homotopy class of each of its elements. For such a family Λ, we can define the width associated to Λ as

$$W(\Lambda) = \inf_{\{\Gamma_t\}\in\Lambda} \mathbf{L}(\{\Gamma_t\}).$$

We notice that when M has no boundary, $W(\Lambda) > 0$ for any Λ (see Proposition 0.5 in [**7**]).

If Λ is a homotopically closed family of sweep-outs and the sequence $(\{\Gamma_t^k\}_t)_{k\in\mathbb{N}}$ of sweep-outs is such that $\mathbf{L}(\{\Gamma_t^k\}_t) \xrightarrow[k\to\infty]{} W(\Lambda)$, a min–max sequence is a sequence $(\Gamma_{t_k}^k)$ (or a subsequence of this sequence) such that $|\Gamma_{t_k}^k| \xrightarrow[k\to\infty]{} W(\Lambda)$. The main existence-result about the min–max theory in this setting is (see Theorem 0.7 [**7**] and Theorem 2.7 [**32**])

Theorem 5 (De Lellis, Tasnady [**7**], Zhou [**32**])**.** *Let M be a compact Riemannian $(n+1)$-manifold ($2 \le n \le 6$). Let Λ be a homotopically closed family of continuous sweep-outs of M. If M has no boundary, there is a min–max sequence that converges (in the varifold sense) to an integral varifold whose support is a finite collection of embedded connected disjoint minimal hypersurfaces of M. As a consequence*

$$W(\Lambda) = \sum_{i=1}^{p} n_i |S_i|,$$

where $\cup_{i=1}^p S_i$ is the support of the limit varifold.

If M has boundary, the same result is true if we assume that the mean curvature vector of ∂M does not vanish and points into M and $W(\Lambda) > |\partial M|$.

We refer to [**29**] for the definition of the convergence in the varifold sense.

Remark 1. One consequence of this result that we will use is that if we have some continuous sweep-out $\{\Gamma_t\}_t$ of M ($\partial M = \emptyset$) then there is some connected minimal hypersurface S in M with $|S| \leq \mathbf{L}(\{\Gamma_t\})$.

3.2. The discrete setting.

Here we recall some aspects of the Almgren–Pitts min–max theory which deals with discrete families of elements of $\mathcal{Z}_n(M)$, *i.e.*, integral rectifiable n-currents in M with no boundary. For definitions about currents, we refer to [**9**, **29**].

If $I = [0,1]$, we first introduce some cell complex structure on I and I^2.

Definition 6. Let j be an integer, we define $I(1,j)$ to be the cell complex of I, whose 0-cells are the points $[\frac{i}{3^j}]$ for $i = 0, \dots, 3^j$ and the 1-cells are the intervals $[\frac{i}{3^j}, \frac{i+1}{3^j}]$ for $i = 0, \dots, 3^j - 1$.

We also define a cell complex $I(2,j)$ on I^2 by $I(2,j) = I(1,j) \otimes I(1,j)$. Similarly $I(m,j)$ can be defined on I^m.

Let us introduce some notations about these cell complexes

- $I_0(1,j)$ denotes the set of the boundary 0-cells $\{[0],[1]\}$.
- $I(m,j)_0$ denotes the set of 0-cells of $I(m,j)$.
- The distance between two elements of $I(m,j)_0$ is
$$\mathbf{d} : I(m,j)_0 \times I(m,j)_0 \to \mathbb{N} \ ; \ (x,y) \mapsto 3^j \sum_{i=1}^{m} |x_i - y_i|.$$
- The projection map $n(i,j) : I(m,i)_0 \to I(m,j)_0$ is defined such that $n(i,j)(x)$ is the unique element in $I(m,j)_0$ such that
$$\mathbf{d}(x, n(i,j)(x)) = \inf\{\mathbf{d}(x,y), y \in I(1,j)_0\}.$$

We are going to look at maps $\varphi : I(m,j)_0 \to \mathcal{Z}_n(M)$. For such a map φ, we define its fineness by
$$\mathbf{f}(\varphi) = \sup \left\{ \frac{\mathbf{M}(\varphi(x) - \varphi(y))}{\mathbf{d}(x,y)}, x, y \in I(m,j)_0 \text{ and } x \neq y \right\},$$
where $\mathbf{M}$ denotes the mass of a current.

When we write $\varphi : I(1,j)_0 \to (\mathcal{Z}_n(M), \{0\})$, we mean $\varphi(I(1,j)_0) \subset \mathcal{Z}_n(M)$ and $\varphi(I_0(1,j)) = \{0\}$.

Definition 7. Let δ be a positive real number and $\varphi_i : I(1,k_i)_0 \to (\mathcal{Z}_n(M), \{0\})$, $i = 1,2$. We say that φ_1 and φ_2 are 1*-homotopic in* $(\mathcal{Z}_n(M), \{0\})$ *with fineness* δ if there are $k_3 \in \mathbb{N}$, $k_3 \geq \max(k_1,k_2)$, and a map
$$\psi : I(2,k_3)_0 \to \mathcal{Z}_n(M),$$
such that

- $\mathbf{f}(\psi) \leq \delta$;
- $\psi([i-1], x) = \varphi_i(n(k_3,k_i)(x))$ for all $x \in I(1,k_3)_0$;
- $\psi(I(1,k_3)_0 \times \{[0],[1]\}) = 0$.

Let us now define the equivalent of generalized smooth family in the discrete setting.

Definition 8. A $(1,\mathbf{M})$-*homotopy sequence of maps into* $(\mathcal{Z}_n(M), \{0\})$ is a sequence of maps $\{\varphi_i\}_{i\in\mathbb{N}}$,

$$\varphi_i : I(1,k_i)_0 \to (\mathcal{Z}_n(M),\{0\}),$$

such that φ_i is 1-homotopic to φ_{i+1} in $(\mathcal{Z}_n(M),\{0\})$ with fineness δ_i and

- $\lim_{i\to\infty}\delta_i = 0$;
- $\sup_i\{\mathbf{M}(\varphi_i(x)), x\in I(1,k_i)_0\} < +\infty$.

As in the continuous setting, two $(1,\mathbf{M})$-homotopy sequences can be said to be homotopic and this defines an equivalence relation (see Section 4.1 in [**24**] or Definition 4.4 in [**32**]). The set of all equivalence classes is denoted by $\pi_1^{\#}(\mathcal{Z}_n(M),\mathbf{M},\{0\})$. One of the main results of the Almgren–Pitts theory says that $\pi_1^{\#}(\mathcal{Z}_n(M),\mathbf{M},\{0\})$ is naturally isomorphic to the homology group $H_{n+1}(M,\mathbb{Z})$ (Theorem 4.6 in [**24**], see also [**1**]).

If $S=\{\varphi_i\}_i$ is a $(1,\mathbf{M})$-homotopy sequence, we define the quantity

$$\mathbf{L}(S) = \limsup_{i\to\infty}\max\{\mathbf{M}(\varphi_i(x)), x\in I(1,k_i)_0\}.$$

Now, if $\Pi\in\pi_1^{\#}(\mathcal{Z}_n(M),\mathbf{M},\{0\})$ is an equivalence class, we can define the width associated to Π by

$$W(\Pi) = \inf\{\mathbf{L}(S), S\in\Pi\}.$$

The class that corresponds to the fundamental class in $H_{n+1}(M)$ by the Almgren–Pitts isomorphism is denoted Π_M. If $S=\{\varphi_i\}_i\in\Pi_M$, we say that S is a *discrete sweep-out* of M. The width $W(\Pi_M)$ is denoted by W_M and is called the width of the manifold M.

The theory tells us that there is $S\in\Pi_M$ such that $\mathbf{L}(S)=W(\Pi_M)=W_M$. If $S=\{\varphi_i\}_i$, we then say that $\varphi_{i_j}(x_j)$ is a min–max sequence ($x_j\in I(1,k_{i_j})$) if $\mathbf{M}(\varphi_{i_j}(x_j))\to W_M$. The min–max theorem of the Almgren–Pitts theory says the following (see [**24**] for $n\le 5$ and [**27**] for $n=6$):

Theorem 9 (Pitts [**24**], Schoen–Simon [**27**]). *Let M be a closed Riemannian $(n+1)$-manifold ($2\le n\le 6$). There is an $S=\{\varphi_i\}_i\in\Pi_M$ with $\mathbf{L}(S)=W_M$ and a min–max sequence $\{\varphi_{i_j}(x_j)\}_j$ that converges (in the varifold sense) to an integral varifold whose support is a finite collection of embedded connected disjoint minimal hypersurfaces of M. As a consequence*

$$W_M = \sum_{i=1}^{p} n_i|S_i|,$$

where $\cup_{i=1}^{p} S_i$ is the support of the limit varifold.

A limit varifold as in the above theorem will be called a min–max varifold associated to the fundamental class of $H_{n+1}(M)$ and by extension we say that its support is a min–max minimal hypersurface associated to the fundamental class of $H_{n+1}(M)$.

Remark 2. In [**32**], Zhou gives some precisions about the multiplicities that appear in the above theorem. He proved that if S_i is a non-orientable minimal hypersurface then its multiplicity n_i has to be even (Proposition 6.1 in [**32**]).

3.3. From continuous to discrete. It is easy to construct a continuous sweep-out of a manifold: we can just look at the level sets of a Morse function on the manifold M. The construction of a discrete sweep-out is not as clear even if the Almgren–Pitts isomorphism tells us that they exist.

In order to make a link between continuous and discrete sweep-outs, we use the following result (see Theorem 13.1 in [**17**] and Theorems 5.5 and 5.8 in [**32**]):

Theorem 10. *Let* $\{\Omega_t\}_{t\in[a,b]}$ *be a family of open subsets of* M *satisfying* $(sw2)$, $(sw3)$ *and*

- $\Phi(t)=\partial[\Omega_t]\in\mathcal{Z}_n(M)$;
- $\sup\{\mathbf{M}(\Phi(t)),t\in[a,b]\}<+\infty$;
- $\mathbf{m}(\Phi,r)=\sup\{\|\Phi(t)\|B(p,r),p\in M$ *and* $t\in[a,b]\}\to 0$ *as* $r\to 0$ *where* $B(p,r)$ *is the geodesic ball of* M *of center* p *and radius* r *and* $\|\cdot\|$ *denote the Radon measure on* M *associated to a current.*

Then there is a $(1,\mathbf{M})$*-homotopy sequence* $S\in\Pi_M$ *such that*

$$\mathbf{L}(S)\le\sup\{\mathbf{M}(\Phi(t)),t\in[a,b]\}.$$

Remark 3. Actually, the estimate on $\mathbf{L}(S)$ comes from a much stronger property of the construction. Let $\widetilde\Phi(t)=\Phi(a+t(b-a))$. The $(1,\mathbf{M})$ homotopy sequence $S=\{\varphi_i\}_{i\in\mathbb{N}}$ has the following property: there are sequences $\delta_i\to 0$ and $l_i\to\infty$ such that

$$\mathbf{M}(\varphi_i(x))\le\sup\{\mathbf{M}(\widetilde\Phi(y)),\ x,y\in\alpha\text{ for some 1-cell }\alpha\in I(1,l_i)\}+\delta_i. \tag{1}$$

Another property of S is that $\mathcal{F}(\varphi_i(x)-\widetilde\Phi(x))\le\delta_i$ for any $x\in I(1,k_i)_0$ where $\mathcal{F}$ is the flat norm on the space of currents and $\varphi_i: I(1,k_i)_0\to\mathcal{Z}_n(M)$.

Remark 4. The hypothesis about $\mathbf{m}(\Phi,r)$ is a no concentration property of the family $\{\Phi(t)\}_t$. Actually, the above theorem is used to produce discrete sweep-outs from continuous ones. This can be done since the hypotheses on $\mathbf{m}(\Phi,r)$ is satisfied if $\Phi(t)=[\Gamma_t]$ where $\{\Gamma_t\}_t$ is a continuous sweep-out (see Proposition 5.1 in [**32**]).

4. Stable minimal hypersurfaces

Among all minimal hypersurfaces, the stable ones play an important role since they appear when certain minimization arguments are done among some class of hypersurfaces. As a consequence, they are natural candidates for a minimal hypersurface with least area.

In this section, we study these minimization arguments and look at a stable minimal hypersurface with least area.

4.1. Non-separating hypersurfaces. We first look at hypersurfaces that do not separate M in two connected components.

Proposition 11. *Let M be a compact Riemannian $(n+1)$-manifold $(2 \le n \le 6)$ with mean-convex boundary (i.e., non-outward pointing mean curvature vector). Let Σ be an oriented hypersurface in M that is not homologous to 0. Then there is a connected orientable stable minimal hypersurface Σ' which is non-vanishing in homology and such that $|\Sigma'| \le |\Sigma|$. Moreover, if Σ is not a stable minimal hypersurface then $|\Sigma'| < |\Sigma|$.*

Typically, this proposition will be applied to non-separating hypersurfaces.

Proof. Σ represents a non-vanishing homology class in $H_n(M, \mathbb{Z})$. In terms of geometric measure theory, Σ can be seen as an integral n-cycle $[\Sigma]$. We can then minimize the mass among all integral cycles in the homology class of $[\Sigma]$ (see 5.1.6 in [**9**]). This produces an integral cycle homologous to $[\Sigma]$ whose support is made of several smooth connected orientable stable minimal hypersurfaces (see 5.4.15 in [**9**] or [**29**]). Since $[\Sigma] \neq 0$, there is one connected component Σ' of this support that does not vanish in homology, this component satisfies the properties of the above proposition.

If Σ is not a stable minimal hypersurface, it is clear that there are hypersurfaces homologous to Σ with area strictly less that $|\Sigma|$; so $|\Sigma'| < |\Sigma|$. q.e.d.

Let us fix a definition.

Definition 12. Let N and M be two n-manifolds with boundary and $\varphi : N \to M$ a smooth map. φ is said to be *locally invertible* if, for any point p in N, $d\varphi(p)$ is invertible and there is a neighborhood V of p in N such that φ is bijective from V to $\varphi(V)$ with smooth inverse map.

This definition mainly deals with properties of the map at boundary points of N: for example, boundary points of N are not necessarily sent to boundary points of M. The inclusion $[-1, 1] \hookrightarrow [-2, 2]$ is locally invertible, the map $[-\pi, \pi] \to \mathbb{S}^1; t \mapsto (\cos t, \sin t)$ is also locally invertible.

Proposition 13. *Let Σ be a connected closed oriented non-separating hypersurface in the interior of a manifold M with boundary. Then there is a manifold $\widetilde{M}$ with boundary with two particular boundary components Σ_1 and Σ_2 and a locally invertible smooth map $\varphi : \widetilde{M} \to M$ such that $\varphi : \widetilde{M} \setminus (\Sigma_1 \cup \Sigma_2) \to M \setminus \Sigma$ is a diffeomorphism and for $i = 1, 2$ $\varphi : \Sigma_i \to \Sigma$ is a diffeomorphism.*

Proof. Let us fix some complete Riemannian metric on M. Let ν be some unit normal vector field along Σ. The map $\Phi : \Sigma \times (-2\varepsilon, 2\varepsilon) \to M; (p, t) \mapsto \exp_p(t\nu(p))$ is a diffeomorphism on its image for small ε. Let ε be so. Let M_ε be $M \setminus \Phi(\Sigma \times [-\varepsilon, \varepsilon])$. We then define $\widetilde{M}$ as the quotient of the disjoint union of M_ε, $\Sigma \times [0, 2\varepsilon)$ and $\Sigma \times (-2\varepsilon, 0]$ by the identifications $(p, t) \simeq \Phi(p, t) \in M_\varepsilon$ for (p, t) in $\Sigma \times (-2\varepsilon, -\varepsilon)$ or $\Sigma \times (\varepsilon, 2\varepsilon)$.

The map φ is then defined as the identity on M_ε and by Φ on $\Sigma \times (-2\varepsilon, 0]$ and $\Sigma \times [0, 2\varepsilon)$. Σ_1 and Σ_2 are the two copies of $\Sigma \times \{0\}$. The map φ clearly satisfies the expected properties. q.e.d.

In the following, we will say that $\widetilde{M}$ is obtained by opening M along Σ. In general, there will be a metric on M so we always lift this metric to $\widetilde{M}$ so that φ is a local isometry.

4.2. Non-orientable hypersurfaces. In this section, we look at the area of non-orientable minimal hypersurfaces in M.

Proposition 14. *Let M be a closed orientable Riemannian $(n+1)$-manifold ($2 \leq n \leq 6$) with mean-convex boundary. Let Σ be a non-orientable hypersurface in M. Then there is a connected stable minimal hypersurface Σ' such that $|\Sigma'| \leq |\Sigma|$. Moreover, if Σ is not a stable minimal hypersurface then $|\Sigma'| < |\Sigma|$.*

Proof. Since M is orientable and Σ is non-orientable, Σ is not 2-sided. Thus Σ represents a non-vanishing element in $H_n(M, \mathbb{Z}/2\mathbb{Z})$. In the geometric measure theory setting, Σ can also be seen as a flat chain modulo 2 denoted $[\Sigma]$ (see 4.2.26 in [**9**]). We can then minimize the mass among all flat chains modulo 2 that are homologous to $[\Sigma]$. We then get a flat chain T modulo 2 which is homologous to $[\Sigma]$ and minimizes the mass. The support of T is then made of a finite union of disjoint smooth minimal hypersurfaces (the regularity theory for area-minimizing flat chains modulo 2 can be found in [**21**] Corollary 2.5 and Remark 1; it uses also Lemma 4.2 in [**20**]). Let Σ' be one of these minimal hypersurfaces; it could be orientable or not but in both cases the area-minimizing property of T implies that Σ' is stable.

If Σ is not a stable minimal hypersurface, it is clear that there is a hypersurface homologous to Σ with area strictly less that $|\Sigma|$; so $|\Sigma'| < |\Sigma|$. q.e.d.

As in the preceding section, we can open a manifold along a non-orientable hypersurface.

Proposition 15. *Let Σ be a connected closed non-orientable hypersurface in the interior of a manifold M with boundary. Then there is a manifold $\widetilde{M}$ with boundary with a particular boundary component $\widetilde{\Sigma}$ and a locally invertible smooth map $\varphi : \widetilde{M} \to M$ such that $\varphi : \widetilde{M}\backslash\widetilde{\Sigma} \to M\backslash\Sigma$ is a diffeomorphism and $\varphi : \widetilde{\Sigma} \to \Sigma$ is an orientable double cover of Σ.*

The proof is similar to the orientable case (Proposition 13, see also Proposition 3.7 in [**32**]).

Proof. As in the preceding subsection, we consider a complete metric on M. Let $\pi : \widetilde{\Sigma} \to \Sigma$ be an orientable double cover of Σ and let σ be the non-trivial deck transformation of π. π defines an immersion of $\widetilde{\Sigma}$ to M so we can consider ν a unit normal vector field along $\widetilde{\Sigma}$ we have $\nu(\sigma(p)) = -\nu(p)$. Let us consider the map $\Phi : \widetilde{\Sigma} \times [0, 2\varepsilon) \to M : (p,t) \mapsto \exp_{\pi(p)}(t\nu(p))$. We can chose ε so that Φ is a diffeomorphism from $\widetilde{\Sigma} \times (0, 2\varepsilon)$ to a tubular 2ε-neighborhood of Σ with Σ removed. Let M_ε be $M\backslash\Phi(\widetilde{\Sigma}\times[0,\varepsilon])$. We then define $\widetilde{M}$ as the quotient of the disjoint union of M_ε and $\widetilde{\Sigma} \times [0, 2\varepsilon)$ by the identifications $(p,t) \simeq \Phi(p,t) \in M_\varepsilon$ for (p,t) in $\widetilde{\Sigma} \times (\varepsilon, 2\varepsilon)$.

The map φ is then defined as the identity on M_ε and by Φ on $\widetilde{\Sigma} \times [0, 2\varepsilon)$. The map φ clearly satisfies the expected properties. q.e.d.

As an example, if M is $\mathbb{R}P^3$ and Σ is an equatorial $\mathbb{R}P^2$ then $\widetilde{M}$ is a hemisphere of $\mathbb{S}^3$ bounded by an equator $\widetilde{\Sigma}$.

4.3. The number $\mathcal{A}_\mathcal{S}$. Let M be a compact orientable Riemannian $(n+1)$-manifold with mean convex boundary ($2 \le n \le 6$). If M contains a non-orientable or non-separating hypersurface then Propositions 11 and 14 give the existence of some stable minimal hypersurface in M. So let us assume that M contains some stable minimal hypersurface, we define $\mathcal{O}_\mathcal{S}$ the collection of connected orientable stable minimal hypersurfaces and $\mathcal{U}_\mathcal{S}$ the collection of connected non-orientable stable minimal hypersurfaces. We then define

$$\mathcal{A}_\mathcal{S}(M) = \inf(\{|\Sigma|, \Sigma \in \mathcal{O}_\mathcal{S}\} \cup \{2|\Sigma|, \Sigma \in \mathcal{U}_\mathcal{S}\}).$$

This number is the "least area" of stable minimal hypersurfaces in M. If $\mathcal{O}_\mathcal{S} \cup \mathcal{U}_\mathcal{S} = \emptyset$, $\mathcal{A}_\mathcal{S}(M) = +\infty$.

The main result of this section is that this number is realized.

Proposition 16. *The number $\mathcal{A}_\mathcal{S}(M)$ is realized if it is finite: either there exists $\Sigma \in \mathcal{O}_\mathcal{S}$ such that $|\Sigma| = \mathcal{A}_\mathcal{S}(M)$ or $\Sigma \in \mathcal{U}_\mathcal{S}$ such that $2|\Sigma| = \mathcal{A}_\mathcal{S}(M)$.*

Proof. We can assume that there exists a sequence $(\Sigma_n)_{n\in\mathbb{N}}$ in $\mathcal{O}_{\mathcal{S}}$ (or in $\mathcal{U}_{\mathcal{S}}$) such that $|\Sigma_n| \to \mathcal{A}_{\mathcal{S}}(M)$ (or $2|\Sigma_n| \to \mathcal{A}_{\mathcal{S}}(M)$).

If the sequence is in $\mathcal{O}_{\mathcal{S}}$, this is a sequence of stable minimal hypersurfaces whose areas are uniformly bounded. By compactness results as Theorem 1.3 in [**7**] (see also [**27**]), a subsequence (Σ_{n_i}) converges as varifolds to a minimal hypersurface Σ and, locally, (Σ_{n_i}) converges smoothly as a multi-graph in the tubular neighborhood of Σ. As Σ_n is orientable and embedded, this implies that Σ_n is an entire graph over Σ if Σ is orientable or an entire two-sheeted graph if Σ is non-orientable. In the first case $|\Sigma| = \lim|\Sigma_n| = \mathcal{A}_{\mathcal{S}}(M)$ and, moreover, Σ is stable. In the second case, $\mathcal{A}_{\mathcal{S}}(M) \le 2|\Sigma| = \lim|\Sigma_n| = \mathcal{A}_{\mathcal{S}}(M)$, so $\mathcal{A}_{\mathcal{S}}(M) = 2|\Sigma|$ and Proposition 14 implies that Σ is stable.

If the sequence is in $\mathcal{U}_{\mathcal{S}}$, we can still apply the compactness result. Indeed, for any ball B of radius less than the injectivity radius of M, $\Sigma_n \cap B$ is orientable and stable in the 2-sided sense. In that case, $(\Sigma_n)_n$ converges to a non-oriented stable minimal hypersurface with multiplicity 1. We then have $\mathcal{A}_{\mathcal{S}}(M) \le 2|\Sigma| = \lim 2|\Sigma_n| = \mathcal{A}_{\mathcal{S}}(M)$, so $\mathcal{A}_{\mathcal{S}}(M) = 2|\Sigma|$. q.e.d.

5. Minimal hypersurfaces with area less than $\mathcal{A}_{\mathcal{S}}(M)$

In this section, we study minimal hypersurfaces whose areas are less than $\mathcal{A}_{\mathcal{S}}(M)$. Actually, we are going to prove that such a minimal hypersurface can be seen as the leaf of maximal area in some continuous sweep-out of the ambient manifold M.

Let Σ be a minimal hypersurface in M. If Σ is oriented and $|\Sigma| < \mathcal{A}_{\mathcal{S}}(M)$, Proposition 11 tells us that Σ separates M and it is unstable. If Σ is non-orientable, Proposition 14 implies that $2|\Sigma| \ge \mathcal{A}_{\mathcal{S}}(M)$. So we are going to look at orientable, unstable, separating minimal hypersurfaces.

Proposition 17. *Let M be a closed orientable Riemannian $(n+1)$-manifold ($2 \le n \le 6$). Let Σ be a connected oriented minimal hypersurface which is unstable and $|\Sigma| \le \mathcal{A}_{\mathcal{S}}(M)$. Then there is a continuous sweep-out $\{\Sigma_t\}_{t\in[-1,1]}$ of M such that $\Sigma_0 = \Sigma$, $\mathbf{L}(\{\Sigma_t\}) = |\Sigma|$ and, for any $\varepsilon > 0$, there is $\delta > 0$ such that $|\Sigma_t| \le |\Sigma| - \delta$ if $|t| \ge \varepsilon$.*

Moreover, if u_1 is the first eigenfunction of the Jacobi operator on Σ and ν is a unit normal vector field along Σ, the hypersurface Σ_t is given by $\Phi(\Sigma, t)$ for t close to zero where

$$\Phi : \Sigma \times \mathbb{R} \to M; (p,t) \mapsto \exp_p(tu_1(p)\nu(p)).$$

The proof of Proposition 17 consists in gluing together two continuous sweep-outs given by the following proposition:

Proposition 18. *Let M be a compact Riemannian $(n+1)$-manifold ($2 \le n \le 6$) with $\partial M = \Sigma$ connected, minimal and unstable. Moreover,* *we assume that* $|\Sigma| \le \mathcal{A}_{\mathcal{S}}(M)$*. Then there is a continuous sweep-out* $\{\Sigma_t\}_{t\in[0,1]}$ *of* M *such that* $\mathbf{L}(\{\Sigma_t\}) = |\Sigma|$ *and, for any* $\varepsilon > 0$*, there is* $\delta > 0$ *such that* $|\Sigma_t| \le |\Sigma| - \delta$ *if* $t \ge \varepsilon$*.*

Moreover, if u_1 *is the first eigenfunction of the Jacobi operator on* Σ *and* ν *is the inward unit normal vector field along* Σ*, the hypersurface* Σ_t *is given by* $\Phi(\Sigma, t)$ *for* t *close to zero where*

$$\Phi : \Sigma \times [0, \varepsilon] \to M; (p, t) \mapsto \exp_p(tu_1(p)\nu(p)).$$

Proof. Since Σ is unstable, the first eigenvalue λ_1 associated to u_1 is negative. u_1 is a positive function. For $\varepsilon > 0$ small enough, the map $\Phi : \Sigma \times [0, \varepsilon] \to M; (p, t) \mapsto \exp_p(tu_1(p)\nu(p))$ is well defined.

We then define $\Sigma_t = \Phi(\Sigma, t)$ and $M_t = M \setminus \Phi(\Sigma \times [0, t))$. If ε is chosen small enough, the family $\{\Sigma_t\}_{t\in[0,\varepsilon]}$ defines a foliation of a neighborhood of Σ and satisfies the property (sw3'). All the leaves Σ_t ($t > 0$) have non-vanishing mean curvature vector pointing towards M_t. Also $|\Sigma_t|$ decreases for t close to 0 and $|\Sigma_\varepsilon| \le |\Sigma| - \delta$ for some $\delta > 0$. So in order to construct the sweep-out announced in the proposition, it is sufficient to construct a sweep-out $\{\Sigma_t\}_{t\in[\varepsilon,1]}$ of M_ε such that $\mathbf{L}(\{\Sigma_t\}_{t\in[\varepsilon,1]}) \le |\Sigma| - \delta/2$: indeed, we can glue such a sweep-out with the foliation $\{\Sigma_t\}_{t\in[0,\varepsilon]}$ to produce the continuous sweep-out of M.

So let us assume by contradiction that any continuous sweep-out $\{\Sigma_t\}_{t\in[\varepsilon,1]}$ of M_ε satisfies $\mathbf{L}(\{\Sigma_t\}_{t\in[\varepsilon,1]}) \ge |\Sigma| - \delta/2 \ge |\Sigma_\varepsilon| + \delta/2$. Then the min–max theorem for manifolds with boundary (Theorem 5 or Theorem 2.7 in [**32**]) implies the existence of a connected minimal hypersurface S in M_ε. Let us now look at properties of this hypersurface S.

Claim 1. *The hypersurface* S *is orientable.*

If S is not orientable, we can consider the manifold $\widetilde{M_\varepsilon}$ constructed by opening M_ε along S by Proposition 15 with a map $\varphi : \widetilde{M_\varepsilon} \to M_\varepsilon$ and the induced metric. The boundary of $\widetilde{M_\varepsilon}$ has two connected components: one is $\widetilde{\Sigma}_\varepsilon$ which is isometric to Σ_ε and its mean curvature vector points into $\widetilde{M_\varepsilon}$ and the other is $\widetilde{S}$ which is a double cover of S and is minimal. Since S is not orientable and $\widetilde{S}$ is a double cover, Proposition 14 gives

$$|\widetilde{S}| = 2|S| \ge \mathcal{A}_{\mathcal{S}}(M) > |\Sigma_\varepsilon| = |\widetilde{\Sigma}_\varepsilon|. \tag{2}$$

Since the boundary of $\widetilde{M_\varepsilon}$ is mean convex and the homology class $[\widetilde{\Sigma}_\varepsilon]$ is non-zero in $H_n(\widetilde{M_\varepsilon})$, Proposition 11 applies. So there is a connected orientable stable minimal hypersurface S' in $\widetilde{M_\varepsilon}$ with area less than $|\widetilde{\Sigma}_\varepsilon| = |\Sigma_\varepsilon|$. S' could be equal to $\widetilde{S}$, but this would imply that $|\widetilde{\Sigma}_\varepsilon| > |\widetilde{S}|$ which is not the case by (2). Thus, S' is in the interior of $\widetilde{M_\varepsilon}$. Then $\varphi(S')$ is an embedded orientable stable minimal hypersurface in M_ε with $|\varphi(S')| \le |\Sigma_\varepsilon|$. We then have the following inequalities $|\mathcal{A}_{\mathcal{S}}(M)| \le |\varphi(S')| \le |\Sigma_\varepsilon| \le |\Sigma| - \delta \le |\mathcal{A}_{\mathcal{S}}(M)| - \delta$ which gives us a contradiction. Claim 1 is proved.

Claim 2. *The hypersurface S separates M_ε.*

If S does not separate, Proposition 11 produces a non-separating stable minimal hypersurface S' in M_ε (S' does not separate since it does not vanish in homology and M_ε has only one connected component). By Proposition 13, we have a manifold $\widetilde{M_\varepsilon}$ with three boundary components S'_1 and S'_2 isometric to S' and $\widetilde{\Sigma}_\varepsilon$ isometric to Σ_ε.

The argument is then similar to the one of Claim 1. Since the boundary of $\widetilde{M_\varepsilon}$ is mean convex, Proposition 11 applies to the homology class $[\widetilde{\Sigma}_\varepsilon]$ which is non-zero and gives a connected orientable stable minimal hypersurface S'' in $\widetilde{M_\varepsilon}$ whose area is less than $|\widetilde{\Sigma}_\varepsilon| = |\Sigma_\varepsilon|$. S'' could be equal to S'_i ($i = 1, 2$), but this would imply that $|\widetilde{\Sigma}_\varepsilon| > |S'_i| = |S'| \geq \mathcal{A}_\mathcal{S}(M)$ which is not the case. Thus S'' is in the interior of $\widetilde{M_\varepsilon}$. Then $\varphi(S'')$ is an embedded orientable stable minimal hypersurface in M_ε with $|\varphi(S'')| \leq |\Sigma_\varepsilon|$. We then have the following inequalities $|\mathcal{A}_\mathcal{S}(M)| \leq |\varphi(S'')| \leq |\Sigma_\varepsilon| \leq |\Sigma| - \delta \leq \mathcal{A}_\mathcal{S}(M) - \delta$ which gives us a contradiction. Claim 2 is proved.

Thus the hypersurface S is orientable and separates; let M' be the piece of M_ε whose boundary is made of S and Σ_ε. If $|S| \geq |\Sigma_\varepsilon|$, we can apply Proposition 11 to produce a stable minimal hypersurface S' in the interior of M with area less than $|\Sigma_\varepsilon|$ (we notice that S' cannot be equal to S since $|S'| < |\Sigma_\varepsilon|$). We get the contradiction $\mathcal{A}_\mathcal{S}(M) \leq |S'| < |\Sigma_\varepsilon| < \mathcal{A}_\mathcal{S}(M)$.

If $|S| < |\Sigma_\varepsilon|$, we have $|S| < \mathcal{A}_\mathcal{S}(M)$. Thus S is unstable and we can apply Proposition 11 to produce a stable minimal hypersurface S' in the interior of M with area less that $|S| < |\Sigma_\varepsilon|$ which still leads to a contradiction as above.

So we have proved that any minimal hypersurfaces S produced by the min–max theorem in M_ε leads to a contradiction; thus there is a continuous sweep-out as in the statement of Proposition 18. q.e.d.

Let us now give the proof of Proposition 17.

Proof of Proposition 17. Since Σ is unstable and $|\Sigma| \leq \mathcal{A}_\mathcal{S}(M)$, Σ separates. Let M_1 and M_2 be the two sides of Σ in M: $M = M_1 \cup M_2$ and $M_1 \cap M_2 = \Sigma$. Proposition 18 gives a continuous sweep-out $\{\Sigma_t^1\}_{t\in[0,1]}$ of M_1 and a continuous sweep-out $\{\Sigma_t^2\}_{t\in[0,1]}$ of M_2. We also have families $\{\Omega_t^1\}$ and $\{\Omega_t^2\}$ of open subdomains of M_1 and M_2.

Let us define $\{\Sigma_t\}_{t\in[-1,1]}$ and $\{\Omega_t\}_{t\in[-1,1]}$ by $\Sigma_t = \Sigma_{-t}^1$ and $\Omega_t = M_1 \setminus \overline{\Omega_{-t}^1}$ if $t \leq 0$ and $\Sigma_t = \Sigma_t^2$ and $\Omega_t = M_1 \cup \Omega_t^2$ if $t \geq 0$. $\{\Sigma\}_{t\in[-1,1]}$ is then a sweep-out which satisfies the properties stated in Proposition 17. q.e.d.

A consequence of Proposition 17 is the following estimate of the width of a manifold M:

Proposition 19. *Let M be a closed Riemannian $(n+1)$-manifold ($2 \le n \le 6$). Let Σ be an orientable minimal hypersurface which is unstable and $|\Sigma| \le \mathcal{A}_{\mathcal{S}}(M)$. Then the width of M satisfies $W_M \le |\Sigma|$.*

Proof. By Proposition 17, there is a continuous sweep-out $\{\Sigma_t\}_{t\in[-1,1]}$ of M with $\mathbf{L}(\{\Sigma_t\}) = |\Sigma|$. By Theorem 10, there is a discrete sweep-out $S \in \Pi_M$ with $\mathbf{L}(S) \le \mathbf{L}(\{\Sigma_t\}) = |\Sigma|$. Then $W_M \le |\Sigma|$. q.e.d.

6. Proof of Theorem A

This section is entirely devoted to the proof of Theorem A. The first step is to prove that $\mathcal{A}_1(M)$ is realized by some particular minimal hypersurfaces satisfying some properties. The second step consists in estimating the index of these particular minimal hypersurfaces. Let us just recall Theorem A.

Theorem A. *Let M be an oriented closed Riemannian $(n+1)$-manifold ($2 \le n \le 6$). Then $\mathcal{A}_1(M)$ is equal to one of the following possibilities:*

1) *$|\Sigma|$ where $\Sigma \in \mathcal{O}$ is a min–max hypersurface of M associated to the fundamental class of $H_{n+1}(M)$ and has index 1.*
2) *$|\Sigma|$ where $\Sigma \in \mathcal{O}$ is stable.*
3) *$2|\Sigma|$ where $\Sigma \in \mathcal{U}$ is stable and its orientable 2-sheeted cover has index 0 or 1; if the index is 1, $2|\Sigma| = W_M$.*

Moreover, if $\Sigma \in \mathcal{O}$ satisfies $|\Sigma| = \mathcal{A}_1(M)$, then Σ is of type 1 or 2 and if $\Sigma \in \mathcal{U}$ satisfies $2|\Sigma| = \mathcal{A}_1(M)$, then Σ is of type 3.

So we fix some closed orientable $(n+1)$-manifold ($2 \le n \le 6$) and we look at the number $\mathcal{A}_1(M)$.

6.1. $\mathcal{A}_1(M)$ is realized.

In this section, we prove that $\mathcal{A}_1(M)$ is realized either by a stable minimal hypersurface or by an orientable min–max hypersurface. We begin by a remark about the min–max hypersurfaces.

The Almgren–Pitts theory tells that the width W_M of the manifold is equal to $\sum_{i=1}^p n_i|S_i|$ where $S_1, \cdots, S_p$ is a finite collection of connected minimal hypersurfaces and $n_1, \cdots, n_p$ are integers (Theorem 9). The following proposition makes this writing more precise when $W_M \le \mathcal{A}_{\mathcal{S}}(M)$.

Proposition 20. *Let us consider a writing $W_M = \sum_{i=1}^p n_i|S_i|$ given by Theorem 9. If $W_M \le \mathcal{A}_{\mathcal{S}}(M)$ then*

- *either $W_M = |S_1|$ with $S_1 \in \mathcal{O}$,*
- *or $W_M = 2|S_1|$ with $S_1 \in \mathcal{U}$.*

Moreover, if $W_M < \mathcal{A}_{\mathcal{S}}(M)$, the second case is not possible.

Proof. We know $W_M = \sum_{i=1}^p n_i|S_i|$. Let us first assume that S_1 is an orientable minimal hypersurface. If S_1 is stable then $\mathcal{A}_{\mathcal{S}}(M) \le |S_1| \le$

$\sum_{i=1}^{p} n_i|S_i| = W_M \leq \mathcal{A}_{\mathcal{S}}(M)$. So we have equality in all the inequalities and $n_1 = 1$ and $p = 1$. If S_1 is unstable, we have $|S_1| \leq W_M \leq \mathcal{A}_{\mathcal{S}}(M)$ and, by Proposition 19, $W_M \leq |S_1| \leq \sum_{i=1}^{p} n_i|S_i| = W_M$ so $n_1 = 1$ and $p = 1$.

Let us now assume that S_1 is non-orientable, we then know by Proposition 6.1 in [**32**] that n_1 is at least 2. This implies that $\mathcal{A}_{\mathcal{S}}(M) \leq 2|S_1| \leq W_M \leq \mathcal{A}_{\mathcal{S}}(M)$ and then $n_1 = 2$ and $p = 1$. q.e.d.

The proof of Theorem A consists in proving that

$$\mathcal{A}_1(M) = \min(\mathcal{A}_{\mathcal{S}}(M), W_M). \tag{3}$$

Because of Propositions 16 and 20, the above inequality implies that $\mathcal{A}_1(M)$ is realized. So let us prove (3). By Proposition 16, $\mathcal{A}_{\mathcal{S}}(M)$ is realized (if it is finite); so assume that $\mathcal{A}_{\mathcal{S}}(M) > \mathcal{A}_1(M)$. By Propositions 11 and 14, it means that there is some orientable unstable minimal hypersurface Σ with $|\Sigma| < \mathcal{A}_{\mathcal{S}}(M)$. By Proposition 19, $\mathcal{A}_{\mathcal{S}}(M) > |\Sigma| \geq W_M$ so Proposition 20 applies and W_M is realized by a connected minimal hypersurface $\overline{S}$. We have then proved that any minimal hypersurface Σ with $|\Sigma| < \mathcal{A}_{\mathcal{S}}(M)$ is such that $|\overline{S}| = W_M \leq |\Sigma|$; so (3) is proved.

Now let us consider a minimal hypersurface Σ that realizes $\mathcal{A}_1(M)$ but not of type 2 or 3, *i.e.*, not stable. We want to prove that Σ is an orientable min–max hypersurface. By Proposition 14, Σ is orientable. By Proposition 17, there is a continuous sweep-out $\{\Sigma_t\}_{t\in[-1,1]}$ of M with $\Sigma_0 = \Sigma$ and $\mathbf{L}(\{\Sigma_t\}) = |\Sigma|$. By Theorem 10, there is a discrete sweep-out $S = \{\varphi_i\}$ associated to $\{\Sigma_t\}$ with $\mathbf{L}(S) \leq \mathbf{L}(\{\Sigma_t\})$. As a consequence, $W_M \leq \mathbf{L}(S) \leq \mathbf{L}(\{\Sigma_t\}) = |\Sigma| = \mathcal{A}_1(M) \leq W_M$; thus, S realizes the width of M. So there is a min–max sequence $\{\varphi_{i_j}(x_j)\}$ that converges in the varifold sense to a minimal hypersurface that realizes the width of M. We want to prove that Σ is this limit minimal hypersurface.

In order to use Remark 3, let us denote $\widetilde{\Phi}(t) = \Sigma_{-1+2t}$. We know that $\lim_j \mathbf{M}(\varphi_{i_j}(x_j)) = W_M = |\Sigma| = \mathbf{M}(\widetilde{\Phi}(1/2))$. So, because of (1) and the properties of the continuous sweep-out $\{\Sigma_t\}_t$, $x_j \to 1/2$. By Remark 3, this implies that $\varphi_{i_j}(x_j)$ converges to $\widetilde{\Phi}(1/2) = \Sigma$ in the flat topology. Since $|\Sigma| = \lim_j \mathbf{M}(\varphi_{i_j}(x_j))$, this implies that we also have convergence in the varifold sense. So Σ is the limit of a min–max sequence and then a min–max hypersurface.

In order to finish the proof of the Theorem, we still have to control the index of these hypersurfaces.

6.2. Index in the orientable case.

Let us now prove that a type 1 hypersurface has index 1 (see also [**16**]).

Let Σ be an orientable unstable minimal hypersurface with $|\Sigma| = W_M = \mathcal{A}_1(M)$. We want to prove that its index is at most 1. So let us assume it has index at least 2. We then denote by u_1 and u_2 the first two eigenfunctions of the Jacobi operator on Σ. By Proposition 17, there is a sweep-out $\{\Sigma_t\}_{t\in[-1,1]}$ of M such that $\Sigma_0=\Sigma$, $\mathbf{L}(\{\Sigma_t\})=|\Sigma|$ and $|\Sigma_t|\le|\Sigma|-\delta(\varepsilon)$ for any $|t|\ge\varepsilon$. Moreover, we have $\Sigma_t=\Phi(\Sigma,t)$ for t close to 0 where

$$\Phi:\Sigma\times\mathbb{R}\to M;(p,t)\mapsto\exp_p(tu_1(p)\nu(p)).$$

Let us change the definition of the map Φ by adding one variable and consider the new definition

$$\Phi:\Sigma\times\mathbb{R}\times\mathbb{R}\to M;(p,t,s)\mapsto\exp_p((tu_1(p)+su_2(p))\nu(p)).$$

For t and s small, we define $\Sigma_{t,s}=\Phi(\Sigma,t,s)$. These are embedded hypersurfaces living in a tubular neighborhood of Σ. The volume functional $A(t,s)=|\Sigma_{t,s}|$ is smooth for t,s small and its differential at $(0,0)$ vanishes since Σ is minimal. Its Hessian is negative definite since u_1 and u_2 are associated to negative eigenvalues of the Jacobi operator on Σ. So for ε small enough, we have $A(\varepsilon\sin\theta,\varepsilon\cos\theta)\le|\Sigma|-c\varepsilon^2$ for some $c>0$ and all $\theta\in\mathbb{R}$.

Let us define a new continuous sweep-out $\{\Sigma'_t\}_{t\in[-1,1]}$ of M by the following choices:

$$\Sigma'_t=\begin{cases}\Sigma_t \text{ if } t\le-\varepsilon,\\ \Sigma_{\varepsilon\sin\frac{t\pi}{2\varepsilon},\varepsilon\cos\frac{t\pi}{2\varepsilon}} \text{ if } -\varepsilon\le t\le\varepsilon,\\ \Sigma_t \text{ if } t\ge\varepsilon.\end{cases}$$

The family of open subsets $\{\Omega'_t\}_t$ associated to $\{\Sigma'_t\}_t$ can be adapted from the original family $\{\Omega_t\}_t$.

Because of the properties of the original sweep-out and the control on the function A, we see that $|\Sigma'_t|\le|\Sigma|-\delta$ for some $\delta>0$ and any $t\in[-1,1]$. By Theorem 10, there exists a discrete sweep-out $S\in\Pi_M$ with $\mathbf{L}(S)\le|\Sigma|-\delta$. This implies that $W_M\le|\Sigma|-\delta=W_M-\delta$ and gives a contradiction. So the index of Σ is at most 1.

6.3. Index in the non-orientable case. In this section, we control the index of the double cover of a type 3 non-orientable minimal hypersurface that realizes $\mathcal{A}_1(M)$. We want to prove that it has index at most 1.

Let Σ be a type 3 non-orientable minimal hypersurface. We thus have $2|\Sigma|=\mathcal{A}_1(M)\le W_M$. We open M along Σ by Proposition 15 and get $\varphi:\widetilde{M}\to M$ where $\varphi:\widetilde{\Sigma}=\partial\widetilde{M}\to\Sigma$ is a double cover. We lift the metric of M to $\widetilde{M}$. Let σ denote the non-trivial deck transformation of $\varphi:\widetilde{\Sigma}\to\Sigma$.

We assume that the Jacobi operator on $\widetilde{\Sigma}$ has index at least 2. We know that Σ is a stable minimal hypersurface means that the Jacobi operator on $\widetilde{\Sigma}$ is positive on the space of σ-odd functions. So the first two eigenfunctions u_1 and u_2 on $\widetilde{\Sigma}$ must be σ-even since their eigenvalues are negative. As a consequence, u_1 and u_2 can be seen as functions on Σ.

Since φ is a local isometry from the interior of $\widetilde{M}$ to $M\setminus\Sigma$, $\mathcal{A}_{\mathcal{S}}(\widetilde{M}) \geq \mathcal{A}_{\mathcal{S}}(M)$ and thus $|\widetilde{\Sigma}| = \mathcal{A}_1(M) \leq \mathcal{A}_{\mathcal{S}}(M) \leq \mathcal{A}_{\mathcal{S}}(\widetilde{M})$. Thus Proposition 18 gives a continuous sweep-out $\{\widetilde{\Sigma}_t\}_{t\in[0,1]}$ of $\widetilde{M}$ with $\widetilde{\Sigma}$ of maximum area.

If $\Phi : \widetilde{\Sigma}\times[0,\varepsilon] \to \widetilde{M}; (p,t) \mapsto \exp_p(tu_1(p)\tilde{\nu}(p))$ ($\tilde{\nu}$ the inward unit normal vector field to $\widetilde{\Sigma}$), we know that $\widetilde{\Sigma}_t = \Phi(\widetilde{\Sigma},t)$ for t close to 0. Moreover, for $t>0$, we have $\widetilde{\Sigma}_t = (\partial\widetilde{\Omega}_t \setminus \widetilde{\Sigma}) \cup \widetilde{P}_t$ where $\{\widetilde{\Omega}_t\}$ is a family of open subsets of $\widetilde{M}$ with $\widetilde{\Sigma}\subset\widetilde{\Omega}_t$ and $\{\widetilde{P}_t\}$ is a family of finite subsets.

Let us consider $\Omega_t = \varphi(\widetilde{\Omega}_t)$ and $P_t = \varphi(\widetilde{P}_t)$ for $t\in[0,1]$. We have $\Omega_0=\emptyset$ and, for $t>0$, Ω_t is a domain in M that contains Σ and whose boundary is $\Sigma_t\setminus P_t = \varphi(\widetilde{\Sigma}_t\setminus\widetilde{P}_t)$. For t close to 0, Ω_t is contained in a tubular neighborhood of Σ.

Let $N\Sigma$ be the normal bundle to Σ, it is a twisted line bundle over Σ. We notice that the map $\varphi : \widetilde{\Sigma}$ extends to a double cover $\pi : N\widetilde{\Sigma} \to N\Sigma$ where the normal bundle $N\widetilde{\Sigma}$ to $\widetilde{\Sigma}$ is trivial. For a non-negative function u on Σ, we can consider

$$N_u\Sigma = \{(p,n)\in N\Sigma \mid \|n\| < u(p)\}.$$

If $\varepsilon>0$, we have $N_\varepsilon\Sigma$ for the constant function $u\equiv\varepsilon$. The map $\Psi : N_\varepsilon\Sigma \to M; (p,n(p)) \mapsto \exp_p(n(p))$ is a diffeomorphism on the ε-tubular neighborhood of Σ when ε is small enough. For a continuous non-negative function u on Σ with $u\leq\varepsilon$, $D_u=\Psi(N_u\Sigma)$ is an open subset of the ε-tubular neighborhood of Σ. With this notation, if $0<t<\varepsilon'$ for ε' small enough, we have $\Omega_t = D_{tu_1}$ (here the σ-even function u_1 is seen as a function on Σ).

In order to construct a particular sweep-out, we are going to change the domains Ω_t for t small. Let ε' be such that $\varepsilon'(\|u_1\|_\infty + \|u_2\|_\infty) \leq \varepsilon$. Then if $t\leq\varepsilon'$ and $\theta\in\mathbb{R}$, the domain $O_{t,\theta} = D_{t(\cos\theta u_1+\sin\theta u_2)^+}$ (where $u^+=\max(0,u)$ denotes the positive part of u) is well defined and is included in the ε-tubular neighborhood D_ε of Σ.

Let us remark that u_2 does not have a fixed sign so $\cos\theta u_1+\sin\theta u_2$ can be negative somewhere and then Σ can be not included in $O_{t,\theta}$. The boundary of $O_{t,\theta}$ is included in an immersed hypersurface $S_{t,\theta}$ which is the image of $\{p, t(\cos\theta u_1(p)+\sin\theta u_2(p))\tilde{\nu}(p), p\in\widetilde{\Sigma}\} \in N\widetilde{\Sigma}$ by $\Psi\circ\pi$. This implies that $O_{t,\theta}$ is a domain with rectifiable boundary. Moreover, we can estimate $\mathcal{H}^n(\partial O_{t,\theta})$ in two different ways.

The first estimation is just the fact that $\partial O_{t,\theta}\subset S_{t,\theta}$ so

$$\mathcal{H}^n(\partial O_{t,\theta}) \leq |S_{t,\theta}| \tag{4}$$

($|S_{t,\theta}|$ computes the volume of an immersed hypersurface so multiplicities may appear.)

The second estimation uses the fact that $\cos\theta u_1 + \sin\theta u_2$ can be negative somewhere. So, in order to compute the $\mathcal{H}^n$-measure of $\partial O_{t,\theta}$, we just have to take care of the part of $S_{t,\theta}$ that correspond to point where $\cos\theta u_1 + \sin\theta u_2$ is positive. This implies that

$$\mathcal{H}^n(\partial O_{t,\theta}) \le 2\mathcal{H}^n(\{p\in\Sigma \mid \cos\theta u_1(p) + \sin\theta u_2(p) > 0\}) + ct, \tag{5}$$

for some constant c that does not depend on t and θ.

As in Section 6.2, the fact that u_1 and u_2 are eigenfunctions associated to negative eigenvalues of the Jacobi operator on $\widetilde{\Sigma}$ implies that there is some positive constant c' such that, for t small,

$$|S_{t,\theta}| \le |\widetilde{\Sigma}| - c't^2 = 2|\Sigma| - c't^2. \tag{6}$$

Let us define our particular "sweep-out". So choose some small $\eta > 0$ such that, for $0 < t < \eta$, the subdomains $O_{t,\theta}$ are well defined and the estimates (4), (5) and (6) are true. For $t\in[\eta,1]$, we define $\Omega'_t = \Omega_t$ and, for $t\in[-\pi/2+\eta,\eta]$, we define $\Omega'_t = O_{\eta,\eta-t}$ (both definitions coincide at $t=\eta$, see Figure 1). We then have $\Omega'_{-\pi/2+\eta} = O_{\eta,\pi/2}$. Finally, for $t\in[-\pi/2,-\pi/2+\eta]$, we define $\Omega'_t = O_{t+\pi/2,\pi/2}$, we notice that both definitions agree at $t=-\pi/2+\eta$ and $\Omega'_{-\pi/2}=\emptyset$. We notice that the family of open subsets $\{\Omega'_t\}_{t\in[-\pi/2,1]}$ satisfies (sw2) and (sw3).

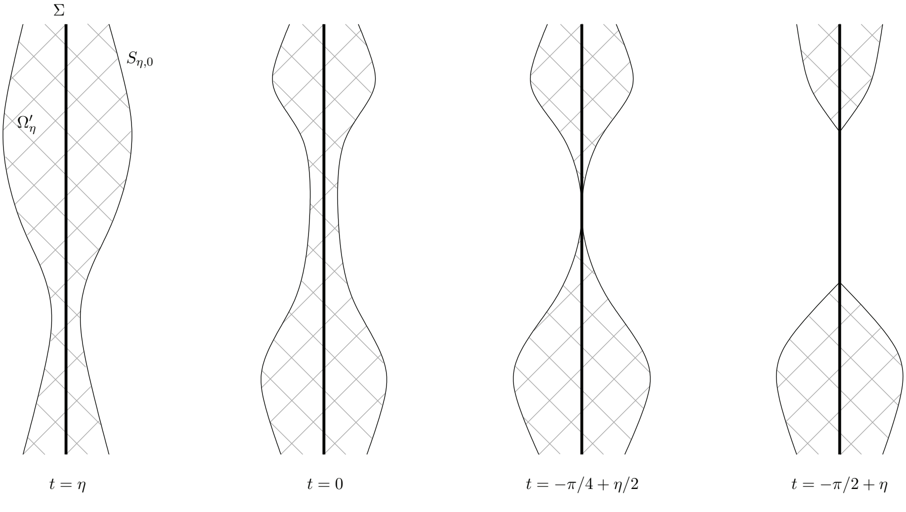

Figure 1. The evolution of Ω'_t for $t\in[-\pi/2+\eta,\eta]$.

Let us estimate the mass of $\partial[\Omega'_t]$ for $t\in[-\pi/2,1]$. If $t\ge\eta$, $\Omega'_t=\Omega_t$ so we know by Proposition 18 that there is δ such that

$$\mathbf{M}(\partial[\Omega'_t]) = |\partial\Omega_t| = |\Sigma_t| \le |\widetilde{\Sigma}| - \delta = 2|\Sigma| - \delta. \tag{7}$$

For $t\in[-\pi/2+\eta,\eta]$, we use (4) and (6) to obtain

$$\mathbf{M}(\partial[\Omega'_t]) = \mathcal{H}_n(\partial\Omega'_t) \le |S_{\eta,\eta-t}| \le 2|\Sigma| - c'\eta^2. \tag{8}$$

For $t \in [-\pi/2, -\pi/2 + \eta]$, we use (4), (5) and (6) to obtain

$$\begin{aligned} \mathbf{M}(\partial[\Omega'_t]) &= \mathcal{H}_n(\partial\Omega'_t) \\ &\le \min\{2|\Sigma| - c'(t+\pi/2)^2, 2\mathcal{H}^n(\{u_2 > 0\}) + c(t+\pi/2)\}. \end{aligned} \tag{9}$$

Since $\mathcal{H}^n(\{u_2 > 0\}) < |\Sigma|$, the estimates (7), (8) and (9) imply that there is some $\delta' > 0$ such that $\mathbf{M}(\partial[\Omega'_t]) \le 2|\Sigma| - \delta'$ for any $t \in [-\pi/2, 1]$. We can now apply Theorem 10 to obtain a discrete sweep-out $S \in \Pi_M$ with $\mathbf{L}(S) \le 2|\Sigma| - \delta'$ (the hypothesis on $\mathbf{m}(\Phi, r)$ is fulfilled since the supports of $\partial[\Omega'_t]$ are contained in a continuous family of immersed hypersurfaces so arguments in the proof of Proposition 5.1 in [**32**] apply, see also Remark 4). As a consequence, this implies that $W_M \le 2|\Sigma| - \delta'$ which contradicts that $2|\Sigma| \le W_M$. We have then proved that the Jacobi operator on $\widetilde{\Sigma}$ has index at most 1.

If Σ has index 1, as above, Σ_t and Ω_t can be constructed since they only depend on the first eigenvalue. By construction $\sup\{\mathbf{M}(\partial[\Omega_t]), t \in [0,1]\} = 2|\Sigma|$. So by Theorem 10, $W_M \le 2|\Sigma|$ and then $W_M = 2|\Sigma|$.

Remark 5. We just proved that $W_M = 2|\Sigma|$ not that Σ is a min–max hypersurface, *i.e.*, a varifold limit of a min–max sequence. With respect to the orientable case, the difference comes from the fact that, as currents, the limit is here 0. In fact, it seems possible, looking at the proof of Theorem 10 to control the support of the discrete sweep-out from the support of the continuous one and prove that actually Σ is a min–max hypersurface associated to the fundamental class of $H_{n+1}(M)$.

7. The 3-dimensional case

In this section, we give some improvements to Theorem A when the ambient manifold has dimension 3.

7.1. Some topology of 3-manifolds.

Let us recall some definitions about the topology of 3-manifolds.

A *compression body* is a 3-manifold B with boundary with a particular boundary component $\partial_+ B = \Sigma \times \{0\}$ such that B is obtained from $\Sigma \times [0,1]$ by attaching 2-handles and 3-handles, where no attachments are performed along $\partial_+ B = \Sigma \times \{0\}$ (see [**3**] for related definitions).

A compression body with only one boundary component, *i.e.*, $\partial B = \partial_+ B$, is called a *handlebody*. A handlebody can also be seen as a closed ball with 1-handles attached along the boundary.

Let M be a compact 3-manifold with maybe non-empty boundary. A separating orientable surface Σ in the interior of M is a *Heegaard splitting* if both sides of Σ are compression bodies B_1 and B_2 with $\partial_+ B_1 = \Sigma = \partial_+ B_2$. Let us notice that Heegaard splittings always exist. If M has no boundary B_1 and B_2 are handlebodies.

If M is a compact 3-manifold, the *Heegaard genus* of M (denoted by $g_H(M)$) is defined as the minimum of the genera of all surfaces that are Heegaard splittings.

In the following, we will use the following characterization of handlebodies which is due to Meeks, Simon and Yau (see Proposition 1 in [**19**]):

Proposition 21. *Let M be a compact Riemannian 3-manifold with one boundary component. M is a handlebody if and only if the isotopy class of a parallel surface to ∂M contains surfaces of arbitrary small area.*

Let M be a compact 3-manifold with boundary, a proper embedding of $\mathbb{S}^1 \times [0,1]$ is an incompressible annulus if the inclusion is π_1 injective (by proper we mean $\partial(\mathbb{S}^1 \times [0,1])$ is sent to ∂M).

7.2. An improvement in the 3-dimensional case. The improvement that we obtain in the 3-dimensional case is that we can control the genus of the min–max surfaces that appear in Theorem A. So we have the following result:

Theorem B. *Let M be an oriented closed Riemannian 3-manifold. Then $\mathcal{A}_1(M)$ is equal to one of the following possibilities:*

1) *$|\Sigma|$ where $\Sigma \in \mathcal{O}$ is a min–max surface of M associated to the fundamental class of $H_3(M)$, Σ has index 1 and $g_\Sigma \geq g_H(M)$.*
2) *$|\Sigma|$ where $\Sigma \in \mathcal{O}$ is stable.*
3) *$2|\Sigma|$ where $\Sigma \in \mathcal{U}$ is stable and its orientable 2-sheeted cover has index 0 or 1. Moreover, if the double cover $\widetilde{\Sigma}$ has index 1, we have $g_{\widetilde{\Sigma}} \geq g_H(M) - 1$ and $W_M = 2|\Sigma|$.*

Moreover, if $\Sigma \in \mathcal{O}$ satisfies $|\Sigma| = \mathcal{A}_1(M)$, then Σ is of type 1 or 2 and if $\Sigma \in \mathcal{U}$ satisfies $2|\Sigma| = \mathcal{A}_1(M)$, then Σ is of type 3.

The proof is based on the following lemma where we use ideas similar to the proof of Proposition 18.

Lemma 22. *Let M be a compact Riemannian 3-manifold with $\partial M = \Sigma$ connected, minimal and unstable. Moreover, we assume that $|\Sigma| \leq \mathcal{A}_\mathcal{S}(M)$. Then M is a handlebody.*

Proof. Since Σ is unstable, using the notations of the proof of Proposition 18, the manifold $M_t = M \setminus \Phi(\Sigma \times [0,t)])$ has mean convex boundary for t small. Let t, t_0 be small with $t < t_0$ and look at the quantity

$$A = \inf\{|S|, S \text{ isotopic to } \Sigma_{t_0} \text{ in } M_t\}.$$

If $A = 0$, then M_t and thus M are handlebodies by Proposition 21. If $A \neq 0$, A is realized by a union of stable minimal surfaces with multiplicities (see Theorem 1' in [**19**]). Let S be one of these stable minimal surfaces. If $S \in \mathcal{O}$, then $\mathcal{A}_\mathcal{S}(M) \leq |S| \leq |\Sigma_{t_0}| < |\Sigma| \leq \mathcal{A}_\mathcal{S}(M)$

which is a contradiction. If $S \in \mathcal{U}$, Theorem 1' in [**19**] tells that its multiplicity is at least 2, so the same contradiction as above occurs. So we have $A = 0$ and M is a handlebody. q.e.d.

Proof of Theorem B. The only thing we need is the control on the genus of the surface Σ.

Let Σ be a type 1 surface. So Σ is a non-stable minimal surface and $|\Sigma| = \mathcal{A}_1(M) \leq \mathcal{A}_{\mathcal{S}}(M)$. By Proposition 11, Σ separates M and, by Lemma 22, both sides of Σ in M are handlebodies. Σ is then a Heegaard splitting and then $g_\Sigma \geq g_H(M)$.

Let Σ be a type 3 surface whose double cover $\widetilde{\Sigma}$ is not stable. Let us open M along Σ (Proposition 15) to obtain a 3-manifold $\widetilde{M}$ with boundary $\widetilde{\Sigma}$. Since Σ realizes $\mathcal{A}_1(M)$ we have $|\widetilde{\Sigma}| \leq \mathcal{A}_{\mathcal{S}}(\widetilde{M})$. By Lemma 22, $\widetilde{M}$ is a handlebody. So M can be seen as a handlebody where points on the boundary are identified through a fixed point free involution that reverses the orientation. Actually, it is possible to control the Heegaard genus of M in terms of the genus of $\widetilde{\Sigma}$: there is an argument attributed to Rubinstein by Shalen (see 4.5 in [**28**]) which implies that $g_H(M) \leq g_{\widetilde{\Sigma}}+1$. The argument works as follows. Let M_ε be the outside of an ε-tubular neighborhood of Σ. Since $\widetilde{M}$ is a handlebody, M_ε is also a handlebody. Choose a point p on Σ and consider γ the normal geodesic to Σ with length 2ε and p as middle point. The end points of γ are in ∂M_ε. Let H be the union of M_ε with a small tubular neighborhood of γ. H can be seen as M_ε to which a 1-handle is attached so it is a handlebody. In fact, the complement of H is also a handlebody since the complement of a point in a closed surface continuously retracts to a bouquet of circles. Now the genus of ∂H is just $g_{\widetilde{\Sigma}} + 1$. q.e.d.

8. Minimal surfaces in hyperbolic 3-manifolds

In this section, we prove a lower bound for the area of minimal surfaces in hyperbolic 3-manifolds.

8.1. Area and genus. In a hyperbolic 3-manifold, the area of a minimal surface Σ is always bounded above by its topology, we have $|\Sigma| \leq -2\pi\chi(\Sigma)$ (it is a classical consequence of the Gauss and Gauss–Bonnet formulas). If its index is at most 1, we can also obtain a lower bound in terms of its genus.

Lemma 23. *Let Σ be an immersed orientable closed minimal surface in an oriented hyperbolic 3-manifold.*

- *If Σ is stable, then $|\Sigma| \geq \pi|\chi(\Sigma)| = 2\pi(g_\Sigma - 1)$.*
- *If Σ has index 1, then $|\Sigma| \geq 2\pi\Big(g_\Sigma - 2 - \Big[\frac{g_\Sigma+1}{2}\Big]\Big)$.*

The first estimate tells that the area of an orientable stable minimal surface is well controlled by its topology $\pi|\chi(\Sigma)| \leq |\Sigma| \leq 2\pi|\chi(\Sigma)|$.

This estimate was observed by K. Uhlenbeck but not published (see Hass [**11**]). The second estimate can be derived from Proposition 3.1 in [**8**] by El Soufi and Ilias but we prefer to give a proof here for the sake of completeness.

Proof. If Σ is stable, we can use the constant function 1 as a test function in the stability operator and obtain

$$\int_\Sigma -(\mathrm{Ric}(\nu,\nu)+\|A\|^2)\geq 0.$$

The Gauss formula implies that $\|A\|^2=-2(K_\Sigma+1)$ with K_Σ the sectional curvature of Σ. So, using the Gauss–Bonnet formula, we obtain

$$|\Sigma|\geq -\frac{1}{2}\int_\Sigma K_\Sigma=-\pi\chi(\Sigma)=2\pi(g_\Sigma-1).$$

The study of the index 1 case is based on what is called the Hersch trick. Let u_1 be the first eigenfunction of the Jacobi operator on Σ. Let φ be a conformal map from Σ to $\mathbb{S}^2\subset\mathbb{R}^3$ and look at the following integral:

$$\int_\Sigma u_1(p)\times h\circ\varphi(p)dp\in\mathbb{R}^3,$$

where h is a Möbius transformation of $\mathbb{S}^2$. Since u_1 is non-negative, we can find h such that the above integral vanishes (see [**15**]). Let (f_1,f_2,f_3) be the three coordinates of $h\circ\varphi$. f_i is then orthogonal to u_1 and Σ has index 1, so

$$\int_\Sigma \|\nabla f_i\|^2-(\mathrm{Ric}(\nu,\nu)+\|A\|^2)f_i^2\geq 0.$$

Summing these three inequalities and using that $h\circ\varphi$ is conformal we get

$$\begin{aligned}0&\leq\int_\Sigma\|\nabla h\circ\varphi\|^2-(\mathrm{Ric}(\nu,\nu)+\|A\|^2)\\&=8\pi\deg(h\circ\varphi)-\int_\Sigma(\mathrm{Ric}(\nu,\nu)+\|A\|^2)\\&=8\pi\deg(\varphi)-\int_\Sigma(\mathrm{Ric}(\nu,\nu)+\|A\|^2).\end{aligned}$$

As in [**26**], we can choose φ such that $\deg(\varphi)\leq 1+\left[\frac{g_\Sigma+1}{2}\right]$. So computations similar to the stable case give

$$|\Sigma|\geq 2\pi\Big(-1-\left[\frac{g_\Sigma+1}{2}\right]\Big)-\frac{1}{2}\int_\Sigma K_\Sigma=2\pi\Big(g_\Sigma-2-\left[\frac{g_\Sigma+1}{2}\right]\Big).$$

q.e.d.

We remark that in the above proof we only use the fact that the sectional curvature of the ambient manifold is bounded below by -1.

We can also remark that, in the stable case, the equality cannot occur. Indeed, if $|\Sigma| = 2\pi(g_\Sigma - 1)$, the proof tells that the constant function 1 is in the kernel of the Jacobi operator so $\mathrm{Ric}(\nu,\nu) + \|A\|^2 = 0$ and then $K_\Sigma = -2$. So the lift of Σ to $\mathbb{H}^3$ gives a complete immersion with constant sectional curvature -2 which is not possible by Theorem 12 in [**10**].

8.2. The compact case. We can now state our lower bound for the area of minimal surfaces in hyperbolic 3-manifolds.

Theorem C. *Let M be a closed orientable hyperbolic 3-manifold. If $g_H(M) \geq 7$ then $\mathcal{A}_1(M) \geq 2\pi$. In other words, any orientable minimal surface in M has area at least 2π and any non-orientable minimal surface has area at least π.*

Proof. Since M has negative sectional curvature, any immersed closed minimal surface in M has negative Euler characteristic. By Theorem B, $\mathcal{A}_1(M)$ is realized by some minimal surface Σ.

If Σ is of type 2, Lemma 23 gives $|\Sigma| \geq 2\pi(g_\Sigma - 1) \geq 2\pi$ since Σ has negative Euler characteristic.

If Σ is of type 1, Lemma 23 gives

$$|\Sigma| \geq 2\pi\Big(g_\Sigma - 2 - \Big[\frac{g_\Sigma + 1}{2}\Big]\Big) \geq 2\pi\Big(g_H(M) - 2 - \Big[\frac{g_H(M) + 1}{2}\Big]\Big) \geq 2\pi.$$

If Σ is of type 3, let $\widetilde{\Sigma}$ be its orientable double cover. If $\widetilde{\Sigma}$ is stable, we get $2|\Sigma| = |\widetilde{\Sigma}| \geq 2\pi$ as above. If $\widetilde{\Sigma}$ has index 1, Theorem B gives us $g_{\widetilde{\Sigma}} \geq g_H(M) - 1$ and we have

$$\begin{aligned} 2|\Sigma| = |\widetilde{\Sigma}| &\geq 2\pi\Big(g_{\widetilde{\Sigma}} - 2 - \Big[\frac{g_{\widetilde{\Sigma}} + 1}{2}\Big]\Big) \\ &\geq 2\pi\Big(g_H(M) - 3 - \Big[\frac{g_H(M)}{2}\Big]\Big) \geq 2\pi. \end{aligned}$$

So in all cases, we have $\mathcal{A}_1(M) \geq 2\pi$. q.e.d.

Remark 6. If we know that there is no non-orientable surface in M, then the conclusion of the above theorem is true if we only assume $g_H(M) \geq 6$.

We can also remark that the same result is true if we only assume that the sectional curvature of M satisfies $-1 \leq K_M < 0$.

We also notice that the existence of hyperbolic 3-manifolds with arbitrarily large Heegaard genus is given by a result of Souto (see Theorem 4.1 in [**31**] and [**23**]).

If the hypothesis on the Heegaard genus is dropped, the monotonicity formula and the thin-thick decomposition of M tells us that any minimal surface in a closed hyperbolic 3-manifold has area at least some $c > 0$ that does not depend on M (see [**5**]) (this is also true for closed immersed

H-surfaces with $H < 1$). So this leads us to ask: is there a closed orientable hyperbolic 3-manifold M that minimizes $\mathcal{A}_1(M)$ among such 3-manifolds? What is a minimal surface that realizes this $\mathcal{A}_1(M)$ in M? We ask the same question for properly embedded minimal surfaces in complete hyperbolic 3-manifolds of finite volume M (see the following section). We believe an answer is a Seifert surface (a once punctured torus) of the figure eight knot, made minimal in the hyperbolic structure of the complement of the figure eight knot.

8.3. The finite volume case. In this section, we extend Theorem C to the case where M is a complete non-compact hyperbolic 3-manifold with finite volume. Notice that such a manifold has closed minimal surfaces (see [**5**]).

If M is such a manifold, M is diffeomorphic to the interior of a compact manifold $\overline{M}$ with boundary whose boundary components are tori. Moreover, each end E of M can be isometrically parametrized by N_{v_1,v_2}, the quotient of $\{(x,y,z)\in\mathbb{R}^2\times\mathbb{R}_+^*, z\ge 1/2\}$ by the group generated by the translations by the independent horizontal vectors v_1 and v_2, endowed with the Riemannian metric

$$g_{\mathbb{H}} = \frac{1}{z^2}(dx^2+dy^2+dz^2).$$

We notice that the z coordinate is well defined on N_{v_1,v_2}.

In the following, we denote $\Lambda(E) = \Lambda(N_{v_1,v_2}) = \max(\|v_1\|,\|v_2\|)$ ($\|\cdot\|$ the Euclidean norm) and we notice that by parameterizing a smaller part of E we can always choose a chart with $\Lambda(E)$ as small as we want.

We will use other metrics on N_{v_1,v_2} to change the metric on M. More precisely, we will use the following metric:

$$g_\Psi = \frac{1}{\Psi^2(z)}(dx^2+dy^2+dz^2),$$

where Ψ is a function satisfying

- $\Psi(z) = z$ on $[1/2, 1]$,
- Ψ is non-decreasing.

The first condition means that this metric can be glued to the original hyperbolic metric. The second one gives that the foliation by the tori $T(c) = \{z = c\}$ has a mean curvature vector pointing in the ∂_z direction.

In [**5**], Collin, Hauswirth and the authors proved the following result:

Proposition 24. *Let* $t_0 \in (0, 1/2)$ *and* Ψ *be as above. There is a* $\Lambda_0 = \Lambda(t_0,\Psi)$ *such that if* $\Lambda(N_{v_1,v_2}) \le \Lambda_0$ *and* Σ *is a compact embedded minimal surface in* (N_{v_1,v_2}, g_Ψ) *with* $\partial\Sigma \in T(1-t_0)$ *then* $\Sigma \subset \{z \le 1\}$.

As said above, in a finite volume hyperbolic 3-manifold M, we can choose a chart N_{v_1,v_2} of each end E with $\Lambda(E) \le \Lambda(1/3, z\mapsto z)$. The above proposition says that any compact minimal surface in M never enters in $\{z > 1\}$ inside the ends. Thus all compact minimal surfaces in M stay in a compact piece of M; this compact piece will be denoted $C(M)$. In the following, all modifications on M will be made outside of $C(M)$.

We need a topological property of M.

Lemma 25. *Let M be a complete non-compact hyperbolic 3-manifold with finite volume; M is the interior of some manifold $\overline{M}$. $\overline{M}$ has no incompressible annulus.*

Proof. Let $E_1, \dots, E_p$ be the ends of M and $N_1, \dots, N_p$ the associated charts with function z_i on each end. Let Ψ be a function as above with $\Psi'' \le 0$ and $\Psi'(2) = 0$ and $\Psi'(t) > 0$ for $t < 2$. This implies that g_Ψ has negative sectional curvature on $\{1/2 < z < 2\}$ and $T(2)$ is totally geodesic. We endow each E_i with this new metric and we cut $\{z_i > 2\}$ for each end. We get a manifold diffeomorphic to $\overline{M}$ with a Riemannian metric with negative sectional curvature on the inside and totally geodesic boundary. This defines a metric on $\overline{M}$.

Let A be an incompressible annulus in $\overline{M}$ endowed with the above metric; we can deform it isotopically such that its boundary consists of geodesic circles in $\partial\overline{M}$. By Theorem 6.12 in [**12**], there is a minimal surface S isotopic to A with the same boundary. Since the boundary of $\overline{M}$ is totally geodesic, ∂S is geodesic inside S. By the Gauss formula $K_S \le K_{\overline{M}} < 0$. So the Gauss–Bonnet formula $0 = 2\pi\chi_S = \int_S K_S < 0$ gives a contradiction. q.e.d.

Let $\overline{M}$ be a compact 3-manifold whose boundary components are tori. Let T be one of these tori. By fixing a basis of the homology of T, we define a chart on $T \simeq \mathbb{S}^1 \times \mathbb{S}^1$ such that the basis of the homology is $(\mathbb{S}^1 \times \{p\}, \{q\} \times \mathbb{S}^1)$. Let $\mathbb{S}^1 \times D$ (D the unit disk) be a solid torus, we then can glue $\overline{M}$ and the solid torus by identifying the boundary using the chart on T. The topology of this Dehn filling depends on the choice of the homology basis we made. By making Dehn filling on each boundary component of $\overline{M}$, we get a closed manifold $D(\overline{M})$. One can easily see that, concerning the Heegaard genus, we have the following inequality $g_H(D(\overline{M})) \le g_H(\overline{M})$.

If $\overline{M}$ comes from a complete non-compact hyperbolic 3-manifold M with finite volume, Rieck and Sedgwick [**25**] proved that the Dehn fillings can always be done (by choosing particular homology basis) such that $g_H(D(\overline{M})) = g_H(\overline{M})$ (the acylindrical hypothesis in their theorem is satisfied because of Lemma 25) (see also Moriah and Rubinstein [**22**]).

We can now state our result concerning finite volume hyperbolic 3-manifolds.

Theorem 26. *Let M be a complete non-compact hyperbolic 3-manifold with finite volume, we denote by $\overline{M}$ the associated compact 3-manifold with boundary. If $g_H(\overline{M}) \ge 7$, then any closed orientable* *minimal surface in M has area at least 2π and any closed non-orientable minimal surface has area at least π.*

The proof is based on ideas that appear in [**5**].

Proof. Let $T_1, \dots, T_p$ be the boundary tori of $\overline{M}$, because of the above discussion, there are bases of the homology of $T_1, \dots, T_p$ such that the associated Dehn filling $D(\overline{M})$ has the same Heegaard genus as $\overline{M}$.

We are going to construct some Riemannian metric on $D(\overline{M})$ to estimate the areas of minimal surfaces in M. Let Ψ be a function on $[1/2, \infty)$ such that

- $\Psi(z) = z$ on $[1/2, 1]$,
- $\Psi'(z) > 0$,
- $\lim_\infty \Psi = 2$.

Let $C(M)$ be the compact part of M that contains all compact minimal surfaces in M. Now, for each end E_i, we can find a chart $N_{v_1^i, v_2^i}$ such that $E_i \cap C(M) = \emptyset$, $\Lambda(E_i) < \Lambda_0$ where Λ_0 is given by Proposition 24 for g_Ψ and $t_0 = 1/3$. We also assume that the curves $t \mapsto (tv_1^i, 1)$ and $t \mapsto (tv_2^i, 1)$ in $T_i(1)$ give the homology basis of T_i that we have fixed above.

Let us fix some large $L > 0$, we are going to change the metric on $\{L \le z_i \le L + 1\}$ in order to perform the Dehn filling. The tori $T_i(c)$ is parametrized by $(u\frac{v_1^i}{2\pi} + v\frac{v_2^i}{2\pi})$ where $(u, v) \in \mathbb{S}^1 \times \mathbb{S}^1$. Then the metric g_Ψ on $N_{v_1^i, v_2^i}$ can be written

$$g_\Psi = \frac{1}{\Psi^2(z_i)}(a^2 \mathrm{d}u^2 + 2b\mathrm{d}u\mathrm{d}v + c^2\mathrm{d}v^2 + \mathrm{d}z_i^2),$$

for some $a, b, c \in \mathbb{R}$. Let η be a smooth non-increasing function on $[L, L + 1]$ such that $\eta(z) = 1$ near L and $\eta(z) = ((L + 1) - z)/a$ near $L + 1$. We then change the metric on $\{L \le z_i \le L + 1\}$ by

$$\frac{1}{\Psi^2(z_i)}(\mathrm{d}z_i^2 + \eta^2(z_i)a^2\mathrm{d}u^2 + 2\eta(z_i)b\mathrm{d}u\mathrm{d}v + c^2\mathrm{d}v^2).$$

This new metric is singular at $z_i = L + 1$ in the u direction but it is not if we make the identification $(u, v, L + 1) \sim (u', v, L + 1)$ for any u, u'. To see this, Let $(r, \theta) \in [0, 1] \times \mathbb{S}^1$ be the polar coordinates on the unit disk and h be the map $D \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1 \times [L, L + 1]$ defined by $(r, \theta, v) \mapsto (\theta, v, L + 1 - r)$. The induced metric by h on $D \times \mathbb{S}^1$ near $r = 0$ is then

$$\frac{1}{\Psi^2(L + 1 - r)}(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + 2\frac{b}{a}r\mathrm{d}\theta\mathrm{d}v + c^2\mathrm{d}v^2),$$

which is well defined on $D \times \mathbb{S}^1$. Actually, it consists in cutting $\{z_i \ge L\}$ from the end E_i and gluing a solid torus along $T(L)$. The map h tells us that we have performed the Dehn filling we want. We also notice that the tori $T_i(c) = \{z_i = c\}$ have positive mean curvature with respect to ∂_{z_i} for $L \leq c < L + 1$.

Once all Dehn fillings are done, we have constructed a metric on $D(\overline{M})$ (it depends on the parameter L that we need to adjust). Let us study the area of minimal surfaces in $D(\overline{M})$ with that metric. Let Σ be a minimal surface in $D(\overline{M})$, first it can stay outside of all the $\{z_i \geq 1\}$, these correspond to minimal surfaces living in the original hyperbolic part of $D(\overline{M})$ so in M. These surfaces are the ones whose areas we wish to bound from below. Since the foliation $\{T_i(c)\}_{c\in[1,L+1)}$ is mean convex with respect to ∂_{z_i}, there is no minimal surface inside an end $\{z_i \geq 1\}$. Proposition 24 tells us that a minimal surface that intersects $\{z_i \geq 1/2\}$ but does not reach $T_i(L)$ never enters into $\{z_i > 1\}$. So it stays in the original hyperbolic part. So a minimal surface that meets $\{z_i = 1\}$ meets necessarily $\{z_i = L\}$. Thus it meets all tori $T_i(c)$ for $1 \leq c \leq L$. Since $\lim_\infty \Psi = 2$, for large z_i the metric g_Ψ is close to the Euclidean metric and then there is some constant k that does not depend on L such that

$$|\Sigma \cap \{1 \leq z_i \leq L\}| \geq kL.$$

So if L is chosen large, the area of Σ is large. More precisely, we choose L such that $kL \geq \mathcal{A}_1(M) + 1$.

Since $C(M)$ is isometrically contained in $D(\overline{M})$, we have $\mathcal{A}_1(M) \geq \mathcal{A}_1(D(\overline{M}))$. The above discussion implies that any minimal surface Σ in $D(\overline{M})$ either is contained in $C(M)$ or has area $|\Sigma| \geq kL \geq \mathcal{A}_1(M) + 1$. So $\mathcal{A}_1(M) = \mathcal{A}_1(D(\overline{M})$. Moreover, $\mathcal{A}_1(D(\overline{M}))$ is realized by a minimal surface contained in $C(M)$ where the metric is hyperbolic and where we can apply the same reasoning as in the proof of Theorem C and using the fact that $g_H(D(\overline{M})) = g_H(\overline{M}) \geq 7$. q.e.d.

In a finite volume hyperbolic 3-manifold, it is also interesting to find a good lower bound of the area of non-compact minimal surfaces. We notice that in this case the estimates $\pi|\chi(\Sigma)| \leq |\Sigma| \leq 2\pi|\chi(\Sigma)|$ are still valid for properly embedded stable minimal surfaces (see [**5**]).

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