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UNIFORM HYPERBOLICITY OF INVARIANT CYLINDER

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Abstract

For a positive definite Hamiltonian system $H = h(p) + \epsilon P(p,q)$ with $(p,q) \in \mathbb{R}^3 \times \mathbb{T}^3$, large normally hyperbolic invariant cylinders exist along the whole resonant path, except for the $\epsilon^{\frac{1}{2}+d}$-neighborhood of finitely many double resonant points. It allows one to construct diffusion orbits to cross double resonance.

1. Introduction and the main result

In this paper, we study small perturbations of integrable Hamilton systems with three degrees of freedom

$$H(p,q) = h(p) + \epsilon P(p,q), \qquad (p,q) \in \mathbb{R}^3 \times \mathbb{T}^3, \tag{1.1}$$

where $\partial^2 h(p)$ is positive definite, both h and P are C^r-functions with $r \geq 6$. In the energy level set $H^{-1}(E)$ with $E > \min h$, we search for invariant cylinders along resonant path. An irreducible integer vector $k' \in \mathbb{Z}^3 \backslash \{0\}$ determines a resonant path

$$\Gamma' = \{p \in h^{-1}(E) : \langle \partial h(p), k' \rangle = 0\}.$$

A point $p'' \in \Gamma'$ is called double resonant if $\exists$ another irreducible vector $k'' \in \mathbb{Z}^3 \backslash \{0\}$, independent of k', such that $\langle k'', \partial h(p'') \rangle = 0$ holds as well. There are infinitely many double resonant points, but only strong double resonance causes trouble. A double resonance is called strong if $|k''|$ is not so large.

To make things simpler we introduce a symplectic coordinate transformation

$$\mathfrak{M}: \qquad u = M^t q, \qquad v = M^{-t} p,$$

where the matrix is made up by three integer vectors $M = (k'', k', k_3)$. As both k' and k'' are irreducible, $\exists\ k_3 \in \mathbb{Z}^3$ such that $\det M = 1$. There are infinitely many choices for k_3, we choose that k_3 so that $|k_3|$ is the smallest one. For simplicity of notation, we assume that the Hamiltonian of (1.1) is already under such transformation and denote the canonical coordinates by (p,q) still. So we have $\partial h(p'') = (0, 0, \omega_3)$.

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To get a normal form around a double resonance, we introduce a coordinate transformation $\Phi_{\epsilon F}$ which is defined as the time-2π-map $\Phi_{\epsilon F}=\Phi^t_{\epsilon F}|_{t=2\pi}$ of the Hamiltonian flow generated by the function $\epsilon F(p,q)$. This function solves the homological equation

$$\Big\langle\frac{\partial h}{\partial p}(p''),\frac{\partial F}{\partial q}\Big\rangle=-P(p,q)+Z(p,q), \tag{1.2}$$

where

$$Z(p,q)=\sum_{\ell\in\mathbb{Z}^3,\ell_3=0}P_\ell(p)e^{i(\ell_1q_1+\ell_2q_2)},$$

in which P_ℓ represents the Fourier coefficient of P, $\ell=(\ell_1,\ell_2,\ell_3)$. Expanding F into Fourier series and comparing both sides of the equation we obtain

$$F(p,q)=\sum_{\ell\in\mathbb{Z}^3,\ell_3\neq0}\frac{iP_\ell(p)}{\langle\ell,\partial h(p'')\rangle}e^{i\langle\ell,q\rangle}.$$

Under the transformation $\Phi_{\epsilon F}$ we obtain a new Hamiltonian

$$\begin{aligned}\Phi^*_{\epsilon F}H=&h(p)+\epsilon Z(p,q)+\epsilon\Big\langle\frac{\partial h}{\partial p}(p)-\frac{\partial h}{\partial p}(p''),\frac{\partial F}{\partial q}\Big\rangle\\&+\frac{\epsilon^2}{2}\int_0^1(1-t)\{\{H,F\},F\}\circ\Phi^t_{\epsilon F}dt.\end{aligned}$$

To solve Equation (1.2), we do not have the problem of small divisor because $|\langle\ell,\partial h(p'')\rangle|=|\ell_3\omega_3|$, where $\omega_3=\partial_3h(p'')\neq0$ since $h(p'')>\min h$.

The function $\Phi^*_{\epsilon F}H(p,q)$ determines its Hamiltonian equation

$$\frac{dq}{dt}=\frac{\partial}{\partial p}\Phi^*_{\epsilon F}H,\qquad\frac{dp}{dt}=-\frac{\partial}{\partial q}\Phi^*_{\epsilon F}H. \tag{1.3}$$

For this equation we introduce another transformation (call it homogenization)

$$\tilde G_\epsilon=\frac{1}{\epsilon}\Phi^*_{\epsilon F}H,\qquad\tilde y=\frac{1}{\sqrt{\epsilon}}\Big(p-p''\Big),\qquad\tilde x=q,\qquad s=\sqrt{\epsilon}t,$$

with $\tilde x=(x,x_3)$, $\tilde y=(y,y_3)$, $x=(x_1,x_2)$, $y=(y_1,y_2)$. In the new canonical variables $(\tilde x,\tilde y)$ and the new time s, Equation (1.3) turns out to be the Hamiltonian equation with the generating function as the following:

$$\tilde G_\epsilon=\frac{1}{\epsilon}\Big(h(p''+\sqrt{\epsilon}\tilde y)-h(p'')\Big)-V(x)+\sqrt{\epsilon}\tilde R_\epsilon(\tilde x,\tilde y), \tag{1.4}$$

where $V=-Z(p'',x)$ and

$$
\begin{aligned}
&\tilde R_\epsilon=\tilde R_1+\tilde R_2+\tilde R_3,\\
&\tilde R_1=\frac{1}{\sqrt{\epsilon}}\Big[Z(p''+\sqrt{\epsilon}\tilde y,x)-Z(p'',x)\Big],\\
&\tilde R_2=\frac{1}{\sqrt{\epsilon}}\Big\langle\frac{\partial h}{\partial p}\Big(p''+\sqrt{\epsilon}\tilde y\Big)-\frac{\partial h}{\partial p}(p''),\frac{\partial F}{\partial q}\Big\rangle,\\
&\tilde R_3=\frac{\sqrt{\epsilon}}{2}\int_0^1(1-t)\{\{H,F\},F\}\circ\Phi^t_{\epsilon F}dt.
\end{aligned}
\tag{1.5}
$$

To choose the neighborhood where we study the normal form (1.4), we notice the following two points.

1) there are finitely many double resonant points $\{p''_i\}\subset\Gamma'$ such that Γ' is covered by the disks $\{\|p-p''_i\|<K_i^{-1}\epsilon^{\sigma'}\}$, where $\sigma'<\frac{1}{6}$, $K_i\le K_0\epsilon^{-\frac{1}{3}(1-3\sigma')}$ is the period of the double resonance at p''_i, i.e., $K_i\partial h(p''_i)\in\mathbb{Z}^3$ and $K\partial h(p''_i)\notin\mathbb{Z}^3$ for any $K<K_i$ (see Chapter 3 of [**Lo**]). Therefore, the size of each disk is between $O(\epsilon^{\frac{1}{7}})$ and $O(\epsilon^{\frac{1}{3}})$;
2) one is unable to use the KAM technique in $K\sqrt{\epsilon}$-neighborhood of strong double resonance to obtain invariant cylinder, even with large $K>0$.

Therefore, we will study the normal form (1.4) in the domain

$$
\Omega_\epsilon=\Big\{(\tilde x,\tilde y):|\tilde y|\le\epsilon^{\sigma-\frac{1}{2}},\tilde x\in\mathbb{T}^3\Big\},\quad\text{with}\quad 0<\sigma<\frac{1}{2},
$$

where the term $|\sqrt{\epsilon}\tilde R_i|_{C^{r-2}}$ is bounded by a small number of order $O(\epsilon^\sigma)$ (for $i=1,2,3$). If we introduce a symplectic coordinate transformation further

$$
\mathfrak{S}:\qquad I=\frac{\omega_3}{\sqrt{\epsilon}}y_3,\qquad \theta=\frac{\sqrt{\epsilon}}{\omega_3}x_3,
\tag{1.6}
$$

then $\frac{\partial\tilde G_\epsilon}{\partial I}=1+O(\epsilon^\sigma)$ holds in the domain $\mathfrak{S}\Omega_\epsilon$. Therefore, there is a unique function $I=-G_\epsilon(x,y,\theta)$ which solves the equation

$$
\tilde G_\epsilon\Big(x,\frac{\omega_3}{\sqrt{\epsilon}}\theta,y,-\frac{\sqrt{\epsilon}}{\omega_3}G_\epsilon(x,y,\theta)\Big)=0,
\tag{1.7}
$$

where we use the fact that $h(p'')=E$. Since the Hamiltonian (1.4) is nearly integrable, we shall see later that the function $G_\epsilon(x,y,\theta)$ takes the form

$$
G_\epsilon(x,y,\theta)=\bar G_\epsilon(x,y)+\epsilon^\sigma R_\epsilon(x,y,\theta),\qquad \text{for } \|y\|\le O(\epsilon^{\sigma-\frac{1}{2}}),
\tag{1.8}
$$

where $\bar G_\epsilon$ solves the equation $h(p''+\sqrt{\epsilon}(y,-\frac{\sqrt{\epsilon}}{\omega_3}\bar G_\epsilon(x,y)))-h(p'')=\epsilon V(x)$. By assuming $\min V=0$, the energy of $\bar G_\epsilon$ ranges from zero to very high level $E'\epsilon^{2\sigma-1}$. Restricted on the energy level set $\tilde G_\epsilon^{-1}(0)$, the dynamics of $\tilde{G}_\epsilon$ is equivalent to the dynamics of G_ϵ where θ plays the role of time. Let $\Phi^\theta_{G_\epsilon}$ denote the Hamiltonian flow of G_ϵ.

To state our results, we introduce some notations. A manifold with boundary is called cylinder if it is homeomorphic to the standard cylinder $\mathbb{T}\times[0,1]$. A typical case is the cylinder made up by periodic orbits of an autonomous Hamiltonian system where different orbit lies in different energy level set. The cylinder is denoted by $\Pi_{E_1,E_2,g}$ if all orbits are associated with the same first homology class g and they lie in the level set with energy E_1 to the set with E_2. The cylinder is invariant for the Hamiltonian flow. If the system is under small time-periodic perturbation, the time-periodic map generated by the Hamiltonian flow is a small perturbation of the original map. The cylinder will survive small perturbations of the map with small deformation, denoted by $\Pi^\epsilon_{E_1,E_2,g}=\Pi^{\epsilon,\theta}_{E_1,E_2,g}|_{\theta=0}$. Let $\tilde{\Pi}^\epsilon_{E_1,E_2,g}=\cup_{\theta\in\mathbb{T}}(\Pi^{\epsilon,\theta}_{E_1,E_2,g},\theta)$, the small deformation of $\Pi_{E_1,E_2,g}\times\mathbb{T}$. We also call it cylinder.

The Tonelli Hamiltonian G_ϵ determines a Tonelli Lagrangian through the Legendre transformation. So, the α- and β-function are well defined, denoted by α_{G_ϵ} and β_{G_ϵ}, respectively. They define the Legendre–Fenchel duality $\mathscr{L}_{\beta_{G_\epsilon}}$ between the first homology and the first cohomology: a first cohomology class $c\in\mathscr{L}_{\beta_{G_\epsilon}}(g)$ if $\alpha_{G_\epsilon}(c)+\beta_{G_\epsilon}(g)=\langle c,g\rangle$. By adding a constant to G_ϵ we can assume $\min\alpha_{G_\epsilon}=0$. Once a Lagrangian L is fixed, we also use $\mathscr{L}_{\beta_L}$ to denote the Legendre–Fenchel duality.

The Hamiltonian G_ϵ produces a map $\mathscr{L}_{G_\epsilon}$: $T^*\mathbb{T}^2\times\mathbb{T}\to T\mathbb{T}^2\times\mathbb{T}$: $(x,y,\theta)\to(x,\dot{x},\theta)$ where $\dot{x}=\partial_yG_\epsilon(x,y,\theta)$. In this paper, a set in $T^*\mathbb{T}^2\times\mathbb{T}$ as well as its time-2π-section is called Mather set (Aubry set or Mañé set) if its image under the map $\mathscr{L}_{G_\epsilon}$ is a Mather set (Aubry set or Mañé set) in the usual definition. Let NHIC be the abbreviation of normally hyperbolic invariant cylinder, the following theorem is the main result of this paper:

Theorem 1.1. *For a class $g\in H_1(\mathbb{T}^2,\mathbb{R})$, there is an open-dense set $\mathfrak{V}\subset C^r(\mathbb{T}^2,\mathbb{R})$ $(r\geq 5)$. For each $V\in\mathfrak{V}$, there exists $\epsilon_0>0$ such that for each $\epsilon\in(0,\epsilon_0)$*

1) *there are finitely many NHICs for the map $\Phi^{2\pi}_{G_\epsilon}$: $\Pi^\epsilon_{\epsilon^d,E_0+\delta,g}$, $\Pi^\epsilon_{E_0-\delta,E_1+\delta,g}\cdots\Pi^\epsilon_{E_{i_0-1}-\delta,E_{i_0}+\delta,g}$, $\Pi^\epsilon_{E_{i_0}-\delta,\epsilon^{2\sigma-1},g}$ where the integer i_0, the numbers $E_{i_0}>\cdots>E_1>E_0>0$, the small numbers $\delta,d>0$ and the normal hyperbolicity of each cylinder are all independent of ϵ;*
2) *for each $c\in\mathscr{L}_{\beta_{G_\epsilon}}(\lambda g)$ with $\lambda>0$*
   - *$\exists\ N>1$ such that if $\alpha_{G_\epsilon}(c)\in(N\epsilon^d,E_0)$, the Aubry set lies on $\Pi^\epsilon_{\epsilon^d,E_0+\delta,g}$;*
   - *if $\alpha_{G_\epsilon}(c)\in(E_i,E_{i+1})$, the Aubry set lies on $\Pi^\epsilon_{E_i-\delta,E_{i+1}+\delta,g}$ where the subscript i ranges over the set $\{0,1,\cdots,i_0-1\}$;*
   - *if $\alpha_{G_\epsilon}(c)\in(E_{i_0},\epsilon^{2\sigma-1})$, the Aubry set lies on $\Pi^\epsilon_{E_{i_0}-\delta,\epsilon^{2\sigma-1},g}$;*

- *if* $\alpha_{G_\epsilon}(c) = E_i$*, the Aubry set has two connected components, one is on* $\Pi^\epsilon_{E_{i-1}-\delta,E_i+\delta,g}$ *and another one is on* $\Pi^\epsilon_{E_i-\delta,E_{i+1}+\delta,g}$.

Let us see what the theorem implies for the Hamiltonian H if we return back to the original coordinates. Since h is integrable, $\mathscr{L}_{\beta_h}(\partial h(\Gamma')) \subset H^1(\mathbb{T}^3,\mathbb{R})$ is a smooth curve. Through the Legendre–Fenchel duality induced by β_H, the β-function for H, one obtains a channel $\mathscr{L}_{\beta_H}(\partial h(\Gamma')) \subset H^1(\mathbb{T}^3,\mathbb{R})$. If we denote a-neighborhood of the set S by $S+a=\{x : d(x,S)\le a\}$, it follows from Theorem 1.1 that

Theorem 1.2. *Given a resonant path* $\Gamma' \subset h^{-1}(E)$ *and a potential* $V \in \mathfrak{V}$*, some small numbers* $\epsilon_0, d > 0$ *exist such that for each* $\epsilon \in (0,\epsilon_0)$*, only finitely many frequencies* $\{\omega_k \in \Gamma'\}$ *need to be treated as strong double resonance, the number is independent of* ϵ*. For each class* $c \in \mathscr{L}_{\beta_H}(\partial h(\Gamma'))\backslash \cup(\mathscr{L}_{\beta_H}(\omega_k)+\epsilon^{\frac{1}{2}+d})$ *the Aubry set* $\tilde{\mathcal{A}}(c)$ *lies on some NHIC. The number of NHICs is finite, independent of* ϵ*, these NHICs extend to* $\epsilon^{\frac{1}{2}+d}$*-neighborhood of strong double resonant points.*

The result in [**CZ2**] plays important role in this paper. It is for the minimal periodic orbit of Tonelli Lagrangian of two degrees of freedom. Let L be a Tonelli Lagrangian and let $\mathfrak{M}(L)$ be the set of Borel probability measures on $T\mathbb{T}^2$, which are invariant for the Lagrange flow ϕ^t_L produced by L. Each $\mu \in \mathfrak{M}(L)$ is associated with a rotation vector $\rho(\mu) \in H_1(\mathbb{T}^2,\mathbb{R})$ s.t. for every closed 1-form η on $\mathbb{T}^2$ one has

$$\langle[\eta],\rho(\mu)\rangle = \int \eta d\mu.$$

Let $\mathfrak{M}_\omega(L)=\{\mu\in\mathfrak{M}(L):\rho(\mu)=\omega\}$, an invariant measure μ is called minimal with the rotation vector ω if

$$\int Ld\mu = \inf_{\nu\in\mathfrak{M}_\omega(L)} \int Ld\nu.$$

A rotation vector $\omega \in H_1(\mathbb{T}^2,\mathbb{R})$ is called resonant if there exists a non-zero integer vector $k\in\mathbb{Z}^2$ such that $\langle\omega,k\rangle = 0$. For two-dimensional torus, it uniquely determines an irreducible element $g \in H_1(\mathbb{T}^2,\mathbb{Z})$ and a positive number $\lambda>0$ such that $\omega=\lambda g$ if ω is resonant. Each orbit in the support of minimal measure μ is periodic if and only if $\rho(\mu)$ is resonant. Let $E=\alpha(\mathscr{L}_{\beta_L}(\lambda g))$, such periodic orbit is called (E,g)-*minimal.* The following result (Theorem 2.1 of [**CZ2**]) has been proved for Tonelli Lagrangian with two degrees of freedom $L: T\mathbb{T}^2\to\mathbb{R}$:

Theorem 1.3. *Given a class* $g\in H_1(\mathbb{T}^2,\mathbb{Z})$ *and two positive numbers* $E''>E'>0$*, there exists an open-dense set* $\mathfrak{V}\subset C^r(\mathbb{T}^2,\mathbb{R})$ *with* $r\ge 5$ *such that for each* $V\in\mathfrak{V}$*, it holds simultaneously for all* $E\in[E',E'']$ *that every* (E,g)*-minimal periodic orbit of* $L+V$ *is hyperbolic. Indeed, except for finitely many* $E_i\in[E',E'']$*, there is only one* (E,g)*-minimal orbit for* $E\neq E_i$ *and there are two* (E,g)*-minimal orbits for* $E=E_i$.

Therefore, these (E,g)-minimal periodic orbits make up finitely many pieces of NHICs.

Applying this theorem to the Hamiltonian $\bar{G}_\epsilon$ in (1.8), one immediately obtains the existence of NHICs which extend from the level set with energy of order $O(1)$ to the level set with very high energy $E \gg 1$ but independent of ϵ. In Section 2, we show the NHICs which extend from the level set with energy of order $O(1)$ to the level set with very lower energy $E = O(\epsilon^d)$ and show in Section 3 the NHICs which extends from level set with high energy $E \gg 1$ to extremely high energy $E = O(\epsilon^{2\sigma-1})$. Although we are searching for NHICs ranging from the level with very lower energy to the level with energy approaching infinity as $\epsilon \to 0$, in Section 4, we show that, for generic potential V, the number of the NHICs is finite, independent of ϵ. It allows us to apply the theorem of normally hyperbolic invariant manifold to obtain the existence of NHICs for the time-periodic map of $\Phi_{G_\epsilon} = \Phi^\theta_{G_\epsilon}|_{\theta=\omega_3^{-1}2\pi\sqrt{\epsilon}}$. Since these cylinders may be overflow, we show in Section 5 which Aubry sets remain in these cylinders.

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2. NHIC around double resonant point

The main result in this section is Theorem 2.3. It verifies that a NHIC for the map Φ_{G_ϵ} extends from ϵ^d-neighborhood of the double resonant point to a place which is of order $O(1)$-away from the double resonant point.

We do not try to find a formulation of G_ϵ in (1.8) which is valid for the whole region Ω_ϵ. Instead, we are satisfied with getting a local expression when it is restricted on $\{|p-p'_i| \le O(\sqrt{\epsilon})\}$ where $p'_i \in \Gamma' \cap \{|p-p''| \le \epsilon^\sigma\}$. Along the path $\Gamma' \cap \{|p-p''| \le \epsilon^\sigma\}$ we choose points $\{p'_i\}$ such that $p'_0 = p''$, $\partial_1 h(p'_i) = Ki\sqrt{\epsilon}$, where $K > 0$ is an integer, independent of ϵ. Let $\omega_{3,i} = \partial_3 h(p'_i)$, we introduce coordinate rescaling and translation

$$\Big(y, \frac{\sqrt{\epsilon}}{\omega_{3,i}} I\Big) = \frac{1}{\sqrt{\epsilon}}(p - p'_i), \qquad \theta = \frac{\sqrt{\epsilon}}{\omega_{3,i}} x_3. \tag{2.1}$$

Let $Ki=\Omega_i$, we expand $\tilde G_\epsilon$ of (1.4) in $O(\sqrt{\epsilon})$ neighborhood of p_i' and get

(2.2)
$$\begin{aligned}\tilde G_\epsilon =&I+\Omega_iy_1+\frac12\Big\langle\tilde A_i\Big(y,\frac{\sqrt{\epsilon}}{\omega_{3,i}}I\Big),\Big(y,\frac{\sqrt{\epsilon}}{\omega_{3,i}}I\Big)\Big\rangle\\ &-V(x)+\sqrt{\epsilon}\tilde R_h\Big(y,\frac{\sqrt{\epsilon}}{\omega_{3,i}}I\Big)+\sqrt{\epsilon}\tilde R_\epsilon\Big(x,\frac{\omega_{3,i}}{\sqrt{\epsilon}}\theta,\Big(\sqrt{\epsilon}y,\frac{\epsilon}{\omega_{3,i}}I\Big)+p_i'\Big),\end{aligned}$$

where $\tilde A_i=\frac{\partial^2h}{\partial p^2}(p_i')$ and term $\tilde R_h$ represents the Taylor remainder

$$\begin{aligned}\tilde R_h =&\frac{1}{\sqrt{\epsilon}^3}\Big\{h\Big(p_i'+\Big(\sqrt{\epsilon}y,\frac{\epsilon}{\omega_{3,i}}I\Big)\Big)\\ &-\Big[h(p_i')+I+\Omega_iy_1+\frac12\Big\langle\tilde A_i\Big(y,\frac{\sqrt{\epsilon}}{\omega_{3,i}}I\Big),\Big(y,\frac{\sqrt{\epsilon}}{\omega_{3,i}}I\Big)\Big\rangle\Big]\Big\}.\end{aligned}$$

For $|p_i'|\le O(\epsilon^\sigma)$, $|y|\le O(1)$ and $|I|\le O(\frac{\omega_3}{\sqrt{\epsilon}})$ both $\sqrt{\epsilon}\tilde R_h$ and $\sqrt{\epsilon}\tilde R_\epsilon$ are bounded by a quantity of order $O(\epsilon^\sigma)$ in C^{r-2}-topology, where θ variable is not taken derivatives. From the expression of $\tilde G_\epsilon$ in (2.2) we get a local solution of the equation (cf. Equation 1.7)

$$\tilde G_\epsilon\Big(x,\frac{\omega_{3,i}}{\sqrt{\epsilon}}\theta,y,-\frac{\sqrt{\epsilon}}{\omega_{3,i}}G_{\epsilon,i}(x,y,\theta)\Big)=0, \tag{2.3}$$

which takes the form

$$\begin{aligned}G_{\epsilon,i}(x,y,\theta)&=G_i(x,y)+\epsilon^\sigma R_{\epsilon,i}(x,y,\theta),\\ G_i(x,y)&=\Omega_iy_1+\frac12\langle Ay,y\rangle-V(x),\end{aligned} \tag{2.4}$$

where A is a 2×2 matrix obtained from $\tilde A_0$ by eliminating the third row and the third column. At first view, the matrix A should come from $\tilde A_i$ in the same way. However, using the property $|p_i'-p''|\le O(\epsilon^\sigma)$ we can put the difference term into the remainder.

Let us compare the Hamiltonian $G_{\epsilon,i}$ of (2.4) with the Hamiltonian G_ϵ of (1.8). It follows from the transformations (1.6) and (2.1) that for $\sqrt{\epsilon}y-\hat p_i'\le O(\sqrt{\epsilon})$ one has

$$\frac{\omega_{3,i}}{\omega_3}G_\epsilon\Big(x,y-\frac{1}{\sqrt{\epsilon}}(\hat p''-\hat p_i'),\frac{\omega_{3,i}}{\omega_3}\theta\Big)-\frac{\omega_{3,i}}{\epsilon}(p_3''-p_{i,3}')=G_{\epsilon,i}(x,y,\theta), \tag{2.5}$$

where the notations $p''=(p_1'',p_2'',p_3'')=(\hat p'',p_3'')$, $p_i'=(p_{i,1}',p_{i,2}',p_{i,3}')=(\hat p_i',p_{i,3}')$ are used. So, up to a translation, we have

$$\Phi_{G_\epsilon}:=\Phi^\theta_{G_\epsilon}|_{\theta=\frac{\sqrt{\epsilon}}{\omega_3}2\pi}=\Phi^\theta_{G_{\epsilon,i}}|_{\theta=\frac{\sqrt{\epsilon}}{\omega_{3,i}}2\pi}:=\Phi_{G_{\epsilon,i}}. \tag{2.6}$$

Therefore, we only need to study the Hamiltonian map $\Phi_{G_{\epsilon,i}}$ determined by $G_{\epsilon,i}$ when y is restricted in the domain where $\sqrt{\epsilon}y-\hat p_i'\le O(\sqrt{\epsilon})$.

If we ignore the small term $\epsilon^\sigma R_{\epsilon,i}$ in (2.4), the truncated system G_i has two degrees of freedom only. Let L_i be the Lagrangian obtained from G_i by the Legendre transformation, we get periodic orbit with rotation vector λg by searching for the minimizer $\gamma(\cdot, E, g, x)$ of the Lagrange action

$$F_i(x,E,g) = \inf_{\substack{\gamma(0)=\gamma(\frac{2\pi}{\lambda})=x \\ [\gamma]=g}} \int_0^{\frac{2\pi}{\lambda}} L_i(\gamma(t),\dot\gamma(t))dt, \tag{2.7}$$

where $g \in H_1(\mathbb{T}^2, \mathbb{Z})$ is irreducible, $E = \alpha(\mathscr{L}_{\beta_L}(\lambda g))$. If $F_i(\cdot, E, g)$ reaches its minimum at x^*, then $(\gamma(\cdot, E, g, x^*), \dot\gamma(\cdot, E, g, x^*))$ is the minimal periodic orbit we are looking for [**CZ2**].

Let us study the case $i = 0$ in this section, i.e., the system is restricted in $K\sqrt{\epsilon}$-neighborhood of the double resonant point, in which the Hamiltonian takes the special form

$$\begin{aligned} G_{\epsilon,0}(x,y,\theta) &= G_0(x,y) + \epsilon^\sigma R_{\epsilon,0}(x,y,\theta), \\ G_0(x,y) &= \frac{1}{2}\langle Ay,y\rangle - V(x). \end{aligned} \tag{2.8}$$

A minimizer $\gamma(\cdot, E, g, x^*)$ of $F_0(\cdot, E, g)$ determines a periodic orbit $z_{E,g}(t) = (x_{E,g}(t), y_{E,g}(t))$ of the Hamiltonian flow $\Phi^t_{G_0}$, where $x_{E,g}(t) = \gamma(t, E, g, x^*)$, $y_{E,g}(t) = \partial_{\dot x} L(\gamma(t, E, g, x^*), \dot\gamma(t, E, g, x^*))$. As the Hamiltonian is autonomous, the orbit $z_{E,g}(t)$ lies in the energy level $G_0^{-1}(E)$. When the energy E decreases, λ also decreases. We assume $\min V = 0$, then there are two possibilities:

(1), $\lambda \downarrow \lambda_0 > 0$ as $E \downarrow 0$. In this case, certain periodic orbit $z^*(t) \subset G_0^{-1}(0)$ such that $z_{E,g}(t) \to z^*(t)$. It is possible, we have an example. Let

$$L = \frac{1}{2}\dot x_1^2 + \frac{1}{2}\dot x_2^2 + V(x),$$

where V satisfies the conditions: $x = 0$ is the minimal point of V only; there exist two numbers $d > d' > 0$ such that for any closed curve γ: $[0,1] \to \mathbb{T}^2$ passing through the origin with $[\gamma] \neq 0$ one has

$$\int_0^1 V(\gamma(s))ds \geq d;$$

$V = d' + (x_2 - a)^2$ when it is restricted a neighborhood of circle $x_2 = a$. For $g = (1,0)$, $\lambda \downarrow \sqrt{2d'}$ as $E \downarrow 0$, $z^*(t) = (\frac{t}{\sqrt{2d'}}, a)$. No problem of double resonance appears in this case.

(2), $\lambda \downarrow 0$ as $E \downarrow 0$. It is typical that V attains its minimum at one point which correspond a fixed point of the Hamiltonian flow $\Phi^t_{G_0}$. As the period $2\lambda^{-1}\pi$ approaches infinity, the orbit $z_{E,g}(t)$ approaches homoclinic orbit(s) as E approaches zero. It is possible that there are two irreducible classes $g_1, g_2 \in H_1(\mathbb{T}^2, \mathbb{Z})$ and two non-negative integers k_1, k_2 such that $g = k_1 g_1 + k_2 g_2$. It is a difficult part of the problem of double resonance, $\{z_{E,g}\}_{E>0}$ makes up a cylinder which takes homoclinic orbits as its boundary. The cylinder cannot survive any small perturbation. In this section, we are going to study how close some invariant cylinder of $\Phi_{G_{\epsilon,0}}$ of (2.6) can extend to the double resonant point.

2.1. Hyperbolicity of minimal periodic orbit around double resonant point. At the double resonant point we have $p'_0 = p''$ and $\omega'_1 = \partial h(p'_0) = 0$. In this case, the Lagrangian determined by G_0 takes the form

$$L_0 = \frac{1}{2}\langle A^{-1}\dot{x}, \dot{x}\rangle + V(x).$$

For Hamiltonian system G_0, the minimal point of V determines a stationary solution which corresponds to a minimal measure of the Lagrangian L_0. Up to a translation of coordinates $x \to x + x_0$, it is open-dense condition that

(**H1**): *V attains its minimum at $x = 0$ only, the Hessian matrix of V at $x = 0$ is positive definite. All eigenvalues of the matrix*

$$\begin{pmatrix} 0 & A \\ \partial_x^2 V & 0 \end{pmatrix}$$

are different: $-\lambda_2 < -\lambda_1 < 0 < \lambda_1 < \lambda_2$.

If we denote by $\Lambda_i^+ = (\Lambda_{xi}, \Lambda_{yi})$ the eigenvector corresponding to the eigenvalue λ_i, where Λ_{xi} and Λ_{yi} are for the x- and y-coordinate respectively, then the eigenvector for $-\lambda_i$ will be $\Lambda_i^- = (\Lambda_{xi}, -\Lambda_{yi})$.

By the assumption (**H1**), the fixed point $z = (x, y) = 0$ has its stable manifold W^+ and its unstable manifold W^-. They intersect each other along homoclinic orbit. Since each homoclinic orbit entirely stays in the stable and the unstable manifolds, the intersection cannot be transversal in the standard definition, but in the sense that

$$T_zW^- \oplus T_zW^+ = T_zG_0^{-1}(0)$$

holds for any point z along homoclinic orbit. Without danger of confusion, we also call the intersection transversal.

Being treated as a closed curve, a homoclinic orbit $(\gamma(t), \dot{\gamma}(t))$ is associated with a homological class $[\gamma] = g \in H_1(\mathbb{T}^2, \mathbb{Z})$. A homoclinic orbit $(\gamma, \dot{\gamma})$ is called *minimal* if

$$\int_{-\infty}^{\infty} L_{G_0}(\gamma(t), \dot{\gamma}(t))dt = \inf_{[\zeta]=[\gamma]} \int_{-\infty}^{\infty} L_{G_0}(\zeta(t), \dot{\zeta}(t))dt.$$

For convenience, we call γ homoclinic curve if $(\gamma, \dot{\gamma})$ is a homoclinic orbit.

We claim that each $g \in H_1(\mathbb{T}^2, \mathbb{Z})$ is associated with an open-dense set in $\mathfrak{O}_g \subset C^r(M, \mathbb{R})$ such that for each $V \in \mathfrak{O}_g$ there is only one minimal homoclinic curve γ with $[\gamma] = g$. In fact, for a minimal homoclinic curve γ_1 one constructs a potential $\delta V_1 \geq 0$ so that suppδV_1 is away from the point $x = 0$ and suppδV_1 looks like a tubular neighborhood of a piece of the homoclinic curve γ_1. Since all minimal homoclinic curves with the same class do not cross each other, all minimal homoclinic curve of the perturbed system $L - \delta V_1$ must pass through suppδV_1. Pick up a minimal homoclinic orbit γ_2 of the perturbed system, one introduces δV_2 so that suppδV_2 lies in a more narrow tubular neighborhood of γ_2. Step by step, one obtains a sequence of potential perturbations δV_i and a sequence of curves γ_i such that $\gamma_i \to \gamma_\infty$, supp$\delta V_i$ shrinks to a piece of γ_∞ and γ_∞ is the unique minimal homoclinic curve of the Lagrangian $L - \Sigma\delta V_i$ such that $[\gamma_\infty] = g$ and $\Sigma\delta V_i$ is small. Therefore, a residual set $\mathfrak{R} = \cap_g \mathfrak{O}_g$ exists such that for each $V \in \mathfrak{R}$, there is only one minimal homoclinic orbit for each class in $g \in H_1(\mathbb{T}^2, \mathbb{Z})$, although there are infinitely many homoclinic orbits associated with the same class [**Z2**, **CC**]. So, some residual set $\mathfrak{R} \subset C^r(M, \mathbb{R})$ exists such that for each $V \in \mathfrak{R}$ one has

(**H2**): *The stable manifold intersects the unstable manifold transversally along each minimal homoclinic orbit. These minimal homoclinic orbits approach the fixed point along the direction* Λ_1*:* $\dot\gamma(t)/\|\dot\gamma(t)\| \to \Lambda_{x1}$ *as* $t \to \pm\infty$*.*

Once fixing the homological class g, we denote the periodic curve $x_E(t) = x_{E,g}(t)$ which determines a periodic orbit $z_E = (x_E, y_E)$ in the phase space. As it stays in the energy level set $G_0^{-1}(E)$, let us show how the period T_E is related to the energy E.

Lemma 2.1. *Assume the hypothesis* (**H2**)*. For* $g = k_1 g_1 + k_2 g_2$ *and suitably small* $E > 0$*, the period* T_E *of the orbit* z_E *is related to the energy* E *through the formula*

$$T_E = T(E, g) = \tau_{E,g} - \frac{1}{\lambda_1}(k_1 + k_2)\ln E, \tag{2.9}$$

where $\tau_{E,g}$ *is uniformly bounded as* $E \downarrow 0$*.*

Proof. By the condition, there are two minimal homoclinic curves $\gamma_1(t), \gamma_2(t)$ such that $[\gamma_1] = g_1$, $[\gamma_2] = g_2$. Let $z_1(t), z_2(t)$ denote the homoclinic orbit determined by γ_1, γ_2 in the Hamiltonian formalism, $z_j(t) = (\gamma_j(t), \partial_{\dot x} L(\gamma_j(t), \dot\gamma_j(t)))$ $(j = 1, 2)$, the periodic orbit $z_E(t)$ approaches these homoclinic orbits as $E \downarrow 0$. By the hypotheses (**H1**, **H2**), these homoclinic orbits approach the fixed point $z = 0$ along the direction $\Lambda_1^\pm$.

Let B_δ be a sphere centered at $z = 0$ with small radius $\delta > 0$. Since z_E approaches the homoclinic orbits, it passes through the ball if $E > 0$ is small. Denote by $t^+_{E,i}$ the time when z_E enters the ball, $t^-_{E,i}$ the subsequent time when z_E leaves the ball, namely, $z_E(t) \in B_\delta$ for $t \in [t^+_{E,i}, t^-_{E,i}]$ and $z_E(t) \notin B_\delta$ for $t \in (t^-_{E,i}, t^+_{E,i+1})$. Since $g = k_1 g_1 + k_2 g_2$, we have

$$t^+_{E,1} < t^-_{E,1} < t^+_{E,2} < \cdots < t^+_{E,k_1+k_2} < t^-_{E,k_1+k_2} < t^+_{E,k_1+k_2+1} = t^+_{E,1} + T_E.$$

Recall the notation $z_E(t)=(x_E(t),y_E(t))$. As the minimal homoclinic orbits approach the fixed point $z=0$ along the direction $\Lambda_1^{\pm}$.

$$\Big\|\frac{\dot{x}_E(t^{\pm}_{E,i})}{\|\dot{x}_E(t^{\pm}_{E,i})\|}-\frac{\Lambda^{\pm}_{x1}}{\|\Lambda^{\pm}_{x1}\|}\Big\|<\frac{1}{4} \tag{2.10}$$

holds if $E>0,\delta>0$ are suitably small.

In a suitably small neighborhood of $z=0$, we use a Birkhoff normal form

$$G_0=\frac{1}{2}(y_1^2-\lambda_1^2x_1^2)+\frac{1}{2}(y_2^2-\lambda_2^2x_2^2)+P_3(x,y),$$

where $P_3(x,y)=O(|(x,y)|^3)$. In such coordinates, the eigenvector for the eigenvalue $\pm\lambda_1$ is $\Lambda_1^{\pm}=(1,0,\pm\lambda_1,0)$ and that for the eigenvalue $\pm\lambda_2$ reads $\Lambda_2^{\pm}=(0,1,0,\pm\lambda_2)$.

By the method of variation of constants, we obtain the solution of the Hamilton equation generated by G_0

$$\begin{aligned}x_\ell(t)=&e^{-\lambda_\ell t}(b_\ell^-+F_\ell^-)+e^{\lambda_\ell t}(b_\ell^++F_\ell^+),\\ y_\ell(t)=&-\lambda_\ell e^{-\lambda_\ell t}(b_\ell^-+F_\ell^-)+\lambda_\ell e^{\lambda_\ell t}(b_\ell^++F_\ell^+),\end{aligned} \tag{2.11}$$

where $\ell=1,2$, $b_\ell^{\pm}$ are constants determined by boundary condition and

$$\begin{aligned}F_\ell^-=&\frac{1}{2\lambda_\ell}\int_0^t e^{\lambda_\ell s}(\lambda_\ell\partial_{y_\ell}P_3+\partial_{x_\ell}P_3)(x(s),y(s))ds,\\ F_\ell^+=&\frac{1}{2\lambda_\ell}\int_0^t e^{-\lambda_\ell s}(\lambda_\ell\partial_{y_\ell}P_3-\partial_{x_\ell}P_3)(x(s),y(s))ds.\end{aligned}$$

Substituting (x,y) with the formula (2.11) into G_0 we obtain a constraint condition for the constants $b_\ell^{\pm}$:

$$G_0(x(t),y(t))=-2(\lambda_1^2b_1^-b_1^++\lambda_2^2b_2^-b_2^+)+P_3((b_\ell^++b_\ell^-),\lambda_\ell(b_\ell^+-b_\ell^-)). \tag{2.12}$$

If $(x(\pm T),y(\pm T))\in\partial B_\delta$, we obtain from the theorem of Grobman–Hartman that

$$\begin{aligned}x_\ell(-T)=&b_\ell^-e^{\lambda_\ell T}+b_\ell^+e^{-\lambda_\ell T}+o(\delta),\\ x_\ell(T)=&b_\ell^-e^{-\lambda_\ell T}+b_\ell^+e^{\lambda_\ell T}+o(\delta).\end{aligned} \tag{2.13}$$

Let $x_E(t)=(x_{E,1}(t),x_{E,2}(t))$. As Formula (2.10) holds for (x_E,y_E), the first component of $x_E(t)$ satisfies

$$|x_{E,1}(t^{\pm}_{E,i})|\geq\frac{\delta}{2\sqrt{1+\lambda_1^2}},\qquad i=1,\cdots,k_1+k_2. \tag{2.14}$$

Let $2T=t^-_{E,i}-t^+_{E,i}$. The time translation, $t^+_{E,i}\to -T$ induces $t^-_{E,i}\to T$. For sufficiently large $T>0$, it deduces from Equation (2.13) and Assumption (2.14) that

$$\frac{\delta}{3\sqrt{1+\lambda_1^2}}e^{-\lambda_1T}\leq|b_1^{\pm}|\leq 2e^{-\lambda_1T},\qquad |b_2^{\pm}|\leq 2\delta e^{-\lambda_2T},$$

and

$$b_1^- b_1^+ < 0, \qquad |P_3((b_\ell^+ + b_\ell^-), \lambda_\ell(b_\ell^+ - b_\ell^-))| \le Ce^{-3\lambda_1 T},$$

where the constant C depends only on the function P_3. So, for suitably small $\delta > 0$ and sufficiently large $|t_{E,i}^- - t_{E,i}^+|$, we obtain from (2.12) that

$$\begin{aligned} E \ge & \frac{2\lambda_1^2\delta^2}{9(1+\lambda_1^2)} e^{-\lambda_1|t_{E,i}^- - t_{E,i}^+|} - 8\lambda_2^2\delta^2 e^{-\lambda_2|t_{E,i}^- - t_{E,i}^+|} - Ce^{-3\lambda_1|t_{E,i}^- - t_{E,i}^+|/2} \\ \ge & \frac{\lambda_1^2\delta^2}{9(1+\lambda_1^2)} e^{-\lambda_1|t_{E,i}^- - t_{E,i}^+|}. \end{aligned}$$

The quantity $|t_{E,i}^- - t_{E,i}^+|$ becomes sufficiently large if $E > 0$ is sufficiently small. On the other hand, E is obviously upper bounded by

$$\begin{aligned} E \le & 8\lambda_1^2\delta^2 e^{-\lambda_1|t_{E,i}^- - t_{E,i}^+|} + 8\lambda_2^2\delta^2 e^{-\lambda_2|t_{E,i}^- - t_{E,i}^+|} + Ce^{-3\lambda_1|t_{E,i}^- - t_{E,i}^+|/2} \\ \le & 9\lambda_1^2\delta^2 e^{-\lambda_1|t_{E,i}^- - t_{E,i}^+|}. \end{aligned}$$

Therefore, we find the dependence of speed on the energy

$$|t_{E,i}^- - t_{E,i}^+| = \frac{1}{\lambda_1}|\ln E| - \frac{2}{\lambda_1}|\ln\delta| + \tau_{E,i}, \tag{2.15}$$

where $\tau_{E,i}$ is uniformly bounded for each $i \le k_1 + k_2$:

$$\frac{1}{\lambda_1}\Big(2\ln\lambda_1 + \ln\frac{1}{9(1+\lambda_1^2)}\Big) \le \tau_{E,i} \le \frac{1}{\lambda_1}(2\ln\lambda_1 + 3\ln 3).$$

For $t \in (t_{E,i}^-, t_{E,i+1}^+)$, the point $z_E(t)$ does not fall into the ball B_δ. So, the quantity $t_{E,i+1}^+ - t_{E,i}^-$ is uniformly bounded as $E \downarrow 0$. Set

$$\tau_{E,g} = \sum_{i=1}^{k_1+k_2}\Big(\tau_{E,i} + (t_{E,i+1}^+ - t_{E,i}^-) - \frac{2}{\lambda_1}|\ln\delta|\Big),$$

we obtain the formula (2.9). q.e.d.

Next, we study the hyperbolicity of the periodic orbit $z_E(t) = (x_E(t), y_E(t))$. Since the Hamilton flow $\Phi_{G_0}^t$ preserves the energy, we take a two-dimensional section $\Sigma_E \subset G_0^{-1}(E)$, which is transversal to the periodic orbit $z_E(t)$ at $z_{E,0}$ in the sense that

$$T_{z_{E,0}}G_0^{-1}(E) = \mathrm{span}\{J\nabla G_0(z_{E,0}), T\Sigma_E\}.$$

The Hamiltonian flow produces a Poincaré map, for which $z_{E,0}$ is periodic point (the orbit intersects the section at several points). We study the hyperbolicity of periodic point for the Poincaré map. If a periodic orbit is associated with the homological class g and it stays in the energy level of E, we call it (E,g)-periodic orbit.

Lemma 2.2. *Assume the hypotheses* (**H1**, **H2**)*. Given a class* $g \in H_1(\mathbb{T}^2, \mathbb{Z})$ *we assume that, as* $E \downarrow 0$*, there is* (E, g)*-periodic orbit* $z_E(t)$ *approaching the minimal homoclinic orbits* z_1 *and* z_2 *such that* $g = k_1[\gamma_1] + k_2[\gamma_2]$*. Then, there exists small* $E' > 0$ *such that for each* $E \in (0, E']$*, there is a two-dimensional disk* $\Sigma_E \subset G_0^{-1}(E)$ *which intersects the orbit* $z_E(t)$ *transversally. Restricted on the section, the Hamiltonian flow* $\Phi^t_{G_0}$ *induces a Poincaré return map* Φ_E*:* $\Sigma_E \to \Sigma_E$*, and there exists some* $\lambda > 1, C > 1$ *independent of* $E \le E'$ *such that*

$$\|D\Phi_E(z_{E,0})v^-\| \ge CE^{-\lambda}\|v^-\|, \qquad \forall\ v^- \in T_{z_{E,0}}W_E^-;$$

$$\|D\Phi_E(z_{E,0})v^+\| \le C^{-1}E^{\lambda}\|v^-\|, \qquad \forall\ v^+ \in T_{z_{E,0}}W_E^+,$$

where $z_{E,0}$ *is the point where the periodic orbit intersects* Σ_E*,* $W_E^\pm$ *denotes the stable* (*unstable*) *manifold of the periodic orbit.*

Proof. To study the dynamics around the homoclinic orbits $(\gamma_j(t), \dot\gamma_j(t))$, we use new canonical coordinates (x, y) such that, restricted in a small neighborhood of $z = 0$, one has the form

$$G_0 = \frac{1}{2}(y_1^2 - \lambda_1^2 x_1^2) + \frac{1}{2}(y_2^2 - \lambda_2^2 x_2^2) + P_3(x, y),$$

with $P_3(x, y) = O(\|x, y\|^3)$. In such coordinates, we use $z_j = (x_j, y_j)$ to denote the homoclinic orbit ($j = 1, 2$). We can assume $x_{j,1}(t) \downarrow 0$ as $t \to -\infty$, $x_{j,1}(t) \uparrow 0$ as $t \to \infty$ and $\dot x_j(t)/\|\dot x_j(t)\| \to (1, 0)$ as $t \to \pm\infty$. Here the notation is taken as granted: $x_j = (x_{j,1}, x_{j,2})$. We choose 2-dimensional disk lying in $G_0^{-1}(E)$

$$\Sigma^\mp_{E,\delta} = \{(x, y) \in \mathbb{R}^4 : \|(x, y)\| \le d, G_0(x, y) = E, x_1 = \pm\delta\}.$$

Because of the special form of G_0, one has the disk lying on $G_0^{-1}(0)$ as

$$\Sigma^\mp_{0,\delta} = \{x_1 = \pm\delta, y_1^2 + y_2^2 - \lambda_2^2 x_2^2 = \lambda_1^2\delta^2 - 2P_3(\pm\delta, x_2, y), \|(x, y)\| \le d\}.$$

Let W^- (W^+) denote the unstable (stable) manifold of the fixed point which entirely stays in the energy level set $G_0^{-1}(0)$. If $P_3 = 0$, the tangent vector of $W^- \cap \Sigma^-_{0,\delta}$ has the form $(0, \pm1, 0, \pm\lambda_2)$. Then, for general $P_3 \ne 0$, the tangent vector of $W^- \cap \Sigma^-_{0,\delta}$ takes the form

$$v^-_\delta = (v_{x_1}, v_{x_2}, v_{y_1}, v_{y_2}) = (0, \pm1, y_{1,\delta}, \pm\lambda_2 + y_{2,\delta}) \in T_{z^-_\delta}(W^- \cap \Sigma^-_{0,\delta}),$$

where both $y_{1,\delta}$ and $y_{2,\delta}$ are small.

Let $T^\pm_{\delta,j}$ be the time when the homoclinic orbit $z_j(t)$ passes through $\Sigma^\pm_{0,\delta}$. Because $\partial_{y_1}G_0 > 0$ holds at the point $z_j(t) \cap \{x_1 = \mp\delta\}$, both homoclinic orbits $z_1(t)$ and $z_2(t)$ approach the fixed point in the same direction, the section $\Sigma^\pm_{0,\delta}$ intersects these two homoclinic orbits transversally. Let $z^\pm_{\delta,j}$ denote the intersection point. In a small neighborhood $B_\varepsilon(z^-_{\delta,j})$ of the point $z^-_{\delta,j}$, one obtains a map $\Psi_{0,\delta}$: $\Sigma^-_{0,\delta} \cap B_\varepsilon(z^-_{\delta,j}) \to \Sigma^+_{0,\delta}$ in following way, starting from a point z in this neighborhood, there is a unique orbit which moves along $z_j(t)$ and comes to a point $\Psi_{0,\delta}(z)\in\Sigma^+_{0,\delta}$ after a time approximately equal to $T^+_{\delta,j}-T^-_{\delta,j}$.

Let us fix small $D>0$, the quantities such as $T^{\pm}_{D,j}$, $z^{\pm}_{D,j}$, $\Sigma^{\pm}_{E,D}$ and $\Psi_{0,D}$ are well-defined in the same way as the quantities $T^{\pm}_{\delta,j}$, $z^{\pm}_{\delta,j}$, $\Sigma^{\pm}_{E,\delta}$ and $\Psi_{0,\delta}$ are defined. There exists $C_1>1$ (depending on D) such that

$$C_1^{-1}\le\|D\Psi_{0,D}(z^-_{D,j})|_{T(W^-\cap\Sigma^-_{0,D})}\|,\|D\Psi^{-1}_{0,D}(z^+_{D,j})|_{T(W^+\cap\Sigma^+_{0,D})}\|\le C_1$$

holds for $j=1,2$. Clearly, one has $C_1\to\infty$ as $D\to 0$.

As the homoclinic curves approach to the origin in the direction of $(1,0)$ in x-space, for $\delta\ll D$, there is a constant $\mu_1>0$ with $\mu_1\ll\lambda_1$ such that $\mu_1\downarrow 0$ as $D\to 0$ and

$$\frac{1}{\lambda_1+\mu_1}\ln\Big(\frac{D}{\delta}\Big)\le T^-_{D,j}-T^-_{\delta,j},T^+_{\delta,j}-T^+_{D,j}\le\frac{1}{\lambda_1-\mu_1}\ln\Big(\frac{D}{\delta}\Big). \tag{2.16}$$

The Hamiltonian flow $\Phi^t_{G_0}$ defines two maps $\Psi^-_{0,\delta,D}$: $\Sigma^-_{0,\delta}\to\Sigma^-_{0,D}$, $\Psi^+_{0,\delta,D}$: $\Sigma^+_{0,D}\to\Sigma^+_{0,\delta}$: emanating from a point in $\Sigma^-_{0,\delta}$ ($\Sigma^+_{0,D}$) there exists a unique orbit which arrives $\Sigma^-_{0,D}$ ($\Sigma^+_{0,\delta}$) after a time bounded by the last formula.

Restricted in a small neighborhood of the origin $z=0$, we consider the variational equation of the flow $\Phi^t_{G_0}$ along the orbit $z_j(t)$. It follows from the normal form of the Hamiltonian G_0 that the tangent vector $(\Delta x,\Delta y)=(\Delta x_1,\Delta x_2,\Delta y_1,\Delta y_2)$ satisfies the variational equation

$$\begin{aligned}\Delta\dot x_\ell&=\Delta y_\ell+\sum_{k=1}^{2}\Big(\frac{\partial^2P}{\partial x_k\partial y_\ell}\Delta x_k+\frac{\partial^2P}{\partial y_k\partial y_\ell}\Delta y_k\Big),\\ \Delta\dot y_\ell&=\lambda_\ell^2\Delta x_\ell-\sum_{k=1}^{2}\Big(\frac{\partial^2P}{\partial x_k\partial x_\ell}\Delta x_k+\frac{\partial^2P}{\partial y_k\partial x_\ell}\Delta y_k\Big),\qquad \ell=1,2.\end{aligned} \tag{2.17}$$

For the initial value $\Delta z(T^+_{D,j})=(\Delta x(T^+_{D,j}),\Delta y(T^+_{D,j}))$ satisfying the condition

$$|\langle\Delta z(T^+_{D,j}),v^-_\delta\rangle|\ge\frac{2}{3}\|\Delta z(T^+_{D,j})\|\|v^-_\delta\|$$

($v^-_\delta=(0,\pm1,y_{1,\delta},\pm\lambda_2+y_{2,\delta})$) one obtains from the hyperbolicity that

$$C_2^{-1}e^{(\lambda_2-\mu_1)(T^+_{\delta,j}-T^+_{D,j})}\le\frac{\|\Delta z(T^+_{\delta,j})\|}{\|\Delta z(T^+_{D,j})\|}\le C_2e^{(\lambda_2+\mu_1)(T^+_{\delta,j}-T^+_{D,j})}$$

holds for some constant $C_2>1$ depending on λ_1,λ_2 and on P. Thus, for each vector $v\in T_{z^+_D}\Sigma^+_{0,D}$ nearly parallel to $T_{z^+_D}(W^-\cap\Sigma^+_{0,D})$ in the sense that $|\langle v,v'\rangle|\ge\frac{2}{3}\|v\|\|v'\|$ holds for $v'\in T_{z^+_D}(W^-\cap\Sigma^+_{0,D})$ we obtain from the last formula and (2.16) that

$$C_2^{-1}\Big(\frac{D}{\delta}\Big)^{\frac{\lambda_2}{\lambda_1}-\mu_2}\le\lim_{\|v\|\to0}\frac{\|D\Psi^+_{0,\delta,D}(z^+_{D,\ell})v\|}{\|v\|}\le C_2\Big(\frac{D}{\delta}\Big)^{\frac{\lambda_2}{\lambda_1}+\mu_2},$$

where $\mu_2 > 0$ and $\mu_2 \to 0$ as $D \to 0$. Similarly, one has

$$C_3^{-1}\Big(\frac{D}{\delta}\Big)^{\frac{\lambda_2}{\lambda_1}-\mu_2} \le \|D\Psi^-_{0,\delta,D}(z^-_{\delta,\ell})|_{T_{z^-_\delta}(W^-\cap\Sigma^-_{0,\delta})}\| \le C_3\Big(\frac{D}{\delta}\Big)^{\frac{\lambda_2}{\lambda_1}+\mu_2},$$

where $C_3 > 1$ also depends on λ_1, λ_2 and on P.

By the construction, the 2-dimensional disk $\Sigma^-_{0,\delta}$ intersects the unstable manifold W^- along a curve. Let $\Gamma^-_{\delta,j} \subset W^- \cap \Sigma^-_{0,\delta}$ be a very short segment of the curve, passing through the point $z^-_{\delta,j}$. Pick up a point z^*_j on the homoclinic orbit z_j far away from the fixed point and take a 2-dimensional disk $\Sigma^*_j \subset G_0^{-1}(0)$ containing the point z^*_j and transversal to the flow $\Phi^t_{G_0}$ in the sense that $T_{z^*_j}G_0^{-1}(0) = \mathrm{span}(T_{z^*_j}\Sigma^*_j, J\nabla G_0(z^*_j))$. The Hamiltonian flow $\Phi^t_{G_0}$ brings a point of $\Gamma^-_{\delta,j}$ to this disk provided it is close to $z^-_{\delta,j}$. In this way, one obtains a map $\Psi^{-,*}_{\delta,j}$: $\Sigma^-_{0,\delta} \to \Sigma^*_j$. Let $\Gamma^{-,*}_{\delta,j} = \Psi^{-,*}_{\delta,j}\Gamma^-_{\delta,j}$. According to the assumption (**H2**), one has $T_{z^*_j}G_0^{-1}(0) = \mathrm{span}(T_{z^*_j}W^+, T_{z^*_j}W^-)$. Thus, one also has $T_{z^*_j}G_0^{-1}(0) = \mathrm{span}(T_{z^*_j}W^+, T_{z^*_j}\Gamma^{-,*}_{\delta,j})$. It follows from the λ-lemma that $\Psi_{0,\delta}(\Gamma^-_{\delta,j})$ keeps C^1-close to $W^- \cap \Sigma^+_{0,\delta}$ at the point $z^+_{\delta,j}$ and $\Psi^{-1}_{0,\delta}(\Gamma^+_{\delta,j})$ keeps C^1-close to $W^+ \cap \Sigma^-_{0,\delta}$ at the point $z^-_{\delta,j}$ provided $\delta > 0$ is sufficiently small. As $\Psi_{0,\delta} = \Psi^-_{0,\delta,D} \circ \Psi_{0,D} \circ \Psi^+_{0,\delta,D}$, one obtains

$$C_4^{-1}\left(\frac{D}{\delta}\right)^{2(\frac{\lambda_2}{\lambda_1}-\mu_2)} \le \|D\Psi_{0,\delta}(z^-_{\delta,j})|_{T_{z^-_{\delta,j}}(W^-\cap\Sigma^-_{0,\delta})}\| \le C_4\left(\frac{D}{\delta}\right)^{2(\frac{\lambda_2}{\lambda_1}+\mu_2)},$$

and

$$C_4^{-1}\left(\frac{D}{\delta}\right)^{2(\frac{\lambda_2}{\lambda_1}-\mu_2)} \le \|D\Psi^{-1}_{0,\delta}(z^+_{\delta,j})|_{T_{z^+_{\delta,j}}(W^+\cap\Sigma^+_{0,\delta})}\| \le C_4\left(\frac{D}{\delta}\right)^{2(\frac{\lambda_2}{\lambda_1}+\mu_2)},$$

where $C_4 = C_1C_2C_3 > 1$. See Figure 1.

By the definition, $\Sigma^\pm_{E,\delta}$ is a two-dimensional disk lying in the energy level set $G_0^{-1}(E)$. For $E > 0$ sufficiently small, $\Sigma^\pm_{E,\delta}$ is C^{r-1}-close to $\Sigma^\pm_{0,\delta}$, respectively. Let $z_E(t) = (x_E(t), y_E(t))$ be the minimal periodic orbit staying in the energy level set $G_0^{-1}(E)$, it approaches to the homoclinic orbit as E decreases to zero. Thus, for sufficiently small $E > 0$, it passes through the section $\Sigma^-_{E,\delta}$ as well as $\Sigma^+_{E,\delta}$ $k_1 + k_2$ times for one period. We number these points as $z^\pm_{E,i}$ $(i = 1, 2, \cdots k_1 + k_2)$ by the role that emanating from the point $z^-_{E,i}$, the orbit reaches to the point $z^+_{E,i+1}$ after time $\Delta t^-_{E,i} = t^+_{E,i+1} - t^-_{E,i}$, then to the point $z^-_{E,i+1}$ and so on. Note that $\Delta t^-_{E,i}$ remains bounded uniformly for any $E > 0$. Restricted on small neighborhoods of these points, denoted by $B_\varepsilon(z^\pm_{E,i})$, the flow $\Phi^t_{G_0}$ defines a local diffeomorphism $\Psi_{E,\delta}$: $\Sigma^-_{E,\delta} \supset B_\varepsilon(z^-_{E,i}) \to \Sigma^+_{E,\delta}$. Because of the smooth dependence of ODE solutions on initial data, a small $d > 0$ exists

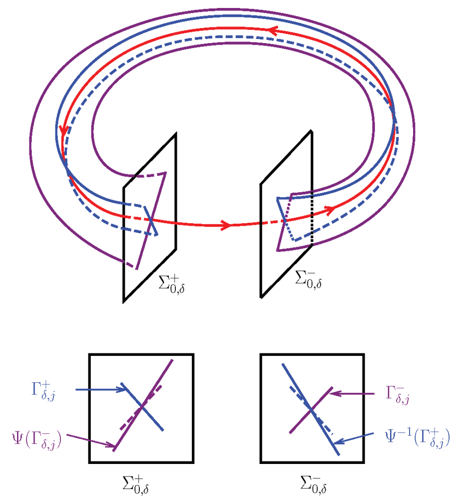

Figure 1. The unstable manifold (in purple) intersects the stable manifold (in blue) along the homoclinic orbit (in red).

such that, for the vector $v^{\pm}$ d-parallel to $T_{z^{\pm}_{\delta}}(W^{\pm}\cap\Sigma^{\pm}_{0,\delta})$ in the sense that $|\langle v^{\pm}, v^{*\pm}\rangle| \geq (1-d)\|v^{\pm}\|\|v^{*\pm}\|$ holds for some $v^{*\pm} \in T_{z^{\pm}_{\delta}}(W^{\pm}\cap\Sigma^{\pm}_{0,\delta})$, we obtain from the hyperbolicity of $\Psi_{0,\delta}$ (see the formulae above the figure) that

$$C_5^{-1}\left(\frac{D}{\delta}\right)^{2(\frac{\lambda_2}{\lambda_1}-\mu_3)} \leq \frac{\|D\Psi_{E,\delta}(z^-_{E,k})v^-\|}{\|v^-\|} \leq C_5\left(\frac{D}{\delta}\right)^{2(\frac{\lambda_2}{\lambda_1}+\mu_3)},$$

and

$$C_5^{-1}\left(\frac{D}{\delta}\right)^{2(\frac{\lambda_2}{\lambda_1}-\mu_3)} \leq \frac{\|D\Psi^{-1}_{E,\delta}(z^+_{E,k})v^+\|}{\|v^+\|} \leq C_5\left(\frac{D}{\delta}\right)^{2(\frac{\lambda_2}{\lambda_1}+\mu_3)},$$

where $C_5 \geq C_4 > 1$, $0 < \mu_3 \to 0$ as $D \to 0$. If the vector v^- is chosen d-parallel to $T_{z^-_{\delta}}(W^-\cap\Sigma^-_{0,\delta})$ then the vector $D\Psi_{E,\delta}(z^-_{E,k})v^-$ is d-parallel to $T_{z^+_{\delta}}(W^+\cap\Sigma^+_{0,\delta})$.

For $E > 0$, the Hamiltonian flow $\Phi^t_{G_0}$ defines a local diffeomorphism $\Psi^+_{E,\delta,\delta}$: $\Sigma^+_{E,\delta} \supset B_\varepsilon(z^+_{E,i}) \to \Sigma^-_{E,\delta}$. To make sure $\Psi^+_{E,\delta,\delta}(B_\varepsilon(z^+_{E,i})) \subset \Sigma^-_{E,\delta}$ one has $\varepsilon \to 0$ as $E \to 0$. According to Formula (2.15), starting from $\Sigma^+_{E,\delta}$, the periodic orbit comes to $\Sigma^-_{E,\delta}$ after a time approximately equal to

$$T = \frac{1}{\lambda_1}\Big|\ln\Big(\frac{\delta^2}{E}\Big)\Big| + \tau_\delta,$$

in which τ_δ is uniformly bounded as $\delta \to 0$. Given a vector v, we use v_ℓ denote the (x_ℓ, y_ℓ)-component. For a vector v^+ d-parallel to $T_{z^+_{\delta,j}}(W^-\cap\Sigma^+_{0,\delta})$ with small $d > 0$, there is $C > 0$ such that $\|v^+_2\| \geq$

$C\|v_1^+\|$. From Eq. (2.17) one obtains

$$\begin{aligned}\|v_2^+\|e^{(\lambda_2-\mu_4)T} \leq & \|D\Psi^+_{E,\delta,\delta}(z^+_{E,i})v_2^+\| \leq \|v_2^+\|e^{(\lambda_2+\mu_4)T},\\ \|v_1^+\|e^{(\lambda_1-\mu_4)T} \leq & \|D\Psi^+_{E,\delta,\delta}(z^+_{E,i})v_1^+\| \leq \|v_1^+\|e^{(\lambda_1+\mu_4)T},\end{aligned} \tag{2.18}$$

where $0<\mu_4\to 0$ as $\delta\to 0$, $\lambda_2>\lambda_1>0$. It follows that the vector $D\Psi^+_{E,\delta,\delta}(z^+_{E,i})v^+$ is d-parallel to $T_{z^-_{\delta,j}}(W^-\cap\Sigma^-_\delta)$ and

$$C_6^{-1}\Big(\frac{\delta^2}{E}\Big)^{\frac{\lambda_2}{\lambda_1}-\mu_5}\leq\frac{\|D\Psi^+_{E,\delta,\delta}(z^+_{E,i})v^+\|}{\|v^+\|}\leq C_6\Big(\frac{\delta^2}{E}\Big)^{\frac{\lambda_2}{\lambda_1}+\mu_5},$$

where $C_6>1$ and $\mu_5\downarrow 0$ as $\delta\downarrow 0$. Similarly, for a vector v^- d-parallel to $T_{z^-_{0,\delta}}(W^+\cap\Sigma^-_{0,\delta})$, one sees that the vector $D\Psi^+_{E,\delta,\delta}(z^-_{E,i})^{-1}v^-$ is d-parallel to $T_{z^-_{\delta,j}}(W^-\cap\Sigma^-_{0,\delta})$ and

$$C_6^{-1}\Big(\frac{\delta^2}{E}\Big)^{\frac{\lambda_2}{\lambda_1}-\mu_5}\leq\frac{\|D\Psi^+_{E,\delta,\delta}{}^{-1}(z^-_{E,i})v^-\|}{\|v^-\|}\leq C_6\Big(\frac{\delta^2}{E}\Big)^{\frac{\lambda_2}{\lambda_1}+\mu_5}.$$

The composition of these two maps is a Poinćare map $\Phi_{E,\delta}=\Psi^+_{E,\delta,\delta}\circ\Psi_{E,\delta}$, it maps a small neighborhood of the point $z^-_{E,i}$ in $\Sigma^-_{E,\delta}$ to a small neighborhood of the point $z^-_{E,i+1}$ in $\Sigma^-_{E,\delta}$. For a vector v^- d-parallel to $T_{z^-_{0,\delta}}(W^-\cap\Sigma^-_{0,\delta})$ the vector $D\Phi_{E,\delta}(z^-_{E,i})v^-$ is still d-parallel to $T_{z^-_{\delta,j}}(W^-\cap\Sigma^-_{0,\delta})$ and

$$\Lambda^{-1}\left(\frac{D^2}{E}\right)^{\frac{\lambda_2}{\lambda_1}-\mu_6}\leq\frac{\|D\Phi_{E,\delta}(z^-_{E,i})v^-\|}{\|v^-\|}\leq\Lambda\left(\frac{D^2}{E}\right)^{\frac{\lambda_2}{\lambda_1}+\mu_6}, \tag{2.19}$$

and for a vector v^+ d-parallel to $T_{z^-_{0,\delta}}(W^+\cap\Sigma^-_{0,\delta})$ the vector $D\Phi^{-1}_{E,\delta}(z^-_{E,k})v^+$ is still d-parallel to $T_{z^-_{0,\delta}}(W^+\cap\Sigma^-_{0,\delta})$ and

$$\Lambda^{-1}\left(\frac{D^2}{E}\right)^{\frac{\lambda_2}{\lambda_1}-\mu_6}\leq\frac{\|D\Phi^{-1}_{E,\delta}(z^-_{E,i})v^+\|}{\|v^+\|}\leq\Lambda\left(\frac{D^2}{E}\right)^{\frac{\lambda_2}{\lambda_1}+\mu_6} \tag{2.20}$$

holds for each i, where $\Lambda\geq C_5C_6>1$, $0<\mu_6\to 0$ as $D\to 0$. Therefore, each point $z^-_{E,i}$ is a hyperbolic fixed point for the map $\Phi^{k_1+k_2}_{E,\delta}$, $\{z^-_{E,i}:i=1,\cdots,k_1+k_2\}$ is a hyperbolic orbit of $\Phi_{E,\delta}$. It will be proved in [**C17**] that these points are uniquely ordered, k_1+k_2 is the minimal period. We complete the proof. q.e.d.

Corollary 2.1. *The (E,g)-minimal periodic orbit lying in the energy level set $G_0^{-1}(E)$ with $E\leq E'$ has a continuation of hyperbolic periodic orbits which approach the homoclinic orbits z_1 and z_2. They make up an invariant cylinder which takes the homoclinic orbits as its boundary.*

Proof. According to Lemma 2.2, the hyperbolicity of (E,g)-minimal orbit becomes very strong when $E\downarrow 0$. Such hyperbolic property is gained if the periodic orbit approaches the homoclinic orbits, the minimal property is not used. By the theorem of implicit function, this (E,g)-minimal orbit has a continuation of hyperbolic periodic orbits arbitrarily close to the homoclinic orbits z_1 and z_2. q.e.d.

Let $E_1'=h(p_1')$. As we increase the energy from E' to E_1', it follows from Theorem 1.3 that there are finitely many $E_i\in[E',E_1']$ only such that for $E\in[E',E_1']\backslash\{E_i\}$, the energy level $G_0^{-1}(E)$ contains only one (E,g)-minimal orbit and $G_0^{-1}(E_i)$ contains two minimal periodic orbits. We call these $\{E_i\}$ bifurcation points. Therefore, these hyperbolic orbits make up finitely many pieces of invariant cylinder, normally hyperbolic for the time-2π-map $\Phi^{2\pi}_{G_0}$, produced by the Hamiltonian flow $\Phi^t_{G_0}$.

In the next step, we study whether these cylinders survive the map $\Phi_{G_{\epsilon,0}}$ defined in (2.6), induced by the flow $\Phi^t_{G_{\epsilon,0}}$, where $G_{\epsilon,0}$, defined in (2.8), is a small time-periodic perturbation of G_0.

2.2. Invariant splitting of the tangent bundle: near double resonance. As shown in the last section, there is a cylinder made up by periodic orbits $(x_E(t),y_E(t))$ of $\Phi^t_{G_0}$ which extends from the energy level $G_0^{-1}(E')$ to the homoclinic orbits, denoted by

$$\Pi_{0,E',g}=\{(x_E(t),y_E(t)):[x_E]=g,E\in(0,E'],t\in\mathbb{R}\}.$$

Let $T(E)$ denote the period of the periodic orbit in $G_0^{-1}(E)$, for any $0<a<b\le E'$ one has

$$\int_{\Pi_{a,b,g}}\omega=\int_a^b\int_0^{T(E)}dE\wedge dt>0.$$

The cylinder might be slant and crumpled, we want to know how the symplectic area is related to the usual area of the cylinder. We notice that the cylinder is made up by periodic orbits $\{z_E(t)\}$. If the orbit $z_E(t)$ intersects the section $x_1=\delta$ at the point $(\delta,y_1(E),x_2(E),y_2(E))$, then $(x_2(E),y_2(E))$ is a fixed point of the Poincaré return map $\Phi_{E,\delta}$, i.e., $\Phi_{E,\delta}(x_2(E),y_2(E),y_1)=(x_2(E),y_2(E))$ and $\Phi^{-1}_{E,\delta}(x_2(E),y_2(E),y_1)=(x_2(E),y_2(E))$. Since $\partial_{y_1}G_0>0$, the value of y_1 uniquely determines the energy level set $G_0^{-1}(E)$ where the periodic orbit lies. Write $\Phi_{E,\delta}=(\Phi_{E,\delta,x_2},\Phi_{E,\delta,y_2})$, then we have

$$\begin{aligned}\Big(M_E-\mathrm{id}\Big)\Big(\frac{\partial x_2}{\partial y_1},\frac{\partial y_2}{\partial y_1}\Big)^t&=-\frac{\partial\Phi_{E,\delta}}{\partial y_1},\\ \Big(M_E^{-1}-\mathrm{id}\Big)\Big(\frac{\partial x_2}{\partial y_1},\frac{\partial y_2}{\partial y_1}\Big)^t&=-\frac{\partial\Phi^{-1}_{E,\delta}}{\partial y_1},\end{aligned}\tag{2.21}$$

where $\frac{\partial\Phi_{E,\delta}}{\partial y_1}=(\frac{\partial\Phi_{E,\delta,x_2}}{\partial y_1},\frac{\partial\Phi_{E,\delta,y_2}}{\partial y_1})^t$ and

$$M_E=\begin{bmatrix}\frac{\partial\Phi_{E,\delta,x_2}}{\partial x_2} & \frac{\partial\Phi_{E,\delta,x_2}}{\partial y_2}\\ \frac{\partial\Phi_{E,\delta,y_2}}{\partial x_2} & \frac{\partial\Phi_{E,\delta,y_2}}{\partial y_2}\end{bmatrix}.$$

Since the Hamiltonian flow preserves the symplectic structure, the matrix M_E is area-preserving. One eigenvalue is large, denoted by σ_1, bounded by (2.19), another will be $\sigma_2 = \sigma_1^{-1}$. Let ζ_ℓ be the eigenvector of M_E for the eigenvalue σ_ℓ for $\ell = 1, 2$, then ζ_1 is the eigenvector of M_E^{-1} for the eigenvalue $\sigma_2 = \sigma_1^{-1}$ and ζ_2 is the eigenvector of M_E^{-1} for the eigenvalue σ_1. Let $\frac{\partial\Phi_{E,\delta}}{\partial y_1} = a_1\zeta_1 + a_2\zeta_2$ be a decomposition, where ζ_1 and ζ_2 are normalized $\|\zeta_1\| = \|\zeta_2\| = 1$, one obtains from the equations in (2.21) that

$$\begin{aligned}(\sigma_1 - 1)a_1\zeta_1 + \Big(\frac{1}{\sigma_1} - 1\Big)a_2\zeta_2 &= -\frac{\partial\Phi_{E,\delta}}{\partial y_1},\\ \Big(\frac{1}{\sigma_1} - 1\Big)a_1\zeta_1 + (\sigma_1 - 1)a_2\zeta_2 &= -\frac{\partial\Phi^{-1}_{E,\delta}}{\partial y_1}.\end{aligned}$$

If both $\|\frac{\partial\Phi_{E,\delta}}{\partial y_1}\|$ and $\|\frac{\partial\Phi^{-1}_{E,\delta}}{\partial y_1}\|$ are bounded by $C_7E^{-\frac{\lambda_2}{\lambda_1}-\mu_7}$, one obtains from the (2.19) that both $|a_1|$ and $|a_2|$ are bounded by $2C_7\Lambda D^{-2(\lambda_2/\lambda_1-\mu_6)}E^{-(\mu_6+\mu_7)}$ if $E > 0$ is suitably small. Therefore, to make sure that there exists a constant $C_8 > 0$ such that

$$\Big|\frac{\partial x_2}{\partial y_1}\Big|, \Big|\frac{\partial y_2}{\partial y_1}\Big| \le C_8E^{-(\mu_6+\mu_7)}, \tag{2.22}$$

let us study the quantity $\|\frac{\partial\Phi_{E,\delta}}{\partial y_1}\|$. To do it, we recall Figure 1. Emanating from a point $(\delta, y_1, x_2, y_2) \in G_0^{-1}(E)$ the orbit reach a point z in the section $\{x_1 = -\delta\}$ after a time $\tau(E, \delta)$. Let $z^* \in \{x_1 = -\delta\}$ be the point corresponding to $(\delta, y_1^*, x_2, y_2) \in G_0^{-1}(E^*)$, obtained in the same way. Since $\tau(E, \delta)$ remains bounded as $E \downarrow 0$, the difference of the (x_2, y_2)-coordinate of z and z^* is bounded by $d_0|y_1 - y_1^*|$ where d_0 depends on $\tau(E, \delta)$. Let $(\Delta x, \Delta y)$ be the solution of the variational Equation (2.17) along the (E, g)-minimal periodic solution $(x_E(t), y_E(t))$, let $t_0 < t_1$ be the time such that $x_{E,1}(t_0) = -\delta$ and $x_{E,1}(t_1) = \delta$ if we use the notation $x_E = (x_{E,1}, x_{E,2})$, the quantity $t_1 - t_0$ is bounded by (2.15). In virtue of the formula (2.18), some constant $C_9 > 0$, small $\mu_7 > 0$ exists so that the following holds for suitably small $E > 0$

$$\begin{aligned}\|(\Delta x, \Delta y)(t_1)\| &\le e^{(\lambda_2+\mu_4)(t_1-t_0)}\|(\Delta x, \Delta y)(t_0)\|\\ &\le C_9E^{-\frac{\lambda_2}{\lambda_1}-\mu_7}\|(\Delta x, \Delta y)(t_0)\|,\end{aligned}$$

where $0 < \mu_7 \to 0$ as $\delta \to 0$. It implies that certain constant $C_{10} > 0$ exists such that

$$\Big\|\frac{\partial\Phi_{E,\delta}}{\partial y_1}\Big\| \le C_{10}E^{-\frac{\lambda_2}{\lambda_1}-\mu_7}$$

holds for suitably small $E > 0$, as $\|(\Delta x, \Delta y)(t_1 + \tau(E, \delta))\|\|(\Delta x, \Delta y)(t_1))\|^{-1}$ is uniformly bounded as E decreases to zero, which follows from the fact that $\tau(E, \delta)$ is uniformly bounded. This estimate obviously holds for $\|\frac{\partial\Phi_{E,\delta}^{-1}}{\partial y_1}\|$ also. It guarantees the validity of the estimate (2.22), which provides a lower bound for the symplectic area $\omega|_{\Pi_{E,b,g}} = \Sigma dx_\ell \wedge dy_\ell|_{\Pi_{E,b,g}}$ with respect to the usual area S of the cylinder $\Pi_{E,b,g}$

$$|\omega| \geq C_{11}E^{(\mu_6+\mu_7)}|S|, \tag{2.23}$$

where $C_{11} > 0$ is independent of E when $E > 0$ is suitably small.

Next, we study the invariant splitting of the tangent bundle over the cylinder $\Pi_{0,E',g}$, where $E' > 0$ is the energy so that Lemma 2.2 holds. Recall T_E defined in (2.9) denotes the period of minimal periodic orbit lying on $G_0^{-1}(E)$

Theorem 2.1. *Let $E_d \in (0, E')$. With the hypotheses* (**H1**), (**H2**), *the invariant cylinder $\Pi_{E_d,E',g}$ is normally hyperbolic for the map Φ^s_{G0} provided $s \geq T_{E_d}$. The tangent bundle of $\mathbb{T}^2$ over $\Pi_{E_d,E',g}$ admits the invariant splitting:*

$$T_zT\mathbb{T}^2 = T_zN^+ \oplus T_z\Pi_{E_d,E',g} \oplus T_zN^-$$

some $\Lambda_1 \geq 1$, $\Lambda_2 \geq 1$ and small $\nu > 0$ exist such that $\lambda_2/\lambda_1 - \nu > 1 + \nu$

$$\begin{aligned} \Lambda_1^{-1}E_d^{1+\nu} < \frac{\|D\Phi^s_{G_0}(z)v\|}{\|v\|} &< \Lambda_1 E_d^{-1-\nu}, \qquad \forall\, v \in T_z\Pi_{E_d,E',g}, \\ \frac{\|D\Phi^s_{G_0}(z)v\|}{\|v\|} &\leq \Lambda_2 E_d^{\frac{\lambda_2}{\lambda_1}-\nu}, \qquad \forall\, v \in T_zN^+, \\ \frac{\|D\Phi^s_{G_0}(z)v\|}{\|v\|} &\geq \Lambda_2^{-1} E_d^{-\frac{\lambda_2}{\lambda_1}+\nu}, \qquad \forall\, v \in T_zN^-. \end{aligned} \tag{2.24}$$

Proof. The cylinder $\Pi_{0,E',g}$ is a 2-dimensional symplectic sub-manifold, invariant for the Hamiltonian flow $\Phi^s_{G_0}$. However, it is not clear whether this cylinder admits the invariant splitting such that Formula (2.24) holds for the time-2π-map $\Phi_{G_0} = \Phi^s_{G_0}|_{s=2\pi}$. It is possible that

$$m(D\Phi_{G_0}|_{T\Pi_{E_d,E',g}}) = \inf\{|D\Phi_{G_0}v| : v \in T\Pi_{E_d,E',g}, |v| = 1\} < 1,$$
$$\|D\Phi_{G_0}|_{T\Pi_{E_d,E',g}}\| > 1,$$

and we do not know the norm of $D\Phi_{G_0}$ when it acts on the normal bundle.

By Formulae (2.19) and (2.20), one sees that the smaller the energy reaches, the stronger hyperbolicity the map $\Phi_{E,\delta}$ obtains. The strong hyperbolicity is obtained by passing through small neighborhood of the fixed point. However, on the other hand, the smaller the energy decreases, the longer the return time becomes.

For small $E > 0$, emanating from any point z on the minimal periodic orbit $z_E(s)$ and after a time T_E, $\Phi^s_{G_0}(z)$ passes through a neighborhood of the fixed point at least once. Therefore, the map $\Phi^s_{G_0}|_{s\geq T_E}$ obtains strong hyperbolicity on normal bundle such as (2.19) and (2.20).

To see how the map $D\Phi^s_{G_0}$ acts on the tangent bundle, let us study how it elongates or shortens small arc of the periodic orbit $z_E(t)$. To pass through δ-neighborhood of the origin along the orbit $z_E(t)$, it needs a time approximately equal to $|\lambda_1^{-1}\ln\delta^{-2}E|$. Restricted in δ-neighborhood of the origin, there exists small $\mu_8>0$ such that

$$|x(0)|e^{-(\lambda_1+\mu_8)t}\le|x(t)|\le|x(0)|e^{(\lambda_1+\mu_8)t}.$$

Therefore, it follows from (2.15) that the variation of the length of short arc is between $O(E^{1+\mu_8})$ and $O(E^{-1-\mu_8})$ where $\mu_8\downarrow 0$ as $\delta\downarrow 0$. Because of the relation (2.23) between the symplectic area ω and the usual area S, the variation of $\|D\Phi^s_{G_0}\|$, restricted on the tangent bundle of the cylinder, is between $O(E^{1+\mu_8+\mu_7+\mu_6})$ and $O(E^{-1-\mu_8-\mu_7-\mu_6})$, where we use the property that Hamiltonian flow preserves the symplectic structure. Due to periodicity, this lower and upper bound is independent of s. Therefore, the theorem is proved if we set $\nu=\mu_6+\mu_7+\mu_8$. q.e.d.

Let $E'_1=O(1)>E'$. For cylinder $\Pi_{E_i,E_{i+1},g}$ with $E'\le E_i<E_{i+1}\le E'_1$, the normal hyperbolicity is obvious.

Theorem 2.2. *For $E'\le E_i<E_{i+1}\le E'_1$ and typical V, there exists $s_i>0$ depending on E', E'_1 and V, such that the tangent bundle over the invariant cylinder $\Pi_{E_i,E_{i+1},g}$ admits $D\Phi^s_{G_0}$-invariant splitting*

$$T_zM=T_zN^+\oplus T_z\Pi_{E_i,E_{i+1},g}\oplus T_zN^-$$

some $\Lambda_2>\Lambda_1\ge 1$ such that the following hold for $s\ge s_i$

$$\begin{aligned}\Lambda_1^{-1}<\frac{\|D\Phi^s_{G_0}(z)v\|}{\|v\|}&<\Lambda_1, &&\forall\ v\in T_z\Pi_{E_i,E_{i+1},g},\\ \frac{\|D\Phi^s_{G_0}(z)v\|}{\|v\|}&\le\Lambda_2, &&\forall\ v\in T_zN^+,\\ \frac{\|D\Phi^s_{G_0}(z)v\|}{\|v\|}&\ge\Lambda_2^{-1}, &&\forall\ v\in T_zN^-.\end{aligned}\tag{2.25}$$

Proof. The cylinder is a symplectic sub-manifold, made up by minimal periodic orbits. Therefore, some $\Lambda_1\ge 1$ exists such that

$$\Lambda_1^{-1}\|v\|\le\|\Phi^s_{G_0}(z_E(t))v\|\le\Lambda_1\|v\|$$

holds for any $s>0$ if v is a vector tangent to z_E at $z_E(t)$. Since the Hamiltonian flow preserves the symplectic form ω, restricted on the cylinder which is an area element. Clearly, $|\omega|$ is lower bounded by usual area element $|S|$. It follows that the last formula holds for any vector tangent to the cylinder at $z_E(t)$ which implies the first formula in (2.25). Let $\Sigma_{E,z}\subset G_0^{-1}(E)$ be a two-dimensional disk, transversally intersects the periodic orbit $z_E(t)$. The flow $\Phi^s_{G_0}$ defines a Poincaré return map Φ_E, the fixed point corresponds to the periodic orbit. Let $\lambda_{1,E}$ and $\lambda_{2,E}$ be the eigenvalues of the matrix $D\Phi_E$, it depends on the energy E. According to Theorem 1.3, each of these orbits is hyperbolic, namely, some $\lambda>1$ exists such that

$$\min\{|\lambda_{1,E}|,|\lambda_{2,E}|\}\le\lambda^{-1}<\lambda\le\max\{|\lambda_{1,E}|,|\lambda_{2,E}|\},\qquad\forall\, E\in[E_i,E_{i+1}].$$

Let $\Lambda_2=\lambda([\frac{\Lambda_1}{\lambda}]+2)$, then $\Lambda_2>\Lambda_1$. Let T_E be the period of the orbit $z_E(t)$ and set

$$s_i=\max_{E\in[E_i,E_{i+1}]}T_E\Big(\Big[\frac{\Lambda_1}{\lambda}\Big]+2\Big), \tag{2.26}$$

the second and the third formulae in (2.25) hold for $\Phi^s_{G_0}$ with $s\ge s_i$. q.e.d.

The cylinder $\Pi_{0,E',g}$ may extend to the energy level $G_0^{-1}(E_1+\Delta)$, where $\Pi_{E',E_1,g}$ is made up by (E,g)-minimal orbits for $E\in[E',E_1]$, Formula (2.25) instead of Formula (2.24) applies to $\Pi_{E',E_1+\Delta,g}$. One can see that the whole cylinder $\Pi_{E_d,E_1+\Delta,g}$ is normal hyperbolic for $D\Phi^s_{G_0}$ for $s\ge\max\{T_{E_d},s'\}$ where s' is defined so that (2.25) holds for $\Pi_{E',E_1+\Delta,g}$ (cf. (2.26)).

2.3. Bifurcation point. Let $E_i<E_{i+1}$ be two adjacent bifurcation points, then each $G_0^{-1}(E)$ contains only one (E,g)-minimal orbit for $E\in(E_i,E_{i+1})$, denoted by z_E. Let $z^-_{E_i}=\lim_{E\downarrow E_i}z_E$, $z^+_{E_{i+1}}=\lim_{E\uparrow E_{i+1}}z_E$. These orbits make up an invariant cylinder

$$\Pi_{E_i,E_{i+1},g}=\{(x_E(t),y_E(t)):[x_E]=g,E\in[E_i,E_{i+1}],t\in\mathbb{R}\}.$$

In typical case, at the bifurcation point E_i, there exist two minimal periodic orbits lying in the energy level $G_0^{-1}(E_i)$, denoted by $z^+_{E_i}(t)$ and $z^-_{E_i}(t)$. The orbit $z^+_{E_i}(t)$ makes up the upper boundary of $\Pi_{E_{i-1},E_i,g}$ and the orbit $z^-_{E_i}(t)$ makes up the lower boundary of $\Pi_{E_i,E_{i+1},g}$. Because of the implicit function theorem, there is a continuation of hyperbolic periodic orbits which extends from $z^+_{E_i}(t)$ to higher energy, denoted by $z^+_E(t)$, and hyperbolic orbits extending from $z^-_{E_i}(t)$ to lower energy, denoted by $z^-_E(t)$. Those orbits $\{z^-_E(t),z^+_E(t)\}$ are not in the Mather set unless $E=E_i$, the action along these orbits reaches local minimum instead of global minimum. In this way, we get two cylinders $\Pi_{E_{i-1}-\delta,E_i+\delta,g}$, $\Pi_{E_i-\delta,E_{i+1}+\delta,g}$ which ranges from the energy level $G_0^{-1}(E_{i-1}-\delta)$ to $G_0^{-1}(E_i+\delta)$, from the energy level $G_0^{-1}(E_i-\delta)$ to $G_0^{-1}(E_{i+1}+\delta)$, respectively. The normally hyperbolic invariant splitting (2.25) applies to the extended cylinders $\Pi_{E_{i-1}-\delta,E_i+\delta,g}$ and $\Pi_{E_i-\delta,E_{i+1}+\delta,g}$.

By the definition of F in (2.7), we have $\frac{\partial F}{\partial E}(x^+_{E_i}(0),E_i)\ge\frac{\partial F}{\partial E}(x^-_{E_i}(0),E_i)$. It is obviously a generic condition that

(**H3**): $\frac{\partial F}{\partial E}(x^+_{E_i}(0),E_i)>\frac{\partial F}{\partial E}(x^-_{E_i}(0),E_i)$.

2.4. Persistence of NHICs: near double resonance. We apply the theorem of normally hyperbolic manifold [**HPS**] to obtain NHIC for the Hamiltonian $G_{\epsilon,0}$ of (2.8). We need the following preliminary lemma.

Lemma 2.3. *Let the equation* $\dot z = F_\epsilon(z,t)$ *be a perturbation of* $\dot z = F_0(z,t)$, *let* Φ_ϵ^t *and* Φ_0^t *denote the flow determined by these two equations, respectively. Then*

$$\|\Phi_\epsilon^t - \Phi_0^t\|_{C^1} \le \frac{B}{A}\Big(\frac{1}{3}e^{At} + \frac{1}{2}\Big)e^{3At},$$

where $A = \max_{t,\lambda=\epsilon,0} \|F_\lambda(\cdot,t)\|_{C^2}$ *and* $B = \max_t \|(F_\epsilon - F_0)(\cdot,t)\|_{C^1}$.

Proof. Let $z_\lambda(t)$ be the solution of the equations $\dot z = F_\lambda(z,t)$ for $\lambda = \epsilon, 0$, respectively. Along each orbit $z_\lambda(t)$, the differential of the flow Φ_λ^t satisfies the equation

$$\frac{d}{dt}D\Phi_\lambda^t = \partial_z F_\lambda(z_\lambda(t),t)D\Phi_\lambda^t, \qquad \lambda = \epsilon, 0.$$

Therefore, for each tangent vector v attached to $z_\lambda(0)$ one has

$$\|D\Phi_\lambda^t v\| \le \|v\|e^{At}. \tag{2.27}$$

To study the differential of $\Phi_\epsilon^t - \Phi_0^t$, we consider the equation of secondary variation. If δz_λ solves the variational equation $\delta\dot z_\lambda = \partial_z F_\lambda(z_\lambda(t),t)\delta z_\lambda$ for $\lambda = \epsilon, 0$, respectively, let $\Delta\delta z(t) = \delta z_\epsilon(t) - \delta z_0(t)$, $\Delta z(t) = z_\epsilon(t) - z_0(t)$. Then

$$\begin{aligned}\frac{d}{dt}\Delta\delta z = {} & \partial_z F_\epsilon(z_\epsilon(t),t)\Delta\delta z + \partial_z^2 F_\epsilon(z_0(t) + \nu(t)(z_\epsilon(t) \\ & - z_0(t)),t)\Delta z(t)\delta z_0(t) + \partial_z(F_\epsilon - F_0)(z_0(t),t)\delta z_0(t),\end{aligned}$$

where $\nu(t) \in (0,1)$, $\Delta z(t)$ solves the equation

$$\Delta\dot z = \partial_z F_\epsilon((\nu z_0 + (1-\nu)z_\epsilon)(t),t)\Delta z + (F_\epsilon - F_0)(z(t),t),$$

with the initial condition $\Delta z(0) = 0$.

To obtain an estimate on $\|\Delta\delta z(t)\|$, we recall the method of variation of constants to solve ODE. For a linear ODE $\dot z = C(t)z + f(t)$ with $z \in \mathbb{R}^n$ with the initial condition $z(0) = 0$, one has the solution

$$z(t) = e^{\int_0^t C(s)ds}\int_0^t e^{-\int_0^s C(\tau)d\tau} f(s)ds. \tag{2.28}$$

Applying this formula to the equation just above, we find that

$$\|\Delta z(t)\| \le \frac{B}{A}e^{2At}.$$

By Formula (2.27) one has $\|\delta z_0(t)\| \le \|\delta z_0(0)\|e^{At}$. Using Formula (2.28) for $\Delta\delta z(t)$, it follows $\delta z_\epsilon(0) = \delta z(0)$ that

$$\|\Delta\delta z(t)\| \le \|\delta z_0(0)\|\frac{B}{A}\Big(\frac{1}{3}e^{At} + \frac{1}{2}\Big)e^{3At}.$$

It completes the proof. q.e.d.

Let us apply this lemma to study the invariant cylinders of the Hamiltonian $G_{\epsilon,0}$. A sub-manifold N is called overflowing invariant for a flow Φ^s if, for each $z \in \text{int}N$, the orbit $\Phi^s(z)$ either stays in N forever, or by passing through ∂N to leave. We use $\Phi^{s_0,s}$ to denote the map from the time s_0-section to the time s-section. A sub-manifold N' is called a δ-deformation of another sub-manifold N if $d_H(N,N') \le \delta$, where d_H denotes Hausdorff distance.

Recall the Poincaré return map defined in (2.6), one has

Theorem 2.3. *In the extended phase space* $T^*\mathbb{T}^2 \times \frac{\sqrt{\epsilon}}{\omega_3}\mathbb{T}$, *the Hamiltonian flow* $\Phi^s_{G_{\epsilon,0}}$ *admits overflowing invariant cylinders* $\tilde{\Pi}^{\epsilon}_{E_i-\delta+\epsilon^d,E_{i+1}+\delta-\epsilon^d,g}$ *and* $\tilde{\Pi}^{\epsilon}_{\epsilon^d,E_1+\delta-\epsilon^d,g}$, *which are the* ϵ^σ*-deformation of* $\Pi_{E_i-\delta+\epsilon^d,E_{i+1}+\delta-\epsilon^d,g} \times \frac{\sqrt{\epsilon}}{\omega_3}\mathbb{T}$ *and* $\Pi_{\epsilon^d,E_1+\delta-\epsilon^d,g} \times \frac{\sqrt{\epsilon}}{\omega_3}\mathbb{T}$, *respectively, if*

$$0 < d < \min\left\{\frac{\lambda_1}{24\max_x\sqrt{\|A\|^2+\|\partial^2V\|^2}},\frac{1}{4}\right\}\sigma, \tag{2.29}$$

and $\epsilon \ge 0$ *is sufficiently small. The cylinder* $\tilde{\Pi}^{\epsilon}_{\epsilon^d,E_1+\delta-\epsilon^d,g}$ *admits normally hyperbolic invariant splitting of* (2.24) *for the map* $\Phi^{s_0,s}_{G_\epsilon}$ *with*

$$s - s_0 = \frac{2(k_1+k_2)}{\lambda_1}|\ln\epsilon^{3d}|;$$

the cylinder $\tilde{\Pi}^{\epsilon}_{E_i-\delta+\epsilon^d,E_{i+1}+\delta-\epsilon^d,g}$ *admits normally hyperbolic invariant splitting of* (2.25) *for* $\Phi^{s_0,s}_{G_\epsilon}$ *where* $s-s_0$ *is given by* (2.26), *independent of* ϵ.

Proof. Let $G_{\epsilon,0}$ be the Hamiltonian defined in (2.8). Considering $R_{\epsilon,0}$ as the function of (x,y) and treating θ as parameter, we find that there exists some constant $C_{12} = \max_\theta \|R_{\epsilon,0}(\cdot,\theta)\|_{C^1}$ such that

$$\max_\theta \|J\nabla G_{\epsilon,0} - J\nabla G_0\|_{C^1} \le C_{12}\epsilon^\sigma.$$

Let $C_{13} = \max_x\sqrt{\|A\|^2+\|\partial^2V\|^2}$, for $s-s_0 = \frac{2}{\lambda_1}|\ln\epsilon^{3d}|$ one obtains from Lemma 2.3 that

$$\|\Phi^{s_0,s}_{G_{\epsilon,0}} - \Phi^{s_0,s}_{G_0}\|_{C^1} \le \frac{C_{12}}{C_{13}}\epsilon^{\sigma-\frac{24C_{12}d}{\lambda_1}}.$$

If the condition $0 < d < \frac{\lambda_1\sigma}{24C_{13}}$ holds, then $\|\Phi^{s,s_0}_{G_0} - \Phi^{s,s_0}_{G_{\epsilon,0}}\|_{C^1} \to 0$ as $\epsilon \to 0$.

On the other hand, let $E_d = \epsilon^{3d}$. By the definition of (2.9), one has $s - s_0 \ge T_{E_d}$ if $\epsilon > 0$ is suitably small. So, Theorem 2.1 holds, which allows one to apply the theorem of normally hyperbolic invariant manifold to obtain the existence of invariant cylinder.

We consider a piece of hyperbolic cylinder $\Pi_{E_d,E_1+\delta,g} \subset \Pi_{0,E_1+\delta,g}$. Since G_0 is autonomous, $\Phi^{s_0,s}_{G_0} = \Phi^{s-s_0}_{G_0}$. Note that $\Pi_{E_d,E_1+\delta,g}$ is a cylinder with boundary, normally hyperbolic and invariant for $\Phi^{s-s_0}_{G_0}$, we do not expect that the whole cylinder survives small perturbation, it may lose some part close to the boundary.

As the first step to measure how much the cylinder survives, we modify the Hamiltonian $G_{\epsilon,0}$. Let ρ be a C^2-function such that $\rho(\mu)=1$ for $\mu\ge 1$ and $\rho(\mu)=0$ for $\mu\le 0$ and let $\rho_2=1-\rho(\frac{G_0(x,y)-(E_1+\delta-\epsilon^d)}{\epsilon^d})$, $\rho_1(x,y)=\rho(2\frac{G_0(x,y)-\epsilon^{3d}}{\epsilon^d-2\epsilon^{3d}})$. We introduce

$$G'_{\epsilon,0}=\begin{cases}G_0+\epsilon^\sigma\rho_1R_{\epsilon,0}, & \text{if } G_0(x,y)\in[\epsilon^{3d},\frac{1}{2}\epsilon^d],\\ G_0+\epsilon^\sigma\rho_2R_{\epsilon,0}, & \text{if } G_0(x,y)\in[E_1+\delta-\epsilon^d,E_1+\delta],\\ G_{\epsilon,0}, & \text{elsewhere.}\end{cases} \tag{2.30}$$

Clearly, $\|G'_{\epsilon,0}-G_0\|_{C^2}\ll 1$ if $d<\sigma$ and $\epsilon\ll 1$. It follows that the cylinder $\Pi_{E_d,E_1+\delta,g}$ survives the perturbation $\Phi^{s_0,s}_{G_0}\to\Phi^{s_0,s}_{G'_{\epsilon,0}}$ and the boundary of $\Pi_{E_d,E_1+\delta,g}$ remains unchanged for $\Phi^{s_0,s}_{G'_{\epsilon,0}}$. The survived cylinder in the extended phase space $\mathbb{T}^2\times\mathbb{R}^2\times\frac{\sqrt{\epsilon}}{\omega_3}\mathbb{T}$ is denoted by $\tilde\Pi^\epsilon_{E_d,E_1+\delta,g}$.

Since $G_{\epsilon,0}=G'_{\epsilon,0}$ when they are restricted on $\tilde\Pi^\epsilon_{E_d,E_1+\delta,g}\cap\{(x,y,\theta): G_0\in[\epsilon^d,E_1+\delta-\epsilon^d]\}$ and $\tilde\Pi^\epsilon_{E_i-\delta,E_{i+1}+\delta,g}\cap\{(x,y,\theta): G_0\in[E_i-\delta+\epsilon^d,E_{i+1}+\delta-\epsilon^d]\}$, one then obtains the overflowing invariant cylinders $\tilde\Pi^\epsilon_{\epsilon^d,E_1+\delta-\epsilon^d,g}$ and $\tilde\Pi^\epsilon_{E_i-\delta+\epsilon^d,E_{i+1}+\delta-\epsilon^d,g}$ for $\Phi^s_{G_{\epsilon,0}}$. The normally hyperbolic invariant splitting of the invariant cylinders is an application of the theorem of normally hyperbolic invariant manifold. q.e.d.

3. Transition of NHIC from double to single resonance

Along the resonant path $\Gamma'\cap\{|p-p''|\le\epsilon^\sigma\}$ we have chosen the points $\{p'_i\}$ such that $p'_0=p''$, $\partial_1h(p'_i)=Ki\sqrt{\epsilon}$, where $K\in\mathbb{Z}$. As ∂^2h is positive definite, the number of such points is bounded by a quantity of $O([K^{-1}\epsilon^{\sigma-\frac{1}{2}}])$. What we studied in the last section is about the disk which is centered at $p=p''$, the double resonant point, where the normal hyperbolicity is obtained for the Poincaré return map. In this section, we consider the disks which are "quite away from" the double resonant point in the sense that $Ki\gg 1$. Let $Ki=\Omega_i$, we recall the Hamiltonian of (2.4) $G_{\epsilon,i}(x,y,\theta)=G_i(x,y)+\epsilon^\sigma R_{\epsilon,i}(x,y,\theta)$ where

$$G_i(x,y)=\Omega_iy_1+\frac{1}{2}\langle Ay,y\rangle-V(x). \tag{3.1}$$

We are going to show that the invariant cylinders in the such disks look more and more like the cylinders in the case of single resonance when $\Omega_i\to\infty$.

As the first step, we consider the Hamiltonian G_i. Applying Theorem 1.3 proved in [**CZ2**], we find that all (E,g)-minimal periodic orbits make up some pieces of NHIC. However, it is not enough to study the persistence of these NHICs under the small perturbation $G_i \to G_{\epsilon,i} = G_i + \epsilon^\sigma R_{\epsilon,i}$, because the number of cubes approaches infinity as $\epsilon \to 0$. We need to show that, for a typical potential V, the normal hyperbolicity of all cylinders are uniformly lower bounded away from zero.

For a function $V \in C^r(\mathbb{T}^2,\mathbb{R})$, we define

$$[V](x_2) = \frac{1}{2\pi}\int_0^{2\pi} V(x_1,x_2)dx_1. \tag{3.2}$$

A set $\mathfrak{V}_\infty \subset C^r(\mathbb{T}^2,\mathbb{R})$ is defined such that $\forall\, V \in \mathfrak{V}_\infty$, the function $[V]$ has a unique minimal point which is non-degenerate, i.e., $\frac{d^2}{dx_2^2}[V](x_2) > 0$ holds at its minimal point. Obviously, the set $\mathfrak{V}_\infty$ is open-dense in $C^r(\mathbb{T}^2,\mathbb{R})$ with $r \ge 2$.

To denote a cylinder for G_i and for $G_{\epsilon,i}$, respectively, we add superscript ^{i} and $^{\epsilon,i}$ to the notation of cylinder $\Pi_{a,b,g}, \Pi^\epsilon_{a,b,g}, \tilde\Pi^\epsilon_{a,b,g} \to \Pi^i_{a,b,g}, \Pi^{\epsilon,i}_{a,b,g}, \tilde\Pi^{\epsilon,i}_{a,b,g}$.

Theorem 3.1. *Given a potential $V \in \mathfrak{V}_\infty$ and a number $\bar K > 1$, there exists some suitably large $\Omega^* > 0$ so that for $\Omega_i \ge \Omega^*$, the Hamiltonian flow $\Phi^t_{G_i}$ of (3.1) admits a unique invariant cylinder $\Pi^i_{0,\bar K\Omega_i,g}$ made up by (E,g)-minimal orbits which lie on the energy level $G_i^{-1}(E\Omega_i)$ with $E \in [0,\bar K]$.*

Moreover, the tangent bundle of $\mathbb{T}^2$ over $\Pi^i_{0,\bar K\Omega_i,g}$ admits the invariant splitting:

$$T_zT\mathbb{T}^2 = T_zN^+ \oplus T_z\Pi_{0,\bar K\Omega_i,g} \oplus T_zN^-,$$

some numbers $\Lambda > \lambda \ge 1$, and an integer $k \ge 1$ exist such that

$$\begin{aligned} \lambda^{-1}\|v\| < \|D\Phi^{2k\pi}_{G_i}(z)v\| < \lambda\|v\|, &\qquad \forall\, v \in T_z\Pi^i_{0,\bar K\Omega_i,g},\\ \|D\Phi^{2k\pi}_{G_i}(z)v\| \le \Lambda^{-1}\|v\|, &\qquad \forall\, v \in T_zN^+,\\ \|D\Phi^{2k\pi}_{G_i}(z)v\| \ge \Lambda\|v\|, &\qquad \forall\, v \in T_zN^-. \end{aligned} \tag{3.3}$$

holds for any large $\Omega_i \ge \Omega^*$.

Proof. For large Ω_i, the energy of the Hamiltonian G_i ranges over from almost zero to order $O(\Omega_i)$ if $\|y\| \le O(1)$. Under the coordinate transformation

$$(x_1,x_2,y_1,y_2) \to \Big(\frac{x_1}{\Omega_i}, x_2, \Omega_i y_1, y_2\Big), \tag{3.4}$$

let A_{ij} be the ij-th entry of the matrix A in (2.4), the Hamiltonian G_i turns out to be

$$G_i' = y_1 + \frac{1}{2\Omega_i^2}A_{11}y_1^2 + \frac{A_{12}}{\Omega_i}y_1y_2 + \frac12 A_{22}y_2^2 - V\Big(\Omega_i x_1, x_2\Big), \tag{3.5}$$

The equation $G_i'(x_1,x_2,y_1(x_1,x_2,y_2),y_2)=E\Omega_i$ is solved by the function

(3.6)
$$\begin{aligned}y_1=&\frac{\Omega_i^2}{A_{11}}\Big\{-\Big(1+\frac{A_{12}}{\Omega_i}y_2\Big)+\sqrt{\Delta}\Big\}\\=&E\Omega_i-\frac{1}{2}A_{11}E-\frac{1}{2}A_{22}y_2^2-A_{12}Ey_2+V(\Omega_ix_1,x_2)+\Omega_i^{-1}R_H,\end{aligned}$$

where $\Delta=\Big(1+\frac{A_{12}}{\Omega_i}y_2\Big)^2-\frac{A_{11}}{\Omega_i^2}(A_{22}y_2^2-2V-2E\Omega_i)$, E ranges over an interval $[0,\bar K]$ where $\bar K$ is independent of Ω_i, the remainder $\Omega_i^{-1}R_H$ is of order $O(\Omega_i^{-1})$ in C^r-topology. Let $\tau=x_1$ be the new "time", the Hamiltonian $-y_1$ produces a Lagrangian up to an additive constant

$$L_1=\frac{1}{2A_{22}}\Big(\frac{dx_2}{d\tau}\Big)^2-\frac{A_{12}E}{A_{22}}\frac{dx_2}{d\tau}+V+\frac{1}{\Omega_i}R_L,$$

where R_L is C^r-bounded for any large Ω_i. The periodic orbit with rotation vector $(\nu,0)$ for $\Phi^t_{G_i}$ is converted to be periodic orbit of $\phi^\tau_{L_1}$. Since $\Omega_i\in\mathbb{N}$, the hyperbolicity of such minimal periodic orbit is uniquely determined by the nondegeneracy of the minimal point of the following function (see [**CZ2**])

$$F(x_2,\Omega_i,E)=\inf_{\gamma(0)=\gamma(2\pi)=x_2}\int_0^{2\pi}L_1\Big(\dot\gamma(\tau),\gamma(\tau),\Omega_i\tau,E\Big)d\tau.$$

Let $\gamma_{\Omega_i,E}(\tau,x_2)$ be the minimizer of $F(x_2,\Omega_i,E)$, i.e., along which the action is equal to $F(x_2,\Omega_i,E)$. Then, $|\dot\gamma_{\Omega_i,E}(\tau,x_2)|$ is uniformly bounded for any large Ω_i. Since the system has one degree of freedom, $\frac{2\pi}{\Omega_i}$-periodical in τ, the minimum of F determines an $\frac{2\pi}{\Omega_i}$-periodic curve $\gamma^*_{\Omega_i,E}$, because each minimal periodic curve does not intersect its $k\frac{2\pi}{\Omega_i}$-translation. We shall see later that $|\dot\gamma^*_{\Omega_i,E}(\tau)|\to 0$ as $\Omega_i\to\infty$.

Because of the condition $\gamma(0)=\gamma(2\pi)=x_2$, the term $\frac{A_{12}E}{A_{22}}\dot x_2$ does not contribute to F (it is an exact form), so it can be dropped.

Although the potential V and then the Lagrangian L_1 depend on Ω_i in a singular way as $\Omega_i\to\infty$, the function F appears regular in Ω_i^{-1} as $\Omega_i\to\infty$. To see it let us decompose the action

$$F(x_2,\Omega_i,E)=F_0(x_2,\Omega_i,E)+\frac{1}{\Omega_i}F_R(x_2,\Omega_i,E),$$

where $[V]$ is defined in (3.2) and

$$\begin{aligned}F_0=&\int_0^{2\pi}\Big(\frac{1}{2A_{22}}(\dot\gamma_{\Omega_i,E}(\tau,x_2))^2+[V](\gamma_{\Omega_i,E}(\tau,x_2))\Big)d\tau,\\F_R=&\int_0^{2\pi}\Omega_i(V-[V])(-\Omega_i\tau,\gamma_{\Omega_i,E}(\tau,x_2))d\tau\\&+\int_0^{2\pi}R_L(\gamma_{\Omega_i,E}(\tau,x_2),\dot\gamma_{\Omega_i,E}(\tau,x_2),\Omega_i\tau)d\tau.\end{aligned}$$

Lemma 3.1. *Assume the potential $V \in C^4(\mathbb{T}^2,\mathbb{R})$. Then, F_R is uniformly bounded in C^2-topology as $\Omega_i \to \infty$ when x_2 is restricted in a small neighborhood $F^{-1}(\min F)$.*

Proof. As the first step, we show that F_R is uniformly C^0-bounded. For the first integral of F_R, we expand V into a Fourier series

$$V(-\Omega_i\tau, x_2) = [V](x_2) + \sum_{k\neq 0} V_k(x_2)e^{ik\Omega_i\tau}.$$

With the periodic boundary condition $\gamma_{\Omega_i,E}(0,x_2) = \gamma_{\Omega_i,E}(2\pi,x_2)$, the condition that $V \in C^r$ $(r \geq 4)$ and doing integration by parts we obtain,

$$\begin{aligned}
&|\text{The first integral of } F_R| \\
&= \Big|\Omega_i \sum_{k\neq 0}\int_0^{2\pi} V_k(\gamma_{\Omega_i,E}(\tau,x_2))e^{ik\Omega_i\tau}d\tau\Big| \\
&= \Big|\sum_{k\neq 0}\frac{1}{ik}\int_0^{2\pi} \dot V_k(\gamma_{\Omega_i,E}(\tau,x_2))\dot\gamma_{\Omega_i,E}(\tau,x_2)e^{ik\Omega_i\tau}d\tau\Big| \\
&\leq \sum_{k\neq 0}\frac{1}{|k|}\int_0^{2\pi}|\dot V_k||\dot\gamma_{\Omega_i,E}|d\tau \leq B\sum_{k\neq 0}\frac{1}{|k|^r},
\end{aligned}$$

where $B = \|V\|_{C^r}\max_\tau|\dot\gamma_{\Omega_i,E}(\tau,x_2)|$. As $\gamma_{\Omega_i,E}$ is a minimizer, $|\dot\gamma_{\Omega_i,E}(\tau,x_2)|$ keeps uniformly bounded as $\Omega_i \to \infty$. The second integral of F_R is obviously bounded in C^0-topology. It finishes the proof of the first step. q.e.d.

Proposition 3.1. *Let x_2 be a minimal point of $F(\cdot,\Omega_i,E)$, then, the minimal curve $\gamma^*_{\Omega_i,E}(\cdot,x_2)$ of $F(x_2,\Omega_i,E)$ approaches the constant solution in C^1-topology: as $\Omega_i \to \infty$ we have*

$$|\gamma^*_{\Omega_i,E}(\tau,x_2) - x_2| \to 0, \qquad |\dot\gamma^*_{\Omega_i,E}(\tau,x_2)| \to 0.$$

Proof. As each minimizer determines a solution of the equation produced by (3.5)

$$\begin{cases} \dot x_1 = 1 + \frac{A_{11}}{\Omega_i^2}y_1 + \frac{A_{12}}{\Omega_i}y_2, & \dot y_1 = \Omega_i\frac{\partial V}{\partial x_1}, \\ \dot x_2 = \frac{A_{12}}{\Omega_i}y_1 + A_{22}y_2, & \dot y_2 = \frac{\partial V}{\partial x_2}, \end{cases} \tag{3.7}$$

the second derivative of x_2 in x_1 is bounded for any large Ω_i, as one has the following calculation

$$\begin{aligned}
\frac{d^2x_2}{dx_1^2} =& \frac{d}{dt}\Big(\frac{\dot x_2}{\dot x_1}\Big)\dot x_1^{-1} = \frac{d}{dt}\Big(\frac{\frac{A_{12}}{\Omega_i}y_1 + A_{22}y_2}{1+\frac{A_{11}}{\Omega_i^2}y_1 + \frac{A_{12}}{\Omega_i}y_2}\Big)\dot x_1^{-1} \\
=& \frac{(A_{12}\frac{\partial V}{\partial x_1} + A_{22}\frac{\partial V}{\partial x_2}) - (\frac{A_{11}}{\Omega_i}\frac{\partial V}{\partial x_1} + \frac{A_{12}}{\Omega_i}\frac{\partial V}{\partial x_2})\frac{dx_2}{dx_1}}{(1+\frac{A_{11}}{\Omega_i^2}y_1 + \frac{A_{12}}{\Omega_i}y_2)^2}.
\end{aligned}$$

By adding a constant to V, we assume $\min[V]=0$. In this case, the action of L_1 along $x_2=x_2^*$ with $x_2^*\in[V]^{-1}(0)$ is bounded by the quantity $O(\frac{1}{\Omega_i})$. Consequently, the action along the minimizer $\gamma^*_{\Omega_i,E}(\tau_{\Omega_i},x_2)$ approaches to zero as $\Omega_i\to\infty$. If there exists $d>0$ as well as some τ_{Ω_i} such that $|\dot\gamma^*_{\Omega_i,E}(\tau_{\Omega_i},x_2)|\ge d>0$ holds for any large Ω_i, the action of F_0 along the curve $\gamma^*_{\Omega_i,E}(\tau,x_2)$ would be lower bounded away from zero as $\Omega_i\to\infty$. It is guaranteed by that $|\frac{d^2x_2}{d\tau^2}|$ is uniformly bounded for large Ω_i. q.e.d.

Continued proof of Lemma 3.1. To show the boundedness of $\partial^\ell_{x_2}F_R$ for $\ell=1,2$, let us study the dependence of $\gamma_{\Omega_i,E}(\tau,x_2)$ and $\dot\gamma_{\Omega_i,E}(\tau,x_2)$ on x_2. The Hamiltonian equation generated by (3.6) is the following:

$$\begin{aligned}\frac{dx_2}{d\tau}=&\Omega_i\frac{A_{12}}{A_{11}}\Big(-1+\frac{1}{\sqrt{\Delta}}\Big)+\frac{(A_{12}^2-A_{11}A_{22})y_2}{A_{11}\sqrt{\Delta}},\\ \frac{dy_2}{d\tau}=&-\frac{1}{\sqrt{\Delta}}\frac{\partial V}{\partial x_2}.\end{aligned}\tag{3.8}$$

Treating the term $\Omega_i(1-\sqrt{\Delta}^{-1})$ as a function y_2 and V, we see that it remains bounded in C^2-topology as $\Omega_i\to\infty$. Therefore, the right hand side of Equation (3.8) is smooth and bounded in C^2-topology for any large Ω_i and bounded y_2.

It is proved in [**CZ2**] that there is a small neighborhood of the minimal point of $F(\cdot,\Omega_i,E)$ where the minimal curve $\gamma_{\Omega_i,E}(\cdot,x_2)$ is uniquely determined by x_2. Since Equation (3.8) is equivalent to the Lagrange equation determined by L_1, it implies that boundary value problem $\{x_2(0)=x_2(2\pi)=x_2'\}$ of Equation (3.8) is well defined provided x_2' is in the neighborhood. Therefore, there is a smooth dependence of $y_2=y_2(x_2')$ such that the solution of the initial value problem $\{x_2(0)=x_2',y_2(0)=y_2(x_2')\}$ is the same as the boundary value problem. Applying the theorem of the smooth dependence of solution of ODE on its initial value, we find the first and the second derivatives of $(\gamma_{\Omega_i,E}(\tau,x_2),\partial_{\dot x_2}L_1(\gamma_{\Omega_i,E}(\tau,x_2),\dot\gamma_{\Omega_i,E}(\tau,x_2),\tau))$ with respect to x_2 is smooth and bounded for any large Ω_i. As L_1 is positive definite in $\dot x_2$, the first and the second derivatives of $\dot\gamma_{\Omega_i,E}(\tau,x_2)$ in x_2 is also bounded.

By direct calculations (doing integration by parts) we find:

$$\begin{aligned}\frac{\partial F_R}{\partial x_2}=&-\sum_{k\neq0}\frac{1}{ik}\int_0^{2\pi}\Big(\frac{d^2V_k}{dx_2^2}\frac{\partial\gamma_{\Omega_i,E}}{\partial x_2}\dot\gamma_{\Omega_i,E}+\frac{dV_k}{dx_2}\frac{\partial\dot\gamma_{\Omega_i,E}}{\partial x_2}\Big)e^{ik\Omega_i\tau}d\tau\\ &+\int_0^{2\pi}\Big(\frac{\partial R_L}{\partial\dot x_2}\frac{\partial\dot\gamma_{\Omega_i,E}}{\partial x_2}+\frac{\partial R_L}{\partial x_2}\frac{\partial\gamma_{\Omega_i,E}}{\partial x_2}\Big)d\tau,\\ \frac{\partial^2F_R}{\partial x_2^2}=&-\sum_{k\neq0}\frac{1}{ik}\int_0^{2\pi}\Big(\frac{d^3V_k}{dx_2^3}\Big(\frac{\partial\gamma_{\Omega_i,E}}{\partial x_2}\Big)^2\dot\gamma_{\Omega_i,E}+\frac{dV_k}{dx_2}\frac{\partial^2\dot\gamma_{\Omega_i,E}}{\partial x_2^2}\Big)e^{ik\Omega_i\tau}d\tau\end{aligned}$$

$$
\begin{aligned}
&-\sum_{k\neq 0}\frac{1}{ik}\int_0^{2\pi}\frac{d^2V_k}{dx_2^2}\Big(2\frac{\partial\gamma_{\Omega_i,E}}{\partial x_2}\frac{\partial\dot\gamma_{\Omega_i,E}}{\partial x_2}+\frac{\partial^2\gamma_{\Omega_i,E}}{\partial x_2^2}\dot\gamma_{\Omega_i,E}\Big)e^{ik\Omega_i\tau}d\tau\\
&+\int_0^{2\pi}\Big(\frac{\partial^2R_L}{\partial\dot x_2^2}\Big(\frac{\partial\dot\gamma_{\Omega_i,E}}{\partial x_2}\Big)^2+\frac{\partial^2R_L}{\partial x_2^2}\Big(\frac{\partial\gamma_{\Omega_i,E}}{\partial x_2}\Big)^2\Big)d\tau\\
&+\int_0^{2\pi}\Big(\frac{\partial R_L}{\partial\dot x_2}\frac{\partial^2\dot\gamma_{\Omega_i,E}}{\partial x_2^2}+\frac{\partial R_L}{\partial x_2}\frac{\partial^2\gamma_{\Omega_i,E}}{\partial x_2^2}\\
&+2\frac{\partial^2R_L}{\partial x_2\partial\dot x_2}\frac{\partial\dot\gamma_{\Omega_i,E}}{\partial x_2}\frac{\partial\gamma_{\Omega_i,E}}{\partial x_2}\Big)d\tau.
\end{aligned}
$$

It follows from these formulae that F_R is C^2-bounded if $V\in C^4$. q.e.d.

Let us calculate the second derivative of F_0 with respect to x_2:

$$
\begin{aligned}
\frac{\partial^2F_0}{\partial x_2^2}=&\int_0^{2\pi}\Big(\frac{1}{A_{22}}\Big(\frac{\partial\dot\gamma_{\Omega_i,E}}{\partial x_2}\Big)^2+\frac{d^2}{dx_2^2}[V](\gamma^*_{\Omega_i,E})\Big(\frac{\partial\gamma_{\Omega_i,E}}{\partial x_2}\Big)^2\Big)d\tau\\
&+\int_0^{2\pi}\Big(\frac{1}{A_{22}}\dot\gamma_{\Omega_i,E}\frac{\partial^2\dot\gamma_{\Omega_i,E}}{\partial x_2^2}+\frac{d}{dx_2}[V](\gamma^*_{\Omega_i,E})\frac{\partial^2\gamma_{\Omega_i,E}}{\partial^2x_2}\Big)d\tau.
\end{aligned}
$$

The second integral approaches zero as $\Omega_i\to\infty$ if $\gamma_{\Omega_i,E}=\gamma^*_{\Omega_i,E}$. Indeed, it follows from Proposition 3.1 that $|\dot\gamma^*_{\Omega_i,E}(\tau)|\to 0$ and $|\gamma^*_{\Omega_i,E}(\tau)-x_2^*|\to 0$ as $\Omega_i\to\infty$, where x_2^* is a minimal point of $[V]$. Therefore, $\frac{d[V]}{dx_2}(\gamma^*_{\Omega_i,E}(\tau))\to 0$ as $\Omega_i\to\infty$. To estimate the first integral, we note that the minimizer $\gamma^*_{\Omega_i,E}(\tau)$ stays in a small neighborhood of the minimal point of $[V]$ provided Ω_i is sufficiently large. Given a generic $V\in\mathfrak{V}_\infty$, certain $d>0$ exists such that $\frac{d^2}{dx_2^2}[V](\gamma^*_{\Omega_i,E})\geq d$ holds for all $\tau\in[0,2\pi]$. The linearized variational equation of (3.8) with the boundary condition $\frac{\partial\gamma_{\Omega_i,E}}{\partial x_2}(0)=\frac{\partial\gamma_{\Omega_i,E}}{\partial x_2}(2\pi)=1$ admits a unique solution

$$
\Big(\frac{\partial}{\partial x_2}\gamma_{\Omega_i,E}(\tau,x_2),\frac{\partial}{\partial x_2}\frac{\partial L_1}{\partial\dot x_2}\Big(\gamma_{\Omega_i,E}(\tau,x_2),\dot\gamma_{\Omega_i,E}(\tau,x_2),\tau\Big)\Big),
$$

and the right hand side of (3.8) is C^2-smooth and uniformly bounded for any large Ω_i. Therefore, certain $T>0$ exists, uniformly lower bounded for any large Ω_i, such that $\frac{\partial\gamma_{\Omega_i,E}}{\partial x_2}(\tau)>\frac{1}{2}$ for all $\tau\in[0,T]\cup[2\pi-T,2\pi]$. As the minimizer is $\frac{2\pi}{\Omega_i}$-periodic, we find $\frac{\partial\gamma_{\Omega_i,E}}{\partial x_2}(\tau)>\frac{1}{2}$ for all $\tau\in[0,2\pi]$ for large Ω_i. These arguments lead to the conclusion that certain $\mu>0$ and suitably large $\Omega^*>0$ exist such that $\partial^2_{x_2}F_0(\gamma^*_{\Omega_i,E}(0),\Omega_i,E)\geq 2\mu$ if $\Omega_i\geq\Omega^*$. As the function of action F is a $O(\frac{1}{\Omega_i})$-perturbation of F_0 we have

$$
\frac{\partial^2}{\partial x_2^2}F(\gamma^*_{\Omega_i,E}(0),\Omega_i,E)\geq\mu,\qquad \Omega_i\geq\Omega^*.
$$

Let x_2^* be the minimal point of $F(\cdot,\Omega_i,E)$. In this case we have

$$
F(x_2,\Omega_i,E)-F(x_2^*,\Omega_i,E)\geq\mu(x_2-x_2^*)^2,
$$

if $|x_2 - x_2^*|$ is suitably small, Let $B_E := u^- - u^+$ denote the barrier function where $u^\pm$ are the backward and forward weak KAM solutions, as it was shown in [**CZ2**], one has

$$B_E(x_2) - B_E(x_2^*) \geq F(x_2, \Omega_i, E) - F(x_2^*, \Omega_i, E).$$

As barrier function is semi-concave, there exists a number $C_L > 2\mu$ such that

$$B_E(x_2) - B_E(x_2^*) \leq C_L(x_2 - x_2^*)^2.$$

It follows that the hyperbolicity of the minimizer is not weaker than $\frac{1}{\kappa} = \sqrt{1 - \frac{2\mu}{C_L}}$. Let us assume the contrary, denote by $(\gamma_E^*(\tau), \dot{\gamma}_E^*(\tau))$ the minimal periodic orbit and denote by $(\gamma^\pm(\tau), \dot{\gamma}^\pm(\tau))$ the orbit such that $\gamma^-(0) = \gamma^+(0)$ and they asymptotically approaches to the orbit $(\gamma^*(\tau), \dot{\gamma}^*(\tau))$ as $\tau \to \pm\infty$, we then have

$$|\gamma_E^*(\pm j) - \gamma^\pm(\pm j)| > \frac{1}{\kappa}|\gamma_E^*(\pm(j-1)) - \gamma^\pm(\pm(j-1))|,$$

if $|\gamma_E^*(0) - \gamma^\pm(0)|$ is suitably small. The computation below leads to a contradiction:

$$\begin{aligned} C_L(\gamma^\pm(0) - \gamma_E^*(0))^2 &\geq B_E(\gamma^\pm(0)) - B_E(\gamma_E^*(0)) \\ &\geq \sum_{j=1}^{\infty} \Big(F(\gamma^-(-j)) - F(\gamma_E^*(0))\Big) + \Big(F(\gamma^+(j)) - F(\gamma_E^*(0))\Big) \\ &> 2\mu \frac{(\gamma^\pm(0) - \gamma_E^*(0))^2}{1 - \kappa^2} = C_L(\gamma^\pm(0) - \gamma_E^*(0))^2. \end{aligned}$$

We observe a fact that such hyperbolicity holds for all $E \in [0, \bar{K}]$. Therefore, these (E, g)-minimal periodic orbits make up a cylinder $\Pi^i_{0,\bar{K}\Omega_i,g}$.

Let us return back to the coordinates before the transformation (3.4). That the new coordinate x_1 goes around the circle $\mathbb{T}$ once amounts to that the old coordinate x_1 sweeps out an angle of Ω_i. In the original coordinate system, we have $\frac{dx_1}{d\theta} = \Omega_i + O(1)$. Therefore, the normal hyperbolicity we obtain for $\tau = 2\pi$-map is almost the same as the time $\theta = 2\pi$-map determined by the Hamiltonian flow $\Phi^\theta_{G_i}$.

To investigate whether the tangent space of $\mathbb{T}^2$ over the cylinder admits an invariant splitting, we consider the tangent bundle of $\Pi^i_{0,\bar{K}\Omega_i,g}$. The tangent space at a point $z \in \Pi^i_{0,\bar{K}\Omega_i,g}$ is two dimensional, spanned by the a vector v'_z tangent to the minimal orbit passing through this point and an orthogonal vector, denoted by v''_z. Because the map $\Phi^{2\pi}_{G_i}$ preserves the symplectic structure, the cylinder $\Pi^i_{0,\bar{K}\Omega_i,g}$ is an invariant symplectic sub-manifold made up by periodic orbits, there exists a number $\lambda \geq 1$ such that

$$\lambda^{-1}\|v_z\| \leq \|D\Phi^{2k\pi}_{G_{\Omega_i}}(z)v_z\| \leq \lambda\|v_z\|$$

holds for any $v_z \in \mathrm{Span}(v_z', v_z'')$ and for any $k \in \mathbb{Z}$. Let $\Lambda = \kappa^k$ with $k = [\frac{\lambda}{\kappa}] + 1$, then Formula (3.3) holds. From (3.7) we see that the number λ is uniformly bounded for any large Ω_i. This completes the proof of Theorem 3.1. q.e.d.

Although the Hamiltonian equation for $G_{\epsilon,i}$ contains one term Ω_i which appears to be singular as $\Omega_i \to \infty$

$$\begin{cases} \dot{x}_1 = \Omega_i + A_{11}y_1 + A_{12}y_2 + \epsilon^\sigma \partial_{y_1} R_{\epsilon,i}, & \dot{y}_1 = \frac{\partial V}{\partial x_1} - \epsilon^\sigma \partial_{x_1} R_{\epsilon,i}, \\ \dot{x}_2 = A_{12}y_1 + A_{22}y_2 + \epsilon^\sigma \partial_{y_2} R_{\epsilon,i}, & \dot{y}_2 = \frac{\partial V}{\partial x_2} - \epsilon^\sigma \partial_{x_2} R_{\epsilon,i}, \end{cases}$$

that term does not contribute to the variational equation. The right hand side of its variational equation

$$\Delta \dot{z} = J \frac{\partial^2 G_{\epsilon,i}}{\partial z^2} \Delta z, \qquad z = (x, y)$$

is uniformly bounded in C^1-topology as $\Omega_i \to \infty$. It allows us to apply the theorem of normally hyperbolic invariant manifold to the Hamiltonian flow $\Phi^\theta_{G_{\epsilon,i}}$. Thus, there exists $\epsilon_{i_0} > 0$ such that for $\epsilon \le \epsilon_{i_0}$ the map $\Phi_{G_{\epsilon,i}}$ defined in (2.6) also admits a NHIC $\Pi^{\epsilon,i}_{0,\bar{K}\Omega_i,g}$ which is a small perturbation of $\Pi^i_{0,\bar{K}\Omega_i,g}$.

4. Finiteness of invariant cylinders

Recall that along the resonant path $\Gamma' \cap \{|p - p''| \le \epsilon^\sigma\}$ we choose points $\{p_i'\}$ so that $p_0' = p''$, $\partial_1 h(p_i') = Ki\sqrt{\epsilon}$, where $K \in \mathbb{Z}$. We claim that there exists a positive constant $K_1 > 0$, independent of ϵ, such that

$$\|p'_{i+1} - p'_i\| \le K_1 \sqrt{\epsilon}. \tag{4.1}$$

Indeed, because

$$\det \begin{bmatrix} \frac{\partial^2 h}{\partial p_2^2} & \frac{\partial^2 h}{\partial p_2 \partial p_3} \\ \frac{\partial h}{\partial p_2} & \frac{\partial h}{\partial p_3} \end{bmatrix} = \frac{\partial^2 h}{\partial p_2^2} \frac{\partial h}{\partial p_3} \ne 0$$

holds along the resonant path Γ', it follows from the implicit function theorem that there are smooth functions $p_2(p_1)$ and $p_3(p_1)$ which solve the equations $h(p_1, p_2(p_1), p_3(p_1)) \equiv$ constant and $\partial_2 h(p_1, p_2(p_1), p_3(p_1)) \equiv 0$, where the notation $p = (p_1, p_2, p_3)$ is used. Let $\dot{p}_j$ be the derivative of p_j in p_1 for $j = 2, 3$, it follows from the equation $h(p_1, p_2(p_1), p_3(p_1)) \equiv$ constant that $\dot{p}_3 = 0$ holds at the double resonant point $p = p''$.

Consider the variation of $\partial h = \omega$ along the resonant path. Let $\dot{\omega}_j$ be the derivative of ω_j in p_1 for $j = 1, 2, 3$, one has

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = \frac{\partial^2 h}{\partial p^2} \begin{bmatrix} 1 \\ \dot{p}_2 \\ \dot{p}_3 \end{bmatrix}.$$

Since $\dot\omega_2\equiv 0$ and $\dot p_3=0$ holds at the double resonant point, one has $\dot\omega_1\neq 0$ at $p=p''$ otherwise one would have $0=\langle\dot\omega,\dot p\rangle=\langle\partial^2h\dot p,\dot p\rangle$ which contradicts the fact ∂^2h is positive definite, where $\dot\omega=(\dot\omega_1,\dot\omega_2,\dot\omega_3)$ and $\dot p=(1,\dot p_2,\dot p_3)$. As we are restricted on the domain $\{|p-p''|\le\epsilon^\sigma\}$, one obtains (4.1) from the relations $\partial_1h(p'_{i+1})-\partial_2h(p'_i)=K\sqrt{\epsilon}$, $\dot\omega_1(p'')\neq 0$, $|p'_i-p''|\le\epsilon^\sigma$ and $\partial h(p'_{i+1})-\partial h(p'_i)=\partial^2h(p'_i+\lambda(p'_{i+1}-p'_i))(p'_{i+1}-p'_i)$ for $\lambda\in(0,1)$.

Around the $\bar K\sqrt{\epsilon}$-neighborhood of p'_i ($\bar K$ is independent of ϵ), the Hamiltonian is resaled to the form $G_{\epsilon,i}(x,y,\theta)=G_i(x,y)+\epsilon^\sigma R_{\epsilon,i}(x,y,\theta)$ with

$$G_i(x,y)=Kiy_1+\frac{1}{2}\langle Ay,y\rangle-V(x).$$

The subscript i ranges from 0 to $i_1=O([\frac{1}{K}\epsilon^{\sigma-\frac{1}{2}}])$. We fix a potential $V\in\mathfrak{V}_\infty$. By the result of the last section, there is i_0 independent of ϵ such that, for each $i\ge i_0$, $Ki\ge\Omega^*$ holds and the Hamiltonian flow $\Phi^\theta_{G_i}$ admits a unique normally hyperbolic invariant cylinder $\Pi^i_{0,\bar K\Omega_i,g}$ of (2.4).

We claim that for all $i\ge i_0$, each cylinder $\Pi^i_{0,\bar K\Omega_i,g}$ is just a part of some large cylinder. Indeed, for two adjacent subscripts $i,i+1$, it follows from (4.1) that the energy level $G_i^{-1}(\bar K\Omega_i)$ is contained in the set where $G_{i+1}>\Omega_i$ provided $\bar K$ is suitably large. To see why, we recall that the rescaling $\sqrt{\epsilon}y_\ell=p_\ell-p'_{\ell,i}$ ($\ell=1,2$) in (2.1) is introduced to obtain the Hamiltonian $G_{\epsilon,i}$.

Due to the normal hyperbolicity, in the region where both $G_{\epsilon,i}$ and $G_{\epsilon,i+1}$ remains valid, there is a unique cylinder $\tilde\Pi^{i,\epsilon}_{0,\bar K\Omega_i,g}\cap\tilde\Pi^{i+1,\epsilon}_{0,\bar K\Omega_{i+1},g}$ containing the relevant Aubry sets. Due to the coordinate rescaling (2.1), there exists a unique NHIC of $\Phi^\theta_{G_\epsilon}$ which extends from the energy level $G_\epsilon^{-1}(\omega_3(p'_{3,i_0}-p''_3)\epsilon^{-1})$ to the energy level $G_\epsilon^{-1}(\omega_3(p'_{3,i_1}-p''_3)\epsilon^{-1})$, the subscript i_1 is chosen so that p'_{i_1} is the largest one satisfying the condition $|p'_{i_1}-p''|\le\epsilon^\sigma$, namely, one has $|p'_{i_1+1}-p''|>\epsilon^\sigma$. Since $\frac{dp_3}{dp_1}=0$ holds at the double resonant point p'', one has $\omega_3(p'_{3,i_1}-p''_3)\epsilon^{-1}=O(|p'_{1,i_1}-p''_1|^2)\epsilon^{-1}=E'\epsilon^{2\sigma-1}$. The tangent space of $\mathbb{T}^2$ over the whole $\Pi^\epsilon_{E_{i_0},E_{i_1},g}$ admits normally hyperbolic invariant splitting of (3.3). For $i<0$, the situation is the same, instead of considering the class g, we consider the class $-g$.

Back to the original coordinates, for the class g as well as for $-g$, there is a NHIC which extends from $\Omega_{i_0}\sqrt{\epsilon}$-neighborhood of the double resonant point p'' to the border of the disk $\{|p-p''|\le\epsilon^\sigma\}$.

Since $\|p'_{i+1}-p'_i\|\le K_1\sqrt{\epsilon}$, there are as many as $O([\epsilon^{\sigma-\frac{1}{2}}])$ points $\{p'_i\}$ along the resonant path Γ'. Around at most $2i_0+1$ points (the number is independent of ϵ), the situation need to be handled in the way treated in [**CZ2**]. For each of them, there is an open-dense set

$\mathfrak{V}_i \subset C^r(\mathbb{T}^2,\mathbb{R})$. For each $V \in \mathfrak{V}_i$, the Hamiltonian flow $\Phi^\theta_{G_i}$ admits NHICs in the domain with certain normal hyperbolicity independent of ϵ. Therefore, certain $\epsilon_i > 0$ exists such that for each $\epsilon \le \epsilon_i$, the cylinders survive the time-periodic perturbation $\Phi^\theta_{G_i} \to \Phi^\theta_{G_{\epsilon,i}}$. Note the Hamiltonian $G_{\epsilon,i}$ is a local expression of G_ϵ.

Now the situation becomes clear. One cylinder extends from $G_\epsilon^{-1}(\epsilon^d)$ to $G_\epsilon^{-1}(E_0)$, another cylinder extends from $G_\epsilon^{-1}(E_{i_0})$ to $G_\epsilon^{-1}(\epsilon^{2\sigma-1})$. Between the energy level $G_\epsilon^{-1}(E_0)$ and $G_\epsilon^{-1}(E_{i_0})$ there are finitely many pieces of NHICs. Each energy level intersects these NHIC's along one or two circles. Let

$$\mathfrak{V} = \Big(\bigcap_{|j|<i_0} \mathfrak{V}_j\Big) \cap \mathfrak{V}_\infty,$$

for each $V \in \mathfrak{V}$ we choose $\epsilon_V = \min\{\epsilon_0, \cdots, \epsilon_{\pm i_0},\}$. Therefore, the first part of Theorem 1.1 is proved for $V \in \mathfrak{V}$ and $\epsilon \le \epsilon_V$.

5. Aubry sets along resonant path: near double resonance

Since the NHICs obtained may be overflowing, we need to identify whether the Aubry sets along resonant path remain in the cylinder.

An irreducible class $g \in H_1(\mathbb{T}^2,\mathbb{R})$ determines a channel of first cohomology classes

$$\mathbb{C}_{g,G} = \bigcup_{\lambda\in\mathbb{R}^+} \mathscr{L}_{\beta_G}(\lambda g), \qquad \mathbb{C}_{E',E'_1,g,G} = \Big\{c \in \mathbb{C}_{g,G} : \alpha_G(c) \in [E', E'_1]\Big\}.$$

Theorem 5.1. *For the Hamiltonian $G_{\epsilon,0}$ of* (2.8)*, a class $g \in H_1(\mathbb{T}^2,\mathbb{Z})$ and a large positive number $E'_1 > 0$, there exists a residual set $\mathfrak{V} \subset C^r(\mathbb{T}^2,\mathbb{R})$ $(r \ge 5)$. For each $V \in \mathfrak{V}$ there are numbers $N > 1$, $\epsilon_0 > 0$ and $d > 0$ such that for $\epsilon \le \epsilon_0$ it holds for each $c \in \mathbb{C}_{N\epsilon^d,E'_1,g}$ that the Aubry set $\tilde{\mathcal{A}}(c)$ of $G_{\epsilon,0}$ lies on the invariant cylinder.*

The proof of this theorem is built on the following preliminary works.

Lemma 5.1. *For the Hamiltonian $G_{\epsilon,0}$ of* (2.8)*, if an orbit $z(s)$ remains in a bounded region $\Omega \subset T^*M$ for $s \in [s_0,s_1]$, some constant $K > 0$ exists, independent of ϵ but may depend on Ω, such that the variation of energy along the orbit $z(s)$ is bounded by*

$$|G_{\epsilon,0}(z(s_1),s_1) - G_{\epsilon,0}(z(s_0),s_0)| \le K|s_1 - s_0 + 1|\epsilon^\sigma.$$

Proof. Along an orbit $z(s)$ of the Hamiltonian flow $\Phi^\theta_{G_{\epsilon,0}}$, the variation of the energy is controlled by

$$\frac{d}{d\theta}G_{\epsilon,0}(z(\theta),\theta) = \frac{\partial}{\partial\theta}G_{\epsilon,0}(z(\theta),\theta) = \omega_3\epsilon^{\sigma-\frac12}\frac{\partial R_{\epsilon,0}}{\partial x_3}, \tag{5.1}$$

where $x_3 = \omega_3\frac{\theta}{\sqrt{\epsilon}}$. Recall $R_{\epsilon,0}$ is regular and 2π-periodic in $\omega_3\frac{\theta}{\sqrt{\epsilon}}$, see (1.7), we expend R into Fourier series

$$\partial_\theta G_{\epsilon,0}(z,\theta) = \omega_3\epsilon^{\sigma-\frac{1}{2}}\sum_{k\neq 0} R_k(z)e^{ik\frac{\omega_3\theta}{\sqrt{\epsilon}}}.$$

Integrating by parts, we have

$$\begin{aligned}\epsilon^{\sigma-\frac{1}{2}}\int_{s_0}^{s_1} R_k(z(\theta))e^{ik\frac{\omega_3\theta}{\sqrt{\epsilon}}}d\theta =& \frac{\epsilon^\sigma}{i\omega_3 k}R_k(z(\theta))e^{ik\frac{\omega_3\theta}{\sqrt{\epsilon}}}\Big|_{s_0}^{s_1}\\ &-\frac{\epsilon^\sigma}{i\omega_3 k}\int_{s_0}^{s_1}\langle\partial R_k,\dot{z}(\theta)\rangle e^{ik\frac{\omega_3\theta}{\sqrt{\epsilon}}}d\theta.\end{aligned} \tag{5.2}$$

Because the perturbation term R is C^4-smooth, we have

$$|R_k| \le \frac{\|R_{\epsilon,0}\|_{C^3}}{2\pi|k|^3}, \qquad |\partial R_k| \le \frac{\|\partial_z R_{\epsilon,0}\|_{C^3}}{2\pi|k|^3}.$$

Since $\dot{z} = J\mathrm{diag}(Ay,\partial V)$, by setting

$$K = \frac{1}{\omega_3\pi}\max_{z\in\Omega}\Big\{\|R_{\epsilon,0}\|_{C^3},\Big(|Ay| + \Big|\frac{\partial V}{\partial x}\Big|\Big)\|\partial_z R_{\epsilon,0}\|_{C^3}\Big\}\sum_{k\neq 0}\frac{1}{|k|^4},$$

which is independent of ϵ, it follows from (5.1) and (5.2) that,

$$\begin{aligned}|G_{\epsilon,0}(z(s_1),s_1) - G_{\epsilon,0}(z(s_0),s_0)| &= \Big|\int_{s_0}^{s_1}\frac{d}{d\theta}G_{\epsilon,0}(z(\theta),\theta)d\theta\Big|\\ \le \Big|\sum_{k\neq 0}\frac{\epsilon^\sigma}{i\omega_3 k}R_k(z(\theta))e^{ik\frac{\omega_3\theta}{\sqrt{\epsilon}}}\Big|_{s_0}^{s_1} &- \sum_{k\neq 0}\frac{\epsilon^\sigma}{i\omega_3 k}\int_{s_0}^{s_1}\langle\partial R_k,\dot{z}(\theta)\rangle e^{ik\frac{\omega_3\theta}{\sqrt{\epsilon}}}d\theta\Big|,\end{aligned}$$

the right hand side is not bigger than $K(s_1 - s_0 + 1)\epsilon^\sigma$. q.e.d.

Since $\Pi_{E_i,E_{i+1},g}$ is a NHIC, the channel $\mathbb{C}_{E_i,E_{i+1},g,G_0}$ admits a foliation of lines (one-dimensional flat), denoted by $\{I_E\}$. Restricted on each I_E, α_{G_0} keeps constant, while restricted on a line Γ_g orthogonal to these flats, the function is smooth since G_0 can be treated as a Hamiltonian with one degree of freedom when it is restricted on the cylinder. Therefore, the function α_{G_0} is smooth in $\mathbb{C}_{0,E_1,g,G_0}$ and $\mathbb{C}_{E_i,E_{i+1},g,G_0}$.

Proposition 5.1. *There exists a number $N > 1$, independent of ϵ, so that the Aubry set for $c \in \Gamma_g \cap \alpha^{-1}_{G_{\epsilon,0}}(E)$ with $E \ge N\epsilon^d$ lies in the NHIC, each orbit in this set does not hit the energy level set $G^{-1}_{\epsilon,0}(E)$ with $E \le \epsilon^d$.*

Proof. We only need to prove the conclusion for the Hamiltonian $G'_{\epsilon,0}$ of (2.30), because $G'_{\epsilon,0} = G_{\epsilon,0}$ when it is restricted on the set where $G'_{\epsilon,0} \in [\epsilon^d, E_1]$. So, each orbit in the Aubry set lies in the cylinder forever. If the proposition does not hold, there would exist an orbit $z(s)$ in the Aubry set for $c \in \Gamma_g \cap \alpha^{-1}_{G'_{\epsilon,0}}(N\epsilon^d)$ hitting the energy level

$G'^{-1}_{\epsilon,0}(\epsilon^d)$ at the time s_0, i.e., $G'_{\epsilon,0}(z(s_0),s_0)=\epsilon^d$. Due to Lemma 5.1, it returns to a neighborhood of $z(s_0)$ after a time $s'=O(|\ln\epsilon^d|)$ (cf. formula (2.9)) and

$$|G'_{\epsilon,0}(z(s'+s_0),s'+s_0)-G'_{\epsilon,0}(z(s_0),s_0)|\le K(s'+1)\epsilon^\sigma. \tag{5.3}$$

As $G_0^{-1}(E)\cap\Pi_{E_d,E_1,g}$ is an invariant circle for $\Phi^t_{G_0}$, the perturbed cylinder is $O(\epsilon^\sigma)$-close to the original one and the cylinder may be crumpled but at most up to the order $O(\epsilon^{d(-\mu_6-\mu_7)})$ (cf. (2.22)), so there is a time $S=O(|\ln\epsilon^d|)$ and a small number $\mu'=d(\mu_6+\mu_7)>0$ such that $\sigma-\mu'>0$ and

$$\|z(S+s_0)-z(s_0)\|\le C_{14}(S+1)\epsilon^{\sigma-\mu'}.$$

Since $z(s)$ is in the Aubry set for the class c, the curve $x(s)$ is c-static. Let $\alpha_{G'_{\epsilon,0}}$ and α_{G_0} denote the α-function for $G'_{\epsilon,0}$ and G_0, respectively, one has

$$\Big|\int_{s_0}^{S+s_0}(L_{G'_{\epsilon,0}}(x(s),\dot x(s),s)-\langle c,\dot x(s)\rangle+\alpha_{G'_{\epsilon,0}}(c))ds\Big|\le C_{15}(S+1)\epsilon^{\sigma-\mu'}. \tag{5.4}$$

As the cylinder $\Pi_{E_d,E_1,g}\times\frac{\sqrt{\epsilon}}{\omega_3}\mathbb{T}$ is ϵ^σ-close to $\tilde\Pi^\epsilon_{E_d,E_1,g}$, $\exists$ a c'-minimal orbit $z'(s)$ of $\Phi^s_{G_0}$ on $\Pi_{E_d,E_1,g}$ such that $\alpha_{G_0}(c')=\epsilon^d$ and $\|z'(s_0)-z(s_0)\|\le O(\epsilon^{\sigma-\mu'})$. Let $\Gamma_x=\bigcup_{s=s_0}^{s_0+S}(x(s),y(s))$ and $\Gamma_{x'}=\bigcup_{s=s_0}^{s_0+S'}(x'(s),y'(s))$ where S' is the period of $x'(s)$, we have an estimate on the Hausdorff distance $d_H(\Gamma_x,\Gamma_{x'})\le O((S+1)\epsilon^{\sigma-\mu'})$. So,

$$\int_{\Gamma_x}\langle y,dx\rangle-\int_{\Gamma_{x'}}\langle y,dx\rangle=O((S+1)\epsilon^{\sigma-\mu'}).$$

Because of $G_0(x'(s),y'(s))\equiv\alpha_{G_0}(c')$ we have

$$\begin{aligned}0&=\int_0^{S'}(L_{G_0}(x'(t),\dot x'(t))-\langle c',\dot x'(t)\rangle+\alpha_{G_0}(c'))dt\\&=\int_0^{S'}\langle y'(s)-c',\dot x'(s)\rangle ds.\end{aligned}$$

Let $\bar x(s)$ be the lift of $x(s)$ to the universal covering space, it follows that

$$\begin{aligned}&\int_{s_0}^{S+s_0}\langle y(s)-c,\dot x(s)\rangle ds\\&=\int_{s_0}^{S+s_0}\langle y(s)-c',\dot x(s)\rangle ds-\int_{s_0}^{S'+s_0}\langle y'(s)-c',\dot x'(s)\rangle ds\\&\quad-\langle c-c',\bar x(S+s_0)-\bar x(s_0)\rangle\\&=\int_{\Gamma_x}\langle y,dx\rangle-\int_{\Gamma_{x'}}\langle y',dx'\rangle+O((S+1)\epsilon^{\sigma-\mu'})\end{aligned} \tag{5.5}$$

$$- \langle c - c', \bar{x}(S + s_0) - \bar{x}(s_0)\rangle$$
$$= - \langle c - c', \bar{x}(S + s_0) - \bar{x}(s_0)\rangle + O((S + 1)\epsilon^{\sigma-\mu'}),$$

where the last equality follows from the estimate $d_H(\Gamma_x, \Gamma_{x'}) \le O((S + 1)\epsilon^{\sigma-\mu'})$. Since $G'_\epsilon(z(s_0), s_0) = \alpha_{G_0}(c')$, it follows from (5.3) that, for all $s \in [s_0, S + s_0]$, we have

$$\alpha_{G'_{\epsilon,0}}(c) - G'_{\epsilon,0}(x(s), y(s), s) \ge \alpha_{G'_\epsilon}(c) - \alpha_{G_0}(c') - O((S + 1)\epsilon^{\sigma-\mu'}).$$

Consequently, by using the formulae (5.3) and (5.5) we have

(5.6)
$$\int_{s_0}^{S+s_0} (L_{G'_{\epsilon,0}}(x(s), \dot{x}(s), s) - \langle c, \dot{x}(s)\rangle + \alpha_{G'_{\epsilon,0}}(c))ds$$
$$= \int_{s_0}^{S+s_0} \Big(\langle y(s) - c, \dot{x}(s)\rangle + (\alpha_{G'_{\epsilon,0}}(c) - G'_{\epsilon,0}(x(s), y(s), s))\Big)ds$$
$$\ge (\alpha_{G'_{\epsilon,0}}(c) - \alpha_{G_0}(c'))S - \langle c - c', \bar{x}(S + s_0) - \bar{x}(s_0)\rangle - O((S + 1)\epsilon^{\sigma}).$$

To derive contradiction between the right-hand-side of above inequality and (5.4), we note that the function α_{G_0} keeps constant along each flat in the channel $\mathbb{C}_{0,E_1,g,G_0}$. The frequency vector $\omega(c)$ is, therefore, parallel to the direction of Γ_g. To get the norm of $\omega(c)$, we assume the general case $g = k_1g_1 + k_2g_2$ and consider the Hamiltonian in the finite covering space $\bar{M} = \bar{k}_1\mathbb{T} \times \bar{k}_2\mathbb{T}$ where $\bar{k}_m = k_1g_{1m} + k_2g_{2m}$ for $m = 1, 2$ if we write $g_j = (g_{j1}, g_{j2})$ for $j = 1, 2$. In $T^*\bar{M}$ there are $k_1 + k_2$ fixed points for the return map. According to Formula (2.9), for small $E > 0$ the period of the frequency λg is $T_{\lambda g} = \tau_{E,g} - \lambda_1^{-1}(k_1 + k_2)\ln E$ where $\tau_{E,g}$ is uniformly bounded as $E \to 0$. Since $\partial\alpha_{G_0} = \omega = \lambda g$, one has

$$\frac{\lambda_1}{\lambda_1\tau_{E,g} - (k_1 + k_2)\ln E} = |\omega|, \qquad \forall\ c \in \Gamma_g. \tag{5.7}$$

Let c^* be the class such that $\alpha_{G_0}(c^*) = \alpha_{G'_{\epsilon,0}}(c)$, then $\alpha_{G_0}(c^*)-\alpha_{G_0}(c') = (N - 1)\epsilon^d > 0$. Since α_{G_0} is convex and $\alpha_{G_0}(c^*) > \alpha_{G_0}(c')$,

$$\langle c^* - c', \partial\alpha_{G_0}(c^*)\rangle > \alpha_{G_0}(c^*) - \alpha_{G_0}(c').$$

Since $c^*, c' \in \Gamma_g$, $c^* - c'$ is parallel to $\partial\alpha_{G_0}(c^*)$. It follows from (5.7) and the relation $E = O(\epsilon^d)$ that some constant $C_{16} > 0$ exists such that

$$|c^* - c'| \ge \frac{1}{\|\partial\alpha_{G_0}(c^*)\|}\Big(\alpha_{G_0}(c^*) - \alpha_{G_0}(c')\Big) \ge C_{16}\epsilon^d|\ln\epsilon^d|. \tag{5.8}$$

To measure the distance between c^* and c, we exploit the convexity of the α-function and get $|c - c^*||\omega(c^*)| = |\langle c - c^*, \omega(c^*)\rangle| \le |\alpha_{G_0}(c^*) - \alpha_{G_0}(c)| = |\alpha_{G_{\epsilon,0}}(c) - \alpha_{G_0}(c)| = \epsilon^\sigma$. It follows from the fact that the α-function undergoes small variation: $|\alpha_L(c) - \alpha_{L'}(c)| \le \varepsilon$ for small perturbation $L' \to L$ with $\|L' - L\|_{C^1} \leq \varepsilon$ ([**C11**]). Therefore, some number $C_{17} > 0$ exists such that

$$|c^* - c| \leq C_{17}\epsilon^{\sigma}|\ln \epsilon^d|. \tag{5.9}$$

Be aware that α_{G_0} is smooth and strictly convex when it is restricted on the line Γ_g, the first cohomology classes $c', c^* \in \Gamma_g$ are uniquely determined so that $\alpha_{G_0}(c') = \epsilon^d$, $\alpha_{G_0}(c^*) = N\epsilon^d$ where the number N is chosen such that

$$\ln(N-1) = 3\max\{|\sup \tau_g(E)|, 1\}.$$

Let $c'' \in \Gamma_g$ such that $\alpha_0(c'') = (N-1)\epsilon^d$, we find that

$$\alpha_{G_0}(c^*) - \alpha_{G_0}(c'') > \langle \omega'', c^* - c''\rangle, \qquad \alpha_{G_0}(c'') - \alpha_{G_0}(c') > \langle \omega', c'' - c'\rangle,$$

where $\omega' = \partial\alpha_{G_0}(c')$ and $\omega'' = \partial\alpha_{G_0}(c'')$. It follows that

$$\alpha_{G_0}(c^*) - \alpha_{G_0}(c') > \langle c^* - c', \omega'\rangle + \langle c^* - c'', \omega'' - \omega'\rangle. \tag{5.10}$$

In the way to get (5.8) one obtains that

$$|c^* - c''| \geq C_{18}\epsilon^d|\ln \epsilon^d|, \tag{5.11}$$

where the number C_{18} depends on N. One obtains from (2.9) that

$$\begin{aligned}|\omega'' - \omega'| =& \frac{1}{T_{(N-1)\epsilon^d}} - \frac{1}{T_{\epsilon^d}} \\ \geq& \frac{(k_1 + k_2)\ln(N-1) + \lambda_1(\tau_{\epsilon^d,g} - \tau_{(N-1)\epsilon^d,g})}{\lambda_1 T_{(N-1)\epsilon^d}T_{\epsilon^d}} \\ \geq& C_{19}|\ln \epsilon^d|^{-2}.\end{aligned} \tag{5.12}$$

Since $\langle \omega'' - \omega', c^* - c''\rangle = |\omega'' - \omega'||c^* - c''|$ (restricted on the cylinder, the system has one degree of freedom, so they are treated as scalers, not vectors), one obtains

$$\begin{aligned}&\alpha_{G'_{\epsilon,0}}(c) - \alpha_{G_0}(c') - \langle c - c', \omega'\rangle \\ =&\alpha_{G_0}(c^*) - \alpha_{G_0}(c') - \langle c^* - c', \omega'\rangle - \langle c - c^*, \omega'\rangle \\ \geq&\langle c^* - c'', \omega'' - \omega'\rangle - \langle c - c^*, \omega'\rangle \geq C_{20}\frac{\epsilon^d}{|\ln \epsilon^d|},\end{aligned}$$

where the first inequality is obtained by applying (5.10), the second one is obtained by applying (5.9), (5.11) and (5.12). It follows that the right hand side of (5.6) is lower bounded by $C_{21}\epsilon^d$ because $S = O(|\ln \epsilon^d|)$, where $C_{21} > 0$ is a constant. Because μ' is very small, the formula (5.6) contradicts (5.4) provided $\sigma > d + \mu'$. It completes the proof. q.e.d.

Proposition 5.2. *If the Aubry set for $c \in \Gamma_g \cap \alpha^{-1}_{G_{\epsilon,0}}(E)$ is contained in the NHIC, and $E > 0$ is independent of ϵ, each orbit in this set does not hit the energy level set $G^{-1}_{\epsilon,0}(E \pm \epsilon^{\frac{1}{3}\sigma})$ if $\epsilon > 0$ is sufficiently small.*

Proof. If an orbit $z(s)$ of the Aubry set for $c \in \Gamma_g \cap \alpha^{-1}_{G_{\epsilon,0}}(E)$ touches the energy level $G^{-1}_{\epsilon,0}(E-\epsilon^{\frac{1}{3}\sigma})$, following the proof of Proposition 5.1 we also have (5.4) and (5.6). Again, we are going to show the contradiction between them.

Let c^* be the class such that $\alpha_{G_0}(c^*) = \alpha_{G_{\epsilon,0}}(c)$ and let $c' \in \Gamma_g$ such that $\alpha_{G_0}(c') = E - \epsilon^{\frac{1}{3}\sigma}$, then $\alpha_{G_0}(c^*) - \alpha_{G_0}(c') = \epsilon^{\frac{1}{3}\sigma}$. Similar to the way to get (5.8), note the period is of order one, we obtain

$$|c^* - c'| \geq C_{22}\epsilon^{\frac{1}{3}\sigma}. \tag{5.13}$$

Since α_{G_0} is smooth and strictly convex when it is restricted on Γ_g, one obtains from (5.13) and (5.9) that

$$\begin{aligned}\alpha_{G_0}(c^*) - \alpha_{G_0}(c') - \langle c^* - c', \omega'\rangle &= \frac{1}{2}|\partial^2\alpha_{G_0}(\nu c + (1-\nu)c')||c' - c^*|^2 \\ &\geq C_{23}\epsilon^{\frac{2}{3}\sigma},\end{aligned}$$

from which we see that the right hand side of (5.6) is lower bounded by $O(\epsilon^{\frac{2}{3}\sigma})$. As μ' is very small, the formula (5.6) contradicts (5.4) provided $\sigma > d + \mu'$. The proof for $E + \epsilon^{\frac{1}{3}\sigma}$ is the same. q.e.d.

Proof of Theorem 5.1. For the Hamiltonian G_0 with $V \in \mathfrak{V}$, there are at most finitely bifurcation points $0 < E_1, E_2, \cdots E_k \leq E_1'$. The Aurby set $\tilde{\mathcal{A}}(c)$ for G_0 is a (E,g)-minimal periodic orbit if $c \in \mathscr{L}_{\beta_{G_0}}(\lambda g)$ and $\alpha_{G_0}(c) \neq E_i$ for $i = 1, 2, \cdots k$. At each bifurcation point the Aubry set consists of exactly two (E,g)-minimal periodic orbits. These periodic orbits make up several pieces of NHICs which admit a continuation to the energy level of $E_i \pm \delta$, denoted by $\Pi_{0,E_1+\delta,g}$ and $\Pi_{E_i-\delta,E_{i+1}+\delta,g}$, respectively. The continuation is made up by local (E,g)-minimal periodic orbits. Restricted on the cylinder $\Pi_{E_i-\delta,E_{i+1}+\delta,g}$, the Hamiltonian has one degree of freedom, associated with a smooth α-function denoted by α_i: $c_1 \in [c_1^i - \delta c_1, c_1^{i+1} + \delta c_1] \to \mathbb{R}$. The first cohomology class c_1 determines uniquely a flat $I_E \subset \mathbb{C}_{E_i,E_{i+1},g,G_0}$ such that $\alpha_i(c_1) = \alpha_{G_0}(I_E)$ if $c_1 \in [c_1^i, c_1^{i+1}]$. Indeed, one has $\alpha_{G_0}(c_1,c_2) = \alpha_i(c_1)$ if $(c_1,c_2) \in I_{\alpha_{G_0}(c_1,c_2)}$ and we use certain coordinates so that $g = (1,0)$. By the definition, we have $\alpha_{i-1}(c_1^i) = \alpha_i(c_1^i)$, $\alpha_{i-1}(c_1) \geq \alpha_i(c_1)$ for $c_1 \in [c_1^i, c_1^i + \delta c_1]$ and $\alpha_i(c_1) \leq \alpha_{i+1}(c_1)$ for $c_1 \in [c_1^{i+1} - \delta c_1, c_1^{i+1}]$. It follows from the generic condition (**H3**) that

$$\frac{d}{dc_1}\alpha_{i-1}(c_1^i) > \frac{d}{dc_1}\alpha_i(c_1^i), \qquad \forall\ i.$$

Under the perturbation $\epsilon^\sigma R$, large part of NHICs survive, such as $\tilde{\Pi}^\epsilon_{\epsilon^d,E_1+\delta-\epsilon^d}$ and $\tilde{\Pi}^\epsilon_{E_i-\delta+\epsilon^d,E_{i+1}+\delta-\epsilon^d}$. The former is ϵ^σ-close to $\Pi_{\epsilon^d,E_1+\delta-\epsilon^d,g} \times \frac{\sqrt{\epsilon}}{\omega_3}\mathbb{T}$, the latter is ϵ^σ-close to $\Pi_{E_i-\delta+\epsilon^d,E_{i+1}+\delta-\epsilon^d,g} \times \frac{\sqrt{\epsilon}}{\omega_3}\mathbb{T}$.

For the Hamiltonian G_0 and the class $c^i \in \mathbb{C}_{g,G_0}$ with $\alpha_{G_0}(c^i) = E_i$, the Aubry set consists of two $\lambda_i g$-minimal periodic orbits, the Mañé set contains these two periodic orbits plus some orbits connecting them (hetroclinic orbits). For the Hamiltonian $G_{\epsilon,0}$ and the class $c \in \mathbb{C}_{\epsilon^d,E_1',g}$ so that $|\alpha_{G_{\epsilon,0}}(c) - E_i| \le \epsilon^\sigma$, the Mañé set $\tilde{\mathcal{N}}(c)$ stays in a small neighborhood of cylinders $\tilde{\Pi}^\epsilon_{E_{i-1}-\delta+\epsilon^d,E_i+\delta-\epsilon^d,g}$ and $\tilde{\Pi}^\epsilon_{E_i-\delta+\epsilon^d,E_{i+1}+\delta-\epsilon^d,g}$. It is due to the upper semi-continuity of Mañé on small perturbations. So, it follows from the hyperbolic structure that each ergodic minimal measure for this class has its support in the cylinder either $\tilde{\Pi}^\epsilon_{E_{i-1}-\delta+\epsilon^d,E_i+\delta-\epsilon^d,g}$ or $\tilde{\Pi}^\epsilon_{E_i-\delta+\epsilon^d,E_{i+1}+\delta-\epsilon^d,g}$.

Since the energy level set $G^{-1}_{\epsilon,0}(E)$ is in ϵ^σ-neighborhood of $G_0^{-1}(E)$, we obtain from Proposition 5.2 and the condition (**H3**) that for $c \in \mathbb{C}_{g,G_{\epsilon,0}}$ such that $\alpha_{G_{\epsilon,0}}(c)$ is close to E_i we have

$$\tilde{\mathcal{A}}(c) \subset \tilde{\Pi}^\epsilon_{E_{i-1},E_i+\xi\epsilon^{\frac{\sigma}{3}},g} \cup \tilde{\Pi}^\epsilon_{E_i-\xi\epsilon^{\frac{\sigma}{3}},E_{i+1},g},$$

where $\xi \ge 2\max\{(\frac{d}{dc_1}\alpha_{i-1}(c_1^i) - \frac{d}{dc_1}\alpha_i(c_1^i))^{-1}, 1\}\|R\|_\infty$. Since $\delta > 0$ is independent of ϵ, the Aubry set completely lies on the cylinders if $\epsilon > 0$ is suitably small.

To verify that the Aubry set $\tilde{\mathcal{A}}(c)$ with $\alpha_{G_{\epsilon,0}}(c) = N\epsilon^d$ is contained in the cylinder, we apply Theorem 2.3. The invariant cylinder $\tilde{\Pi}^\epsilon_{\frac{1}{2}\epsilon^d,E_1+\delta-\epsilon^d,g}$ for $\Phi^\theta_{G'_{\epsilon,0}}$ lies in $O(\epsilon^\sigma)$-neighborhood of the cylinder $\Pi_{\frac{1}{2}\epsilon^d,E_1+\delta-\epsilon^d,g} \times \frac{\sqrt{\epsilon}}{\omega_3}\mathbb{T}$. By the choice of the number d in (2.29), we see that $G_{\epsilon,0}(z,\theta) \ge \epsilon^d$ if (z,θ) stays in that Aubry set $\tilde{\mathcal{A}}(c)$ with $\alpha_{G_{\epsilon,0}}(c) = N\epsilon^d$. Applying Lemma 5.1, we then complete the proof. q.e.d.

The second part of Theorem 1.1 follows from Theorem 5.1 and the result obtained in Section 4 (the finiteness of NHICs).

6. Criterion for strong and weak double resonance

Given a perturbation $\epsilon P(p,q)$, it is natural to ask, along the resonant path Γ', how many double resonances need to be treated as strong double resonance. The argument below is, in fact, the proof of Theorem 1.2.

Proof of Theorem 1.2. On the path Γ' the resonance condition $\langle \partial h(p), k'\rangle = 0$ is always satisfied and at each double resonant point some other $k'' \in \mathbb{Z}^3$, independent of k', exists such that $\langle \partial h(p), k''\rangle = 0$ holds. Recall the process of KAM iteration, the main part of the resonant term is obtained by averaging the perturbation over a circle determined by these two resonant relations. It takes the form

$$Z = Z_{k'}(p, \langle k', q\rangle) + Z_{k',k''}(p, \langle k', q\rangle, \langle k'', q\rangle),$$

where

$$Z_{k'} = \sum_{j\in\mathbb{Z}\setminus\{0\}} P_{jk'}(p)e^{j\langle k',q\rangle i},$$

$$Z_{k',k''} = \sum_{(j,l)\in\mathbb{Z}^2,l\neq 0} P_{jk'+lk''}(p)e^{(j\langle k',q\rangle+l\langle k'',q\rangle)i}.$$

Since the perturbation P is C^r-function, the coefficient $P_{jk'+lk''}$ is bounded by

$$|P_{jk'+lk''}| \leq 8\pi^3\|P\|_{C^r}\|jk'+lk''\|^{-r}.$$

Therefore, some constant $\vartheta = \vartheta(k')$ depending on k' exists such that

$$\|Z_{k',k''}\|_{C^2} \leq \vartheta\|P\|_{C^r}\|k''\|^{-r+2}. \tag{6.1}$$

Recall the procedure we did in the second section, after the rescaling and linear coordinate transformation we obtain the main part of the system

$$G_0 = \frac{1}{2}\langle Ay,y\rangle - V_{k'}(x_2) - V_{k',k''}(x_1,x_2).$$

We assume that $V_{k'}$ has a non-degenerate minimal point at x_2^*, i.e., $\ddot{V}_{k'}(x_2^*) = \lambda_3 > 0$, the system $\frac{1}{2}\langle Ay,y\rangle - V_{k'}(\langle k',q\rangle)$ possesses a NHIC

$$\Pi^0_{k',k''} = \{y = \xi y_0, \xi\in\mathbb{R}, x_2 = x_2^*, x_1\in\mathbb{T}\},$$

where y_0 solves the equation $(1,0)^t = Ay_0$. Applying the normally hyperbolic invariant manifold theorem, one obtains from the estimate (6.1) that some positive number $\vartheta_1 = \vartheta_1(\lambda_3) > 0$ exists such that $\Phi^t_{G_0}$ also admits a normally hyperbolic and invariant cylinder $\Pi_{k',k''}$ close to $\Pi^0_{k',k''}$ provided

$$\|k''\|^{r-2} \geq \frac{\vartheta(k')}{\vartheta_1(\lambda_3)}\|P\|_{C^r}. \tag{6.2}$$

It is a criterion, if the integer vector k'' satisfies this condition, the double resonance is thought as weak resonance and can be treated in the way for *a priori* unstable system.

We notice that the potential V is obtained by fixing a double resonant point $y = y''$. It seems that the non-degeneracy of the minimal point depends on the position of double resonant point on the resonant path Γ', namely, the number λ_3 depends on the $y\in\Gamma'$. Because the set of double resonant points is dense along the resonant path, it appears necessary to ask whether it holds simultaneously for all $p\in\Gamma'$ that the minimal point of $Z_{k'}(p,x)$ is non-degenerate when it is treated as a function of $x = \langle k',q\rangle$. Fortunately, we have the following result [**CZ1**]

Theorem 6.1. *Assume M is a closed manifold with finite dimensions, $F_\zeta\in C^r(M,\mathbb{R})$ with $r\geq 4$ for each $\zeta\in[\zeta_0,\zeta_1]$ and F_ζ is Lipschitz in the parameter ζ. Then, there exists an open-dense set $\mathfrak{V}\subset C^r(M,\mathbb{R})$ so that for each $V\in\mathfrak{V}$, it holds simultaneously for all $\zeta\in[\zeta_0,\zeta_1]$ that the minimal point of $F_\zeta+V$ is non-degenerate. In fact, given $V\in\mathfrak{V}$*

there are finitely many $\zeta_i \in [\zeta_0, \zeta_1]$ *such that* $F_\zeta + V$ *has only one global minimal point for* $\zeta \neq \zeta_i$ *and has two global minimal point if* $\zeta = \zeta_i$.

So, once one has a generic single resonant term $Z_{k'}$, the non-degeneracy λ_3 is lower bounded from zero for all double resonant points. There are finitely many $k'' \in \mathbb{Z}^3$ which do not satisfy the condition (6.2), thus need to be treated as strong double resonance. Obviously, the number of such points is independent of ϵ.

It follows from Theorem 6.1 that there are finitely many point $p = p'_j \in \Gamma'$ where the single resonant term $Z_{k'}$ has two global minimal points when it is treated as the function $\langle k', q\rangle$. It is clearly generic that the condition (**H3**) holds for $Z_{k'}$. It implies that as one move p along Γ', the Mather set varies along one cylinder and jump to another cylinder when it crosses the point p'_j which is called bifurcation point. It is also generic that none of these bifurcation points is strong double resonant point. q.e.d.

Remark. Given a resonant path determined by a class $g \in H_1(\mathbb{T}^2, \mathbb{Z})$, we have a channel $\mathbb{C}_g = \cup_\lambda \mathscr{L}_{\beta_{G_{\epsilon,0}}}(\lambda g) \subset H^1(\mathbb{T}^2, \mathbb{R})$. By the result we get in this paper, this channel has certain width except the place very close to the disk $\mathbb{F}_i$ which corresponds to strong double resonance. For each $c \in \mathrm{int}\mathbb{C}_g$ with $d(c, \mathbb{F}_i) > O(\epsilon^{\frac{1}{2}+d})$, the Aubry set is located in certain NHIC. By using the method of [**CY1**, **CY2**, **LC**], this Aubry set can be connected to other Aubry set nearby also lying on the cylinder. Some local connecting orbit looks like heteroclinic orbit (Arnold's mechanism), some other orbits are constructed by using cohomology equivalence. Because certain Hölder modulus continuity of weak KAM solutions is established in [**Z1**] for the whole cylinder, not only restricted on the set of invariant circles, one can always connects such Aubry set to another one via Arnold's mechanism.

In conclusion, we find transition chain along the resonant path except for finitely many gaps around the strong double resonant points. The size of the gaps is so small that these pieces of transition chain are connected by paths of cohomology equivalence on energy levels slightly above zero around double resonance, shown in a subsequent paper [**C17**]. Another way to cross double resonance was suggested in [**Mat**], elaborated in [**KZ**], [**Mar**] using cylinders with holes entering the negative energy region.

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