WIDTH, RICCI CURVATURE, AND MINIMAL HYPERSURFACES

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Abstract

Let (M, g_0) be a closed Riemannian manifold of dimension n, for $3 \leq n \leq 7$, and non-negative Ricci curvature. Let $g = \phi^2 g_0$ be a metric in the conformal class of g_0. We show that there exists a smooth closed embedded minimal hypersurface in (M, g) of volume bounded by $C(n) V^{\frac{n-1}{n}}$, where V is the total volume of (M, g). When $Ric(M, g_0) \geq -(n-1)$ we obtain a similar bound with constant C depending only on n and the volume of (M, g_0).

Our second result concerns manifolds (M, g) of positive Ricci curvature and dimension at most seven. We obtain an effective version of a theorem of F. C. Marques and A. Neves on the existence of infinitely many minimal hypersurfaces on (M, g). We show that for any such manifold there exists k minimal hypersurfaces of volume at most $C_n V \left(\text{sys}_{n-1}(M) \right)^{-\frac{1}{n-1}} k^{\frac{1}{n-1}}$, where V denotes the volume of (M, g_0) and $\text{sys}_{n-1}(M)$ is the smallest volume of a non-trivial minimal hypersurface.

1. Introduction

1.1. Results. In [**32**] Pitts proved existence of a smooth closed embedded minimal hypersurface in any closed Riemannian manifold M of dimension n, for $3 \leq n \leq 6$. This result was extended to manifolds of dimension $n \leq 7$ by Schoen and Simon [**34**]. Our first main result is a bound on the volume of this hypersurface for certain conformal classes of Riemannian metrics.

Theorem 1.1. *Suppose M_0 is a closed Riemannian manifold of dimension n, for $3 \leq n \leq 7$. If M is in the conformal class of M_0 then M contains a smooth closed embedded minimal hypersurface Σ with volume bounded above by $C(M_0) \operatorname{Vol}(M)^{\frac{n-1}{n}}$. When $\operatorname{Ricci}(M_0) \geq 0$ the constant $C(M_0)$ is an absolute constant that depends only on n. In general, for M_0 with $\operatorname{Ricci}(M_0) \geq -(n-1)a^2$ we can take $C(M_0) = C(n) \max\{1, a \operatorname{Vol}(M_0)^{\frac{1}{n}}\}$.*

If $n > 7$ the same upper bound will hold for the $(n-1)$-volume of a closed minimal hypersurface with singularities of dimension at most $n - 8$.

Theorem 1.1 follows from a bound on the width of M. One can find background and many results about widths of manifolds in [16, App.1F], [18], [3], [25]. Informally, the width $W(M)$ of a manifold M is the smallest number such that every sweep-out of M by hypersurfaces contains a hypersurface of volume at least $W(M)$. We give a precise definition of width in Section 2.

To state our bound on the width of manifolds it will be convenient to define a conformal invariant called the min–conformal volume. This invariant was recently introduced in a work of Hassannezhad [20].

Definition 1.2. Let M be a compact Riemannian manifold. Define the min-conformal volume of M to be: $\mathrm{MCV}(M) = \inf\{\mathrm{Vol}(M')\}$, where the infimum is taken over all manifolds M' in the conformal class of M with $\mathrm{Ricci}(M') \geq -(n-1)$.

Theorem 1.3. *Let M be a closed Riemannian manifold of dimension n then*

$$\mathrm{W}(M) \leq C(n) \max\{1, \mathrm{MCV}(M)^{\frac{1}{n}}\} \mathrm{Vol}(M)^{\frac{n-1}{n}}.$$

Corollary 1.4. *Let M be a closed Riemannian manifold of dimension n, $a \geq 0$ and suppose that $\mathrm{Ricci}(M) \geq -(n-1)a^2$. Then*

$$\mathrm{W}(M) \leq C(n) \max\{1,\ a\,\mathrm{Vol}(M)^{\frac{1}{n}}\} \mathrm{Vol}(M)^{\frac{n-1}{n}}.$$

More generally, Theorem 1.3 holds if we replace the min-conformal volume in the estimate by the infimum of $\{d\,\mathrm{Vol}(M')\}$, where d is any positive integer, such that M admits a degree d conformal branched covering onto a manifold M' with $\mathrm{Ricci}(M') \geq -(n-1)$.

The conformal invariant MCV is somewhat reminiscent of (but different from) the conformal volume studied by Li and Yau in [22]. $\mathrm{MCV}(M)$ can be computed explicitly when M is a Riemannian surface of genus g. By the uniformization theorem M is conformally equivalent to a surface M_0 of constant curvature 1 (if $g = 0$), 0 (if $g = 1$) or -1 (if $g \geq 2$). When the genus is 0 or 1 it follows that $\mathrm{MCV}(M) = 0$. When $g \geq 2$ and the Gaussian curvature of a surface M' satisfies $K \geq -1$ then by Gauss–Bonnet theorem $\mathrm{Area}(M') \geq 4\pi(g-1)$ with equality holding exactly when $K = -1$ everywhere. We conclude that $\mathrm{MCV}(M) = 4\pi(g-1)$

Theorem 1.3 then implies that for any surface M of genus g we have

$$\mathrm{W}(M) \leq C\sqrt{(g+1)\,\mathrm{Area}(M)}.$$

This result was previously obtained by Balacheff and Sabourau in [3] with constant $C = 10^8$. Using a slightly modified version of our proof and invoking the Riemann–Roch theorem we can get a somewhat better constant for orientable surfaces.

Theorem 1.5. *Any closed orientable Riemannian manifold S_g of dimension 2 and genus g satisfies*

$$W(S_g) \leq 220\sqrt{(g+1)\operatorname{Area}(S_g)}.$$

Upper bounds on the higher parametric versions of width $W^k(M)$ for all Riemannian surfaces were recently obtained by the second author [**23**].

In [**4**] Brooks constructed hyperbolic surfaces of large genus and Cheeger constant bounded away from zero. These surfaces have width $W(M)$ bounded below by $c\sqrt{g\operatorname{Area}(M)}$ for some constant $c > 0$. Hence, the inequality in Theorem 1.3 is optimal up to the value of the constant $C(n)$.

It follows from the works of Almgren [**2**], Pitts [**32**], and Schoen and Simon [**34**] that estimates on width yield upper bounds on the volume of smooth embedded minimal hypersurfaces in manifolds of dimension less than or equal to 7. In higher dimensions, we obtain bounds on the volume of stationary integral varifolds, which are smooth hypersurfaces everywhere except possibly for a closed set of Hausdorff dimension at most $n - 8$.

It is possible to obtain more minimal hypersurfaces if one considers parametric families of sweep-outs. In Section 2 we define families of hypersurfaces that correspond to cohomology classes of mod 2 $(n - 1)$-cycles on M. To each such p-dimensional family we assign the corresponding min–max quantity $W^p(M)$. Let S^n be the round unit n-sphere. In [**12**, 4.2.B] Gromov showed that there are constants $0 < c(n) < C(n)$ so that $W^p(S^n)$ satisfies:

$$c(n)p^{\frac{1}{n}}\operatorname{Vol}(S^n)^{\frac{n-1}{n}} \leq W^p(S^n) \leq C(n)p^{\frac{1}{n}}\operatorname{Vol}(S^n)^{\frac{n-1}{n}}.$$

Guth [**19**] derived similar bounds for min–max quantities corresponding to the Steenrod algebra generated by the fundamental class λ. Marques and Neves [**25**], building on the work of Gromov and Guth, proved existence of infinitely many minimal hypersurfaces on a manifold M of dimension n, for $3 \leq n \leq 7$, under the assumption that M has positive Ricci curvature.

In Section 7 we show that if M has non-negative Ricci curvature then $W^p(M) \leq C(n)p^{\frac{1}{n}}\operatorname{Vol}(M)^{\frac{n-1}{n}}$. We use this bound to derive an effective version of the theorem of Marques and Neves. Let $\operatorname{sys}_{n-1}(M)$ be the infimum of volumes of smooth closed embedded minimal hypersurfaces in M.

Theorem 1.6. *Suppose M is a closed Riemannian manifold of dimension n, $3 \leq n \leq 7$, and positive Ricci curvature. For every k there exists k smooth closed embedded minimal hypersurfaces of volume*

bounded above by $C(n)k^{\frac{1}{n-1}} \operatorname{Vol}(M) \left(\operatorname{sys}_{n-1}(M)\right)^{-\frac{1}{n-1}}$, *where* $C(n)$ *depends only on* n.

1.2. Previous work. The main estimates in our paper were motivated by similar estimates on the spectrum of the Laplace operator on Riemannian manifolds.

Let M be a closed Riemannian manifold in the conformal class of M_0. In [21] Korevaar constructed a decomposition of M into annuli (and other regions) which measures the 'volume concentration' of the metric M with respect to the base metric of M_0. This annular decomposition is then used to estimate Rayleigh quotients, thus bounding the spectrum of the Laplacian of M. Korevaar's method was further developed by Grigor'yan–Yau in [11] and Grigor'yan–Netrusov–Yau in [10] to obtain upper bounds on the eigenvalues of elliptic operators on various metric spaces. In [13] Gromov used a different approach (based on Kato's inequality) to obtain upper bounds for the spectrum of the Laplacian on Kähler manifolds [13].

In [20] Hassannezhad, combining methods of [6] and [10], obtained upper bounds for eigenvalues of the Laplacian in terms of the conformal invariant MCV (see Definition 1.2) and the volume of the manifold.

As suggested by Gromov in [12] the problem of bounding width $W(M)$ and its parametric version W^k can be thought of as a non-linear analogue of finding the spectrum of the Laplacian on M. In this paper we were guided by this analogy.

In [18] Guth constructed sweep-outs of open subsets of Euclidean space by k-cycles of controlled volume for all k, $1 \le k \le n-1$. In particular, he proved the following

Theorem 1.7 (Guth [18])**.** *For every open subset* $U \subset \mathbb{R}^n$, $W(U) \le C(n) \operatorname{Vol}(U)^{\frac{n-1}{n}}$.

In dimension $n \ge 3$ one can find a family of parallel hyperplanes in $\mathbb{R}^n$ yielding the desired sweep-out. This follows from the work of Falconer [7] on the (n, k)-Besicovitch conjecture. In dimension 2, however, it may happen that any slicing of U by parallel lines contains an arbitrarily large segment. To surpass problems of this kind Guth developed a method of sweeping out regions by 'bending planes' around the skeleton of the unit lattice in $\mathbb{R}^n$. This method was developed further in [19] to bound higher parametric versions of width.

It has been observed in [18, Appendix 5] that on any manifold of dimension greater than two there exists a Riemannian metric of small volume and arbitrarily large $(n-1)$-width. Our results show that this does not happen for certain conformal classes of manifolds. In particular, this does not happen in the presence of a Ricci curvature lower bound.

In [**30**] Nabutovsky and Rotman showed that any closed Riemannian manifold possesses a stationary 1-cycle of mass bounded by $C(n) \operatorname{Vol}(M)^{\frac{1}{n}}$. One wonders if this result can be generalized to the case of minimal surfaces on Riemannian manifolds.

Some results in this direction were obtained by A. Nabutovsky and R. Rotman in [**31**] where they bounded volumes of minimal surfaces on Riemannian manifolds in terms of homological filling functions of M. The k-th homological filling function $\operatorname{FH}_k : \mathbb{R}_+ \to \mathbb{R}_+$ is defined as the smallest number $\operatorname{FH}_k(x)$, such that every k-cycle of mass at most x can be filled by a $(k+1)$-chain of mass at most $\operatorname{FH}_k(x) + \epsilon$.

Theorem 1.8 (Nabutovsky–Rotman [**30**])**.** *Let M be a closed Riemannian manifold of dimension n, for $3 \leq n \leq 7$, such that the first $n-1$ homology groups are trivial, $H_1(M) = ... = H_{n-1}(M) = 0$. There exists a smooth, closed, embedded minimal hypersurface of volume bounded by* $C(n) \operatorname{FH}_{n-1}(C(n) \operatorname{FH}_{n-2}(\ldots \operatorname{FH}_2(C(n) \operatorname{Vol}(M)^{\frac{1}{n}}) \ldots)).$

Their proof uses a combination of Almgren–Pitts min–max method and other techniques. In particular, a bound on the width of M in terms of homological filling functions does not follow from their argument. It would be interesting to know whether such a bound exists. It is also interesting to know whether homological filling functions can be controlled in terms of Ricci curvature of M.

Other important results are contained in a paper of Marques and Neves [**26**] where, among other things, they prove a *sharp* upper bound on $W(M)$, when M is a Riemannian 3-sphere with Ricci > 0 and scalar curvature $R \geq 6$.

1.3. Plan of the Paper. The structure of the proof of Theorem 1.3 is as follows: To construct a sweep-out of M, we subdivide M repeatedly, using an isoperimetric inequality adapted to our context. Once we have subdivided M into a collection of small volume open subsets, we construct a sweep-out of each small volume piece using the fact that at small scales M is locally Euclidean. We then assemble these local sweep-outs into a global sweep-out of M.

In Section 2 we define what it means for a family of $(n-1)$-cycles to sweep-out M. We define the width $W(M)$ and its higher parametric version $W^k(M)$. We also prove Proposition 2.3, which gives us control of the width of M in terms of widths of its open subsets.

In Section 3 we use an idea of Colbois and Maerten from [**6**] together with the length-area method to prove an isoperimetric inequality (Theorem 3.4) which allows us to partition any open set in M in two parts with both parts satisfying a lower volume bound. The subdividing surface satisfies an upper bound on area which depends on the volume of the open set.

In Section 4 we estimate the width of small volume submanifold $M' \subset M$ in terms of $(n-1)$-volume of its boundary. The proof proceeds by covering M' with small balls, which are $(1+\epsilon_0)$-bilipschitz diffeomorphic to balls in Euclidean space. We construct a sequence of nested open subsets U_i of M' with volumes tending to 0, such that the difference $U_i \setminus U_{i+1}$ is contained in a small ball. Since the ball is almost Euclidean, we can sweep out $U_i \setminus U_{i+1}$ by cycles of controlled volume. We then use Proposition 2.3 to assemble a sweep-out of M'.

In Section 5 we prove Theorem 1.3 by inductively constructing sweep-outs of larger and larger subsets of M. The result of Section 4 serves as the base of the induction.

In Section 6 we prove Theorem 1.5. We also describe how to obtain a version of Theorem 1.3 for manifolds, which admit a conformal mapping into some nice space M_0.

In Section 7 we show that a manifold with non-negative Ricci curvature can be covered by balls of small n-volume, small $(n-1)$-volume of the boundary, and such that the cover has controlled multiplicity. We use this decomposition to bound the volume of k-parametric sweep-outs of M and, consequently, volumes of stationary integral varifolds or minimal hypersurfaces in M.

Remark 1.9. In [**33**] Stephane Sabourau independently obtained upper bounds on the width and volume of the smallest minimal hypersurface on Riemannian manifolds with Ricci ≥ 0.

2. Width of Riemannian manifolds

Let G be an abelian group. We denote the space of flat G-chains in M by $F_k(M;G)$ and the space of flat G-cycles by $Z_k(M;G)$. The space of integral flat chains was defined in [**9**]. For flat chains with coefficients in an abelian group G see [**8**, (4.2.26)]. The deformation theorem of Federer and Fleming states that a flat chain of finite mass and boundary mass can be approximated by a piecewise linear polyhedral chain (see [**8**, (4.2.20),(4.2.20)$^\nu$]). The deformation theorem will be used throughout this paper. Often we will abuse notation and use the same letter for a flat G-chain and a polyhedral chain approximating it. We will denote the mass of a k-chain c by $\mathrm{Vol}_k(c)$.

In [**1**] F. Almgren constructed an isomorphism

$$F_A : \pi_k(Z_{n-1}(M;G);0) \to H_{n+k}(M;G).$$

For $k = 1$ the map F can be described as follows. Let $c_t \in Z_{n-1}(M;G)$, $t \in S^1$, be a continuous family of cycles. Pick a fine subdivision $t_0, ..., t_m$

of S^1 and let C_i be a (nearly) volume minimizing n-chain filling $c_i - c_{i-1}$ (for $i \in \mathbb{Z}_m$). Then $C = \sum_{i \in \mathbb{Z}_m} C_i$ is an n-cycle. It turns out that homology class of C is independent of the choice of the subdivision and filling chains C_i as long as the subdivision is fine enough and the mass of chains C_i is close to the mass of a minimal filling.

If M is a manifold with boundary we may also consider the space of flat cycles relative to the boundary of M. Let q be a quotient map $q : F_k(M; G) \to F_k(M, \partial M; G) = F_k(M; G)/F_k(\partial M; G)$. The boundary map on $F_k(M; G)$ descends to a boundary map ∂ on the quotient. This allows us to define the space of relative cycles $Z_k(M, \partial M; G)$. Cycles in this space can be represented by $(n-1)$-chains with boundary in ∂M. Almgren's map then defines an isomorphism $\pi_1(Z_{n-1}(M, \partial M; G), \{0\}) \cong H_{n-1}(M, \partial M; G)$.

For simplicity from now on we assume everywhere that group $G = \mathbb{Z}_2$. Henceforth we drop the reference to the group G from our notation. $\mathbb{Z}_2$ coefficients will suffice for all applications to volumes of minimal surfaces that we obtain in this paper. When manifold M is orientable the bound in Theorem 1.3 holds for sweep-outs with integer coefficients. The proof is essentially the same with some minor modifications to account for orientation of cycles.

Definition 2.1. We define the following two notions:

1) For a closed manifold M a map $f : S^1 \to Z_{n-1}(M)$ is called a *sweep-out* of M if it is not contractible, i.e., $F_A([f]) \neq 0$. Similarly, if M has a boundary we call $f : S^1 \to Z_{n-1}(M, \partial M)$ a sweep-out if the image of $[f]$ under Almgren's isomorphism is non-zero.
2) The *width of M* is

$$\mathrm{W}(M) = \inf_{\{f\}} \sup_t \mathrm{Vol}_{n-1}(f(t)),$$

where the infimum is taken over the set of all sweep-outs of M.

For manifolds with boundary it will be convenient to consider a particular type of sweep-outs that start on a trivial cycle and end on ∂M. We will call them ∂-sweep-outs.

Definition 2.2.   1) Let M be a manifold with boundary. A ∂-*sweep-out of M* is a map $f : [0, 1] \to Z_k(M)$, such that:
   a) $f(0)$ is a trivial k-cycle and $f(1) = \partial M$,
   b) Let $q \circ f : [0, 1] \to Z_{n-1}(M, \partial M)$ be the composition of f with the quotient map q. When we identify $q \circ f(0)$ and $q \circ f(1)$ we obtain a sweep-out of M.
2) The ∂-*width of M* is:

$$\mathrm{W}^\partial(M) = \inf_{\{f\}} \sup_t \mathrm{Vol}_{n-1}(f(t)),$$

where the infimum is taken over the set of all ∂-sweep-outs of M.

From the definition we have inequalities $W(M) \leq W^\partial(M)$ and $W^\partial(M) \geq \text{Vol}_{n-1}(\partial M)$.

Definition 2.2 is motivated by the following proposition.

Proposition 2.3. *Let $U_0 \subset ... \subset U_{m-1} = M$ be a sequence of nested open subsets of M, and let A_i denote the closure of $U_i \setminus U_{i-1}$ for $1 \leq i \leq m-1$ and A_0 denote the closure of U_0. Then $W^\partial(M) \leq \sup\{W^\partial(A_0), W^\partial(A_1) + \text{Vol}_{n-1}(\partial U_0), ..., W^\partial(A_{m-1}) + \text{Vol}_{n-1}(\partial U_{m-2})\}$.*

Proof. By the definition of ∂-width for each i there exists a map $c_i : [0,1] \to Z_{n-1}(M)$ that starts on a trivial cycle, ends on ∂A_i and is bounded in volume by $W^\partial(A_i) + \epsilon$. By definition of A_i, $\partial A_i \subset \partial U_i \cup \partial U_{i-1}$ and $\partial U_i + c_{i+1}(1) = \partial U_{i+1}$.

We define a sweep-out $c : [0,1] \to Z_{n-1}(M)$ as follows. For $0 \leq t \leq \frac{1}{m}$ we set $c(t) = c_0(t/m)$ and for $\frac{i}{m} \leq t \leq \frac{i+1}{m}$, $i = 1, ..., m-1$ we set $c(t) = c_i(m(t - \frac{i}{m})) + \partial U_{i-1}$.

Let F_A be the Almgren's isomorphism. We can represent the homology class $F_A(c)$ by a sum of n-chains $\sum_{i=0}^{m-1} C_i$, such that $\partial C_i = c(\frac{i}{m}) - c(\frac{i-1}{m})$. Moreover, since each c_i is a ∂-sweep-out of A_i we may assume that C_i represents a non-trivial homology class in $H_n(M, \overline{M \setminus A_i}) \cong H_n(A_i, \partial A_i)$.

We claim that the sum $\sum_{i=0}^k C_i$ represents a non-trivial homology class in $H_n(M, M \setminus U_k)$. Indeed, assume this to hold for $\sum_{i=0}^{k-1} C_i$. Let V_1 denote a small tubular neighbourhood of the set U_{k-1} inside U_k. Let V_2 be a small tubular neighbourhood of $U_k \setminus U_{k-1}$ inside U_k. Let $V_3 = V_1 \cap V_2 \subset U_k$. The pair $(V_3, V_3 \cap \partial U_k)$ is homotopy equivalent to $(\partial U_{k-1}, \partial U_{k-1} \cap \partial U_k)$. From the Mayer–Vietoris sequence we have an isomorphism $H_n(\overline{U_k}, \partial U_k) \to^\partial H_{n-1}(\partial U_{k-1}, \partial U_{k-1} \cap \partial U_k)$. This map sends $[\sum_{i=0}^{k-1} C_i + C_k]$ to the fundamental class $[\overline{\partial U_{k-1} \setminus \partial U_k}]$.

We conclude that $c(t)$ is a ∂-sweep-out of M.   q.e.d.

In the last section of this paper we will obtain upper bounds on the k-parametric sweep-outs $W^k(M)$ of M. By Almgren's isomorphism theorem we have $\pi_m(Z_{n-1}(M; Z); 0) = 0$ for $m > 1$ and $\pi_1(Z_{n-1}(M; \mathbb{Z}_2); 0) \cong \mathbb{Z}_2$. Hence, the connected component Z_{n-1}^0 of $Z_{n-1}(M; \mathbb{Z}_2)$ that contains the 0-cycle is weakly homotopy equivalent to the Eilenberg–MacLane space $K(\mathbb{Z}_2, 1) \simeq \mathbb{RP}^\infty$.

Let K be a k-dimensional polyhedral complex and $\sigma : K \to Z_{n-1}^0(M)$ be continuous and assume that $\sigma(x)$ has finite mass for all x. Following [25] we define k-parametric width W^k as follows.

Definition 2.4. We introduce the following parametric version of Definition 2.1:

1) For a closed manifold M we say that σ is a k-parametric sweep-out of M if $\sigma(K)$ represents the non-zero class in $H_k(Z_{n-1}^0, \mathbb{Z}_2) \cong \mathbb{Z}_2$.

2) Define the k-parametric width to be $\mathrm{W}^k(M) =$ $\inf_\sigma \sup_{t\in K} \mathrm{Vol}_{n-1}(f(t))$, where the infimum is taken over the set of all k-parametric sweep-outs σ.

It follows from the definition that $\mathrm{W}^1(M) = \mathrm{W}(M)$ and $\mathrm{W}^k(M) \leq \mathrm{W}^{k+1}(M)$.

Using Almgren–Pitts min–max theory it is possible to obtain minimal hypersurfaces from sweep-outs of M. In [**25**] Marques and Neves proved the following results.

Theorem 2.5 (Marques–Neves)**.** *Let M be a closed Riemannian manifold of dimension n, $3 \leq n \leq 7$.*

1) *There exists a smooth, closed, embedded minimal hypersurface in M of volume $\leq \mathrm{W}(M)$.*
2) *If $\mathrm{W}^k(M) = \mathrm{W}^{k+1}(M)$ then there exists infinitely many smooth, closed, embedded minimal hypersurfaces in M of volume $\leq \mathrm{W}^k(M)$.*
3) *Suppose M is a manifold of positive Ricci curvature and there exists only finitely many minimal hypersurfaces of volume $\leq \mathrm{W}^k(M)$. Then there exists a smooth, closed, embedded minimal hypersurface Σ_k and $a_k \in \mathbb{N}$, such that $a_k \mathrm{Vol}_{n-1}(\Sigma_k) = \mathrm{W}^k(M)$.*

Remark 2.6. In the proof of these results Marques and Neves impose an additional technical condition on W^k. Namely, they require that the infimum in the definition of W^k is taken over only those maps $f : K \to Z_{n-1}(M)$ that have no concentration of mass. This is defined as follows. Using the notation of [**8**] let $\|c\|$ denote the Radon measure associated with the flat chain c. Then a map f is said to have no concentration of mass if

$$\lim_{r\to 0} \sup\{\|f(x)\|(B_r(a)) : \ x \in K, a \in M\} = 0.$$

All estimates on W^k in our paper come from explicit constructions of families of flat cycles (in fact, polyhedral cycles), which have no concentration of mass. Therefore we can safely combine our estimates with the conclusions of Theorem 2.5.

3. Isoperimetric inequality

Let (M, g_0) be a closed Riemannian n-manifold with Ricci $\geq -(n-1)$. Let $g = \phi^2 g_0$ be a Riemannian metric on (M, g_0) in the conformal class of g_0. Here $\phi : M_0 \to \mathbb{R}_+$ is a smooth function on (M, g_0).

Notation 3.1. We write M_0 for (M, g_0) and M for (M, g).

Below we use the convention that geometric structures measured with respect to g_0 have a superscript zero in their notation. Geometric structures measured with respect to g have no superscript.

Notation 3.2. Let $\mathrm{Vol}_k(U)$, $d(x,y)$, dV, $B(x,r)$ and ∇ denote the k-volume function, distance function, volume element, closed metric ball of radius r about x, and gradient with respect to g. Let $\mathrm{Vol}_k^0(U)$, $d^0(x,y)$, dV^0, $B^0(x,r)$, ∇^0 denote the corresponding quantities with respect to g_0.

Let W be a subset of U and let $N_l^0(W)$ denote the set $\{x \in U \mid d^0(W,x) \le l\}$.

Lemma 3.3. *There exists a set $W \subset U$ and $l \in (0, \frac{1}{2}]$, such that*

1) $\mathrm{Vol}_n(U)/25^n \le \mathrm{Vol}_n(W) \le 2\,\mathrm{Vol}_n(U)/25^n$,

2) $\mathrm{Vol}_n(N_l^0(W)) \le (1 - \frac{1}{25^n})\,\mathrm{Vol}_n(U)$,

3) $\mathrm{Vol}_n^0(N_l^0(W) \setminus W) \le l^n \max\{2\,\mathrm{Vol}_n^0(U), c(n)\}$.

Proof. The argument is essentially the same as the proof of Lemma 2.2 in the work of Colbois and Maerten [**6**]. Let r be the smallest radius with the property that $\mathrm{Vol}(B^0(a,r) \cap U) = \frac{\mathrm{Vol}(U)}{25^n}$ for some $a \in M$.

We consider two cases. If $r \le 1$ we define $W = B^0(a,r) \cap U$ and $l = \frac{r}{2}$.

We observe, using curvature comparison for the space M_0, that the l-neighbourhood of $B^0(a,r)$ can be covered by at most 24.4^n balls of radius r. Indeed, let $\{B^0(x_i, r/2)\}_{i=1}^N$ be a maximal collection of disjoint balls with centres in $B^0(a, \frac{3r}{2})$. Since the collection is maximal, the union $\bigcup B(x_i, r)$ covers $B^0(a, \frac{3r}{2})$. Using the Bishop–Gromov comparison theorem we can estimate the number N. Let $\mathrm{Vol}_n^0(B(x_j, \frac{r}{2})) = \min_i\{\mathrm{Vol}_n^0(B(x_i, \frac{r}{2}))\}$.

$$N \le \frac{\mathrm{Vol}_n^0(B^0(a, \frac{3r}{2}))}{\mathrm{Vol}_n^0(B(x_j, \frac{r}{2}))} \le \frac{\mathrm{Vol}_n^0(B(x_j, \frac{5r}{2}))}{\mathrm{Vol}_n^0(B(x_j, \frac{r}{2}))} \le \frac{V(\frac{5r}{2})}{V(\frac{r}{2})},$$

where $V(r)$ denotes the volume of a ball of radius r in n-dimensional hyperbolic space. When $r \in (0,1]$ this quantity is maximized for $r = 1$. We conclude that $B^0(a, \frac{3r}{2})$ can be covered by

$$N \le \frac{\int_0^{\frac{5}{2}} \sinh^{n-1}(s)ds}{\int_0^{\frac{1}{2}} \sinh^{n-1}(s)ds} \le (2e^{\frac{5}{2}})^n \le 24.4^n$$

balls, such that each of them has $\mathrm{Vol}_n(B^0(x_i, r) \cap U)$ at most $\frac{\mathrm{Vol}_n(U)}{25^n}$. This proves inequalities (1) and (2) for the case $r \le 1$.

Volume of a unit ball in hyperbolic n-space satisfies $V(1) \le \omega_n e^{n-1}$, where ω_n denotes the volume of a unit n-ball in Euclidean space. Hence, $\mathrm{Vol}_n^0(B^0(a, \frac{3r}{2}) \setminus B^0(a,r)) \le 25^n e^{n-1} \omega_n r^n = c(n)$. This proves (3) for the case $r \le 1$.

Suppose $r > 1$. Let k be the smallest number, such that there exists a collection of k balls of radius 1 $\{B^0(x_i, 1)\}_{i=1}^k$ with $\mathrm{Vol}(\bigcup B^0(x_i, 1) \cap U) \ge \frac{\mathrm{Vol}_n(U)}{25^n}$. Let $\{B^0(x_i, 1)\}_{i=1}^k$ be a collection of k balls with the

property that if $\{B^0(y_i, 1)\}_{i=1}^k$ is any other collection of k balls then $\mathrm{Vol}(\bigcup B^0(x_i, 1) \cap U) \geq \mathrm{Vol}(\bigcup B^0(y_i, 1) \cap U)$. We set $W = \bigcup B^0(x_i, 1) \cap U$. Note that by our definition of k we have $\mathrm{Vol}_n(W) < \frac{2 \, \mathrm{Vol}_n(U)}{25^n}$.

Consider $1/2$-neighbourhood of W and note that it can be covered by at most $(24.4)^n$ sets B_j, where each B_j is a union of k balls $B^0(y_i, 1)$ of radius 1. By definition of W we have $\mathrm{Vol}(B_j) \leq \mathrm{Vol}(W)$, so $\mathrm{Vol}_n(N_l^0(W)) \leq \frac{24.4^n + 1}{25^n} \mathrm{Vol}(U)$. Finally, we observe that $\frac{\mathrm{Vol}^0(N_l^0(W))}{1/2} \leq 2 \, \mathrm{Vol}^0(U)$.
$\hspace{10cm}$ q.e.d.

Theorem 3.4. *There exists a constant $c(n)$ such that the following holds: Let $U \subseteq M$ be an open subset. There exists an $(n-1)$-submanifold $\Sigma \subset U$ subdividing U into two open sets U_1 and U_2 such that $\mathrm{Vol}_n(U_i) \geq (\frac{1}{25^n}) \mathrm{Vol}_n(U)$ and $\mathrm{Vol}_{n-1}(\Sigma) \leq c(n) \max\{1, \mathrm{Vol}_n^0(U)^{\frac{1}{n}}\} \mathrm{Vol}_n(U)^{\frac{n-1}{n}}$.*

Proof. We use the length-area method (see [**16**, p. 4]) to find a small volume hypersurface in $N_l^0(W) \setminus W$, where W and l are as in Lemma 3.3.

Let $f(x) = d^0(W, x)|_U : U \to \mathbb{R}^+$ be the d^0 distance form x to W restricted to the set U. By Rademacher's theorem, f is differentiable almost everywhere. By applying the co-area formula we have:

$$\int_0^l \mathrm{Vol}_{n-1}(f^{-1}(t)) dt \;=\; \int_{f^{-1}(0,l)} ||\nabla f|| dV$$

$$(\text{Hölder's inequality}) \;\leq\; \left(\int_{f^{-1}(0,l)} ||\nabla f||^n dV \right)^{\frac{1}{n}} \left(\mathrm{Vol}_n(f^{-1}(0,l)) \right)^{\frac{n-1}{n}}$$

$$=\; \left(\mathrm{Vol}_n^0(f^{-1}(0,l)) \right)^{\frac{1}{n}} \left(\mathrm{Vol}_n(f^{-1}(0,l)) \right)^{\frac{n-1}{n}}.$$

The last equality holds since $||\nabla f||^n dV = ||\nabla^0 f||^n dV^0$ is a conformal invariant. By Lemma 3.3 we have $\mathrm{Vol}_n^0(f^{-1}(0,l))^{\frac{1}{n}} \leq c(n) l \max\{\mathrm{Vol}_n^0(U)^{\frac{1}{n}}, 1\}$. For the second factor we apply the bound $\mathrm{Vol}_n(f^{-1}(0,l)) \leq \mathrm{Vol}_n(U)$. It follows that

$$\min_{r < t < 2r} \mathrm{Vol}_{n-1}(f^{-1}(t)) \leq c(n) \max\{\mathrm{Vol}(U)^{\frac{1}{n}}, 1\} \mathrm{Vol}_n(U)^{\frac{n-1}{n}}.$$

Thus for some regular value of t the level set $f^{-1}(t)$ with area no larger than average, is the desired submanifold Σ. We take $U_1 = f^{-1}([0,t))$ and $U_2 = f^{-1}((t, \infty))$.

Since $W \subseteq U_1$ by Lemma 3.3 we have $\mathrm{Vol}(U_1) \geq \frac{\mathrm{Vol}_n(U)}{25^n}$. On the other hand, $U_1 \subseteq N_l^0(W)$ of volume at most $1 - \frac{\mathrm{Vol}_n(U)}{25^n}$ so $\mathrm{Vol}(U_2) \geq \frac{\mathrm{Vol}_n(U)}{25^n}$.
$\hspace{10cm}$ q.e.d.

4. The width of small submanifolds

In this section we will show that if a submanifold M' of a Riemannian manifold M has small enough volume then its ∂-width can be bounded from above in terms of $\mathrm{Vol}_{n-1}(\partial M')$. First we show this for a submanifold that is contained in a very small ball.

Definition 4.1. For a closed Riemannian manifold M and $\epsilon_0 \in (0,1)$ define $\epsilon(M, \epsilon_0)$ to be the largest radius r such that: for every $x \in M$ we have that $B(x, r)$ is $(1 + \epsilon_0)$-bilipschitz diffeomorphic to the Euclidean ball of radius r.

Lemma 4.2. *If $M' \subset M$ is contained in a ball of radius $\epsilon(M, \epsilon_0)$ then $\mathrm{W}^\partial(M') \leq (1 + \epsilon_0) \mathrm{Vol}_{n-1}(\partial M')$.*

Proof. A 2-dimensional version of the lemma appeared in [**24**]. Let $U \subset \mathbb{R}^n$ be the image of M' under $(1 + \epsilon_0)$-bilipschitz diffeomorphism F. An argument similar to that in [**27**, §6] shows that for a generic line $l \in \mathbb{R}^n$ the projection of ∂U onto l is a Morse function. Let p denote such a projection map and assume that $p(U) = [0, c]$.

Define $f : [0, c] \to Z_{n-1}(U, \mathbb{Z})$ by setting

$$f(t) = \partial(p^{-1}([0, t]) \cap U).$$

Open subsets of hyperplanes in $\mathbb{R}^n$ are volume minimizing regions. Therefore we have $\mathrm{Vol}_{n-1}(f(t)) \leq \mathrm{Vol}(U)$ for all t. Composing f with F^{-1} we obtain the desired sweep-out.   q.e.d.

We extend the result of the lemma to submanifolds of small volume.

Proposition 4.3. *There exist a constant $C_1(n) > 0$, such that for every closed Riemannian n-manifold M, $\epsilon_0 > 0$ and every embedded submanifold $M' \subset M$ of dimension n and volume $\mathrm{Vol}_n(M') \leq \epsilon(M, \epsilon_0)^n/C_1$ the following bound holds:*

$$\mathrm{W}^\partial(M') \leq 3(1 + \epsilon_0) \mathrm{Vol}_{n-1}(\partial M').$$

The proof of Proposition 4.3 somewhat resembles a high dimensional analog of the Birkhoff curve shortening process. We cover M' by a finite collection of small balls B_i such that balls of $1/4$ of the radius still cover M'. Since M' has very small volume it will not contain any of the balls B_i. Hence, we can cut away the part of $\partial M'$ that is contained in B_i and replace it with a minimal surface that does not intersect $(1/4)B_i$. As a result we obtain a new submanifold $M'' \subset M'$ that does not intersect $(1/4)B_i$. Moreover, we can do this in such a way that volume of the boundary does not increase. The difference $M' \setminus M''$ is contained in a small ball, so we can sweep it out by Lemma 4.2. After finitely many iterations we obtain a submanifold that is entirely contained in one of the small balls. We then apply Proposition 2.3 to assemble a sweep-out of M' from sweep-outs in small balls.

In the proof of Proposition 4.3 we will need the following isoperimetric inequality:

Theorem 4.4 (Federer–Fleming). *There exists a constant $C_2(n) > 1$, such that every k-cycle A in a closed unit ball in $B \subset \mathbb{R}^n$ can be filled by a $(k+1)$-chain D in B, such that: (i) $\mathrm{Vol}(D) \leq C_2(n) \mathrm{Vol}(A)^{\frac{k+1}{k}}$, and (ii) D is contained in the $C_2(n) \mathrm{Vol}(A)^{\frac{1}{k}}$-neighbourhood of A.*

To show Proposition 4.3 we first need to prove the following lemma.

Definition 4.5. A k-chain A will be called δ-minimizing if $\mathrm{Vol}(A) - \delta \leq \inf\{A' \in C_k(M, \mathbb{Z}) : \partial A' = \partial M\}$.

Lemma 4.6. *There is a constant $C_3(n)$ such that the following holds: Let B be a ball of radius $r_0 \leq \epsilon(\epsilon_0, M)$ and $A \subset \partial B$ be an $(n-1)$-chain satisfying $\mathrm{Vol}(A) \leq C_3(n) \mathrm{Vol}(\partial B)$. For every $\delta > 0$ there exists δ-minimal filling D of ∂A in $B(x, r_0)$, such that $D \cap B(x, r_0/2) = \emptyset$. We may take $C_3(n) \leq \omega_{n-1}^{-1}(10C_2(n))^{-n}$.*

The proof of Lemma 4.6 is a variation of an argument in [**16**, §4.2-3]. See also [**17**, Lemma 6].

Proof. Fix $\delta' < \delta r_0/100C_2(n)$. Let D_1 be some δ'-minimal filling of ∂A in B. We claim that D_1 is contained in an $r_0/4$-neighbourhood of ∂A except for a subset of volume at most δ'.

Since B is 2-bilipschitz homeomorphic to a Euclidean ball, we may apply the Federer–Fleming isoperimetric inequality (with a worse constant) inside B. We obtain that every $(n-2)$-cycle S can be filled in B by an $(n-1)$-chain of mass at most $2C_2(n) \mathrm{Vol}(S)^{\frac{n-1}{n-2}}$.

Let $A(r) = \mathrm{Vol}_{n-2}(\{x \in D_1 : d(x, \partial A) = r\})$ and $V(r) = \mathrm{Vol}_{n-1}(\{x \in D_1 : d(x, \partial A) > r\})$. The co-area inequality implies that $|V'(r)| \geq A(r)$.

It follows by the δ'-minimality of D_1 that every open subset $U \subset D_1$ not meeting ∂A must have volume at most:
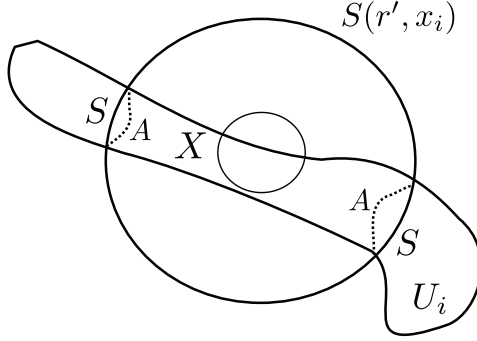
$$\mathrm{Vol}_{n-1}(U) \leq 2C_2(n) \mathrm{Vol}_{n-2}(\partial U)^{\frac{n-1}{n-2}} + \delta'.$$

In particular, we have: $V(r) \leq 2C_2(n)A(r)^{\frac{n-1}{n-2}} + \delta'$. Applying the co-area inequality again we obtain:

$$\frac{d}{dr}\left([V(r) - \delta']^{\frac{1}{n-1}}\right) \leq \frac{-1}{(n-1)(2C_2(n))^{\frac{n-1}{n-2}}}.$$

Hence, $V(r) \leq \delta'$ for some

$$
\begin{aligned}
r &\leq (n-1)(2C_2(n))^{\frac{n-1}{n-2}} \mathrm{Vol}(D_1)^{\frac{1}{n-1}} \\
&\leq (n-1)(2C_2(n))^{\frac{n-1}{n-2}} \mathrm{Vol}(A)^{\frac{1}{n-1}} \\
&\leq 2(n-1)(2C_2(n))^{\frac{n-1}{n-2}} \left(C_3(n)n\omega_n r_0^{n-1}\right)^{\frac{1}{n-1}} \leq r_0/4.
\end{aligned}
$$

Figure 1. Proof of Proposition 4.3.

We will now cut off the piece of D_1 that lies outside of $(r_0/4)$-neighbourhood of ∂D_1. Again, by the co-area inequality we have that: $A(r') \leq \frac{8}{r_0}\delta'$ for some $(1/4)r_0 \leq r' \leq (3/8)r_0$. The Federer–Fleming isoperimetric inequality gives a filling of $\{d(x, \partial D_1) = r'\}$ by an $(n-1)$-chain D_2 satisfying:

$$\mathrm{Vol}(D_2) \leq 2C_2(n)\left(\frac{8}{r_0}\delta'\right)^{\frac{n-1}{n-2}} \leq \delta/2.$$

Moreover, the filling has the property that the distance from $\{d(x, \partial D_1) = r'\}$ to every point of D_2 is at most $2C_2(n)\left(\frac{8}{r_0}\delta'\right)^{\frac{1}{n-1}} \leq r_0/8$. This gives the desired filling.   q.e.d.

Now we prove Proposition 4.3. We will construct a decomposition of M into open sets and then apply Proposition 2.3.

Proof. Set $C_1(n) = 4^n\omega_{n-1}(10C_3(n))^n$. Let $\epsilon = \epsilon(M, \epsilon_0)$ and assume that $M' \subset M$ has volume bounded by $1/C_1(n)\epsilon^n$. Let $B_i = B(x_i, \epsilon)$ for $i = 1, ..., N$, be a collection of balls such that M' is contained in the interior of $\bigcup B(x_i, \epsilon/4)$. Fix $\delta > 0$. We will construct a collection of open subsets $U_1 \subset ... \subset U_N$, with the following properties:

1) $U_N = M'$.
2) $\mathrm{Vol}(\partial U_i) \leq \mathrm{Vol}(\partial U_{i+1}) + \delta/2^i$.
3) $U_i \cap \bigcup_{j=i+1}^N B(x_j, \epsilon/4)$ is empty.

Assume that $U_{i+1}, ..., U_N$ have been defined. If $U_{i+1} \cap B(x_i, \epsilon/4)$ is empty we set $U_i = U_{i+1}$. Otherwise, to construct U_i we proceed as follows.

By the co-area inequality we can find $S(x_i, r') = \partial B(x_i, r')$, with $\frac{3}{4}\epsilon < r' < \epsilon$, such that $S = U_{i+1} \cap S(r', x_i)$ satisfies $\mathrm{Vol}_{n-1}(S) \leq 4\,\mathrm{Vol}_n(U_{i+1} \cap B(x_i, \epsilon))^{1-1/n}$. By Lemma 4.6 there exists an $(n-1)$-chain $A \subset B(x_i, r')$ with $\partial A = \partial S$ which is $(\delta/2^i)$-minimizing and A does not intersect

$B(x_i, \epsilon/4)$. Let X denote the union of the connected components of $U_{i+1} \setminus A$ that intersect $B(x_i, \epsilon/4)$. We define $U_i = U_{i+1} \setminus X$. Note that the volume decreased and by $\delta/2^i$-minimality of A and the volume of the boundary could not have increased by more than $\delta/2^i$.

By Lemma 4.2 we have $W^{\partial}(X) \leq 2(1 + \epsilon_0) \operatorname{Vol}_{n-1}(\partial M') + \delta$. By Proposition 2.3 we have $W^{\partial}(M) \leq 3(1 + \epsilon_0) \operatorname{Vol}_{n-1}(\partial M') + 2\delta$. Since δ can be chosen arbitrarily small this concludes the proof of Proposition 4.3. q.e.d.

5. Proof of the width inequality

In this section we prove Theorem 1.3.

Theorem 5.1. *Let M_0 be a manifold with* $\operatorname{Ricci} \geq -(n-1)$ *and let M be in the conformal class of M_0. Let $M' \subseteq M$ be an n-dimensional submanifold. There exists a constant $C(n)$ that depends on the dimension, such that:*

$$W^{\partial}(M') \leq C(n) \max\{1, \operatorname{Vol}_n^0(M')^{\frac{1}{n}}\} \operatorname{Vol}_n(M')^{\frac{n-1}{n}} + 3 \operatorname{Vol}_{n-1}(\partial M').$$

Theorem 1.3 follows as a special case.

Proof. Pick the constant $C(n) = 4 \cdot 25^n c(n)$, where $c(n)$ is the constant in Theorem 3.4.

Let $\epsilon > 0$ be small enough that every submanifold of volume at most $25^n \epsilon$ satisfies conclusions of Theorem 4.3. Suppose that $M' \subseteq M$, and pick k so that: $k\epsilon < \operatorname{Vol}(M') \leq (k+1)\epsilon$ and $k > 25^n$. We proceed by induction on k.

Assume the desired sweep-out exists for every open subset of volume at most $k\epsilon$. By Lemma 3.4 we can find an $(n-1)$-submanifold Σ subdividing M' into M_1 and M_2 of volume at most $c(n) \max\{1, \operatorname{Vol}_n^0(M')^{\frac{1}{n}}\} \times \operatorname{Vol}_n(M')^{\frac{n-1}{n}}$, such that $\operatorname{Vol}_n(M_i) \leq (1 - 1/25^n) \operatorname{Vol}_n(M')$. Since $k > 25^n$ the inductive hypothesis is applicable to both halves M_i.

By inductive hypothesis we have

$$W^{\partial}(M_i) \leq 3(\operatorname{Vol}(\partial M' \cap M_i) + \operatorname{Vol}(\Sigma))$$
$$+ C(n) \max\{1, \operatorname{Vol}_n(M)^{\frac{1}{n}}\} \operatorname{Vol}_n(U_i)^{\frac{n-1}{n}}.$$

We apply Proposition 2.3 with $U_0 = M' \setminus M_2$ and $U_1 = M'$. We obtain

$$W^{\partial}(M') \leq 3 \operatorname{Vol}(\partial M') + 4 \operatorname{Vol}(\Sigma)$$
$$+ C(n) \max\{1, \operatorname{Vol}_n^0(M')^{\frac{1}{n}}\} \max_{i=1,2}\{\operatorname{Vol}^n(M_i)^{\frac{n-1}{n}}\}.$$

We use bounds $\operatorname{Vol}_n(M_i)^{\frac{n-1}{n}} \leq \frac{25^n - 1}{25^n} \operatorname{Vol}_n(M')$ and

$$\operatorname{Vol}_{n-1}(\Sigma) \leq c(n) \max\{1, \operatorname{Vol}_n^0(M')^{\frac{1}{n}}\} \operatorname{Vol}_n(M')^{\frac{n-1}{n}}.$$

We compute that the resulting expression satisfies the desired bound.
                                                              q.e.d.

Theorem 1.3 follows from Theorem 5.1 by taking the infimum of the total volume of M_0 over all manifolds M_0 that are conformally equivalent to M and have Ricci $\geq -(n-1)$.

6. The width of surfaces

In this section we prove a theorem of Balacheff and Sabourau [3] with an improved constant. Note that the result also follows as an immediate corollary of Theorem 1.3 with a worse constant. However, we observed that one can use a slightly modified version of our proof and invoke the Riemann–Roch theorem to get a somewhat better constant.

Below we prove a version of Theorem 1.3 which allows us to bound width of a manifold M if M admits a conformal map into some nice space M_0 with a small number of pre-images. We will then estimate the width of surfaces by applying uniformization theorem and the Riemann–Roch theorem. Our argument is parallel to the analogous arguments of Yang and Yau [35] and [21, §4] for eigenvalues of the Laplacian on Riemann surfaces.

Definition 6.1. Define $\tau = \tau(M_0)$ and $\nu = \nu(M_0)$ as follows: τ is the least number such that any annulus $B^0(x, 2r) \setminus B^0(x, r)$ in M_0 can be covered by τ balls of radius r. We let $\nu(M_0)$ be the least constant such that $\mathrm{Vol}_n^0(B^0(x, r)) \leq \nu r^n$ for all $r > 0$ and all $x \in M_0$.

Theorem 6.2. *Let $\Phi : (M, g) \to (M_0, g_0)$ be a conformal map. Suppose the following holds: (i) Any point $x \in M_0$ has at most K pre-images, (ii) The set $\{x \in M, d\Phi(x) = 0\}$ is of measure 0. It follows that:*

$$\mathrm{W}(M) \leq \frac{8\nu^{\frac{1}{n}} K^{\frac{1}{n}}}{1 - \left(\frac{\tau+1}{\tau+2}\right)^{\frac{n-1}{n}}} \mathrm{Vol}(M)^{\frac{n-1}{n}}.$$

Proof. First, we prove an analog of our isoperimetric inequality, Theorem 3.4.

Let U be an open set in M. We show that there is an $(n-1)$-submanifold $\Sigma \subset U$ such that $U \setminus \Sigma = U_1 \sqcup U_2$ with $\mathrm{Vol}_n(U_i) \geq \frac{1}{\tau+2} \mathrm{Vol}_n(U)$ and $\mathrm{Vol}_{n-1}(\Sigma) \leq 2\nu^{\frac{1}{n}} K^{\frac{1}{n}} \mathrm{Vol}_n(U)^{\frac{n-1}{n}}$.

Let $p \in M$ and u and v be vectors in the tangent space T_pM. Since Φ is conformal we have

$$\langle \Phi_* u, \Phi_* v \rangle_{g_0} = \phi(x)\langle u, v \rangle_g,$$

for some non-negative function ϕ. In a neighbourhood of a point $p \in M \setminus \{x \in M, d\Phi(x) = 0\}$ map Φ is a local diffeomorphism and

$$\|\nabla(f \circ \Phi)\| = \phi^{1/2}\|\nabla f\| \quad dV_g = \phi^{-n/2}dV_{g_0},$$

where $f : M_0 \to \mathbb{R}$ is a smooth function and dV_g, dV_{g_0} are volume elements.

The fact that the measure of the set $\{x \in M, d\Phi(x) = 0\}$ is zero guarantees that $\lim_{r \to 0} \text{Vol}_n(\Phi^{-1}(B^0(a, r))) = 0$ for all $a \in M_0$. Let r be the smallest radius, such that there exists a ball $B(r, a)$ with $\text{Vol}(\Phi^{-1}(B^0(a, r)) \cap U) = \text{Vol}_n(U)/(\tau + 2)$.

Let d^0 be the distance function on M_0 and define $f(x) = d^0(a, x)|_{\Phi(U)} : \Phi(U) \to \mathbb{R}^+$ to be the distance from $x \in \Phi(U) \subset M$ to a.

$$
\int_r^{2r} \text{Vol}_{n-1}((f \circ \Phi)^{-1}(t)) dt
$$

$$
= \int_{(f \circ \Phi)^{-1}(r, 2r)} ||\nabla(f \circ \Phi)|| dV_g
$$

$$
\leq \left(\int_{(f \circ \Phi)^{-1}(r, 2r)} ||\nabla(f \circ \Phi)||^n dV_g \right)^{\frac{1}{n}} \left(\text{Vol}_n((f \circ \Phi)^{-1}(r, 2r)) \right)^{\frac{n-1}{n}}
$$

$$
\leq K^{\frac{1}{n}} \left(\int_{f^{-1}(r, 2r)} ||\nabla^0 f||^n dV_{g_0} \right)^{\frac{1}{n}} \left(\text{Vol}_n((f \circ \Phi)^{-1}(r, 2r)) \right)^{\frac{n-1}{n}}
$$

$$
\leq 2r \nu^{\frac{1}{n}} K^{\frac{1}{n}} \text{Vol}(U)^{\frac{n-1}{n}}.
$$

It follows that the average of $\text{Vol}_{n-1}((f \circ \Phi)^{-1}(t))$ is smaller than $2\nu^{\frac{1}{n}} K^{\frac{1}{n}} \text{Vol}(U)^{\frac{n-1}{n}}$. We then take $\Sigma = (f \circ \Phi)^{-1}(t)$, with area at most average. This finishes the proof of the analog of Theorem 3.4.

The rest of the proof of Theorem 6.2 proceeds exactly as in Section 5 with $c(n) \max\{1, \text{Vol}_n^0(U)^{\frac{1}{n}}\}$ replaced by $2r\nu^{\frac{1}{n}} K^{\frac{1}{n}}$.   q.e.d.

We now recover Theorem 1.5. Let S_g denote a genus g closed surface with a complete Riemannian metric. We write h for the metric on S_g.

The uniformization theorem for Riemannian surfaces guarantees that there is a metric ϕh of constant sectional curvature in the conformal class of h. If $g = 0$ or $g = 1$ then the result follows from Theorem 6.2 by taking M_0 to be S^2, $\mathbb{RP}^2$, T^2 or the Klein bottle K with the standard metric. In all of these cases we have $\nu = \pi$ and $\tau = 6$ (see Remark 6.3 below).

Suppose now that the surface is orientable and the genus $g > 1$. Take ϕh to have constant sectional curvature $\kappa = -1$. We now apply the Riemann–Roch theorem which gives a meromorphic function $\Phi : S_g \to S^2$ of degree at most $g + 1$. Since Φ is a ramified conformal covering map, it has at most $g+1$ points in each fibre and there are finitely many points where $d\Phi = 0$. Applying Theorem 6.2 to Φ gives a width volume inequality for surfaces of genus $g > 1$, we obtain:

$$
\mathrm{W}(S_g) \leq \frac{8\sqrt{\nu(S^2)}}{1 - \sqrt{\frac{\tau(S^2)+1}{\tau(S^2)+2}}} \sqrt{(g + 1) \, \text{Area}(S_g)}.
$$

Remark 6.3. Clearly $\nu(S^2) = \nu(\mathbb{R}^2) = \pi$. It is well known that the smallest number of discs of radius 1 required to cover an annulus $B(2) \setminus B(1) \subset \mathbb{R}^2$ is 6. A similar covering also works on S^2 so $\tau(S^2) = \tau(\mathbb{R}^2) = 6$. With these values of τ and ν we compute $\dfrac{8\sqrt{\nu(S^2)}}{1 - \sqrt{\frac{\tau(S^2)+1}{\tau(S^2)+2}}} \leq 220$

which improves the upper bound $C \leq 10^8$ from [**3**].

7. Volumes of hypersurfaces

In this section we prove Theorem 1.6.

Theorem 7.1. *If M is a manifold with non-negative Ricci curvature then $\mathrm{W}^k(M) \leq C(n)k^{\frac{1}{n}} \mathrm{Vol}(M)^{\frac{n-1}{n}}$.*

Note that Theorem 7.1 is consistent with the conjecture that the sequence of numbers $\mathrm{W}^k(M)$ obeys a Weyl type asymptotic formula (see [**12**] and the discussion in [**25**, §9]).

To prove Theorem 7.1 we will need to decompose M into open subsets of small sizes. Similar arguments for bounding W^k have been used by Gromov [**12**],[**14**] and Guth [**19**].

Lemma 7.2. *Let M be a closed Riemannian manifold with $\mathrm{Ricci}(M) \geq 0$. There exists a constant $C_4(n)$, such that for any p there exists $p' \leq p$ and a collection of open balls $\{U_i\}_{i=1}^{p'}$ with $\bigcup U_i = M$, $\mathrm{Vol}_n(U_i) \leq C_4(n)\frac{\mathrm{Vol}_n(M)}{p}$ and $\mathrm{Vol}_{n-1}(\partial U_i) \leq C_4(n)\left(\frac{\mathrm{Vol}_n(M)}{p}\right)^{\frac{n-1}{n}}$.*

Proof. It is a standard fact in comparison geometry that for any ball $B(x,r) \subset M$ we have $\mathrm{Vol}_n(B(x,3r)) \leq 3^n \mathrm{Vol}(B(x,r))$ and $\mathrm{Vol}_{n-1}(\partial B(x,3r)) \leq 3^{n-1}n\omega_n^{\frac{1}{n}} \mathrm{Vol}_n(B(x,r))^{\frac{n-1}{n}}$.

Both of these bounds can be deduced, for example, from the Bishop–Gromov inequality

$$\frac{\mathrm{Vol}_n(B(x,r-\epsilon))}{\omega_n(r-\epsilon)^n} \geq \frac{\mathrm{Vol}_n(B(x,r))}{\omega_n r^n},$$

where ω_n denotes the volume of a unit ball in Euclidean n-space.

To prove the second bound observe that $\mathrm{Vol}_n(B(x,r) \setminus B(r - \epsilon)) \leq \frac{n\epsilon}{r} \mathrm{Vol}_n(B(x,r)) + O(\epsilon^2)$. Since $\mathrm{Vol}_n(B(x,r)) \leq \omega_n r^n$ we can bound the volume of the annulus by $n\omega_n^{\frac{1}{n}}\epsilon(\mathrm{Vol}_n(B(r)))^{\frac{n-1}{n}} + O(\epsilon^2)$. Since $\mathrm{Vol}(3B_i) \leq 3^n \mathrm{Vol}(B_i)$ we obtain that for every $\epsilon > 0$ the volume of the annulus $B(x_i, 3r_i) \setminus B(x_i, 3r_i - \epsilon)$ is bounded by $3^{n-1}n\omega_n^{\frac{1}{n}}\epsilon \mathrm{Vol}(B(x_i,r_i))^{\frac{n-1}{n}} + O(\epsilon^2)$. Hence, there must exist a sphere $S(x,r')$ in the annulus, $3r - \epsilon \leq r' \leq 3r$, with $\mathrm{Vol}_{n-1}(S(x,r')) \leq 3^{n-1}n\omega_n^{\frac{1}{n}} \mathrm{Vol}_n(B(x,r))^{\frac{n-1}{n}} + O(\epsilon^2)$. By curvature comparison again the

volume of a sphere cannot suddenly jump up. Since ϵ was arbitrary we conclude $\mathrm{Vol}_{n-1}(\partial S(x,3r)) \le 3^{n-1} n \omega_n^{\frac{1}{n}} \mathrm{Vol}_n(B(x,r))^{\frac{n-1}{n}}$.

Now we construct a covering of M by disjoint balls of volume $\frac{\mathrm{Vol}_n(M)}{p}$, such that balls of three times the radius cover M. This is also standard (see [**15**]). For each x choose $r_x > 0$ to be the radius of a ball $B(x, r_x)$, such that $\mathrm{Vol}_n(B(x, r_x)) = \frac{\mathrm{Vol}_n(M)}{p}$. By compactness there exists a finite subcollection of balls $B(x, r_x)$ that cover M. By the Vitali covering lemma we can further choose a subcollection of disjoint balls $B_1, \dots, B_k$ with radii $r_1, \dots, r_{p'}$, such that balls of three times the radius cover M. Note that we must have $p' \le p$. Theorem now follows by taking $U_i = 3B_i$.   q.e.d.

By Theorem 5.1 we have the following: for each open subset U of M there exists a family of cycles X_t, for $0 \le t \le 1$, sweeping-out U. Moreover, we have that X_0 is a trivial cycle, $X_1 = \partial U$ and $\mathrm{Vol}(X_t) \le \mathrm{Vol}(\partial U) + C(n)\mathrm{Vol}(U)^{\frac{n-1}{n}}$. For each i we let X_t^i be the family of cycles with the above properties for the submanifold with boundary $U_i \setminus (\bigcup_{j=1}^{i-1} U_j)$. Let $V_i = \bigcup_{j=1}^{i} U_j$ for $1 \le i \le p'$ and $V_i = \emptyset$ otherwise. Define a family of mod 2 cycles Z_t for $0 \le t \le p'$ by setting $Z_t = \partial V_{i-1} + X_{t-[t]}^i$ for $i - 1 \le t \le i$, here $[t]$ denotes the integer part of t. We identify the endpoints (which are trivial cycles) and rescale so that Z_t is parametrized by a unit circle.

Observe that for each t cycle Z_t can be decomposed into two $(n-1)$-cycles $Z_t = Z_t^1 + Z_t^2$ with $Z_t^1 \subset \bigcup \partial U_i$ and $\mathrm{Vol}(Z_t^2) \le C_4(n)\left(\frac{\mathrm{Vol}(M)}{p}\right)^{\frac{n-1}{n}}$.

Following Gromov [**12**] and Guth [**19**] we will now define a p-cycle $F : \mathbb{RP}^p \to Z_{n-1}(M, \mathbb{Z}_2)$ which detects the cohomology element λ^p. Consider a truncated symmetric product $TP^p(S^1)$. Recall that the symmetric product is defined as a quotient of $(S^1)^p$ by the symmetric group S_p. The truncated symmetric product is then defined as a quotient of the symmetric product by an equivalence relation $1s_j + 1s_k = 0$ if $s_j = s_k$ and $j \ne k$. In [**29**] Mostovoy proved that $TP^p(S^1)$ is homeomorphic to $\mathbb{RP}^p$ (we learnt about this from [**19**]). We define $F(\sum_{i=1}^{p} a_i t_i) = \sum_{i=1}^{p} a_i Z_{t_i}$. Alternatively, we could define a map from $\mathbb{RP}^p$ to $Z_{n-1}(M, \mathbb{Z}_2)$ with the desired property using zeros of real polynomials of one variable with degree at most p as in [**12**, 4.2B] and [**19**, Section 5].

We claim that $\mathrm{Vol}(F(x)) \le C(n)p^{\frac{1}{n}}\mathrm{Vol}(M)^{\frac{n-1}{n}}$. Indeed, we may decompose each of the p summands $Z_{t_i} = Z_{t_i}^1 + Z_{t_i}^2$ with $Z_{t_i}^1$ contained in the union of the boundaries of U_i's and the volume of $Z_{t_i}^2$ bounded from above by a constant times $(\frac{\mathrm{Vol}(M)}{p})^{\frac{n-1}{n}}$. Since we are dealing with mod 2 cycles the sum of all $Z_{t_i}^1$ cannot be greater than

$\mathrm{Vol}(\bigcup \partial U_i) \leq C(n)p^{\frac{1}{n}} \mathrm{Vol}(M)^{\frac{n-1}{n}}$. We also have that the sum $\sum Z_{t_i}^2 \leq C(n)p^{\frac{1}{n}} \mathrm{Vol}(M)^{\frac{n-1}{n}}$.

Finally, we show that $F^*(\lambda^p) = F^*(\lambda)^p \neq 0$. Observe that $S = \{1 \cdot t : t \in S^1\} \subset TP^p(S^1)$ represents a non-trivial homology class in $H_1(TP^p(S^1), \mathbb{Z}_2)$ and $F(S) = \{Z_t\}_{t \in S^1}$ is a sweep-out of M by Proposition 2.3. It follows that $F^*(\lambda) = 1$. This finishes the proof of Theorem 7.1.

We use this result to bound volumes of minimal hypersurfaces in a space with positive Ricci curvature. These minimal hypersurfaces arise from Almgren–Pitts min–max theory as supports of stationary almost minimizing integral varifolds. Pitts [**32**] and Schoen and Simon [**34**] proved that these hypersurfaces are smooth embedded submanifolds when $n \leq 7$. In higher dimensions they may have singular sets of dimension at most $n - 8$.

Marques and Neves [**25**] showed that every manifold M of dimension n, for $3 \leq n \leq 7$, with positive Ricci curvature possesses infinitely many embedded minimal hypersurfaces. Theorem 1.6 is an effective version of their result. Note that for 2-dimensional surfaces an analogous result for periodic geodesics is false. Morse showed that for an ellipsoid of area 1 with distinct but very close semiaxes the length of the fourth shortest geodesic becomes uncontrollably large ([**28**]).

Proof of Theorem 1.6. Let V_k be the infimum of numbers such that there exists k distinct minimal hypersurfaces of volume less or equal to V_k. By [**25**, Prop. 4.8] and results in Section 2 of the same paper we may assume that each parametric width W^k can be written as a finite linear combination $\mathrm{W}^k = \sum a_j \mathrm{V}_j$, where a_j are integer coefficients. Moreover, when M has positive Ricci curvature (or, more generally whenever M has the property that any two embedded minimal hypersurfaces in M intersect) we have $\mathrm{W}^k = a_{j_k} \mathrm{V}_{j_k}$ for some positive integer a_{j_k}.

Let $C = C(n)$ be the constant from Theorem 7.1 and define $C' = 2^{\frac{1}{n-1}} C^{\frac{n}{n-1}}$. We proceed by contradiction. Suppose

$$\mathrm{V}_k > C' \mathrm{Vol}(M) \left(\mathrm{sys}_{n-1}(M)\right)^{-\frac{1}{n-1}} k^{\frac{1}{n-1}},$$

for some k. Let $A(N) = \{\mathrm{W}^i \leq N\}$. It follows from the proof of Theorem 6.1 in [**25**] that if $\mathrm{W}^i = \mathrm{W}^{i+1}$ for some i then there exists infinitely many hypersurfaces of volume at most W^i. Hence, we may assume that $\mathrm{W}^i < \mathrm{W}^{i+1}$ for all $i < k$. By Theorem 7.1 we have that the number of elements in the set $A(N)$ satisfies $\#A(N) \geq \frac{N^n}{C^n \mathrm{Vol}(M)^{n-1}} - 1$. In particular, we compute $\#A(\mathrm{V}_k) \geq 2\frac{k\mathrm{V}_k}{V_1} - 1$. On the other hand, the set $\{a_i \mathrm{V}_i : a_i \in \mathbb{N}, a_i \mathrm{V}_i \leq \mathrm{V}_k\}$ has at most $\frac{k\mathrm{V}_k}{\mathrm{V}_1}$ elements, which is a contradiction.

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