

ISOPERIMETRIC STRUCTURE OF ASYMPTOTICALLY CONICAL MANIFOLDS

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Abstract

We study the isoperimetric structure of Riemannian manifolds that are asymptotic to cones with non-negative Ricci curvature. Specifically, we generalize to this setting the seminal results of G. Huisken and S.-T. Yau [23] on the existence of a canonical foliation by volume-preserving stable constant mean curvature surfaces at infinity of asymptotically flat manifolds as well as the results of the second-named author with S. Brendle [6] and J. Metzger [14, 15] on the isoperimetric structure of asymptotically flat manifolds. We also include an observation on the isoperimetric cone angle of such manifolds. This result is a natural analogue of the positive mass theorem in this setting.

1. Introduction

The seminal results of G. Huisken and S.-T. Yau [23] on the existence of a canonical foliation at infinity of asymptotically flat manifolds and the uniqueness of the leaves of this foliation (which were refined by J. Qing and G. Tian [31]) has had considerable repercussions on the study of such manifolds, both from a physical and a geometric point of view. We refer the reader to the introductions of [20, 13, 14, 15] for recent accounts of these developments. The goal of this paper is to extend many of these results to certain asymptotically conical manifolds.

Let $m \geq 2$ be an integer and (L, g_L) be a connected closed Riemannian manifold of dimension $m - 1$. If (M, g) is an m -dimensional Riemannian manifold such that there exists a compact set $K \subset M$ and a diffeomorphism

$$M \setminus K \cong (1, \infty) \times L,$$

with

$$(1) \quad g = dr \otimes dr + r^2 g_L + o(1) \quad \text{as } r \rightarrow \infty,$$

then we say that (M, g) is asymptotically conical with link (L, g_L) .

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If $k \geq 0$ is an integer, we say that the expansion (1) holds in C^k if

$$\sum_{\ell=0}^k r^\ell |\nabla^\ell (g - g_C)| = o(1) \quad \text{as } r \rightarrow \infty.$$

Here, ∇ is the Levi-Civita connection of the cone (C, g_C) where

$$C = (0, \infty) \times L \quad \text{and} \quad g_C = dr \otimes dr + r^2 g_L$$

and norms are computed with respect to g_C . Similarly, given $\alpha \in (0, 1)$, we say the expansion holds in $C^{k,\alpha}$ if the weighted $C^{k,\alpha}$ norms of $g - g_C$ tend to zero as $r \rightarrow \infty$.

Given $r > 1$, we denote by B_r the region in M consisting of K and the diffeomorphic image of the set $(1, r) \times L$. It is convenient to introduce a smooth positive function $|\cdot| : M \rightarrow \mathbb{R}$ that extends the coordinate function r on $M \setminus B_{2r}$ to all of M .

In this paper, we study the isoperimetric structure of asymptotically conical manifolds (M, g) of dimension m whose link (L, g_L) satisfies the conditions

$$(2) \quad \text{Ric}_L \geq (m-2)g_L,$$

$$(3) \quad \text{area}(L, g_L) < \omega_{m-1},$$

where ω_{m-1} is the volume of the unit sphere. In view of (2) and R. Bishop's theorem, (3) is equivalent to the requirement that (L, g_L) is not isometric to the $(m-1)$ -dimensional unit sphere. Our results show that conditions (2) and (3) are in many ways analogous to positivity of mass and non-negativity of scalar curvature for asymptotically flat manifolds.

The existence of a foliation through volume-preserving stable CMC surfaces asserted in our first result below follows from the inverse function theorem, cf. [42]. An analogous result in the asymptotically flat setting was first proven by G. Huisken and S.-T. Yau in [23] using volume-preserving mean curvature flow.

Theorem 1 (Canonical foliation by CMC surfaces). *Let (M, g) be an asymptotically conical manifold of dimension m such that the expansion (1) holds in $C^{2,\alpha}$ and with*

$$\lambda_1(-\Delta_L) > (m-1).$$

There are $\delta > 0$ and $V_0 > 0$ such that the following hold. Let $V > V_0$. Let $r > 1$ be such that $\mathcal{L}^m(B_r) = V$. There is exactly one $u_V \in C^{2,\alpha}(L)$ with

$$-\delta \leq u_V \leq \delta,$$

such that

$$\Sigma_V = \{(r(1 + u_V(x)), x) : x \in L\}$$

is a CMC surface that encloses volume V . Moreover, Σ_V is volume-preserving stable, the surfaces

$$\{\Sigma_V\}_{V>V_0},$$

form a foliation of the complement of a compact subset of M , and the $C^{2,\alpha}$ norms of u_V tend to zero as $V \rightarrow \infty$.

Remark 2. If (L, g_L) satisfies conditions (2) and (3), then

$$\lambda_1(-\Delta_L) > m - 1$$

by eigenvalue estimate [9, Theorem 9 of Chapter III] of A. Lichnerowicz.

Theorem 3 (Existence and uniqueness of large isoperimetric regions). *Let (M, g) be an asymptotically conical Riemannian manifold of dimension m whose link (L, g_L) satisfies conditions (2) and (3) and such that (1) holds in $C^{1,\alpha}$. There is $V_0 > 0$ with the following property. For every $V > V_0$ there is an isoperimetric region of volume V . Every such region Ω_V is regular and close to B_r where $r > 1$ is such that*

$$\mathcal{L}_g^m(B_r) = V.$$

If the expansion (1) holds in $C^{2,\alpha}$, then

$$\partial\Omega_V = \Sigma_V,$$

where Σ_V is as in Theorem 1. In particular, Ω_V is the unique isoperimetric region of volume V .

Remark 4. In Appendix B we show that condition (2) in Theorem 3 may be replaced by a weaker, purely isoperimetric condition on the link (L, g_L) .

An analogous result in the asymptotically flat setting was proven by the second-named author and J. Metzger [14, 15]. Their contribution builds on ideas of H. Bray [4] who characterized isoperimetric regions in spatial Schwarzschild. Subsequently, the first-named author [10] established related results for asymptotically hyperbolic three-manifolds. The proof of Theorem 3 is modeled on these results and uses the classification [28] due to F. Morgan and M. Ritoré of isoperimetric regions of cones whose link satisfies (2) and (3).

Definition 5. Let (M, g) be asymptotically conical. Given a subset $S \subset M$, we let

$$\underline{r}(S) = \sup\{r > 1 : B_r \cap S = \emptyset\}$$

and

$$\bar{r}(S) = \inf\{r > 1 : S \subset B_r\}.$$

The following theorem is the analogue for asymptotically conical manifolds of the uniqueness results for volume-preserving stable constant mean curvature surfaces in asymptotically flat manifolds due G. Huisken

and S.-T. Yau [23] (and refined by J. Qing and G. Tian [31]) as well as the results of the second-named author with S. Brendle [6] and with J. Metzger [13].

Theorem 6 (Uniqueness of large closed volume-preserving stable CMC surfaces). *Let (M, g) be an asymptotically conical Riemannian 3-manifold with link (L, g_L) such that the expansion (1) holds in $C^{2,\alpha}$ and such that the scalar curvature R of (M, g) satisfies the estimate*

$$R \geq -O(r^{-2-\epsilon}),$$

for some $\epsilon > 0$. We also assume that

$$K_L > 1,$$

on a dense set of L , where K_L is the Gaussian curvature of the link. There are $\alpha, \beta > 0$ with the following property. Let Σ be a connected closed volume-preserving stable CMC surface in (M, g) with $\underline{r}(\Sigma) > \alpha$. We have the following dichotomy. If

$$\bar{r}(\Sigma)H(\Sigma) > \beta,$$

then Σ is close to an outlying geodesic sphere of radius $2/H(\Sigma)$. If

$$\bar{r}(\Sigma)H(\Sigma) \leq \beta,$$

then Σ is part of the canonical foliation $\{\Sigma_V\}_{V>V_0}$ of Theorem 1.

Remark 7. The assumption on the Gaussian curvature of the link here is equivalent to requiring that the exact cone (C, g_C) contains no flat regions. This hypothesis is clearly necessary. The a priori assumption that $\underline{r}(\Sigma) \geq \alpha$ may be relaxed to the assumption

$$\bar{r}(\Sigma) \geq \alpha,$$

if (M, g) has non-negative scalar curvature. This is because the estimate (11) in the proof of Theorem 6 follows directly from Lemma 23 in this case. If the scalar curvature of (M, g) is positive, the condition $\underline{r}(\Sigma) \geq \alpha$ may be relaxed to the assumption

$$\text{area}(\Sigma) \geq \alpha.$$

This follows from the monotonicity formula and (7). We can allow immersed surfaces in Theorem 6 if, in the latter alternative of the conclusion, we include the possibility that the immersion is a cover of the leaf.

Remark 8. The Alexandrov theorem classifying closed embedded CMC hypersurfaces in space forms (with no stability assumption) was extended to a large class of rotationally symmetric metrics by S. Brendle [5].

Theorem 9 (Non-existence of unbounded complete stable minimal immersions). *Let (M, g) be an asymptotically conical 3-manifold such that the expansion (1) holds in C^2 and whose link satisfies $K_L > 1$. There is no complete stable minimal immersion into (M, g) that has unbounded trace.*

Remark 10. In fact, we can rule out minimal immersions with finite index and unbounded trace. The reason is that such an immersion is stable outside a compact set. The proof goes through without change.

Building on ideas of R. Schoen and S.-T. Yau in their proof of the positive mass theorem [38], analogous results in the asymptotically flat setting were obtained by the second-named author and J. Metzger [13], by A. Carlotto [7], and by A. Carlotto and the first- and second-named authors [8]. Analogous results in the asymptotically flat free boundary setting were obtained by the third-named author [41]. The condition that $K_L > 1$ in Theorem 9 takes the role of the assumptions that the scalar curvature (the mean curvature of the support surface in the free boundary setting) is strictly positive or that the mass (the extrinsic mass in the free boundary setting) is — in a certain sense — distributed uniformly at infinity. It is remarkable that no additional condition such as a sign on the scalar curvature of (M, g) is required in Theorem 9. This is in strong contrast with existing results in the asymptotically flat and the free boundary settings.

Our final result on the isoperimetric structure of asymptotically conical manifolds here is an observation related to G. Huisken’s concept of the isoperimetric cone angle.

Definition 11 (G. Huisken [21, 22]). The isoperimetric cone angle $c_{iso}(M, g)$ of a non-compact m -dimensional Riemannian manifold (M, g) is defined as

$$(4) \quad \inf \left\{ \frac{\mathcal{H}_g^{m-1}(\partial\Omega)^m}{m^{m-1}\omega_{m-1}\mathcal{L}_g^m(\Omega)^{m-1}} : \begin{array}{l} \emptyset \neq \Omega \subseteq M \text{ open with } C^2 \\ \text{outward minimizing boundary} \end{array} \right\}.$$

Remark 12. G. Huisken’s original definition differs from (4) by a positive power depending on m .

The following rigidity result for the isoperimetric cone angle is in some form analogous to the positive mass theorem.

Theorem 13. *Let (M, g) be an asymptotically conical Riemannian manifold of dimension $m \geq 2$ with non-negative Ricci curvature. Then*

$$c_{iso}(M, g) \leq 1,$$

with equality if and only if (M, g) is isometric to Euclidean space.

Finally, we mention the following remarkable characterization of the isoperimetric cone angle.

Theorem 14 (G. Huisken [21, 22]). *Let (M, g) be a complete non-compact Riemannian 3-manifold with positive Ricci curvature. Then*

$$(5) \quad c_{iso}(M, g) = \inf \left\{ \frac{1}{16\pi} \int_{\partial\Omega} H^2 : \emptyset \neq \Omega \Subset M \text{ open with } C^2 \text{ outward minimizing boundary} \right\}.$$

If the infimum in either (4) or (5) is attained by a non-empty, open subset $\Omega \Subset M$ with C^2 outward minimizing boundary, then $(M \setminus \Omega, g)$ is isometrically contained in a cone.

Remark 15. It was shown by F. Morgan and M. Ritoré in [28] that centered coordinate balls are uniquely isoperimetric in cones whose link satisfies conditions (2) and (3). Analogous results for cones over certain two- and three-dimensional tori were established by F. Morgan [27].

F. Duzaar and K. Steffen showed that the large isoperimetric regions of the product

$$(\mathbb{R} \times L, dt \otimes dt + g_L)$$

are slabs [12, Proposition 2.11] when (L, g_L) is a connected closed Riemannian manifold. M. Ritoré and E. Vernadakis [34] and J.G. Pérez [30] extended this characterization of large isoperimetric regions to Riemannian products of (L, g_L) with \mathbb{R}^k for any $k \geq 1$. The optimizers are of the form $D \times L$ where $D \subset \mathbb{R}^k$ is a ball. M. Ritoré and E. Vernadakis [33, 35] studied relative isoperimetric regions in convex regions that are asymptotically conical, and in convex cylinders and cylindrically bounded convex bodies.

A. Mondino and S. Nardulli [25] proved the existence of isoperimetric regions of all volumes in Riemannian manifolds that satisfy a lower Ricci curvature bound and whose geometry at infinity is suitably tame.

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2. Preliminary lemmas

Let (M, g) be a homogeneously regular Riemannian manifold of dimension m . Given $V > 0$ smaller than the volume of the manifold, we let

$$A(V) = \inf \left\{ \mathcal{H}_g^{m-1}(\partial^* \Omega) : \begin{array}{l} \Omega \subset M \text{ a bounded open} \\ \text{region with } \mathcal{L}_g^m(\Omega) = V \end{array} \right\}.$$

A bounded open region $\Omega \subset M$ with

$$\mathcal{L}_g^m(\Omega) = V \quad \text{and} \quad \mathcal{H}_g^{m-1}(\partial^* \Omega) = A(V)$$

is called an isoperimetric region. The reduced boundary of an isoperimetric region is a regular hypersurface that is relatively open in its closure and whose complement in its closure has Hausdorff dimension at most $m - 8$ in (M, g) .

The lemma below follows from the general existence result [29]. We briefly sketch the proof, following the adaptation to the asymptotically flat setting in [14, Proposition 4.2] and [15, Appendix E] of [32, Theorem 2.1] and of [28, Theorems 2.1 and 2.2].

Lemma 16. *Let (M, g) be an asymptotically conical Riemannian manifold of dimension m . Let $V > 0$. There is $\rho > 0$ and an isoperimetric region $\Omega \subset M$ such that the following hold:*

- (i) $\omega_{m-1}\rho^m/m + \mathcal{L}_g^m(\Omega) = V$;
- (ii) $\omega_{m-1}\rho^{m-1} + \mathcal{H}_g^{m-1}(\partial^* \Omega) = A(V)$.

Proof. The idea is to consider a minimizing sequence $\{\Omega_i\}_{i=1}^\infty$ for the isoperimetric problem of volume V . Let

$$V_0 = \lim_{r \rightarrow \infty} \liminf_{i \rightarrow \infty} \mathcal{L}_g^m(\Omega_i \cap B_r).$$

Standard arguments from geometric measure theory show that there is an isoperimetric region Ω in (M, g) of volume V_0 . Let $\rho \geq 0$ be such that

$$\omega_{m-1}\rho^m/m = V - V_0.$$

Using cut and paste arguments we may replace the original minimizing sequence for volume V by a sequence of the form $\{\Omega \cup R_i\}_{i=1}^\infty$ where $R_i \subset M$ are regular regions of volume $\omega_{m-1}\rho^m/m$ which diverge to infinity. Using [28, Theorems 2.1 and 2.2] and a scaling argument, we conclude that

$$\lim_{i \rightarrow \infty} \mathcal{H}_g^{m-1}(\partial R_i) = \omega_{m-1}\rho^{m-1}. \quad \text{q.e.d.}$$

Lemma 17 ([28, Corollary 3.9]). *Assume that the link (L, g_L) satisfies conditions (2) and (3). Let $r > 0$. The set $(0, r) \times L$ uniquely minimizes perimeter for its volume in the cone (C, g_C) .*

J. Barbosa and M. do Carmo [1] classified closed volume-preserving stable CMC immersions in Euclidean space; they are spheres. The following lemma describes an extension of their result to the case where the target manifold is a cone with non-negative Ricci curvature.

Lemma 18 (Essentially [28, Theorem 3.6]). *Let (L, g_L) be a connected closed Riemannian manifold of dimension $m - 1$ such that*

$$(6) \quad \text{Ric}_L > (m - 2)g_L,$$

on a dense set. Let

$$C = (0, \infty) \times L \quad \text{and} \quad g_C = dr \otimes dr + r^2 g_L$$

be the cone on (L, g_L) . We consider an immersion of a connected $(m - 1)$ -dimensional manifold S without boundary into (C, g_C) with the following properties:

- (i) *The immersion is complete away from the tip of the cone.*
- (ii) *The immersion has finite area.*
- (iii) *The immersion is two-sided.*
- (iv) *The immersion has constant non-zero mean curvature.*
- (v) *The immersion is volume-preserving stable in the sense that*

$$\int_S |\nabla f|^2 \geq \int_S (|h|^2 + \text{Ric}(\nu, \nu)) f^2,$$

for all $f \in C_c^\infty(S)$ with

$$\int_S f = 0.$$

Here, ν is a unit normal field along the immersion.

Such an immersion is a covering of $\{r\} \times L$ for some $r > 0$.

Proof. The proof is almost exactly the same as for embedded surfaces in [28, Theorem 3.6]. One uses volume-preserving variations obtained from unit normal perturbation and homothetic rescaling to conclude from stability that the immersion is totally umbilical and that $\text{Ric}(\nu, \nu) = 0$. The “logarithmic cut-off trick” is used to deal with the cone point. For the characterization of totally umbilical immersions in (C, g_C) , we present an argument slightly different from [28, Lemma 3.8]. The discussion in Appendix A shows that the normal direction is radial at points along the immersion where (6) is strict. Such points lie dense in the open subset of S where the immersion is transverse to the radial direction. Finally, note that the immersion cannot be everywhere tangent to the radial direction since this would entail that its trace contains an entire ray, implying that the immersion has infinite area by the monotonicity formula. q.e.d.

Lemma 19 (Essentially [36, Theorem 18]). *Let (M, g) be a homogeneously regular Riemannian 3-manifold. Every immersed complete volume-preserving stable CMC surface of sufficiently large mean curvature is a perturbation of a geodesic sphere.*

Lemma 20 (Essentially [13, Proposition 2.2]). *Let (M, g) be a homogeneously regular Riemannian 3-manifold. For every $\alpha > 0$ there is $\beta > 0$ such that the principal curvatures of every immersed complete volume-preserving stable CMC surface whose mean curvature is bounded in absolute value by α are bounded by β .*

Remark 21. The thresholds and estimates in Lemmas 19 and 20 only depend on the injectivity radius and curvature bounds for the ambient manifold (M, g) .

Lemma 22 (Essentially [13, Proposition 2.3]). *Let (M, g) be an asymptotically conical Riemannian 3-manifold such that the expansion (1) holds in C^2 . For every $\alpha > 0$ and $r_0 > 0$ there is $\beta > 0$ such that for every connected complete volume-preserving stable CMC immersion $\varphi : \Sigma \rightarrow (M, g)$ whose mean curvature is bounded in absolute value by α and whose trace intersects B_{r_0} we have the estimate*

$$\sup_{x \in \Sigma} |\varphi(x)| |h(x)| \leq \beta.$$

Lemma 23 ([11, (4)]). *Let (M, g) be a Riemannian 3-manifold. Then*

$$(7) \quad \int_{\Sigma} H^2 + 2|h|^2 + 2(R \circ \varphi) \leq 64\pi,$$

for every connected closed volume-preserving stable CMC immersion $\varphi : \Sigma \rightarrow (M, g)$. Here, R is the scalar curvature of (M, g) and H and h are, respectively, the scalar mean curvature and the second fundamental form of the immersion. The bound on the right hand side may be lowered to 48π when the genus of Σ is zero.

Lemma 24 (Essentially [23, (5.6), (5.7)]). *Let (M, g) be an asymptotically conical Riemannian 3-manifold such that the expansion (1) holds in C^1 and such that*

$$R \geq -O(r^{-2-\epsilon}),$$

for some $\epsilon > 0$ where R is the scalar curvature of (M, g) . There exists a constant $\gamma > 0$ such that

$$\int_{\Sigma} |h|^2 \leq \gamma,$$

for every connected closed volume-preserving stable CMC surface Σ in (M, g) provided that $\underline{r}(\Sigma) > 1$ is sufficiently large.

Remark 25. The assumption on the scalar curvature is satisfied when $K_L > 1$ and (1) holds in C^2 . Recall that K_L is the Gaussian curvature of the link. It is also satisfied when $K_L \geq 1$ and when

$$g = g_C + o(r^{-\epsilon}),$$

in C^2 as $r \rightarrow \infty$.

Remark 26. The arguments in [13] show that

$$\underline{r}(\Sigma) \rightarrow \infty \quad \text{as} \quad \text{area}(\Sigma) \rightarrow \infty,$$

when the scalar curvature of (M, g) is positive.

Proof. The vector field

$$r\partial_r \in \mathfrak{X}(C),$$

in (C, g_C) shares all the properties of the position vector field of Euclidean space that are needed in the derivation of (5.6) in [23]. The idea is then to combine (7) with the estimate

$$\underline{r}(\Sigma)^\epsilon \int_\Sigma r^{-2-\epsilon} \lesssim \int_\Sigma H^2,$$

which is valid provided that $\underline{r}(\Sigma) > 1$ is sufficiently large. q.e.d.

Lemma 27 (Essentially [19] and [16]). *Let (L, g_L) be a 2-dimensional Riemannian manifold with Gaussian curvature*

$$K_L \geq 1.$$

Let (C, g_C) be the 3-dimensional Riemannian cone on (L, g_L) . Thus

$$C = (0, \infty) \times L \quad \text{and} \quad g_C = dr \otimes dr + r^2 g_L.$$

Consider a stable minimal immersion of a connected two-dimensional manifold S into (C, g_C) that is complete away from the tip of the cone. Then S with its induced metric is conformally equivalent to either the cylinder or the plane. Moreover, the immersion is totally geodesic and

$$\text{Ric}(\nu, \nu) = 0,$$

where Ric is the Ricci tensor of (C, g_C) and ν is a unit normal field along the immersion.

Proof. The condition on the Gaussian curvature of the link (L, g_L) implies that

$$\text{Ric} \geq 0.$$

Following an idea of R. Gulliver and B. Lawson [19], we consider the conformally related cylindrical metric

$$\tilde{g}_C = r^{-2} g_C = (d \log r)^2 + g_C.$$

The scalar curvature of \tilde{g}_C is positive. Moreover, the immersion is complete with respect to this metric. A computation using the stability of the immersion with respect to g_C , the Gauss equation, and the identity

$$\nabla(r\partial_r) = \text{id},$$

in (C, g_C) leads to the estimate

$$-\tilde{\Delta} - \tilde{K} \geq 0,$$

where $\tilde{\Delta}$ and \tilde{K} are the (non-positive) Laplace–Beltrami operator and the Gaussian curvature of the immersion with respect to \tilde{g}_C . The results of D. Fischer-Colbrie and R. Schoen [16, Theorem 3] imply that S is conformally equivalent to either the cylinder or the plane. Either way, as in [16], we may find a sequence $f_j \in C_c^\infty(S)$ that converges to 1 pointwise as $j \rightarrow \infty$ while their Dirichlet energies tend to 0. Using these test functions in the stability inequality for the original immersion and passing to the limit, we obtain that

$$\int_S |h|^2 + \text{Ric}(\nu, \nu) \leq 0. \quad \text{q.e.d.}$$

The following result is an immediate consequence of the characterization of Ricci curvature bounds via optimal transport, cf. [24, 39, 40]. We include an elementary and essentially well-known proof for convenient reference. We are grateful to A. Petrunin and G. Wei for valuable discussions related to this result.

Lemma 28. *Let g, g_1, g_2, \dots be Riemannian metrics on a manifold M such that $g_i \rightarrow g$ in C^0 . If each g_i has non-negative Ricci curvature, then so does g .*

Proof. Fix $p \in M$. By the Laplacian comparison theorem,

$$\Delta_{g_i} \text{dist}_{g_i}(p, \cdot) \leq \frac{m-1}{\text{dist}_{g_i}(p, \cdot)}$$

holds weakly on $M \setminus \{p\}$ where m is the dimension of M . These inequalities pass to the limit

$$(8) \quad \Delta_g \text{dist}_g(p, \cdot) \leq \frac{m-1}{\text{dist}_g(p, \cdot)},$$

as $i \rightarrow \infty$. Conversely, recall that $\Delta_g \text{dist}_g(p, \cdot)$ is the mean curvature of the geodesic sphere $S_r(p)$ for $r > 0$ sufficiently small. The expansion of this mean curvature at $q = \exp r\theta \in S_r(p)$ where θ is a unit tangent vector at p is given by

$$\frac{m-1}{r} - \frac{r}{3} \text{Ric}(\theta, \theta) + O(r^2),$$

as $r \searrow 0$, cf. [37, p. 211]. It follows that $\text{Ric} \geq 0$. q.e.d.

3. Proofs

Proof of Theorem 1. We consider the continuously differentiable map that takes a (small) function $u \in C^{2,\alpha}(L)$ to the mean curvature

$$H(g, u) \in C^{0,\alpha}(L),$$

of the graph

$$\{(1 + u(x), x) : x \in L\} \subset (0, \infty) \times L,$$

with respect to a $C^{2,\alpha}$ metric g on $(0, \infty) \times L$. The (partial) linearization of this operator at the metric $g = dr \otimes dr + g_L$ and at $u = 0$ with respect to u is the linear map $C^{2,\alpha}(L) \rightarrow C^{0,\alpha}(L)$ given by

$$v \mapsto -\Delta_L v - (m-1)v.$$

This operator is invertible. In fact, its lowest eigenvalue is $-(m-1)$ with eigenspace given by the constant functions. The next eigenvalue is

$$\lambda_1(-\Delta_L) - (m-1) > 0.$$

By the implicit function theorem, there is $\sigma > 0$ so that for all g close to g_C and constant functions H close to $(m-1)$, the prescribed mean curvature equation

$$(9) \quad H(g, u) = H$$

has a unique solution $u \in C^{2,\alpha}(L)$ with $\|u\|_{2,\alpha} < \sigma$. Let H_1, H_2 be two constant functions with $H_2 < H_1$ that are close to $(m-1)$ and let u_1, u_2 be the corresponding solutions of (9). Standard analysis of the linear equation satisfied by their difference $u_2 - u_1$ gives that

$$u_1 < u_2.$$

Conversely, by the maximum principle and standard estimates (e.g., [17, Corollary 16.7]), there is $\delta > 0$ such that every solution $u \in C^{2,\alpha}(L)$ of

$$H(g, u) = \text{constant},$$

with $|u| < \delta$ and g sufficiently close to g_C satisfies

$$\|u\|_{2,\alpha} < \sigma.$$

The theorem follows from this and a scaling argument. q.e.d.

Proof of Theorem 3. Let $V_i \rightarrow \infty$. Using Lemma 16, we find $\rho_i > 0$ and $\Omega_i \subset M$ isoperimetric such that

$$\omega_{m-1}\rho_i^m/m + \mathcal{L}_g^m(\Omega_i) = V_i$$

and

$$\omega_{m-1}\rho_i^{m-1} + \mathcal{H}_g^{m-1}(\partial^*\Omega_i) = A(V_i).$$

We may think of $\Omega_i \setminus B_2$ as a region in $(1, \infty) \times L \subset C$. We take the union of Ω_i with B_2 and a far out geodesic ball of volume $\omega_{m-1}\rho_i^m/m$

to account for the stray volume. Upon scaling down homothetically by a factor close to

$$r_i = ((mV_i)/\text{area}(L, g_L))^{1/m},$$

we obtain a minimizing sequence $\tilde{\Omega}_i$ in (C, g_C) for the isoperimetric problem of volume

$$\text{area}(L, g_L)/m.$$

By Lemma 17, the unique solution of this problem is given by $(0, 1) \times L$. It follows that the stray volumes $\omega_{m-1}\rho_i^m/m$ are negligible when compared to V_i as $i \rightarrow \infty$, i.e., that

$$(10) \quad \rho_i \ll r_i.$$

Standard compactness arguments for isoperimetric boundaries show that the reduced boundaries of the regions $\tilde{\Omega}_i$ converge to the cross-section

$$\{1\} \times L,$$

locally as measures in C . Allard's regularity theorem shows that this convergence is in fact in $C^{1,\alpha}$. A hole filling argument as in [14, p. 175] gives that $\tilde{\Omega}_i$ has no other boundary than that near $\{1\} \times L$ for i sufficiently large. These results can now be lifted to Ω_i . Finally, if $\rho_i > 0$ for large i , an argument using the first variation of area shows that the mean curvature of $\partial\Omega_i \sim (m-1)/r_i$ is equal to $(m-1)/\rho_i$. This contradicts (10). q.e.d.

Proof of Theorem 6. Let Σ_k be a connected closed volume-preserving stable CMC surface in (M, g) with $\underline{r}(\Sigma_k) \geq k$. We homothetically rescale

$$\Sigma_k \setminus \bar{B}_2 \quad \text{to} \quad S_k$$

and

$$(M \setminus \bar{B}_2 \cong (2, \infty) \times L, g) \quad \text{to} \quad (M_k \cong (2/\bar{r}(\Sigma_k), \infty) \times L, g_k),$$

so that

$$\bar{r}(S_k) = 1,$$

in (M_k, g_k) . Note that

$$(M_k, g_k) \rightarrow (C, g_C),$$

locally in $C^{2,\alpha}$. The maximum principle shows that

$$H_k \geq 2 - o(1),$$

as $k \rightarrow \infty$ where H_k is the mean curvature of S_k in (M_k, g_k) . The estimate

$$(11) \quad \int_{S_k} |h_k|^2 \leq \gamma$$

follows from scaling and Lemma 24. Using also the lower bound on the mean curvature, we see that the area of S_k is a priori bounded. There is $\beta > 0$ depending only on (C, g_C) such that if $H_k > \beta$ and k

is sufficiently large, then S_k is close to a geodesic coordinate sphere in (C, g_C) by Lemma 19 (or rather its proof). Assume that $H_k \leq \beta$ for all large k . By Lemma 22, we have bounds

$$\sup_{k \geq 1} \sup_{x \in S_k} |x| |h_k(x)| < \infty,$$

for the second fundamental form. Using the geometric version of the Arzelà–Ascoli theorem, we may extract a subsequential limit immersion S of S_k in (C, g_C) that satisfies the conditions of Lemma 18. The trace of this immersion is equal to $\{1\} \times L$. It follows that Σ_k is close in scale to a centered sphere in (M, g) for all k sufficiently large. The assertion follows from the uniqueness part of Theorem 1. q.e.d.

Proof of Theorem 9. We argue by contradiction. Let Σ be as in the statement of the theorem. We may assume that Σ is connected. As in the proof of Theorem 6 above, we homothetically rescale with respect to points whose images in M diverge and then pass to a non-trivial stable minimal limiting immersion $\varphi : S \rightarrow (C, g_C)$ that is complete away from the tip of the cone. Lemma 27 implies that the Ricci tensor of (C, g_C) vanishes in the normal direction along this immersion. Using this, our assumption that $K_L > 1$, and the discussion in Appendix A, we see ν is radial. Hence the trace of the limiting immersion must be a slice in the cone. This contradicts the minimality of the limiting immersion. q.e.d.

Proof of Theorem 13. By Lemma 28, the asymptotic cone (C, g_C) of (M, g) has non-negative Ricci curvature. This is equivalent to the estimate $\text{Ric}_L \geq (m - 2)g_L$ for the link (L, g_L) of (C, g_C) . By Bishop’s theorem, $\text{area}(L, g_L) \leq \omega_{m-1}$ with equality only for the unit sphere. The characterization of isoperimetric regions of (C, g_C) in Lemma 17 gives that

$$c_{iso}(C, g_C) = \text{area}(L, g_L) / \omega_{m-1}.$$

In fact, the infimum in (4) is achieved by slabs $(0, r) \times L$ for every $r > 0$. Together with a comparison argument for slabs of large volume, we obtain that $c_{iso}(M, g) \leq c_{iso}(C, g_C)$. Assume now that $c_{iso}(M, g) = 1$. It follows that (L, g_L) is the unit sphere so that (M, g) is asymptotically flat. A standard application of Bishop’s theorem gives that (M, g) is isometric to Euclidean space. q.e.d.

Appendix A. Geometry of cones

Let (L, g_L) be an $(m - 1)$ -dimensional Riemannian manifold. We consider the Riemannian cone (C, g_C) where

$$C = (0, \infty) \times L \quad \text{and} \quad g_C = dr \otimes dr + r^2 g_L.$$

For every $r > 0$, the slice

$$\{r\} \times L$$

is umbilical with constant mean curvature $(m-1)/r$. We have that

$$\nabla_Y(r\partial_r) = Y,$$

for all $Y \in \mathfrak{X}(C)$ where ∇ is the Levi-Civita connection of (C, g_C) . We have that

$$\begin{aligned} \operatorname{Ric}_C(\partial_r, \partial_r) &= 0, \\ \operatorname{Ric}_C(\partial_r, X) &= 0, \\ \operatorname{Ric}_C(X, X) &= (\operatorname{Ric}_L - (m-2)g_L)(X, X), \end{aligned}$$

for all $X \in \mathfrak{X}(C)$ that are tangent to L . The condition

$$(12) \quad \operatorname{Ric}_L \geq (m-2)g_L,$$

on the link implies that (C, g_C) has non-negative Ricci curvature, and conversely. At points where (12) is strict, the radial direction is the unique eigendirection of the Ricci tensor of (C, g_C) with vanishing eigenvalue.

Appendix B. A remark on conditions (2) and (3)

In this section, we observe that Theorem 3 holds in a much broader class of cones than just those whose link (L, g_L) satisfies (2) and (3). To this end, we define the isoperimetric profile of a closed $(m-1)$ -dimensional Riemannian manifold (L, g_L) as follows. Given $\beta \in (0, 1)$, we let

$$\mathcal{I}_{(L, g_L)}(\beta) = \inf \left\{ \frac{\mathcal{H}_{g_L}^{m-2}(\partial^* \Omega)}{\mathcal{L}_{g_L}^{m-1}(L)} : \begin{array}{l} \Omega \text{ open with finite perimeter} \\ \text{and } \mathcal{L}_{g_L}^{m-1}(\Omega) = \beta \mathcal{L}_{g_L}^{m-1}(L) \end{array} \right\}.$$

Note that $\mathcal{I}_{(L, g_L)}$ is symmetric with respect to $1/2$. It is convenient to set $\mathcal{I}_{(L, g_L)}(0) = \mathcal{I}_{(L, g_L)}(1) = 0$.

Lemma 29 (Levy–Gromov [18, p. 518]). *Let (L, g_L) be a closed Riemannian manifold of dimension $m-1$. If*

$$\operatorname{Ric}_L \geq (m-2)g_L$$

and (L, g_L) is not the unit sphere, then

$$\mathcal{I}_{(L, g_L)}(\beta) > \mathcal{I}_{(\mathbb{S}^{m-1}, g_{\mathbb{S}^{m-1}})}(\beta),$$

for all $\beta \in (0, 1)$.

As such, the condition

$$(13) \quad \mathcal{I}_{(L, g_L)}(\beta) > \mathcal{I}_{(\mathbb{S}^{m-1}, g_{\mathbb{S}^{m-1}})}(\beta) \quad \text{for all } \beta \in (0, 1)$$

is a generalization of conditions (2) and (3).

Note that

$$\mathcal{I}_{(L, \rho^2 g_L)}(\beta) = \rho^{-1} \mathcal{I}_{(L, g_L)}(\beta),$$

for every $\rho > 0$. Recall that up to fudge factors, the Euclidean isoperimetric inequality holds for small volumes in every homogeneously regular Riemannian manifold. It follows that there exists $\rho_0 = \rho_0(L) > 0$ so that

$$(L, \rho^2 g_L)$$

satisfies (13) for all $\rho \in (0, \rho_0)$. The analogous statement for conditions (2) and (3) clearly fails. An estimate for the largest possible ρ_0 is given in terms of a (possibly negative) lower bound for the Ricci curvature and the diameter of (L, g_L) in [2].

In the following lemma, we record the observation of F. Morgan [26, Section 3.2] that the isoperimetric product theorem [36, Proposition 8] of A. Ros also holds in warped products.

Lemma 30 ([26, 36]). *Suppose that (13) holds and let $r > 0$. Then $(0, r) \times L$ is the unique isoperimetric region of its volume in the cone (C, g_C) with link (L, g_L) .*

The following generalization of A. Lichnerowicz' eigenvalue estimate is due to P. Bérard and D. Meyer [3].

Lemma 31 ([3]). *If (13) holds, then $\lambda_1(L, g_L) > m - 1$.*

From this, it is easy to see that in Theorem 3 conditions (2) and (3) may be replaced by (13).

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