ON THE ANTI-CANONICAL GEOMETRY OF $\mathbb{Q}$-FANO THREEFOLDS

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Dedicated to the memory of Professor Gang Xiao

Abstract

For a $\mathbb{Q}$-Fano 3-fold X on which K_X is a canonical divisor, we investigate the geometry induced from the linear system $|-mK_X|$ and prove that the anti-m-canonical map φ_{-m} is birational onto its image for all $m \geq 39$. By a weak $\mathbb{Q}$-Fano 3-fold X we mean a projective one with at worst terminal singularities on which $-K_X$ is $\mathbb{Q}$-Cartier, nef and big. For weak $\mathbb{Q}$-Fano 3-folds, we prove that φ_{-m} is birational onto its image for all $m \geq 97$.

1. Introduction

Throughout we work over any algebraically closed field k of characteristic 0 (for instance, $k = \mathbb{C}$). We adopt the standard notation in Kollár–Mori [**16**] and will freely use them.

A normal projective variety X is called a *weak $\mathbb{Q}$-Fano variety* if X has at worst $\mathbb{Q}$-factorial terminal singularities and the anti-canonical divisor $-K_X$ is nef and big. A weak $\mathbb{Q}$-Fano variety is said to be $\mathbb{Q}$-*Fano* if $-K_X$ is $\mathbb{Q}$-ample and the Picard number $\rho(X) = 1$. According to Minimal Model Program, $\mathbb{Q}$-Fano varieties form a fundamental class in birational geometry.

Given a $\mathbb{Q}$-Fano n-fold X (resp. weak $\mathbb{Q}$-Fano n-fold X), the *anti-m-canonical map* φ_{-m} is the rational map defined by the linear system $|-mK_X|$. By definition, φ_{-m} is birational onto its image when m is sufficiently large. Therefore it is interesting to find such a practical number m_n, independent of X, which stably guarantees the birationality of φ_{-m_n}. Such a number m_3 exists due to the boundedness of $\mathbb{Q}$-Fano 3-folds, which was proved by Kawamata [**11**], and the boundedness of weak $\mathbb{Q}$-Fano 3-folds proved by Kollár–Miyaoka–Mori–Takagi [**15**]. It is natural to consider the following problem.

Problem 1.1. *Find the optimal constant c such that φ_{-m} is birational onto its image for all $m \geq c$ and for all (weak) $\mathbb{Q}$-Fano 3-folds.*

The following example tells us that $c \geq 33$.

Example 1.2 ([**10**, List 16.6, No.95]). The general weighted hypersurface $X_{66} \subset \mathbb{P}(1,5,6,22,33)$ is a $\mathbb{Q}$-Fano 3-fold. It is clear that φ_{-m} is birational onto its image for $m \geq 33$, but φ_{-32} fails to be birational.

It is worthwhile to compare the birational geometry induced from $|mK|$ on varieties of general type with the geometry induced from $|-mK|$ on (weak) $\mathbb{Q}$-Fano varieties. An obvious feature on Fano varieties is that the behavior of φ_{-m} is not necessarily birationally invariant. For example, consider degree 1 (rational) del Pezzo surface S_1 and $\mathbb{P}^2$, $|-K_{\mathbb{P}^2}|$ gives a birational map but $|-K_{S_1}|$ does not. This causes difficulties in studying Problem 1.1. In fact, even if in dimension 3, there is no known practical upper bound for c in written records. The motivation of this paper is to systematically study φ_{-m} on (weak) $\mathbb{Q}$-Fano 3-folds.

When X is nonsingular, we may take $c = 4$ according to Ando [**3**] and Fukuda [**9**]. When X has terminal singularities, Problem 1.1 was treated by the first author in [**8**], where an effective upper bound of c in terms of the Gorenstein index of X is proved (cf. [**8**, Theorem 1.1]). Since, however, the Gorenstein index of a weak $\mathbb{Q}$-Fano 3-fold can be as large as "840" (see Proposition 2.4), the number "$3 \times 840 + 10 = 2530$" obtained in [**8**, Theorem 1.1] is far from being optimal. It turns out that Problem 1.1 is closely related to the following problem (cf. [**8**, Theorem 4.5]).

Problem 1.3. *Given a (weak) $\mathbb{Q}$-Fano 3-fold X, can one find the minimal positive integer $\delta_1 = \delta_1(X)$ such that $\dim \overline{\varphi_{-\delta_1}(X)} > 1$?*

Problem 1.3 is parallel to the following question on 3-folds of general type:

> *Let Y be a 3-fold of general type on which $|nK_Y|$ is composed with a pencil of surfaces for some fixed integer $n > 0$. Can one find an integer m (bounded from above by a function in terms of n) so that $|mK_Y|$ is not composed with a pencil any more?*

This question was solved by Kollár [**13**] who proved that one may take $m \leq 11n + 5$. The result is a direct application of the semi-positivity of $f_* \omega_{Y/B}^l$ since, modulo birational equivalence, one may assume that there is a fibration $f : Y \longrightarrow B$ onto a curve B. As far as we know, there is still no known analogy of Kollár's method in treating $\mathbb{Q}$-Fano varieties.

Firstly, we shall prove the following theorem.

Theorem 1.4. *Let X be a $\mathbb{Q}$-Fano 3-fold. Then there exists an integer $n_1 \leq 10$ such that $\dim \overline{\varphi_{-n_1}(X)} > 1$.*

Theorem 1.4 is close to be optimal due to the following example.

Example 1.5 ([**10**, List 16.7, No.85])**.** Consider the general codimension 2 weighted complete intersection $X := X_{24,30} \subset \mathbb{P}(1,8,9,10,12,15)$ which is a $\mathbb{Q}$-Fano 3-fold. Then $\dim \overline{\varphi_{-9}(X)} > 1$ while $\dim \overline{\varphi_{-8}(X)} = 1$.

In fact, theoretically, there are only 4 possible weighted baskets for which we need to take $n_1 = 10$ (see Remark 3.13 and Subsection 3.6 for more details and discussions). Theorem 1.4 allows us to prove the following result.

Theorem 1.6. *Let X be a $\mathbb{Q}$-Fano 3-fold. Then φ_{-m} is birational onto its image for all $m \geq 39$.*

A key point in proving Theorem 1.4 is that we have $\rho(X) = 1$, which is not the case for arbitrary weak $\mathbb{Q}$-Fano 3-folds. Therefore we should study weak $\mathbb{Q}$-Fano 3-folds in an alternative way. Our result is as follows.

Theorem 1.7. *Let X be a weak $\mathbb{Q}$-Fano 3-fold. Then $\dim \overline{\varphi_{-n_2}(X)} > 1$ for all $n_2 \geq 71$.*

Theorem 1.7 allows us to study the birationality.

Theorem 1.8. *Let X be a weak $\mathbb{Q}$-Fano 3-fold. Then φ_{-m} is birational onto its image for all $m \geq 97$.*

Remark 1.9. We remark that Theorems 1.7 and 1.8 are true even if X has canonical singularities instead of $\mathbb{Q}$-factorial terminal singularities, which is not difficult to see.

This paper is organized as follows. In Section 2, we recall some basic knowledge. In Section 3, we consider Problem 1.3 on $\mathbb{Q}$-Fano 3-folds. We generalize a result of Alexeev and reduce the problem to the numerical behavior of anti-plurigenera, then we apply a method developed by J. A. Chen and the first author to analyze the possible weighted baskets. Section 4 is devoted to proving Theorem 1.7 for weak $\mathbb{Q}$-Fano 3-folds. We reduce the problem to the numerical behavior of Hilbert functions and use Reid's formula to estimate the lower bound of Hilbert functions. Finally we study the birationality in Section 5. We give an effective criterion for the birationality of φ_{-m}. As applications, we prove Theorems 1.6 and 1.8 in the last part.

2. Preliminaries

Let X be a weak $\mathbb{Q}$-Fano 3-fold. Denote by r_X the Gorenstein index of X, i.e. the Cartier index of K_X. For any positive integer m, the number $P_{-m}(X) := h^0(X, \mathcal{O}_X(-mK_X))$ is called the *m-th anti-plurigenus* of X. Clearly, since $-K_X$ is nef and big, Kawamata–Viehweg vanishing theorem [**12**, Theorem 1-2-5] implies

$$h^i(-mK_X) = h^i(X, K_X - (m+1)K_X) = 0$$

for all $i > 0$ and $m \geq 0$.

For two linear systems $|A|$ and $|B|$, we write $|A| \preceq |B|$ if there exists an effective divisor F such that

$$|B| \supset |A| + F.$$

In particular, if $A \leq B$ as divisors, then $|A| \preceq |B|$.

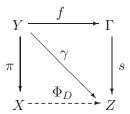### 2.1. Rational map defined by a Weil divisor.

Consider an effective $\mathbb{Q}$-Cartier Weil divisor D on X with $h^0(X, D) \geq 2$. We study the rational map defined by $|D|$, say

$$X \overset{\Phi_D}{\dashrightarrow} \mathbb{P}^{h^0(D)-1}$$

which is not necessarily well-defined everywhere. By Hironaka's big theorem, we can take successive blow-ups $\pi : Y \to X$ such that:

(i) Y is nonsingular projective;
(ii) the movable part $|M|$ of the linear system $|\lfloor \pi^*(D) \rfloor|$ is base point free and, consequently, the rational map $\gamma := \Phi_D \circ \pi$ is a morphism;
(iii) the support of the union of $\pi_*^{-1}(D)$ and the exceptional divisors of π is of simple normal crossings.

Let $Y \overset{f}{\longrightarrow} \Gamma \overset{s}{\longrightarrow} Z$ be the Stein factorization of γ with $Z := \gamma(Y) \subset \mathbb{P}^{h^0(D)-1}$. We have the following commutative diagram.

$$
\begin{array}{ccc}
Y & \overset{f}{\longrightarrow} & \Gamma \\
{\scriptstyle \pi} \downarrow & {\scriptstyle \gamma} \searrow & \downarrow {\scriptstyle s} \\
X & \overset{\Phi_D}{\dashrightarrow} & Z
\end{array}
$$

Case (f_{np}). If $\dim(\Gamma) \geq 2$, a general member S of $|M|$ is a nonsingular projective surface by Bertini's theorem. We say that $|D|$ *is not composed with a pencil of surfaces.*

Case (f_{p}). If $\dim(\Gamma) = 1$, i.e. $\dim \overline{\Phi_D(X)} = 1$, then $\Gamma \cong \mathbb{P}^1$ since $g(\Gamma) \leq q(Y) = q(X) := h^1(\mathcal{O}_X) = 0$. Furthermore, a general fiber S of f is an irreducible nonsingular projective surface by Bertini's theorem. We may write

$$M = \sum_{i=1}^n S_i \sim nS$$

where S_i is a nonsingular fiber of f for all i and $n = h^0(D) - 1$. We can write

$$|D| = |nS'| + E,$$

where $|S'| = |\pi_* S|$ is an irreducible rational pencil, $|nS'|$ is the movable part and E is the fixed part. In this case, $|D|$ is said to *be composed with a rational pencil of surfaces*. We collect a couple of basic facts about rational pencils as follows.

Lemma 2.1. *Keep the same notation as above. If $|D| = |nS'| + E$ is composed with a rational pencil of surfaces, then $n = h^0(D) - 1$.*

Lemma 2.2. *If $|D_1| = |k_1 S_1| + E_1$ and $|D_2| = |k_2 S_2| + E_2$ are composed with rational pencils of surfaces and $D_1 \leq D_2$, then $|S_1| = |S_2|$.*

Proof. Since $D_1 \leq D_2$, we have $\mathrm{Mov}|D_1| \preceq \mathrm{Mov}|D_2|$. Hence $|S_1| \preceq |k_2 S_2|$. Thus $|S_1| \preceq |S_2|$ by the irreducibility of $|S_1|$. Then by $h^0(S_1) = h^0(S_2) = 2$ and $|S_1|, |S_2|$ are movable, we have $|S_1| = |S_2|$.   q.e.d.

For another Weil $\mathbb{Q}$-Cartier divisor D' satisfying $h^0(X, D') > 1$, we say that $|D|$ and $|D'|$ are *composed with the same pencil* if $|D|$ and $|D'|$ are composed with pencils and they define the same fibration structure $Y \to \mathbb{P}^1$ on some model Y. In particular, $|D|$ and $|D'|$ are not composed with the same pencil if one of them is not composed with a pencil.
Define

$$\iota = \iota(D) := \begin{cases} 1, & \text{Case } (f_{\mathrm{np}}); \\ n, & \text{Case } (f_{\mathrm{p}}). \end{cases}$$

Clearly, in both cases, $M \equiv \iota S$ with $\iota \geq 1$.

Definition 2.3. For both Case (f_{np}) and Case (f_{p}), we call S *a generic irreducible element of $|M|$*.

We may also define "a generic irreducible element" of a moving linear system on a surface in the similar way.

Restricting our interest to special cases, we fix an effective Weil divisor $D \sim -m_0 K_X$ at the very beginning assuming that $P_{-m_0} \geq 2$ for some integer $m_0 > 0$. We would like to study the geometry of X induced by Φ_D.

2.2. Reid's formula.

A *basket* B is a collection of pairs of integers (permitting weights), say $\{(b_i, r_i) \mid i = 1, \cdots, s; b_i \text{ is coprime to } r_i\}$. For simplicity, we will alternatively write a basket as follows, say

$$B = \{(1,2), (1,2), (2,5)\} = \{2 \times (1,2), (2,5)\}.$$

Let X be a weak $\mathbb{Q}$-Fano 3-fold. According to Reid [**18**], for a Weil divisor D on X,

$$\chi(D) = 1 + \frac{1}{12}D(D - K_X)(2D - K_X) + \frac{1}{12}(D \cdot c_2) + \sum_Q c_Q(D),$$

where the last sum runs over Reid's basket of orbifold points. If the orbifold point Q is of type $\frac{1}{r}(1, -1, b)$ and $i = i_D$ is the local index of divisor D at Q (i.e. $D \sim iK_X$ around Q, $0 \leq i < r$), then

$$c_Q(D) = -\frac{i(r^2 - 1)}{12r} + \sum_{j=0}^{i-1} \frac{\overline{jb}(r - \overline{jb})}{2r}.$$

Here the symbol $\bar{\cdot}$ means the smallest residue mod r and $\sum_{j=0}^{-1} := 0$. Write

$$\chi_{\text{sing}}(D) := \sum_Q c_Q(D) \text{ and}$$

$$\chi_{\text{reg}}(D) := 1 + \frac{1}{12}D(D - K_X)(2D - K_X) + \frac{1}{12}(D \cdot c_2).$$

We make some remarks here on how to compute the term $c_Q(D)$:

(1) If $D = nK_X$ for $n \in \mathbb{Z}$, we take $i = \overline{n}$ (modulo r) and then

$$c_Q(nK_X) = c_Q(iK_X) = -\frac{i(r^2 - 1)}{12r} + \sum_{j=0}^{i-1} \frac{\overline{jb}(r - \overline{jb})}{2r}.$$

(2) If $D = tK_X$ for $t \in \mathbb{Z}^+$, then it is easy to see

$$c_Q(tK_X) = -\frac{t(r^2 - 1)}{12r} + \sum_{j=0}^{t-1} \frac{\overline{jb}(r - \overline{jb})}{2r}.$$

(3) By Reid's formula, Kawamata–Viehweg vanishing theorem, and Serre duality, we have, for any $n > 0$,

$$P_{-n}(X) = -\chi(\mathcal{O}_X((n+1)K_X))$$
$$= \frac{1}{12}n(n+1)(2n+1)(-K_X^3) + (2n+1) - l(-n)$$

where $l(-n) = l(n+1) = \sum_i \sum_{j=1}^n \frac{\overline{jb_i}(r_i - \overline{jb_i})}{2r_i}$ and the sum runs over Reid's basket of orbifold points

$$B_X = \left\{ (b_i, r_i) \mid i = 1, \cdots, s; 0 < b_i \leq \frac{r_i}{2}; b_i \text{ is coprime to } r_i \right\}.$$

The above formula can be rewritten as:

$$P_{-1} = \frac{1}{2}\Big(-K_X^3 + \sum_i \frac{b_i^2}{r_i}\Big) - \frac{1}{2}\sum_i b_i + 3,$$

$$P_{-m} - P_{-(m-1)} = \frac{m^2}{2}\Big(-K_X^3 + \sum_i \frac{b_i^2}{r_i}\Big) - \frac{m}{2}\sum_i b_i + 2 - \Delta^m$$

where $\Delta^m = \sum_i \Big(\frac{\overline{b_i m}(r_i - \overline{b_i m})}{2r_i} - \frac{b_i m(r_i - b_i m)}{2r_i}\Big)$ for any $m \geq 2$.

2.3. Upper bound of Gorenstein indices.

The following fact might be known to experts. We will apply it in our argument.

Proposition 2.4. *Let X be a weak $\mathbb{Q}$-Fano 3-fold. Then either $r_X = 840$ or $r_X \leq 660$. (This means that the Gorenstein index of a weak $\mathbb{Q}$-Fano 3-fold is bounded from above by 840. Among known $\mathbb{Q}$-Fano 3-folds, the maximal Gorenstein index is 420. For example, so is the general weighted hypersurface $X_{19} \subset \mathbb{P}(1,3,4,5,7)$ (cf. [**10**, List 16.6, No.40]). We do not know if this bound is optimal.)*

Proof. Write Reid's basket

$$B_X = \Big\{(b_i, r_i) \mid i = 1, \cdots, s; 0 < b_i \leq \frac{r_i}{2}; b_i \text{ is coprime to } r_i\Big\}.$$

Then, by definition, $r_X = \text{l.c.m.}\{r_i \mid i = 1, \cdots, s\}$.

By [**15**], we know that $(-K_X \cdot c_2(X)) \geq 0$. Therefore Reid [**18**, 10.3] gives the inequality

$$(2.1) \qquad \sum_i \Big(r_i - \frac{1}{r_i}\Big) \leq 24.$$

Now for the sequence $\mathcal{R} = (r_i)_i$, we define a new set $\mathcal{P} = \{s_j\}_j$ as following: if we factor r_i into its prime factors such that $r_i = p_1^{a_{1i}} p_2^{a_{2i}} \cdots p_k^{a_{ki}}$, then we take $\mathcal{P} = \{p_j^{a_{ji}}\}_{1 \leq j \leq k, i}$. It is easy to show that if $a, b > 1$ and coprime, then

$$(2.2) \qquad ab - \frac{1}{ab} \geq a - \frac{1}{a} + b - \frac{1}{b} + 2.$$

So

$$(2.3) \qquad \sum_j \Big(s_j - \frac{1}{s_j}\Big) \leq \sum_i \Big(r_i - \frac{1}{r_i}\Big) \leq 24.$$

and we also have $\text{l.c.m.}(s_j)_j = \text{l.c.m.}(r_i)_i = r_X$. So the problem is reduced to treat the sequence $(s_j)_j$ instead. Clearly, for each j,

$$s_j \in \{2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19\}.$$

Now we may assume that $r_X > 660$.

Denote by s_1 the largest value in $\mathcal{P}$, by s_2 the second largest value, by s_3, s_4 the third, the forth, and so on. For instance, if $\mathcal{P} = \{2, 3, 4, 5\}$,

then $s_1 = 5$, $s_2 = 4$, $s_3 = 3$, and $s_4 = 2$. If the value s_j does not exist by definition, then we set $s_j = 1$. In the previous example, we have $s_5 = 1$.

Since l.c.m.$(2, 3, 4, 5, 7) = 420$ and l.c.m.$(2, 3, 4, 5, 7, 8) = 840$, if $s_1 \leq 8$, then $3, 5, 7, 8 \in \mathcal{P}$. In this case $\mathcal{P} = \{3, 5, 7, 8\}$ or $\{2, 3, 5, 7, 8\}$ by inequality (2.3) and $\mathcal{R} = (3, 5, 7, 8)$ or $(2, 3, 5, 7, 8)$ by inequality (2.2). In a word, $r_X = 840$.

If $s_1 \geq 16$, then

$$\sum_{j>1} \left(s_j - \frac{1}{s_j} \right) \leq 8 + \frac{1}{16}.$$

Then $s_2 \leq 8$. Also $s_2 \geq 5$ since, otherwise, l.c.m.$(2, 3, 4, s_1) \leq 228 < r_X$ (a contradiction). Hence

$$\sum_{j>2} \left(s_j - \frac{1}{s_j} \right) \leq 3 + \frac{1}{16} + \frac{1}{5}.$$

So $s_3 \leq 3$, but 2 and 3 can not be in $\mathcal{P}$ simultaneously. Then l.c.m.$(s_j)_j \leq 3 \times 8 \times 19 < r_X$, a contradiction.

If $s_1 = 13$, then $s_2 \geq 5$ since, otherwise, l.c.m.$(2, 3, 4, s_1) = 12 s_1 < r_X$ (a contradiction). Then

$$\sum_{j>2} \left(s_j - \frac{1}{s_j} \right) \leq 11 - s_2 + \frac{1}{13} + \frac{1}{s_2}.$$

If $s_2 = 11$, then $s_j = 1$ for any $j > 2$ and $r_X = 11 \times 13$, a contradiction. If $s_2 = 9$, then $s_3 \leq 2$ and l.c.m.$(s_j)_j \leq 2 \times 9 \times 13 < r_X$, a contradiction. If $s_2 = 8$, then $s_3 \leq 3$, but 2 and 3 can not be in $\mathcal{P}$ simultaneously. So l.c.m.$(s_j)_j \leq 3 \times 8 \times 13 < r_X$, a contradiction. If $s_2 = 7$, then $s_3 \leq 4$, but 3 and 4 can not be in $\mathcal{P}$ simultaneously. So l.c.m.$(s_j)_j \leq 6 \times 7 \times 13 < r_X$, a contradiction. If $s_2 = 5$, then 3 and 4 can not be in $\mathcal{P}$ simultaneously. So l.c.m.$(s_j)_j \leq 6 \times 5 \times 13 < r_X$, a contradiction.

If $s_1 = 11$, then $9 \geq s_2 \geq 7$ since, otherwise, l.c.m.$(2, 3, 4, 5, s_1) = 60 s_1 < r_X$ (a contradiction). Then

$$\sum_{j>2} \left(s_j - \frac{1}{s_j} \right) \leq 6 + \frac{1}{11} + \frac{1}{7}.$$

Hence $s_3 \leq 5$. If $s_3 = 5$, then $s_j = 1$ for any $j > 3$ and l.c.m.$(s_j)_j \leq 5 \times 9 \times 11 < r_X$, a contradiction. If $s_3 = 4$, then $s_4 \leq 2$ and l.c.m.$(s_j)_j \leq 4 \times 9 \times 11 < r_X$, a contradiction. If $s_3 \leq 3$, then l.c.m.$(s_j)_j \leq 2 \times 3 \times 9 \times 11 < r_X$, a contradiction.

If $s_1 = 9$, then $8 \geq s_2 \geq 7$ since, otherwise, l.c.m.$(2, 3, 4, 5, 9) = 180 < r_X$ (a contradiction). Consider firstly the case $s_2 = 8$. We have

$$\sum_{j>2} \left(s_j - \frac{1}{s_j} \right) \leq 7 + \frac{1}{9} + \frac{1}{8}.$$

If $s_3 = 7$, then $s_j = 1$ for any $j > 3$ and l.c.m.$(s_j)_j \leq 7 \times 8 \times 9 < r_X$, a contradiction. If $s_3 \leq 5$, then l.c.m.$(s_j)_j \leq$ lcm$(2, 3, 4, 5, 8, 9) = 360 <$

r_X, a contradiction. Next we consider the case $s_2 = 7$. Then

$$\sum_{j>2} \left(s_j - \frac{1}{s_j} \right) \le 8 + \frac{1}{9} + \frac{1}{7}.$$

If $s_3 = 5$, then $s_4 \le 3$ and l.c.m.$(s_j)_j \le 2{\times}5{\times}7{\times}9 < r_X$, a contradiction. If $s_3 \le 4$, then l.c.m.$(s_j)_j \le 4 \times 7 \times 9 < r_X$, a contradiction. So we conclude the statement.

From the proof we also know that $r_X = 840$ only happens when $\mathcal{R} = (3, 5, 7, 8)$ or $(2, 3, 5, 7, 8)$.   q.e.d.

3. When is $|-mK_X|$ not composed with a pencil? (Part I)

The most important part of this paper is to find a minimal positive integer m so that $|-mK_X|$ is not composed with a pencil of surfaces. For the convenience of expression, we fix the notation first.

Definition 3.1. Let X be a weak ℚ-Fano 3-fold. For any $0 \le i \le 2$, define

$$\delta_i(X) := \min\{m \in \mathbb{Z}^+ \mid \dim \overline{\varphi_{-m}(X)} > i\}.$$

We will mainly treat ℚ-Fano 3-folds in this section.

3.1. Two key theorems.
We prove two theorems here which are crucial in proving Theorem 1.4.

Theorem 3.2. *Let X be a ℚ-Fano 3-fold with the basket B of singularities. Fix a positive integer m such that $P_{-m} > 0$. Assume that, for each pair $(b, r) \in B$, one of the following conditions is satisfied:*

(1) $m \equiv 0, \pm 1 \mod r$;
(2) $m \equiv -2 \mod r$ and $b = \lfloor \frac{r}{2} \rfloor$;
(3) $m \equiv 2 \mod r$ and $3b \ge r$;
(4) $m \equiv 3 \mod r$ and $4b \ge r$;
(5) $m \equiv 4 \mod r$, $\overline{b}(r - \overline{b}) \ge \overline{4b}(r - \overline{4b})$, and

$$\overline{b}(r - \overline{b}) + \overline{2b}(r - \overline{2b}) \ge \overline{3b}(r - \overline{3b}) + \overline{4b}(r - \overline{4b}).$$

Then one of the following holds:

(I) *$P_{-m} = 1$ and $-mK_X \sim E$ is an effective prime divisor;*
(II) *$P_{-m} = 2$, $|-mK_X|$ does not have fixed part, and is composed with an irreducible rational pencil of surfaces;*
(III) *$P_{-m} \ge 3$, $|-mK_X|$ does not have fixed part, and is not composed with a pencil of surfaces.*

Proof. We generalize the argument of Alexeev [1, 2.18] where the case $m = 1$ is treated.

Assume that none of the conclusions holds, then there exists a strictly effective divisor E such that $-mK_X - E$ is strictly effective and

$$h^0(-mK_X) - h^0(-mK_X - E) - h^0(E) + h^0(\mathcal{O}_X) = 0.$$

In fact, if $P_{-m} = 1$ and $-mK_X \sim D$ is not a prime divisor, then we take E to be one irreducible component of D; if $P_{-m} \geq 2$ and $|-mK_X|$ has fixed part, then we take E to be one component in the fixed part; if $P_{-m} \geq 3$, $|-mK_X|$ does not have fixed part, but is composed with a (rational) pencil of surfaces, then $|-mK_X| = |nS|$ with $n \geq 2$ and we can take $E = S$.

By Kawamata–Viehweg vanishing theorem and $\rho(X) = 1$, all higher cohomologies vanish for $\mathcal{O}_X(-mK_X)$, $\mathcal{O}_X(-mK_X - E)$, $\mathcal{O}_X(E)$, and $\mathcal{O}_X$. Hence

$$\Delta\Delta_\chi(-mK_X, -mK_X - E, E, 0) = 0,$$

where the *double difference* of a function f is defined by

$$\Delta\Delta_f(a, a - d, b, b - d) = f(a) - f(a - d) - f(b) + f(b - d).$$

Then we have

$$\Delta\Delta_{\chi,\text{reg}}(-mK_X, -mK_X - E, E, 0)$$

$$+\Delta\Delta_{\chi,\text{sing}}(-mK_X, -mK_X - E, E, 0) = 0.$$

It is clear to see that

$$\Delta\Delta_{\chi,\text{reg}}(-mK_X, -mK_X - E, E, 0) = \frac{m+1}{2}(-K_X)(-mK_X - E)E > 0,$$

since E and $-mK_X - E$ are ample by the construction and $\rho(X) = 1$. To get a contradiction, it is sufficient to show that

$$\Delta\Delta_{\chi,\text{sing}}(-mK_X, -mK_X - E, E, 0) \geq 0$$

under the assumption of this theorem. Thus it suffices to show that, for every single point $Q = (b, r) \in B$,

(3.1) $$c_Q(-mK_X) - c_Q(-mK_X - E) - c_Q(E) \geq 0.$$

Set $F(x) := \frac{\overline{x}(r - \overline{x})}{2r}$ for any integer x and $l := \overline{m}$. We may assume that the local index of E at Q is i ($0 \leq i < r$).

Then

$$c_Q(-mK_X) - c_Q(-mK_X - E) - c_Q(E)$$

$$= \left(-\frac{(2r - l)(r^2 - 1)}{12r} + \sum_{j=0}^{2r-l-1} F(jb)\right)$$

$$- \left(-\frac{(2r - l - i)(r^2 - 1)}{12r} + \sum_{j=0}^{2r-l-i-1} F(jb)\right)$$

$$-\left(-\frac{i(r^2-1)}{12r}+\sum_{j=0}^{i-1}F(jb)\right)$$

$$=\sum_{j=0}^{2r-l-1}F(jb)-\sum_{j=0}^{2r-l-i-1}F(jb)-\sum_{j=0}^{i-1}F(jb)$$

$$=\sum_{j=2r-l-i}^{2r-l-1}F(jb)-\sum_{j=0}^{i-1}F(jb)$$

$$=\sum_{j=l+1}^{l+i}F(jb)-\sum_{j=0}^{i-1}F(jb)$$

$$=\sum_{j=i}^{l+i}F(jb)-\sum_{j=0}^{l}F(jb)$$

$$(3.2)\qquad =\sum_{j=0}^{l}F(ib+jb)-\sum_{j=0}^{l}F(jb).$$

Then to prove inequality (3.1), it suffices to prove that

$$G(x):=\sum_{j=0}^{l}F(x+jb)-\sum_{j=0}^{l}F(jb)\geq 0$$

for arbitrary integer x.

Note that $G(x)$ is a periodic piecewise quadratic function with negative leading coefficients. Hence the minimal value can only be reached at end points of each piece. It is easy to see that the set of end points is $\{nr-jb \mid n \in \mathbb{Z}, j=0,1,\ldots,l\}$. Hence $G(x) \geq 0$ is equivalent to $G(-jb) \geq 0$ for all $j=0,1,\ldots,l$. Note that $G(0)=G(-lb)=0$.

If $m \equiv 0,1 \mod r$, there is nothing to prove.

If $m \equiv 2 \mod r$, then $G(-b)=F(b)-F(2b)$. It is easy to see that $F(b)-F(2b) \geq 0$ is equivalent to $3b \geq r$.

If $m \equiv 3 \mod r$, then $G(-b)=G(-2b)=F(b)-F(3b)$. It is easy to see that $F(b)-F(3b) \geq 0$ is equivalent to $4b \geq r$.

If $m \equiv 4 \mod r$, then $G(-b)=G(-3b)=F(b)-F(4b)$ and $G(-2b)=F(b)+F(2b)-F(3b)-F(4b)$.

If $m \equiv -1 \mod r$, then $G(x)=\sum_{j=0}^{r-1}F(x+jb)-\sum_{j=0}^{r-1}F(jb)=0$.

If $m \equiv -2 \mod r$, then $G(x)=\sum_{j=0}^{r-2}F(x+jb)-\sum_{j=0}^{r-2}F(jb)=F(b)-F(x+(r-1)b)$. It is easy to see that $F(b)-F(x+(r-1)b) \geq 0$ for all x if and only if $b=\lfloor\frac{r}{2}\rfloor$.

So we have proved the theorem.   q.e.d.

As a special case of Theorem 3.2, Alexeev proved the following theorem.

Theorem 3.3 ([**1**, 2.18]). *Let X be a $\mathbb{Q}$-Fano 3-fold. If $P_{-1} \geq 3$, then $|-K_X|$ has no fixed part and is not composed with a pencil of surfaces.*

Hence we only need to deal with the case when $P_{-1} < 3$. For this purpose, we prove the following theorem.

Theorem 3.4. *Let X be a $\mathbb{Q}$-Fano 3-fold. Fix a positive integer m. Assume that one of the following holds:*

(i) $P_{-m} = 1$ *and* $E \in |-mK_X|$ *is an effective prime divisor;*
(ii) $P_{-m} = 2$ *and* $|-mK_X|$ *does not have fixed part.*

Write $n_0 := \min\{n \in \mathbb{Z}^+ \mid P_{-nm} \geq 2\}$. For any integer $l \geq n_0$, write $l = sn_0 + t$ with $s \in \mathbb{Z}$ and $0 \leq t \leq n_0 - 1$. Take

$$l_0 = \min\{l \in \mathbb{Z}_{\geq n_0} \mid P_{-lm} > s+1\}.$$

Then $|-l_0 m K_X|$ does not have fixed part and is not composed with a pencil of surfaces.

Proof. First we assume that $|-l_0 m K_X|$ has a base component E_{l_0}. It follows that $P_{-m} = 1$ and $E_{l_0} = E$. Thus, by definition, we have $l_0 > 1$. Hence

$$
\begin{aligned}
P_{-(l_0-1)m} &= h^0(-l_0 m K_X - (-m K_X)) \\
&= h^0(-l_0 m K_X - E_{l_0}) = h^0(-l_0 m K_X) > s+1,
\end{aligned}
$$

which contradicts the minimality of l_0. The similar argument implies that $|-n_0 m K_X|$ does not have fixed part.

Now assume that $|-l_0 m K_X|$ is composed with a (rational) pencil of surfaces, i.e.

$$|-l_0 m K_X| = |(P_{-l_0 m} - 1)S|,$$

where $|S|$ is an irreducible rational pencil. Write $l_0 = sn_0 + t$. Since $P_{-n_0 m} \geq 2$, we have $P_{-sn_0 m} \geq s+1$.

If $t > 0$, by the minimality of l_0 we get $P_{-sn_0 m} = s+1$. So we can write $|-sn_0 m K_X| = |sS|$ by Lemma 2.2 since $|-n_0 m K_X|$ does not have fixed part and $|-sn_0 m K_X| \preceq |-l_0 m K_X|$. Now

$$
\begin{aligned}
-tm K_X &\sim -l_0 m K_X - (-sn_0 m K_X) \sim (P_{-l_0 m} - 1)S - sS \\
&= (P_{-l_0 m} - 1 - s)S \geq S.
\end{aligned}
$$

This implies that $P_{-tm} \geq 2$, which contradicts the minimality of n_0. Hence $t = 0$ and $l_0 = sn_0$.

If $s \geq 2$, by the minimality of l_0 we get $P_{-(s-1)n_0 m} = s \geq 2$. We can write $|-(s-1)n_0 m K_X| = |(s-1)S|$ by Lemma 2.2. Hence

$$
\begin{aligned}
-n_0 m K_X &\sim -l_0 m K_X - (-(s-1)n_0 m K_X) \sim (P_{-l_0 m} - 1)S - (s-1)S \\
&= (P_{-l_0 m} - s)S \geq 2S.
\end{aligned}
$$

This implies that $P_{-n_0 m} \geq 3$, which contradicts the minimality of l_0.

Hence $s = 1$ and $l_0 = n_0$. By $P_{-n_0 m} \geq 3$, we have $n_0 > 1$. This implies, by assumption, $P_{-m} = 1$ and $-mK_X \sim E$ is a fixed prime divisor. Since $E \leq (P_{-Nm} - 1)S \sim -n_0 m K_X$ and E is reduced and irreducible, $E \leq S_0$ for certain surface $S_0 \in |S|$. Hence

$$-(n_0 - 1)mK_X \sim -n_0 mK_X - (-mK_X) \sim (P_{-n_0 m} - 1)S - E$$
$$\geq (P_{-n_0 m} - 2)S + (S_0 - E) \geq S.$$

This implies that $P_{-(n_0 - 1)m} \geq 2$, which contradicts the minimality of n_0. We are done.   q.e.d.

Now let us explain the strategy to prove Theorem 1.4. Firstly, we divide all $\mathbb{Q}$-Fano 3-folds into several families, roughly speaking, by the value of P_{-1}. Then in each family, we may take a suitable m satisfying the condition of Theorem 3.2. Applying Theorem 3.4 to m, we are able to find the number l_0 and so $\delta_1(X) \leq l_0 m$. In order to find such l_0, or an upper bound of l_0, we may assume that l_0 is sufficiently large, say, $l_0 \geq 9$, then by the assumption of Theorem 3.4, we know the value of $P_{-m}, P_{-2m}, P_{-3m}, \ldots, P_{-8m}$. Then, by Chen–Chen's method ([4]) on the analysis of baskets, we can recover all possibilities for baskets of singularities, of which each possibility can be proved to be either impossible or very easy to treat. For this purpose, we need to recall relevant materials on baskets, packings, the canonical sequence, and so on.

3.2. Weighted baskets.

All contents of this subsection are mainly from Chen–Chen [4, 5]. We list them as follows:

1) Let $B = \{(b_i, r_i) \mid i = 1, \cdots, s; 0 < b_i \leq \frac{r_i}{2}; b_i$ is coprime to $r_i\}$ be a basket. We set $\sigma(B) := \sum_i b_i$, $\sigma'(B) := \sum_i \frac{b_i^2}{r_i}$, and $\Delta^n(B) = \sum_i \left(\frac{\overline{b_i n}(r_i - \overline{b_i n})}{2r_i} - \frac{b_i n(r_i - b_i n)}{2r_i} \right)$ for any integer $n > 1$.

2) The new (generalized) basket

$$B' := \{(b_1 + b_2, r_1 + r_2), (b_3, r_3), \cdots, (b_s, r_s)\}$$

is called a *packing* of B, denoted as $B \succeq B'$. Note that $\{(2, 4)\} = \{(1, 2), (1, 2)\}$. We call $B \succ B'$ a *prime packing* if $b_1 r_2 - b_2 r_1 = 1$. A composition of finite packings is also called a packing. So the relation "$\succeq$" is a partial ordering on the set of baskets.

3) Note that for a weak $\mathbb{Q}$-Fano 3-fold X, all the anti-plurigenera P_{-n} can be determined by Reid's basket B_X and $P_{-1}(X)$. This leads to the notion of "weighted basket". We call a pair $\mathbb{B} = (B, \tilde{P}_{-1})$ a *weighted basket* if B is a basket and $\tilde{P}_{-1}$ is a non-negative integer. We write $(B, \tilde{P}_{-1}) \succeq (B', \tilde{P}_{-1})$ if $B \succeq B'$.

4) Given a weighted basket $\mathbb{B} = (B, \tilde{P}_{-1})$, define $\tilde{P}_{-1}(\mathbb{B}) := \tilde{P}_{-1}$ and the volume

$$-K^3(\mathbb{B}) := 2\tilde{P}_{-1} + \sigma(B) - \sigma'(B) - 6.$$

For all $m \geq 1$, we define the "anti-plurigenus" in the following inductive way:

$$\tilde{P}_{-(m+1)} - \tilde{P}_{-m}$$
$$= \frac{1}{2}(m+1)^2(-K^3(\mathbb{B}) + \sigma'(B)) + 2 - \frac{m+1}{2}\sigma - \Delta^{m+1}(B).$$

Note that, if we set $\mathbb{B} = (B_X, P_{-1}(X))$ for a given weak $\mathbb{Q}$-Fano 3-fold X, then we can verify directly that $-K^3(\mathbb{B}) = -K_X^3$ and $\tilde{P}_{-m}(\mathbb{B}) = P_{-m}(X)$ for all $m \geq 1$.

Property 3.5 ([**5**, Section 3]). *Assume* $\mathbb{B} := (B, \tilde{P}_{-1}) \succeq \mathbb{B}' := (B', \tilde{P}_{-1})$. *Then*

(i) $\sigma(B) = \sigma(B')$ *and* $\sigma'(B) \geq \sigma'(B')$;
(ii) *For all integer* $n \geq 1$, $\Delta^n(B) \geq \Delta^n(B')$;
(iii) $-K^3(\mathbb{B}) + \sigma'(B) = -K^3(\mathbb{B}') + \sigma'(B')$;
(iv) $-K^3(\mathbb{B}) \leq -K^3(\mathbb{B}')$;
(v) $\tilde{P}_{-m}(\mathbb{B}) \leq \tilde{P}_{-m}(\mathbb{B}')$ *for all* $m \geq 2$.

Next we recall the "canonical" sequence of a basket B. Set $S^{(0)} := \{\frac{1}{n} \mid n \geq 2\}$, $S^{(5)} := S^{(0)} \cup \{\frac{2}{5}\}$, and inductively for all $n \geq 5$,

$$S^{(n)} := S^{(n-1)} \cup \left\{ \frac{b}{n} \mid 0 < b < \frac{n}{2},\ b \text{ is coprime to } n \right\}.$$

Each set $S^{(n)}$ gives a division of the interval $(0, \frac{1}{2}] = \bigcup_i [\omega_{i+1}^{(n)}, \omega_i^{(n)}]$ with $\omega_i^{(n)}, \omega_{i+1}^{(n)} \in S^{(n)}$. Let $\omega_{i+1}^{(n)} = \frac{q_{i+1}}{p_{i+1}}$ and $\omega_i^{(n)} = \frac{q_i}{p_i}$ with g.c.d$(q_l, p_l) = 1$ for $l = i, i+1$. Then it is easy to see that $q_i p_{i+1} - p_i q_{i+1} = 1$ for all n and i (cf. [**5**, Claim A]).

Now given a basket $B = \{(b_i, r_i) \mid i = 1, \cdots, s\}$, we define new baskets $\mathcal{B}^{(n)}(B)$, where $\mathcal{B}^{(n)}(\cdot)$ can be regarded as an operator on the set of baskets. For each $(b_i, r_i) \in B$, if $\frac{b_i}{r_i} \in S^{(n)}$, then we set $\mathcal{B}_i^{(n)} := \{(b_i, r_i)\}$. If $\frac{b_i}{r_i} \notin S^{(n)}$, then $\omega_{l+1}^{(n)} < \frac{b_i}{r_i} < \omega_l^{(n)}$ for some l. We write $\omega_l^{(n)} = \frac{q_l}{p_l}$ and $\omega_{l+1}^{(n)} = \frac{q_{l+1}}{p_{l+1}}$ respectively. In this situation, we can unpack (b_i, r_i) to $\mathcal{B}_i^{(n)} := \{(r_i q_l - b_i p_l) \times (q_{l+1}, p_{l+1}), (-r_i q_{l+1} + b_i p_{l+1}) \times (q_l, p_l)\}$. Adding up those $\mathcal{B}_i^{(n)}$, we get a new basket $\mathcal{B}^{(n)}(B)$, which is uniquely defined according to the construction and $\mathcal{B}^{(n)}(B) \succeq B$ for all n. Note that, by the definition, $B = \mathcal{B}^{(n)}(B)$ for sufficiently large n.

Moreover, we have

$$\mathcal{B}^{(n-1)}(B) = \mathcal{B}^{(n-1)}(\mathcal{B}^{(n)}(B)) \succeq \mathcal{B}^{(n)}(B)$$

for all $n \geq 1$ (cf. [**5**, Claim B]). Therefore we have a chain of baskets

$$\mathcal{B}^{(0)}(B) \succeq \mathcal{B}^{(5)}(B) \succeq \cdots \succeq \mathcal{B}^{(n)}(B) \succeq \cdots \succeq B.$$

The step $\mathcal{B}^{(n-1)}(B) \succeq \mathcal{B}^{(n)}(B)$ can be achieved by a number of successive prime packings. Let $\epsilon_n(B)$ be the number of such prime packings. For any $n > 0$, set $B^{(n)} := \mathcal{B}^{(n)}(B)$.

The following properties are essential to represent $B^{(n)}$.

Lemma 3.6 ([**5**, Lemma 2.16]). *For the above sequence $\{B^{(n)}\}$, the following statements hold:*

(i) $\Delta^j(B^{(0)}) = \Delta^j(B)$ *for $j = 3, 4$;*
(ii) $\Delta^j(B^{(n-1)}) = \Delta^j(B^{(n)})$ *for all $j < n$;*
(iii) $\Delta^n(B^{(n-1)}) = \Delta^n(B^{(n)}) + \epsilon_n(B)$.

It follows that $\Delta^j(B^{(n)}) = \Delta^j(B)$ for all $j \leq n$ and

$$\epsilon_n(B) = \Delta^n(B^{(n-1)}) - \Delta^n(B^{(n)}) = \Delta^n(B^{(n-1)}) - \Delta^n(B).$$

Moreover, given a weighted basket $\mathbb{B} = (B, \tilde{P}_{-1})$, we can similarly consider $\mathcal{B}^{(n)}(\mathbb{B}) := (B^{(n)}, \tilde{P}_{-1})$. It follows that

$$\tilde{P}_{-j}(\mathcal{B}^{(n)}(\mathbb{B})) = \tilde{P}_{-j}(\mathbb{B})$$

for all $j \leq n$. Therefore we can realize the canonical sequence of weighted baskets as an approximation of weighted baskets via antiplurigenera.

We now recall the relation between weighted baskets and anti-plurigenera more closely. For a given weighted basket $\mathbb{B} = (B, \tilde{P}_{-1})$, we start by computing the non-negative number ϵ_n and $B^{(0)}$, $B^{(5)}$ in terms of $\tilde{P}_{-m}$. From the definition of $\tilde{P}_{-m}$ we get

$$\sigma(B) = 10 - 5\tilde{P}_{-1} + \tilde{P}_{-2},$$

$$\Delta^{m+1} = (2 - 5(m+1) + 2(m+1)^2) + \frac{1}{2}(m+1)(2 - 3m)\tilde{P}_{-1}$$

$$+ \frac{1}{2}m(m+1)\tilde{P}_{-2} + \tilde{P}_{-m} - \tilde{P}_{-(m+1)}.$$

In particular, we have

$$\Delta^3 = 5 - 6\tilde{P}_{-1} + 4\tilde{P}_{-2} - \tilde{P}_{-3};$$

$$\Delta^4 = 14 - 14\tilde{P}_{-1} + 6\tilde{P}_{-2} + \tilde{P}_{-3} - \tilde{P}_{-4}.$$

Assume $B^{(0)} = \{n_{1,r}^0 \times (1, r) \mid r \geq 2\}$. By Lemma 3.6, we have

$$\sigma(B) = \sigma(B^{(0)}) = \sum n_{1,r}^0;$$

$$\Delta^3(B) = \Delta^3(B^{(0)}) = n_{1,2}^0;$$

$$\Delta^4(B) = \Delta^4(B^{(0)}) = 2n_{1,2}^0 + n_{1,3}^0.$$

Thus we get $B^{(0)}$ as follows:

$$\begin{cases} n_{1,2}^0 = 5 - 6\tilde{P}_{-1} + 4\tilde{P}_{-2} - \tilde{P}_{-3}; \\ n_{1,3}^0 = 4 - 2\tilde{P}_{-1} - 2\tilde{P}_{-2} + 3\tilde{P}_{-3} - \tilde{P}_{-4}; \\ n_{1,4}^0 = 1 + 3\tilde{P}_{-1} - \tilde{P}_{-2} - 2\tilde{P}_{-3} + \tilde{P}_{-4} - \sigma_5; \\ n_{1,r}^0 = n_{1,r}^0, r \geq 5, \end{cases}$$

where $\sigma_5 := \sum_{r \geq 5} n_{1,r}^0$. A computation gives

$$\epsilon_5 = 2 + \tilde{P}_{-2} - 2\tilde{P}_{-4} + \tilde{P}_{-5} - \sigma_5.$$

Therefore we get $B^{(5)} = \{n_{1,r}^5 \times (1,r), n_{2,5}^5 \times (2,5) \mid r \geq 2\}$ as follows:

$$\begin{cases} n_{1,2}^5 = 3 - 6\tilde{P}_{-1} + 3\tilde{P}_{-2} - \tilde{P}_{-3} + 2\tilde{P}_{-4} - \tilde{P}_{-5} + \sigma_5; \\ n_{2,5}^5 = 2 + \tilde{P}_{-2} - 2\tilde{P}_{-4} + \tilde{P}_{-5} - \sigma_5; \\ n_{1,3}^5 = 2 - 2\tilde{P}_{-1} - 3\tilde{P}_{-2} + 3\tilde{P}_{-3} + \tilde{P}_{-4} - \tilde{P}_{-5} + \sigma_5; \\ n_{1,4}^5 = 1 + 3\tilde{P}_{-1} - \tilde{P}_{-2} - 2\tilde{P}_{-3} + \tilde{P}_{-4} - \sigma_5; \\ n_{1,r}^5 = n_{1,r}^0, r \geq 5. \end{cases}$$

Because $B^{(5)} = B^{(6)}$, we see $\epsilon_6 = 0$ and on the other hand

$$\epsilon_6 = 3\tilde{P}_{-1} + \tilde{P}_{-2} - \tilde{P}_{-3} - \tilde{P}_{-4} - \tilde{P}_{-5} + \tilde{P}_{-6} - \epsilon = 0$$

where $\epsilon := 2\sigma_5 - n_{1,5}^0 \geq 0$.

Going on a similar calculation, we get

$$\epsilon_7 = 1 + \tilde{P}_{-1} + \tilde{P}_{-2} - \tilde{P}_{-5} - \tilde{P}_{-6} + \tilde{P}_{-7} - 2\sigma_5 + 2n_{1,5}^0 + n_{1,6}^0;$$

$$\epsilon_8 = 2\tilde{P}_{-1} + \tilde{P}_{-2} + \tilde{P}_{-3} - \tilde{P}_{-4} - \tilde{P}_{-5} - \tilde{P}_{-7} + \tilde{P}_{-8}$$
$$- 3\sigma_5 + 3n_{1,5}^0 + 2n_{1,6}^0 + n_{1,7}^0.$$

A weighted basket $\mathbb{B} = (B, \tilde{P}_{-1})$ is said to be *geometric* if $\mathbb{B} = (B_X, P_{-1}(X))$ for a $\mathbb{Q}$-Fano 3-fold X. Geometric baskets are subject to some geometric properties. By [**11**], we have that $(-K_X \cdot c_2(X)) > 0$. Therefore [**18**, 10.3] gives the inequality

$$(3.3) \qquad\qquad \gamma(B) := \sum_i \frac{1}{r_i} - \sum_i r_i + 24 > 0.$$

For packings, it is easy to see the following lemma.

Lemma 3.7. *Given a packing of baskets $B_1 \succeq B_2$, we have $\gamma(B_1) \geq \gamma(B_2)$. In particular, if inequality (3.3) does not hold for B_1, then it does not hold for B_2.*

Lemma 3.7 implies that, for two weighted baskets $\mathbb{B}_1 \succeq \mathbb{B}_2$, if $\mathbb{B}_1$ is non-geometric, then neither is $\mathbb{B}_2$.

Furthermore, $-K^3(\mathbb{B}) = -K_X^3 > 0$ gives the inequality

$$(3.4) \qquad\qquad \sigma'(B) < 2P_{-1} + \sigma(B) - 6.$$

Finally, by [**14**, Lemma 15.6.2], if $P_{-m} > 0$ and $P_{-n} > 0$, then

$$(3.5) \qquad P_{-m-n} \geq P_{-m} + P_{-n} - 1.$$

Notation. For the connivence of readers who are not familiar with Chen–Chen's method, we collect in the following the notation that will be frequently used in the rest of this paper.

$B = \{(b_i, r_i)\}$: a basket.

$\sigma(B) = \sum_i b_i$.

$\sigma'(B) = \sum_i \frac{b_i^2}{r_i}$.

$\mathbb{B} = (B, \tilde{P}_{-1})$: a weighted basket.

$-K^3(\mathbb{B})$: the volume of $\mathbb{B}$.

$\tilde{P}_{-m}(\mathbb{B})$: the m-th anti-plurigenus of $\mathbb{B}$. We just write P_{-m} instead if $\mathbb{B}$ is geometric.

$\{B^{(m)}\}$: the canonical sequence of B.

$B^{(m)} = \{n_{b,r}^m \times (b, r)\}$: expression of $B^{(m)}$.

$\epsilon_m(B)$: the number of prime packings between $B^{(m-1)}$ to $B^{(m)}$.

$\sigma_5 = \sum_{r \geq 5} n_{1,r}^0$.

$\epsilon = 2\sigma_5 - n_{1,5}^0$.

$\gamma(B) = \sum_i \frac{1}{r_i} - \sum_i r_i + 24$.

Note that usually we will omit B in the symbols if B is clear enough.

3.3. $\mathbb{Q}$-Fano 3-folds with $h^0(-K) = 2$.

In this subsection we prove the following theorem.

Theorem 3.8. *Let X be a $\mathbb{Q}$-Fano 3-fold with $P_{-1} = 2$. Then for any integer $m \geq 6$, $\dim \overline{\varphi_{-m}(X)} > 1$. In particular, $\delta_1(X) \leq 6$.*

Theorem 3.8 is optimal due to the following example.

Example 3.9 ([**10**, List 16.6, No.88])**.** Consider the general weighted hypersurface $\underline{X_{42} \subset \mathbb{P}(1^2, 6, 14, 21)}$, which is a $\mathbb{Q}$-Fano 3-fold with $P_{-1} = 2$. Then $\dim \overline{\varphi_{-6}(X_{42})} > 1$ while $\dim \overline{\varphi_{-5}(X_{42})} = 1$. So $\delta_1(X_{42}) = 6$.

Proof of Theorem 3.8. Since $P_{-1} > 0$, it is sufficient to prove that there exists an integer $m \leq 6$ such that $\dim \overline{\varphi_{-m}(X)} > 1$.

Assume, to the contrary, that $\delta_1(X) > 6$. Then, by applying Theorems 3.2 and 3.4 to the case $m = 1$, we have

$$P_{-1} = 2, P_{-2} = 3, P_{-3} = 4, P_{-4} = 5, P_{-5} = 6, P_{-6} = 7.$$

Now by those formulae in Subsection 3.2, we have $n_{1,2}^0 = 1$, $n_{1,3}^0 = 1$, $n_{1,4}^0 = \epsilon_5 = 1 - \sigma_5$, and $0 = \epsilon_6 = 1 - \epsilon$. Hence $\epsilon = 1$, and this implies $\sigma_5 = n_{1,5}^0 = 1$. Hence the basket $B^{(5)} = B^{(0)} = \{(1, 2), (1, 3), (1, 5)\}$ by $\epsilon_5 = 0$. Since $B^{(5)}$ admits no prime packings, $B = B^{(5)}$ and $-K_X^3 = -K^3(\mathbb{B}(X)) = -1/30 < 0$, a contradiction. \qquad q.e.d.

3.4. $\mathbb{Q}$-Fano 3-folds with $h^0(-K) = 1$.
We are going to prove the following theorem.

Theorem 3.10. *Let X be a $\mathbb{Q}$-Fano 3-fold with $P_{-1} = 1$. Then, for any integer $m \geq 9$, $\dim \overline{\varphi_{-m}(X)} > 1$. In particular, $\delta_1(X) \leq 9$.*

This result is optimal as well due to the following example.

Example 3.11 ([**10**, List 16.7, No.85])**.** Consider the general codimension 2 weighted complete intersection $X = X_{24,30} \subset \mathbb{P}(1, 8, 9, 10, 12, 15)$ which is a $\mathbb{Q}$-Fano 3-fold with $P_{-1} = 1$. Then $\dim \overline{\varphi_{-9}(X)} > 1$ and $\dim \overline{\varphi_{-8}(X)} = 1$ since $P_{-8} = 2$. So $\delta_1(X) = 9$.

Proof of Theorem 3.10. Since $P_{-1} > 0$, it is sufficient to prove that there exists an integer $m \leq 9$ such that $\dim \overline{\varphi_{-m}(X)} > 1$. Assume, to the contrary, that $\delta_1(X) > l$ for some integer $l \leq 9$. We will deduce a contradiction.

Applying Theorems 3.2 and 3.4 to the case $m = 1$, we discuss on the number n_0 (defined in Theorem 3.4). By Chen–Chen [**4**, Theorem 1.1], we have $n_0 \leq 8$.

If $n_0 = 2$ and set $l = 6$, then Theorem 3.4(i)($m = 1$) implies that

$$P_{-1} = 1, \ P_{-2} = P_{-3} = 2, \ P_{-4} = P_{-5} = 3, \ P_{-6} = 4.$$

Then $n^0_{1,2} = 5$, $n^0_{1,3} = 1$, $n^0_{1,4} = \epsilon_5 = 1 - \sigma_5$, $0 = \epsilon_6 = 1 - \epsilon$. Hence $\epsilon = 1$, and this implies $\sigma_5 = n^0_{1,5} = 1$. Hence the basket $B^{(5)} = B^{(0)} = \{5 \times (1, 2), (1, 3), (1, 5)\}$ by $\epsilon_5 = 0$. Since $B^{(5)}$ admits no further prime packings, $B = B^{(5)}$ and $-K^3(\mathbb{B}) = -\frac{1}{30} < 0$, a contradiction. Thus $\delta_1(X) \leq 6$.

If $n_0 = 3$ and set $l = 6$, then Theorem 3.4(i)($m = 1$) implies that

$$P_{-1} = P_{-2} = 1, \ P_{-3} = P_{-4} = P_{-5} = 2, \ P_{-6} = 3.$$

Then $n^0_{1,2} = 1$, $n^0_{1,3} = 4$, $n^0_{1,4} = \epsilon_5 = 1 - \sigma_5$, $0 = \epsilon_6 = 1 - \epsilon$. Hence $\epsilon = 1$, and this implies $\sigma_5 = n^0_{1,5} = 1$. Hence the basket $B^{(5)} = B^{(0)} = \{(1, 2), 4 \times (1, 3), (1, 5)\}$ by $\epsilon_5 = 0$. Since $B^{(5)}$ admits no further prime packings, $B = B^{(5)}$ and $-K^3(\mathbb{B}) = -\frac{1}{30} < 0$, a contradiction. Thus $\delta_1(X) \leq 6$.

If $n_0 = 4$ and set $l = 6$, then Theorem 3.4(i)($m = 1$) implies that

$$P_{-1} = P_{-2} = P_{-3} = 1, P_{-4} = P_{-5} = P_{-6} = 2.$$

Then $n^0_{1,2} = 2$, $n^0_{1,3} = 1$, $n^0_{1,4} = 3 - \sigma_5$, $\epsilon_5 = 1 - \sigma_5$, $0 = \epsilon_6 = 1 - \epsilon$. Hence $\epsilon = 1$, and this implies $\sigma_5 = n^0_{1,5} = 1$. Hence $B^{(5)} = \{2 \times (1, 2), (1, 3), 2 \times (1, 4), (1, 5)\}$ by $\epsilon_5 = 0$. Hence $\epsilon_7 \leq 1$ and $\epsilon_8 = 0$ by considering possible prime packings of $B^{(5)}$. On the other hand, $\epsilon_7 = P_{-7} - 1$ and $\epsilon_8 = P_{-8} - P_{-7}$. So $P_{-8} = \epsilon_7 + 1 \leq 2$. But this contradicts $P_{-4} = 2$ and inequality (3.5). So $\delta_1(X) \leq 6$.

If $n_0 = 5$ and set $l = 7$, then Theorem 3.4(i)($m = 1$) implies that

$$P_{-1} = P_{-2} = P_{-3} = P_{-4} = 1, \ P_{-5} = P_{-6} = P_{-7} = 2.$$

Then $n^0_{1,2} = 2$, $n^0_{1,3} = 2$, $n^0_{1,4} = 2 - \sigma_5$, $\epsilon_5 = 3 - \sigma_5$, $0 = \epsilon_6 = 2 - \epsilon$, $\epsilon_7 = 1 - 2\sigma_5 + 2n^0_{1,5} + n^0_{1,6}$. Hence $\epsilon = 2$, and this implies $(\sigma_5, n^0_{1,5}) = (1,0)$ or $(2,2)$. If $(\sigma_5, n^0_{1,5}) = (1,0)$, then $n^0_{1,6} = 1$ by $\epsilon_7 \geq 0$. Hence $\epsilon_5 = 2$ and $B^{(5)} = \{2 \times (2,5), (1,4), (1,6)\}$. Since $B^{(5)}$ admits no further prime packings, $B = B^{(5)}$ and $-K^3(\mathbb{B}) = -\frac{1}{60} < 0$, a contradiction. If $(\sigma_5, n^0_{1,5}) = (2,2)$, then $\epsilon_5 = 1$, $\epsilon_7 = 1$, and $B^{(7)} = \{(3,7), (1,3), 2 \times (1,5)\}$. Since $B^{(7)}$ admits no further prime packings, $B = B^{(7)}$ and $-K^3(\mathbb{B}) = -\frac{2}{105} < 0$, a contradiction. So $\delta_1(X) \leq 7$.

If $n_0 = 6$ and set $l = 8$, then Theorem 3.4(i)($m = 1$) implies that

$$P_{-1} = P_{-2} = P_{-3} = P_{-4} = P_{-5} = 1, \ P_{-6} = P_{-7} = P_{-8} = 2.$$

Then $n^0_{1,2} = 2$, $n^0_{1,3} = 2$, $n^0_{1,4} = 2 - \sigma_5$, $\epsilon_5 = 2 - \sigma_5$, $0 = \epsilon_6 = 3 - \epsilon$. Hence $\epsilon = 3$ and $\sigma_5 \leq 2$, and this implies $(\sigma_5, n^0_{1,5}) = (2,1)$. Then $\epsilon_5 = 0$ and $B^{(5)} = \{2 \times (1,2), 2 \times (1,3), (1,5), (1,s')\}$ for some $s' \geq 6$. This implies $\epsilon_7 = \epsilon_8 = 0$ since there are no further packings. On the other hand, $\epsilon_7 = 2 - 2\sigma_5 + 2n^0_{1,5} + n^0_{1,6}$ and $\epsilon_8 = 2 - 3\sigma_5 + 3n^0_{1,5} + 2n^0_{1,6} + n^0_{1,7}$. Hence $n^0_{1,6} = 0$, $n^0_{1,7} = 1$, and $B^{(7)} = \{2 \times (1,2), 2 \times (1,3), (1,5), (1,7)\}$. Since $B^{(7)}$ is minimal, $B = B^{(7)}$ and $-K^3(\mathbb{B}) = -\frac{1}{105} < 0$, a contradiction. Thus $\delta_1(X) \leq 8$.

If $n_0 \geq 7$ and set $l = 9$, then Theorem 3.4(i)($m = 1$) implies that

$$P_{-1} = P_{-2} = P_{-3} = P_{-4} = P_{-5} = P_{-6} = 1, \ P_{-8} = P_{-9} = 2.$$

Then $n^0_{1,2} = 2$, $n^0_{1,3} = 2$, $n^0_{1,4} = 2 - \sigma_5$, $\epsilon_5 = 2 - \sigma_5$, $0 = \epsilon_6 = 2 - \epsilon$. Hence $\epsilon = 2$ and $\sigma_5 \leq 2$, and this implies $(\sigma_5, n^0_{1,5}) = (1,0)$ or $(2,2)$. If $(\sigma_5, n^0_{1,5}) = (2,2)$, then $B^{(5)} = \{2 \times (1,2), 2 \times (1,3), 2 \times (1,5)\}$ by $\epsilon_5 = 0$. Since $B^{(5)}$ admits no further prime packings, $B = B^{(5)}$ and $-K^3(\mathbb{B}) < 0$, a contradiction.

Thus we are left to consider the case: $(\sigma_5, n^0_{1,5}) = (1,0)$. Then we have $B^{(5)} = \{(1,2), (2,5), (1,3), (1,4), (1,s')\}$ with $s' \geq 6$ by $\epsilon_5 = 1$. Assume that $s' = 6, 7$. Clearly any basket B, with such a given $B^{(5)}$, dominates one of the following minimal ones:

$$B_1 = \{(3,7), (2,7), (1,s')\};$$
$$B_2 = \{(1,2), (3,8), (1,4), (1,s')\}.$$

Since $\sigma'(B) \geq \sigma'(B_i) \geq 2$ where $s' = 6, 7$ and $i = 1, 2$, inequality (3.4) fails for all B, which says that this case does not happen. Hence $s' \geq 8$,

then the expression of ϵ_8 gives

$$P_{-8} - P_{-7} = \epsilon_8 + 1.$$

Hence $P_{-7} = P_{-6} = 1$ and $\epsilon_7 = \epsilon_8 = 0$ since $P_{-8} = 2$. We have $B^{(8)} = B^{(5)} = \{(1,2),(2,5),(1,3),(1,4),(1,s')\}$ with $s' \geq 8$. Since $B^{(8)}$ admits no further prime packings, $B = B^{(8)}$. By inequalities (3.3) and (3.4), s' can only be $9,10,11$. But then direct calculations show that $P_{-9} = 3$ in all these three cases, a contradiction. We have proved $\delta_1(X) \leq 9$.

So we conclude the theorem.   q.e.d.

3.5. $\mathbb{Q}$-Fano 3-folds with $h^0(-K) = 0$.

In this subsection we prove the following theorem.

Theorem 3.12. *Let X be a $\mathbb{Q}$-Fano 3-fold with $P_{-1} = 0$. Then there exists an integer $m_1 \leq 11$ such that $\dim \overline{\varphi_{-m_1}(X)} > 1$. Moreover, we can take such a number $m_1 \leq 8$ except for the following baskets of singularities:*

No.1.	$\{2 \times (1,2), 3 \times (2,5), (1,3), (1,4)\}$;
No.2.	$\{5 \times (1,2), 2 \times (1,3), (2,7), (1,4)\}$;
No.3.	$\{5 \times (1,2), 2 \times (1,3), (3,11)\}$;
No.4.	$\{5 \times (1,2), (1,3), (3,10), (1,4)\}$;
No.A.	$\{7 \times (1,2), (3,7), (1,5)\}$;
No.B.	$\{6 \times (1,2), (4,9), (1,5)\}$;
No.C.	$\{5 \times (1,2), (5,11), (1,5)\}$;
No.D.	$\{4 \times (1,2), (6,13), (1,5)\}$;
No.E.	$\{7 \times (1,2), (3,8), (1,5)\}$;
No.F.	$\{5 \times (1,2), (4,9), (1,3), (1,5)\}$.

Remark 3.13. We do not know if this result is optimal since very few examples with $P_{-1} = 0$ are known. There are 4 known examples due to Iano-Fletcher [**10**, List 16.7, No.60] and Altınok–Reid [**2**], [**19**, Example 9.14]. For these examples we can see that $\dim \overline{\varphi_{-8}(X)} > 1$ by our theorem. Moreover, in next subsection we will treat the exceptional cases. If one can confirm either the existence or non-existence of type No.1–No.4, the result becomes optimal and so does Theorem 1.4.

Before proving Theorem 3.12, we recall a result by J. A. Chen and the first author.

Proposition 3.14 ([**4**, Theorem 3.5]). *Any geometric basket of weak $\mathbb{Q}$-Fano 3-folds with $P_{-1} = P_{-2} = 0$ is among the following list:*

	B	$-K^3$	P_{-3}	P_{-4}	P_{-5}	P_{-6}	P_{-7}	P_{-8}
No.1.	$\{2 \times (1,2), 3 \times (2,5), (1,3), (1,4)\}$	1/60	0	0	1	1	1	2
No.2.	$\{5 \times (1,2), 2 \times (1,3), (2,7), (1,4)\}$	1/84	0	1	0	1	1	2
No.3.	$\{5 \times (1,2), 2 \times (1,3), (3,11)\}$	1/66	0	1	0	1	1	2
No.4.	$\{5 \times (1,2), (1,3), (3,10), (1,4)\}$	1/60	0	1	0	1	1	2
No.5.	$\{5 \times (1,2), (1,3), 2 \times (2,7)\}$	1/42	0	1	0	1	2	3
No.6.	$\{4 \times (1,2), (2,5), 2 \times (1,3), 2 \times (1,4)\}$	1/30	0	1	1	2	2	4
No.7.	$\{3 \times (1,2), (2,5), 5 \times (1,3)\}$	1/30	1	1	1	3	3	4
No.8.	$\{2 \times (1,2), (3,7), 5 \times (1,3)\}$	1/21	1	1	1	3	4	5
No.9.	$\{(1,2), (4,9), 5 \times (1,3)\}$	1/18	1	1	1	3	4	5
No.10.	$\{3 \times (1,2), (3,8), 4 \times (1,3)\}$	1/24	1	1	1	3	3	5
No.11.	$\{3 \times (1,2), (4,11), 3 \times (1,3)\}$	1/22	1	1	1	3	3	5
No.12.	$\{3 \times (1,2), (5,14), 2 \times (1,3)\}$	1/21	1	1	1	3	3	5
No.13.	$\{2 \times (1,2), 2 \times (2,5), 4 \times (1,3)\}$	1/15	1	1	2	4	5	7
No.14.	$\{(1,2), (3,7), (2,5), 4 \times (1,3)\}$	17/210	1	1	2	4	6	8
No.15.	$\{2 \times (1,2), (2,5), (3,8), 3 \times (1,3)\}$	3/40	1	1	2	4	5	8
No.16.	$\{2 \times (1,2), (5,13), 3 \times (1,3)\}$	1/13	1	1	2	4	5	8
No.17.	$\{(1,2), 3 \times (2,5), 3 \times (1,3)\}$	1/10	1	1	3	5	7	10
No.18.	$\{4 \times (1,2), 5 \times (1,3), (1,4)\}$	1/12	1	2	2	5	6	9
No.19.	$\{4 \times (1,2), 4 \times (1,3), (2,7)\}$	2/21	1	2	2	5	7	10
No.20.	$\{4 \times (1,2), 3 \times (1,3), (3,10)\}$	1/10	1	2	2	5	7	10
No.21.	$\{3 \times (1,2), (2,5), 4 \times (1,3), (1,4)\}$	7/60	1	2	3	6	8	12
No.22.	$\{3 \times (1,2), 7 \times (1,3)\}$	1/6	2	3	4	9	12	17
No.23.	$\{2 \times (1,2), (2,5), 6 \times (1,3)\}$	1/5	2	3	5	10	14	20

Proof of Theorem 3.12. In the proof, we will always take a suitable integer m satisfying one of the conditions in Theorem 3.2. If necessary, we apply Theorem 3.4 on m and take $m_1 = l_0 m$.

Case I. $P_{-2} = 0$.

The basket $B = B_X$ of the singularities of X is among the list of Proposition 3.14. We just discuss it case by case.

If B is of type No.1, take $m = 5$. Since $P_{-5} = 1$ and $P_{-10} = 4$, we can take $m_1 = 10$.

If B is of type No.2, take $m = 11$. Since $P_{-11} = 4$, we can take $m_1 = 11$.

If B is of type No.3, take $m = 10$. Since $P_{-10} = 3$, we can take $m_1 = 10$.

If B is of type No.4, take $m = 11$. Since $P_{-11} = 4$, we can take $m_1 = 11$.

If B is of type No.5, take $m = 8$. Since $P_{-8} = 3$, we can take $m_1 = 8$.

If B is of type No.6, take $m = 8$. Since $P_{-8} = 4$, we can take $m_1 = 8$.

If B is of type No.7–No.21, take $m = 3$. Since $P_{-3} = 1$ and $P_{-6} \geq 3$, we can take $m_1 = 6$.

If B is of type No.22–No.23, take $m = 3$. Since $P_{-3} = 2$ and $P_{-6} \geq 9$, we can take $m_1 = 6$.

Case II. $P_{-2} > 0$.

Since $P_{-1} = 0$, the basket $B^{(0)}$ has datum

$$
\begin{cases}
n^0_{1,2} = 5 + 4P_{-2} - P_{-3}; \\
n^0_{1,3} = 4 - 2P_{-2} + 3P_{-3} - P_{-4}; \\
n^0_{1,4} = 1 - P_{-2} - 2P_{-3} + P_{-4} - \sigma_5.
\end{cases}
$$

By Lemma 3.7, $B^{(0)}$ satisfies inequality (3.3) and thus

$$
0 < \gamma(B^{(0)}) = \sum_{r \geq 2} \left(\frac{1}{r} - r \right) n^0_{1,r} + 24
$$

$$
\leq \sum_{r=2,3,4} \left(\frac{1}{r} - r \right) n^0_{1,r} - \frac{24}{5}\sigma_5 + 24
$$

$$
= \frac{25}{12} + \frac{37}{12}P_{-2} + P_{-3} - \frac{13}{12}P_{-4} - \frac{21}{20}\sigma_5.
$$

Hence, by $n^0_{1,3} \geq 0$ and $n^0_{1,4} \geq 0$, we have

(3.6)
$$
\begin{cases}
\frac{25}{12} + \frac{37}{12}P_{-2} + P_{-3} - \frac{13}{12}P_{-4} - \frac{21}{20}\sigma_5 > 0;
\end{cases}
$$

(3.7)
$$
4 - 2P_{-2} + 3P_{-3} - P_{-4} \geq 0;
$$

(3.8)
$$
1 - P_{-2} - 2P_{-3} + P_{-4} - \sigma_5 \geq 0.
$$

Considering the inequality "(3.6)+(3.7)+2×(3.8)":

(3.9)
$$
\frac{97}{12} - \frac{11}{12}P_{-2} - \frac{1}{12}P_{-4} - \frac{61}{20}\sigma_5 > 0,
$$

we obtain $\sigma_5 \leq 2$.

Subcase II-1. $\sigma_5 = 0$.

At first, we consider the case $P_{-3} = 0$. By inequality (3.7), we have $2P_{-2} + P_{-4} \leq 4$. Since $1 \leq P_{-2} \leq P_{-4}$, it follows that $(P_{-2}, P_{-4}) = (1, 1)$ or $(1, 2)$. If $(P_{-2}, P_{-4}) = (1, 1)$, then $B^{(0)} = \{9 \times (1, 2), (1, 3), (1, 4)\}$ with $-K^3(B^{(0)}) = -\frac{1}{12} < 0$. By considering a minimal basket $B_{\min}$ dominated by $B^{(0)}$, then either $B_{\min} = \{(10, 21), (1, 4)\}$ with $-K^3(B_{\min}) = -\frac{1}{84} < 0$ or $B_{\min} = \{9 \times (1, 2), (2, 7)\}$ with $-K^3(B_{\min}) = -\frac{1}{14} < 0$. Thus $-K^3(B) \leq -K^3(B_{\min}) < 0$, a contradiction. If $(P_{-2}, P_{-4}) = (1, 2)$, then $B^{(0)} = \{9 \times (1, 2), 2 \times (1, 4)\}$. Since $B^{(0)}$ admits no prime packings anymore, $B = B^{(0)}$ and $-K^3(B) = 0$, a contradiction.

Let us consider the case $P_{-3} \geq 1$. Since $\sigma_5 = 0$, $B^{(0)}$ is composed of $(1, 2), (1, 3), (1, 4)$. In particular, $4b \geq r$ holds for every pair $(b, r) \in B^{(0)}$. As an easy conclusion, after packings, $4b \geq r$ holds for every pair $(b, r) \in B$. So $m = 3$ satisfies the condition of Theorem 3.2. By Theorem 3.4, we can take $m_1 = 3$ or 6 unless $(P_{-3}, P_{-6}) = (1, 1), (1, 2), (2, 3)$. By inequality (3.8),

(3.10)
$$
P_{-4} \geq 2P_{-3} + P_{-2} - 1 \geq 2P_{-3}.
$$

By $P_{-2} > 0$, $P_{-6} \geq P_{-4}$. Thus we only need to consider the case $(P_{-3}, P_{-6}) = (1, 2)$. By inequality (3.10), $P_{-2} = 1$ and $P_{-4} = 2$. On the other hand,

$$0 = \epsilon_6 = 3P_{-1} + P_{-2} - P_{-3} - P_{-4} - P_{-5} + P_{-6} - \epsilon = -P_{-5}.$$

This implies $P_{-5} = 0$ which contradicts $P_{-2} = P_{-3} = 1$.

Subcase II-2. $\sigma_5 = 2$.

By inequality (3.9) and $P_{-4} \geq 2P_{-2} - 1$, we have $P_{-2} \leq 1$. Hence $P_{-2} = 1$ and, by inequalities (3.6)–(3.8), we have inequalities:

$$
\begin{cases}
\dfrac{46}{15} + P_{-3} - \dfrac{13}{12}P_{-4} > 0; & \text{(3.11)} \\[2mm]
2 + 3P_{-3} - P_{-4} \geq 0; & \text{(3.12)} \\[2mm]
-2 - 2P_{-3} + P_{-4} \geq 0. & \text{(3.13)}
\end{cases}
$$

Considering the inequality "$2 \times (3.11) + (3.13)$", we have $P_{-4} \leq 3$. Hence $P_{-3} = 0$ by inequality (3.13), and $P_{-4} = 2$ by inequalities (3.12) and (3.13). Then $B^{(0)} = \{9 \times (1,2), (1, s_1), (1, s_2)\}$ with $5 \leq s_1 \leq s_2$. If $s_2 > 5$, then $\gamma(B^{(0)}) \leq 9 \times (\frac{1}{2} - 2) + (\frac{1}{5} - 5) + (\frac{1}{6} - 6) + 24 < 0$, a contradiction. Thus $B^{(0)} = \{9 \times (1,2), 2 \times (1,5)\}$. Since $B^{(0)}$ admits no further prime packings, $B = B^{(0)}$. Take $m = 5$. Since $P_{-5} = 3$ by $\epsilon_5 = 0$, we can take $m_1 = 5$ by Theorem 3.4.

Subcase II-3. $\sigma_5 = 1$.

By inequalities (3.6)–(3.8), we have

$$
\begin{cases}
12 + 37P_{-2} + 12P_{-3} - 13P_{-4} \geq 0; & \text{(3.14)} \\[1mm]
4 - 2P_{-2} + 3P_{-3} - P_{-4} \geq 0; & \text{(3.15)} \\[1mm]
-P_{-2} - 2P_{-3} + P_{-4} \geq 0. & \text{(3.16)}
\end{cases}
$$

Considering the inequality "$(3.14) + 13 \times (3.16)$", we have

$$7P_{-3} \leq 12P_{-2} + 6. \tag{3.17}$$

Considering the inequality "$(3.15) + (3.16)$", we have

$$3P_{-2} \leq P_{-3} + 4. \tag{3.18}$$

Inequalities (3.17) and (3.18) imply $P_{-2} \leq 3$.

Subsubcase II-3-i. $(\sigma_5, P_{-2}) = (1, 3)$.

By inequalities (3.17) and (3.18), $5 \leq P_{-3} \leq 6$.

If $P_{-3} = 6$, by inequalities (3.14) and (3.16), $P_{-4} = 15$. Then $B^{(0)} = \{11 \times (1,2), (1,3), (1,s)\}$ for some integer $s \geq 5$. By $\gamma(B^{(0)}) > 0$, we have $s = 5$. Since the one-step packing $B_1 = \{10 \times (1,2), (2,5), (1,5)\}$ has negative $\gamma(B_1)$, $B = B^{(0)} = \{11 \times (1,2), (1,3), (1,5)\}$. Take $m = 4$. Since $P_{-4} = 15$, we can take $m_1 = 4$ by Theorem 3.4.

If $P_{-3} = 5$, by inequalities (3.15) and (3.16), $P_{-4} = 13$. Then $B^{(0)} = \{12 \times (1,2), (1,s)\}$ for some integer $s \geq 5$. By $\gamma(B^{(0)}) > 0$, we have

$s = 5, 6$. Clearly $B = B^{(0)}$. Take $m = 5$. Since $P_{-5} = 22$, we can take $m_1 = 5$ by Theorem 3.4.

Subsubcase II-3-ii. $(\sigma_5, P_{-2}) = (1, 2)$.

By inequalities (3.17) and (3.18), $2 \leq P_{-3} \leq 4$.

If $P_{-3} = 4$, by inequalities (3.14) and (3.16), $P_{-4} = 10$. Then $B^{(0)} = \{9 \times (1, 2), 2 \times (1, 3), (1, s)\}$ for some integer $s \geq 5$. By $\gamma(B^{(0)}) > 0$, we have $s = 5$. Since the one-step packing $B_1 = \{8 \times (1, 2), (2, 5), (1, 3), (1, 5)\}$ has negative $\gamma(B_1)$, $B = B^{(0)} = \{9 \times (1, 2), 2 \times (1, 3), (1, 5)\}$. Take $m = 4$. Since $P_{-4} = 10$, we can take $m_1 = 4$ by Theorem 3.4.

If $P_{-3} = 3$, by inequalities (3.15) and (3.16), $8 \leq P_{-4} \leq 9$. Firstly let us consider the case $P_{-4} = 9$. Clearly $B^{(0)} = \{10 \times (1, 2), (1, 4), (1, s)\}$ for some integer $s \geq 5$. By $\gamma(B^{(0)}) > 0$, we have $s = 5$. If $B = B^{(0)}$, we may take $m = 4$. Since $P_{-4} \geq 9$, we can take $m_1 = 4$ by Theorem 3.4. If $B \neq B^{(0)}$, we have $B = \{10 \times (1, 2), (2, 9)\}$. Take $m = 8$. Since $P_{-8} \geq 3$, we can take $m_1 = 8$ by Theorem 3.4. Now we consider the case $P_{-4} = 8$. We have $B^{(0)} = \{10 \times (1, 2), (1, 3), (1, s)\}$ for some integer $s \geq 5$. Since $\gamma(B^{(0)}) > 0$, we have $5 \leq s \leq 6$. For the case $(P_{-4}, s) = (8, 6)$, we get $B = \{10 \times (1, 2), (1, 3), (1, 6)\}$ since any possible packing of $B^{(0)}$ has negative γ. Take $m = 5$. Since $P_{-5} = 13$, we can take $m_1 = 5$ by Theorem 3.4. For the case $(P_{-4}, s) = (8, 5)$, we get either $B = \{10 \times (1, 2), (1, 3), (1, 5)\}$ or $B = \{9 \times (1, 2), (2, 5), (1, 5)\}$ or $B = \{8 \times (1, 2), (3, 7), (1, 5)\}$ by $\gamma > 0$. For all these cases, take $m = 6$. Since $P_{-6} \geq 3$, we can take $m_1 = 6$ by Theorem 3.4.

If $P_{-3} = 2$, we have $P_{-4} = 6$ by inequalities (3.15) and (3.16). Then $B^{(0)} = \{11 \times (1, 2), (1, s)\}$ for some integer $s \geq 5$. Similarly, $\gamma(B^{(0)}) > 0$ implies $5 \leq s \leq 7$. Since $B^{(0)}$ admits no further prime packings, $B = B^{(0)}$. Take $m = 6$. Since $P_{-6} \geq 3$, we can take $m_1 = 6$ by Theorem 3.4.

Subsubcase II-3-iii. $(\sigma_5, P_{-2}) = (1, 1)$.

By inequality (3.17), $P_{-3} \leq 2$.

If $P_{-3} = 2$, we have $P_{-4} = 5$ by inequalities (3.14) and (3.16). Then $B^{(0)} = \{7 \times (1, 2), 3 \times (1, 3), (1, s)\}$ for some integer $s \geq 5$. Similarly, $\gamma(B^{(0)}) > 0$ implies $s = 5$. Furthermore, we have either $B = \{7 \times (1, 2), 3 \times (1, 3), (1, 5)\}$ or $B = \{6 \times (1, 2), (2, 5), 2 \times (1, 3), (1, 5)\}$ by $\gamma > 0$. Take $m = 4$. Since $P_{-4} = 5$, we can take $m_1 = 4$ by Theorem 3.4.

If $P_{-3} = 1$, we have $3 \leq P_{-4} \leq 4$ by inequalities (3.14) and (3.16). Consider the case $(P_{-3}, P_{-4}) = (1, 4)$. We have

$$B^{(0)} = \{8 \times (1, 2), (1, 3), (1, 4), (1, s)\}$$

for some integer $s \geq 5$. Again we have $s = 5$ since $\gamma(B^{(0)}) > 0$. With the property $\gamma > 0$ and considering all possible baskets with $B^{(0)}$, we see that B must be one of the following baskets:

$$B_1 = \{8 \times (1,2), (1,3), (1,4), (1,5)\},$$
$$B_2 = \{8 \times (1,2), (2,7), (1,5)\},$$
$$B_3 = \{8 \times (1,2), (1,3), (2,9)\},$$
$$B_4 = \{7 \times (1,2), (2,5), (1,4), (1,5)\}.$$

For B_2, take $m = 6$. Since $P_{-6}(B_2) \geq 3$, we can take $m_1 = 6$ by Theorem 3.4. For B_3, take $m = 8$. Since $P_{-8}(B_3) \geq 3$, we can take $m_1 = 8$ by Theorem 3.4. For B_1 and B_4, take $m = 4$. Similarly we can take $m_1 = 4$ by Theorem 3.4. Consider the case $(P_{-3}, P_{-4}) = (1, 3)$. We have $B^{(0)} = \{8 \times (1,2), 2 \times (1,3), (1,s)\}$ for some integer $s \geq 5$. Similarly, $\gamma(B^{(0)}) > 0$ implies $5 \leq s \leq 6$. If $s = 6$, we see either $B = \{8 \times (1,2), 2 \times (1,3), (1,6)\}$ or $B = \{7 \times (1,2), (2,5), (1,3), (1,6)\}$ since $\gamma(B) > 0$. Take $m = 5$. Since $P_{-5} \geq 3$, we can take $m_1 = 5$ by Theorem 3.4. If $s = 5$, by considering all possible packings dominated by $B^{(0)}$ and using the property $\gamma > 0$, we see that B must be one of the following baskets:

$$B_i = \{8 \times (1,2), 2 \times (1,3), (1,5)\},$$
$$B_{ii} = \{7 \times (1,2), (2,5), (1,3), (1,5)\},$$
$$B_{iii} = \{6 \times (1,2), 2 \times (2,5), (1,5)\},$$
$$B_{iv} = \{6 \times (1,2), (3,7), (1,3), (1,5)\},$$
$$B_v = \{5 \times (1,2), (4,9), (1,3), (1,5)\},$$
$$B_{vi} = \{7 \times (1,2), (3,8), (1,5)\},$$
$$B_{vii} = \{5 \times (1,2), (3,7), (2,5), (1,5)\}.$$

For B_v (corresponding to No.F) and B_{vi} (corresponding to No.E), take $m = 9$. Since $P_{-9} \geq 3$, we can take $m_1 = 9$ by Theorem 3.4. For other cases, take $m = 6$. Since $P_{-6} \geq 3$, we can take $m_1 = 6$ by Theorem 3.4.

If $P_{-3} = 0$, by inequality (3.15), $P_{-4} \leq 2$. Firstly, consider the case $(P_{-3}, P_{-4}) = (0, 2)$. We have $B^{(0)} = \{9 \times (1,2), (1,4), (1,s)\}$ for some integer $s \geq 5$. In fact, $5 \leq s \leq 6$ by $\gamma(B^{(0)}) > 0$. When $s = 6$, $B = B^{(0)}$ since $B^{(0)}$ admits no further packings. Take $m = 7$. Since $P_{-7} = 6$, we can take $m_1 = 7$ by Theorem 3.4. When $s = 5$, the property $\gamma > 0$ implies that $B^{(0)}$ admits at most one further packings. Thus either $B = \{9 \times (1,2), (1,4), (1,5)\}$ (take $m = 4$) or $B = \{9 \times (1,2), (2,9)\}$ (take $m = 8$). For the first basket, $P_{-4} = 2$ and $P_{-8} = 7$, we can take $m_1 = 8$ by Theorem 3.4. For the second basket, $P_{-8} = 7$ and we can take $m_1 = 8$ by Theorem 3.4.

Finally we consider the case $(P_{-3}, P_{-4}) = (0, 1)$. We have $B^{(0)} = \{9 \times (1,2), (1,3), (1,s)\}$ for some integer $s \geq 5$. Similarly, $\gamma(B^{(0)}) > 0$ implies $5 \leq s \leq 7$. When $s = 7$, the property $\gamma > 0$ implies that either $B = \{9 \times (1,2), (1,3), (1,7)\}$ or $B = \{8 \times (1,2), (2,5), (1,7)\}$. Take

$m = 8$. Since $P_{-8} \geq 3$, we can take $m_1 = 8$ by Theorem 3.4. When $s = 6$, the inequalities $\gamma > 0$ and $-K^3 > 0$ imply that B must be one of the following baskets:

$$\{8 \times (1,2), (2,5), (1,6)\},$$
$$\{7 \times (1,2), (3,7), (1,6)\},$$
$$\{6 \times (1,2), (4,9), (1,6)\}.$$

Take $m = 7$. Since $P_{-7} \geq 3$, we can take $m_1 = 7$ by Theorem 3.4. When $s = 5$, inequalities $\gamma > 0$ and $-K^3 > 0$ imply that B is among one of the following baskets:

$$B_a = \{7 \times (1,2), (3,7), (1,5)\},$$
$$B_b = \{6 \times (1,2), (4,9), (1,5)\},$$
$$B_c = \{5 \times (1,2), (5,11), (1,5)\},$$
$$B_d = \{4 \times (1,2), (6,13), (1,5)\}.$$

For B_d (corresponding to No.D), take $m = 11$. Since $P_{-11} \geq 3$, we can take $m_1 = 11$ by Theorem 3.4. For other baskets (corresponding to No.A–No.C), take $m = 9$. Since $P_{-9} \geq 3$, we can take $m_1 = 9$ by Theorem 3.4. So the theorem is proved.   q.e.d.

3.6. Exceptional cases.

In this subsection, we treat the exceptional cases in Theorem 3.12.

Theorem 3.15. *Let X be a $\mathbb{Q}$-Fano 3-fold with basket of singularities B.*

(i) *If B is of type No.1–No.4 as in Theorem 3.12, then $\dim \overline{\varphi_{-10}(X)} > 1$.*

(ii) *If B is of type No.A–No.D as in Theorem 3.12, then $\dim \overline{\varphi_{-8}(X)} > 1$.*

(iii) *If B is of type No.E–No.F as in Theorem 3.12, then $\dim \overline{\varphi_{-6}(X)} > 1$.*

Proof. (i). Recall the proof in Case I of Theorem 3.12. We may only consider the two cases with No.2 and No.4. Since $P_{-9} = 2$, $\delta_1(X) \geq 10$. We want to show that $\delta_1(X) = 10$ in both cases. In fact, we have $P_{-4} = P_{-6} = 1$, $P_{-8} = 2$, $P_{-10} \geq 3$. Note that the conditions of Theorem 3.2 are all satisfied with $m = 4$. It follows that $-4K_X \sim E$ is a prime divisor. Assume that $\dim \varphi_{-10}(X) = 1$, then we can write $|-10K_X| = |nS| + E'$ with $n \geq 2$, $|S|$ is an irreducible rational pencil of surfaces and E' is the fixed part. By $P_{-6} > 0$, we have $E \leq |nS| + E'$. Since E is reduced and irreducible, either $E \leq |S|$ or $E \leq E'$ holds. Then

$$P_{-6} = h^0(-10K_X - E) = h^0(nS + E' - E) \geq h^0(S) = 2,$$

a contradiction.

(ii). Recall the last part of Subsubcase II-3-iii in the proof of Theorem 3.12. If B is of type No.A–No.D, we have $P_{-2} = P_{-4} = 1$, $P_{-6} = 2$, and $P_{-8} = 3$. Assume, to the contrary, that $\dim \varphi_{-8}(X) = 1$.

Write $-2K_X \sim D$ for some effective Weil divisor. By Theorem 3.4(i) (with $m = 2$), D must be either reducible or non-reduced. As in the proof of Theorem 3.2, take E to be any strictly effective divisor such that $E < D$. Then inequality (3.1) must fail for some singularity Q in B_a–B_d. Clearly, such an offending singularity Q must be "(1,5)". By equality (3.2), the local index $i_Q(E)$ of E should be 4 since inequality (3.1) holds for $i \in \{0, 1, 2, 3\}$ and $(b, r) = (1, 5)$, that is, $E \sim -K_X$ at Q. Since E is arbitrary such that $0 < E < D$ and $i_Q(-2K_X) = 3$, we conclude that $D = E_1 + E_2$ where E_i is fixed prime divisor with $i_Q(E_i) = 4$ for $i = 1, 2$.

If $E_1 = E_2$, then $2(-K_X - E_1) \sim 0$. By [**17**, Proposition 2.9] and the fact that $-K_X - E_1$ is Cartier at Q, we conclude that $-K_X - E_1$ is not 2-torsion. Hence $-K_X - E_1 \sim 0$, which contradicts $P_{-1} = 0$. Thus E_1 and E_2 are different prime divisors.

Since $|-6K_X| \preceq |-8K_X|$, by Lemma 2.2 we can write

$$|-6K_X| = |S| + a_6 E_1 + b_6 E_2,$$
$$|-8K_X| = |2S| + a_8 E_1 + b_8 E_2,$$

where $|S|$ is an irreducible rational pencil of surfaces, $a_i E_1 + b_i E_2$ is the fixed part, $a_i, b_i \in \mathbb{N}$ for $i = 6, 8$.

Claim 1. $a_6 b_6 = a_8 b_8 = 0$.

Proof. Assume that $a_6, b_6 \geq 1$, then

$$P_{-4} = h^0(-6K_X - E_1 - E_2) \geq h^0(S) = 2,$$

a contradiction. Similarly, we have $a_8 b_8 = 0$.   q.e.d.

We may assume that $b_6 = 0$. Then

(3.19) $3E_1 + 3E_2 \in |S + a_6 E_1| = |S| + a_6 E_1.$

It follows that $a_6 \leq 3$.

Case ii.1. $b_8 = 0$.
In this case

$$2S + a_8 E_1 \sim -8K_X \sim -6K_X + E_1 + E_2 \sim S + (a_6 + 1)E_1 + E_2.$$

Since $a_8 E_1$ is the fixed part of $|2S + a_8 E_1|$, $a_8 \leq a_6 + 1$. Then

(3.20) $S \sim (a_6 + 1 - a_8)E_1 + E_2.$

By relations (3.19) and (3.20),

(3.21) $(2a_6 + 1 - a_8)E_1 + E_2 \sim 3E_1 + 3E_2.$

Clearly, $2a_6 + 1 - a_8 \leq 3$ is absurd. Thus $2a_6 + 1 - a_8 \geq 4$. On the other hand $2a_6 + 1 - a_8 \leq 7$ since $a_6 \leq 3$. Locally at Q, since $i_Q(E_1) = i_Q(E_2) = 4$, we have

$$2a_6 + 1 - a_8 \equiv 0 \mod 5.$$

So $2a_6 + 1 - a_8 = 5$. Then relation (3.21) implies $2E_1 \sim 2E_2$. By [**17**, Proposition 2.9], we conclude that $E_1 \sim E_2$, a contradiction.

Case ii.2. $a_8 = 0$ and $b_8 > 0$.
In this case

$$2S + b_8 E_2 \sim -8K_X \sim -6K_X + E_1 + E_2 \sim S + (a_6 + 1)E_1 + E_2.$$

This implies that $b_8 \leq 1$. Hence $b_8 = 1$ and

(3.22) $$S \sim (a_6 + 1)E_1.$$

By relations (3.19) and (3.22),

(3.23) $$(2a_6 + 1)E_1 \sim 3E_1 + 3E_2.$$

Clearly $2a_6 + 1 \geq 4$ and $2a_6 + 1 \leq 7$ since $a_6 \leq 3$. Locally at Q, since $i_Q(E_1) = i_Q(E_2) = 4$, we have

$$2a_6 + 1 \equiv 1 \mod 5.$$

Since $4 \leq 2a_6 + 1 \leq 7$, this is impossible.

(iii). Recall the cases with B_v (No.F) and B_{vi} (No. E) (see Subsubcase II-3-iii in the proof of Theorem 3.12). We have $P_{-2} = 1$, $P_{-4} = 3$, $P_{-6} = 9$. Assume, to the contrary, that $\dim \overline{\varphi_{-6}(X)} = 1$.

We can write $-2K_X \sim D$ for some effective divisor D. By the same argument as (ii), $D = E_1 + E_2$ with E_i reduced and irreducible and $i_Q(E_i) = 4$ for $i = 1, 2$ where Q is the singularity "(1,5)". Note that, however, we do not know if E_1 and E_2 are different.

Since $|-4K_X| \preceq |-6K_X|$, by Lemma 2.2 we can write

$$|-4K_X| = |2S| + a_4 E_1 + b_4 E_2,$$
$$|-6K_X| = |8S| + a_6 E_1 + b_6 E_2,$$

where $|S|$ is an irreducible rational pencil of surfaces and $a_i E_1 + b_i E_2$ is the fixed part, $a_i, b_i \in \mathbb{N}$ for $i = 4, 6$. Hence

$$2S + a_4 E_1 + b_4 E_2 \sim -4K_X \sim 2(-2K_X) \sim 2E_1 + 2E_2.$$

Since $a_4 E_1 + b_4 E_2$ is the fixed part of $|2S + a_4 E_1 + b_4 E_2|$, we may assume $a_4 \leq b_4 \leq 2$.

If $b_4 = 2$, then $2S \sim (2 - a_4)E_1$. Hence $E_1 \leq S$ by the irreducibility of E_1. Then

$$1 = h^0(E_1) \geq h^0(2S - E_1) \geq h^0(S) = 2,$$

a contradiction.

If $b_4 = 1 \geq a_4$, then $2S \sim (2 - a_4)E_1 + E_2 \geq E_1 + E_2$. Hence $E_1 \leq S$ by the irreducibility of E_1. Then

$$1 = h^0(E_1 + E_2) \geq h^0(2S - E_1) \geq h^0(S) = 2,$$

a contradiction.

Hence $a_4 = b_4 = 0$ and $2S \sim 2E_1 + 2E_2$. Then

$$0 \sim -6K_X - 3E_1 - 3E_2$$
$$\sim 4(2E_1 + 2E_2) + a_6E_1 + b_6E_2 - 3E_1 - 3E_2$$
$$\geq 5E_1 + 5E_2,$$

a contradiction. So we have proved the theorem.   q.e.d.

To make the summary, Theorems 3.12 and 3.15 directly imply the following:

Corollary 3.16. *Let X be a $\mathbb{Q}$-Fano 3-fold with $P_{-1} = 0$.   Then $\delta_1(X) \leq 8$ except for the following cases:*

No.1.	$\{2 \times (1,2), 3 \times (2,5), (1,3), (1,4)\}$	$\delta_1(X) = 10;$
No.2.	$\{5 \times (1,2), 2 \times (1,3), (2,7), (1,4)\}$	$\delta_1(X) = 10;$
No.3.	$\{5 \times (1,2), 2 \times (1,3), (3,11)\}$	$\delta_1(X) = 10;$
No.4.	$\{5 \times (1,2), (1,3), (3,10), (1,4)\}$	$\delta_1(X) = 10;$
No.A.	$\{7 \times (1,2), (3,7), (1,5)\}$	$\delta_1(X) = 8;$
No.B.	$\{6 \times (1,2), (4,9), (1,5)\}$	$\delta_1(X) = 8;$
No.C.	$\{5 \times (1,2), (5,11), (1,5)\}$	$\delta_1(X) = 8;$
No.D.	$\{4 \times (1,2), (6,13), (1,5)\}$	$\delta_1(X) = 8;$
No.E.	$\{7 \times (1,2), (3,8), (1,5)\}$	$\delta_1(X) \leq 6;$
No.F.	$\{5 \times (1,2), (4,9), (1,3), (1,5)\}$	$\delta_1(X) \leq 6.$

Theorem 1.4 follows directly from Theorems 3.3, 3.8, and 3.10, and Corollary 3.16.

4. When is $|-mK_X|$ not composed with a pencil? (Part II)

As we have seen in the last section, the condition $\rho(X) = 1$ is crucial to proving Theorem 3.2. For arbitrary weak $\mathbb{Q}$-Fano 3-folds, we have to study in an alternative way. Naturally what we can prove is weaker than Theorem 1.4.

Let X be a weak $\mathbb{Q}$-Fano 3-fold. We are going to estimate $\delta_1(X)$ from above. The main idea is to relate this problem to the value distribution of the Hilbert function $\chi_{-m} = P_{-m}$.

Lemma 4.1. *Keep the same notation as in Subsection* 2.1.   *The number $r_X(\pi^*(-K_X)^2 \cdot S)_Y$ is a positive integer.*

Proof. Note that the number $(\pi^*(-K_X)^2 \cdot S)_Y$ is positive since

$$(\pi^*(-K_X)^2 \cdot S)_Y = (\pi^*(-K_X)|_S)_S^2$$

and $\pi^*(-K_X)|_S$ is nef and big on S. It is independent of the choice of π according to the projection formula of the intersection theory. So

we may choose such a modification π that dominates a resolution of singularities $\tau : \hat{W} \to X$. Then we see $(\pi^*(-K_X)^2 \cdot S)_Y = (\tau^*(-K_X)^2 \cdot S_1)_{\hat{W}}$ where $S_1 = \theta_*(S)$ is a divisor on $\hat{W}$ and $\theta : Y \to \hat{W}$ is a birational morphism. Note that, however, S_1 is a generic element in an algebraic family though it is not necessarily nonsingular.

We may write $K_{\hat{W}} = \tau^*(K_X) + \Delta_\tau$ where Δ_τ is an exceptional effective $\mathbb{Q}$-divisor over those isolated terminal singularities on X. Now, by intersection theory, we have

$$(r_X \tau^*(-K_X) \cdot \tau^*(-K_X) \cdot S_1)_{\hat{W}} = (r_X \tau^*(-K_X) \cdot (-K_{\hat{W}}) \cdot S_1)_{\hat{W}}$$

is an integer.   q.e.d.

Corollary 4.2. *Let X be a weak $\mathbb{Q}$-Fano 3-fold. If*

$$P_{-m} > r_X(-K_X)^3 m + 1$$

for some integer m, then $|-mK_X|$ is not composed with a pencil.

Proof. Assume that $|-mK_X|$ is composed with a pencil. Set $D := -mK_X$ and keep the same notation as in Subsection 2.1. Then we have $m\pi^*(-K_X) \geq M_{-m} \equiv (P_{-m} - 1)S$. Thus

$$m(-K_X)^3 \geq (P_{-m} - 1)(\pi^*(-K_X)^2 \cdot S) \geq \frac{1}{r_X}(P_{-m} - 1)$$

by Lemma 4.1, a contradiction.   q.e.d.

Next we estimate the number m which satisfies Corollary 4.2. We will do this in two steps as follows.

Proposition 4.3. *Let X be a weak $\mathbb{Q}$-Fano 3-fold. Take an arbitrary real number $0 < t \leq 37$. Denote $r_{\max} := \max\{r_i \in B_X\}$ the maximum of local indices of singularities. If*

$$m \geq \max\left\{37, \frac{r_{\max}t}{3}, \sqrt{6r_X + \frac{12}{t(-K_X^3)}}\right\},$$

then $P_{-m} \geq r_X(-K_X^3)m + 2$. In particular, $|-mK_X|$ is not composed with a pencil.

Proof. By Reid's formula, there exists a basket of singularities

$$B_X = \left\{(b_i, r_i) \mid i = 1, \ldots, s; 0 < b_i \leq \frac{r_i}{2}; b_i \text{ is coprime to } r_i\right\}$$

such that we have the formula

$$P_{-n} = \frac{1}{12}n(n+1)(2n+1)(-K_X^3) + 2n + 1 - l(-n)$$

for any $n > 0$, where

$$l(-n) = \sum_i \sum_{j=1}^{n} \frac{\overline{jb_i}(r_i - \overline{jb_i})}{2r_i}.$$

To estimate the lower bound of P_{-n}, we need to bound $l(-n)$ from above.

For any pair $(b, r) \in B_X$, we have $r \leq 24$ by inequality (2.1). In fact, we have the following estimation.

1) If $r = 2$, then

$$\frac{\overline{jb}(r - \overline{jb})}{2r} = \begin{cases} \frac{1}{4} & \text{when } j \text{ odd}; \\ 0 & \text{when } j \text{ even}. \end{cases}$$

2) If r is odd, then $\frac{\overline{jb}(r-\overline{jb})}{2r} \leq \frac{r^2-1}{8r}$.

3) If r is even and $r > 2$, then

$$\frac{\overline{jb}(r - \overline{jb})}{2r} \leq \begin{cases} \frac{\frac{r-2}{2}\frac{r+2}{2}}{2r} = \frac{r^2-4}{8r} & \text{when } \overline{jb} \neq r/2; \\ \frac{r^2}{8r} & \text{when } \overline{jb} = r/2. \end{cases}$$

Clearly, $b \neq r/2$ under the same situation. Since $\overline{jb} = r/2$ and $\overline{(j-1)b} = r/2$ can not hold simultaneously, we have

$$\frac{\overline{(j-1)b}(r - \overline{(j-1)b})}{2r} + \frac{\overline{jb}(r - \overline{jb})}{2r} \leq \frac{r^2-4}{8r} + \frac{r^2}{8r} \leq \frac{2 \cdot (r^2-1)}{8r}.$$

Hence, when r is even and $r > 2$, we have

$$(4.1) \qquad \sum_{j=1}^{n} \frac{\overline{jb}(r - \overline{jb})}{2r} \leq n \cdot \frac{r^2-1}{8r}.$$

By the way, inequality (4.1) also holds when r is odd.

Recall that we have

$$\sum_{j=1}^{r} \frac{\overline{jb}(r - \overline{jb})}{2r} = \frac{r^2-1}{12}.$$

Hence, whenever $r > 2$ and $n \geq \frac{r_{\max}t}{3}$, we have

$$\sum_{j=1}^{n} \frac{\overline{jb}(r - \overline{jb})}{2r} = \left\lfloor \frac{n}{r} \right\rfloor \frac{r^2-1}{12} + \sum_{j=1}^{\overline{n}} \frac{\overline{jb}(r - \overline{jb})}{2r}$$

$$\leq \left\lfloor \frac{n}{r} \right\rfloor \frac{r^2-1}{12} + \min\left\{ \overline{n} \cdot \frac{r^2-1}{8r}, \frac{r^2-1}{12} \right\}$$

$$(4.2) \qquad \leq \frac{r^2-1}{12r}\left(n + \frac{r}{3}\right)$$

$$\leq \frac{r^2-1}{12r} \cdot \frac{(t+1)n}{t}.$$

We prove the second inequality here. Assume, to the contrary, that

$$(4.3) \qquad \left\lfloor \frac{n}{r} \right\rfloor \frac{r^2-1}{12} + \overline{n} \cdot \frac{r^2-1}{8r} > \frac{r^2-1}{12r}\left(n + \frac{r}{3}\right),$$

and

$$(4.4) \qquad \left\lfloor \frac{n}{r} \right\rfloor \frac{r^2-1}{12} + \frac{r^2-1}{12} > \frac{r^2-1}{12r}\left(n + \frac{r}{3}\right).$$

Inequality (4.3) implies $\overline{n} > \frac{2r}{3}$. But from inequality (4.4), we have $\overline{n} < \frac{2r}{3}$, a contradiction.

Since X is weak $\mathbb{Q}$-Fano, recall that we have inequality

$$\sum_i \left(r_i - \frac{1}{r_i}\right) \le 24$$

by inequality (2.1). Denote by N_2 the number of $r_i = 2$ in B_X. Then, if $n \ge \frac{r_{\max}t}{3}$,

$$
\begin{aligned}
l(-n) &= \sum_i \sum_{j=1}^n \frac{\overline{jb_i}(r_i - \overline{jb_i})}{2r_i} \\
&= \frac{N_2}{4}\left\lfloor \frac{n+1}{2} \right\rfloor + \sum_{r_i>2}\sum_{j=1}^n \frac{\overline{jb_i}(r_i - \overline{jb_i})}{2r_i} \\
&\le \frac{N_2}{4}\left\lfloor \frac{n+1}{2} \right\rfloor + \frac{(t+1)n}{t}\sum_{r_i>2}\frac{r_i^2-1}{12r_i} \\
&\le \frac{N_2}{4}\left\lfloor \frac{n+1}{2} \right\rfloor + \frac{(t+1)n}{t}\cdot\frac{24 - \frac{3}{2}N_2}{12} \\
&\le \frac{2(t+1)n}{t} - N_2\left(\frac{(t+1)n}{8t} - \frac{1}{4}\left\lfloor \frac{n+1}{2} \right\rfloor\right) \\
&\le \frac{2(t+1)n}{t}
\end{aligned}
$$

where $\frac{(t+1)n}{8t} - \frac{1}{4}\left\lfloor \frac{n+1}{2} \right\rfloor \ge 0$ whenever $n \ge t$. Hence

$$P_{-n} = \frac{1}{12}n(n+1)(2n+1)(-K_X^3) + 2n + 1 - l(-n)$$

$$\ge \frac{1}{6}n^3(-K_X^3) + \frac{n^2}{4}(-K_X^3) + 1 - \frac{2n}{t}.$$

By [4], $-K_X^3 \ge \frac{1}{330}$. Hence $\frac{n^2}{4}(-K_X^3) \ge 1$ if $n \ge 37$. If $m \ge \sqrt{6r_X + \frac{12}{t(-K_X^3)}}$, then

$$
\begin{aligned}
P_{-m} &\ge \frac{1}{6}m^3(-K_X^3) + 2 - \frac{2m}{t} \\
&\ge \frac{1}{6}\left(6r_X + \frac{12}{t(-K_X^3)}\right)m(-K_X^3) + 2 - \frac{2m}{t} \\
&= r_X(-K_X^3)m + 2.
\end{aligned}
$$

We complete the proof. \hfill q.e.d.

In practice, we will take a suitable t to apply Proposition 4.3. Note that $r_{\max} \leq 24$.

Proposition 4.4. *Let X be a weak $\mathbb{Q}$-Fano 3-fold.*

(i) *If $r_X \leq 660$, then $\sqrt{6r_X + \frac{3}{2(-K_X^3)}} < 67$. In particular, $P_{-m} \geq r_X(-K_X^3)m + 2$ for $m \geq 67$.*

(ii) *If $r_X > 660$, then $r_X = 840$, and $P_{-m} \geq r_X(-K_X^3)m + 2$ for $m \geq 71$.*

Proof. Statement (i) is clear since $-K_X^3 \geq \frac{1}{330}$ by [**4**] and take $t = 8$ in Proposition 4.3. We mainly prove (ii) here.

First of all, by Proposition 2.4, $r_X = 840$ and $\mathcal{R} = (3,5,7,8)$ or $(2,3,5,7,8)$.

For $r > 2$, we use the inequality (4.2) (in the proof of Proposition 4.3) that

$$\sum_{j=1}^{n} \frac{\overline{jb}(r - \overline{jb})}{2r} \leq \frac{r^2 - 1}{12r}\left(n + \frac{r}{3}\right).$$

Then

$$l(-n) = \sum_{i}\sum_{j=1}^{n} \frac{\overline{jb_i}(r_i - \overline{jb_i})}{2r_i}$$

$$\leq \frac{N_2}{4}\left\lfloor \frac{n+1}{2} \right\rfloor + \sum_{r_i > 2} \frac{r_i^2 - 1}{12r_i}\left(n + \frac{r_i}{3}\right)$$

$$\leq \frac{n+1}{8} + \frac{3^2 - 1}{12 \cdot 3}(n + 1) + \frac{5^2 - 1}{12 \cdot 5}\left(n + \frac{5}{3}\right)$$

$$+ \frac{7^2 - 1}{12 \cdot 7}\left(n + \frac{7}{3}\right) + \frac{8^2 - 1}{12 \cdot 8}\left(n + \frac{8}{3}\right)$$

$$= \frac{19907n}{10080} + \frac{295}{72}$$

$$\leq 2n + \frac{7}{3}$$

as long as $n \geq 71$.

Hence

$$P_{-n} = \frac{1}{12}n(n+1)(2n+1)(-K_X^3) + 2n + 1 - l(-n)$$

$$\geq \frac{1}{6}n^3(-K_X^3) + \left(\frac{n^2}{4}(-K_X^3) - \frac{10}{3}\right) + 2.$$

By [**4**], $-K_X^3 \geq \frac{1}{330}$. Hence $\frac{n^2}{4}(-K_X^3) \geq \frac{10}{3}$ whenever $n \geq 71$. If $m \geq 71 > \sqrt{6r_X}$, then

$$P_{-m} \geq \frac{1}{6}m^3(-K_X^3) + 2$$

$$\geq \frac{1}{6}(6r_X)m(-K_X^3) + 2$$
$$= r_X(-K_X^3)m + 2.$$

We finish the proof.   q.e.d.

Theorem 1.7 directly follows from Corollary 4.2 and Proposition 4.4.

5. Birationality

In this section, we consider the birationality of anti-pluricanonical maps φ_{-m}.

5.1. Main reduction.

In this subsection, we reduce the birationality problem on X to that on Y.

Lemma 5.1 (cf. [**8**, Lemma 2.5]). *Let W be a normal projective variety on which there is an integral Weil $\mathbb{Q}$-Cartier divisor D. Let $h : V \longrightarrow W$ be any resolution of singularities. Assume that E is an effective exceptional $\mathbb{Q}$-divisor on V with $h^*(D) + E$ a Cartier divisor on V. Then*

$$h_*\mathcal{O}_V(h^*(D) + E) = \mathcal{O}_W(D)$$

where $\mathcal{O}_W(D)$ is the reflexive sheaf corresponding to the Weil divisor D.

Lemma 5.2 (cf. [**8**, 2.6]). *Let X be a weak $\mathbb{Q}$-Fano 3-fold and $\pi : Y \longrightarrow X$ the same resolution as in Subsection 2.1. Then, for any $m > 0$, φ_{-m} is birational if and only if so is $\Phi_{|K_Y + \lceil (m+1)\pi^*(-K_X) \rceil|}$.*

Proof. Recall that

$$K_Y = \pi^*(K_X) + E_\pi$$

where E_π is an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor since X has at worst terminal singularities. We have

$$K_Y + \lceil (m+1)\pi^*(-K_X) \rceil$$
$$= \pi^*(K_X) + E_\pi + \pi^*(-(m+1)K_X) + E_{m+1}$$
$$= \pi^*(-mK_X) + E_\pi + E_{m+1}$$

where $E_\pi + E_{m+1}$ is an effective $\mathbb{Q}$-divisor on Y exceptional over X. Lemma 5.1 implies

$$\pi_*\mathcal{O}_Y(K_Y + \lceil (m+1)\pi^*(-K_X) \rceil) = \mathcal{O}_X(-mK_X).$$

Hence φ_{-m} is birational if and only if so is $\Phi_{|K_Y + \lceil (m+1)\pi^*(-K_X) \rceil|}$. q.e.d.

Noting that

$$H^0(\mathcal{O}_X(-mK_X)) \cong H^0(\mathcal{O}_Y(\lfloor -m\pi^*(K_X) \rfloor))$$
$$\cong H^0(\mathcal{O}_Y(K_Y + \lceil (m+1)\pi^*(-K_X) \rceil)),$$

we denote by $|M_{-m}|$ the movable part of $|\lfloor -m\pi^*(K_X)\rfloor|$. We have the equality:

$$(5.1) \qquad\qquad -m\pi^*(K_X) = M_{-m} + F_m$$

where F_m is an effective $\mathbb{Q}$-divisor. Another direct consequence is that we may write:

$$K_Y + \lceil (m+1)\pi^*(-K_X)\rceil \sim M_{-m} + N_{-m}$$

where N_{-m} is the fixed part of $|K_Y + \lceil (m+1)\pi^*(-K_X)\rceil|$.

5.2. Key theorem.

Let X be a weak $\mathbb{Q}$-Fano 3-fold on which $P_{-m_0} \geq 2$ for some integer $m_0 > 0$. Suppose that $m_1 \geq m_0$ is another integer with $P_{-m_1} \geq 2$ and that $|-m_1 K_X|$ and $|-m_0 K_X|$ are not composed with the same pencil. Recall that, for any $m > 0$ with $P_{-m} > 1$,

$$\iota(m) = \begin{cases} 1, & \text{if } |-mK_X| \text{ is not composed with a pencil;} \\ P_{-m} - 1, & \text{if } |-mK_X| \text{ is composed with a pencil.} \end{cases}$$

Set $D := -m_0 K_X$ and keep the same notation as in Subsection 2.1. We may modify the resolution π in Subsection 2.1 such that the movable part $|M_{-m}|$ of $|\lfloor \pi^*(-mK_X)\rfloor|$ is base point free for all $m_0 \leq m \leq m_1$. Pick a generic irreducible element S of $|M_{-m_0}|$. By equality (5.1), we have

$$m_0\pi^*(-K_X) = \iota(m_0)S + F_{m_0}$$

for some effective $\mathbb{Q}$-divisor F_{m_0}. In particular, we see that

$$\frac{m_0}{\iota(m_0)}\pi^*(-K_X) - S \sim_{\mathbb{Q}} \text{ effective } \mathbb{Q}\text{-divisor.}$$

Define the real number

$$\mu_0 = \mu_0(|S|) := \inf\{t \in \mathbb{Q}^+ \mid t\pi^*(-K_X) - S \sim_{\mathbb{Q}} \text{ effective } \mathbb{Q}\text{-divisor}\}.$$

Remark 5.3. Clearly, we have $0 < \mu_0 \leq \frac{m_0}{\iota(m_0)}$. If $|-m_0 K_X|$ is composed with a pencil, for all k such that $|-kK_X| \succeq |-m_0 K_X|$ and $|-kK_X|$ is also composed with a pencil, we have

$$k\pi^*(-K_X) = \iota(k)S + F_k$$

for some effective $\mathbb{Q}$-divisor F_k by Lemma 2.2, and hence $\mu_0 \leq \frac{k}{\iota(k)}$.

By the assumption on $|-m_1 K_X|$, we know that $|G| = |M_{-m_1}|_S|$ is a base point free linear system on S and $h^0(S, G) \geq 2$. Denote by C a generic irreducible element of $|G|$. Since $m_1\pi^*(-K_X) \geq M_{-m_1}$, we have

$$m_1\pi^*(-K_X)|_S \equiv C + H$$

where H is an effective $\mathbb{Q}$-divisor on S.

We define two numbers which will be the key invariants accounting for the birationality of φ_{-m}. They are

$$\zeta := (\pi^*(-K_X) \cdot C)_Y = (\pi^*(-K_X)|_S \cdot C)_S \text{ and}$$
$$\varepsilon(m) := (m + 1 - \mu_0 - m_1)\zeta.$$

Note that ζ and $\varepsilon(m)$ are invariants under taking higher model of the resolution Y by projection formula. Hence we can modify π if necessary.

While studying the birationality of φ_{-m}, we always need to check that the linear system $\Lambda_m := |K_Y + \lceil (m+1)\pi^*(-K_X) \rceil|$ satisfies the following assumption for some integer $m > 0$.

Assumption 5.4. Keep the notation as above.

(1) The linear system Λ_m distinguishes different generic irreducible elements of $|M_{-m_0}|$ (namely, $\Phi_{\Lambda_m}(S') \neq \Phi_{\Lambda_m}(S'')$ for two different generic irreducible elements S', S'' of $|M_{-m_0}|$).

(2) The linear system $\Lambda_m|_S$ distinguishes different generic irreducible elements of the linear system $|G| = |M_{-m_1}|_S|$ on S.

The following is the key theorem in this section.

Theorem 5.5 (cf. [**8**, Theorem 3.5]). *Let X be a weak $\mathbb{Q}$-Fano 3-fold. Keep the notation as above. Let $m > 0$ be an integer. If Assumption 5.4 is satisfied and $\varepsilon(m) > 2$, then φ_{-m} is birational onto its image.*

Proof. By Lemma 5.2, we only need to prove the birationality of Φ_{Λ_m}. Since Assumption 5.4(1) is satisfied, the usual birationality principle (see, for instance, [**6**, 2.7]) reduces the birationality of Φ_{Λ_m} to that of $\Phi_{\Lambda_m}|_S$ for a generic irreducible element S of $|M_{-m_0}|$. Similarly, due to Assumption 5.4(2), we only need to prove the birationality of $\Phi_{\Lambda_m}|_C$ for a generic irreducible element C of $|G|$. Now we show how to restrict the linear system Λ_m to C.

Now assume $\varepsilon(m) > 0$. We can find a sufficiently large integer n so that there exists a number $\mu_0^{(n)} \in \mathbb{Q}^+$ with $0 \leq \mu_0^{(n)} - \mu_0 \leq \frac{1}{n}$, $\lceil \varepsilon(m, n) \rceil = \lceil \varepsilon(m) \rceil$ where $\varepsilon(m, n) := (m + 1 - \mu_0^{(n)} - m_1)\zeta$ and

$$\mu_0^{(n)} \pi^*(-K_X) \sim_{\mathbb{Q}} S + E^{(n)}$$

for an effective $\mathbb{Q}$-divisor $E^{(n)}$. In particular, $\varepsilon(m, n) > 0$, and $\varepsilon(m, n) > 2$ if $\varepsilon(m) > 2$. Re-modify the resolution π in Subsection 2.1 so that $E^{(n)}$ has simple normal crossing support.

For the given integer $m > 0$, we have

(5.2)
$$|K_Y + \lceil (m+1)\pi^*(-K_X) - E^{(n)} \rceil| \preceq |K_Y + \lceil (m+1)\pi^*(-K_X) \rceil|.$$

Since $\varepsilon(m, n) > 0$, the $\mathbb{Q}$-divisor

$$(m+1)\pi^*(-K_X) - E^{(n)} - S \equiv (m + 1 - \mu_0^{(n)})\pi^*(-K_X)$$

is nef and big and thus

$$H^1(Y, K_Y + \lceil (m+1)\pi^*(-K_X) - E^{(n)} \rceil - S) = 0$$

by Kawamata–Viehweg vanishing theorem. Hence we have surjective map

(5.3) $H^0(Y, K_Y + \lceil (m+1)\pi^*(-K_X) - E^{(n)} \rceil) \longrightarrow H^0(S, K_S + L_{m,n})$

where

(5.4) $L_{m,n} := (\lceil (m+1)\pi^*(-K_X) - E^{(n)} \rceil - S)|_S \geq \lceil \mathcal{L}_{m,n} \rceil$

and $\mathcal{L}_{m,n} := ((m+1)\pi^*(-K_X) - E^{(n)} - S)|_S$. Moreover, we have

$$m_1\pi^*(-K_X)|_S \equiv C + H$$

for an effective ℚ-divisor H on S by the setting. Thus the ℚ-divisor

$$\mathcal{L}_{m,n} - H - C \equiv (m+1 - \mu_0^{(n)} - m_1)\pi^*(-K_X)|_S$$

is nef and big by $\varepsilon(m,n) > 0$. By Kawamata–Viehweg vanishing theorem again,

$$H^1(S, K_S + \lceil \mathcal{L}_{m,n} - H \rceil - C) = 0.$$

Therefore, we have surjective map

(5.5) $H^0(S, K_S + \lceil \mathcal{L}_{m,n} - H \rceil) \longrightarrow H^0(C, K_C + D_{m,n})$

where

(5.6) $D_{m,n} := \lceil \mathcal{L}_{m,n} - H - C \rceil|_C \geq \lceil \mathcal{D}_{m,n} \rceil$

and $\mathcal{D}_{m,n} := (\mathcal{L}_{m,n} - H - C)|_C$ with $\deg \lceil \mathcal{D}_{m,n} \rceil \geq \lceil \varepsilon(m,n) \rceil$.

Now by relations (5.2)–(5.6), to prove the birationality of $\Phi_{\Lambda_m}|_C$, it is sufficient to prove that $|K_C + \lceil \mathcal{D}_{m,n} \rceil|$ gives a birational map. Clearly this is the case whenever $\varepsilon(m) > 2$, which in fact implies $\deg(\lceil \mathcal{D}_{m,n} \rceil) \geq \lceil \varepsilon(m,n) \rceil \geq 3$ and $K_C + \lceil \mathcal{D}_{m,n} \rceil$ is very ample.

We complete the proof.   q.e.d.

Corollary 5.6. *Keep the same notation as above. For any integer* $m > 0$, *set*

$$\varepsilon(m, 0) := \left(m + 1 - \frac{m_0}{\iota(m_0)} - m_1 \right)\zeta.$$

If $\varepsilon(m, 0) > 0$, *then*

$$\Lambda_m|_S \succeq |K_S + L_m|$$

where $L_m := (\lceil (m+1)\pi^*(-K_X) - \frac{1}{\iota(m_0)}F_{m_0} \rceil - S)|_S$.

Proof. First of all, relation (5.2) reads

(5.7)
$$|K_Y + \lceil (m+1)\pi^*(-K_X) - \frac{1}{\iota(m_0)}F_{m_0} \rceil| \preceq |K_Y + \lceil (m+1)\pi^*(-K_X) \rceil|.$$

In fact, as long as $\varepsilon(m, 0) > 0$, the front part of the proof of Theorem 5.5 is valid. In explicit, surjective map (5.3) reads the following surjective map

(5.8)
$$H^0(Y, K_Y + \lceil(m+1)\pi^*(-K_X) - \frac{1}{\iota(m_0)}F_{m_0}\rceil) \longrightarrow H^0(S, K_S + L_m)$$

where

(5.9) $$L_m := (\lceil(m+1)\pi^*(-K_X) - \frac{1}{\iota(m_0)}F_{m_0}\rceil - S)|_S.$$

Hence we have proved the statement.   q.e.d.

5.3. Applications.

In order to apply Theorem 5.5, we need to verify Assumption 5.4 and $\varepsilon(m) > 2$ in advance, for which one of the crucial steps is to estimate the lower bound of ζ.

Proposition 5.7 (cf. [**8**, Theorem 3.2]). *Let $m > 0$ be an integer. Keep the same notation as in Subsection 5.2.*

  (i) *If $g(C) > 0$ and $\varepsilon(m) > 1$, then $\zeta \geq \frac{2g(C)-2+\lceil\varepsilon(m)\rceil}{m}$;*
  (ii) *Moreover, if $g(C) > 0$, then*
$$\zeta \geq \frac{2g(C)-1}{\mu_0 + m_1};$$

  (iii) *If $g(C) = 1$, then $\zeta \geq \frac{1}{r_{\max}}$, where $r_{\max} = \max\{r_i \in B_X\}$ is the maximum of local indices of singularities;*
  (iv) *If $g(C) = 0$, then $\zeta \geq 2$;*
  (v) *If $h^0(-\nu K_X) > 0$ for some integer ν, then $\zeta \geq \frac{1}{\nu r_{\max}}$.*

Proof. (i). In the proof of Theorem 5.5, if $g(C) > 0$ and $\varepsilon(m) > 1$ then $|K_C + \lceil\mathcal{D}_{m,n}\rceil|$ is base point free with

$$\deg(K_C + \lceil\mathcal{D}_{m,n}\rceil) \geq 2g(C) - 2 + \lceil\varepsilon(m,n)\rceil = 2g(C) - 2 + \lceil\varepsilon(m)\rceil.$$

Denote by $\mathcal{N}_m$ the movable part of $|K_S + \lceil\mathcal{L}_{m,n} - H\rceil|$. Noting the relations (5.2)–(5.6) while applying [**7**, Lemma 2.7], we get

$$m\pi^*(-K_X)|_S \geq M_{-m}|_S \geq \mathcal{N}_m$$

and $\mathcal{N}_m|_C \geq K_C + \lceil\mathcal{D}_{m,n}\rceil$ since the latter one is base point free. So we have

$$m\zeta = m\pi^*(-K_X)|_S \cdot C \geq \mathcal{N}_m \cdot C \geq \deg(K_C + \lceil\mathcal{D}_{m,n}\rceil).$$

Hence

$$m\zeta \geq 2g(C) - 2 + \lceil\varepsilon(m)\rceil.$$

(ii). Take $m' = \min\{m \mid \varepsilon(m) > 1\}$, then (i) implies $\zeta \geq \frac{2g(C)}{m'}$. We may assume that $m' > \mu_0 + m_1$ otherwise $\zeta \geq \frac{2g(C)}{\mu_0 + m_1}$. Hence

$$\varepsilon(m' - 1) = (m' - 1 + 1 - \mu_0 - m_1)\zeta$$

$$\geq (m' - \mu_0 - m_1)\frac{2g(C)}{m'}.$$

By the minimality of m', it follows that $\varepsilon(m' - 1) \leq 1$. Hence $m' \leq \frac{2g(C)}{2g(C)-1}(\mu_0 + m_1)$. Then

$$\zeta \geq \frac{2g(C)}{m'} \geq \frac{2g(C) - 1}{\mu_0 + m_1}.$$

(iii). If $g(C) = 1$, then

$$\zeta = (\pi^*(-K_X) \cdot C)_Y = ((-K_Y + E_\pi) \cdot C)_Y$$

$$= (-(K_Y + S) \cdot C + S \cdot C + E_\pi \cdot C)_Y$$

$$= (-K_S \cdot C)_S + (S \cdot C + E_\pi \cdot C)_Y$$

$$= (C^2)_S + (S \cdot C + E_\pi \cdot C)_Y.$$

Since C is free on surface S, $(C^2)_S$, $(S \cdot C)_Y$, and $(E_\pi \cdot C)_Y$ are non-negative. Since $(C^2)_S$ and $(S \cdot C)_Y$ are integers, we may assume $(C^2)_S = (S \cdot C)_Y = 0$ otherwise $\zeta \geq 1$. Hence $\zeta = E_\pi \cdot C$.

On the other hand, take $q : W \to X$ is the resolution of isolated singularities and we may assume that Y dominates W by $p : Y \to W$. Then we write

$$K_W = q^* K_X + \Delta.$$

Here

$$\Delta = \sum \frac{a_i}{r_i} E_i$$

where E_i is the exceptional divisor over an isolated singular point of index r_i for some $r_i \in B_X$ and a_i is a positive integer. Then

$$E_\pi = K_Y - p^* K_W + p^* \Delta.$$

Take $r_{\max} = \max\{r_i\}$. Then all the coefficients of E_π are at least $\frac{1}{r_{\max}}$ since $K_Y - p^* K_W$ is integral effective and

$$\text{Supp}(E_\pi) = \text{Supp}(K_Y - p^* K_W + p_*^{-1}\Delta).$$

By $E_\pi \cdot C = \zeta > 0$, we know that there is at least one component E of E_π such that $E \cdot C > 0$. Then $E_\pi \cdot C \geq \frac{1}{r_{\max}} E \cdot C \geq \frac{1}{r_{\max}}$.

(iv). If $g(C) = 0$, then

$$\zeta = (\pi^*(-K_X)|_S \cdot C)_S = ((-K_Y + E_\pi)|_S \cdot C)_S$$

$$\geq (-K_Y|_S \cdot C)_S \geq (-K_S \cdot C)_S \geq -\deg(K_C) = 2.$$

(v). If $h^0(-\nu K_X) > 0$ for some integer ν, then $-\nu K_X \sim D$ for some effective Weil divisor D. Similarly as (iii), $\pi^* D$ is an effective $\mathbb{Q}$-divisor with all the coefficients at least $\frac{1}{r_{\max}}$. By $\pi^* D \cdot C = \nu \zeta > 0$, we know

that there is at least one component D_1 of π^*D such that $D_1 \cdot C > 0$. Then $\zeta = \frac{1}{\nu}\pi^*D \cdot C \geq \frac{1}{\nu r_{\max}}D_1 \cdot C \geq \frac{1}{\nu r_{\max}}$.   q.e.d.

To verify Assumption 5.4(1), we have the following proposition.

Proposition 5.8 (cf. [**8**, Proposition 3.6]). *Let X be a weak $\mathbb{Q}$-Fano 3-fold. Keep the same notation as Subsection 5.2. Then Assumption 5.4(1) is satisfied for all*

$$m \geq \begin{cases} m_0 + 6, & \text{if } m_0 \geq 2; \\ 2, & \text{if } m_0 = 1. \end{cases}$$

Proof. We have

$$K_Y + \lceil (m+1)\pi^*(-K_X) \rceil$$
$$\geq K_Y + \lceil (m-m_0+1)\pi^*(-K_X) + M_{-m_0} \rceil$$
$$= (K_Y + \lceil (m-m_0+1)\pi^*(-K_X) \rceil) + M_{-m_0}$$
$$\geq M_{-m_0}.$$

The last inequality is due to

$$h^0(K_Y + \lceil (m-m_0+1)\pi^*(-K_X) \rceil) = h^0(-(m-m_0)K_X) > 0$$

by Lemma 5.2 and [**8**, Appendix], since $m - m_0 \geq 6$ whenever $m_0 \geq 2$ (resp. ≥ 1 whenever $m_0 = 1$).

When $f : Y \to \Gamma$ is of type (f_{np}), [**20**, Lemma 2] implies that Λ_m can distinguish different generic irreducible elements of $|M_{-m_0}|$. When f is of type (f_{p}), since the rational (i.e. $\Gamma \cong \mathbb{P}^1$) pencil $|M_{-m_0}|$ can already separate different fibers of f, Λ_m can naturally distinguish different generic irreducible elements of $|M_{-m_0}|$.   q.e.d.

It is slightly more complicated to verify Assumption 5.4(2).

Lemma 5.9 (cf. [**8**, Lemma 3.7]). *Let T be a nonsingular projective surface with a base point free linear system $|G|$. Let Q be an arbitrary $\mathbb{Q}$-divisor on T. Denote by C a generic irreducible element of $|G|$. Then the linear system $|K_T + \lceil Q \rceil + G|$ can distinguish different generic irreducible elements of $|G|$ under one of the following conditions:*

(i) *$|G|$ is not composed with an irrational pencil of curves and $K_T + \lceil Q \rceil$ is effective;*
(ii) *$|G|$ is composed with an irrational pencil of curves, $g(C) > 0$, and Q is nef and big;*
(iii) *$|G|$ is composed with an irrational pencil of curves, $g(C) = 0$, Q is nef and big, and $Q \cdot G > 1$.*

Proof. The statement corresponding to (i) follows from [**20**, Lemma 2] and the fact that a rational pencil can automatically separate its different generic irreducible elements.

For situations (ii) and (iii), we pick a generic irreducible element C of $|G|$. Then, since $h^0(S, G) \geq 2$, $G \equiv sC$ for some integer $s \geq 2$ and $C^2 = 0$. Denote by C_1 and C_2 two irreducible elements of $|G|$ such that $C_1 + C_2 \leq |G|$. Then Kawamata–Viehweg vanishing theorem gives the surjective map

$$H^0(T, K_T + \lceil Q \rceil + G) \longrightarrow H^0(C, K_{C_1} + D_1) \oplus H^0(C_2, K_{C_2} + D_2)$$

where $D_i := (\lceil Q \rceil + G - C_i)|_{C_i}$ with $\deg(D_i) \geq Q \cdot C_i > 0$ for $i = 1, 2$.

If $g(C) > 0$, Riemann–Roch formula gives $h^0(C_i, K_{C_i} + D_i) > 0$ for $i = 1, 2$. Thus $|K_T + \lceil Q \rceil + G|$ can distinguish C_1 and C_2.

If $g(C) = 0$ and $Q \cdot C > 1$, then $h^0(C_i, K_{C_i} + D_i) > 0$ for $i = 1, 2$. So $|K_T + \lceil Q \rceil + G|$ can also distinguish C_1 and C_2.   q.e.d.

Proposition 5.10 (cf. [**8**, Propositions 3.8, 3.9]). *Let X be a weak $\mathbb{Q}$-Fano 3-fold. Keep the same notation as in Subsection 5.2. Then Assumption 5.4(2) is satisfied for all*

$$m \geq \begin{cases} m_0 + m_1 + 6, & \text{if } m_0 \geq 2; \\ m_1 + 2, & \text{if } m_0 = 1. \end{cases}$$

Proof. Assuming $m \geq m_0 + m_1$, we have $\varepsilon(m, 0) > 0$, and Corollary 5.6 implies that

$$\Lambda_m|_S \succeq |K_S + L_m|.$$

It suffices to prove that $|K_S + L_m|$ can distinguish different generic irreducible elements of $|G|$.

For a suitable integer $m > 0$, we have the following relations:

$$K_S + L_m = (K_Y)|_S + \lceil (m+1)\pi^*(-K_X) - \frac{1}{\iota(m_0)} F_{m_0} \rceil|_S$$

$$\geq (K_Y + \lceil (m + 1 - m_0 - m_1)\pi^*(-K_X) \rceil)|_S + M_{-m_1}|_S.$$

Thus, if $|G|$ is not composed with an irrational pencil of curves, $|K_S + L_m|$ can distinguish different irreducible elements provided that

$$K_Y + \lceil (m + 1 - m_0 - m_1)\pi^*(-K_X) \rceil$$

is effective, which holds for $m - m_0 - m_1 \geq 6$ whenever $m_0 \geq 2$ (resp. ≥ 1 whenever $m_0 = 1$) by [**8**, Appendix].

Assume $|G|$ is composed with an irrational pencil of curves. we have

$$K_S + L_m \geq K_S + \lceil ((m+1)\pi^*(-K_X) - \frac{1}{\iota(m_0)} F_{m_0} - S)|_S \rceil$$

$$\geq K_S + \lceil ((m - m_1 + 1)\pi^*(-K_X) - \frac{1}{\iota(m_0)} F_{m_0} - S)|_S \rceil + M_{-m_1}|_S.$$

We can take $Q = ((m - m_1 + 1)\pi^*(-K_X) - \frac{1}{\iota(m_0)} F_{m_0} - S)|_S$ in Lemma 5.9 since $\varepsilon(m, 0) > 0$.

If $g(C) > 0$, Lemma 5.9(ii) implies that Assumption 5.4(2) is satisfied for $m \geq m_0 + m_1$.

If $g(C) = 0$, by Lemma 5.9(iii), we need the condition $\varepsilon(m, 0) = (m + 1 - \frac{m_0}{\iota} - m_1)\zeta = Q \cdot C > 1$. But this holds automatically for $m \geq m_0 + m_1$ by Proposition 5.7(iv).

We complete the proof.   q.e.d.

Now we can treat the birationality of φ_{-m} using Theorem 5.5.

Theorem 5.11 (cf. [**8**, Theorems 4.1, 4.2, 4.5]). *Let X be a weak $\mathbb{Q}$-Fano 3-fold. Let ν_0 be an integer such that $h^0(-\nu_0 K_X) > 0$. Keep the same notation as in Subsection* 5.2. *Then φ_{-m} is birational onto its image if one of the following holds:*

(i) $m \geq \max\{m_0 + m_1 + a(m_0), \lfloor 3\mu_0 \rfloor + 3m_1\}$;
(ii) $m \geq \max\{m_0 + m_1 + a(m_0), \lfloor \frac{5}{3}\mu_0 + \frac{5}{3}m_1 \rfloor, \lfloor \mu_0 \rfloor + m_1 + 2r_{\max}\}$;
(iii) $m \geq \max\{m_0 + m_1 + a(m_0), \lfloor \mu_0 \rfloor + m_1 + 2\nu_0 r_{\max}\}$,

where $a(m_0) = \begin{cases} 6, & \text{if } m_0 \geq 2; \\ 1, & \text{if } m_0 = 1. \end{cases}$

Proof. By Propositions 5.8 and 5.10, Assumption 5.4 is satisfied if $m \geq m_0 + m_1 + a(m_0)$.

By Proposition 5.7(v), $\zeta \geq \frac{1}{\nu_0 r_{\max}}$. If $m \geq \lfloor \mu_0 \rfloor + m_1 + 2\nu_0 r_{\max}$, then $\varepsilon(m) = (m + 1 - \mu_0 - m_1)\zeta > 2$, which implies (iii).

For (i) and (ii), we will discuss on the value of $g(C)$.

Case 1. $g(C) = 0$.
By Proposition 5.7(iv), $\zeta \geq 2$. If $m \geq \lfloor \mu_0 \rfloor + m_1 + 1$, then $\varepsilon(m) = (m + 1 - \mu_0 - m_1)\zeta > 2$.

Case 2. $g(C) \geq 2$.
By Proposition 5.7(ii), $\zeta \geq \frac{3}{\mu_0 + m_1}$. If $m \geq \lfloor \frac{5}{3}\mu_0 + \frac{5}{3}m_1 \rfloor$ then $\varepsilon(m) \geq (m + 1 - \mu_0 - m_1)\zeta > 2$.

Case 3. $g(C) = 1$.
By Proposition 5.7(ii), $\zeta \geq \frac{1}{\mu_0 + m_1}$. If $m \geq \lfloor 3\mu_0 \rfloor + 3m_1$, then $\varepsilon(m) = (m + 1 - \mu_0 - m_1)\zeta > 2$. So we have proved (i). On the other hand, by Proposition 5.7(iii), $\zeta \geq \frac{1}{r_{\max}}$. If $m \geq \lfloor \mu_0 \rfloor + m_1 + 2r_{\max}$, then $\varepsilon(m) = (m + 1 - \mu_0 - m_1)\zeta > 2$. Thus (ii) is proved.   q.e.d.

In practice, usually we just use the fact $\mu_0 \leq \frac{m_0}{\iota(m_0)} \leq m_0$. For very few cases, we will utilize a precise upper bound of μ_0 rather than m_0 by Remark 5.3.

Theorem 5.11 is optimal in some cases due to the following examples.

Example 5.12 ([**10**, List 16.6]). Consider general weighted hypersurface $X_{6d} \subset \mathbb{P}(1, a, b, 2d, 3d)$ where $1 \leq a \leq b$ and $d = a + b$ such that X_{6d} is a $\mathbb{Q}$-Fano 3-fold with $r_{\max} = d$. By [**10**, List 16.6], there are exactly 12 such examples. Then φ_{-3d} is birational onto its image but $\varphi_{-(3d-1)}$ is not by the structure. On the other hand, We can take

$\nu_0 = 1$, $m_0 = \mu_0 = a$, and $m_1 = b$, then

$$3d = \lfloor 3\mu_0 \rfloor + 3m_1$$
$$= \lfloor \mu_0 \rfloor + m_1 + 2r_{\max}$$
$$= \lfloor \mu_0 \rfloor + m_1 + 2\nu_0 r_{\max}.$$

Hence Theorem 5.11 tells that φ_{-m} is birational onto its image for all $m \geq 3d$.

Theorem 5.11 directly implies the following result which generalizes a result of Fukuda [**9**, Main theorem].

Corollary 5.13. *Let X be a weak $\mathbb{Q}$-Fano 3-fold with Gorenstein singularities. Then φ_{-m} is birational onto its image for all $m \geq 4$.*

Proof. By Reid's formula, $P_{-1} = \frac{1}{2}(-K_X^3) + 3 > 3$. Hence we can take $m_0 = \nu_0 = 1$.

If $|-K_X|$ is not composed with a pencil, then we can take $m_1 = 1$ and $\mu_0 \leq m_0 = 1$. The result follows directly from Theorem 5.11(iii).

If $|-K_X|$ is composed with a pencil, then $\mu_0 \leq \frac{m_0}{\iota(m_0)} < \frac{1}{2}$. By Reid's formula again, $P_{-2} = \frac{5}{2}(-K_X^3) + 5 > r_X(-K_X^3)2 + 1$. We can take $m_1 = 2$ by Corollary 4.2. The result follows directly from Theorem 5.11(iii). q.e.d.

5.4. Proof of Theorems 1.6 and 1.8.

Now we prove the main results on the birationality of φ_{-m}.

Proof of Theorem 1.6. To apply Theorem 5.11, we always use the fact $\mu_0 \leq m_0$. By [**4**, Theorem 1.1] and Theorem 1.4, we can take $m_0 \leq 8$ and $m_1 \leq 10$ to apply Theorem 5.11(i) and (ii). Hence $m_0 + m_1 + 6 \leq 24$ and $\frac{5}{3}(m_0 + m_1) \leq 30$. By Theorem 5.11, it is sufficient to prove that either $3m_0 + 3m_1 \leq 39$ or $m_0 + m_1 + 2r_{\max} \leq 39$ holds if we choose suitable m_0 and m_1. (Note that ν_0 is not used in this proof.)

Case 1. $P_{-1} \geq 2$.

In this case, we can take $m_0 = 1$ and $m_1 \leq 6$ by Theorem 3.8. Hence $3m_0 + 3m_1 \leq 21$.

Case 2. $P_{-1} = 1$.

Recall the proof of Theorem 3.10. We take $m_0 = n_0$. If $m_0 \leq 5$, then we can take $m_1 \leq 7$ and hence $3m_0 + 3m_1 \leq 36$. Similarly, if $m_0 = 6$ and if we can take $m_1 \leq 7$, then $3m_0 + 3m_1 \leq 39$.

If $m_0 = n_0 = 6$ and $\delta_1(X) = 8$, we can take $m_1 = 8$. Theorem 3.4 implies that

$$P_{-1} = P_{-2} = P_{-3} = P_{-4} = P_{-5} = 1, P_{-6} = P_{-7} = 2.$$

Then $n_{1,2}^0 = 2$, $n_{1,3}^0 = 2$, $n_{1,4}^0 = 2 - \sigma_5$, $\epsilon_5 = 2 - \sigma_5$, $0 = \epsilon_6 = 3 - \epsilon$. Hence $\epsilon = 3$ and $\sigma_5 \leq 2$, and this implies $(\sigma_5, n_{1,5}^0) = (2, 1)$. Then $\epsilon_5 = 0$

and $B^{(5)}(B) = \{2 \times (1,2), 2 \times (1,3), (1,5), (1,s)\}$ for some $s \geq 6$. This implies $\epsilon_7 = 0$ since there are no further packings. On the other hand, $\epsilon_7 = 2 - 2\sigma_5 + 2n^0_{1,5} + n^0_{1,6}$. Hence $n^0_{1,6} = 0$ and $B^{(7)} = \{2 \times (1,2), 2 \times (1,3), (1,5), (1,s)\}$ with $s \geq 7$. Since $B^{(7)}$ admits no prime packings, $B = B^{(7)}$. By inequalities (3.3) and (3.4), s can only be $8, 9, 10$. Hence $m_0 + m_1 + 2r_{\max} \leq 6 + 8 + 2 \times 10 = 34$.

If $m_0 = n_0 \geq 7$, then

$$P_{-1} = P_{-2} = P_{-3} = P_{-4} = P_{-5} = P_{-6} = 1.$$

The proof of Theorem 3.10 implies $B^{(5)} = \{(1,2), (2,5), (1,3), (1,4), (1,s)\}$ with $s \geq 6$. Since $\gamma(B^{(5)}) > 0$, we have $s \leq 11$. Noting that B is dominated by $B^{(5)}$, we see $r_{\max} \leq 11$. By Theorem 3.10, we can take $m_0 \leq 8$ and $m_1 \leq 9$. Hence $m_0 + m_1 + 2r_{\max} \leq 8 + 9 + 2 \times 11 = 39$.

Case 3. $P_{-1} = P_{-2} = 0$.

By the proof of Theorems 3.12 and 3.15, if B is of type No.1, No.2 or No.4, then we have $r_{\max} \leq 10$ and may take $m_0 = 8$, $m_1 = 10$. Hence $m_0 + m_1 + 2r_{\max} \leq 8 + 10 + 2 \times 10 = 38$. If B is of type No.5–No.6, then we have $r_{\max} \leq 7$ and may take $m_0 = 7$, $m_1 = 8$. Hence $m_0 + m_1 + 2r_{\max} \leq 7 + 8 + 2 \times 7 = 29$. If B is of type No.7–No.23, then we can take $m_0 = m_1 = 6$. Hence $3m_0 + 3m_1 \leq 36$. Now the remaining case is type No.3:

$$\{5 \times (1,2), 2 \times (1,3), (3,11)\}.$$

Recall that $P_{-8} = P_{-9} = 2$ and $-4K_X \sim E$ is a prime divisor by the proof of Theorem 3.15(i). By the proof of Theorem 3.4, $|-8K_X|$ has no fixed part. If $|-8K_X|$ and $|-9K_X|$ are composed with a same pencil, we can write

$$|-8K_X| = |S'|,$$
$$|-9K_X| = |S'| + F,$$

where F is the fixed part. This implies that

$$-K_X \sim -9K_X - (-8K_X) = F,$$

which contradicts $P_{-1} = 0$. Hence $|-8K_X|$ and $|-9K_X|$ are composed with different pencils, and we can take $m_0 = 8$, $m_1 = 9$, and $m_0 + m_1 + 2r_{\max} = 39$.

Case 4. $P_{-1} = 0, P_{-2} > 0$.

By [**4**, Proposition 3.10, Case 1], we can take $m_0 = 6$. We can take m_1 the same as in the proof of Theorems 3.12 and 3.15. If $m_1 \leq 6$, then $3m_0 + 3m_1 \leq 36$. If $m_1 \geq 7$, observing Subsubcases II-3-ii and II-3-iii in the proof of Theorem 3.12, we can see that $r_{\max} \leq 11$ holds for any such basket except

$$B_d = \{4 \times (1,2), (6,13), (1,5)\}.$$

Except for B_d, we have $m_0 + m_1 + 2r_{\max} \leq 6 + 8 + 2 \times 11 = 36$. Now we deal with B_d. We claim that we can take $m_1 = 7$. Recall that

$$P_{-1} = P_{-3} = 0, P_{-2} = P_{-4} = P_{-5} = 1, P_{-6} = P_{-7} = 2.$$

Clearly $|-6K_X|$ and $|-7K_X|$ are both composed with pencils. We only need to show that they are composed with different pencils. To the contrary, we assume that $|-6K_X|$ and $|-7K_X|$ are composed with the same pencil. If $-2K_X \sim D$ is a prime divisor, then by the proof of Theorem 3.4, $|-6K_X|$ has no fixed part. By assumption, we can write

$$|-6K_X| = |S'|,$$
$$|-7K_X| = |S'| + F,$$

where F is the fixed part. This implies that

$$-K_X \sim -7K_X - (-6K_X) = F,$$

a contradiction. Hence $-2K_X \sim D$ is not a prime divisor. By the proof of Theorem 3.15(ii), $D = E_1 + E_2$ with E_1 and E_2 different prime divisors. Also we can write

$$|-6K_X| = |S'| + a_6 E_1,$$
$$|-7K_X| = |S'| + F,$$

where $a_6 E_1$ and F are the fixed parts with $a_6 \leq 3$. If $a_6 \leq 1$, then

$$S' \sim 3(E_1 + E_2) - a_6 E_1 \geq 2E_1 + 2E_2 \sim -4K_X.$$

This implies $|-7K_X| \succeq |-4K_X|$, which contradicts $P_{-3} = 0$. If $a_6 = 3$, as in the proof of Theorem 3.2, take $m = 6$ and $E = E_1$ or $2E_1$ or $3E_1$, inequality (3.1) must fail for some singularity P in B_d. Clearly, such an offending singularity P must be "$(6, 13)$". By equality (3.2), the local index $i_P(E)$ of E can only be 9 or 11 since inequality (3.1) holds for other $0 \leq i \leq 12$ and $(b, r) = (6, 13)$. But clearly the local index $i_P(E_1)$, $i_P(2E_1)$, and $i_P(3E_1)$ can not be in the set $\{9, 11\}$ simultaneously, a contradiction. Finally we consider the case $a_6 = 2$. Write $-5K_X \sim B$ a fixed divisor. Then

$$B + S' + 2E_1 \sim -5K_X - 6K_X \sim -4K_X - 7K_X \sim 2E_1 + 2E_2 + S' + F,$$

that is, $B \sim 2E_2 + F$. Obviously, $F \neq 0$. As in the proof of Theorem 3.2, take $m = 5$ and $E = E_1$ or $2E_1$, inequality (3.1) must fails for some singularity P in B_d. Clearly, such an offending singularity P must be "$(6, 13)$". By equality (3.2), the local index $i_P(E)$ of E can only be 10 or 11 since inequality (3.1) holds for other $0 \leq i \leq 12$ and $(b, r) = (6, 13)$. But clearly the local index $i_P(E_1)$, $i_P(2E_1)$ can not be in the set $\{10, 11\}$ simultaneously, a contradiction.

We complete the proof.   q.e.d.

Proof of Theorem 1.8. We shall apply Theorem 5.11 to treat arbitrary weak $\mathbb{Q}$-Fano 3-folds. We will choose suitable m_0 and m_1. Unless otherwise specified, we will use the fact $\mu_0 \leq m_0$.

Case I. $P_{-2} = 0$.

In this case, the possible baskets are classified in Proposition 3.14. From the list we can take $m_0 = 8$. We have $r_X \leq 210$, $-K_X^3 \geq \frac{1}{84}$, and $r_{\max} \leq 14$. By Proposition 4.3 with $t = 8$, we can take $m_1 = 38$. Hence by Theorem 5.11(ii), φ_{-m} is birational onto its image for all $m \geq 76$.

Case II. $r_{\max} \geq 14$.

Write Reid's basket B_X as

$$\left\{ (b_i, r_i) \mid i = 1, \cdots, s; 0 < b_i \leq \frac{r_i}{2}; b_i \text{ is coprime to } r_i \right\}.$$

Recall that $r_X = \text{l.c.m.}\{r_i \mid i = 1, \cdots, s\}$ and that

$$\sum_i \left(r_i - \frac{1}{r_i} \right) \leq 24$$

by inequality (2.1). We recall the sequence $\mathcal{R} = (r_i)_i$ from the proof of Proposition 2.4. Denote by $\tilde{r}_1 = r_{\max}$ the largest value in $\mathcal{R}$, by $\tilde{r}_2$ the second largest value, and by $\tilde{r}_3$, $\tilde{r}_4$ the third, the forth, and so on. For instance, if $\mathcal{R} = (2, 3, 4, 4, 5, 5)$, then $\tilde{r}_1 = 5$, $\tilde{r}_2 = 4$, $\tilde{r}_3 = 3$, and $\tilde{r}_4 = 2$. If the value $\tilde{r}_j$ does not exist by definition, then we set $\tilde{r}_j = 1$. In the previous example, we have $\tilde{r}_5 = 1$.

Clearly $r_{\max} \leq 24$. We will compute an explicit bound for r_X.

If $r_{\max} \geq 23$, then by inequality (2.1), there are no more values in $\mathcal{R}$. Hence $r_X \leq 24$.

If $20 \leq r_{\max} \leq 22$, then by inequality (2.1), $\tilde{r}_2 \leq 4$. Hence

$$r_X \leq \text{l.c.m}(r_{\max}, 4, 3, 2) = 132.$$

If $r_{\max} = 19$, then by inequality (2.1), $\tilde{r}_2 \leq 5$, and at most one of $3, 4, 5$ can be in $\mathcal{R}$. Hence $r_X \leq 19 \times 5 \times 2 = 190$.

If $r_{\max} = 18$, then by inequality (2.1), $\tilde{r}_2 \leq 6$, and at most one of $3, 4, 5, 6$ can be in $\mathcal{R}$. Hence $r_X \leq 18 \times 5 = 90$.

If $r_{\max} = 17$, then by inequality (2.1), $\tilde{r}_2 \leq 7$. If $\tilde{r}_2 \geq 5$, then by inequality (2.1), $\tilde{r}_3 \leq 2$ and hence $\tilde{r}_X \leq 17 \times 7 \times 2 = 238$. If $\tilde{r}_2 \leq 4$, then $r_X \leq \text{l.c.m}(17, 4, 3, 2) = 204$.

If $r_{\max} = 16$, then by inequality (2.1), $\tilde{r}_2 \leq 8$. If $\tilde{r}_2 \geq 6$, then by inequality (2.1), $\tilde{r}_3 \leq 2$ and hence $r_X \leq 16 \times 7 = 112$. If $\tilde{r}_2 \leq 5$, then $r_X \leq \text{l.c.m}(16, 5, 4, 3, 2) = 240$.

If $r_{\max} = 15$, then by inequality (2.1), $\tilde{r}_2 \leq 9$. If $\tilde{r}_2 \geq 6$, then by inequality (2.1), $\tilde{r}_3 \leq 3$ and hence $r_X \leq \text{l.c.m}(r_{\max}, \tilde{r}_2, 3, 2) \leq 15 \times 7 \times 2 = 210$. If $\tilde{r}_2 \leq 5$, then $r_X \leq \text{l.c.m}(15, 5, 4, 3, 2) = 60$.

If $r_{\max} = 14$, then by inequality (2.1), $\tilde{r}_2 \leq 10$. If $\tilde{r}_2 \geq 8$, then by inequality (2.1), $\tilde{r}_3 \leq 2$ and hence $\tilde{r}_X \leq 14 \times 9 = 126$. If $\tilde{r}_2 \leq 7$, then r_X

divides l.c.m$(14, 6, 5, 4, 3, 2) = 420$. But by inequality (2.1), $5, 4, 3$ can not be in $\mathcal{R}$ simultaneously, hence $r_X < 420$. In particular, $r_X \leq 210$.

In summary, when $r_{\max} \geq 14$, we have $r_X \leq 240$.

We can take $m_0 = 8$ by [$\mathbf{4}$, Theorem 1.1]. We have $r_X \leq 240$, $-K_X^3 \geq \frac{1}{240}$ (note that $r_X K_X^3$ is an integer), and $r_{\max} \leq 24$. If $r_{\max} \leq 22$, by Proposition 4.3 with $t = 6$, we can take $m_1 = 44$. Hence by Theorem 5.11(ii), φ_{-m} is birational onto its image for all $m \geq 96$. If $r_{\max} = 23$ or 24, by Proposition 4.3 with $t = 2$, $r_X \leq 24$, $-K_X^3 \geq \frac{1}{24}$, we can take $m_1 = 37$. Hence by Theorem 5.11(ii), φ_{-m} is birational onto its image for all $m \geq 93$.

Case III. $r_{\max} < 14$ and $P_{-1} > 0$.

In this case, $\nu_0 = 1$ and by [$\mathbf{4}$, Theorem 1.1], we can take $m_0 = 8$.

If $r_X \leq 660$ and $r_{\max} \leq 12$, then by Proposition 4.3 with $t = 15$, $r_{\max} \leq 12$, and $-K_X^3 \geq \frac{1}{330}$, we can take $m_1 = 65$. Hence by Theorem 5.11(iii), φ_{-m} is birational onto its image for all $m \geq 97$.

If $r_X \leq 660$ and $r_{\max} = 13$, Then $\tilde{r}_2 \leq 11$. If $\tilde{r}_2 \geq 9$, then $\tilde{r}_3 \leq 2$ and $r_X \leq 286$. If $\tilde{r}_2 = 8$, then $\tilde{r}_3 \leq 3$ and $r_X \leq 312$. If $\tilde{r}_2 = 7$, then $\tilde{r}_3 \leq 4$ and $3, 4$ can not be in $\mathcal{R}$ simultaneously, hence $r_X \leq 546$. If $\tilde{r}_2 \leq 6$, then r_X divides 780 and hence $r_X \leq 390$ by Proposition 2.4. In summary, $r_X \leq 546$. By Proposition 4.3 with $t = 10$, $r_{\max} = 13$, and $-K_X^3 \geq \frac{1}{330}$, we can take $m_1 = 61$. Hence by Theorem 5.11(iii), φ_{-m} is birational onto its image for all $m \geq 95$.

If $r_X > 660$, then $r_X = 840$ and $r_{\max} = 8$. By Theorem 1.7, we can take $m_1 = 71$. Hence by Theorem 5.11(iii), φ_{-m} is birational onto its image for all $m \geq 95$.

Case IV. $r_{\max} < 14$, $P_{-1} = 0$, and $P_{-2} > 0$.

In this case, $\nu_0 = 2$ and by [$\mathbf{4}$, Proposition 3.10, Case 1], we can take $m_0 = 6$.

If $P_{-4} = 1$, then $P_{-2} = 1$. By the proof of Theorem 3.12 (note that the arguments on baskets are valid without assuming $\rho = 1$), we are exactly in the situation $(P_{-3}, P_{-4}) = (0, 1)$, corresponding to the last paragraph of Subsubcase II-3-iii of Theorem 3.12. In fact, the possible baskets are classified in the following list:

$$\{9 \times (1, 2), (1, 3), (1, 7)\},$$
$$\{8 \times (1, 2), (2, 5), (1, 7)\},$$
$$\{8 \times (1, 2), (2, 5), (1, 6)\},$$
$$\{7 \times (1, 2), (3, 7), (1, 6)\},$$
$$\{6 \times (1, 2), (4, 9), (1, 6)\},$$
$$\{7 \times (1, 2), (3, 7), (1, 5)\},$$
$$\{6 \times (1, 2), (4, 9), (1, 5)\},$$
$$\{5 \times (1, 2), (5, 11), (1, 5)\},$$

$$\{4 \times (1, 2), (6, 13), (1, 5)\}.$$

Hence in this case $r_X \leq 130$, $-K_X^3 \geq \frac{1}{130}$, and $r_{\max} \leq 13$. By Proposition 4.3 with $t = 7$, we can take $m_1 = 37$. Hence by Theorem 5.11(iii), φ_{-m} is birational onto its image for all $m \geq 95$.

Hence, from now on, we assume that $P_{-4} > 1$. So we may take $m_0 = 4$.

If $r_{\max} \leq 8$, then r_X divides l.c.m$(8, 7, 6, 5, 4, 3, 2) = 840$. Suppose $r_X < 840$, then $r_X \leq 420$. By Proposition 4.3 with $t = 20$ and $-K_X^3 \geq \frac{1}{330}$, we can take $m_1 = 54$. Hence by Theorem 5.11(iii), φ_{-m} is birational onto its image for all $m \geq 90$. Suppose $r_X = 840$, then $\mathcal{R} = (3, 5, 7, 8)$ or $(2, 3, 5, 7, 8)$ as we have seen in the proof of Proposition 2.4. However,

$$
\begin{aligned}
P_{-1} &= \frac{1}{2}(-K_X^3) - \sum \frac{b_i(r_i - b_i)}{2r_i} + 3 \\
(5.10) \qquad &> 3 - \frac{1}{4} - \frac{2}{6} - \frac{6}{10} - \frac{12}{14} - \frac{15}{16} > 0,
\end{aligned}
$$

a contradiction.

The above argument reminds us to find a condition corresponding to $P_{-1} = 0$. Assume that 2 is not in $\mathcal{R}$, then

$$
\begin{aligned}
P_{-1} &= \frac{1}{2}(-K_X^3) - \sum \frac{b_i(r_i - b_i)}{2r_i} + 3 \\
&> 3 - \frac{1}{8} \sum \left(r_i - \frac{1}{r_i}\right) \geq 0,
\end{aligned}
$$

a contradiction. Hence, $2 \in \mathcal{R}$.

Consider the case $r_{\max} = 9$. If $\tilde{r}_2 \leq 6$, then $r_X \leq$ l.c.m$(9, 6, 5, 4, 3, 2) = 180$. If $\tilde{r}_2 = 8$, then by inequality (2.1) and $2 \in \mathcal{R}$, $\tilde{r}_2 \leq 5$ and $r_X \leq$ l.c.m$(9, 8, 5, 4, 3, 2) = 360$. If $\tilde{r}_2 = 7$ and $5 \notin \mathcal{R}$, then

$$r_X \leq \text{l.c.m.}(9, 7, 6, 4, 3, 2) = 252.$$

If $\tilde{r}_2 = 7$ and $5 \in \mathcal{R}$, then $6 \notin \mathcal{R}$ and r_X divides l.c.m$(9, 7, 5, 4, 3, 2) = 630$. In summary, $r_X \leq 360$ or $r_X = 630$. Whenever $r_X \leq 360$, by Proposition 4.3 with $t = 12$ and $-K_X^3 \geq \frac{1}{330}$, we can take $m_1 = 50$. Hence by Theorem 5.11(iii), φ_{-m} is birational onto its image for all $m \geq 90$. Whenever $r_X = 630$, then $2, 5, 7, 9$ must be in $\mathcal{R}$. Hence $\mathcal{R} = (2, 5, 7, 9)$ or $(2, 2, 5, 7, 9)$ by inequality (2.1). In this case, by arguing as inequality (5.10), B_X can only be $\{2 \times (1, 2), (2, 5), (3, 7), (4, 9)\}$. We will choose suitable m_1 and modify the upper bound of μ_0. Since $P_{-4} = 2$, $|-4K_X|$ is composed with a pencil. Note that $P_{-7} = 10$ and $P_{-3} = 1$. If $|-7K_X|$ is not composed with a pencil, then we can take $m_1 = 7$. By Theorem 5.11(ii), φ_{-m} is birational onto its image for all $m \geq 29$. If $|-7K_X|$ is also composed with a pencil, then we know $\mu_0 \leq \frac{7}{9}$ by Remark 5.3. Also we can see $P_{-61} = 5294 > r_X(-K_X^3)61 + 1$ by direct computation using Reid's formula where $-K_X^3 = \frac{43}{315}$. Hence we can

take $m_1 = 61$ by Corollary 4.2. Hence by Theorem 5.11(iii), φ_m is birational for all $m \geq 97$.

Consider the case $r_{\max} = 10$. If $\tilde{r}_2 \leq 6$, then

$$r_X \leq \text{l.c.m}(10, 6, 5, 4, 3, 2) = 60.$$

If $\tilde{r}_2 = 7$, then r_X divides l.c.m$(10, 7, 5, 4, 3, 2) = 420$, but $3, 4$ can not be in $\mathcal{R}$ simultaneously, hence $r_X \leq 210$. If $\tilde{r}_2 = 8$, then $r_3 \leq 4$ and $r_X \leq$ l.c.m$(10, 8, 4, 3, 2) = 120$. If $\tilde{r}_2 = 9$, then $\tilde{r}_3 \leq 3$ and $r_X \leq$ l.c.m$(10, 9, 3, 2) = 90$. Hence in summary, $r_X \leq 210$. By Proposition 4.3 with $t = 10$ and $-K_X^3 \geq \frac{1}{210}$, we can take $m_1 = 39$. Hence by Theorem 5.11(ii), φ_{-m} is birational onto its image for all $m \geq 71$.

Consider the case $r_{\max} = 11$. If $\tilde{r}_2 = 10$, then $\tilde{r}_3 \leq 2$ and $r_X \leq 110$. If $\tilde{r}_2 = 9$ or 8, then $\tilde{r}_3 \leq 3$ and $r_X \leq 264$. If $\tilde{r}_2 = 7$, then $\tilde{r}_3 \leq 4$ and $3, 4$ can not be in $\mathcal{R}$ simultaneously, hence $r_X \leq 308$ or $r_X =$ l.c.m$(11, 7, 3, 2) = 462$. If $\tilde{r}_2 = 6$, then $5, 4$ can not be in $\mathcal{R}$ simultaneously, hence $r_X \leq$ l.c.m$(11, 6, 5, 3, 2) = 330$. If $\tilde{r}_2 \leq 5$, then r_X divides l.c.m$(11, 5, 4, 3, 2) = 660$. In summary, $r_X \leq 330$ or $r_X = 462$ or $r_X = 660$. Whenever $r_X = 660$, then $2, 3, 4, 5, 11$ must be in $\mathcal{R}$. Hence $\mathcal{R} = (2, 3, 4, 5, 11)$ by inequality (2.1). By arguing as inequality (5.10), this implies $P_{-1} > 0$, a contradiction. Whenever $r_X \leq 330$, by Proposition 4.3 with $t = 13$ and $-K_X^3 \geq \frac{1}{330}$, we can take $m_1 = 48$. Hence by Theorem 5.11(ii), φ_{-m} is birational onto its image for all $m \geq 86$. If $r_X = 462$, then $2, 3, 7, 11$ must be in $\mathcal{R}$. Hence $\mathcal{R} = (2, 3, 7, 11)$ or $(2, 2, 3, 7, 11)$ by inequality (2.1). By arguing as inequality (5.10), B_X can only be $\{2 \times (1, 2), (1, 3), (3, 7), (5, 11)\}$. In this case we can prove $P_{-52} = 2612 > r_X(-K_X^3)52 + 1$ by direct computation using Reid's formula where $-K_X^3 = \frac{50}{462}$. Hence we can take $m_1 = 52$. By Theorem 5.11(ii), φ_{-m} is birational onto its image for all $m \geq 93$.

Consider the case $r_{\max} = 12$. Then $\tilde{r}_2 \leq 10$ and at most one of $5, 6, 7, 8, 9, 10$ will be in $\mathcal{R}$. Hence $r_X \leq 84$. By Proposition 4.3 with $t = 5$ and $-K_X^3 \geq \frac{1}{84}$, we can take $m_1 = 37$. Hence by Theorem 5.11(ii), φ_{-m} is birational onto its image for all $m \geq 68$.

Finally, consider the case $r_{\max} = 13$. Then $\tilde{r}_2 \leq 9$. If $\tilde{r}_2 = 9$ or 8, then $\tilde{r}_3 \leq 2$ and $r_X \leq 234$. If $\tilde{r}_2 = 7$, then $\tilde{r}_3 \leq 3$ and $r_X = 546$ or 182. If $\tilde{r}_2 \leq 6$, then r_X divides 780 and hence $r_X \leq 390$ by Proposition 2.4. In summary, $r_X \leq 390$ or $r_X = 546$. Whenever $r_X \leq 390$, by Proposition 4.3 with $t = 12$ and $-K_X^3 \geq \frac{1}{330}$, we can take $m_1 = 52$. Hence by Theorem 5.11(ii), φ_{-m} is birational onto its image for all $m \geq 93$. Whenever $r_X = 546$, then $\mathcal{R} = (2, 3, 7, 13)$. Argue as inequality (5.10), B_X can only be $\{(1, 2), (1, 3), (3, 7), (6, 13)\}$. We will choose suitable m_1 and modify the upper bound of μ_0. Since $P_{-4} = 2$, $|-4K_X|$ is composed with a pencil. Note that $P_{-10} = 21$ and $P_{-6} = 5$. If $|-10K_X|$ is not composed with a pencil, then we can take $m_1 = 10$. By Theorem 5.11(ii), φ_{-m} is birational onto its image for all $m \geq 40$. If $|-10K_X|$ is

also composed with a pencil, then we know $\mu_0 \leq \frac{1}{2}$ by Remark 5.3. Also we can prove $P_{-57} = 3540 > r_X(-K_X^3)57 + 1$ by direct computation using Reid's formula where $-K_X^3 = \frac{61}{546}$. Hence we can take $m_1 = 57$. By Theorem 5.11(ii), φ_{-m} is birational onto its image for all $m \geq 95$.

We complete the proof.   q.e.d.

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