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MIN-MAX MINIMAL HYPERSURFACES IN NON-COMPACT MANIFOLDS

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Abstract

In this work we prove the existence of embedded closed minimal hypersurfaces in non-compact manifolds containing a bounded open subset with smooth and strictly mean-concave boundary and a natural behavior on the geometry at infinity. For doing this, we develop a modified min-max theory for the area functional following Almgren and Pitts' setting, to produce minimal hypersurfaces with intersecting properties. In particular, we prove that any strictly mean-concave region of a compact Riemannian manifold without boundary intersects a closed minimal hypersurface.

1. Introduction

There is no immersed closed minimal surface in the Euclidean space $\mathbb{R}^3$. This fact illustrates the existence of simple geometric conditions creating obstructions for a Riemannian manifold to admit closed minimal surfaces. In the Euclidean space, we can see the obstruction coming in the following way: by the Jordan-Brouwer separation theorem every connected smooth closed surface $\Sigma^2 \subset \mathbb{R}^3$ divides $\mathbb{R}^3$ into two components, one of them bounded, which we denote Ω. Start contracting a large Euclidean ball containing Ω until it touches Σ the first time. Let $p \in \Sigma$ be a first contact point; then the maximum principle says that the mean curvature vector of Σ at p is non-zero and points inside Ω. In particular, $\Sigma^2 \subset \mathbb{R}^3$ is not minimal.

In this work we consider two natural and purely geometric properties that imply that a complete non-compact Riemannian manifold N admits a smooth closed embedded minimal hypersurface. In order to state the result, we introduce the following notation: we say that N has the $\star_k$-*condition* if there exists $p \in N$ and $R_0 > 0$, such that

$$\sup_{q \in B(p,R)} |\mathrm{Sec}_N|(q) \leq R^k$$

and

$$\inf_{q \in B(p,R)} inj_N(q) \geq R^{-\frac{k}{2}},$$

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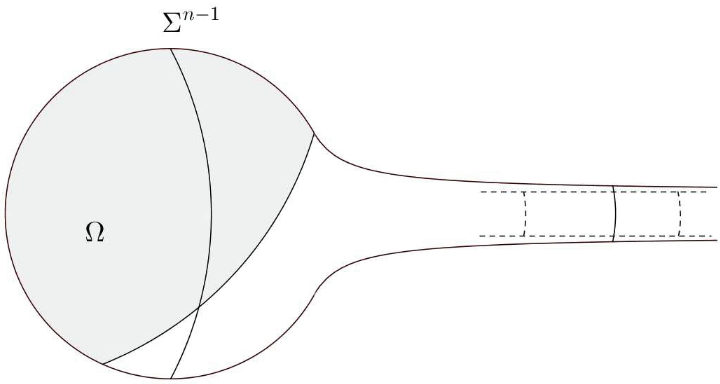

Figure 1. A Riemannian manifold with one end, which is asymptotic to a cylinder, and containing a mean-concave set. In this case, Theorem 1.1 could be applied.

for every $R \geq R_0$, where $|\mathrm{Sec}_N|(q)$ and $inj_N(q)$ denote, respectively, the maximum sectional curvature for 2-planes contained in the tangent space T_qN and the injectivity radius of N at q. For instance, if N has bounded geometry, then the $\star_k$-condition holds for every positive k.

Our main result is:

Theorem 1.1. *Let (N^n, g) be a complete non-compact Riemannian manifold of dimension $3 \leq n \leq 7$. Suppose that it:*

- *contains a bounded open subset Ω, such that $\overline{\Omega}$ is a manifold with smooth and strictly mean-concave boundary;*
- *satisfies the $\star_k$-condition, for some $k \leq \frac{2}{n-2}$.*

Then, there exists a closed embedded minimal hypersurface $\Sigma^{n-1} \subset N$ that intersects Ω.

Remark 1. The argument gives the natural extension of this result to the case of dimension higher than 7; namely, it guarantees the existence of a closed, embedded minimal hypersurface with a singular set of Hausdorff codimension at least 7. This regularity assumption relies on the curvature estimates for stable minimal hypersurfaces of Schoen-Simon-Yau [**20**] and Schoen-Simon [**21**].

In a recent work, Collin, Hauswirth, Mazet, and Rosenberg [**6**] prove that any complete non-compact hyperbolic three-dimensional manifold of finite volume admits a closed embedded minimal surface. These spaces have a different behavior at infinity from those in Theorem 1.1.

In our arguments, the geometric behavior of the ends involving the $\star_k$-condition is used together with the monotonicity formula to provide a lower bound for the $(n-1)$-dimensional volume of connected minimal hypersurfaces in N that, simultaneously, intersect Ω and contain points very far from it. More precisely, the $\star_k$-condition guarantees that a connected minimal hypersurface that intersects a compact set and goes very far must have large area. When this happpens Gromov [**10**] says that the manifold is thick at infinity.

The hypothesis involving the mean-concave bounded domain Ω comes from the theory of closed geodesics in non-compact surfaces. In 1980, Bangert proved the existence of infinitely many closed geodesics in a complete Riemannian surface M of finite area and homeomorphic to either the plane, the cylinder, or the Möbius band; see [**3**]. The first step in his argument is to prove that the finite area assumption implies the existence of locally convex neighborhoods of the ends of M.

As a motivation for our approach, we briefly discuss how Bangert uses the Lusternik-Schnirelmann theory in the case that M is homeomorphic to the plane. If $C \subset M$ is a locally convex neighborhood of the infinity of M whose boundary $\partial C \neq \varnothing$ is not totally geodesic, he proves that M contains infinitely many closed geodesics intersecting $M - C$. To obtain one such curve, the idea is to apply the Lusternik-Schnirelmann technique for a class Π of paths β defined on $[0, 1]$ and taking values in a finite-dimensional subspace of the space of piecewise C^1 closed curves, with the properties that β_0 and β_1 have images in the interior of C, being β_0 non-contractible in C and β_1 contractible in C. The min-max invariant in this case is the number

$$L(\Pi) = \inf_{\beta\in\Pi} \sup\{E(\beta_t) : \beta_t(S^1) \cap (M - \mathring{C}) \neq \varnothing\},$$

where $E(\gamma)$ denotes the energy of $\gamma : S^1 \to M$, which is defined by

$$E(\gamma) = \int_0^1 |\gamma'(s)|^2 ds.$$

Then, he achieves this min-max width, $L(\Pi)$, as the energy of a closed geodesic that intersects the region $M - C$ of the surface M.

In order to prove Theorem 1.1, we develop a min-max method that is adequate to produce minimal hypersurfaces with intersecting properties in manifolds of dimension greater than or equal to three. Although our main result concern existence of minimal hypersurfaces in non-compact manifolds, our argument proves also that the same conclusion of Theorem 1.1 holds for compact manifolds. Let us briefly describe our technique. Let (M^n, g) be a closed Riemannian manifold and Ω be an open subset of M. Consider a homotopy class Π of one-parameter sweepouts of M by codimension-one submanifolds. For each given sweepout $S = \{\Sigma_t\}_{t\in[0,1]} \in \Pi$, we consider the number

$$L(S, \Omega) = \sup\{\mathcal{H}^{n-1}(\Sigma_t) : \Sigma_t \cap \overline{\Omega} \neq \varnothing\},$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure associated with the Riemannian metric. Define the width of Π with respect to Ω to be

$$L(\Pi, \Omega) = \inf\{L(S, \Omega) : S \in \Pi\}.$$

More precisely, our min-max technique is inspired by the discrete setting of Almgren and Pitts. The original method was introduced in [**2**] and [**19**] between the 1960's and 1980's, and has been used recently by Marques and Neves to answer deep questions in geometry; see [**14**] and [**15**]. The method consists of applications of variational techniques for the area functional. It is a powerful tool in the production of unstable minimal surfaces in closed manifolds. For instance, Marques and Neves, in the proof of the Willmore conjecture, proved that the Clifford Torus in the three-sphere is a min-max minimal surface. The min-max technique for the area functional appears also in a different setting, as introduced by Simon and Smith in the unpublished work [**18**], or in the survey paper [**4**] by Colding and De Lellis. Other recent developments on this theory can be found in [**7**], [**8**], [**12**], [**13**], and [**23**].

In the Almgren and Pitts discrete setting, Π is a homotopy class in $\pi_1^\#(\mathcal{Z}_{n-1}(M; \mathbf{M}), \{0\})$. We define the width of Π with respect to Ω, $\mathbf{L}(\Pi, \Omega)$, following the same principle as above. Then, we prove:

Theorem 1.2. *Let (M^n, g) be a closed Riemannian manifold, $3 \leq n \leq 7$, and $\Pi \in \pi_1^\#(\mathcal{Z}_{n-1}(M; \boldsymbol{M}), \{0\})$ be a non-trivial homotopy class. Suppose that M contains an open subset Ω, such that $\overline{\Omega}$ is a manifold with smooth and strictly mean-concave boundary. There exists a stationary integral varifold Σ whose support is a smooth embedded closed minimal hypersurface intersecting Ω and with $||\Sigma||(M) = \boldsymbol{L}(\Pi, \Omega)$.*

Remark 2. In the same spirit of Remark 1, Theorem 1.2 can be extended to higher dimensions, allowing the type of singularities which are already present in the solutions of the Plateau problem.

Consider the unit three-sphere $S^3 \subset \mathbb{R}^4$ and assume that points in $\mathbb{R}^4$ have normal coordinates (x, y, z, w). Let Ω be a mean-concave subset of S^3. The min-max minimal surface Σ produced from our method, starting with the homotopy class Π of the standard sweepout $\{\Sigma_t\}$ of S^3, $\Sigma_t = \{w = t\}$ for $t \in [-1, 1]$, is a great sphere. Indeed, we have

$$\mathbf{L}(\Pi, \Omega) \leq \max\{\mathcal{H}^2(\Sigma_t) : t \in [-1, 1]\} = 4\pi.$$

This allows us to conclude that Σ must be a great sphere, because these are the only minimal surfaces in S^3 with area less than or equal to 4π.

The intersecting condition in Theorem 1.2 is optimal in the sense that it is possible that the support of Σ is not entirely in $\overline{\Omega}$. We illustrate this with two examples of mean-concave subsets of S^3 containing no great sphere. The first example is the complement of three spherical geodesic balls, which can be seen in Figure 2.

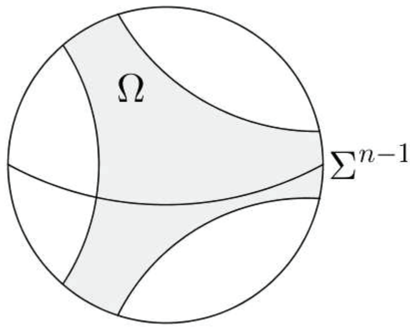

Figure 2. Example of mean-concave domain $\Omega \subset S^3$ for which Σ^{n-1} given by Theorem 1.2 is not entirely inside Ω.

In order to introduce the second example, for each $0 < t < 1$, consider the subset of S^3 given by

$$\Omega(t) = \{(x, y, z, w) \in S^3 : x^2 + y^2 > t^2\}.$$

It is not hard to see that no $\Omega(t)$ contains great spheres. Moreover, the boundary of $\Omega(t)$ is a constant mean-curvature torus in S^3 and, if $t < 1/\sqrt{2}$, its mean-curvature vector points outside $\Omega(t)$. In this case, $\Omega = \Omega(t)$ is a mean-concave subset of S^3 that contains no great spheres.

In future work, we will adapt our methods to the set up of Simon and Smith. This different approach is important for a better understanding of the geometrical and topological behavior of the produced surfaces.

Organization. The content of this paper is organized as follows:

In Section 2, we outline the main ideas to prove Theorem 1.2, avoiding the technical details and reducing the use of the language of geometric measure theory to a minimum.

In Section 3, we recall some definitions from geometric measure theory, state the maximum principle for varifolds, and introduce the discrete maps in the space of currents.

In Section 4, we develop the min-max theory for intersecting slices. We give the definitions, state the main results of our method, and compare them with the results obtained by the classical setting of Almgren and Pitts. The proofs of these results are done in Sections 5 to 11.

In Section 5, we describe the interpolation results that we use. They are a powerful tool to deal with discrete maps in the space of currents.

In Section 6, we develop a deformation argument for boundaries of integral n-currents on compact n-dimensional manifolds with boundary. This is used to construct critical sequences in our min-max setting.

In Sections 7 and 8, we prove a deformation lemma for discrete maps in the space of codimension-one integral currents. This result plays an important role in the subsequent sections. A key ingredient to the argument of these sections is the so called small mass deformation lemma, referred to as Lemma 7.1.

In Section 9, we describe how to create discrete sweepouts out of a continuous one controlling the intersecting slices.

In Section 10, we adapt the pull-tight procedure to our setting.

In Section 11, we prove the existence of intersecting and almost minimizing critical varifolds in a given homotopy class.

In Section 12, we apply our method to prove the existence of closed minimal surfaces in complete non-compact manifolds, Theorem 1.1.

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2. Main ideas of the min-max theory for intersecting slices

In this section, we outline the proof of our min-max result, Theorem 1.2. We present the main ideas, omitting the technical issues and reducing the use of the language of geometric measure theory.

The min-max minimal hypersurface of Theorem 1.2 arises as a limit in the weak sense of varifolds. We consider sweepouts of a closed manifold by codimension-one integral currents without boundary. For this reason, a minimum of this language is unavoidable to introduce our main ideas.

Let us give a brief idea about these structures. Currents and varifolds are measure-theoretic generalized submanifolds. They come with notions of dimension, support, and mass. Currents also come with orientation and a notion of boundary, which is a current of one dimension lower. We use ∂T to denote the boundary of a current T and $\mathcal{Z}_{n-1}(M)$ to denote the space of integral $(n-1)$-currents without boundary and supported in M. The mass of $T \in \mathcal{Z}_{n-1}(M)$, $\mathbf{M}(T)$, is a generalized $(n-1)$-dimensional Hausdorff measure with multiplicities. On the other hand, the mass of a varifold V is a Radon measure $||V||$ on M. The space of varifolds with uniformly bounded masses has the remarkable property of being compact with respect to the weak varifold convergence. A class of varifolds that plays a key role in this work is the class of stationary varifolds; those are the varifolds that are critical with respect to the mass in M, $||V||(M)$. This generalizes the idea of minimal submanifolds of M. We can obtain a varifold $|T|$ out of a current $T \in \mathcal{Z}_{n-1}(M)$, simply by forgetting the orientation.

2.1. Min-max Theorem. Let M^n be an orientable closed Riemannian manifold of dimension n and Ω be a connected open subset of M with smooth and strictly mean-concave boundary; i.e., $\partial\Omega$ is smooth and its mean-curvature vector is everywhere non-zero and points outside Ω.

We consider continuous sweepouts $S = \{\Sigma_t\}_{t\in[0,1]}$ of M by integral $(n-1)$-currents with no boundary, $\Sigma_t \in \mathscr{Z}_{n-1}(M)$, and such that the slices Σ_t degenerate to the zero current at $t = 0$ and 1. For example, given a Morse function $f : M \to [0,1]$, we can consider the sweepout of the level sets $\Sigma_t = \partial(\{x \in M : f(x) < t\})$.

Two such sweepouts S^1 and S^2 are homotopic to each other if there is a continuous map ψ, defined on $[0,1]^2$, for which $\{\psi(s,t)\}_{t\in[0,1]}$ is a sweepout of M, for each $s \in [0,1]$, being $S^1 = \{\psi(0,t)\}_{t\in[0,1]}$ and $S^2 = \{\psi(1,t)\}_{t\in[0,1]}$. Once a homotopy class Π of sweepouts is fixed, we can run the min-max.

The key difference between our method and the original one by Almgren and Pitts is that we see only the slices Σ_t that intersect $\overline{\Omega}$. Given $S = \{\Sigma_t\}_{t\in[0,1]} \in \Pi$, we use the following notation:

$$\mathrm{dmn}_\Omega(S) = \{t \in [0,1] : \mathrm{spt}(\Sigma_t) \cap \overline{\Omega} \neq \varnothing\},$$

where $\mathrm{spt}(\Sigma_t)$ denotes the support of $\Sigma_t \in \mathscr{Z}_{n-1}(M)$, and

$$\mathbf{L}(S,\Omega) = \sup\{\mathbf{M}(\Sigma_t) : t \in \mathrm{dmn}_\Omega(S)\}.$$

We define the width of Π with respect to Ω to be

$$\mathbf{L}(\Pi,\Omega) = \inf\{\mathbf{L}(S,\Omega) : S \in \Pi\}.$$

The standard setting of Almgren and Pitts coincides with ours when Ω is M. In this case, the min-max philosophy is to obtain sequences $S^k = \{\Sigma_t^k\}_{t\in[0,1]} \in \Pi$, $k = 1, 2, \ldots$, and $\{t_k\}_{k\in\mathbb{N}} \subset [0,1]$, for which

$$\mathbf{L}(\Pi, M) = \lim_{k\to\infty} \mathbf{L}(S^k, M) = \lim_{k\to\infty} \mathbf{M}(\Sigma_{t_k}^k)$$

and such that $|\Sigma_{t_k}^k|$ converges, as varifolds, to a disjoint union of closed embedded smooth minimal hypersurfaces, possibly with integer multiplicities, denoted by V. In our approach, we show that it is possible to realize the width $\mathbf{L}(\Pi,\Omega)$ via a min-max sequence of slices $\Sigma_{t_k}^k$, as before, and with the extra properties that $t_k \in \mathrm{dmn}_\Omega(S^k)$, for every $k \in \mathbb{N}$, and the support of the min-max limit V intersects Ω.

2.2. Main steps. The organization of our proof is as follows:

1. If Π is a non-trivial homotopy class, then $\mathbf{L}(\Pi,\Omega) > 0$.
2. We can always find $\mathscr{S} = \{S^k\}_{k\in\mathbb{N}} \subset \Pi$ such that

$$\lim_{k\to\infty} \mathbf{L}(S^k,\Omega) = \mathbf{L}(\Pi,\Omega)$$

and

$$\sup\{\mathbf{M}(\Sigma) : \Sigma \text{ is a slice of } S^k, \text{ for some } k \in \mathbb{N}\} < \infty.$$

Such sequences of sweepouts are said to be critical with respect to Ω. In this case, a sequence of intersecting slices $\Sigma^k \in S^k$ satisfying

$$\lim_{k\to\infty} \mathbf{M}(\Sigma^k) = \mathbf{L}(\Pi, \Omega)$$

is said to be a min-max sequence in $\mathscr{S}$. We use $\mathcal{C}(\mathscr{S}, \Omega)$ to denote the set of limits (as varifolds) of min-max sequences in $\mathscr{S}$.

3. There exists $\mathscr{S} \subset \Pi$ critical with respect to Ω for which $V \in \mathcal{C}(\mathscr{S}, \Omega)$ implies either that V is stationary or $\operatorname{spt}(V) \cap \Omega = \varnothing$.
4. Given $\mathscr{S} \subset \Pi$ as in item 3, we can find a non-trivial intersecting min-max limit $V \in \mathcal{C}(\mathscr{S}, \Omega)$ that is almost minimizing in small annuli (as introduced by Pitts; see the precise definition in 4.9).
5. In conclusion, the obtained V intersects Ω, and is stationary in M. Since V is almost minimizing in small annuli and stationary, Pitts' Regularity Theorem implies that V is smooth.

2.3. The role of the mean-concavity. Let $U \subset \Omega$ be a subset with the property that the difference $\Omega - U$ is contained in a small tubular neighborhood of $\partial\Omega$ in M. Since $U \subset \Omega$, we easily see that $\mathbf{L}(\Pi, U) \leq \mathbf{L}(\Pi, \Omega)$, for all homotopy classes of sweepouts Π. We use the mean-concavity assumption to prove that the inequality above is indeed an equality: $\mathbf{L}(\Pi, U) = \mathbf{L}(\Pi, \Omega)$. The main tool to prove this claim is the maximum principle for general varifolds of Brian White, [**22**].

More precisely, if we had the strict inequality $\mathbf{L}(\Pi, U) < \mathbf{L}(\Pi, \Omega)$ for some Π, we would be able to find $0 < \varepsilon < \mathbf{L}(\Pi, \Omega) - \mathbf{L}(\Pi, U)$ and a sweepout $S = \{\Sigma_t\}_{t\in[0,1]} \in \Pi$, such that $\mathbf{L}(S, U) < \mathbf{L}(\Pi, \Omega) - \varepsilon$. In particular, if $t \in \operatorname{dmn}_\Omega(S)$ and $\mathbf{M}(\Sigma_t) \geq \mathbf{L}(\Pi, \Omega) - \varepsilon$, then $\Sigma_t \subset M - \overline{U}$. We use the maximum principle to deform those Σ_t, without increasing the mass, to a new slice Σ'_t, supported outside Ω. Then, we obtain $S' \in \Pi$ with $\mathbf{L}(S', \Omega) < \mathbf{L}(\Pi, \Omega) - \varepsilon$. This is a contradiction.

Proof of Theorem 1.2. The existence of non-trivial homotopy classes goes back to Almgren. He proved, in [**1**], that the set of homotopy classes of sweepouts is isomorphic to the top dimensional homology group of M, $H_n(M^n, \mathbb{Z})$. Moreover, it is possible to prove the existence of positive constants $\alpha_0 = \alpha_0(M)$ and $r_0 = r_0(M)$ with the property that: given $p \in M$, $0 < r \leq r_0$ and a sweepout $S = \{\Sigma_t\}_{t\in[0,1]}$ in a non-trivial homotopy class, we have

$$\sup_{t\in[0,1]} \mathbf{M}((\Sigma_t) \llcorner B(p,r)) \geq \alpha_0 r^{n-1},$$

where $(\Sigma_t) \llcorner B(p,r)$ is the restriction of Σ_t to the geodesic ball $B(p,r)$ of M, of radius r and centered at p. If we take $B(p,r) \subset\subset \Omega$, then

$$\mathbf{L}(S, \Omega) = \sup_{t\in \operatorname{dmn}_\Omega(S)} \mathbf{M}(\Sigma_t) \geq \sup_{t\in[0,1]} \mathbf{M}((\Sigma_t) \llcorner B(p,r)) \geq \alpha_0 r^{n-1}.$$

Therefore, $\mathbf{L}(\Pi, \Omega) > 0$, for non-trivial homotopy classes Π. The existence of such numbers α_0 and r_0 is due to Gromov; see Section 4.2.B in [**9**]. These bounds appear also in [**11**] and Section 8 of [**15**].

We can construct a critical sequence of sweepouts with respect to Ω as follows. Take a minimizing sequence $\mathscr{S} = \{S^k\}_{k\in\mathbb{N}} \subset \Pi$, i.e., a sequence for which $\mathbf{L}(S^k, \Omega)$ tends to $\mathbf{L}(\Pi, \Omega)$ as k tend to infinity. In order to obtain a critical sequence out of $\mathscr{S}$, we have to uniformly control the masses of the non-intersecting slices of the S^k's. Let $k \in \mathbb{N}$ and $[t_1, t_2] \subset [0, 1]$ be such that $[t_1, t_2] \cap \mathrm{dmn}_\Omega(S^k) = \{t_1, t_2\}$. We homotopically deform the path $\{\Sigma^k_t\}$, $t \in [t_1, t_2]$, keeping the ends fixed and without entering Ω to a new continuous path $\{\tilde{\Sigma}^k_t\}$, $t \in [t_1, t_2]$, in such a way that

$$\sup_{t\in[t_1,t_2]} \mathbf{M}(\tilde{\Sigma}^k_t) \leq C \left(\mathbf{M}(\Sigma^k_{t_1}) + \mathbf{M}(\Sigma^k_{t_2})\right),$$

where $C > 0$ is a constant depending only on M. In this deformation, it is important that $(\Sigma^k_{t_2} - \Sigma^k_{t_1})$ is the boundary of an integral n-current supported in $M - \Omega$. Since the masses of the extremal slices are already accounted for in $\mathbf{L}(S^k, \Omega)$, we have the desired uniform control.

Let $\mathscr{S} = \{S^k\}_{k\in\mathbb{N}} \subset \Pi$ be a critical sequence with respect to Ω. The third step of our list is called the pull-tight argument and it is inspired by Theorem 4.3 in [**19**] and Proposition 8.5 in [**14**]. We deform $\mathscr{S}$ to obtain a strictly better critical sequence $\overline{\mathscr{S}} = \left\{\overline{S}^k\right\}_{k\in\mathbb{N}} \subset \Pi$ with the extra property that:

$$V \in C\left(\overline{\mathscr{S}}, \Omega\right) \Rightarrow \text{ either } V \text{ is stationary or } \mathrm{spt}(V) \cap \overline{\Omega} = \varnothing.$$

More precisely, we continuously deform each S^k to $\overline{S}^k$ via a homotopy map $\{H(s,t)\}_{s,t\in[0,1]}$ that starts with $\{H(0,t)\}_t = S^k$ and ends with $\{H(1,t)\}_t = \overline{S}^k$, whose key property is that

$$\mathbf{M}(H(1,t)) < \mathbf{M}(H(0,t)),$$

unless $H(0,t)$ is either stationary or $\mathrm{spt}(H(0,t)) \cap \Omega = \varnothing$, for which it must be true that $H(s,t) = H(0,t)$, for every $s \in [0,1]$. The main improvement of our pull-tight deformation H is that it keeps unmoved the non-intersecting slices, allowing us to conclude that $\overline{\mathscr{S}}$ is also critical with respect to Ω. For simplicity, let us keep using $\mathscr{S} = \{S^k\}_{k\in\mathbb{N}} \subset \Pi$ to denote a critical sequence with respect to Ω, but now with the further property that all critical varifolds $V \in C\left(\mathscr{S}, \Omega\right)$ are either stationary or do not intersect the mean-concave subset.

The final part is to prove that some $V \in \mathcal{C}(\mathscr{S}, \Omega)$ is at the same time intersecting and almost minimizing in small annuli. For the precise notation, see Definition 4.9 and Theorem 4.10 in Section 4, or Theorem 4.10 in Pitts [**19**]. The almost minimizing varifolds are natural objects in the theory, they can be approximated by locally mass-minimizing currents. The proof of the regularity of stationary almost minimizing varifolds, in dimensions $3 \leq n \leq 7$, uses the curvature estimates for stable minimal hypersurfaces of [**20**] and [**21**].

Assuming that no $V \in \mathcal{C}(\mathscr{S}, \Omega)$ satisfies our assumption, we deform the critical sequence $\mathscr{S} \subset \Pi$ to obtain strictly better competitors, i.e., we obtain a sequence of sweepouts $\tilde{S}^k \in \Pi$ out of S^k such that

$$\mathbf{L}\left(\tilde{S}^k, U\right) < \mathbf{L}\left(S^k, \Omega\right) - \rho, \tag{1}$$

for some uniform $\rho > 0$ and $U \subset \Omega$ such that the difference set $\Omega - U$ is contained in a small tubular neighborhood of $\partial\Omega$ in M. Since $\mathscr{S}$ is critical with respect to Ω, we have

$$\mathbf{L}\left(\tilde{S}^k, U\right) < \mathbf{L}(\Pi, \Omega) = \mathbf{L}(\Pi, U)$$

for large $k \in \mathbb{N}$, which contradicts the fact that $\tilde{S}^k$ belongs to Π. The equality in the last expression follows from the discussion in Subsection 2.3. In order to obtain a sequence $\tilde{\mathscr{S}} = \{\tilde{S}^k\}_{k\in\mathbb{N}}$ with such a property, recall that we have two types of varifolds $V \in \mathcal{C}(\mathscr{S}, \Omega)$: either

(1) $\mathrm{spt}(V) \cap \Omega \neq \varnothing$ and V is stationary in M, or
(2) $\mathrm{spt}(V) \cap \Omega = \varnothing$.

From the negative assumption, it follows that big intersecting slices of $\mathscr{S}$ are close either to intersecting, stationary, non–almost minimizing varifolds or to non-intersecting varifolds.

The deformation from S^k to $\tilde{S}^k$ is done in two parts. The first is based on the fact that if V is type (1), then it is not almost minimizing in small annuli centered at some $p = p(V) \in \mathrm{spt}(V)$. Then, there exists $\varepsilon(V) > 0$ such that given a slice $\Sigma_t^k \in S^k$ close to V and $\eta > 0$, we can find a continuous path $\{\Sigma_t^k(s)\}_{s\in[0,1]}$ with the following properties:

- $\Sigma_t^k(0) = \Sigma_t^k$
- $\mathrm{spt}(\Sigma_t^k(s) - \Sigma_t^k) \subset a(V)$
- $\mathbf{M}(\Sigma_t^k(s)) \leq \mathbf{M}(\Sigma_t^k) + \eta$
- $\mathbf{M}(\Sigma_t^k(1)) < \mathbf{M}(\Sigma_t^k) - \varepsilon(V)$,

where $a(V)$ is a small annulus centered at $p(V)$. Observe that if Σ_t^k is also close to a type (2) varifold, then it has small mass $||\Sigma_t^k||(U_1)$ in compactly supported subsets $U_1 \subset \Omega$. Let us fix open sets $U \subset U_1 \subset \Omega$, $\overline{U_1} \subset \Omega$ and such that $\Omega - U$ is inside a small tubular neighborhood of $\partial\Omega$. If we take $a(V)$ small enough, we can suppose that either $a(V) \subset U_1$ or $a(V) \subset M - \overline{U}$. Anyway, we have

$$||\Sigma_t^k(s)||(\overline{U}) \leq ||\Sigma_t^k||(U_1) + \eta,$$

for every $s \in [0,1]$. Let $[t_1, t_2] \subset \mathrm{dmn}_\Omega(S^k)$ be a maximal interval of big intersecting slices Σ_t^k close to type (1) varifolds. Replace $\{\Sigma_t^k\}$, $t \in [t_1, t_2]$, by the continuous path $\{\hat{\Sigma}_t^k\}$, $t \in [t_1, t_2]$, obtained as follows:

(i) first, use $\{\Sigma_{t_1}^k(s)\}_{s\in[0,1]}$ to go from $\Sigma_{t_1}^k$ to $\Sigma_{t_1}^k(1)$;

(ii) then use $\{\Sigma_t^k(1)\}$, $t \in [t_1, t_2]$ to reach $\Sigma_{t_2}^k(1)$;

(iii) finally, use the inverse way of $\{\Sigma_{t_2}^k(s)\}_{s\in[0,1]}$ to come back to $\Sigma_{t_2}^k$.

Repeating the process on each maximal interval as above, we obtain $\hat{\mathscr{S}} = \left\{\hat{S}^k\right\}_{k\in\mathbb{N}}$, critical with respect to Ω and such that big intersecting slices have small mass inside $\overline{U}$. This ends the first part of the argument.

The second part is a continuous deformation of such big intersecting slices $\hat{\Sigma}_t^k \in \hat{S}^k$ to ones not intersecting $\overline{U}$. This change is supported inside Ω and the masses along the process do not increase by much. The key ingredient we use to perform this part is the small mass deformation lemma, see Lemma 7.1. Consequently, the product of this part is the desired sequence $\tilde{\mathscr{S}}$, for which big intersecting slices with respect to U are not big intersecting with respect to Ω. In particular, we have proved that expression (1) holds for some positive uniform ρ. q.e.d.

3. Notation and preliminaries

3.1. Definitions from geometric measure theory. In this section we recall some definitions and notations from Geometric Measure Theory. A standard reference is the book of Simon [**17**]. Our approach follows Section 4 in [**14**] and, sometimes, the notation of [**19**].

Let (M^n, g) be an orientable compact Riemannian manifold. We assume that M is isometrically embedded in $\mathbb{R}^L$. We denote by $B(p, r)$ the open geodesic ball in M of radius r and center $p \in M$.

We denote by $\mathbf{I}_k(M)$ the space of k-dimensional integral currents in $\mathbb{R}^L$ supported in M; $\mathcal{Z}_k(M)$ the space of currents $T \in \mathbf{I}_k(M)$ with $\partial T = 0$; and $\mathcal{V}_k(M)$ the closure of the space of k-dimensional rectifiable varifolds in $\mathbb{R}^L$ with support contained in M, in the weak topology.

Given $T \in \mathbf{I}_k(M)$, we denote by $|T|$ and $||T||$ the integral varifold and Radon measure in M associated with T, respectively; given $V \in \mathcal{V}_k(M)$, $||V||$ denotes the Radon measure in M associated with V.

The above spaces come with several relevant metrics. We use the standard notations $\mathcal{F}$ and $\mathbf{M}$ for the flat norm and mass norm on $\mathbf{I}_k(M)$, respectively. The $\mathbf{F}$-*metric* on $\mathcal{V}_k(M)$ is defined in Pitts [**19**]. It induces the varifold weak topology on $\mathcal{V}_k(M)$. Finally, the $\mathbf{F}$-*metric* on $\mathbf{I}_k(M)$ is defined by

$$\mathbf{F}(S, T) = \mathcal{F}(S - T) + \mathbf{F}(|S|, |T|).$$

We also use the k-dimensional Hausdorff measure $\mathcal{H}^k$ of subsets of M.

We assume that $\mathbf{I}_k(M)$ and $\mathcal{Z}_k(M)$ both have the topology induced by the flat metric. When endowed with a different topology, these spaces will be denoted either by $\mathbf{I}_k(M; \mathbf{M})$ and $\mathcal{Z}_k(M; \mathbf{M})$, in the case of the mass norm, or $\mathbf{I}_k(M; \mathbf{F})$ and $\mathcal{Z}_k(M; \mathbf{F})$, if we use the $\mathbf{F}$-metric. If $U \subset M$ is an open set of finite perimeter, the associated current in $\mathbf{I}_n(M)$ is denoted by $[|U|]$. The space $\mathcal{V}_k(M)$ is always considered with the weak topology of varifolds.

Given a C^1-map $F : M \to M$, the push-forwards of $V \in \mathcal{V}_k(M)$ and $T \in \mathbf{I}_k(M)$ are denoted by $F_\#(V)$ and $F_\#(T)$, respectively. Let $\mathcal{X}(M)$ denote the space of smooth vector fields of M with the C^1-topology. The first variation $\delta : \mathcal{V}_k(M) \times \mathcal{X}(M) \to \mathbb{R}$ is defined as

$$\delta V(X) = \frac{d}{dt}\bigg|_{t=0} ||F_{t\#}(V)||(M),$$

where $\{F_t\}_t$ is the flow of X. The first variation is continuous with respect to the product topology of $\mathcal{V}_k(M) \times \mathcal{X}(M)$. A varifold V is said to be *stationary in* M if $\delta V(X) = 0$ for every $X \in \mathcal{X}(M)$.

3.2. The maximum principle. In [**22**], White proved a maximum principle type theorem for general varifolds. In this paper we use an application of this result which we describe now.

Let (M^n, g) be a closed Riemannian manifold and $\Omega \subset M$ be an open domain with smooth and strictly mean-concave boundary $\partial\Omega$. Consider the foliation of a small neighborhood of $\partial\Omega$ given by $\partial\Omega_t$, where $\Omega_t = \{x \in M : d_{\partial\Omega}(x) < t\}$, $d_{\partial\Omega}$ is the signed distance function to $\partial\Omega$ negative in Ω, and let ν be the unit vector field normal to all $\partial\Omega_t$ pointing outside Ω. Use $\overrightarrow{H}(p)$ to denote the mean-curvature vector of $\partial\Omega$ at $p \in \partial\Omega$. White's result and a covering argument in $\partial\Omega$ imply:

Corollary 3.1. *Let $\overrightarrow{H}(p) = H(p)\nu(p)$ and suppose that $H > \eta > 0$ over $\partial\Omega$. There exist $a < 0 < b$ and a smooth vector field X on $M - \Omega_a$ with*

$$X \cdot \nu > 0 \text{ on } \overline{\Omega_b} - \Omega_a$$

and

$$\delta V(X) \leq -\eta \int |X| d\mu_V,$$

for every $(n-1)$-varifold V in $M - \Omega_a$.

We refer to X as the maximum principle vector field. Up to multiplying X by a constant, we can suppose that

$$\Phi(1, M - \Omega_a) \subset M - \overline{\Omega_b}, \tag{2}$$

where $\{\Phi(s, \cdot)\}_{s \geq 0}$ is the flow of X. The key property of this flow is that it is mass-decreasing for $(n-1)$-varifolds.

Remark 3. Actually, $\{\Phi(s, \cdot)\}_{s \geq 0}$ is also mass-decreasing for n-varifolds.

3.3. Cell complexes. We begin by introducing the domains of our discrete maps. More details can be found in [**14**] or [**19**].

- $I^n = [0,1]^n \subset \mathbb{R}^n$ and $I_0^n = \partial I^n = I^n - (0,1)^n$;
- for each $j \in \mathbb{N}$, $I(1,j)$ denote the cell complex of I^1 whose 0-cells and 1-cells are, respectively, $[0], [3^{-j}], \ldots, [1-3^{-j}]$,$[1]$ and $[0, 3^{-j}]$, $[3^{-j}, 2 \cdot 3^{-j}], \ldots, [1 - 3^{-j}, 1]$;
- $I(n,j) = I(1,j) \otimes \cdots \otimes I(1,j)$, n times;

- $I(n,j)_p = \{\alpha_1 \otimes \cdots \otimes \alpha_n : \alpha_i \in I(1,j) \text{ and } \sum_{i=1}^n \dim(\alpha_i) = p\}$;
- $I_0(n,j)_p = I(n,j)_p \cap I_0^n$ are the p-cells in the boundary;
- $\partial : I(n,j) \to I(n,j)$: the boundary homomorphism is defined by

$$\partial(\alpha_1 \otimes \cdots \otimes \alpha_n) = \sum_{i=1}^{n} (-1)^{\sigma(i)} \alpha_1 \otimes \cdots \otimes \partial\alpha_i \otimes \cdots \otimes \alpha_n,$$

  where $\sigma(i) = \sum_{j<i} \dim(\alpha_i)$, $\partial[a,b] = [b] - [a]$ and $\partial[a] = 0$;
- $\mathbf{d} : I(n,j)_0 \times I(n,j)_0 \to \mathbb{N}$ is the grid distance: it is given by

$$\mathbf{d}(x,y) = 3^j \sum_{i=1}^{n} |x_i - y_i|;$$

- $\mathbf{n}(i,j) : I(n,i)_0 \to I(n,j)_0$: the nearest vertex map satisfies

$$\mathbf{d}(x, \mathbf{n}(i,j)(x)) = \min\{\mathbf{d}(x,y) : y \in I(n,j)_0\}.$$

3.4. Discrete maps into $\mathcal{Z}_{n-1}(M)$ and generalized homotopies. Let (M^n, g) denote an orientable compact Riemannian manifold.

Definition 3.2. Given $\phi : I(n,j)_0 \to \mathcal{Z}_{n-1}(M)$, we define its *fineness* as

$$\mathbf{f}(\phi) = \sup\left\{ \frac{\mathbf{M}(\phi(x) - \phi(y))}{\mathbf{d}(x,y)} : x, y \in I(n,j)_0, x \neq y \right\}.$$

REMARK 4. If $\mathbf{M}(\phi(x) - \phi(y)) < \delta$, for every pair of vertices at distance $\mathbf{d}(x,y) = 1$, then we can directly conclude that $\mathbf{f}(\phi) < \delta$.

Definition 3.3. Let $\phi_i : I(1,k_i)_0 \to \mathcal{Z}_{n-1}(M)$, $i = 1,2$, be given discrete maps. We say that ϕ_1 *is* 1*-homotopic to* ϕ_2 *in* $(\mathcal{Z}_{n-1}(M; \boldsymbol{M}), \{0\})$ *with fineness* δ if we can find $k \in \mathbb{N}$ and a map

$$\psi : I(1,k)_0 \times I(1,k)_0 \to \mathcal{Z}_{n-1}(M)$$

with the following properties:

(i) $\mathbf{f}(\psi) < \delta$;
(ii) $\psi([i-1], x) = \phi_i(\mathbf{n}(k,k_i)(x))$, $i = 1,2$ and $x \in I(1,k)_0$;
(iii) $\psi(\tau, [0]) = \psi(\tau, [1]) = 0$, for $\tau \in I(1,k)_0$.

4. Min-max theory for intersecting slices

In this section we describe the min-max theory that we use to prove our main results. The set up that we follow to develop that is similar to the original one introduced by Almgren and Pitts. The crucial difference is that we see only the slices intersecting a fixed closed subset $\overline{\Omega} \subset M$, which is a manifold with boundary. The aim with this is to produce an embedded closed minimal hypersurface intersecting the given domain.

Let (M^n, g) be an orientable closed Riemannian manifold and $\Omega \subset M$ be a connected open subset. We begin with the basic definitions following Almgren, Pitts, and Marques-Neves ([**2**], [**14**], [**15**] and [**19**]).

Definition 4.1. A $(1,\ M)$—homotopy sequence of mappings into $(\mathcal{Z}_{n-1}(M;\mathbf{M}),\{0\})$ is a sequence of maps $\{\phi_i\}_{i\in\mathbb{N}}$

$$\phi_i : I(1,k_i)_0 \to \mathcal{Z}_{n-1}(M),$$

such that ϕ_i is 1-homotopic to ϕ_{i+1} in $(\mathcal{Z}_{n-1}(M;\mathbf{M}),\{0\})$ with fineness δ_i and

(i) $\lim_{i\to\infty}\delta_i = 0$;
(ii) $\sup\{\mathbf{M}(\phi_i(x)) : x \in \mathrm{dmn}(\phi_i) \text{ and } i \in \mathbb{N}\} < \infty$.

The notion of homotopy between two $(1,\mathbf{M})$-homotopy sequences of mappings into $(\mathcal{Z}_{n-1}(M;\mathbf{M}),\{0\})$ is the following:

Definition 4.2. We say that $S^1 = \{\phi_i^1\}_{i\in\mathbb{N}}$ *is homotopic with* $S^2 = \{\phi_i^2\}_{i\in\mathbb{N}}$ if ϕ_i^1 is 1-homotopic to ϕ_i^2 with fineness δ_i and $\lim_{i\to\infty}\delta_i = 0$.

To be "homotopic with" is an equivalence relation on the set of $(1,\mathbf{M})$-homotopy sequences of mappings into $(\mathcal{Z}_{n-1}(M;\mathbf{M}),\{0\})$. An equivalence class is called a $(1,\mathbf{M})$-homotopy class of mappings into $(\mathcal{Z}_{n-1}(M;\mathbf{M}),\{0\})$. We follow the usual notation $\pi_1^{\#}(\mathcal{Z}_{n-1}(M;\mathbf{M}),\{0\})$ for the set of homotopy classes, as it appears in Pitts' book, [**19**].

The key difference is that our width considers intersecting slices only. Given a map $\phi : I(1,k)_0 \to \mathcal{Z}_{n-1}(M)$, we defined its reduced domain by

$$\mathrm{dmn}_\Omega(\phi) = \{x \in I(1,k)_0 : \mathrm{spt}(||\phi(x)||) \cap \overline{\Omega} \neq \varnothing\}. \tag{3}$$

Definition 4.3. Let $\Pi \in \pi_1^{\#}(\mathcal{Z}_{n-1}(M;\mathbf{M}),\{0\})$ be a homotopy class and $S = \{\phi_i\}_{i\in\mathbb{N}} \in \Pi$. We define

$$\mathbf{L}(S,\Omega) = \limsup_{i\to\infty} \max\{\mathbf{M}(\phi_i(x)) : x \in \mathrm{dmn}_\Omega(\phi_i)\}. \tag{4}$$

The *width of* Π *with respect to* Ω is the minimum $\mathbf{L}(S,\Omega)$ among all sequences $S \in \Pi$,

$$\mathbf{L}(\Pi,\Omega) = \inf\{\mathbf{L}(S,\Omega) : S \in \Pi\}. \tag{5}$$

Keeping the notation in the previous definition, we write $V \in \mathbf{K}(S,\Omega)$ if $V = \lim_j |\phi_{i_j}(x_j)|$, for some increasing sequence $\{i_j\}_{j\in\mathbb{N}}$ and $x_j \in \mathrm{dmn}_\Omega(\phi_{i_j})$. Moreover, if $\mathbf{L}(S,\Omega) = \mathbf{L}(\Pi,\Omega)$, we say that S *is critical with respect to* Ω. In this case we consider the *critical set of* S *with respect to* Ω, defined by

$$\mathcal{C}(S,\Omega) = \{V \in \mathbf{K}(S,\Omega) : ||V||(M) = \mathbf{L}(S,\Omega)\}. \tag{6}$$

As in the classical theory, $\mathcal{C}(S,\Omega) \subset \mathcal{V}_{n-1}(M)$ is compact and nonempty, but here it is not clear whether there exists $V \in \mathcal{C}(S,\Omega)$ with $||V||(\overline{\Omega}) > 0$.

In the direction of proving that there are critical varifolds intersecting $\overline{\Omega}$, we construct a deformation process in Section 8, to deal with discrete maps whose big slices enter Ω with very small mass. For doing this, and from now on, we introduce our main geometric assumption:

(7) *Ω has smooth and strictly mean-concave boundary $\partial\Omega$.*

This deformation process is inspired by Pitts' deformation arguments for constructing replacements, Section 3.10 in [**19**], and Corollary 3.1.

We introduce some notation to explain that result, which is precisely stated and proved in Section 8. Following the notation in Subsection 3.2, consider an open subset $U \subset \Omega$ such that $\overline{\Omega_a} \subset U$. Lemma 8.1 guarantees the existence of positive constants η_0 and ε_2 depending on M, Ω and U, and C_1 depending only on M, with the following properties: given a discrete map $\phi : I(1,k)_0 \to \mathcal{Z}_{n-1}(M)$ such that $\mathbf{f}(\phi) \leq \eta_0$ and, for some $L > 0$,

$$\mathbf{M}(\phi(x)) \geq L \Rightarrow ||\phi(x)||(U) < \varepsilon_2,$$

then, up to a discrete homotopy of fineness $C_1\mathbf{f}(\phi)$, we can suppose that

$$\max\{\mathbf{M}(\phi(x)) : x \in \mathrm{dmn}_\Omega(\phi)\} < L + C_1\mathbf{f}(\phi).$$

The proof is quite technical and lengthy, because it involves interpolation results; see Subsection 5.2. Despite the technical objects in that proof, the lemma has several applications in key arguments of this work.

Since continuous maps $\Gamma : [0,1] \to \mathcal{Z}_{n-1}(M; \mathbf{F})$, with $\Gamma(0) = \Gamma(1) = 0$, are more natural, it is important to generate $(1, \mathbf{M})$-homotopy sequences of mappings into $(\mathcal{Z}_{n-1}(M; \mathbf{M}), \{0\})$ out of them. Similarly to the discrete set up, use the notations

$$\mathrm{dmn}_\Omega(\Gamma) = \{t \in [0,1] : \mathrm{spt}(||\Gamma(t)||) \cap \overline{\Omega} \neq \varnothing\} \tag{8}$$

and

$$L(\Gamma, \Omega) = \sup\{\mathbf{M}(\Gamma(t)) : t \in \mathrm{dmn}_\Omega(\Gamma)\}. \tag{9}$$

Theorem 4.4. *Let Γ be as above and suppose that it defines a non-trivial class in $\pi_1(\mathcal{Z}_{n-1}(M; \mathcal{F}), 0)$. Then, there exists a non-trivial homotopy class $\Pi \in \pi_1^\#(\mathcal{Z}_{n-1}(M; \boldsymbol{M}), \{0\})$, such that $\boldsymbol{L}(\Pi, \Omega) \leq L(\Gamma, \Omega)$.*

Remark 5. The proof of this theorem is a nice application of Lemma 8.1, combined with interpolation results. This is done in Section 9.

Observe that item (ii) in the definition of $(1, \mathbf{M})$-homotopy sequences of mappings into $(\mathcal{Z}_{n-1}(M; \mathbf{M}), \{0\})$ requires a uniform control on the masses of all slices. When we try to minimize the width in a given homotopy class, in order to construct a critical sequence with respect to Ω, there is no restriction about the non-intersecting slices. Then, it is possible that item (ii) fails in the limit. More precisely, for a given homotopy class $\Pi \in \pi_1^\#(\mathcal{Z}_{n-1}(M; \mathbf{M}), \{0\})$, via a diagonal sequence argument through a minimizing sequence $\{S^j\}_{j\in\mathbb{N}} \subset \Pi$, we can produce

$S^* = \{\phi_i^*\}_{i\in\mathbb{N}}$, so that ϕ_i^* is 1-homotopic to ϕ_{i+1}^* with fineness tending to zero and

$$\lim_{i\to\infty} \max\{\mathbf{M}(\phi_i^*(x)) : x \in \mathrm{dmn}_\Omega(\phi_i^*)\} = \mathbf{L}(\Pi, \Omega).$$

But, perhaps, in doing this, we do not guarantee that the masses of the non-intersecting slices do not diverge. To overcome this difficulty, we prove the following statement:

Lemma 4.5. *There exists $C = C(M, \Omega) > 0$ with the following property: given a discrete map $\phi : I(1, k)_0 \to \mathcal{Z}_{n-1}(M)$ of small fineness, we can find $\tilde{\phi} : I(1, \tilde{k})_0 \to \mathcal{Z}_{n-1}(M)$ such that:*

(a) *$\tilde{\phi}$ is 1-homotopic to ϕ with fineness $C \cdot \boldsymbol{f}(\phi)$;*
(b) *$\phi(dmn_\Omega(\phi)) = \tilde{\phi}(dmn_\Omega(\tilde{\phi}))$;*
(c)

$$\max_{x\in dmn(\tilde{\phi})} \boldsymbol{M}(\tilde{\phi}(x)) \le C \cdot \left(\max_{x\in dmn_\Omega(\phi)} \boldsymbol{M}(\phi(x)) + \boldsymbol{f}(\phi) \right).$$

To produce a true competitor out of S^*, for each i large enough we replace ϕ_i^* by another discrete map, also denoted by ϕ_i^*, using Lemma 4.5. The new S^* has the same intersecting slices and, as a consequence of items (a) and (c), the additional property of being an element of Π. This concludes the existence of critical S^* for $\mathbf{L}(\Pi, \Omega)$. The proof of Lemma 4.5 is based on a natural deformation of boundaries of n-currents in n-dimensional manifolds with boundary. Briefly, the deformation is the image of the given $(n-1)$-boundary via the gradient flow of a Morse function with no interior local maximum; see Section 6.

Actually, we prove a pull-tight type theorem, as Proposition 8.5 in [**14**]. Precisely, given $\Pi \in \pi_1^\#(\mathcal{Z}_{n-1}(M; \mathbf{M}), 0)$, we obtain:

Proposition 4.6. *There exists a critical sequence $S^* \in \Pi$. For each critical sequence S^*, there exists a critical sequence $S \in \Pi$ such that*

- *$\mathcal{C}(S, \Omega) \subset \mathcal{C}(S^*, \Omega)$, up to critical varifolds Σ with $||\Sigma||(\Omega) = 0$;*
- *every $\Sigma \in \mathcal{C}(S, \Omega)$ is either a stationary varifold or $||\Sigma||(\Omega) = 0$.*

In this statement, critical means critical with respect to Ω. The proof is postponed to Section 10. In the classical set up, the pull-tight gives a critical sequence for which all critical varifolds are stationary in M. In our case, it is enough to know that the intersecting critical varifolds are stationary.

The existence of non-trivial classes was proved by Almgren, in [**1**]. In fact, he provides an isomorphism

$$F : \pi_1^\#(\mathcal{Z}_{n-1}(M; \mathbf{M}), \{0\}) \to H_n(M). \tag{10}$$

Then, we apply the Proposition 8.2 of [**15**] to prove:

Lemma 4.7. *If* $\Pi \in \pi_1^{\#}(\mathcal{Z}_{n-1}(M; \boldsymbol{M}), \{0\})$ *is a non-trivial homotopy class, then* $\boldsymbol{L}(\Pi, \Omega) > 0$.

REMARK 6. Given $S = \{\phi_i\}_{i\in\mathbb{N}} \in \Pi$, the ϕ_i's with sufficient large i can be extended to maps Φ_i continuous in the mass norm, respecting the non-intersecting property; see Theorem 5.1. Proposition 8.2 in [**15**] provides a lower bound on the value of $\sup\{\mathbf{M}(\Phi(\theta)\llcorner B(p,r)) : \theta \in S^1\}$, for continuous maps $\Phi : S^1 \to \mathcal{Z}_{n-1}(M)$ in the flat topology. To obtain Lemma 4.7, apply Proposition 8.2 for those Φ_i and $\overline{B(p,r)} \subset \Omega$.

The next step in the program developed by Almgren and Pitts is to find a critical varifold $V \in \mathcal{C}(S, \Omega)$ with the property of being almost minimizing in small annuli. This is a variational property that enables us to approximate the varifold by arbitrarily close integral cycles which are themselves almost locally area minimizing. The key characteristic of varifolds that are almost minimizing in small annuli is regularity, which is not necessarily the case for general integral stationary varifolds.

Let M^n be a compact Riemannian manifold and U be an open subset of M. Use $B(p,r)$ and $A(p,s,r) = B(p,r) - \overline{B(p,s)}$ to denote the open geodesic ball of radius r and centered at p and the open annulus in M, respectively.

Notation 4.8. *Given $\varepsilon, \delta > 0$, consider the set $\mathcal{A}(U; \varepsilon, \delta)$ of integer cycles $T \in \mathcal{Z}_{n-1}(M)$ for which the following happens: for every finite sequence $T = T_0, T_1, \dots, T_m \in \mathcal{Z}_{n-1}(M)$, such that*

$$spt(T_i - T) \subset U, \quad \boldsymbol{M}(T_i, T_{i-1}) \leq \delta \quad and \quad \boldsymbol{M}(T_i) \leq \boldsymbol{M}(T) + \delta,$$

it must be true that $\boldsymbol{M}(T_m) \geq \boldsymbol{M}(T) - \varepsilon$.

Definition 4.9. Let $V \in \mathcal{V}_{n-1}(M)$ be a rectifiable varifold in M. We say that V *is almost minimizing in* U if for every $\varepsilon > 0$, there exist $\delta > 0$ and $T \in \mathcal{A}(U; \varepsilon, \delta)$ satisfying $\mathbf{F}(V, |T|) < \varepsilon$.

REMARK 7. This definition is basically the same as in Pitts' book; the difference is that we ask $|T|$ to be ε-close to V in the $\mathbf{F}$-metric on M, not only in U. This creates no problem because our definition implies Pitts', and in the next step we prove the existence of almost minimizing varifolds in the sense of 4.9. This is also observed in Remark 6.4 of [**23**].

The goal of our construction is to produce minimal hypersurfaces with intersecting properties. In order to obtain this, we prove the following version of the existence theorem of almost minimizing varifolds:

Theorem 4.10. *Let (M^n, g) be a closed Riemannian manifold, $n \leq 7$, and $\Pi \in \pi_1^{\#}(\mathcal{Z}_{n-1}(M; \boldsymbol{M}), \{0\})$ be a non-trivial homotopy class. Suppose that M contains an open subset Ω with smooth and strictly mean-concave boundary. There exists an integral varifold V such that*

(i) $||V||(M) = \boldsymbol{L}(\Pi, \Omega)$;
(ii) V *is stationary in* M;
(iii) $||V||(\Omega) > 0$;
(iv) *for each* $p \in M$*, there exists a positive number* r *such that* V *is almost minimizing in* $A(p, s, r)$ *for all* $0 < s < r$.

This is similar to Pitts' Theorem 4.10 in [**19**], the difference being that we prove that the almost minimizing and stationary varifold also intersects Ω. We postpone its proof to Section 11; it is a combination of Pitts' argument and Lemma 8.1. This is the last preliminary result to prove Theorem 1.2.

The argument to prove Theorem 1.2 is simple now. Lemma 4.7 provides a non-trivial homotopy class, for which we can apply Theorem 4.10 and obtain an integral critical varifold V that is stationary, almost minimizing in small annuli, and intersects Ω. The Pitts regularity theory, developed in Chapters 5 to 7 in [**19**], guarantees that $\operatorname{spt}(||V||)$ is an embedded smooth hypersurface.

5. Interpolation Results

Interpolation is an important tool for passing from discrete maps of small fineness to continuous maps in the space of integral cycles and vice-versa, the fineness and continuity being with respect to two different topologies. This type of technique appeared already in [**1**], [**2**], [**19**], [**14**], and [**15**]. In our approach, we follow mostly Sections 13 and 14 of [**14**]. We also make a remark that is important for us concerning the supports of interpolating sequences.

We start with conditions under which a discrete map is approximated by a continuous map in the mass norm. The main result is important to prove Lemmas 8.1 and 4.7, and Proposition 4.6.

Let (M^n, g) be a closed Riemannian manifold. We observe from Corollary 1.14 in [**1**] that there exists $\delta_0 > 0$, depending only on M, such that for every

$$\psi : I(2,0)_0 \to \mathcal{Z}_{n-1}(M)$$

with $\mathbf{f}(\psi) < \delta_0$ and $\alpha \in I(2,0)_1$ with $\partial\alpha = [b] - [a]$ (see Figure 3), we can find $Q(\alpha) \in \mathbf{I}_n(M)$ with

$$\partial Q(\alpha) = \psi([b]) - \psi([a]) \text{ and } \mathbf{M}(Q(\alpha)) = \mathcal{F}(\partial Q(\alpha)).$$

Let Ω_1 be a connected open subset of M, such that $\overline{\Omega_1}$ is a manifold with boundary. The first important result in this section is:

Theorem 5.1. *There exists* $C_0 > 0$*, depending only on* M*, such that for every map*

$$\psi : I(2,0)_0 \to \mathcal{Z}_{n-1}(M)$$

with $\boldsymbol{f}(\psi) < \delta_0$*, we can find a continuous map in the mass norm*

$$\Psi : I^2 \to \mathcal{Z}_{n-1}(M; \boldsymbol{M})$$

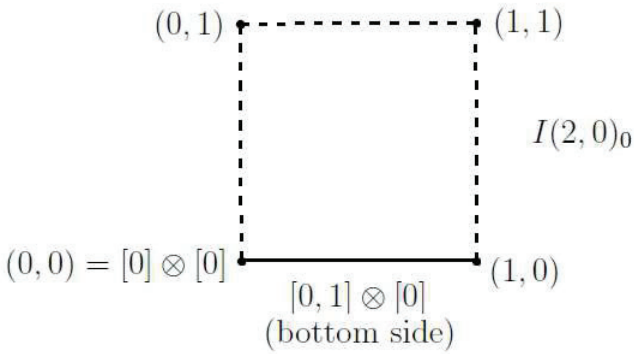

Figure 3. Domain of the discrete map ψ and associated cells.

such that

(i) $\Psi(x) = \psi(x)$, *for all* $x \in I(2,0)_0$*;*
(ii) *for every* $\alpha \in I(2,0)_p$, $\Psi|_\alpha$ *depends only on the values assumed by* ψ *on the vertices of* α*;*
(iii)

$$\sup\{\boldsymbol{M}(\Psi(x) - \Psi(y)) : x, y \in I^2\} \leq C_0 \sup_{\alpha \in I(2,0)_1} \{\boldsymbol{M}(\partial Q(\alpha))\}.$$

Moreover, if $\boldsymbol{f}(\psi) < \min\{\delta_0, \mathcal{H}^n(\Omega_1)\}$ *and*

$$spt(||\psi(0,0)||) \cup spt(||\psi(1,0)||) \subset M - \overline{\Omega_1},$$

we can choose Ψ *with* $spt(||\Psi(t,0)||) \subset M - \overline{\Omega_1}$*, for all* $t \in [0,1]$*.*

Proof. The first part of this result is Theorem 14.1 in Marques and Neves [**14**]. There the authors sketch the proof following the work of Almgren, Section 6 of [**1**], and ideas of Pitts, Theorem 4.6 of [**19**]. To prove our second claim, we follow that sketch. They start with Δ, a differentiable triangulation of M. Hence, if $s \in \Delta$ then the faces of s also belong to Δ. Given $s, s' \in \Delta$, use the notation $s' \subset s$ if s' is a face of s. Let $U(s) = \cup_{s \subset s'} s'$. Let Ω_2 be a small connected neighborhood of $\overline{\Omega_1}$, so that

$$\mathrm{spt}(||\psi(0,0)||) \cup \mathrm{spt}(||\psi(1,0)||) \cap \overline{\Omega_2} = \varnothing.$$

Up to a refinement of Δ, we can suppose that

$$s \in \Delta \text{ and } U(s) \cap (M - \Omega_2) \neq \varnothing \Rightarrow U(s) \subset M - \overline{\Omega_1}.$$

For $Q = Q([0,1] \otimes [0])$, we have $Q \in \mathbf{I}_n(M)$ and $\partial Q = \psi(1,0) - \psi(0,0)$ does not intersect Ω_2. Then, there are two possibilities: either $\mathrm{spt}(||Q||) \cap \Omega_2 = \varnothing$ or $\Omega_2 \subset \mathrm{spt}(||Q||)$. Note that in the second case we would have

$$\mathcal{H}^n(\Omega_1) < \mathcal{H}^n(\Omega_2) \leq ||Q||(\Omega_2) \leq \mathbf{M}(Q) \leq \mathbf{f}(\psi) \leq \mathcal{H}^n(\Omega_1).$$

This is a contradiction and we have $\mathrm{spt}(||Q||) \cap \Omega_2 = \varnothing$. By the construction of Ψ we know that

$$\mathrm{spt}(||\Psi(t,0)||) \subset \bigcup\{U(s) : U(s) \cap \mathrm{spt}(||Q||) \neq \varnothing\}.$$

But $U(s) \cap \mathrm{spt}(Q) \neq \varnothing$ implies that $U(s) \cap (M - \Omega_2) \neq \varnothing$. Then, the choice of the triangulation gives us $U(s) \subset M - \overline{\Omega_1}$. q.e.d.

As in [**14**], we also use the following discrete approximation result for continuous maps. Assume we have a continuous map in the flat topology $\Phi : I^m \to \mathcal{Z}_{n-1}(M)$, with the following properties:

- $\Phi|_{I_0^m}$ is continuous in the $\mathbf{F}$-metric,
- $L(\Phi) = \sup\{\mathbf{M}(\Phi(x)) : x \in I^m\} < \infty$,
- $\limsup_{r\to 0} \mathbf{m}(\Phi, r) = 0$,

where $\mathbf{m}(\Phi, r)$ is the concentration of mass of Φ in balls of radius r, i.e.:

$$\mathbf{m}(\Phi, r) = \sup\{||\Phi(x)||(B(p,r)) : x \in I^m, p \in M\}.$$

Theorem 5.2. *There exist sequences of mappings*

$$\phi_i : I(m,k_i)_0 \to \mathcal{Z}_{n-1}(M) \text{ and } \psi_i : I(1,k_i)_0 \times I(m,k_i)_0 \to \mathcal{Z}_{n-1}(M),$$

with $k_i < k_{i+1}$, $\psi_i([0],\cdot) = \phi_i(\cdot)$, $\psi_i([1],\cdot) = \phi_{i+1}(\cdot)|_{I(m,k_i)_0}$, *and sequences* $\delta_i \to 0$ *and* $l_i \to +\infty$, *such that*

(i) *for every* $y \in I(m,k_i)_0$

$$\boldsymbol{M}(\phi_i(y)) \leq \sup\{\boldsymbol{M}(\Phi(x)) : \alpha \in I(m,l_i)_m, x, y \in \alpha\} + \delta_i$$

and in particular,

$$\max\{\boldsymbol{M}(\phi_i(x)) : x \in I(m,k_i)_0\} \leq L(\Phi) + \delta_i;$$

(ii) $\boldsymbol{f}(\psi_i) < \delta_i$;

(iii)

$$\sup\{\mathcal{F}(\psi_i(y,x) - \Phi(x)) : (y,x) \in dmn(\psi_i)\} < \delta_i;$$

(iv) *if* $x \in I_0(m,k_i)_0$ *and* $y \in I(1,k_i)_0$, *we have*

$$\boldsymbol{M}(\psi_i(y,x)) \leq \boldsymbol{M}(\Phi(x)) + \delta_i.$$

Moreover, if $\Phi|_{\{0\}\times I^{m-1}}$ *is continuous in the mass topology then we can choose* ϕ_i *so that*

$$\phi_i(x) = \Phi(x), \text{ for all } x \in B(m,k_i)_0.$$

Remark 8. In case Φ is continuous in the F-metric on I^m, there is no concentration of mass. This is the content of Lemma 15.2 in [**14**].

In the proof of our main lemma, in Section 8, we apply the following consequence of Theorem 5.2. Let $U, \Omega_1 \subset M$ be open subsets, such that the closure of Ω_1 is a manifolds with boundary, and $\rho > 0$.

Assume that we have a map $\Psi : I^2 \to \mathcal{Z}_{n-1}(M)$ which is continuous in the F-metric. Suppose also that

- $t \in [0,1] \mapsto \Psi(0,t)$ is continuous in the mass norm;
- $\sup\{||\Psi(s,t)||(\overline{U}) : s \in [0,1] \text{ and } t \in \{0,1\}\} < \rho$;
- $\mathrm{spt}(||\Psi(1,t)||) \subset M - \overline{\Omega_1}$, for every $t \in [0,1]$.

Corollary 5.3. *Given $\delta > 0$, there exists $k \in \mathbb{N}$ and a map*

$$\Psi_1 : I(2,k)_0 \to \mathcal{Z}_{n-1}(M),$$

with the following properties:

(i) $\boldsymbol{f}(\Psi_1) < \delta$;
(ii) $\sup\{||\Psi_1(\sigma,\tau)||(\overline{U}) : \sigma \in I(1,k)_0$ *and* $\tau \in \{0,1\}\} < \rho + \delta$;
(iii) $\sup\{||\Psi_1(1,\tau))||(\overline{\Omega_1}) : \tau \in I(1,k)_0\} < \delta$;
(iv) $\boldsymbol{M}(\Psi_1(x)) \leq \boldsymbol{M}(\Psi(x)) + \delta$, *for every* $x \in I(2,k)_0$;
(v) $\Psi_1(0,\tau) = \Psi(0,\tau)$, *if* $\tau \in I(1,k)_0$.

Items (i) and (v) are easy consequences of Theorem 5.2. The other items hold if we choose a sufficiently close discrete approximation. The proof of Corollary 5.3 involves a simple combination of the uniform continuity of $\mathbf{M} \circ \Psi$ on I^2, compactness arguments, and Lemma 4.1 in [**14**], which we state now.

Lemma 5.4. *Let $\mathcal{S} \subset \mathcal{Z}_k(M;\boldsymbol{F})$ be a compact set. For every $\rho > 0$, there exists δ so that for every $S \in \mathcal{S}$ and $T \in \mathcal{Z}_k(M)$*

$$\boldsymbol{M}(T) < \boldsymbol{M}(S) + \delta \text{ and } \mathcal{F}(T - S) \leq \delta \Rightarrow \boldsymbol{F}(S,T) \leq \rho.$$

The last lemma that we discuss in this section is similar to Theorem 5.2, with $m = 1$. It says that the hypothesis about no concentration of mass is not required in this case. See Lemma 3.8 in [**19**].

Lemma 5.5. *Suppose $L, \eta > 0$, K is a compact subset of U and $T \in \mathcal{Z}_k(M)$. There exists $\varepsilon_0 = \varepsilon_0(L,\eta,K,U,T) > 0$, such that whenever*

- $S_1, S_2 \in \mathcal{Z}_k(M)$;
- $\mathcal{F}(S_1 - S_2) \leq \varepsilon_0$;
- $spt(S_1 - T) \cup spt(S_2 - T) \subset K$;
- $\boldsymbol{M}(S_1) \leq L$ *and* $\boldsymbol{M}(S_2) \leq L$,

there exists a finite sequence $S_1 = T_0, T_1, \ldots, T_m = S_2 \in \mathcal{Z}_k(M)$ with

$$spt(T_l - T) \subset U, \quad \boldsymbol{M}(T_l - T_{l-1}) \leq \eta \text{ and } \boldsymbol{M}(T_l) \leq L + \eta.$$

This lemma is useful to approximate continuous maps in flat topology by discrete ones with small fineness in mass norm, but first we have to restrict the continuous map to finer and finer grids $I(1,k)_0$.

6. Natural deformations on manifolds with boundary

In this section, we develop a deformation tool and apply it to give the details of the proof of Lemma 4.5, which concerns deforming non-intersecting slices to uniformly control their masses. First, we state that deformation result.

Lemma 6.1. *Let $(\tilde{M}^n, g)$ be a compact Riemannian manifold with non-empty smooth boundary $\partial\tilde{M}$. There exists $C = C(\tilde{M}) > 0$ with the following property: given $A \in \boldsymbol{I}_n(\tilde{M})$ such that $\boldsymbol{M}(A) + \boldsymbol{M}(\partial A) < \infty$, we can find a map*

$$\phi : [0,1] \to Z_{n-1}(\tilde{M})$$

continuous in the flat topology such that

(i) $\phi(0) = \partial A$ *and* $\phi(1) = 0$*;*
(ii) $\boldsymbol{M}(\phi(t)) \leq C \cdot \boldsymbol{M}(\partial A)$*, for every* $t \in [0,1]$*.*

Moreover, if spt$(||A||) \subset$ *int*$(\tilde{M})$*, there exists a compact subset* $K \subset$ *int*$(\tilde{M})$*, such that spt*$(||\phi(t)||) \subset K$*, for every* $t \in [0,1]$*.*

Remark 9. The construction makes clear that ϕ has no concentration of mass, i.e., it is true that

$$\limsup_{r\to 0} \sup\{||\Phi(x)||(B(p,r)) : x \in I^m, p \in \tilde{M}\} = 0.$$

Let us present the proof of Lemma 4.5 now, and later in this section we come back to the argument of Lemma 6.1.

Proof of Lemma 4.5. Consider a discrete map $\psi : I(1,k)_0 \to \mathcal{Z}_{n-1}(M)$ such that $\mathbf{f}(\psi) \leq \min\{\delta_0, \mathcal{H}^n(\Omega)\}$. Recall the choice of δ_0 in Section 5. We replace ψ with another discrete map of small fineness. Observe, for each $v \in I(1,k)_0 - \{1\}$, that we can find the isoperimetric choice $A(x) \in \mathbf{I}_n(M)$ of $\psi(v + 3^{-k}) - \psi(v)$, i.e., $\partial A(v) = \psi(v + 3^{-k}) - \psi(v)$ and $\mathbf{M}(A(v)) = \mathcal{F}(\partial A(v))$.

Consider $x, y \in I(1,k)_0$ with the following properties:

- $[x,y] \cap \mathrm{dmn}_\Omega(\psi) = \varnothing$;
- $x - 3^{-k}$ and $y + 3^{-k}$ belong to $\mathrm{dmn}_\Omega(\psi)$.

Observe that, for every $z \in [x, y - 3^{-k}] \cap I(1,k)_0$, we have

$$\mathrm{spt}(||A(z)||) \subset M - \Omega. \tag{11}$$

In fact, observe that $\psi(z)$ and $\psi(z+3^{-k})$ have zero mass in $\overline{\Omega}$ and $\psi(z+3^{-k}) = \psi(z) + \partial A(z)$, so the possibilities are that either $\mathrm{spt}(||A(z)||)$ contains Ω or does not intersect it. The first case is not possible because $A(z)$ is an integral current with mass smaller than $\mathbf{f}(\psi) \leq \mathcal{H}^n(\Omega)$. Next, we explain how ψ is modified on each such $[x,y]$, which is called a maximal interval of non-intersecting slices. Observe that $\mathrm{spt}(||\psi(x)||) \cup \mathrm{spt}(||\psi(y)||) \subset M - \overline{\Omega}$. Let $\tilde{M} = M - \Omega$ and consider the integral current

$$A = \sum A(z) \in \mathbf{I}_n(\tilde{M}),$$

where we sum over all $z \in [x, y - 3^{-k}] \cap I(1,k)_0$. Note that $\partial A = \psi(y) - \psi(x)$ and that $\mathrm{spt}(||A||)$ do not intersect $\partial\tilde{M}$. Let $\phi : [0,1] \to \mathcal{Z}_{n-1}(\tilde{M})$ be the map obtained via Lemma 6.1 applied to the chosen A.

To produce a discrete map, we have to discretize ϕ. Since we are dealing with a one-parameter map, we can directly apply Lemma 5.5. Consider a compact set $K \subset M - \overline{\Omega} =: U$, such that

$$\mathrm{spt}(||\phi(t)||) \subset K, \quad \text{for all } t \in [0,1]. \tag{12}$$

This is related to the extra property of ϕ that we have in Lemma 6.1, because $\mathrm{spt}(||A||) \subset M - \overline{\Omega}$. Choose $\varepsilon_0 > 0$ that makes the statement of Lemma 5.5 work with $L = C \cdot \mathbf{M}(\partial A)$, $\eta \leq \mathbf{f}(\psi)$, $T = 0$, and the fixed $K \subset U$. The constant $C = C(M, \Omega) > 0$ is the one in Lemma 6.1. Let $k_1 \in \mathbb{N}$ be large enough so that $\phi_1 := \phi|_{I(1,k_1)_0}$ is ε_0-fine in the flat topology. In this case, for all $\theta \in I(1,k_1)_0$, we have:

- $\mathcal{F}(\phi_1(\theta + 3^{-k_1}) - \phi(\theta)) \leq \varepsilon_0$;
- $\mathrm{spt}(\phi(\theta)) \subset K$;
- $\mathbf{M}(\phi(\theta)) \leq L = C \cdot \mathbf{M}(\partial A)$.

Lemma 5.5 says that we can take $\tilde{k} \geq k_1$, so that ϕ_1 admits extension $\tilde{\phi}$ to $I(1,\tilde{k})_0$ with fineness $\mathbf{f}(\tilde{\phi}) \leq \eta \leq \mathbf{f}(\psi)$, $\mathrm{spt}(\tilde{\phi}(\theta)) \subset M - \overline{\Omega}$, and with uniformly controlled masses

$$\mathbf{M}(\tilde{\phi}(\theta)) \leq L + \eta \leq C \cdot \mathbf{M}(\partial A) + \mathbf{f}(\psi).$$

We replace $\psi|_{[x,y]\cap I(1,k)_0}$ with a discrete map defined in a finer grid

$$\tilde{\psi} : I(1, k+\tilde{k})_0 \cap [x,y] \to \mathcal{Z}_{n-1}(M),$$

such that

$$\tilde{\psi}(w) = \psi(y) - \tilde{\phi}((w-x)\cdot 3^k),$$

if $w \in I(1,k+\tilde{k})_0 \cap [x, x+3^{-k}]$, and $\tilde{\psi}(w) = \psi(y)$, otherwise. This map has the same fineness as $\tilde{\phi}$, and so $\mathbf{f}(\tilde{\psi}) \leq \mathbf{f}(\psi)$. Also, $\tilde{\psi}$ and ψ agree in x and y,

$$\tilde{\psi}(x) = \psi(y) - \tilde{\phi}(0) = \psi(y) - \partial A = \psi(x) \text{ and } \tilde{\psi}(y) = \psi(y).$$

Moreover, no slice $\tilde{\psi}(w)$ intersects $\overline{\Omega}$ and

$$\begin{aligned} \mathbf{M}(\tilde{\psi}(w)) &\leq \mathbf{M}(\psi(y)) + C\cdot \mathbf{M}(\partial A) + \mathbf{f}(\psi) \\ &\leq C\cdot \left(\max\{\mathbf{M}(\psi(z)) : z \in \mathrm{dmn}_\Omega(\psi)\} + \mathbf{f}(\psi)\right). \end{aligned} \tag{13}$$

The second line is possible, because $x - 3^{-k}$ and $y + 3^{-k}$ belong to $\mathrm{dmn}_\Omega(\psi)$. Also, the constant appearing in the last inequality is bigger than the original, but still depending only on M and Ω.

Observe that one can write the original ψ restricted to $[x,y] \cap I(1,k)_0$, similarly to the expression that defines $\tilde{\psi}$, as $\psi(w) = \psi(y) - \hat{\phi}(w/3)$, where $\hat{\phi} : I(1,k+1)_0 \to \mathcal{Z}_{n-1}(M)$ is given by

- $\hat{\phi}(\theta) = \partial A$, if $\theta \in [0, x/3] \cap I(1,k+1)_0$;
- $\hat{\phi}(\theta) = \psi(y) - \psi(3\theta)$, if $\theta \in [x/3, y/3] \cap I(1,k+1)_0$;
- $\hat{\phi}(\theta) = 0$, if $\theta \in [y/3, 1] \cap I(1,k+1)_0$.

We concatenate the inverse direction of $\tilde{\phi}$ with $\hat{\phi}$ to construct

$$\overline{\phi} : I(1, \overline{k}+1)_0 \to \mathcal{Z}_{n-1}(M), \tag{14}$$

where $\overline{k} = \max\{\tilde{k}, k+1\}$, and

$$\overline{\phi}(\theta) = \begin{cases} \tilde{\phi} \circ \mathbf{n}(\overline{k}, \tilde{k})(1-3\theta) & \text{if } 0 \leq \theta \leq 3^{-1} \\ \hat{\phi} \circ \mathbf{n}(\overline{k}, k+1)(3\theta - 1) & \text{if } 3^{-1} \leq \theta \leq 2 \cdot 3^{-1} \\ 0 & \text{if } 2 \cdot 3^{-1} \leq \theta \leq 1. \end{cases}$$

Claim 6.2. *We have that $\boldsymbol{f}(\overline{\phi}) \leq \boldsymbol{f}(\psi)$ and $\overline{\phi}$ is 1-homotopic to zero in $(\mathcal{Z}_{n-1}(M; \boldsymbol{M}), \{0\})$ with fineness $C_1 \cdot \boldsymbol{f}(\psi)$.*

The constant $C_1 > 0$ is uniform, in the sense that it does not depend on ψ. The proof of this claim finishes the argument, because the discrete homotopy between $\overline{\phi}$ and the zero map tells us that the initial map ψ and the $\tilde{\psi}$ we built are 1-homotopic with the same fineness. Claim 6.2 is a consequence of Almgren's Isomorphism. q.e.d.

Lemma 6.1 is about the construction of natural deformations starting with the boundary of a fixed n-dimensional integral current A in a compact manifold with boundary, and contracting it continuously and with controlled masses to the zero current. This generalizes the notion of cones in $\mathbb{R}^n$.

Proof of Lemma 6.1. To make the notation simpler, in this proof we use M instead of $\tilde{M}$ to denote the manifold with boundary.

Step 1: Consider a Morse function $f : M \to [0,1]$ with $f^{-1}(1) = \partial M$ and no interior local maximum. Let $C(f) = \{p_1, \ldots, p_k\} \subset M$ be the critical set of f, with $c_i = f(p_i)$ and $\operatorname{index}(f, p_i) = \lambda_i \in \{0, 1, \ldots, n-1\}$. Suppose, without loss of generality, $0 = c_k < \cdots < c_1 < 1$.

We adapt Morse's theorems to construct a list of homotopies

$$h_i : [0,1] \times M_i \to M_i, \quad \text{for } i = 1, \ldots, k,$$

defined on sublevel sets $M_1 = M$ and $M_i = M^{c_{i-1} - \varepsilon_{i-1}} = \{f \leq c_{i-1} - \varepsilon_{i-1}\}$, for $i = 2, \ldots, k$, with sufficiently small $\varepsilon_i > 0$. Those maps are constructed in such a way that we have the following properties:

- h_i is smooth for all $i = 1, \ldots, k$;
- $h_i(1, \cdot)$ is the identity map of M_i;
- $h_i(0, M_i)$ is contained in M_{i+1} with a λ_i-cell attached, if $i \leq k-1$;
- $h_k(0, M_k) = f^{-1}(0)$.

This is achieved by Theorem A.1, in Appendix A.

Step 2: Recall that $A \in \mathbf{I}_n(M)$ and $h_0(0, M)$ is contained in the union of M_1 with a λ_1-cell. Since $\lambda_1 < n$, the support of $A^1 = h_1(0, \cdot)_\# A$ is a subset of M_2. Indeed, it is an integral n-dimensional current and the λ_1-cell has zero n-dimensional measure. Inductively, we observe:

$$A^i := h_i(0, \cdot)_\# A^{i-1} \text{ has support in } M_{i+1}, \text{ for } i = 2, \ldots, k,$$

with the additional notation $M_{k+1} = f^{-1}(0)$. In particular, this allows us to concatenate the images of A by the sequence of homotopies; i.e., consider the map $\Phi : [0,1] \to \mathbf{I}_n(M)$ that starts with $\Phi(0) = A$ and is inductively defined by

$$\Phi(t) = h_i(i-kt,\cdot)_\# \Phi((i-1)/k), \quad \text{ for } [(i-1)/k, i/k] \text{ and } i = 1, \dots, k.$$

Step 3: The homotopy formula, page 139 in [**17**], tells us

$$\Phi(t) - \Phi(s) = (h_i)_\# \left([|(i-ks, i-kt)|] \times \partial\Phi((i-1)/k) \right), \tag{15}$$

for $(i-1)/k \le t \le s \le i/k$. In fact, the boundary term vanishes,

$$\partial (h_i)_\# \left([|(i-ks, i-kt)|] \times \Phi((i-1)/k) \right) = 0,$$

because $[|(i-ks, i-kt)|] \times \Phi((i-1)/k)$ is $(n+1)$-dimensional, while h_i takes values in a n-dimensional space.

Step 4: The mass of the product current $[|(i-ks, i-kt)|] \times \partial\Phi((i-1)/k)$ is the product measure of Lebesgue's measure on $(i-ks, i-kt)$ and the mass of $\partial\Phi((i-1)/k)$. Since all homotopies $h_1, \dots, h_k$ are smooth, we conclude that Φ is continuous in the mass norm and that there exists a positive constant $C = C(M) > 0$ such that

$$\mathbf{M}(\partial\Phi(t)) \le C \cdot \mathbf{M}(A), \quad \text{for all } t \in [0,1].$$

Step 5: Consider ϕ as the boundary map applied to Φ, i.e., $\phi(t) = \partial\Phi(t)$. Since Φ is continuous in the mass topology, it follows directly that ϕ is continuous in the flat metric. Moreover, all maps $h_i(t,\cdot)$ with $t > 0$ are diffeomorphisms; this implies that ϕ is continuous in the F-metric, up to finitely many points.

The last claim in the statement of Lemma 6.1 follows from construction, because $\mathrm{spt}(||A||) \subset \mathrm{int}(\tilde{M})$ implies there exists a small $\varepsilon > 0$ such that $\mathrm{spt}(||A||)$ is contained in the sublevel set $K := \{f \le 1 - \varepsilon\}$. q.e.d.

7. Deforming currents with small intersecting mass

Let (M^n, g) be a compact Riemannian manifold isometrically embedded in $\mathbb{R}^L$. Consider open subsets $W \subset\subset U$ in M. The goal of this section is to prove that it is possible to deform an integral current T with small mass in U to a current T^* outside W. The deformation is discrete, with support in U; arbitrarily small fineness and the masses along the deformation sequence can not increase much. In this part, the mean-concavity is not used and the statements do not involve Ω.

Lemma 7.1. *There exists $\varepsilon_1 > 0$ with the following property: given*

$$T \in \mathcal{Z}_{n-1}(M^n) \text{ with } ||T||(U) < \varepsilon_1 \text{ and } \eta > 0,$$

it is possible to find a sequence $T = T_1, \dots, T_q \in \mathcal{Z}_{n-1}(M^n)$ such that

(1) *$spt(T_l - T) \subset U$;*
(2) *$\boldsymbol{M}(T_l - T_{l-1}) \le \eta$;*

(3) $\boldsymbol{M}(T_l) \leq \boldsymbol{M}(T) + \eta$;
(4) $spt(T_q) \subset M - \overline{W}$.

REMARK 10. The constant ε_1 depends only on M and $d_g(W, M-U)$.

The main ingredients to prove this result are: a discrete deformation process used by Pitts in the construction of replacements; See section 3.10 in [**19**], and the monotonicity formula.

7.1. Pitts' deformation argument. *Let $K \subset U$ be subsets of M, with K compact and U open. Given $T \in \mathcal{Z}_k(M)$ and $\eta > 0$, there exists a finite $T = T_1, \dots, T_q \in \mathcal{Z}_k(M)$ such that*

(i) $spt(T_l - T) \subset U$;
(ii) $\boldsymbol{M}(T_l - T_{l-1}) \leq \eta$;
(iii) $\boldsymbol{M}(T_l) \leq \boldsymbol{M}(T) + \eta$;
(iv) $\boldsymbol{M}(T_q) \leq \boldsymbol{M}(T)$;
(v) T_q *is locally area-minimizing in* $int(K)$.

The monotonicity formula implies that there are $C, r_0 > 0$, such that for any minimal submanifold $\Sigma^k \subset M$ and $p \in \Sigma$, we have $\mathcal{H}^k(\Sigma \cap B(p,r)) \geq Cr^k$, for all $0 < r < r_0$. As references for the monotonicity formula, see [**5**] or Section 17 of [**17**]. Now, we proceed to the proof of small mass deformation.

Proof of Lemma 7.1*:* Consider a compact subset K with $W \subset\subset K \subset U$, and let

$$T \in \mathcal{Z}_{n-1}(M^n) \text{ with } ||T||(U) < \varepsilon_1 \text{ and } \eta > 0$$

be given, $\varepsilon_1 > 0$ to be chosen. Apply Corollary 7.1 to these $K \subset U$, T, and η, to obtain a finite sequence $T = T_1, \dots, T_q \in \mathcal{Z}_{n-1}(M)$ with properties (i)–(v) of that result; this gives (1)–(3) of Lemma 7.1 automatically. Note that (i) and (iv) imply $||T_q||(U) \leq ||T||(U) \leq \varepsilon_1$.

Note that the induced varifold $|T_q|$ is stationary in $\text{int}(K)$, being locally area-minimizing in this open set. Suppose T_q is not outside W; take $p \in W \cap \text{spt}(T_q)$ and $0 < r = 2^{-1} \min\{r_0, d_g(W, M-K)\}$. Then, applying the monotonicity formula, we have

$$Cr^{n-1} \leq ||T_q||(B(p,r)) \leq ||T_q||(U) \leq \varepsilon_1.$$

In order to conclude the proof, we choose $\varepsilon_1 < Cr^{n-1}$. q.e.d.

8. Deforming bad discrete sweepouts

Let (M^n, g) be a closed embedded submanifold of $\mathbb{R}^L$ and $\Omega \subset M$ be an open subset with smooth and strictly mean-concave boundary $\partial\Omega$. Recall the domains $\Omega_a \subset \Omega \subset \Omega_b$ and the maximum principle vector field X that we considered in Subsection 3.2. Let $U \subset \Omega$ be an open subset such that $\overline{\Omega_a} \subset U$. In this section, we consider discrete maps $\phi : I(1,k)_0 \to \mathcal{Z}_{n-1}(M)$ with small fineness for which slices $\phi(x)$ with mass greater than a given $L > 0$ have small mass in U. The goal here is to extend the construction of Section 7 and deform the map ϕ via a 1-homotopy with small fineness.

In order to state the precise result, consider the constants: $C_0 = C_0(M)$ and $\delta_0 = \delta_0(M)$ coming from the interpolation theorems, as introduced in Section 5, and $\varepsilon_1(U, \Omega_a)$ as given by Lemma 7.1 for $W = \Omega_a$ and the fixed U. Consider also the following combinations:

- $(3 + C_0)\eta_0 = \varepsilon_2 = \min\{\varepsilon_1(U, \Omega_a), 5^{-1}\delta_0, 5^{-1}\mathcal{H}^n(\Omega_a)\}$;
- $C_1 = 3C_0 + 7$.

Observe that $C_1 = C_1(M)$, but ε_2 and η_0 depend also on U and Ω_a.

Assume we have a discrete map $\phi : I(1, k)_0 \to \mathcal{Z}_{n-1}(M)$ with fineness $\mathbf{f}(\phi) \leq \eta_0$ and satisfying the property that, for some $L > 0$,

$$\mathbf{M}(\phi(x)) \geq L \Rightarrow ||\phi(x)||(\overline{U}) < \varepsilon_2. \tag{16}$$

Lemma 8.1. *There exists $\tilde{\phi} : I(1, N)_0 \to \mathcal{Z}_{n-1}(M)$ 1-homotopic to ϕ with fineness $C_1 \cdot \boldsymbol{f}(\phi)$, with the following properties:*

$$L(\tilde{\phi}) = \max\{\boldsymbol{M}(\tilde{\phi}(x)) : x \in dmn_\Omega(\tilde{\phi})\} < L + C_1 \cdot \boldsymbol{f}(\phi)$$

and such that the image of $\tilde{\phi}$ is equal to the union of $\{\phi(x) : \boldsymbol{M}(\phi(x)) < L\}$ with a subset of $\{T \in \mathcal{Z}_{n-1}(M) : ||T||(\overline{U}) \leq 2\varepsilon_2\}$.

Remark 11. If $0 < \varepsilon \leq \varepsilon_2$ and we assume that $||\phi(x)||(\overline{U}) < \varepsilon$, in (16), we can still apply the lemma. Moreover, the image of $\tilde{\phi}$ will coincide with $\{\phi(x) : \mathbf{M}(\phi(x)) < L\}$ up to slices with the property $||\tilde{\phi}(x)||(\overline{U}) \leq \varepsilon + C_1 \cdot \eta$.

8.1. Overview of the proof of Lemma 8.1. Deforming each big slice using the small mass procedure, we obtain a map ψ defined in a 2-dimensional grid. The first difficulty that arises is that the obtained map has fineness of order ε_2 instead of $\mathbf{f}(\phi)$. We correct this using the interpolation results; see Section 5. The fine homotopy that we are able to construct ends with a discrete map $\tilde{\phi}$ whose intersecting slices $\tilde{\phi}(x)$ with mass exceeding L by much do not get very deep in Ω, i.e., $\mathrm{spt}(\tilde{\phi}(x) \llcorner \Omega)$ is contained in a small tubular neighborhood of $\partial\Omega$ in M.

The second part is the application of the maximum principle. The idea is to continuously deform the slices supported in $M - \Omega_a$ of the map $\tilde{\phi}$, produced in the first step, via the flow $\{\Phi(s, \cdot)\}_{s \geq 0}$ of the maximum principle vector field; see Corollary 3.1. Since $\Phi(1, M - \Omega_a) \subset M - \overline{\Omega_b}$, the bad slices $\tilde{\phi}(x)$ end outside $\overline{\Omega_b}$. But this deformation is continuous only with respect to the F-metric and we need a map with small fineness in the mass norm. This problem is similar to the difficulty that arises in the classical pull-tight argument. We produce then a discrete version of the maximum principle deformation that is arbitrarily close, in the F-metric, to the original one. This correction creates one more complication, because the approximation can create very big intersecting slices with small mass in Ω_b. To overcome this we apply the small mass procedure again.

Proof of 8.1. Consider the set

$$\mathcal{K} = \{x \in I(1,k)_0 : \mathbf{M}(\phi(x)) \geq L\}.$$

Let $\alpha, \beta \in \mathcal{K}$. A subset $[\alpha,\beta] \cap I(1,k)_0$ is called a maximal interval on $\mathcal{K}$ if $[\alpha,\beta] \cap I(1,k)_0 \subset \mathcal{K}$ and $\alpha - 3^{-k}, \beta + 3^{-k} \notin \mathcal{K}$.

We describe the construction of the homotopy on each maximal interval. Observe that $||\phi(x)||(\overline{U}) < \varepsilon_2 \leq \varepsilon_1$, for every $x \in [\alpha,\beta] \cap I(1,k)_0$.

Let $N_1 = N_1(\phi)$ be a positive integer so that, for each $x \in [\alpha,\beta] \cap I(1,k)_0$, we can apply Lemma 7.1 to $W = \Omega_a \subset\subset U$ and find sequences

$$\phi(x) = T(0,x), T(1,x), \ldots, T(3^{N_1}, x) \in \mathcal{Z}_{n-1}(M),$$

with fineness at most $\eta = \mathbf{f}(\phi)$, controlled supports and masses

$$\text{spt}(T(l,x) - \phi(x)) \subset U \quad \text{and} \quad \mathbf{M}(T(l,x)) \leq \mathbf{M}(\phi(x)) + \eta, \tag{17}$$

and ending with an integral cycle $T(3^{N_1}, x)$ whose support is contained in $M - \overline{\Omega_a}$. Observe that $||\phi(x)||(\Omega) = 0$ implies that $T(l,x)$ is constant $\phi(x)$. Then, we perform the first step of the deformation. Consider

$$\psi : I(1,N_1)_0 \times ([\alpha,\beta] \cap I(1,k)_0) \to \mathcal{Z}_{n-1}(M)$$

defined by $\psi(l,x) = T(l \cdot 3^{N_1}, x)$, for every $x \in [\alpha,\beta] \cap I(1,k)_0$ and $l \in I(1,N_1)_0$. It follows directly from the construction that the map ψ satisfies:

$$\sup\{\mathbf{M}(\psi(l,x)) : l \in I(1,N_1)_0 \text{ and } x \in \{\alpha,\beta\}\} \leq L + 2\eta, \tag{18}$$

and

$$\text{spt}(||\psi(1,x)||) \subset M - \overline{\Omega_a}, \quad \text{for every } x \in [\alpha,\beta] \cap I(1,k)_0. \tag{19}$$

It is also an easy fact that

$$\mathbf{M}(\psi(3^{-N_1}, x) - \psi(3^{-N_1}, x + 3^{-k})) \leq 3\eta, \tag{20}$$

for $x \in [\alpha, \beta - 3^{-k}] \cap I(1,k)_0$. Observe that (17) implies that

$$\sup\{||\psi(l,x)||(\overline{U}) : l \in I(1,N_1)_0\} \leq ||\phi(x)||(\overline{U}) + \eta \leq \varepsilon_2 + \eta, \tag{21}$$

for every x, and $||\psi(1,x)||(\overline{U}) \leq ||\phi(x)||(\overline{U}) \leq \varepsilon_2$. Moreover,

$$\mathbf{f}(\psi) \leq 3\eta + 4\varepsilon_2 \leq 5\varepsilon_2 \leq \min\{\delta_0, \mathcal{H}^n(\Omega_a)\}. \tag{22}$$

We have to prove that $\mathbf{M}(\psi(l,x) - \psi(l,x')) \leq 5\varepsilon_2$, for $|x - x'| \leq 3^{-k}$. Indeed, rewrite that difference as

$$(\psi(l,x) - \phi(x)) + (\phi(x) - \phi(x')) + (\phi(x') - \psi(l,x')). \tag{23}$$

The first and third terms are similar; if they are non-zero, the analysis follows the steps: $\mathbf{M}(\psi(l,x) - \phi(x)) = ||\psi(l,x) - \phi(x)||(U)$, because $\psi(l,x)$ is constructed in such a way that $\text{spt}(\psi(l,x) - \phi(x)) \subset U$ and

$$\begin{aligned} ||\psi(l,x) - \phi(x)||(U) &\leq ||\psi(l,x)||(U) + ||\phi(x)||(U) \\ &\leq \eta + 2||\phi(x)||(U) \\ &< \eta + 2\varepsilon_2. \end{aligned}$$

In the above estimate, we have used only (21) and that $||\phi(x)||(U) < \varepsilon_2$. The mass of the second term of (23) is at most η, then

$$\mathbf{M}(\psi(l,x) - \psi(l,x')) \leq 3\eta + 4\varepsilon_2 \leq 5\varepsilon_2.$$

The last inequality follows from the choices of $\eta \leq \eta_0$ and $3\eta_0 \leq \varepsilon_2$.

In order to get the desired fineness on the final 1-homotopy, we have to interpolate. For each $l \in I(1,N_1)_0 - \{0,1\}$ and $x \in [\alpha, \beta - 3^{-k}] \cap I(1,k)_0$, we apply Theorem 5.1 to the restriction of ψ to the four corner vertices of $[l, l+3^{-N_1}] \times [x, x+3^{-k}]$. This is allowed because of the expression (22). The result of this step is a continuous map in the mass norm

$$\Psi : [3^{-N_1}, 1] \times [\alpha, \beta] \to \mathcal{Z}_{n-1}(M) \tag{24}$$

that extends ψ. Moreover, in the 1-cells of the form $[l, l+3^{-N_1}] \times \{x\}$, the interpolating elements $\Psi(s,x)$ differ from $\psi(l,x)$ or $\psi(l+3^{-N_1},x)$ in the mass norm at most by a factor of $C_0\mathbf{M}(\psi(l,x) - \psi(l+3^{-N_1},x)) \leq C_0\eta$. This remark implies that

$$\sup\{\mathbf{M}(\Psi(s,x)) : s \in [3^{-N_1}, 1] \text{ and } x \in \{\alpha,\beta\}\} \leq L + (2+C_0)\eta, \tag{25}$$

and, together with expression (21),

$$\sup\{||\Psi(s,x)||(\overline{U}) : s \in [3^{-N_1}, 1] \text{ and } x \in \{\alpha,\beta\}\} \leq (\varepsilon_2+\eta)+C_0\eta. \tag{26}$$

Since $\Psi(1,\alpha) = \psi(1,\alpha)$, we have better estimates $\mathbf{M}(\Psi(1,\alpha)) \leq L + 2\eta$ and $||\Psi(1,\alpha)||(\overline{U}) \leq \varepsilon_2$, at time $s = 1$. The same holds for β. A similar assertion holds for the 1-cells $\{l\} \times [x, x+3^{-k}]$ and gives us

$$\mathbf{M}(\Psi(3^{-N_1},t) - \psi(3^{-N_1},x)) \leq 3C_0\eta, \tag{27}$$

for $x \in [\alpha,\beta] \cap I(1,k)_0$ and $t \in [\alpha,\beta]$ with $|t-x| \leq 3^{-k}$. Property (27) implies that restrictions of $\Psi(3^{-N_1},\cdot)$ to any $[\alpha,\beta] \cap I(1,N)_0$ are 1-homotopic to the original ϕ with fineness at most $(3C_0+1)\mathbf{f}(\phi)$. Finally, since $\mathbf{f}(\psi) \leq \mathcal{H}^n(\Omega_a)$ and $\text{spt}(||\psi(1,x)||) \subset M - \overline{\Omega_a}$, we can suppose that

$$\text{spt}(||\Psi(1,t)||) \subset M - \overline{\Omega_a}, \quad \text{for every } t \in [\alpha,\beta]. \tag{28}$$

Recall that $\{\Phi(s,\cdot)\}_s$ is the flow of the maximum principle vector field X and consider the map

$$(s,t) \in [1,2] \times [\alpha,\beta] \mapsto \Phi(s-1,\cdot)_\#(\Psi(1,t)) =: \Psi(s,t).$$

This map is continuous in the F-metric, because each map $\Phi(s-1,\cdot)$ is diffeomorphism. Since $\Phi(s,\cdot)$ is a mass-decreasing flow, we have

$$\mathbf{M}(\Psi(s,t)) \leq \mathbf{M}(\Psi(1,t)) \leq \max\{\mathbf{M}(\Psi(1,t)) : t \in [\alpha,\beta]\} < \infty, \tag{29}$$

for every $t \in [\alpha,\beta]$ and $s \in [1,2]$. In particular, the estimates (25) and (26) hold on the whole $[3^{-N_1}, 2]$. Moreover, because of (2), we have

$$\text{spt}(||\Psi(2,t)||) \subset M - \overline{\Omega_b}, \quad \text{for every } t \in [\alpha,\beta]. \tag{30}$$

The final homotopy must be a fine discrete map defined on a 2-dimensional grid. In order to attain this we interpolate once more, but now via Corollary 5.3. Let $0 < \varepsilon_3 \leq \varepsilon_1(\Omega_b, \Omega)$ be chosen in such a way that we can apply Lemma 7.1 with the sets Ω and Ω_b, and with $\varepsilon_3 \leq \eta$. Apply Corollary 5.3 for a sufficiently small δ, to obtain a number $N_2 = N_2(\phi) \geq N_1 + k$ and a discrete map

$$\Psi_1 : I(1, N_2)_0 \times ([\alpha, \beta] \cap I(1, N_2)_0) \to \mathcal{Z}_{n-1}(M),$$

such that

(i) $\mathbf{f}(\Psi_1) < \eta$;

(ii)

$$\sup\{||\Psi_1(\sigma, \tau)||(\overline{U}) : \sigma \in I(1, N_2)_0, \tau \in \{\alpha, \beta\}\} < (\varepsilon_2 + (1 + C_0)\eta) + \eta;$$

(iii)

$$\sup\{||\Psi_1(1, \tau))||(\overline{\Omega_b}) : \tau \in [\alpha, \beta] \cap I(1, N_2)_0\} < \varepsilon_3;$$

(iv)

$$\sup\{\mathbf{M}(\Psi_1(\sigma, x)) : \sigma \in I(1, N_2)_0 \text{ and } x \in \{\alpha, \beta\}\} \leq L + (3 + C_0)\eta;$$

(v) $\Psi_1(0, \tau) = \Psi(3^{-N_1}, \tau)$, if $\tau \in [\alpha, \beta] \cap I(1, N_2)_0$.

The next step is a second application of Lemma 7.1, now for the slices $\Psi_1(1, \tau)$. The choice of ε_3 guarantees that there exists $N_3 = N_3(\phi) \in \mathbb{N}$ and an extension of Ψ_1 to the discrete domain

$$\text{dmn}(\Psi_1) = (I(1, N_2)_0 \cup (I(1, N_3)_0 + \{1\})) \times ([\alpha, \beta] \cap I(1, N_2)_0),$$

where $I(1, N_3)_0 + \{1\} = \{\lambda + 1 : \lambda \in I(1, N_3)_0\}$. This map has the following properties: in the same spirit as (22), we have, respectively,

$$\mathbf{f}(\Psi_1) \leq 3\eta + 4\varepsilon_3 \leq 7\eta. \tag{31}$$

Moreover, similarly to (18), (19), and (21), we have, respectively,

$$\sup\{\mathbf{M}(\Psi_1(\sigma, \tau)) : (\sigma, \tau) \in \text{dmn}(\Psi_1) \text{ and } \tau \in \{\alpha, \beta\}\} \leq L + (4 + C_0)\eta, \tag{32}$$

$$\text{spt}(||\Psi_1(2, \tau)||) \subset M - \overline{\Omega}, \quad \text{for every } \tau \in ([\alpha, \beta] \cap I(1, N_2)_0), \tag{33}$$

and

$$\sup\{||\Psi_1(\sigma, \tau)||(\overline{U}) : (\sigma, \tau) \in \text{dmn}(\Psi_1), \tau \in \{\alpha, \beta\}\} < \varepsilon_2 + (2 + C_0)\eta.$$

The last part is the organization of the homotopy. Take $N = N(\phi)$ sufficiently large that it is possible to define a map

$$\Psi' : I(1, N)_0 \times ([\alpha - 3^{-k}, \beta + 3^{-k}] \cap I(1, N)_0) \to \mathcal{Z}_{n-1}(M)$$

in the following way:

- if $j = 0, 1, \ldots, 3^{N_2}$ and $\tau \in ([\alpha, \beta] \cap I(1, N)_0)$,
$$\Psi'(j \cdot 3^{-N}, \tau) = \Psi_1(j \cdot 3^{-N_2}, \mathbf{n}(N, N_2)(\tau));$$
- if $j = 0, 1, \ldots, 3^{N_3}$ and $\tau \in ([\alpha, \beta] \cap I(1, N)_0)$,
$$\Psi'(3^{-N+N_2} + j \cdot 3^{-N}, \tau) = \Psi_1(1 + j \cdot 3^{-N_3}, \mathbf{n}(N, N_2)(\tau));$$
- if $0 < \lambda_2 \leq \lambda_1 \leq (3^{N_2} + 3^{N_3}) \cdot 3^{-N}$,
$$\Psi'(\lambda_1, \alpha - \lambda_2) = \Psi'(\lambda_1 - \lambda_2, \alpha) \text{ and } \Psi'(\lambda_1, \beta + \lambda_2) = \Psi'(\lambda_1 - \lambda_2, \beta);$$

- if $0 < \lambda_1 < \lambda_2 \leq (3^{N_2} + 3^{N_3}) \cdot 3^{-N}$,
$$\Psi'(\lambda_1, \alpha - \lambda_2) = \phi(\alpha) \text{ and } \Psi'(\lambda_1, \beta + \lambda_2) = \phi(\beta);$$
- if $0 \leq \lambda_1 \leq (3^{N_2} + 3^{N_3}) \cdot 3^{-N} < \lambda_2 \leq 3^{-k}$
$$\Psi'(\lambda_1, \alpha - \lambda_2) = \phi(\mathbf{n}(N,k)(\alpha - \lambda_2))$$
and
$$\Psi'(\lambda_1, \beta + \lambda_2) = \phi(\mathbf{n}(N,k)(b + \lambda_2));$$
- if $(3^{N_2} + 3^{N_3}) \cdot 3^{-N} \leq \lambda \leq 1$ and $\tau([\alpha - 3^{-k}, \beta + 3^{-k}] \cap I(1,N)_0)$, put
$$\Psi'(\lambda, \tau) = \Psi'((3^{N_2} + 3^{N_3}) \cdot 3^{-N}, \tau).$$

In order to obtain a 1-homotopy we need to take N such that
$$(3^{N_2} + 3^{N_3}) \cdot 3^{-N} < \frac{1}{2} \cdot 3^{-k}.$$

Extend Ψ' to $I(1,N)_0 \times I(1,N)_0$, using the above construction near each maximal interval on $\mathcal{K}$ and putting $\Psi'(\lambda, \tau) = \phi(n(N,k)(\tau))$ on the complement. This map is a homotopy and has fineness $\mathbf{f}(\Psi') \leq 7\eta$. Then, the obtained map $\tilde{\phi}(\cdot) = \Psi'(1, \cdot)$ is 1-homotopic to the original discrete map ϕ with fineness at most $\max\{7, 3C_0 + 1\}\mathbf{f}(\phi) \leq C_1\mathbf{f}(\phi)$. The $(3C_0 + 1)$ factor comes from the first deformation step; recall (27).

Moreover, if $x \in ([\alpha - 3^{-k}, \beta + 3^{-k}] \cap I(1,N)_0)$ we have three possibilities for $\tilde{\phi}(x)$: it coincides either with some $\Psi_1(\sigma, \alpha)$, or $\Psi_1(\sigma, \beta)$, or $\Psi_1(2, \tau)$. By (32) and (33), we conclude that if $x \in ([\alpha - 3^{-k}, \beta + 3^{-k}] \cap I(1,N)_0)$, then either $\mathbf{M}(\tilde{\phi}(x)) \leq L + C_1\mathbf{f}(\phi)$ or $x \notin \mathrm{dmn}_\Omega(\tilde{\phi})$. In particular, if $x \in ([\alpha - 3^{-k}, \beta + 3^{-k}] \cap \mathrm{dmn}_\Omega(\tilde{\phi}))$, then

$$\mathbf{M}(\tilde{\phi}(x)) \leq L + (4 + C_0)\eta < L + C_1\mathbf{f}(\phi). \tag{34}$$

If $x \in \mathrm{dmn}_\Omega(\tilde{\phi})$ and $x \notin [\alpha - 3^{-k}, \beta + 3^{-k}]$, for any $[\alpha, \beta]$ maximal on $\mathcal{K}$, then $\tilde{\phi}(x)$ also appears in ϕ and $\mathbf{M}(\tilde{\phi}(x)) < L$. This concludes the proof. q.e.d.

9. Construction of discrete sweepouts

Next, we apply Lemma 8.1 to prove Theorem 4.4.

Proof of Theorem 4.4. Let ϕ_i, ψ_i, δ_i be given by Theorem 5.2 applied to the map Γ. It follows from property (iv) of Theorem 5.2 and the fact that $\Gamma(0) = \Gamma(1) = 0$ that, for all $y \in I(1,k_i)_0$ and $x \in \{0,1\}$, we have

$$\mathbf{M}(\psi_i(y,x)) \leq \delta_i. \tag{35}$$

Define $\bar{\psi}_i : I(1,k_i)_0 \times I(1,k_i)_0 \to \mathcal{Z}_{n-1}(M)$ by $\bar{\psi}_i(y,x) = 0$ if $x \in \{0,1\}$ and $\bar{\psi}_i(y,x) = \psi_i(y,x)$ otherwise. Define also $\bar{\phi}_i(x) = \bar{\psi}_i([0],x)$ for $x \in I(1,k_i)_0$. Note that $\mathbf{f}(\bar{\psi}_i) < 2\delta_i$, by (35) and Theorem 5.2 part (ii).

Then, we obtain $\{\bar{\phi}_i\}_{i\in\mathbb{N}}$, that is, a $(1,\mathbf{M})$-homotopy sequence of mappings into $(\mathcal{Z}_{n-1}(M;\mathbf{M}),0)$. But we can not control its width by $L(\Gamma,\Omega)$ yet. To simplify notation, let us keep using ϕ_i and ψ_i.

Since Γ is continuous in the $\mathbf{F}$-metric, $\mathbf{M}\circ\Gamma$ is uniformly continuous on I. Combine this with item (i) of Theorem 5.2 to conclude that

$$\mathbf{M}(\phi_i(y)) \leq \mathbf{M}(\Gamma(y)) + \frac{1}{i} + \delta_i. \tag{36}$$

From Lemma 5.4, property (iii) of Theorem 5.2, and (36), we get

$$\lim_{i\to\infty} \sup\{\mathbf{F}(\phi_i(x) - \Gamma(x)) : x \in \operatorname{dmn}(\phi_i)\} = 0.$$

Recall the domains Ω_t starting with $\Omega = \Omega_0$ defined in Subsection 3.2, and also Ω_a, $a < 0$, fixed by Corollary 3.1.

Let $U \subset\subset \Omega$ be an open subset with $\overline{\Omega_a} \subset U$, and $0 < \varepsilon = \varepsilon_2(U,\Omega_a)$ be given by Lemma 8.1 with respect to the subsets U and Ω_a. The set

$$\mathcal{K} = \{\Gamma(t) : t \in [0,1] \text{ and } ||\Gamma(t)||(\Omega) = 0\}$$

is compact with respect to the $\mathbf{F}$-metric. Since $\overline{U} \subset \Omega$, it is possible to find $\rho > 0$ so that $\mathbf{F}(\mathcal{K},T) < \rho$ implies $||T||(\overline{U}) < \varepsilon_2$. This construction of ρ does not involve the interpolation step, so we can suppose that

$$\sup\{\mathbf{F}(\phi_i(x) - \Gamma(x)) : x \in \operatorname{dmn}(\phi_i)\} < \rho, \tag{37}$$

for large $i \in \mathbb{N}$. If $x \in I(1,k_i)_0$ is such that $x \in \mathcal{T}(\Gamma,\Omega)$, expression (36) gives us

$$\mathbf{M}(\phi_i(x)) \leq L(\Gamma,\Omega) + \frac{1}{i} + \delta_i.$$

Otherwise, $\Gamma(x) \in \mathcal{K}$ and (37) imply $||\phi_i(x)||(\overline{U}) < \varepsilon_2$. If $i \in \mathbb{N}$ is sufficiently large, we apply Lemma 8.1 to obtain $\tilde{\phi}_i$ 1-homotopic to ϕ_i with fineness tending to zero and such that

$$\max\{\mathbf{M}(\tilde{\phi}_i(x)) : x \in \operatorname{dmn}_\Omega(\tilde{\phi}_i)\} \leq L(\Gamma,\Omega) + \frac{1}{i} + \delta_i + C_1 \cdot \mathbf{f}(\phi_i).$$

q.e.d.

10. Pull-tight argument

The classical pull-tight argument is based on the construction of an area decreasing flows letting the stationary varifolds remain unmoved. Flowing all slices of a critical sequence S^*, we produce a better competitor S, for which critical varifolds are stationary in M. In our setting, we use a slightly different flow, because we also leave the non-intersecting varifolds unmoved.

Precisely, let $S^* = \{\phi_i^*\}_{i\in\mathbb{N}}$ be a critical sequence with respect to Ω. Consider the set $A_0 \subset \mathcal{V}_{n-1}(M)$ of varifolds with $||V||(M) \leq 2C$ and with one of the following properties: either V is stationary in M or $||V||(\Omega) = 0$. Here, $C = \sup\{\mathbf{M}(\phi_i^*(x)) : i \in \mathbb{N} \text{ and } x \in \operatorname{dmn}(\phi_i^*)\}$.

Following the same steps as in Section 15 of [**14**], we get a map

$$H : [0,1] \times (\mathcal{Z}_{n-1}(M;\mathbf{F}) \cap \{\mathbf{M} \le 2C\}) \to (\mathcal{Z}_{n-1}(M;\mathbf{F}) \cap \{\mathbf{M} \le 2C\}),$$

whose key properties are

(i) H is continuous in the product topology;
(ii) $H(t,T) = T$ for all $0 \le t \le 1$ if $|T| \in A_0$;
(iii) $||H(1,T)||(M) < ||T||(M)$ unless $|T| \in A_0$.

Direct application of H on the slices of S^*, as in Section 15 of [**14**], does not necessarily provide a better competitor, because this involves discrete approximations. Indeed, it is possible that the approximation near very big non-intersecting slices of ϕ_i^* creates bad intersecting slices. To overcome this difficulty, we make the approximation very close and use Lemma 8.1.

Proof of Proposition 4.6. We follow closely the proof given in Proposition 8.5 of [**14**]. The first claim about existence of critical sequences is proved in Section 4.

We concentrate now in the pull-tight deformation of a given critical sequence $S^* \in \Pi$. Suppose $S^* = \{\phi_i^*\}_{i\in\mathbb{N}}$, and set

$$C = \sup\{\mathbf{M}(\phi_i^*(x)) : i \in \mathbb{N} \text{ and } x \in \mathrm{dmn}(\phi_i^*)\} < +\infty.$$

Consider the following compact subsets of $\mathcal{V}_{n-1}(M)$:

$$A = \{V \in \mathcal{V}_{n-1}(M) : ||V||(M) \le 2C\};$$
$$A_0 = \{V \in A : \text{either } V \text{ is stationary in } M \text{ or } ||V||(\Omega) = 0\}.$$

Following the same steps as in Section 15 of [**14**], we get a map

$$H : [0,1] \times (\mathcal{Z}_{n-1}(M;\mathbf{F}) \cap \{\mathbf{M} \le 2C\}) \to (\mathcal{Z}_{n-1}(M;\mathbf{F}) \cap \{\mathbf{M} \le 2C\}),$$

whose key properties are

(i) H is continuous in the product topology;
(ii) $H(t,T) = T$ for all $0 \le t \le 1$ if $|T| \in A_0$;
(iii) $||H(1,T)||(M) < ||T||(M)$ unless $|T| \in A_0$.

We now proceed to the construction of a sequence $S = \{\phi_i\}_{i\in\mathbb{N}} \in \Pi$ such that $\mathcal{C}(S,\Omega) \subset A_0 \cap \mathcal{C}(S^*,\Omega)$, up to critical varifolds V with $||V||(\Omega) = 0$. Denote the domain of ϕ_i^* by $I(1,k_i)_0$, and let $\delta_i = \mathbf{f}(\phi_i^*)$. Up to subsequence, we can suppose that, for $x \in I(1,k_i)_0$, either $\mathbf{M}(\phi_i^*(x)) < \mathbf{L}(\Pi,\Omega) + i^{-1}$ or $\mathrm{spt}(||\phi_i^*(x)||) \subset M - \overline{\Omega}$.

For sufficiently large i, apply Theorem 5.1 to obtain continuous maps in the mass norm

$$\bar{\Omega}_i : [0,1] \to \mathcal{Z}_{n-1}(M;\mathbf{M}),$$

such that for all $x \in I(1,k_i)_0$ and $\alpha \in I(1,k_i)_1$ we have

$$\bar{\Omega}_i(x) = \phi_i^*(x) \quad \text{and} \quad \sup_{y,z\in\alpha}\{\mathbf{M}(\bar{\Omega}_i(z) - \bar{\Omega}_i(y))\} \le C_0\delta_i. \tag{38}$$

Moreover, for every $x \in [0,1]$, either $\mathbf{M}(\overline{\Omega}_i(x)) < \mathbf{L}(\Pi, \Omega) + i^{-1} + C_0\delta_i$ or $\operatorname{spt}(||\overline{\Omega}_i(x)||) \subset M - \overline{\Omega}$.

Consider the continuous map in the $\mathbf{F}$-metric

$$\Omega_i : [0,1] \times [0,1] \to \mathcal{Z}_{n-1}(M; \mathbf{F}), \quad \Omega_i(t,x) = H(t, \overline{\Omega}_i(x)).$$

Observe that, for every $(t,x) \in [0,1]^2$ and large $i \in \mathbb{N}$, we have either $\mathbf{M}(\Omega_i(t,x)) \leq \mathbf{M}(\overline{\Omega}_i(x)) < \mathbf{L}(\Pi, \Omega) + i^{-1} + C_0\delta_i$ or $\operatorname{spt}(||\Omega_i(t,x)||) \subset M - \overline{\Omega}$.

For each i, the map Ω_i above has no concentration of mass, because it is continuous in the $\mathbf{F}$-metric, and has uniformly bounded masses because of property (iii) of H. Then, we can apply Theorem 5.2 for Ω_i to obtain

$$\bar{\phi}_{ij} : I(1, s_{ij})_0 \times I(1, s_{ij})_0 \to \mathcal{Z}_{n-1}(M)$$

such that

(a) $\mathbf{f}(\bar{\phi}_{ij}) < \frac{1}{j}$;

(b)

$$\sup\{\mathcal{F}(\bar{\phi}_{ij}(t,x) - \Omega_i(t,x)) \,:\, (t,x) \in I(2, s_{ij})_0\} \leq \frac{1}{j};$$

(c)

$$\mathbf{M}(\bar{\phi}_{ij}(t,x)) \leq \mathbf{M}(\Omega_i(t,x)) + \frac{1}{j} \quad \text{for all } (t,x) \in I_0(2, s_{ij})_0;$$

(d) $\bar{\phi}_{ij}([0], x) = \Omega_i(0,x) = \bar{\Omega}_i(x)$ for all $x \in I(1, s_{ij})_0$.

From Lemma 5.4, properties (b) and (c), we get

$$\lim_{j\to\infty} \sup\{\mathbf{F}(\bar{\phi}_{ij}(t,x), \Omega_i(t,x)) : (t,x) \in I_0(2, s_{ij})_0\} = 0.$$

Hence, using a diagonal sequence argument, we can find $\{\bar{\phi}_i = \bar{\phi}_{ij(i)}\}$ such that

$$\lim_{i\to\infty} \sup\{\mathbf{F}(\bar{\phi}_i(t,x), \Omega_i(t,x)) : (t,x) \in I_0(2, s_{ij(i)})_0\} = 0. \tag{39}$$

We define $\hat{\phi}_i : I(1, s_{ij(i)})_0 \times I(1, s_{ij(i)})_0 \to \mathcal{Z}_{n-1}(M)$ to be equal to zero on $I(1, s_{ij(i)})_0 \times \{0,1\}$, and equal to $\bar{\phi}_i$ otherwise.

Recall the domains Ω_t starting with $\Omega = \Omega_0$ defined in Subsection 3.2, as well as the fixed Ω_a, $a < 0$. We hope this notation does not cause confusion with the maps Ω_i. In this argument, we use sets Ω_t with negative t close to zero only.

Let $U_i = \Omega_{-i^{-1}}$, $\overline{\Omega_a} \subset U_i$ be such that $U_i \subset U_{i+1}$ and $U_i \subset\subset \Omega$. Choose $0 < \varepsilon_i < \varepsilon_2(U_i, \Omega_a)$, where $\varepsilon_2(U_i, \Omega_a)$ is the small number that makes Lemma 8.1 work with respect to the sets U_i and Ω_a, and such that $\lim_{i\to\infty} \varepsilon_i = 0$. The sets

$$K_i = \{\Omega_i(1,x) : x \in [0,1] \text{ and } ||\Omega_i(1,x)||(\Omega) = 0\}$$

are compact with respect to the $\mathbf{F}$-metric. Then, for each $i \in \mathbb{N}$, it is possible to find $\rho_i > 0$, so that $\mathbf{F}(\mathcal{K}_i, T) < \rho_i$ implies $||T||(\overline{U_i}) < \varepsilon_i$.

Since this construction of ρ_i does not involve the last interpolation step, we can suppose that

$$\sup\{\mathbf{F}(\hat{\phi}_i(t,x), \Omega_i(t,x)) : (t,x) \in I_0(2, s_{ij(i)})_0\} < \rho_i. \tag{40}$$

If $x \in I(1, s_{ij})_0$ is such that $\mathbf{M}(\Omega_i(1,x)) < \mathbf{L}(\Pi, \Omega) + i^{-1} + C_0\delta_i$, then item (c) above and the choice of $j(i)$ give us

$$\mathbf{M}(\hat{\phi}_i(1,x)) \leq \mathbf{M}(\Omega_i(1,x)) + \frac{1}{j(i)} < \mathbf{L}(\Pi, \Omega) + \frac{2}{i} + C_0\delta_i.$$

Otherwise, we have $\mathrm{spt}(||\Omega_i(1,x)||) \subset M - \overline{\Omega}$ and $\Omega_i(1,x) \in K_i$. By (40), $||\hat{\phi}_i(1,x)||(\overline{U_i}) < \varepsilon_i$. If $i \in \mathbb{N}$ is sufficiently large, we apply Lemma 8.1 to obtain ϕ_i 1-homotopic to $\hat{\phi}_i(1,\cdot)$ with $\mathbf{f}(\phi_i) \to 0$ and

$$\max\{\mathbf{M}(\phi_i(x)) : x \in \mathrm{dmn}_\Omega(\phi_i)\} \leq \mathbf{L}(\Pi, \Omega) + \frac{2}{i} + C_0\delta_i + C_1\mathbf{f}(\hat{\phi}_i(1,\cdot)). \tag{41}$$

This works only for large i, because there exists a constant $\eta_0(U_i, \Omega_a)$, such that $\mathbf{f}(\hat{\phi}_i(1,\cdot)) \leq \eta_0(U_i, \Omega_a)$ is required for the application of Lemma 8.1. Indeed, by Remark 10, the choice of η_0, and $U_i \subset U_{i+1}$, we conclude that $\eta_0(U_i, \Omega_a) \leq \eta_0(U_{i+1}, \Omega_a)$, while $\mathbf{f}(\hat{\phi}_i(1,\cdot))$ decreases to zero as i tends to infinity. Then, if i is sufficiently large we have $\mathbf{f}(\hat{\phi}_i(1,\cdot)) \leq \eta_0(U_1, \Omega_a) \leq \eta_0(U_i, \Omega_a)$.

Since $\mathbf{f}(\hat{\phi}_i)$ tends to zero, we obtain that ϕ_i is 1-homotopic to $\hat{\phi}_i([0],\cdot)$ in $(\mathcal{Z}_{n-1}(M; \mathbf{M}), 0)$ with fineness tending to zero. On the other hand, it follows from (38) and property (d) that $\hat{\phi}_i([0],\cdot)$ is 1-homotopic to ϕ_i^* in $(\mathcal{Z}_{n-1}(M; \mathbf{M}), 0)$ with fineness tending to zero. Hence, $S = \{\phi_i\}_{i\in\mathbb{N}} \in \Pi$. Furthermore, it follows from (41) that S is critical with respect to Ω.

We are left to show that $\mathcal{C}(S, \Omega) \subset A_0 \cap \mathcal{C}(S^*, \Omega)$, up to varifolds with zero mass in Ω. First we compare $\mathcal{C}(S, \Omega)$ with the critical set of $\{\hat{\phi}_i(1,\cdot)\}_i$. By Lemma 8.1 and Remark 11, each $\phi_i(x)$ either coincides with some slice of $\hat{\phi}_i(1,\cdot)$ or $||\phi_i(x)||(U_i) < \varepsilon_i + C_1\mathbf{f}(\hat{\phi}_i(1,\cdot))$. Then, $\mathcal{C}(S, \Omega) \subset \mathcal{C}(\{\hat{\phi}_i(1,\cdot)\}_i, \Omega)$, up to critical varifolds with zero mass in Ω. To conclude the proof, we claim that $\mathcal{C}(\{\hat{\phi}_i(1,\cdot)\}_i, \Omega) \subset A_0 \cap \mathcal{C}(S^*, \Omega)$. Here we omit the proof of this fact, because it is exactly the same as in the end of Section 15 of [**14**]. q.e.d.

11. Existence of intersecting almost minimizing varifolds

In this section we prove Theorem 4.10.

Proof of Theorem 4.10.

Part 1: Let $\Omega_a \subset \Omega \subset \Omega_b$ be fixed as in Subsection 3.2. Consider open subsets $\overline{\Omega_a} \subset U \subset U_1 \subset \Omega$ such that $\overline{U} \subset U_1$ and $\overline{U_1} \subset \Omega$.

Let $S \in \Pi$ be given by Proposition 4.6. Write $S = \{\phi_i\}_{i\in\mathbb{N}}$ and let $I(1,k_i)_0 = \mathrm{dmn}(\phi_i)$. Consider $\varepsilon_2 = \varepsilon_2(U,\Omega_a)$ as given by Lemma 8.1.

As in the original work of Pitts, our argument is by contradiction; we homotopically deform S to decrease its width with respect to Ω. This will create a contradiction with the fact that S is a critical sequence.

Part 2: Observe that $\mathcal{C}_0(S,\Omega) := \{V \in \mathcal{C}(S,\Omega) : ||V||(\Omega) = 0\}$ is compact in the weak sense of varifolds. Then, it is possible to find $\varepsilon > 0$ such that

$$T \in \mathcal{Z}_{n-1}(M) \text{ and } \mathbf{F}(|T|, \mathcal{C}_0(S,\Omega)) < \varepsilon \Rightarrow ||T||(\overline{U_1}) < 2^{-1}\varepsilon_2. \tag{42}$$

Part 3: Given $V \in \mathcal{C}(S,\Omega)$ with $||V||(\Omega) > 0$, by the contradiction assumption, there exists $p = p(V) \in \mathrm{spt}(||V||)$ such that V is not almost minimizing in small annuli centered at p. For $\mu = 1,2$, choose

$$a_\mu(V) = A(p, s_\mu, r_\mu) \text{ and } A_\mu(V) = A(p, \tilde{s}_\mu, \tilde{r}_\mu), \tag{43}$$

with the following properties:

- $\tilde{r}_1 > r_1 > s_1 > \tilde{s}_1 > 3\tilde{r}_2 > 3r_2 > 3s_2 > 3\tilde{s}_2$;
- if $p \in U_1$, then $A_\mu(V) \subset U_1$;
- if $p \notin \overline{U}$, then $A_\mu(V) \cap \overline{U} = \varnothing$;
- V is not almost minimizing in $a_\mu(V)$.

Since V is not almost minimizing in $a_\mu(V)$, for $\mu = 1,2$, there exists $\varepsilon(V) > 0$ with the following property: given

$$T \in \mathcal{Z}_{n-1}(M), \quad \mathbf{F}(|T|, V) < \varepsilon(V), \quad \mu \in \{1,2\} \text{ and } \eta > 0, \tag{44}$$

we can find a finite sequence $T = T_0, T_1, \ldots, T_q \in \mathcal{Z}_{n-1}(M)$ such that

(a) $\mathrm{spt}(T_l - T) \subset a_\mu(V)$;
(b) $\mathbf{M}(T_l - T_{l-1}) \leq \eta$;
(c) $\mathbf{M}(T_l) \leq \mathbf{M}(T) + \eta$;
(d) $\mathbf{M}(T_q) < \mathbf{M}(T) - \varepsilon(V)$.

The properties of those annuli $a_\mu(V)$ concerning the sets U and U_1 make them slightly smaller than in Pitts' original choice. The aim with this is to obtain the property that $||T||(U_1)$ controls $||T_l||(\overline{U})$, for every l. Indeed, (a) implies that $\mathrm{spt}(T_l - T)$ is always contained in either U_1 or $M - \overline{U}$. In the first case, we can use item (c) to prove that $||T_l||(\overline{U}) \leq |T_l||(U_1) \leq ||T||(U_1) + \eta$. In the second case, it follows that $||T_l||(\overline{U}) = ||T||(\overline{U}) \leq ||T||(U_1)$.

Part 4: $\mathcal{C}(S,\Omega)$ is compact. Take $V_1, V_2, \ldots, V_\nu \in \mathcal{C}(S,\Omega)$ such that

$$\mathcal{C}(S,\Omega) \subset \bigcup_{j=1}^{\nu} \{V \in \mathcal{V}_{n-1}(M) : \mathbf{F}(V, V_j) < 4^{-1}\varepsilon(V_j)\}, \tag{45}$$

where $\varepsilon(V_j)$ is the one we chose in Part 3 in the case of intersecting varifolds, $||V_j||(\Omega) > 0$, or $\varepsilon(V_j) = \varepsilon$ as chosen in Part 2, otherwise.

Part 5: Let $\delta > 0$ and $N \in \mathbb{N}$ be such that: given

$$i \geq N, \quad x \in \mathrm{dmn}_\Omega(\phi_i) \quad \text{and} \quad \mathbf{M}(\phi_i(x)) \geq \mathbf{L}(S, \Omega) - 2\delta, \tag{46}$$

there exists $f_1(x) \in \{1, \ldots, \nu\}$ with

$$\mathbf{F}(|\phi_i(x)|, V_{f_1(x)}) < 2^{-1}\varepsilon(V_{f_1(x)}). \tag{47}$$

The existence of such numbers can be seen via a contradiction argument. Moreover, choose δ and N satisfying the following two extra conditions:

- $\delta \leq \min\{2^{-1}\varepsilon(V_j) : j = 1, \ldots \nu\}$;
- $i \geq N$ implies $\mathbf{f}(\phi_i) \leq \min\{\delta, 2^{-1}\varepsilon_2, \delta_0\}$,

where $\delta_0 = \delta_0(M)$ is defined in Section 5 and $\varepsilon_2 = \varepsilon_2(U, \Omega_a)$ in Part 1.

Part 6: In this step of our proof, we use the construction done in Theorem 4.10 of [**19**], Parts 7 to 18, to state a deformation result for discrete maps. Fix $i \geq N$. Let $\alpha, \beta \in \mathrm{dmn}(\phi_i) = I(1, k_i)_0$ be such that

(1) $[\alpha, \beta] \cap \mathrm{dmn}(\phi_i) \subset \mathrm{dmn}_\Omega(\phi_i)$;
(2) if $x \in (\alpha, \beta] \cap \mathrm{dmn}(\phi_i)$, then $\mathbf{M}(\phi_i(x)) \geq \mathbf{L}(S, \Omega) - \delta$;
(3) if $x \in [\alpha, \beta] \cap \mathrm{dmn}(\phi_i)$, then $||V_{f_1(x)}||(\Omega) > 0$.

q.e.d.

Claim 11.1. *There exist $\{\delta_i\}_{i \geq N}$ tending to zero, $N(i) \geq k_i$ and*

$$\psi_i : I(1, N(i))_0 \times ([\alpha, \beta] \cap I(1, N(i))_0) \to \mathcal{Z}_{n-1}(M)$$

with the following properties:

(i) $\lim_{i \to \infty} \boldsymbol{f}(\psi_i) = 0$;
(ii) $\psi_i([0], x) = \phi_i(\boldsymbol{n}(N(i), k_i)(x))$;
(iii) $\boldsymbol{M}(\psi_i(1, \zeta)) < \boldsymbol{M}(\psi_i(0, \zeta)) - \delta + \delta_i$, *for every* $\zeta \in [\alpha, \beta] \cap I(1, N(i))_0$;
(iv) $\psi_i(j, \alpha) = \phi_i(\alpha)$, *for every* $j \in I(1, N(i))_0$;
(v) $\{\psi_i(\lambda, x) : \lambda \in I(1, N(i))_0\}$ *describes the deformation obtained in Part* 3, *starting with* $T = \phi_i(x)$, *supported in some* $a_\mu(V_{f_1(x)})$ *and fineness* $\eta = \delta_i$, *for every* $x \in (\alpha, \beta] \cap dmn(\phi_i)$.

Remark 12. If we include $x = \alpha$ in the hypothesis (2), then (v) also holds for $x = \alpha$, instead of (iv).

Pitts wrote this argument using 27 annuli. It is suggested by [**4**] that when dealing with one-parameter sweepouts it is enough to take only 2.

Part 7: Consider $\alpha, \beta \in \mathrm{dmn}(\phi_i) = I(1, k_i)_0$, such that $[\alpha, \beta]$ is maximal for the property: if $x \in [\alpha + 3^{-k_i}, \beta - 3^{-k_i}] \cap \mathrm{dmn}(\phi_i)$, then $x \in \mathrm{dmn}_\Omega(\phi_i)$, $\mathbf{M}(\phi_i(x)) \geq \mathbf{L}(S, \Omega) - \delta$ and $||V_{f_1(x)}||(\Omega) > 0$.

Let $\{\delta_i\}$ and $N(i)$ be as in Claim 11.1. Set $n_i = N(i) + k_i + 1$ and

$$L_i = \max\{\mathbf{M}(\phi_i(x)) : x \in \mathrm{dmn}_\Omega(\phi_i)\} - \delta + \delta_i. \tag{48}$$

Claim 11.2. *There exists a map*

$$\psi_i : I(1, n_i)_0 \times ([\alpha, \beta] \cap I(1, n_i)_0) \to \mathcal{Z}_{n-1}(M)$$

with the following properties:

(a) $\lim_{i\to\infty} \boldsymbol{f}(\psi_i) = 0$;
(b) $\psi_i([0],\cdot) = \phi_i \circ \boldsymbol{n}(n_i, k_i)$;
(c) $\psi_i(\lambda,\alpha) = \phi_i(\alpha)$ *and* $\psi_i(\lambda,\beta) = \phi_i(\beta)$ *for every* $\lambda \in I(1,n_i)_0$;
(d) $\max\{\boldsymbol{M}(\psi_i(1,\zeta)) : \zeta \in [\alpha + 3^{-k_i}, \beta - 3^{-k_i}] \cap I(1,n_i)_0\} < L_i$.

Moreover, if $\zeta \in [\alpha,\beta] \cap I(1,n_i)_0$ *and* $\boldsymbol{M}(\psi_i(1,\zeta)) \geq L_i$, *then*

$$||\psi_i(1,\zeta)||(\overline{U}) \leq 2^{-1}\varepsilon_2 + \delta_i. \tag{49}$$

Proof of Claim 11.2. Apply Claim 11.1 on $[\alpha + 3^{-k_i}, \beta - 3^{-k_i}]$ to obtain the map

$$\psi_i : I(1,N(i))_0 \times ([\alpha + 3^{-k_i}, \beta - 3^{-k_i}] \cap I(1,N(i))_0) \to \mathcal{Z}_{n-1}(M).$$

In order to perform the extension of this ψ_i to $I(1,n_i)_0 \times ([\alpha,\beta] \cap I(1,n_i)_0)$, we analyze the possibilities for $\phi_i(\alpha)$ and $\phi_i(\beta)$.

If $\alpha \in \mathrm{dmn}_\Omega(\phi_i)$ and $||V_{f_1(\alpha)}||(\Omega) > 0$, we can apply Claim 11.1 directly on $[\alpha, \beta - 3^{-k_i}]$. Then, we have that the map ψ_i is already defined on $I(1,N(i))_0 \times ([\alpha, \beta - 3^{-k_i}] \cap I(1,N(i))_0)$. Extend it to the desired domain simply by $\psi_i \circ \mathbf{n}(n_i, N(i))$. Observe that the choice of $[\alpha,\beta]$ implies that

$$\mathbf{L}(S,\Omega) - 2\delta \leq \mathbf{M}(\phi_i(\alpha)) < \mathbf{L}(S,\Omega) - \delta.$$

By item (iv) of Claim 11.1, $\psi_i(\lambda,\alpha) = \phi_i(\alpha)$, for every $\lambda \in I(1,n_i)_0$. Item (iii) of the same statement implies that

$$\max\{\mathbf{M}(\psi_i(1,\zeta)) : \zeta \in [\alpha, \beta - 3^{-k_i}] \cap I(1,n_i)_0\} < L_i.$$

Suppose now that α satisfies one of the following properties:

(I) either $\alpha \in \mathrm{dmn}_\Omega(\phi_i)$ and $||V_{f_1(\alpha)}||(\Omega) = 0$;
(II) or $\alpha \notin \mathrm{dmn}_\Omega(\phi_i)$.

In both cases, the extension of ψ_i on $I(1,n_i)_0 \times ([\alpha + 3^{-k_i}, \beta - 3^{-k_i}] \cap I(1,n_i)_0)$ is given by $\psi_i \circ \mathbf{n}(n_i, N(i))$. We complete the extension in such a way that $\{\psi_i(\lambda,\zeta) : \zeta \in [\alpha, \alpha + 3^{-k_i}] \cap I(1,n_i)_0\}$ is contained in $\{\psi_i(j, \alpha + 3^{-k_i})\}_{j \in I(1,N(i))_0} \cup \{\phi_i(\alpha)\}$. The motivation for doing this is that $\phi_i(\alpha)$ and all $\psi_i(j, \alpha + 3^{-k_i})$ already have small mass inside $\overline{U}$. Let us first prove this claim about the masses inside $\overline{U}$ and later we conclude the construction of the map ψ_i.

In case (I), we observe that $V_{f_1(\alpha)} \in \mathcal{C}_0(S,\Omega)$ and that

$$\mathbf{F}(|\phi_i(\alpha + 3^{-k_i})|, V_{f_1(\alpha)}) \leq \mathbf{f}(\phi_i) + \mathbf{F}(|\phi_i(\alpha)|, V_{f_1(\alpha)}) < \varepsilon,$$

where the last estimate is a consequence of the choice of N and f_1 in Part 5. In particular, $\mathbf{F}(|\phi_i(\alpha + 3^{-k_i})|, \mathcal{C}_0(S,\Omega)) < \varepsilon$. Item (v) of Claim 11.1 and the comments in the end of Part 3 imply that, for every $j \in I(1,N(i))_0$,

$$||\psi_i(j, \alpha + 3^{-k_i})||(\overline{U}) \leq 2^{-1}\varepsilon_2 + \delta_i. \tag{50}$$

Case (II) is simpler because $||\phi_i(\alpha)||(\overline{\Omega}) = 0$ directly implies that

$$||\phi_i(\alpha + 3^{-k_i})||(\overline{\Omega}) \leq \mathbf{f}(\phi_i) \leq 2^{-1}\varepsilon_2.$$

Then, a similar analysis tells us that (50) also holds in this case.

Finally, define ψ_i on $I(1,n_i)_0 \times ((\alpha, \alpha + 3^{-k_i}) \cap I(1,n_i)_0)$ by

$$\psi_i(\lambda, \alpha + 3^{-k_i} - \zeta \cdot 3^{-n_i}) = \psi_i(\max\{0, \mathbf{n}(n_i, N(i))(\lambda) - \zeta \cdot 3^{-N(i)}\}, \alpha + 3^{-k_i}).$$

Put $\psi_i(\lambda, \alpha) = \phi_i(\alpha)$, for every $\lambda \in I(1,n_i)_0$. Then, Claim 11.2 holds. q.e.d.

Part 8: Let $\psi_i : I(1,n_i)_0 \times I(1,n_i)_0 \to \mathcal{Z}_{n-1}(M)$ be the map obtained in such a way that for each interval $[\alpha, \beta]$ as in Part 7, its restriction to

$$I(1,n_i)_0 \times ([\alpha, \beta] \cap I(1,n_i)_0)$$

is the map of Claim 11.2, and $\psi_i(\lambda, \zeta) = (\phi_i \circ \mathbf{n}(n_i, k_i))(\zeta)$, otherwise.

Take $S^* = \{\phi_i^*\}_{i\in\mathbb{N}}$, where $\phi_i^* = \psi_i(1, \cdot)$ is defined on $I(1,n_i)_0$. Because the maps ψ_i have fineness tending to zero, $S^* \in \Pi$.

Part 9: The sequence S is critical with respect to Ω; this implies that

$$\lim_{i\to\infty} \max\{\mathbf{M}(\phi_i(x)) : x \in \mathrm{dmn}_\Omega(\phi_i)\} = \mathbf{L}(S, \Omega).$$

In particular, $\lim_{i\to\infty} L_i = \mathbf{L}(S, \Omega) - \delta$. Let i be sufficiently large that

$$i \geq N, \quad \mathbf{f}(\phi_i^*) \leq \eta_0, \quad L_i < \mathbf{L}(S, \Omega) - 2^{-1}\delta \quad \text{and} \quad \delta_i < 2^{-1}\varepsilon_2,$$

where $\eta_0 = \eta_0(U, \Omega_a)$ is given by Lemma 8.1. This choice implies that the maps ϕ_i^* have the following property:

$$\mathbf{M}(\phi_i^*(\zeta)) \geq L_i \Rightarrow ||\phi_i^*(\zeta)||(\overline{U}) < \varepsilon_2. \tag{51}$$

Use Lemma 8.1 to produce $\tilde{S} = \{\tilde{\phi}_i\}$ homotopic with S^*, such that

$$\max\{\mathbf{M}(\tilde{\phi}_i(x)) : x \in \mathrm{dmn}_\Omega(\tilde{\phi}_i)\} < L_i + C_1\mathbf{f}(\phi_i^*),$$

where $C_1 = C_1(M)$ is also given by Lemma 8.1. In particular, $\tilde{S} \in \Pi$ and $\mathbf{L}(\tilde{S}, \Omega) \leq \mathbf{L}(S, \Omega) - \delta = \mathbf{L}(\Pi, \Omega) - \delta$. This is a contradiction.

12. Proof of Theorem 1.1

In this section we use the min-max theory for intersecting slices to prove the main result of our work, Theorem 1.1.

Proof of Theorem 1.1. We divide the proof into five steps.

Step 1: Let $p \in N$ be such that items (a) and (b) about the $\star_k$-condition hold for geodesic balls $B(p, R)$ centered at p, as defined in Section 1. Choose a proper Morse function $f : N \to [0, +\infty)$ and let $\{\Sigma_t\}_{t\geq 0}$ be the one-parameter sweepout of integral cycles induced by the level sets of f,

$$\Sigma_t := \partial\big(\{x \in N : f(x) < t\}\big) \in \mathcal{Z}_{n-1}(N). \tag{52}$$

This family has the special property that $\mathcal{H}^{n-1}(\Sigma_t)$ is a continuous function. Since $\overline{\Omega} \subset N$ is compact, we conclude

$$(53)\quad L := L\left(\{\Sigma_t\}_{t\geq 0}\right) = \sup\{\mathcal{H}^{n-1}(\Sigma_t) : \mathrm{spt}(||\Sigma_t||) \cap \overline{\Omega} \neq \varnothing\} < +\infty.$$

Step 2: Consider $t_0 > 0$ such that Σ_{t_0} is a smooth regular level of f and large enough to satisfy the following property: any connected minimal hypersurface $\Sigma^{n-1} \subset N$, intersecting $\overline{\Omega}$, with non-empty boundary and $\inf_{\partial\Sigma} f \geq t_0$, must satisfy $\mathcal{H}^{n-1}(\Sigma) \geq 2L$.

The existence of such t_0 is a consequence of the monotonicity formula for minimal hypersurfaces and the fact that N satisfies the $\star_k$-condition for some $k \leq \frac{2}{n-2}$. We accomplish this argument in Subsection 12.1.

Step 3: Let (M^n, h) be a compact Riemannian manifold without boundary, containing an isometric copy of $\{f \leq t_0\}$ and such that f extends to M as a Morse function. We call this extension f_1. In Subsection 12.2 below we perform a construction that gives one possible M. Since $\overline{\Omega} \subset \{f < t_0\}$, we have a copy of Ω inside M, which we also denote Ω. We can suppose that $f_1(M) = [0, 1]$.

Step 4: Let $\Gamma = \{\Gamma_t\}_{t\in[0,1]}$ be the sweepout of M given by $\Gamma_t = f_1^{-1}(t)$. Consider the set of intersecting times

$$\mathrm{dmn}_\Omega(\Gamma) = \{t \in [0, 1] : \mathrm{spt}(||\Gamma_t||) \cap \overline{\Omega} \neq \varnothing\}.$$

Observe that Γ_t coincides with the slice Σ_t, for every $0 \leq t \leq t_0$, and that $t \notin \mathrm{dmn}_\Omega(\Gamma)$, for $t_0 < t \leq 1$. In particular,

$$L(\Gamma, \Omega) := \sup\{\mathcal{H}^{n-1}(\Gamma_t) : t \in \mathrm{dmn}_\Omega(\Gamma)\}$$

coincides with the number L defined in Step 1. We apply now the min-max Theory developed in Section 4 to produce a closed embedded minimal hypersurface $\Sigma^{n-1} \subset M$ with $\mathcal{H}^{n-1}(\Sigma) \leq L$ and $\Sigma \cap \overline{\Omega} \neq \varnothing$.

Since our min-max methods follow the discrete setting of Almgren and Pitts, we still have to construct out of Γ a non-trivial homotopy class $\Pi \in \pi_1^{\#}(\mathcal{Z}_{n-1}(M; \mathbf{M}), \{0\})$, such that $\mathbf{L}(\Pi, \Omega) \leq L(\Gamma, \Omega) = L$. This is the content of Theorem 4.4. We can apply this result because Γ is continuous in the $\mathbf{F}$-metric and non-trivial.

Step 5: The choice of t_0 in Step 2 guarantees that any component of Σ^{n-1} that intersects $\overline{\Omega}$ can not go outside $\{f \leq t_0\}$. Otherwise, this would imply that $2L \leq \mathcal{H}^{n-1}(\Sigma) \leq L$. In conclusion, any intersecting component of Σ is a closed embedded minimal hypersurface in the open manifold N. q.e.d.

12.1. Choice of a slice far from Ω. Let us prove that the $\star_k$-condition implies that any minimal hypersurface that intersects Ω and, at the same time, goes far from Ω has large Hausdorff measure $\mathcal{H}^{n-1}$. Let $\Sigma^{n-1} \subset N$ be a minimal hypersurface as in Step 2 above. The main tool for this subsection is the following consequence of the monotonicity formula.

Proposition 12.1. *For every $q \in B(p,R)$ and $0 < s < R^{-\frac{k}{2}}$, we have*

$$\mathcal{H}^{n-1}(\Sigma \cap B(q,s)) \geq \frac{\omega_{n-1}}{e^{(n-1)\sqrt{R^k}s}} \cdot s^{n-1}, \tag{54}$$

where ω_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$.

Consider $R_0 \leq R_1$ and $l \in \mathbb{N}$, for which $\overline{\Omega} \subset B(p,R_1)$ and $B(p,R_1+l) \subset \{f < t_0\}$, t_0 to be chosen. Recall that Σ intersects $\overline{\Omega}$ and it is not contained in the sublevel set $\{f < t_0\}$. Then, for every $i \in \{1,2\ldots,l\}$, there are points $q_{ij} \in \Sigma$, $j \in \{1,2,\ldots,\lfloor\sqrt{(R_1+i)^k}\rfloor\}$, whose distance in N to p is given by

$$d(q_{ij},p) = R_1 + i - 1 + \frac{2j-1}{2\sqrt{(R_1+i)^k}}.$$

Observe that $q_{ij} \in B(p,R_1+i)$ and that the balls $B_{ij} = B(q_{ij},2^{-1}(R_1+i)^{-\frac{k}{2}})$ are pairwise disjoint. Apply Proposition 12.1 to conclude that

$$\begin{aligned} \mathcal{H}^{n-1}(\Sigma) &\geq \sum_{i=1}^{l}\sum_{j}\mathcal{H}^{n-1}(\Sigma \cap B_{ij}) \\ &\geq \frac{\omega_{n-1}}{(2\sqrt{e})^{n-1}}\sum_{i=1}^{l}\left(\lfloor(R_1+i)^{\frac{k}{2}}\rfloor \cdot (R_1+i)^{-\frac{k(n-1)}{2}}\right). \end{aligned} \tag{55}$$

Since $k \leq \frac{2}{n-2}$, if we keep R_1 fixed and let $l \in \mathbb{N}$ go to infinity, then the right-hand side of expression (55) also tends to infinity. In particular, it is greater than $2L$ for some large $l \in \mathbb{N}$, where $L = L(\{\Sigma_t\}_{t\geq 0})$ is the number we considered in Step 1 above. This concludes the argument, because we chose t_0 such that $B(p,R_1+l) \subset \{f < t_0\}$.

12.2. Change the metric g far from Ω. Since t_0 is a regular value of f, $f^{-1}([t_0,t_0+3\varepsilon])$ has no critical points for sufficiently small $\varepsilon > 0$. Moreover, there exists a natural diffeomorphism

$$\xi : f^{-1}([t_0,t_0+3\varepsilon]) \to \Sigma_{t_0} \times [0,3\varepsilon]$$

that identifies $f^{-1}(t_0)$ with $\Sigma_{t_0} \times \{0\}$. Suppose that $\Sigma_{t_0} \times [0,3\varepsilon]$ has the product metric $g|_{\Sigma_{t_0}} \times \mathcal{L}$, where $\mathcal{L}$ denotes the Lebesgue measure. Consider the pullback metric $g_1 = \xi^*(g|_{\Sigma_{t_0}} \times \mathcal{L})$ and choose a smooth bump function $\varphi : [t_0,t_0+3\varepsilon] \to [0,1]$, such that

- $\varphi(t) = 1$, for every $t \in [t_0,t_0+\varepsilon]$, and
- $\varphi(t) = 0$, for every $t \in [t_0+2\varepsilon,t_0+3\varepsilon]$.

On $f^{-1}([t_0,t_0+3\varepsilon])$, mix the original g and the product metric g_1 using the smooth function φ, to obtain

$$h_1(x) = (\varphi \circ f)(x)g(x) + (1-(\varphi \circ f)(x))g_1(x). \tag{56}$$

This metric admits the trivial smooth extension $h_1 = g$ over $f^{-1}([0, t_0])$. Summarizing, we produced a Riemannian manifold with boundary

$$M_1 = (\{f \leq t_0 + 3\varepsilon\}, h_1),$$

with the following properties:

(i) M_1 contains an isometric copy of $\{f \leq t_0\}$ with the metric g;
(ii) near $\partial M_1 = f^{-1}(t_0 + 3\varepsilon)$, $h_1 = g_1$ is the product metric.

Then, it is possible to attach two copies of M_1 via the identity map of ∂M_1. Doing this we obtain a closed manifold M with a smooth Riemannian metric h that coincides with h_1 on each half. Precisely,

$$M = M_1 \cup_{\mathcal{I}} M_1,$$

where $\mathcal{I}$ denotes the identity map of ∂M_1. The metric h is smooth because of item (ii). Moreover, item (i) says that M has an isometric copy of $\{f \leq t_0\}$. Finally, let us construct a Morse function f_1 on M that coincides with f in the first piece M_1. On the second half M_1, put $f_1 = 2(t_0 + 3\varepsilon) - f$.

Appendix A.

Let (M^n, g) be a compact Riemannian manifold with smooth non-empty boundary ∂M, and $f : M \to [0, 1]$ be a Morse function on M such that $f^{-1}(1) = \partial M$ and with no interior local maximum. The fundamental theorems in Morse theory describe how the homotopy type of the sublevel sets $M^a = \{x \in M : f(x) \leq a\}$ changes with a; see Theorems 3.1 and 3.2 in [**16**]. In this section we develop a slightly different approach to those results.

Theorem A.1. *Let $0 \leq c < d \leq 1$, $\lambda \in \{0, 1, \ldots, n-1\}$, and $p \in f^{-1}(c)$ be an index λ critical point of f. Suppose $\varepsilon > 0$ is such that $f^{-1}([c - \varepsilon, d])$ contains no critical points other than p. Then, for sufficiently small $\varepsilon > 0$, there exists a smooth homotopy*

$$h : [0, 1] \times M^d \to M^d,$$

with the following properties:

1. *$h(1, \cdot)$ is the identity map of M^d;*
2. *$h(0, M^d)$ is contained in $M^{c-\varepsilon}$ with a λ-cell attached.*

Remark 13. The difference between the above statement and the classical ones is that here we are able to guarantee that the homotopy is smooth by relaxing the condition of $h(1, \cdot)$ being a retraction onto the whole $M^{c-\varepsilon}$ with the λ-cell attached. This is important in the proof of Lemma 6.1.

Proof. By the classical statements 3.1 and 3.2 in [**16**], we know that there exists a smooth homotopy between the identity map of M^d and a retraction of M^d onto $M^{c-\varepsilon} \cup H$, i.e., the sublevel set with a handle H

attached. It is also observed that this set has smooth boundary. The final argument in the proof of Theorem 3.2 in Milnor's book is a vertical projection of the handle in $M^{c-\varepsilon} \cup e^\lambda$, where e^λ is a λ-cell contained in H. See diagram 7 on page 19 of [**16**]. This projection is not adequate for us because it is not smooth. We only adapt this step in that proof by defining a smooth projection.

Following the notation in [**16**], let $u_1, \ldots, u_n$ be a coordinate system in a neighborhood U of p so that the identity

$$f = c - (u_1^2 + \ldots + u_\lambda^2) + (u_{\lambda+1}^2 + \ldots + u_n^2) \tag{57}$$

holds throughout U. We use $\xi = u_1^2 + \ldots + u_\lambda^2$ and $\eta = u_{\lambda+1}^2 + \ldots + u_n^2$. For sufficiently small $\varepsilon > 0$, the λ-cell e^λ can be explicitly given by the points in U with $\xi \leq \varepsilon$ and $\eta = 0$. Consider $\delta > 0$ so that the image of U by the coordinate system contains the set of points with $\xi \leq \varepsilon + \delta$ and $\eta \leq \delta$. Let $\phi : (0, +\infty) \to (0, +\infty)$ be a function such that $\phi \in C^\infty(0, +\infty)$ and:

(a) $\phi(\xi) = 0$, if $\xi \leq \varepsilon$;
(b) $\phi(\xi) = \xi - \varepsilon$, if $\varepsilon + \delta \leq \xi$;
(c) $\phi(\xi) \leq \xi - \varepsilon$, for all $\xi \in (0, +\infty)$.

With these choices we are able to redefine the projection for points classified as case 2 on page 19 of [**16**], i.e., $\varepsilon \leq \xi \leq \eta + \varepsilon$. Consider

$$(t, u_1, \ldots, u_n) \mapsto (u_1, \ldots, u_\lambda, s_t u_{\lambda+1}, \ldots, s_t u_n),$$

where the number $s_t \in [0, 1]$ is defined by

$$s_t = t + (1 - t)\sqrt{\frac{\phi(\xi)}{\eta}}.$$

This map is smooth for points in case 2 because now the set $\phi(\xi) = \eta$ meets the boundary of e^λ smoothly. By (c), we see that the image of each point at time $t = 0$ is inside $M^{c-\varepsilon} \cup e^\lambda$. Using this new projection, the statement follows via the same program as in the proof of Theorem 3.2 in [**16**]. q.e.d.

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