J. DIFFERENTIAL GEOMETRY
102 (2016) 409-484

ON THE B-TWISTED TOPOLOGICAL SIGMA MODEL AND CALABI-YAU GEOMETRY

Qin Li & Si Li

Abstract

We provide a rigorous perturbative quantization of the B-twisted topological sigma model via a first-order quantum field theory on derived mapping space in the formal neighborhood of constant maps. We prove that the first Chern class of the target manifold is the obstruction to the quantization via Batalin-Vilkovisky formalism. When the first Chern class vanishes, i.e. on Calabi-Yau manifolds, the factorization algebra of observables gives rise to the expected topological correlation functions in the B-model. We explain a twisting procedure to generalize to the Landau-Ginzburg case, and show that the resulting topological correlations coincide with Vafa's residue formula.

Contents

1. Introduction 410
   1.1. Calabi-Yau model 411
   1.2. Landau-Ginzburg model 414
2. The classical theory 415
   2.1. The model 415
   2.2. The spaces $(\Sigma_g)_{dR}$ and $X_{\bar\partial}$ 416
   2.3. Classical action functional 419
   2.4. Classical master equation 420
3. Quantization 423
   3.1. Regularization 423
   3.2. Effective renormalization 426
   3.3. Quantization 430
4. Observable theory 445
   4.1. Classical observables 446
   4.2. Quantum observables 447
   4.3. Correlation function 456
5. Landau-Ginzburg Twisting 461

Received 3/18/2014.

5.1. Classical theory 461
5.2. Quantization 463
5.3. Observable theory and correlation functions 464
Appendix A. Proof of Lemma 3.6 466
Appendix B. Some Feynman graph computations 467
Appendix C. One-loop anomaly 474
Appendix D. Chevalley-Eilenberg complex vs de Rham complex of jet bundles 478
References 483

1. Introduction

Mirror symmetry predicts dualities between quantum geometries on Calabi-Yau manifolds. The two sides of the dual theories are called *A-model* and *B-model* respectively. The A-model is related to symplectic geometry, which is mathematically established as the *Gromov-Witten* theory of counting holomorphic maps. The B-model is attached to complex geometry, which could be understood via *Kodaira-Spencer gauge* theory. Such gauge theory is proposed by Bershadsky, Cecotti, Ooguri, and Vafa [4] as a closed string analogue of Chern-Simons theory [28] in the B-model, whose classical theory describes the deformation of complex structures. We refer to [2, 9, 19–21] for some recent mathematical development of its quantum geometry.

Although Kodaira-Spencer gauge theory provides the geometry of the B-model from the point of view of string/gauge duality, a direct mathematical approach to the B-model in the spirit of σ-model is still lacking. The main difficulty is the unknown measure of path integral on the infinite dimensional mapping space. Thanks to supersymmetry, the physics of the B-model path integral is expected to be fully encoded in the small neighborhood of constant maps. This allows us to extract physical quantities via classical geometries, such as Yukawa couplings for genus-0 correlation functions, etc. (See [15] for an introduction.) It is thus desired to have a mathematical theory to reveal the above physics context in the vicinity of constant maps, parallel to the localized space of holomorphic maps in the A-model.

The main purpose of the current paper is to provide a rigorous geometric model to analyze the B-model via mapping space. To illustrate our method, we will focus on topological field theory in this paper, while leaving the topological string for coupling with gravity in future works. In the rest of the introduction, we will sketch the main ideas and explain our construction. A closely related development of the B-model in physics has been communicated recently to us by Losev [22].

The geometry of the B-twisted σ-model (in the spirit of AKSZ-formalism [1]) describes the mapping space

$$(\Sigma_g)_{dR} \to T_X^\vee[1],$$

where $(\Sigma_g)_{dR}$ is the ringed space with the sheaf of the de Rham complex on the Riemann surface Σ_g, and $T_X^\vee[1]$ is the super-manifold associated to the cotangent bundle of X with degree one shifting in the fiber direction. The full mapping space is difficult to analyze. Instead we will consider the mapping space in the formal neighborhood of constant maps. Such consideration is proposed in [7] to fit into the effective renormalization method developed in [5]. Therefore the corresponding perturbative quantum field theory can be rigorously analyzed, which is the main context of the current paper. As we have mentioned above, zooming into the neighborhood of constant maps in the B-model does not lose information in physics due to supersymmetry.

Notations: We will fix some notations that will be used throughout the paper. For a smooth manifold M, we will let $\mathcal{A}_M$ denote the sheaf of the de Rham complex of smooth differential forms on M, and let $\mathcal{A}_M^\sharp$ denote the sheaf of smooth differential forms forgetting the de Rham differential:

$$\mathcal{A}_M := (\mathcal{A}_M^\sharp, d_M).$$

D_M refers to the sheaf of smooth differential operators on M. When M is a complex manifold, $\mathcal{O}_M$ refers to the sheaf of holomorphic functions, and T_M denotes either the holomorphic tangent bundle, or the sheaf of holomorphic tangent vectors, while its meaning should be clear from the context (similarly for the dual $T_M^\vee$). We will use $\Omega_M^\bullet$ to denote the sheaf of the holomorphic de Rham complex on M, and D_M^{hol} the sheaf of holomorphic differential operators. The tensor product $\otimes$ without mentioning its ring means $\otimes_{\mathbb{C}}$.

1.1. Calabi-Yau model. The space of fields describing our B-twisted σ-model is given by

$$\mathcal{E} := \mathcal{A}_{\Sigma_g} \otimes (\mathfrak{g}_X[1] \oplus \mathfrak{g}_X^\vee),$$

where $\mathfrak{g}_X$ is the sheaf of curved L_∞-algebra on X describing its complex geometry [6]. As a sheaf itself,

$$\mathfrak{g}_X = \mathcal{A}_X^\sharp \otimes_{\mathcal{O}_X} T_X[-1], \quad \mathfrak{g}_X^\vee = \mathcal{A}_X^\sharp \otimes_{\mathcal{O}_X} T_X^\vee[1].$$

The Chevalley-Eilenberg complex $C^*(\mathfrak{g}_X)$ is a resolution of the sheaf $\mathcal{O}_X$ of holomorphic functions on X, and the curved L_∞-algebra $\mathfrak{g}_X \oplus \mathfrak{g}_X^\vee[-1]$ describes the derived geometry of $T_X^\vee[1]$ (see section 2 for details).

The Chevalley-Eilenberg differential and the natural symplectic pairing equip $T_X^\vee[1]$ (more precisely its L_∞-enrichment) with the structure of QP-manifold [1]. The action functional is constructed via the AKSZ-formalism in the same fashion as in [6], formally written as

$$S(\alpha+\beta)=\int_{\Sigma_g}\langle d_{\Sigma_g}\alpha,\beta\rangle+\sum_{k\geq 0}\left\langle \frac{l_k(\alpha^{\otimes k})}{(k+1)!},\beta\right\rangle$$

for $\alpha \in \mathcal{A}_{\Sigma_g}\otimes \mathfrak{g}_X[1], \beta \in \mathcal{A}_{\Sigma_g}\otimes \mathfrak{g}_X^\vee$. Here l_k's are the L_∞-products for $\mathfrak{g}_X$. By construction, the action functional satisfies a version of classical master equation (see sections 2.3 and 2.4). One interesting feature is that S contains only one derivative (coming from d_{Σ_g}), and the first-order formulation has been used (e.g. [3, 10, 17]) to describe the twisted σ-model around the large volume limit. We follow the more recent formulation [6, 12], using L_∞-algebra via jet bundles as a coherent way to do perturbative expansion over the target manifold X. In fact, the terms involving L_∞ products exactly represent the curvature of the target (see [6] for an explanation) in terms of jets.

We would like to do perturbative quantization via Feynman diagrams on the infinite dimensional space $\mathcal{E}$ analogous to the ordinary non-linear σ-model [11]. One convenient theory via effective Batalin-Vilkovisky formalism is developed by Costello [5], and we will analyze the quantization problem via this approach.

Theorem 1.1 (Theorem 3.32, Theorem 3.36)**.** *Let X be a complex manifold.*

1) *The obstruction to the existence of perturbative quantization of our B-twisted topological σ-model is given by $(2-2g)c_1(X)$, where g is the genus of the Riemann surface Σ_g and $c_1(X)$ is the first Chern class of X.*
2) *If $c_1(X)=0$, i.e. X being Calabi-Yau, then there exists a canonical perturbative quantization associated to a choice of holomorphic volume form Ω_X.*

We refer to section 3 for the precise meaning of the theorem. The theorem is proved by analyzing Feynman diagrams with the heat kernel on Σ_g associated to the constant curvature metric, and this is consistent with physics that B-twisting can only exist on Calabi-Yau manifolds. Similar results on half-twisted B-model and 2d holomorphic Chern-Simons theory have been obtained in [14, 25] via background field method. Another approach to topological B-model via D-module techniques is communicated to us by Rozenblyum [23].

Given a perturbative quantization, there exists a rich structure of factorization algebra for observables developed by Costello and Gwilliam [8]. In our case of quantum field theory in two dimensions, the factorization product for local observables gives rise to the structure of E_2-algebra. A perturbative quantization of a so-called cotangent field theory (where our Calabi-Yau model belongs) can be viewed as defining a certain *projective volume form* on the space of fields [6]. It allows us to define correlation functions for local observables via the local-to-global factorization product. The next theorem concerns the local and global observables in our model.

Theorem 1.2. *Let X be a compact Calabi-Yau with holomorphic volume form Ω_X.*

1) *The cohomology of local quantum observables on any disk $U \subset \Sigma_g$ is $H^*(X, \wedge^* T_X)[[\hbar]]$.*
2) *The complex of quantum observables on Σ_g is quasi-isomorphic to the de Rham complex of a trivial local system on X concentrated at degree $(2g-2)\dim_{\mathbb{C}} X$.*

See section 4.2 for the explanation.

Instead of the de Rham cohomology for observables in the Gromov-Witten theory, the observables in the B-model are described by polyvector fields. Let $\mu_i \in H^*(X, \wedge^* T_X)$, and let $\amalg_i U_i \subset \Sigma_g$ be the disjoint union of disks on Σ_g. Let O_{μ_i, U_i} be a local observable in U_i representing μ_i via the above theorem. Then the factorization product with respect to the embedding

$$\amalg_i U_i \hookrightarrow \Sigma_g$$

gives a global observable $O_{\mu_1,U_1} \star O_{\mu_2,U_2} \star \cdots \star O_{\mu_k,U_k}$. Following [6], the correlation function of topological field theory is defined by the natural integration

$$\langle O_{\mu_1,U_1}, \cdots, O_{\mu_k,U_k} \rangle_{\Sigma_g} := \int_X [O_{\mu_1,U_1} \star O_{\mu_2,U_2} \star \cdots \star O_{\mu_k,U_k}] \in \mathbb{C}((\hbar)).$$

Here $[-]$ is the de Rham cohomology class represented by the quantum observable as in the second part of the above theorem. The degree shifting implies that the correlation function is zero unless $\sum_i \deg O_{\mu_i,U_i} = \sum_i \deg \mu_i = (2-2g)\dim_{\mathbb{C}} X$. Explicit calculation on the sphere gives

Theorem 1.3 (Theorem 4.29)**.** *Let $\Sigma_g = \mathbb{P}^1$, and let X be a compact Calabi-Yau with holomorphic volume form Ω_X. Then*

$$\langle O_{\mu_1,U_1}, \cdots, O_{\mu_k,U_k} \rangle_{\mathbb{P}^1} = \hbar^{\dim_{\mathbb{C}} X} \int_X (\mu_1 \cdots \mu_k \vdash \Omega_X) \wedge \Omega_X,$$

where $\vdash$ is the contraction map and $\hbar$ is a formal variable.

When Σ_g is an elliptic curve, the only non-trivial topological correlation function is the partition function without inputs.

Theorem 1.4 (Theorem 4.30)**.** *Let $g = 1$; then $\langle 1 \rangle_{\Sigma_g} = \chi(X)$ is the Euler characteristic of X.*

To establish the above computation of correlation functions, we describe a formalism in the spirit of Batalin-Vilkovisky Lagrangian integration, which is equivalent to the above definition of correlation functions for our model (Corollary 4.27). It not only simplifies the computation, but also sheds light on the potential application to theories which are not cotangent. In fact, the Landau-Ginzburg model to be described below is not a cotangent field theory; hence the definition of the correlation function in [6] does not work in this case. However, the Batalin-Vilkovisky Lagrangian integration still makes sense and gives rise to the expected result (Proposition 5.14).

1.2. Landau-Ginzburg model. The Calabi-Yau model described above allows a natural generalization to the Landau-Ginzburg model associated to a pair (X, W), where W is a holomorphic function on X called the *superpotential*. This is accomplished by a *twisting procedure*: at the classical level, the interaction is modified by adding a term I_W (Definition 5.5); at the quantum level, this simple modification is still valid (Proposition 5.9). In particular, a choice of holomorphic volume form Ω_X on X leads to a quantization of our Landau-Ginzburg B-model.

Let us describe the corresponding observable theory. For simplicity, let us assume $X = \mathbb{C}^n$, and that the critical set of the superpotential $Crit(W)$ is finite. We let $\{z^i\}$ be the affine coordinates on $\mathbb{C}^n$, and choose $\Omega_X = dz^1 \wedge \cdots \wedge dz^n$. We consider the quantization associated to the pair (X, Ω_X) with the twisting procedure described above.

Theorem 1.5 (Proposition 5.12)**.** *The cohomology of Landau-Ginzburg B-model local quantum observables on any disk $U \subset \Sigma_g$ is* $\mathrm{Jac}(W)[[\hbar]]$.

Similar to the Calabi-Yau model, we use $O_{f,U}$ to denote a local quantum observable representing $f \in \mathrm{Jac}(W)$ in the above theorem. Let $\amalg_i U_i \subset \Sigma_g$ be the disjoint union of disks on Σ_g. Then the factorization product

$$O_{f_1,U_1} \star \cdots \star O_{f_k,U_k}$$

defines a global quantum observable on Σ_g. However, the Landau-Ginzburg theory is no longer a cotangent theory in the sense of [6], and the projective volume form interpretation of quantization breaks down. Instead, we directly construct an integration map on quantum observables following the interpretation of Batalin-Vilkovisky Lagrangian geometry described above. This allows us to define the correlation function (Definition 5.13)

$$\langle O_{f_1,U_1} \star \cdots \star O_{f_k,U_k} \rangle^W_{\Sigma_g}$$

in the Landau-Ginzburg case.

Theorem 1.6 (Proposition 5.14). *The correlation function of the topological Landau-Ginzburg B-model is*

$$\langle O_{f_1,U_1} \star \cdots \star O_{f_k,U_k} \rangle^W_{\Sigma_g} = \sum_{p \in Crit(W)} \mathrm{Res}_p \left(\frac{f_1 \cdots f_k \det(\partial_i \partial_j W)^g dz^1 \wedge \cdots \wedge dz^n}{\prod_i \partial_i W} \right),$$

where Res_p *is the residue at the critical point* p [13].

This coincides with Vafa's residue formula [26].

Acknowledgments. The authors would like to thank Kevin Costello, Ryan Grady, Owen Gwilliam, and Yuan Shen for valuable discussions on quantum field theories. Part of the work was done while the authors were visiting MSC at Tsinghua University in the summer of 2012 and 2013, and while the first author was visiting Boston University in 2012 and 2013. We would like to thank these universities for their hospitality. The first author is partially supported by Chinese Universities Scientific Fund WK0010000030. The second author is partially supported by NSF DMS-1309118.

2. The classical theory

In this section we will describe the geometry of the B-twisted topological σ-model and set up our theory at the classical level.

2.1. The model. Let X be a complex manifold, and let Σ_g be a closed Riemann surface of genus g. Two-dimensional σ-models are concerned with the space of maps

$$\Sigma_g \to X.$$

One useful way to incorporate interesting information about the geometry and topology of the target X is to enhance ordinary σ-models to supersymmetric ones and apply topological twists. There are two twisted supersymmetric theories that have been extensively studied both in the mathematics and physics literature: the A-model and the B-model. These lead to the famous mirror symmetry between symplectic and complex geometries. In this paper we will mainly focus on the B-model.

One possible mathematical formulation of the quantum field theory of the B-twisted σ-model is proposed by Costello [7] via formal derived geometry, and we will adopt this point of view.

Definition 2.1 ([7]). The (fully twisted) B-model, with source a genus g Riemann surface Σ_g and target a complex manifold X, is the cotangent theory to the elliptic moduli problem of maps

$$(\Sigma_g)_{dR} \to X_{\bar{\partial}}.$$

In the subsequent subsections, we will explain all the notations and geometric data in the above definition. Basically, we have enhanced the mapping as from a dg-space $(\Sigma_g)_{dR}$ to the L_∞-space $X_{\bar\partial}$ to implement supersymmetry. However, the full mapping space is complicated and hard to analyze. Instead, we will focus on the locus in the formal neighborhood of constant maps. Under this reduction, we describe our classical action functional in section 2.3. From the physical point of view, the quantum field theory of the B-twisted σ-model is fully encoded in the neighborhood of constant maps, thanks to supersymmetry. Therefore we do not lose any information via this consideration.

2.2. The spaces $(\Sigma_g)_{dR}$ and $X_{\bar\partial}$.

2.2.1. The dg-space $(\Sigma_g)_{dR}$. We use $(\Sigma_g)_{dR}$ to denote the dg-ringed space

$$(\Sigma_g)_{dR} = \left(\Sigma_g, \mathcal{A}_{\Sigma_g}\right)$$

on the Riemann surface Σ_g, where the structure sheaf is the sheaf of asmooth de Rham complex. $\mathcal{A}_{\Sigma_g}$ is an elliptic complex, and we view $(\Sigma_g)_{dR}$ as an *elliptic ringed space* in the sense of [7].

2.2.2. The L_∞-space $X_{\bar\partial}$. The space $X_{\bar\partial}$ is a derived version of the complex manifold X itself, which is introduced in [6] to describe holomorphic Chern-Simons theory. This is a suitable concept to discuss perturbative quantum field theory invariant under a diffeomorphism group. It consists of a pair

$$X_{\bar\partial} = (X, \mathfrak{g}_X),$$

where $\mathfrak{g}_X$ is the sheaf of curved L_∞-algebras on X that we describe now. As a graded sheaf on X, $\mathfrak{g}_X$ is defined by

$$\mathfrak{g}_X := \mathcal{A}_X^\sharp \otimes_{\mathcal{O}_X} T_X[-1],$$

where $T_X[-1]$ is the sheaf of holomorphic tangent vectors with degree shifting such that it is concentrated at degree 1. To describe the curved L_∞-structure, we consider

$$C^*(\mathfrak{g}_X) := \widehat{\mathrm{Sym}}_{\mathcal{A}_X^\sharp}\left(\mathfrak{g}_X[1]^\vee\right) = \prod_{k\geq 0} \mathrm{Sym}^k_{\mathcal{A}_X^\sharp}\left(\mathfrak{g}_X[1]^\vee\right),$$

where

$$\mathfrak{g}_X[1]^\vee := \mathcal{A}_X^\sharp \otimes_{\mathcal{O}_X} T_X^\vee$$

is the dual sheaf of $\mathfrak{g}_X[1]$ over $\mathcal{A}_X^\sharp$, and $\mathrm{Sym}^k_{\mathcal{A}_X^\sharp}\left(\mathfrak{g}_X[1]^\vee\right)$ is the graded symmetric tensor product of k copies of $\mathfrak{g}_X[1]^\vee$ over $\mathcal{A}_X^\sharp$. When $k=0$, we set $\mathrm{Sym}^0_{\mathcal{A}_X^\sharp}(\mathfrak{g}_X[1]^\vee) \equiv \mathcal{A}_X^\sharp$.

It is easy to see that

$$C^*(\mathfrak{g}_X) = \mathcal{A}_X^\sharp \otimes_{\mathcal{O}_X} \widehat{\mathrm{Sym}}_{\mathcal{O}_X}\left(T_X^\vee\right).$$

Thus $C^*(\mathfrak{g}_X)$ is a sheaf of algebras over $\mathcal{A}_X^\sharp$.

Notation 2.2. Let $\{z^1, \cdots, z^n\}$ denote local holomorphic coordinates on X; we will let $\{\widetilde{\partial_{z^i}}\}$ denote the corresponding basis of $\mathfrak{g}_X$ over $\mathcal{A}_X^\sharp$, and let $\{\widetilde{dz^i}\}$ denote the corresponding basis of $\mathfrak{g}_X^\vee$ over $\mathcal{A}_X^\sharp$ similarly.

A curved L_∞-algebra structure on $\mathfrak{g}_X$ is a differential on $C^*(\mathfrak{g}_X)$ with which it becomes a dg-algebra over the dg-ring $\mathcal{A}_X$. Such a structure is obtained in [16], which is called a weak Lie algebra there. We reformulate the construction for the application in the B-twisted σ-model. Let us first recall

Definition 2.3. Let E be a holomorphic vector bundle on X. We define the holomorphic jet bundle $\mathrm{Jet}_X^{hol}(E)$ as follows: let π_1 and π_2 denote the projections of $X \times X$ onto the first and second component respectively,

$$\begin{array}{ccc} & X \times X & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ X & & X \end{array}$$

then

$$\mathrm{Jet}_X^{hol}(E) := \pi_{1*}\left(\widehat{\mathcal{O}}_\Delta \otimes_{\mathcal{O}_{X\times X}} \pi_2^* E\right),$$

where $\Delta \hookrightarrow X \times X$ is the diagonal, and $\widehat{\mathcal{O}}_\Delta$ is the analytic formal completion of $X \times X$ along Δ. The jet bundle $\mathrm{Jet}_X^{hol}(E)$ has a natural filtration defined by

$$F^k \,\mathrm{Jet}_X^{hol}(E) := I_\Delta^k \,\mathrm{Jet}_X^{hol}(E),$$

where I_Δ is the structure sheaf of Δ.

It is clear that $\mathrm{Jet}_X^{hol}(E)$ inherits a D_X^{hol}-module structure from $\widehat{\mathcal{O}}_\Delta$, and we will let $\Omega_X^*\left(\mathrm{Jet}_X^{hol}(E)\right)$ be the corresponding holomorphic de Rham complex. The natural embedding

$$E \hookrightarrow \Omega_X^*\left(\mathrm{Jet}_X^{hol}(E)\right)$$

induced by taking Taylor expansions of holomorphic sections is a quasi-isomorphism.

Let us consider a smooth map

$$\rho : U \to X \times X,$$

where $U \subset T_X$ is a small neighborhood of the zero section. We require that ρ is a diffeomorphism onto its image, and if we write

$$\rho : (x, v) \mapsto (x, \rho_x(v)),$$

then $\rho_x(-)$ is holomorphic if we fix x. Such a diffeomorphism can be constructed from a Kähler metric on X via the Kähler normal coordinates. Note that in general $\rho_x(-)$ does not vary holomorphically with respect to x. Such a map ρ induces an isomorphism

$$\rho^* : C^\infty(X) \otimes_{\mathcal{O}_X} \pi_{1*}\left(\widehat{\mathcal{O}}_\Delta\right) \xrightarrow{\sim} C^\infty(X) \otimes_{\mathcal{O}_X} \widehat{\mathrm{Sym}}\left(T_X^\vee\right).$$

Tensoring with $\mathcal{A}_X^\sharp$, we find the following identification:

$$\rho^* : \mathcal{A}_X^\sharp \otimes_{\mathcal{O}_X} \mathrm{Jet}_X^{hol}(\mathcal{O}_X) \xrightarrow{\sim} C^*\left(\mathfrak{g}_X\right). \tag{2.1}$$

Let d_{D_X} be the de Rham differential on $\mathcal{A}_X^\sharp \otimes_{\mathcal{O}_X} \mathrm{Jet}_X^{hol}(\mathcal{O}_X)$ induced from the D_X^{hol}-module structure on $\mathrm{Jet}_X^{hol}(\mathcal{O}_X)$. We can define a differential d_{CE} on $C^*\left(\mathfrak{g}_X\right)$ by

$$d_{CE} = \rho^* \circ d_{D_X} \circ \rho^{*-1}.$$

The differential d_{CE} defines a curved L_∞-structure on $\mathfrak{g}_X$, under which d_{CE} is the corresponding Chevalley-Eilenberg differential. We remark that the use of a Kähler metric is only auxiliary: any choice of smooth splitting of the projection

$$F^1\,\mathrm{Jet}_X^{hol}(\mathcal{O}_X) \to F^1\,\mathrm{Jet}_X^{hol}(\mathcal{O}_X)/F^2\,\mathrm{Jet}_X^{hol}(\mathcal{O}_X)$$

can be used to define a curved L_∞-structure on $\mathfrak{g}_X$, and different choices are homotopic equivalent [6]. Therefore we will not refer to a particular choice.

Definition 2.4. $\mathfrak{g}_X$ is the sheaf of curved L_∞-algebras on X defined by the Chevalley-Eilenberg complex $(C^*\left(\mathfrak{g}_X\right), d_{CE})$. We will denote the components of the structure maps (shifted by degree 1) of $\mathfrak{g}_X$ by

$$l_k : \mathrm{Sym}^k_{\mathcal{A}_X^\sharp}\left(\mathfrak{g}_X[1]\right) \to \mathfrak{g}_X.$$

Therefore l_1 defines $\mathfrak{g}_X$ as a dg-module over $\mathcal{A}_X$, l_k's are $\mathcal{A}_X^\sharp$-linear for $k > 1$, and l_0 defines the curving. There is a natural quasi-isomorphic embedding

$$(X, \mathcal{O}_X) \hookrightarrow (X, C^*\left(\mathfrak{g}_X\right))$$

and $X_{\bar\partial}$ is viewed as the derived enrichment of X in this sense.

Classical constructions of vector bundles can be naturally extended to the L_∞-space $X_{\bar\partial}$.

Definition 2.5. Let E be a holomorphic vector bundle on X. The induced vector bundle $E_{\bar\partial}$ on the L_∞-space $X_{\bar\partial}$ is defined by the $\mathfrak{g}_X$-module whose sheaf of Chevalley-Eilenberg complex $C^*\left(\mathfrak{g}_X, E_{\bar\partial}\right)$ is the dg module

$$C^*\left(\mathfrak{g}_X, E_{\bar\partial}\right) := \mathcal{A}_X^\sharp \otimes_{\mathcal{O}_X} \mathrm{Jet}_X^{hol}(E)$$

over the dg algebra $C^*(\mathfrak{g}_X)$.

Example 2.6. The tangent bundle $TX_{\bar\partial}$ is given by the module $\mathfrak{g}_X[1]$, with its naturally induced module structure over $\mathfrak{g}_X$. Similarly, the cotangent bundle $T^*X_{\bar\partial}$ is given by the natural $\mathfrak{g}_X$-module $\mathfrak{g}_X[1]^\vee$.

Symmetric and exterior tensor products of vector bundles are defined in the same fashion. For example,

$$\wedge^k T^* X_{\bar\partial} = \wedge^k \left(\mathfrak{g}_X[1]^\vee\right)$$

and a k-form on $X_{\bar\partial}$ is a section of the sheaf

$$C^*\left(\mathfrak{g}_X, \wedge^k\left(\mathfrak{g}_X[1]^\vee\right)\right) = \mathcal{A}_X^\sharp \otimes_{\mathcal{O}_X} \mathrm{Jet}_X^{hol}(\wedge^k T_X^\vee).$$

In Appendix D, we present the corresponding L_∞ constructions in more detail.

2.2.3. Mapping space as L_∞-space. Let $f : \Sigma_g \to X$ be a smooth map. The sheaf

$$f^*\mathfrak{g}_X \otimes_{f^*\mathcal{A}_X} \mathcal{A}_{\Sigma_g}$$

naturally inherits a curved L_∞-algebra on Σ_g within which Maurer-Cartan elements are defined [6] .

Definition 2.7. A map $(\Sigma_g)_{dR} \to X_{\bar\partial}$ consists of a smooth map $f : \Sigma_g \to X$, together with a Maurer-Cartan element

$$\alpha \in f^*\mathfrak{g}_X \otimes_{f^*\mathcal{A}_X} \mathcal{A}_{\Sigma_g}.$$

We would like to consider those maps which are constant on the underlying manifold. As shown in [6], the space of such maps can be represented by the L_∞-space

$$\left(X, \mathcal{A}_{\Sigma_g} \otimes_{\mathbb{C}} \mathfrak{g}_X\right),$$

which is an enrichment of $X_{\bar\partial}$ by the information from the Riemann surface Σ_g.

2.3. Classical action functional. As in Definition 2.1, our model is defined as the cotangent theory to the elliptic moduli problem of maps

$$(\Sigma_g)_{dR} \to X_{\bar\partial}.$$

The cotangent construction of perturbative field theory is described in [8] as a convenient way to implement Batalin-Vilkovisky quantization. In our case, we consider the enlarged mapping space

$$(\Sigma_g)_{dR} \to T^*X_{\bar\partial}[1].$$

The dg-space $(\Sigma_g)_{dR}$ is equipped with a volume form of degree -2, and $T^*X_{\bar\partial}[1]$ has a natural symplectic form of degree 1. This fits into the AKSZ-construction [1] and leads to an odd symplectic structure of degree -1 on the mapping space as desired for Batalin-Vilkovisky formalism.

We are interested in the locus around constant maps. As explained in section 2.2.3, such locus is represented by the L_∞-space

$$\left(X, \mathcal{A}_{\Sigma_g} \otimes_{\mathbb{C}} \mathfrak{g}_{T^*X_{\bar\partial}[1]}\right),$$

where $\mathfrak{g}_{T^*X_{\bar\partial}[1]} = \mathfrak{g}_X \oplus \mathfrak{g}_X[1]^\vee$ is the curved L_∞-algebra representing $T^*X_{\bar\partial}[1]$.

Definition 2.8. The space of fields of the B-twisted σ-model is the $\mathcal{A}_X^\sharp$-module

$$\mathcal{E} := \mathcal{A}_{\Sigma_g}^\sharp \otimes_{\mathbb{C}} \left(\mathfrak{g}_X[1] \oplus \mathfrak{g}_X^\vee\right).$$

Lemma/Definition 2.9. *There exists a natural graded sympletic pairing $\langle -, - \rangle$ on $\mathcal{E}$ of degree -1.*

The proof is standard and we omit it here. The classical action functional is constructed in a similar way as in [6].

Definition 2.10. The classical action functional is defined as the $\mathcal{A}_X^\sharp$-valued formal function on $\mathcal{E}$

$$S(\alpha+\beta) := \int_{\Sigma_g} \left(\langle d_{\Sigma_g}\alpha, \beta\rangle + \sum_{k\geqslant 0} \frac{1}{(k+1)!} \langle l_k(\alpha^{\otimes k}), \beta\rangle \right),$$

where $\alpha \in \mathcal{A}_{\Sigma_g}^\sharp \otimes \mathfrak{g}_X[1], \beta \in \mathcal{A}_{\Sigma_g}^\sharp \otimes \mathfrak{g}_X^\vee$, d_{Σ_g} is the de Rham differential on Σ_g, and l_k is the L_∞-product for $\mathfrak{g}_X$.

We will let

$$Q = d_{\Sigma_g} + l_1 : \mathcal{E} \to \mathcal{E}$$

and split the classical action S into its free and interaction parts

$$S = S_{free} + I_{cl},$$

where

$$I_{cl}(\alpha+\beta) = \int_{\Sigma_g} \left(\langle l_0, \beta\rangle + \sum_{k\geqslant 2} \frac{1}{(k+1)!} \langle l_k(\alpha^{\otimes k}), \beta\rangle \right)$$

and

$$S_{free}(\alpha+\beta) = \int_{\Sigma_g} \langle Q(\alpha), \beta\rangle.$$

For later discussion, we denote the following functionals by

$$\tilde{l}_k(\alpha+\beta) := \frac{1}{(k+1)!} \int_{\Sigma_g} \langle l_k(\alpha^{\otimes k}), \beta\rangle, \quad \text{for } k \geq 0. \tag{2.2}$$

2.4. Classical master equation. The classical action functional S satisfies the classical master equation, which is equivalent to the gauge invariance in the Batalin-Vilkovisky formalism. We will explain the classical master equation in this section and set up some notations to be used for quantization later.

2.4.1. Functionals on fields. The space of fields $\mathcal{E}$ is an $\mathcal{A}^\sharp_{\Sigma_g}$-module. Let $\mathcal{E}^{\otimes k}$ denote the $\mathcal{A}^\sharp_X$-linear completed tensor product of k copies of $\mathcal{E}$, where the completion is over the products of Riemann surfaces. Explicitly,

$$\mathcal{E}^{\otimes k} := \mathcal{A}_{\Sigma_g\times\cdots\times\Sigma_g} \otimes_{\mathbb{C}} \left(\left(\mathfrak{g}_X[1]\oplus \mathfrak{g}_X^\vee\right) \otimes_{\mathcal{A}^\sharp_X} \cdots \otimes_{\mathcal{A}^\sharp_X} \left(\mathfrak{g}_X[1]\oplus \mathfrak{g}_X^\vee\right)\right).$$

The permutation group S_k acts naturally on $\mathcal{E}^{\otimes k}$ and we will let

$$\mathrm{Sym}^k(\mathcal{E}) := \left(\mathcal{E}^{\otimes k}\right)_{S_k}$$

denote the S_k-coinvariants.

We will use $\overline{\mathcal{A}}_{\Sigma_g}$ to denote the distribution valued de Rham complex on Σ_g. $\overline{\mathcal{E}}$ will be distributional sections of $\mathcal{E}$:

$$\overline{\mathcal{E}} = \overline{\mathcal{A}}_{\Sigma_g} \otimes_{\mathbb{C}} \left(\mathfrak{g}_X[1]\oplus \mathfrak{g}_X^\vee\right).$$

We will also use

$$\mathcal{E}^\vee := \mathrm{Hom}_{\mathcal{A}^\sharp_X}\left(\mathcal{E}, \mathcal{A}^\sharp_X\right)$$

to denote functionals on $\mathcal{E}$ which are linear in $\mathcal{A}^\sharp_X$. The symplectic pairing $\langle -,-\rangle$ gives a natural embedding

$$\mathcal{E} \hookrightarrow \mathcal{E}^\vee[-1],$$

which induces an isomorphism

$$\overline{\mathcal{E}} \cong \mathcal{E}^\vee[-1].$$

Definition 2.11. We define the space of k-homogenous functionals on $\mathcal{E}$ by the linear functional (distribution) on $\Sigma_g\times\cdots\times\Sigma_g$ (k-copies)

$$\mathcal{O}^{(k)}(\mathcal{E}) := \mathrm{Hom}_{\mathcal{A}^\sharp_X}\left(\mathrm{Sym}^k(\mathcal{E}), \mathcal{A}^\sharp_X\right),$$

where our convention is that $\mathcal{O}^{(0)}(\mathcal{E}) = \mathcal{A}^\sharp_X$. We introduce the following notations:

$$\mathcal{O}(\mathcal{E}) := \prod_{k\geq 0}\mathcal{O}^{(k)}(\mathcal{E}), \quad \mathcal{O}^+(\mathcal{E}) := \prod_{k\geq 1}\mathcal{O}^{(k)}(\mathcal{E}).$$

Therefore $\mathcal{O}(\mathcal{E})$ can be viewed as formal power series on $\mathcal{E}$. The isomorphism $\overline{\mathcal{E}} \cong \mathcal{E}^\vee[-1]$ leads to natural isomorphisms

$$\mathcal{O}^{(k)}(\mathcal{E}) = \left(\mathcal{E}^\vee\right)^{\otimes k}_{S_k} \cong \left(\overline{\mathcal{E}}[1]\right)^{\otimes k}_{S_k},$$

where the tensor products are the $\mathcal{A}^\sharp_X$-linear completed tensor products over k copies of Σ_g.

Definition 2.12. Let $P \in \mathrm{Sym}^k(\mathcal{E})$. We define the operator of contraction with P

$$\frac{\partial}{\partial P} : \mathcal{O}^{(m+k)}(\mathcal{E}) \to \mathcal{O}^{(m)}(\mathcal{E})$$

by

$$\left(\frac{\partial}{\partial P}\Phi\right)(\mu_1,\cdots,\mu_m) := \Phi(P,\mu_1,\cdots,\mu_m),$$

where $\Phi \in \mathcal{O}^{(m+k)}(\mathcal{E}), \mu_i \in \mathcal{E}$.

Definition 2.13. We will denote by $\mathcal{O}_{loc}(\mathcal{E}) \subset \mathcal{O}(\mathcal{E})$ the subspace of local functionals, i.e. those of the form given by the integration of a Lagrangian density on Σ_g

$$\int_{\Sigma_g} \mathcal{L}(\mu), \quad \mu \in \mathcal{E}.$$

$\mathcal{O}^+_{loc}(\mathcal{E})$ is defined similarly to local functionals modulo constants.

Example 2.14. The classical action functional S in Definition 2.10 is a local functional.

2.4.2. Classical master equation. As a general fact in symplectic geometry, the Poisson kernel of a symplectic form induces a Poisson bracket on the space of functions. In our case we are dealing with the infinite dimensional symplectic space $(\mathcal{E}, \langle -,-\rangle)$. The Poisson bracket is of the form of δ-function distribution; therefore the Poisson bracket is well-defined on local functionals.

Lemma/Definition 2.15. *The symplectic pairing $\langle -,-\rangle$ induces an odd Poisson bracket of degree* 1 *on the space of local functionals, denoted by*

$$\{-,-\} : \mathcal{O}_{loc}(\mathcal{E}) \otimes_{\mathcal{A}^\sharp_X} \mathcal{O}_{loc}(\mathcal{E}) \to \mathcal{O}_{loc}(\mathcal{E}),$$

which is bilinear in $\mathcal{A}^\sharp_X$.

Lemma 2.16. *Let F_{l_1} be the functional on $\mathcal{E}$ defined as follows:*

$$F_{l_1}(\alpha+\beta) := \langle l_1^2(\alpha), \beta\rangle, \quad \alpha \in \mathcal{A}^\sharp_{\Sigma_g} \otimes \mathfrak{g}_X[1], \beta \in \mathcal{A}^\sharp_{\Sigma_g} \otimes \mathfrak{g}_X^\vee.$$

The classical interaction functional I_{cl} satisfies the following classical master equation:

$$QI_{cl} + \frac{1}{2}\{I_{cl}, I_{cl}\} + F_{l_1} = 0. \tag{2.3}$$

Proof. This follows from the fact that the maps $\{l_k\}_{k\geq 0}$ of $\mathfrak{g}_X$ define a curved L_∞-structure. See [6]. The extra term F_{l_1} describes the curving: $\{F_{l_1}, -\} = Q^2 = l_1^2$. q.e.d.

In particular, Lemma 2.16 implies that the operator $Q + \{I_{cl}, -\}$ defines a differential on $\mathcal{O}_{loc}(\mathcal{E})$.

Definition 2.17. The complex $Ob := \left(\mathcal{O}^+_{loc}(\mathcal{E}), Q + \{I_{cl}, -\}\right)$ is called the deformation-obstruction complex associated to the classical field theory defined by $(\mathcal{E}, S)$.

As established in [5], the complex Ob controls the deformation theory of the perburbative quantization of S, hence the name.

3. Quantization

In this section we establish the quantization of our B-twisted σ-model via Costello's perturbative renormalization method [5]. We show that the obstruction to the quantization is given by $(2-2g)c_1(X)$. When $c_1(X)=0$, i.e. X being Calabi-Yau, every choice of holomorphic volume form on X leads to an associated canonical quantization of the B-twisted σ-model.

3.1. Regularization. Perturbative quantization of the classical action functional S is to model the asymptotic $\hbar$-expansion of the infinite dimensional path integral

$$\int_{L\subset\mathcal{E}} e^{S/\hbar},$$

where L is an appropriate subspace related to some gauge fixing (a BV-Lagrangian in the Batalin-Vilkovisky formalism). A natural formalism based on finite dimensional models is

$$\int_{L\subset\mathcal{E}} e^{S/\hbar} \mapsto \exp\left(\hbar^{-1}W(G,I_{cl})\right),$$

where $W(G,I_{cl})$ is the weighted sum of Feynman integrals over all connected graphs, with G ($=\mathbb{P}_0^\infty$ below) labeling the internal edges, and I_{cl} labeling the vertices. One essential difficulty is the infinite dimensionality of the space of fields which introduces singularities in the propagator G and breaks the naive interpretation of Feynman diagrams. Certain regularization is required to make sense of the theory, which is the celebrated idea of renormalization in quantum field theory. We will use the heat kernel regularization to fit into Costello's renormalization technique [5].

3.1.1. Gauge fixing. We need to choose a gauge fixing operator for regularization. For any Riemann surface Σ_g, we pick the metric on Σ_g of constant curvature $0,1$, or -1, depending on the genus g. In particular, we choose the hyperbolic metric on Σ_g when $g>1$. The gauge fixing operator is

$$Q^{GF} := d^*_{\Sigma_g},$$

where $d^*_{\Sigma_g}$ is the adjoint of the de Rham differential d_{Σ_g} on Σ_g with respect to the chosen metric. It is clear that the Laplacian $H=[Q,Q^{GF}]=d_{\Sigma_g}d^*_{\Sigma_g}+d^*_{\Sigma_g}d_{\Sigma_g}$ is the usual Laplacian on $\mathcal{A}_{\Sigma_g}$. We will let e^{-tH} denote the heat operator acting on $\mathcal{A}_{\Sigma_g}$ for $t>0$.

REMARK 3.1. The operators Q^{GF}, H, and e^{-tH} extend trivially over $\mathfrak{g}_X[1]\oplus\mathfrak{g}_X^\vee$ to define operators on $\mathcal{E}$, and we will use the same notations without confusion.

3.1.2. Effective propagator. To analyze the B-twisted σ-model, we first describe the propagator of the theory.

Definition 3.2. The heat kernel $\mathbb{K}_t$ for $t > 0$ is the element in $\mathrm{Sym}^2(\mathcal{E})$ defined by the equation

$$\langle \mathbb{K}_t(z_1, z_2), \phi(z_2)\rangle = e^{-tH}(\phi)(z_1), \quad \forall \phi \in \mathcal{E}, z_1 \in \Sigma_g.$$

Notation 3.3. The fact that the symplectic pairing on $\mathcal{E}$ is (up to sign) the tensor product of the natural pairings on $\mathcal{A}^\sharp_{\Sigma_g}$ and $\mathfrak{g}_X[1] \oplus \mathfrak{g}_X^\vee$ implies that the heat kernel $\mathbb{K}_t(z_1, z_2)$ is of the following form:

$$\mathbb{K}_t(z_1, z_2) = K_t(z_1, z_2) \otimes (\mathrm{Id}_{\mathfrak{g}_X} + \mathrm{Id}_{\mathfrak{g}_X^\vee}),$$

where K_t is simply the usual heat kernel of e^{-tH} on Σ_g, and $\mathrm{Id}_{\mathfrak{g}_X} + \mathrm{Id}_{\mathfrak{g}_X^\vee}$ is the Poisson kernel corresponding to the natural symplectic pairing on $\mathfrak{g}_X[1] \oplus \mathfrak{g}_X^\vee$. We will call K_t and $\mathrm{Id}_{\mathfrak{g}_X} + \mathrm{Id}_{\mathfrak{g}_X^\vee}$ the analytic and combinatorial parts of $\mathbb{K}_t$ respectively.

The combinatorial part of $\mathbb{K}_t$ can be described locally as follows: pick a local basis $\{X_i\}$ of $\mathfrak{g}_X[1]$ as an $\mathcal{A}^\sharp_X$-module, and let $\{X^i\}$ be the corresponding dual basis of $\mathfrak{g}_X^\vee$. Then we have

$$\mathrm{Id}_{\mathfrak{g}_X} + \mathrm{Id}_{\mathfrak{g}_X^\vee} = \sum_i (X_i \otimes X^i + X^i \otimes X_i).$$

Definition 3.4. For $0 < \epsilon < L < \infty$, we define the effective propagator $\mathbb{P}_\epsilon^L$ as the element in $\mathrm{Sym}^2(\mathcal{E})$ by

$$\mathbb{P}_\epsilon^L(z_1, z_2) = P_\epsilon^L(z_1, z_2) \otimes (\mathrm{Id}_{\mathfrak{g}_X} + \mathrm{Id}_{\mathfrak{g}_X^\vee}),$$

where the analytic part of the propagator P_ϵ^L is given by

$$P_\epsilon^L := \int_\epsilon^L (Q^{GF} \otimes 1) K_t dt.$$

Remark 3.5. In the notations $P_\epsilon^L(z_1, z_2)$ and $K_t(z_1, z_2)$, we have omitted their anti-holomorphic dependence for simplicity.

In other words, $\mathbb{P}_\epsilon^L$ is the kernel representing the operator $\int_\epsilon^L Q^{GF} e^{-tH} dt$ on $\mathcal{E}$. The full propagator $\mathbb{P}_0^\infty$ represents the operator $\frac{Q^{GF}}{H}$, which is formally the inverse of the quadratic pairing S_{free} after gauge fixing. The standard trick of Feynman diagram expansions picks $\mathbb{P}_0^\infty$ as the propagator. However, $\mathbb{P}_0^\infty$ exhibits singularity along the diagonal in $\Sigma_g \times \Sigma_g$, and the above effective propagator with cut-off parameters ϵ, L is viewed as a regularization.

It is known that the heat kernel K_t on a Riemann surface Σ_g has an asymptotic expansion:

$$K_t(z_1, z_2) \sim \frac{1}{4\pi t} e^{-\frac{\rho^2(z_1, z_2)}{4t}} \left(\sum_{i=0}^\infty t^i \cdot a_i(z_1, z_2)\right) \quad \text{as } t \to 0, \tag{3.1}$$

where each $a_i(z_1, z_2)$ is a smooth 2-form on $\Sigma_g \times \Sigma_g$ and $\rho(z_1, z_2)$ denotes the geodesic distance between z_1 and z_2. Similarly, for the propagator P_ϵ^L, we have

Lemma 3.6 (Appendix A). *The propagator on the hyperbolic upper half plane $\mathbb{H}$ is given explicitly by*

(3.2)

$$P_\epsilon^L = \int_\epsilon^L f(\rho, t) dt \cdot \left(\frac{2(x_1 - x_2)}{y_1 y_2} (dy_1 - dy_2) - \frac{(y_1 - y_2)(y_1 + y_2)}{y_1 y_2} \left(\frac{dx_1}{y_1} - \frac{dx_2}{y_2} \right) \right),$$

where $x_i = \operatorname{Re} z_i$ and $y_i = \operatorname{Im} z_i$, for $i = 1, 2$. The function $f(\rho, t)$ is smooth on $\mathbb{R}_{\geqslant 0} \times \mathbb{R}_{>0}$, and has an asymptotic expansion as $t \to 0$:

$$f(\rho, t) \sim \sum_{k=0}^{\infty} t^{-2+k} e^{-\frac{\rho^2}{4t}} b_k(\rho). \tag{3.3}$$

3.1.3. Effective Batalin-Vilkovisky formalism. The heat kernel cut-off also allows us to regularize the Poisson bracket $\{-,-\}$ and extend its definition from local functionals to all distributions.

Definition 3.7. We define the effective BV Laplacian Δ_L at scale $L > 0$

$$\Delta_L := \frac{\partial}{\partial \mathbb{K}_L} : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$$

by contracting with $\mathbb{K}_L$ (see Definition 2.12).

Since the regularized Poisson kernel $\mathbb{K}_L$ is smooth, Δ_L is well-defined on $\mathcal{O}(\mathcal{E})$ and can be viewed as a second-order differential operator in our infinite dimensional setting.

Definition 3.8. We define the effective BV bracket at scale L

$$\{-,-\}_L : \mathcal{O}(\mathcal{E}) \times \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$$

by

$$\{\Phi_1, \Phi_2\}_L := \Delta_L (\Phi_1 \Phi_2) - (\Delta_L \Phi_1) \Phi_2 - (-1)^{|\Phi_1|} \Phi_1 (\Delta_L \Phi_2), \quad \forall \Phi_1, \Phi_2 \in \mathcal{O}(\mathcal{E}).$$

As we will see, Batalin-Vilkovisky structures at different scales will be related to each other via the renormalization group flow.

For two distributions $\Phi_1, \Phi_2 \in \mathcal{O}(\mathcal{E})$, the bracket $\{\Phi_1, \Phi_2\}_L$ will in general diverge as $L \to 0$. However, for $\Phi_1, \Phi_2 \in \mathcal{O}_{loc}(\mathcal{E})$,

$$\lim_{L \to 0} \{\Phi_1, \Phi_2\}_L = \{\Phi_1, \Phi_2\},$$

where on the right hand side $\{-,-\}$ is the Poisson bracket as in Lemma /Definition 2.15. Therefore $\{-,-\}_L$ is a regularization of the classical Poisson bracket.

3.2. Effective renormalization. We discuss Costello's quantization framework [5] in our current set-up.

3.2.1. Renormalization group flow. We start from the definition of graphs:

Definition 3.9. A graph γ consists of the following data:

1) A finite set of vertices $V(\gamma)$.
2) A finite set of half-edges $H(\gamma)$.
3) An involution $\sigma : H(\gamma) \to H(\gamma)$. The set of fixed points of this map is denoted by $T(\gamma)$ and is called the set of tails of γ. The set of two-element orbits is denoted by $E(\gamma)$ and is called the set of internal edges of γ.
4) A map $\pi : H(\gamma) \to V(\gamma)$ sending a half-edge to the vertex to which it is attached.
5) A map $g : V(\gamma) \to \mathbb{Z}_{\geqslant 0}$ assigning a genus to each vertex.

There exists a naturally constructed topological space $|\gamma|$ from the above abstract data (see chapter 2, section 3.1, in [5] for more details on the construction). A graph γ is called *connected* if $|\gamma|$ is connected. A graph is called stable if every vertex of genus 0 is at least trivalent, and every genus 1 vertex is at least univalent. The genus of the graph γ is defined to be

$$g(\gamma) := b_1(|\gamma|) + \sum_{v \in V(\gamma)} g(v),$$

where $b_1(|\gamma|)$ denotes the first Betti number of $|\gamma|$.

Let

$$(\mathcal{O}(\mathcal{E})[[\hbar]])^+ \subset \mathcal{O}(\mathcal{E})[[\hbar]]$$

be the subspace consisting of those functionals which are at least cubic modulo $\hbar$ and modulo the nilpotent ideal $\mathcal{I}$ in the base ring $\mathcal{A}_X^\sharp$. Let $I \in (\mathcal{O}(\mathcal{E})[[\hbar]])^+$ be a functional which can be expanded as

$$I = \sum_{k,i \geq 0} \hbar^k I_i^{(k)}, \quad I_i^{(k)} \in \mathcal{O}^{(i)}(\mathcal{E}).$$

We view $I_i^{(k)}$ as an S_i-invariant linear map

$$I_i^{(k)} : \mathcal{E}^{\otimes i} \to \mathcal{A}_X^\sharp.$$

With the propagator $\mathbb{P}_\epsilon^L$, we will describe the *Feynman weights*

$$W_\gamma(\mathbb{P}_\epsilon^L, I) \in (\mathcal{O}(\mathcal{E})[[\hbar]])^+$$

for any connected stable graph γ: we label every vertex v in γ of genus $g(v)$ and valency i by $I_i^{(g(v))}$, which we denote by:

$$I_v : \mathcal{E}^{\otimes H(v)} \to \mathcal{A}_X^\sharp,$$

where $H(v)$ is the set of half-edges of γ which are incident to v. We label every internal edge e by the propagator

$$\mathbb{P}_e = \mathbb{P}_\epsilon^L \in \mathcal{E}^{\otimes H(e)},$$

where $H(e) \subset H(\gamma)$ is the two-element set consisting of the half-edges forming e. Now we can contract

$$\otimes_{v\in V(\gamma)} I_v : \mathcal{E}^{H(\gamma)} \to \mathcal{A}_X^\sharp$$

with

$$\otimes_{e\in E(\gamma)} \mathbb{P}_e \in \mathcal{E}^{H(\gamma)\setminus T(\gamma)} \to \mathcal{A}_X^\sharp$$

to yield a linear map

$$W_\gamma(\mathbb{P}_\epsilon^L, I) : \mathcal{E}^{\otimes T(\gamma)} \to \mathcal{A}_X^\sharp.$$

We can now define the *renormalization group flow* operator:

Definition 3.10. The renormalization group flow operator from scale ϵ to scale L is the map

$$W(\mathbb{P}_\epsilon^L, -) : (\mathcal{O}(\mathcal{E})[[\hbar]])^+ \to (\mathcal{O}(\mathcal{E})[[\hbar]])^+$$

defined by taking the sum of Feynman weights over all stable connected graphs:

$$I \mapsto \sum_\gamma \frac{1}{|\mathrm{Aut}(\gamma)|} \hbar^{g(\gamma)} W_\gamma(\mathbb{P}_\epsilon^L, I).$$

A collection of functionals

$$\{I[L] \in (\mathcal{O}(\mathcal{E})[[\hbar]])^+ \,|L \in \mathbb{R}_+\}$$

is said to satisfy the renormalization group equation (RGE) if for any $0 < \epsilon < L < \infty$, we have

$$I[L] = W(\mathbb{P}_\epsilon^L, I[\epsilon]).$$

Remark 3.11. Formally, the RGE can be equivalently described as $e^{I[L]/\hbar} = e^{\hbar \frac{\partial}{\partial \mathbb{P}_\epsilon^L}} e^{I[\epsilon]/\hbar}$.

3.2.2. Quantum master equation. Now we explain the quantum master equation as the quantization of the classical master equation. Usually the quantum master equation is associated with the following operator [5] in the Batalin-Vilkovisky formalism

$$Q + \hbar\Delta_L,$$

which can be viewed as a quantization of the differential Q.

However, in our case, the above operator does not define a differential due to the curving

$$(Q + \hbar\Delta_L)^2 = l_1^2.$$

We will modify the construction in [5] to incorporate with the curving.

Definition 3.12. We define the effective curved differential $Q_L : \mathcal{E} \to \mathcal{E}$ by

$$Q_L := Q + l_1^2 \int_0^L Q^{GF} e^{-tH} dt,$$

where $l_1^2 \int_0^L Q^{GF} e^{-tH} dt$ is the composition of the operator $\int_0^L Q^{GF} e^{-tH} dt$ with l_1^2.

It is straightforward to prove the following lemma:

Lemma 3.13. *The quantized operator* $Q_L + \hbar\Delta_L + \frac{F_{l_1}}{\hbar}$ *is compatible with the renormalization group flow in the following sense (recall Lemma* 2.16 *for the definition* F_{l_1}*):*

$$e^{\hbar \frac{\partial}{\partial \mathbb{P}_\epsilon^L}} \left(Q_\epsilon + \hbar\Delta_\epsilon + \frac{F_{l_1}}{\hbar} \right) = \left(Q_L + \hbar\Delta_L + \frac{F_{l_1}}{\hbar} \right) e^{\hbar \frac{\partial}{\partial \mathbb{P}_\epsilon^L}}.$$

Moreover, it squares to zero modulo $\mathcal{A}_X^\sharp$*:*

$$\left(Q_L + \hbar\Delta_L + \frac{F_{l_1}}{\hbar} \right)^2 = C$$

equals the multiplication by some $C \in \mathcal{A}_X^\sharp$.

Therefore we will use $Q_L + \hbar\Delta_L + \frac{F_{l_1}}{\hbar}$ instead of $Q + \hbar\Delta_L$ in order to define the quantum master equation. The constant C does not bother us since the perturbative quantization in [5] is defined modulo the constant terms. Precisely,

Definition 3.14. Let $\{I[L] \in (\mathcal{O}(\mathcal{E})[[\hbar]])^+ \, | L \in \mathbb{R}_+\}$ be a collection of effective interactions that satisfy the renormalization group equation. We say that they satisfy the quantum master equation if for all $L > 0$ the following scale L quantum master equation (QME) is satisfied:

$$\left(Q_L + \hbar\Delta_L + \frac{F_{l_1}}{\hbar} \right) e^{I[L]/\hbar} = R e^{I[L]/\hbar}, \tag{3.4}$$

where $R \in \mathcal{A}_X^\sharp[[\hbar]]$ does not depend on L.

In other words, if we view $Q_L + \hbar\Delta_L + \frac{F_{l_1}}{\hbar}$ as defining a projectively flat connection, then a solution of the quantum master equation defines a projectively flat section.

Lemma 3.15. *The quantum master equation is compatible with the renormalization group flow in the following sense: if the collection* $\{I[L] | L \in \mathbb{R}_+\}$ *satisfies the QME at some scale* $L_0 > 0$*, then the QME holds for any scale.*

Proof. This follows from Lemma 3.13. q.e.d.

Lemma 3.16. *Suppose $I[L]$ satisfies the quantum master equation at scale $L > 0$; then $Q_L + \hbar\Delta_L + \{I[L], -\}_L$ defines a square-zero operator on $\mathcal{O}(\mathcal{E})[[\hbar]]$.*

Proof. Let $U_L = Q_L + \hbar\Delta_L + \{I[L], -\}_L$ and $\Phi \in \mathcal{O}(\mathcal{E})[[\hbar]]$. Then

$$\left(Q_L + \hbar\Delta_L + \frac{F_{l_1}}{\hbar}\right)\left(\Phi e^{I[L]/\hbar}\right) = \left(U_L(\Phi) + R\Phi\right) e^{I[L]/\hbar}.$$

Applying $Q_L + \hbar\Delta_L + \frac{F_{l_1}}{\hbar}$ again to both sides, we find

$$C\Phi e^{I[L]/\hbar} = \left(U_L^2(\Phi) + U_L(R\Phi) + R(U_L(\Phi) + R\Phi)\right) e^{I[L]/\hbar}.$$

Set $\Phi = 1$. We find $C = d_X R + R^2$, while $R^2 = 0$ since R is a 1-form. Here d_X is the de Rham differential on X. On the other hand,

$$U_L(R\Phi) = (d_X R)\Phi - RU_L(\Phi).$$

Comparing the two sides of the above equation, we get $U_L^2(\Phi) = 0$ as desired. q.e.d.

Remark 3.17. From the above proof, we find the following compatibility equation: $C = d_X R$. It is not hard to see that the two form C is given by the contraction between F_{l_1} and Δ_L, describing the curvature l_1^2. In fact C represents $(2-2g)c_1(X)$. The compatibility equation says that C is an exact form, implying that the Calabi-Yau condition is necessary for the quantum consistency (if $g \neq 1$). In section 3.3.3, we will show that the Calabi-Yau condition is also sufficient for anomaly cancellation. Such a phenomenon arising from the curved L_∞-algebra does not play a role in [6], but we expect that it would appear in general.

It is easy to see that the leading $\hbar$-order of the quantum master equation reduces to the classical master equation when $L \to 0$. Therefore the square-zero operator $Q_L + \hbar\Delta_L + \{I[L], -\}_L$ defines a quantization of the classical $Q + \{I_{cl}, -\}$.

3.2.3. $\mathbb{C}^\times$-symmetry. Later, when we study quantum theory of the B-twisted σ-model, we will be interested in quantizations which preserve certain symmetries we describe now: we consider an action of $\mathbb{C}^\times$ on $\mathcal{E}$ as follows:

$$\lambda \cdot (\alpha_1 \otimes \mathfrak{g}_1 + \alpha_2 \otimes \mathfrak{g}_2^\vee) := \alpha_1 \otimes \mathfrak{g}_1 + \lambda^{-1}\alpha_2 \otimes \mathfrak{g}_2^\vee,\ \lambda \in \mathbb{C}^\times.$$

Definition 3.18. We define an action of $\mathbb{C}^\times$ on $\mathcal{O}(\mathcal{E})((\hbar))$ by

$$(\lambda \cdot (\hbar^k F))(v) := \lambda^k \hbar^k F(\lambda^{-1} \cdot v), F \in \mathcal{O}(\mathcal{E}), v \in \mathcal{E}.$$

It is obvious that the classical interaction $I_{cl}/\hbar$ is invariant under this action. The following lemma can be proved by straightforward calculation, which we omit:

Lemma 3.19. *The following operations are equivariant under the action of $\mathbb{C}^\times$:*

1) *the renormalization group flow operator:*

$$W(\mathbb{P}_\epsilon^L, -) : (\mathcal{O}(\mathcal{E})[[\hbar]])^+ \to (\mathcal{O}(\mathcal{E})[[\hbar]])^+,$$

2) *the differential* $Q : \mathcal{O}(\mathcal{E})[[\hbar]] \to \mathcal{O}(\mathcal{E})[[\hbar]]$,
3) *the quantized differential* $Q_L + \hbar\Delta_L + \hbar^{-1}F_{l_1}$,
4) *the BV bracket* $\{-,-\}_L : \mathcal{O}(\mathcal{E})[[\hbar]] \otimes_{\mathcal{A}_X^\sharp[[\hbar]]} \mathcal{O}(\mathcal{E})[[\hbar]] \to \mathcal{O}(\mathcal{E})[[\hbar]]$, *for all* $L > 0$.

The following proposition describes those functionals in $\mathcal{O}(\mathcal{E})[[\hbar]]$ that are $\mathbb{C}^\times$-invariant.

Proposition 3.20. *Let* $I = \sum_{i\geq 0} I^{(i)} \cdot \hbar^i \in \mathcal{O}(\mathcal{E})[[\hbar]]$. *If* I *is invariant under the* $\mathbb{C}^\times$ *action, then* $I^{(i)} = 0$ *for* $i > 1$, *and furthermore* $I^{(1)}$ *lies in the subspace*

$$\mathcal{O}(\mathcal{A}_{\Sigma_g} \otimes \mathfrak{g}_X[1]) \subset \mathcal{O}(\mathcal{E}).$$

Proof. The hypothesis that I is invariant implies that $I^{(i)}$ has weight $-i$ under the $\mathbb{C}^\times$ action. Notice that the weight of the $\mathbb{C}^\times$ action on $\mathcal{O}(\mathcal{E})$ can be only -1 or 0, which implies the first statement. The second statement of the proposition is obvious. q.e.d.

3.3. Quantization. We study the quantization of the B-twisted σ-model in this section. Firstly, let us recall the definition of perturbative quantization of classical field theories in the Batalin-Vilkovisky formalism in [5]:

Definition 3.21. Let $I \in \mathcal{O}_{loc}(\mathcal{E})$ be a classical interaction functional satisfying the classical master equation. A quantization of the classical field theory defined by I consists of a collection $\{I[L] \in (\mathcal{O}(\mathcal{E})[[\hbar]])^+ | L \in \mathbb{R}_+\}$ of effective functionals such that

1) The renormalization group equation is satisfied.
2) The functional $I[L]$ must satisfy a locality axiom, saying that as $L \to 0$ the functional $I[L]$ becomes more and more local.
3) The functional $I[L]$ satisfies the scale L *quantum master equation* (3.4).
4) Modulo $\hbar$, the $L \to 0$ limit of $I[L]$ agrees with the classical interaction functional I.

The strategy for constructing a quantization of a given classical action functional is to run the renormalization group flow from scale 0 to scale L. In other words, we try to define the effective interaction $I[L]$ as the following limit:

$$\lim_{\epsilon\to 0} W(\mathbb{P}_\epsilon^L, I).$$

However, the above limit in general does not exist. Then the technique of counter-terms solves the problem: after the choice of a *Renormalization*

Scheme, there is a unique set of counter-terms $I^{CT}(\epsilon) \in (\mathcal{O}_{loc}(\mathcal{E})[[\hbar]])^+$ such that the limit

$$\lim_{\epsilon\to 0} W(\mathbb{P}_\epsilon^L, I - I^{CT}(\epsilon)) \in (\mathcal{O}(\mathcal{E})[[\hbar]])^+ \tag{3.5}$$

exists. For more details on the Renormalization Scheme and counter-terms, we refer the readers to [5]. It is then natural to define the *naive quantization* $I_{naive}[L]$ of I to be the limit in equation (3.5). For the B-twisted σ-model, we calculate the naive quantization in section 3.3.1. In particular, we show that the counter-terms for our theory actually vanish.

The naive quantization $\{I_{naive}[L]|L > 0\}$ automatically satisfies the renormalization group equation and the locality axiom by construction. However, it does not satisfy the quantum master equation in general. In order to find the genuine quantization, we need to analyze the possible cohomological obstruction to solving the QME, and correct the naive quantization $\{I_{naive}[L]|L > 0\}$ term by term in $\hbar$ accordingly if the obstruction vanishes. The $\mathbb{C}^\times$ symmetry of the classical interaction I_{cl} simplifies this computation to a one-loop anomaly, and in Appendix C we give a formula for a one-loop anomaly for general field theories. This formula, when specialized to the B-twisted σ-model, shows that the condition for anomaly cancellation is exactly the Calabi-Yau condition of the target X. This is done in section 3.3.3. In section 3.3.4, we give an explicit formula for the one-loop correction of the naive quantization when X is Calabi-Yau.

Remark 3.22. In later sections, we will give the details of the analysis for Riemann surfaces of genus $g > 1$; the studies of $\mathbb{P}^1$ and elliptic curves are similar and actually easier, and we omit them.

3.3.1. The naive quantization. Let I_{cl} denote the classical interaction of the B-twisted σ-model. We will show that the following limit exists for all $L > 0$:

$$\lim_{\epsilon\to 0} W(\mathbb{P}_\epsilon^L, I_{cl}). \tag{3.6}$$

The following simple observation simplifies the analysis greatly: for any $L > \epsilon > 0$ and any graph γ, the associated Feynman weight $\frac{\hbar^{|g(\gamma)|}}{|\mathrm{Aut}(\gamma)|} W_\gamma(\mathbb{P}_\epsilon^L, I_{cl})$ is invariant under the $\mathbb{C}^\times$ action defined in section 3.2.3, by Lemma 3.19 and the fact that $I_{cl}/\hbar$ is $\mathbb{C}^\times$-invariant. By Proposition 3.20, we have

$$W_\gamma(\mathbb{P}_\epsilon^L, I_{cl}) = 0$$

for those stable graphs γ with genus greater than 1. Thus we only need to consider Feynman weights for trees and one-loop graphs. For any classical interaction I, the limit (3.6) always exists for trees, but not necessarily for one-loop graphs. Fortunately, for the classical interaction I_{cl} of the B-twisted σ-model, the limit (3.6) exists.

Lemma/Definition 3.23. *Let γ be a graph of genus 1; then the following limit exists:*

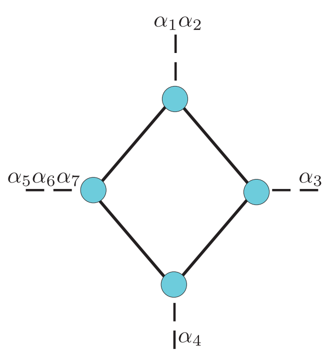
$$\lim_{\epsilon\to 0} W_\gamma(\mathbb{P}_\epsilon^L, I_{cl}).$$

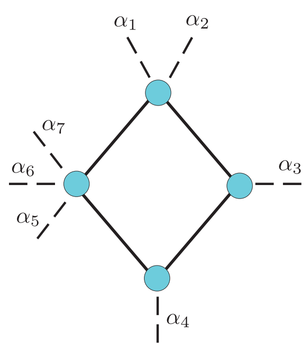
We define the naive quantization at scale L to be

$$I_{naive}[L] := \lim_{\epsilon\to 0} W(\mathbb{P}_\epsilon^L, I_{cl}).$$

REMARK 3.24. As discussed in Definition 3.4, the propagator $\mathbb{P}_\epsilon^L$ consists of an analytic part and a combinatorial part. It is clear that only the analytic part is relevant to the convergence issue. In the following, we will use the notation $W(P_\epsilon^L, I_{cl})$ to denote the analytic part of the RG flow $W(\mathbb{P}_\epsilon^L, I_{cl})$, whose inputs are only differential forms on Σ_g. We will also use similar notations replacing $\mathbb{K}_\epsilon$ by K_ϵ later.

Proof. Since $I_{cl} \in (\mathcal{O}(\mathcal{E}))^+ \subset (\mathcal{O}(\mathcal{E})[[\hbar]])^+$, we only need to consider those genus 1 graphs γ whose vertices are all of genus 0, i.e. $b_1(\gamma) = 1$. Such a graph is called a *wheel* if it can not be disconnected by removing a single edge. Every graph with first Betti number 1 is a wheel with trees attached on it. Since trees do not contribute any divergence, we only need to prove the lemma for wheels. Let γ be a wheel with n tails, and let $\alpha_1\otimes\mathfrak{g}_1, \cdots, \alpha_n\otimes\mathfrak{g}_n \in \mathcal{E}$ be inputs on the tails. If the valency of some vertices of γ is greater than 3, we can combine the analytic part of the inputs on the tails that are attached to the same vertex. More precisely, the convergence property of the following two Feynman weights is the same:





Thus the proof of the lemma can be further reduced to trivalent wheels. Let γ be a trivalent wheel with n vertices; we prove the lemma for the three possibilities:

(1) $n = 1$: in this case, the graph γ contains a self-loop, and the Feynman weight is given by

$$W_\gamma(P_\epsilon^L, I_{cl})(\alpha) = \int_{t=\epsilon}^L dt \int_{z_1\in\Sigma_g} d^*K_t(z_1, z_1)\alpha.$$

Let the Riemann surface Σ_g be of the form $\Sigma_g = \mathbb{H}/\Gamma$, where Γ is a subgroup of isometry acting discretely on $\mathbb{H}$. Let k_t denote the heat kernel on $\mathbb{H}$, and let π denote the natural projection $\mathbb{H} \to \Sigma_g$. The heat kernel on Σ_g can be written as:

$$K_t(\pi(x_1), \pi(x_2)) = \sum_{g\in\Gamma} k_t(x_1, g \cdot x_2),$$

from which it is clear that K_t is regular along the diagonal in $\Sigma_g \times \Sigma_g$: we can pick $x_1 = x_2$ in the above identity. If $g = \mathrm{id}$, then $k_t(x_1, x_1)$ vanishes by Lemma 3.6; otherwise $k_t(x_1, g \cdot x_1)$ is automatically regular since the heat kernel is singular only along the diagonal but $x_1 \neq g \cdot x_1$.

(2) $n \geq 3$: the Feynman weight is given explicitly by:

(3.7)
$$\begin{aligned} &W_\gamma(P_\epsilon^L, I_{cl})(\alpha_1, \cdots, \alpha_n) \\ &= \int_{z_1,\cdots,z_n\in\Sigma_g} P_\epsilon^L(z_1, z_2) P_\epsilon^L(z_2, z_3) \cdots P_\epsilon^L(z_n, z_1) \alpha_1(z_1, \bar{z}_1) \cdots \alpha_n(z_n, \bar{z}_n) \\ &= \int_{t_1,\cdots,t_n=\epsilon}^L dt_1 \cdots dt_n \\ &\left(\int_{z_1,\cdots,z_n\in\Sigma_g} (d^* K_{t_1}(z_1, z_2)) \cdots (d^* K_{t_n}(z_n, z_1))\, \alpha_1(z_1, \bar{z}_1) \cdots \alpha_n(z_n, \bar{z}_n) \right) \end{aligned}$$

Using the same argument as in case (1), there is no difference if we replace Σ_g in equation (3.7) by $\mathbb{H}$ as far as only the convergence property is concerned:

(3.8)
$$\begin{aligned} &\int_{t_1,\cdots,t_n=\epsilon}^L dt_1 \cdots dt_n \\ &\left(\int_{z_1,\cdots,z_n\in\mathbb{H}} (d^* k_{t_1}(z_1, z_2)) \cdots (d^* k_{t_n}(z_n, z_1))\, \alpha_1(z_1, \bar{z}_1) \cdots \alpha_n(z_n, \bar{z}_n) \right) \end{aligned}$$

For simplicity, we keep the notation for the inputs $\alpha_1, \cdots, \alpha_n$ which are now differential forms on $\mathbb{H}$ with compact support. We can write the integral (3.8) as the sum of the following integrals where σ runs over the symmetric group S_n:

(3.9)
$$\begin{aligned} &\int_{\epsilon\leqslant t_{\sigma(1)}\leqslant\cdots\leqslant t_{\sigma(n)}\leqslant L} dt_1 \cdots dt_n \\ &\left(\int_{z_1,\cdots,z_n\in\mathbb{H}} (d^* k_{t_1}(z_1, z_2)) \cdots (d^* k_{t_n}(z_n, z_1))\, \alpha_1(z_1, \bar{z}_1) \cdots \alpha_n(z_n, \bar{z}_n) \right) \end{aligned}$$

We will show that the integral (3.9) converges as $\epsilon \to 0$ for $\sigma = \mathrm{id} \in S_n$; the proof for other permutations σ is the same. Let $(z_1, \cdots, z_n) = (x_1 + iy_1, \cdots, x_n + iy_n)$ denote the standard coordinates on $\mathbb{H} \times \cdots \times \mathbb{H}$. By

Lemma 3.6, the leading term of $d^*k_{t_k}(z_k, z_{k+1})$ is of the form

$$\frac{1}{t_k^2}e^{-\frac{\rho^2(z_k,z_{k+1})}{4t_k}}\Big(Q_1(z_k,\bar{z}_k;z_{k+1},\bar{z}_{k+1})(x_{k+1}-x_k)(dy_{k+1}-dy_k) \\ -Q_2(z_k,\bar{z}_k;z_{k+1},\bar{z}_{k+1})(y_{k+1}-y_k)(\frac{dx_{k+1}}{y_{k+1}}-\frac{dx_k}{y_k})\Big), \tag{3.10}$$

where Q_1 and Q_2 are smooth functions on $\mathbb{H}\times\mathbb{H}$. To show the convergence of the integral (3.9) as $\epsilon\to 0$, we will apply Wick's lemma and show that the integral of the leading term in (3.9) converges. The higher order terms have better convergence property.

Without loss of generality, we can assume that the supports of $\alpha_i's$ lie in a small neighborhood of the small diagonal $\Delta=\{(z,\cdots,z):z\in\mathbb{H}\}$ of $\mathbb{H}\times\cdots\times\mathbb{H}$. Otherwise we can choose a cut-off function supported around Δ and split the integral into parts of the desired form. We consider the following change of coordinates: let $w_0=(u_0,v_0)=(x_1,y_1)\in\mathbb{H}$ and let $w_k=(u_k,v_k)\in\mathbb{R}^2$ be the Riemann normal coordinate of the point (x_{k+1},y_{k+1}) with center (x_k,y_k) for $1\leqslant k\leqslant n-1$ such that on $\Delta_k:=\{(z_1,\cdots,z_n)\in\mathbb{H}\times\cdots\times\mathbb{H}:z_k=z_{k+1}\}$, we have

$$\frac{\partial(x_{k+1}-x_k)}{\partial u_k}\Big|_{\Delta_k}=\frac{\partial(y_{k+1}-y_k)}{\partial v_k}\Big|_{\Delta_k}=\frac{1}{y_k}, \\ \frac{\partial(x_{k+1}-x_k)}{\partial v_k}\Big|_{\Delta_k}=\frac{\partial(y_{k+1}-y_k)}{\partial u_k}\Big|_{\Delta_k}=0. \tag{3.11}$$

By the definition of Riemann normal coordinates, the geodesic distance between z_k and z_{k+1} is $\rho(z_k,z_{k+1})=(u_k^2+v_k^2)^{\frac{1}{2}}=||w_k||$ when they are close. It is obvious that there are smooth positive functions $\{\phi_k,\psi_k,1\leqslant k\leqslant n\}$ on $\mathbb{H}\times\mathbb{R}^{2n-2}$ such that

$$|x_{k+1}-x_k|\leqslant\phi_k\cdot||w_k||,\quad |y_{k+1}-y_k|\leqslant\psi_k\cdot||w_k||,\quad \text{for}\ \ 1\leqslant k\leqslant n-1 \\ |x_n-x_1|\leqslant\phi_n\cdot(\sum_{k=1}^{n-1}||w_k||),\quad |y_n-y_1|\leqslant\psi_n\cdot(\sum_{k=1}^{n-1}||w_k||). \tag{3.12}$$

With the above preparation, we are now ready to show the convergence of the integral of the leading term in (3.9). After plugging equation (3.10) into (3.9) and using the estimate (3.12), it is not difficult to see that there is a smooth positive function Φ on $\mathbb{H}\times\mathbb{R}^{2n-2}$, such that the integral (3.9) with $\sigma=\mathrm{id}$ is bounded above in absolute value by:

$$\int_{\epsilon\leqslant t_1\leqslant\cdots\leqslant t_n\leqslant L}\prod_{i=1}^n dt_i\int_{w_0\in\mathbb{H}}\int_{w_1\cdots,w_{n-1}\in\mathbb{R}^2}\Phi(w_0,\cdots,w_n)\cdot \\ \left(\prod_{i=1}^{n-1}\frac{||w_i||}{t_i^2}e^{-\frac{||w_i||^2}{4t_i}}\right)\cdot\frac{||w_1||+\cdots+||w_{n-1}||}{t_n^2}\cdot e^{-\frac{\rho^2(z_n,z_1)}{4t_n}}\prod_{i=0}^{n-1}d^2w_i$$

$$\leqslant \int_{\epsilon\leqslant t_1\leqslant\cdots\leqslant t_n\leqslant L}\prod_{i=1}^{n}dt_i\int_{w_0\in\mathbb{H}}\int_{w_1\cdots,w_{n-1}\in\mathbb{R}^2}\Phi(w_0,\cdots,w_n)\cdot$$
$$\left(\prod_{i=1}^{n-1}\frac{||w_i||}{t_i^2}e^{-\frac{||w_i||^2}{4t_i}}\right)\cdot\frac{||w_1||+\cdots+||w_{n-1}||}{t_n^2}\prod_{i=0}^{n-1}d^2w_i.$$

The inequality follows simply by dropping the term $e^{-\frac{\rho^2(z_n,z_1)}{4t_n}}$, and the function Φ arises as the product of absolute value of the following functions or differential forms:

1) the functions ϕ_k and ψ_k in (3.12);
2) the Jacobian of the change from the standard coordinates to the Riemann normal coordinates;
3) the functions Q_1, Q_2 in (3.10);
4) the inputs on the tails of the wheel.

From 4) it is clear that we can choose Φ with compact support. Thus we only need to show that the following integral is convergent:

$$\int_{\epsilon\leqslant t_1\leqslant\cdots\leqslant t_n\leqslant L}\prod_{i=1}^{n}dt_i\int_{w_1\cdots,w_{n-1}\in\mathbb{R}^2}\left(\prod_{i=1}^{n-1}\frac{||w_i||}{t_i^2}e^{-\frac{||w_i||^2}{4t_i}}\right)\cdot\frac{||w_1||+\cdots+||w_{n-1}||}{t_n^2}\prod_{i=1}^{n-1}d^2w_i. \tag{3.13}$$

We can further change the coordinates: let

$$\xi_k = w_k\cdot t_k^{-\frac{1}{2}},\quad 1\leqslant k\leqslant n-1.$$

Then (3.13) becomes

$$\int_{\epsilon\leqslant t_1\leqslant\cdots\leqslant t_n\leqslant L}\prod_{i=1}^{n-1}dt_i\int_{\xi_1\cdots,\xi_{n-1}\in\mathbb{R}^2}\left(\prod_{i=1}^{n-1}\frac{||\xi_i||}{t_i^{1/2}}e^{-\frac{||\xi_i||^2}{4}}\right)\cdot\frac{||\xi_1||t_1^{1/2}+\cdots+||\xi_{n-1}||t_{n-1}^{1/2}}{t_n^2}\prod_{i=1}^{n-1}d^2\xi_i,$$

which is bounded above by

$$\left(\int_{\epsilon\leqslant t_1\leqslant\cdots\leqslant t_n\leqslant L}\left(\prod_{i=1}^{n-1}t_i^{-\frac{1}{2}}\right)t_n^{-\frac{3}{2}}\prod_{i=1}^{n}dt_i\right)\cdot\left(\int_{\xi_1,\cdots,\xi_{n-1}\in\mathbb{R}^2}P(||\xi_i||)e^{-\sum_{i=1}^{n-1}||\xi_i||^2/4}\right),$$

where $P(||\xi_i||)$ is a polynomial of $||\xi_i||$'s. It is not difficult to see that the first integral converges as $\epsilon\to 0$ when $n\geqslant 3$, and that the second integral is finite.

(3) $n = 2$: Plugging the leading term of equation (3.10) into the integral (3.8) for $n = 2$, we can see that the integral of the leading term is of the following form:

(3.14)

$$\frac{1}{t_1^2 t_2^2} \int_{t_1, t_2 = \epsilon}^{L} dt_1 dt_2 \int_{(u_0, v_0) \in \mathbb{H}} \int_{(u_1, v_1) \in \mathbb{R}^2} \Phi(u_0, v_0, u_1, v_1) \\ \cdot (x_1 - x_2)(y_1 - y_2) \exp\left(-(u_1^2 + v_1^2)\left(\frac{1}{t_1} + \frac{1}{t_2}\right)\right) du_0 dv_0 du_1 dv_1,$$

where Φ is similar to that in the case where $n \geq 3$. The fact that the functions $(x_1 - x_2)^2$ and $(y_1 - y_2)^2$ do not show up in equation (3.14) follows from the trivial observation that $(dy_1 - dy_2)^2 = (\frac{dx_1}{y_1} - \frac{dx_2}{y_2})^2 = 0$. This simple fact, together with the derivatives of $x_1 - x_2$ and $y_1 - y_2$ in equation (3.11), implies that the leading term in Wick's expansion of the integral

$$\frac{1}{t_1^2 t_2^2} \int_{(u_1, v_1) \in \mathbb{R}^2} \Phi(u_0, v_0, u_1, v_1)(x_1 - x_2)(y_1 - y_2) \\ \cdot \exp\left(-(u_1^2 + v_1^2)\left(\frac{1}{t_1} + \frac{1}{t_2}\right)\right) du_1 dv_1$$

is given by a multiple of

$$\frac{1}{t_1^2 t_2^2} \int_{(u_1, v_1) \in \mathbb{R}^2} u_1^2 v_1^2 \exp\left(-(u_1^2 + v_1^2)\left(\frac{1}{t_1} + \frac{1}{t_2}\right)\right) du_1 dv_1 \propto \frac{t_1 t_2}{(t_1 + t_2)^3}.$$

The integral of $\dfrac{t_1 t_2}{(t_1 + t_2)^3}$ on $[\epsilon, L] \times [\epsilon, L]$ clearly converges as $\epsilon \to 0$. Furthermore, since Φ has compact support on $\mathbb{H} \times \mathbb{R}^2$, it is clear that the integral (3.14) converges as $\epsilon \to 0$. q.e.d.

3.3.2. Obstruction analysis. By construction, the naive quantization $\{I_{naive}[L] | L \in \mathbb{R}_+\}$ satisfies all requirements of a quantization except for the quantum master equation. In general, there exist potential obstructions to solving the quantum master equation known as the anomaly. The analysis of such obstructions is usually very difficult. In [5], Costello has developed a convenient deformation theory to deal with this problem, which we will follow to compute the obstruction space of the B-twisted σ-model.

Recall that $Ob = \left(\mathcal{O}_{loc}^+(\mathcal{E}), Q + \{I_{cl}, -\}\right)$ is the deformation-obstruction complex of our theory. Costello's deformation method says that

$$H^1(Ob)$$

is the obstruction space for solving the quantum master equation, and

$$H^0(Ob)$$

parametrizes the deformation space. Both cohomology groups can be computed via D-module techniques. In our case, we can restrict to a subcomplex of Ob, thanks to the $\mathbb{C}^\times$-symmetry.

Definition 3.25. We define $\tilde{\mathcal{E}} \subset \mathcal{E}$ to be the subspace

$$\tilde{\mathcal{E}} := \mathcal{A}_{\Sigma_g} \otimes \mathfrak{g}_X[1]$$

and $\widetilde{Ob}$ to be the reduced deformation-obstruction complex

$$\widetilde{Ob} := \left(\mathcal{O}^+_{loc}\left(\tilde{\mathcal{E}}\right), Q + \{I_{cl}, -\}\right).$$

Proposition 3.26. *The obstruction space for solving the quantum master equation with prescribed $\mathbb{C}^\times$-symmetry is $H^1\left(\widetilde{Ob}\right)$.*

Proof. This is the same as the holomorphic Chern-Simons theory in [6]. q.e.d.

To describe the complex $\widetilde{Ob}$, we first introduce some notations. Let

$$\mathrm{Jet}_{\Sigma_g}\left(\tilde{\mathcal{E}}\right) := \mathrm{Jet}_{\Sigma_g}\left(\mathcal{A}_{\Sigma_g}\right) \otimes \mathfrak{g}_X[1]$$

be the sheaf of smooth jets of differential forms on Σ_g valued in $\mathfrak{g}_X[1]$, and let D_{Σ_g} be the sheaf of smooth differential operators on Σ_g. $\mathrm{Jet}_{\Sigma_g}\left(\tilde{\mathcal{E}}\right)$ is naturally a D_{Σ_g}-module, and we define its dual

$$\mathrm{Jet}_{\Sigma_g}\left(\tilde{\mathcal{E}}\right)^\vee := \mathrm{Hom}_{C^\infty(\Sigma_g)\otimes\mathcal{A}^\sharp_X}\left(\mathrm{Jet}_{\Sigma_g}\left(\tilde{\mathcal{E}}\right), C^\infty(\Sigma_g) \otimes \mathcal{A}^\sharp_X\right).$$

Equivalently,

$$\mathrm{Jet}_{\Sigma_g}\left(\tilde{\mathcal{E}}\right)^\vee = \mathrm{Jet}_{\Sigma_g}\left(\mathcal{A}_{\Sigma_g}\right)^\vee \otimes \mathfrak{g}_X[1]^\vee,$$

where $\mathrm{Jet}_{\Sigma_g}\left(\mathcal{A}_{\Sigma_g}\right)^\vee$ is the complex of the dual D_{Σ_g}-module of $\mathrm{Jet}_{\Sigma_g}\left(\mathcal{A}_{\Sigma_g}\right)$, with an induced differential which we still denote by d_{Σ_g}. There is a natural identification between complexes of D_{Σ_g}-modules

$$\mathrm{Jet}_{\Sigma_g}\left(\mathcal{A}_{\Sigma_g}\right)^\vee \cong D_{\Sigma_g} \otimes \wedge^* T_{\Sigma_g},$$

where T_{Σ_g} is the smooth tangent bundle, and the right hand side is the usual complex of Spencer's resolution. In particular, we have the quasi-isomorphism

$$(\mathrm{Jet}_{\Sigma_g}\left(\mathcal{A}_{\Sigma_g}\right)^\vee, d_{\Sigma_g}) \simeq C^\infty(\Sigma_g). \tag{3.15}$$

$\mathrm{Jet}_{\Sigma_g}\left(\tilde{\mathcal{E}}\right)^\vee$ is a locally free D_{Σ_g}-module. We will let $\mathcal{A}^{top}_{\Sigma_g}$ denote the right D_{Σ_g}-module of top differential forms on Σ_g. According to the definition of local functionals, $\mathcal{O}^+_{loc}(\tilde{\mathcal{E}})$ is isomorphic to the global sections of the following complex of sheaves on Σ_g:

$$\mathcal{A}^{top}_{\Sigma_g} \otimes_{D_{\Sigma_g}} \prod_{k\geq 1} \mathrm{Sym}^k_{D_{\Sigma_g}\otimes\mathcal{A}^\sharp_X}\left(\mathrm{Jet}_{\Sigma_g}\left(\tilde{\mathcal{E}}\right)^\vee\right) \tag{3.16}$$

with the differential induced from $Q + \{I_{cl}, -\}$. All the sheaves here, including the sheaf of jets, are sheaves over smooth functions on Σ_g. Thus these sheaves are all fine sheaves, a fact which implies that the cohomology we want to compute is nothing but the hypercohomology of the complex (3.16) with respect to the global section functor.

Proposition 3.27. *The cohomology of the deformation-obstruction complex of the B-twisted σ-model is*

$$H^k(\widetilde{Ob}) = \sum_{p+q=k+2} H^p_{dR}(\Sigma_g) \otimes H^q(X, \Omega^1_{cl}),$$

where Ω^1_{cl} is the sheaf of closed holomorphic 1-forms on X. In particular, the obstruction space for the quantization at the one-loop is given by

$$\begin{aligned} H^1(\widetilde{Ob}) = &\left(H^0_{dR}(\Sigma_g) \otimes H^3(X, \Omega^1_{cl})\right) \oplus \left(H^1_{dR}(\Sigma_g) \otimes H^2(X, \Omega^1_{cl})\right) \\ &\oplus \left(H^2_{dR}(\Sigma_g) \otimes H^1(X, \Omega^1_{cl})\right). \end{aligned}$$

Proof. We follow the strategy developed in [6]. The Koszul resolution gives a resolution of the D_{Σ_g}-module

$$\mathcal{A}_{\Sigma_g}(D_{\Sigma_g})[2] \to \mathcal{A}^{top}_{\Sigma_g},$$

where $\mathcal{A}_{\Sigma_g}\left(D_{\Sigma_g}\right)[2]$ is the de Rham complex of D_{Σ_g}. Together with the quasi-isomorphism (3.15) and the fact that the D_{Σ_g}-module

$$\prod_{k\geq 1} \mathrm{Sym}^k_{D_{\Sigma_g}\otimes \mathcal{A}^\sharp_X}\left(\mathrm{Jet}_{\Sigma_g}\left(\tilde{\mathcal{E}}\right)^\vee\right)$$

is flat, we find quasi-isomorphisms

$$\begin{aligned} \widetilde{Ob} &\cong \mathcal{A}^{top}_{\Sigma_g} \otimes^L_{D_{\Sigma_g}} \left(\prod_{k\geq 1} \mathrm{Sym}^k_{D_{\Sigma_g}\otimes \mathcal{A}^\sharp_X}\left(\mathrm{Jet}_{\Sigma_g}\left(\tilde{\mathcal{E}}\right)^\vee\right)\right) \\ &\cong \mathcal{A}_{\Sigma_g}(D_{\Sigma_g}) \otimes^L_{D_{\Sigma_g}} \left(\prod_{k\geq 1} \mathrm{Sym}^k_{D_{\Sigma_g}\otimes \mathcal{A}^\sharp_X}\left(\mathrm{Jet}_{\Sigma_g}\left(\tilde{\mathcal{E}}\right)^\vee\right)\right)[2] \\ &\cong \mathcal{A}_{\Sigma_g} \otimes_{\mathbb{C}} \left(\prod_{k\geq 1} \mathrm{Sym}^k_{\mathcal{A}^\sharp_X}\left(\mathfrak{g}_X[1]^\vee\right)\right)[2] = \mathcal{A}_{\Sigma_g} \otimes_{\mathbb{C}} C^*_{red}\left(\mathfrak{g}_X\right)[2]. \end{aligned}$$

The differential on the last complex is $d_{\Sigma_g} + l_1 + \{I_{cl}, -\} = d_{\Sigma_g} + d_{CE}$, where d_{Σ_g} is the de Rham differential on $\mathcal{A}_{\Sigma_g}$ and d_{CE} is the Chevalley-Eilenberg differential on the reduced Chevalley-Eilenberg complex of $\mathfrak{g}_X$. Therefore

$$H^k\left(\widetilde{Ob}\right) = \sum_{p+q=k+2} H^p_{dR}\left(\Sigma_g\right) \otimes H^q\left(C^*_{red}(\mathfrak{g}_X), d_{CE}\right).$$

Finally, from the following short exact sequence

$$0 \to \mathcal{A}_X \to C^*(\mathfrak{g}_X) \to C^*_{red}(\mathfrak{g}_X) \to 0,$$

we have the quasi-isomorphism of complexes of sheaves

$$C^*_{red}(\mathfrak{g}_X) \simeq [\mathcal{A}_X \to C^*(\mathfrak{g}_X)] \simeq [\mathbb{C} \to \mathcal{O}_X] \simeq \Omega^1_{X,cl},$$

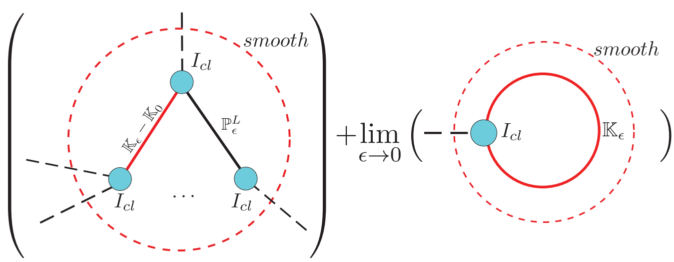
which implies

$$H^q(C^*_{red}(\mathfrak{g}_X), d_{CE}) \cong H^q\left(X, \Omega^1_{X,cl}\right).$$

q.e.d.

3.3.3. Computation of the obstruction. We now compute the obstruction to the quantization of the B-twisted σ-model. In section 3.3.1, we have seen that the $\mathbb{C}^\times$-invariance of $I_{cl}/\hbar$ and the RG flow operator guarantees that the naive quantization $\{I_{naive}[L] | L > 0\}$ only contains the constant term and linear term in the power expansion of $\hbar$. The naive quantization automatically satisfies the quantum master equation modulo $\hbar$ since I_{cl} satisfies the classical master equation. Thus, we only need to take care of the one-loop anomaly. We have the following explicit graphical expression of the one-loop anomaly for general perturbative QFTs:

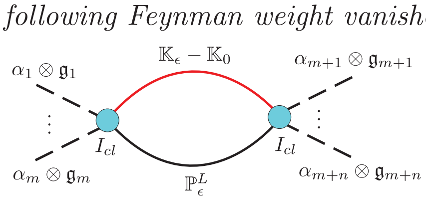
Theorem 3.28. *The one-loop obstruction O_1 to quantizing a classical field theory with classical interaction I_{cl} is given graphically by*

(3.17) $O_1 = \lim_{\epsilon \to 0}$ 

Remark 3.29. After fixing a renormalization scheme, we can define the smooth part of a Feynman weight $W_\gamma(P^L_\epsilon, I_{cl})$ for any graph γ. We take the smooth part of the term in the dashed red circle.

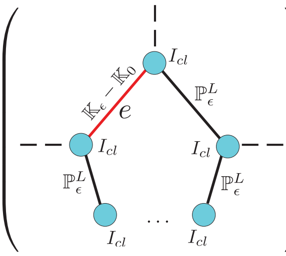
The proof of this theorem is given in Appendix C. For the B-twisted σ-model, the following two lemmas imply that the first term in (3.17) vanishes as $\epsilon \to 0$. We defer the proof of these two lemmas to Appendix B.

Lemma 3.30. *Let γ be a genus 1 graph containing a wheel with 2 vertices. Then the following Feynman weight vanishes:*

(3.18) 

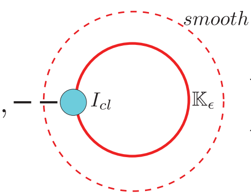
Lemma 3.31. *Let γ be a genus 1 graph containing a wheel with n vertices, and let e be an edge of γ which is part of the wheel. Assume that $n \geq 3$; then we have*

$$\lim_{\epsilon\to 0} W_{\gamma,e}(\mathbb{P}_\epsilon^L, \mathbb{K}_\epsilon - \mathbb{K}_0, I_{cl}) = \lim_{\epsilon \to 0} \left(\text{[diagram]} \right) = 0.$$



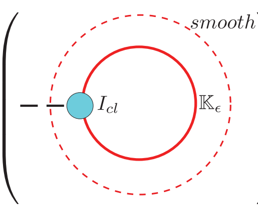
Hence the scale L one-loop obstruction is given by

$$O_1[L] = \sum_{\gamma:tree} \lim_{\epsilon\to 0} \frac{1}{|\mathrm{Aut}(\gamma)|} W_\gamma\Big(\mathbb{P}_\epsilon^L, \text{[diagram]}\Big). \tag{3.19}$$



By the fact that $\lim_{L\to 0}(I_{naive}^{(0)}[L]) = I_{cl}$, we have:

$$O_1 = \lim_{L\to 0} O_1[L] = \lim_{\epsilon\to 0} \left(\text{[diagram]} \right). \tag{3.20}$$



The obstruction O_1 contains an analytic part and a combinatorial part. It is clear that the analytic part is given by the limit of the super trace of the heat kernel along the diagonal in $\Sigma_g \times \Sigma_g$:

$$\lim_{\epsilon\to 0} Str(K_\epsilon(z,z)) = (2-2g)\mathrm{dvol}_{\Sigma_g},$$

where dvol_{Σ_g} is the normalized volume form on Σ_g with respect to the constant curvature metric, and the identity follows from the local index theorem. Similar to the holomorphic Chern-Simons theory [6], the combinatorial factor of equation (3.20) gives the first Chern class of the target manifold X. Thus, we can conclude this section by:

Theorem 3.32. *The obstruction to quantizing the B-twisted σ-model is given by*

$$[(2-2g)\,dvol_{\Sigma_g}] \otimes c_1(X) = c_1(\Sigma_g) \otimes c_1(X) \in H^2_{dR}(\Sigma_g) \otimes H^1(X, \Omega^1_{cl}) \subset H^1(\widetilde{Ob}),$$

and the topological B-twisted σ-model can be quantized (on any Riemann surface Σ_g) if and only if the target X is Calabi-Yau.

3.3.4. One-loop quantum correction. Now let us assume that X is a Calabi-Yau manifold with a holomorphic volume form Ω_X. By Theorem 3.32, the quantization of our topological B-twisted σ-model is unobstructed. This means that there exists some quantum correction $I_{qc}[L]$ to the naive quantization $I_{naive}[L]$ such that $I_{naive}[L] + \hbar I_{qc}[L]$ solves the quantum master equation. In this section we give an explicit description of the one-loop quantum correction which will be used in the next section to compute the quantum correlation functions.

We first have the following lemma:

Lemma 3.33. *Let $I_{qc} \in \mathcal{O}_{loc}(\mathcal{E})$ be a local functional on $\mathcal{E}$ satisfying the equation*

$$QI_{qc} + \{I_{cl}, I_{qc}\} = O_1, \tag{3.21}$$

where O_1 is the one-loop anomaly described in section 3.3.3. *Then the effective functionals*

$$I_{qc}[L] := \lim_{\epsilon\to 0} \sum_{\gamma\in trees, v\in V(\gamma)} W_{\gamma,v}(P_\epsilon^L, I_{cl}, I_{qc})$$

satisfy the equation

$$QI_{qc}[L] + \{I_{naive}^{(0)}[L], I_{qc}[L]\}_L = O_1[L],$$

where $W_{\gamma,v}(P_\epsilon^L, I_{cl}, I_{qc})$ is the Feynman weight associated to the graph γ with the vertex v labeled by I_{qc} and all other vertices labeled by I_{cl}. In particular, $I_{naive}[L] + \hbar I_{qc}[L]$ solves the quantum master equation.

Proof. The proof of the lemma is a simple Feynman graph calculation. See [5]. q.e.d.

The objective is to find a local functional I_{qc} satisfying equation (3.21). Let Δ be the operator on $\mathrm{Sym}^*(\mathfrak{g}_X) \otimes \mathrm{Sym}^*(\mathfrak{g}_X[1]^\vee)$ given by contraction with the identity in $\mathrm{End}_{\mathcal{A}_X}(\mathfrak{g}_X \oplus \mathfrak{g}_X^\vee)$, and let L denote the functional on $\mathfrak{g}_X[1] \oplus \mathfrak{g}_X^\vee$ given by

$$L(\alpha+\beta) := \frac{1}{(n+1)!}\sum_{n\geqslant 0}\langle l_n(\alpha^{\otimes n}), \beta\rangle, \quad \alpha\in\mathfrak{g}_X[1], \beta\in\mathfrak{g}_X^\vee.$$

From the graphical expression of O_1 in equation (3.20), it is not difficult to see that O_1 is only a functional on $C^\infty(\Sigma_g)\otimes\mathfrak{g}_X[1]$ of the following form:

$$\begin{aligned}(O_1)_k&((f_1\otimes g_1)\otimes\cdots\otimes(f_k\otimes g_k))\\ &=(2-2g)(\Delta L)_k(g_1\otimes\cdots\otimes g_k)\int_{\Sigma_g} f_1\cdots f_k \,\mathrm{dvol}_{\Sigma_g},\end{aligned}$$

where $(O_1)_k$ denotes the k-component of O_1 in $\mathcal{O}^{(k)}(\mathcal{E})$, and similarly for $(\Delta L)_k$. We are looking for an I_{qc} which is only a functional on

$C^\infty(\Sigma_g) \otimes \mathfrak{g}_X[1]$ of the form

$$
\begin{aligned}
&(I_{qc})_k((f_1 \otimes g_1) \otimes \cdots \otimes (f_k \otimes g_k)) \\
&\qquad = B_k(g_1 \otimes \cdots \otimes g_k) \int_{\Sigma_g} f_1 \cdots f_k \, \mathrm{dvol}_{\Sigma_g},
\end{aligned}
\tag{3.22}
$$

where $B_k \in \mathrm{Sym}^k(\mathfrak{g}_X[1]^\vee)$. With this ansatz, we have $QI_{qc} = l_1 I_{qc}$ by the type reason and equation (3.21) is reduced to

$$
l_1 I_{qc} + \{I_{cl}, I_{qc}\} = O_1. \tag{3.23}
$$

Letting $B = \sum_{k \geq 0} B_k$, it is clear that

$$
\begin{aligned}
&(l_1 I_{qc} + \{I_{cl}, I_{qc}\})\left((f_1 \otimes g_1) \otimes \cdots \otimes (f_k \otimes g_k)\right) \\
&\qquad = (d_{CE} B)(g_1 \otimes \cdots \otimes g_k) \int_{\Sigma_g} f_1 \cdots f_k \, \mathrm{dvol}_{\Sigma_g}.
\end{aligned}
$$

Equation (3.23) is then reduced to

$$
d_{CE} B = (2 - 2g)\Delta L
$$

which, since the Chevalley-Eilenberg differential d_{CE} is the same as the bracket $\{L, -\}$, can be further reduced to

$$
(2 - 2g)\Delta L - \{L, B\} = 0. \tag{3.24}
$$

Since we only need to solve the equation modulo constant functionals, equation (3.24) is equivalent to the vanishing of the operator $\{(2 - 2g)\Delta L - \{L, B\}, -\}$.

Lemma 3.34. *We have the following two identities for any B:*

$$
\{\{L, B\}, -\} = [\{L, -\}, \{B, -\}],
$$
$$
\{\Delta L, -\} = [\Delta, \{L, -\}].
$$

Proof. The first identity follows directly from the Jacobi identity. The second identity follows from the identity

$$
[\Delta, [\Delta, L]] = 0.
$$

q.e.d.

By Lemma 3.34, to solve equation (3.24), we only need to find $B \in C^*(\mathfrak{g}_X)$ such that the operator $(2 - 2g)\Delta + \{B, -\}$ commutes with the Chevalley-Eilenberg differential $d_{CE} = \{L, -\}$. The following technical proposition transfers the problem to a geometric context:

Proposition 3.35. [6] *There is a natural isomorphism of cochain complexes of $\mathcal{A}_X$-modules*

$$
\tilde{K} : (C^*(\mathfrak{g}_X, Sym^* \mathfrak{g}_X), d_{CE}) \xrightarrow{\sim} \left(\mathcal{A}_X \otimes_{\mathcal{O}_X} Jet_X^{hol}(\wedge^* T_X), d_{D_X}\right), \tag{3.25}
$$

where d_{CE} on the left hand side is the Chevalley-Eilenberg differential of the $\mathfrak{g}_X$-module $Sym^* \mathfrak{g}_X$, and d_{D_X} is the differential of the de Rham complex of the holomorphic jet bundle.

The explicit formula of the above isomorphism is given in Appendix D.

There is a natural second order differential operator on the right hand side of equation (3.25) which commutes with the differential d_{D_X}: let Ω_X be a holomorphic volume form on X which induces an isomorphism between holomorphic polyvector fields and holomorphic differential forms via the contraction map:

$$\begin{aligned} \wedge^* T_X &\xrightarrow{\sim} \Omega_X^* \\ \alpha &\mapsto \alpha \vdash \Omega_X. \end{aligned}$$

This isomorphism transfers the holomorphic de Rham differential ∂ on Ω_X^* to an operator on polyvector fields:

$$\partial_{\Omega_X} : \Gamma(\wedge^* T_X) \to \Gamma(\wedge^{*-1} T_X),$$

which naturally induces a second order operator (denoted by the same symbol)

$$\partial_{\Omega_X} : \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathrm{Jet}_X^{hol}(\wedge^* T_X) \to \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathrm{Jet}_X^{hol}(\wedge^{*-1} T_X),$$

that commutes with d_{D_X}.

To solve equation (3.24), we need to transfer the operator Δ to the de Rham complex of the jet bundle in equation (3.25). For simplicity, we still denote this operator by Δ.

Claim. *The two second order differential operators* Δ *and* ∂_{Ω_X} *on* $\mathcal{A}_X \otimes_{\mathcal{O}_X} Jet_X^{hol}(\wedge^* T_X)$ *have the same symbol.*

Proof. We prove the claim by some local calculation from which we can also find an explicit expression of the functional $B \in C^*(\mathfrak{g}_X)$.

Let $\{z^1, \cdots, z^n\}$ be local holomorphic coordinates on $U \subset X$ where $n = \dim_{\mathbb{C}} X$, such that the holomorphic volume form can be expressed as $\Omega_X|_U = dz^1 \wedge \cdots \wedge dz^n$, and let $\delta z^1, \cdots, \delta z^n$ be the corresponding jet coordinates. The isomorphism $\tilde{K}$ in equation (3.25) gives rise to (recall Notation 2.2):

$$\begin{aligned} \mathcal{A}_X(U)[[\delta z^i, \pi_2^*(\partial_{z^i})]] &= \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathrm{Jet}_X^{hol}(\wedge^* T_X)(U) \\ &\cong \mathcal{A}_X(U)[[\tilde{K}(\widetilde{dz^i}), \tilde{K}(\widetilde{\partial_{z^j}})]]. \end{aligned} \tag{3.26}$$

Let T denote the restriction of ρ^{*-1} in equation (2.1) to Ω_X^1:

$$T : \Omega_X^1 \to C^\infty(X) \otimes_{\mathcal{O}_X} \mathrm{Jet}_X^{hol}(\mathcal{O}_X).$$

Let ∂_{dR} be the internal de Rham differential

$$\partial_{dR} : \mathrm{Jet}_X^{hol}(\Omega_X^*) \to \mathrm{Jet}_X^{hol}(\Omega_X^{*+1}),$$

and let $\partial_{dR} \circ T$ be the composition

$$\partial_{dR} \circ T : \Omega_X^1 \to C^\infty(X) \otimes_{\mathcal{O}_X} \mathrm{Jet}_X^{hol}(\Omega_X^1). \tag{3.27}$$

Let

$$\langle -,-\rangle : \Big(C^\infty(X)\otimes_{\mathcal{O}_X}\mathrm{Jet}_X^{hol}(T_X)\Big)\otimes_{C^\infty(X)}\Big(C^\infty(X)\otimes_{\mathcal{O}_X}\mathrm{Jet}_X^{hol}(\Omega_X^1)\Big) \to C^\infty(X)\otimes_{\mathcal{O}_X}\mathrm{Jet}_X^{hol}(\mathcal{O}_X)$$

be the natural pairing induced from that between T_X and Ω_X^1. By our convention, $T(dz^i)=\tilde{T}(\widetilde{dz^i})$, and

$$\Big\langle \partial_{dR}\circ T(dz^i), \tilde{K}(\widetilde{\partial_{z^j}})\Big\rangle = \delta_j^i, \quad \big\langle \partial_{dR}(\delta z^i), \pi_2^*(\partial_{z^j})\big\rangle = \delta_j^i. \tag{3.28}$$

By construction, there exists an invertible $P\in C^\infty(X)\otimes_{\mathcal{O}_X}\mathrm{Jet}_X^{hol}(\mathcal{O}_X)(U)$ such that

$$\begin{aligned}&\pi_2^*(dz^1\wedge\cdots\wedge dz^n)\\ =&P\cdot\big((\partial_{dR}\circ T)(dz^1)\wedge\cdots\wedge(\partial_{dR}\circ T)(dz^n)\big)\in \mathrm{Jet}_X^{hol}(\Omega_X^*)(U).\end{aligned}\tag{3.29}$$

Under the identification (3.26),

$$\Delta=\sum_i \frac{\partial}{\partial(\tilde{T}(\widetilde{dz^i})}\frac{\partial}{\partial(\tilde{K}(\widetilde{\partial_{z^i}}))}, \quad \partial_{\Omega_X}=\sum_i\frac{\partial}{\partial(\delta z^i)}\frac{\partial}{\partial(\pi_2^*(\partial_{z^i}))}.$$

By (3.28), (3.29), it is not difficult to see that

$$\partial_{\Omega_X}=\Delta+\sum_i\big\langle\partial_{dR}\circ T(dz^i),\log P\big\rangle\frac{\partial}{\partial(\tilde{K}(\widetilde{\partial_{z^i}}))}=\Delta+\{\log P,-\}.\tag{3.30}$$

This proves the claim. q.e.d.

We conclude this section with the following theorems:

Theorem 3.36. *Any pair (X,Ω_X) leads to a canonical quantization of the topological B-twisted σ-model, whose one-loop quantum correction, which will be denoted by I_{qc}, is of the form in equation (*3.22*).*

The theorem follows from the following explicit description of B in equation (3.22). By taking the top wedge product of $\partial_{dR}\circ T$, we define

$$\wedge^n(\partial_{dR}\circ T):\Omega_X^n\to C^\infty(X)\otimes_{\mathcal{O}_X}\mathrm{Jet}_X^{hol}(\Omega_X^n).$$

Proposition 3.37. *The quantum correction associated to the canonical quantization of the pair (X,Ω_X) has the combinatorial part*

$$B=(2-2g)\log\left(\frac{\pi_2^*(\Omega_X)}{\wedge^n(\partial_{dR}\circ T)(\Omega_X)}\right)\in C^\infty(X)\otimes_{\mathcal{O}_X}\mathrm{Jet}_X^{hol}(\mathcal{O}_X)\subset C^*(\mathfrak{g}_X),$$

where π_2 is the same as in Definition 2.3.

Proof. This follows from equation (3.24) and the local calculation (3.30). q.e.d.

REMARK 3.38. The existence of the quantum correction is due to the fact that the curved L_∞ structure on $\mathfrak{g}_X$ requires the choice of a splitting [6], although different choices lead to homotopic equivalent theories. The quantum correction I_{qc} precisely compensates such a choice and links the effective Batalin-Vilkovisky geometry to the canonical Batalin-Vilkovisky structure of polyvector fields associated to the Calabi-Yau structure.

With the one-loop quantum correction term I_{qc}, we can give an explicit formula of the constant term $R \in \mathcal{A}_X$ in quantum master equation (3.4), which will be used later in observable theory:

Lemma 3.39. *Let $(I_{qc})_1$ denote the linear term in the one-loop correction I_{qc}, and let $\tilde{l}_0$ denote the functional on $\mathcal{E}$ given by*

$$\tilde{l}_0(\alpha+\beta) = \langle l_0, \beta\rangle.$$

Then the constant term R is given by:

$$R = \{(I_{qc})_1, \tilde{l}_0\}.$$

Proof. Let $I[L] = I^{(0)}[L] + \hbar I^{(1)}[L]$ be the scale L effective interaction. Then the quantum master equation (3.4) can be expanded as

$$Q_L I[L] + \frac{1}{2}\{I^{(0)}[L] + \hbar I^{(1)}[L], I^{(0)}[L] + \hbar I^{(1)}[L]\}_L + \hbar\Delta_L I[L] + \hbar R + F_{l_1} = 0. \tag{3.31}$$

It is clear by the type reason that the constant term in equation (3.31) other than $\hbar R$ can only live in the bracket $\{I^{(0)}[L], \hbar I^{(1)}[L]\}_L$. Thus we only need to find the linear terms in both $I^{(0)}[L]$ and $I^{(1)}[L]$. On one hand, it is obvious that the only linear term in $I^{(0)}[L]$ is $\tilde{l}_0$ since $\tilde{l}_0$ does not propagate by the type reason. Therefore the only linear term in $I^{(1)}[L]$ that contributes $\{I^{(1)}[L], \tilde{l}_0\}_L$ is $(I_{qc})_1$. It follows that

$$R = \{(I_{qc})_1, \tilde{l}_0\}_L = \{(I_{qc})_1, \tilde{l}_0\}$$

since R does not depend on L. q.e.d.

4. Observable theory

The objective of this section is to study the quantum observables of the B-twisted topological σ-model following the general theory developed by Costello and Gwilliam [8]. In section 4.1, we show that classical and quantum local observables are given by the cohomology of polyvector fields. In section 4.2, we study global topological quantum observables on Riemann surfaces of any genus g. Using the local to global factorization map, we define the topological correlation functions of quantum observables. In section 4.3, we show that the correlation functions on $\mathbb{P}^1$ are given by the trace map on the Calabi-Yau manifold, and the partition function on the elliptic curve reproduces the Euler characteristic of the target manifold. This is in complete agreement with the physics prediction.

4.1. Classical observables. We first recall that classical observables are given by the derived critical locus of the classical action functional [8].

Definition 4.1. The classical observable of the B-twisted σ-model is the graded commutative factorization algebra on Σ_g whose value on an open subset $U \subset \Sigma_g$ is the cochain complex

$$\mathrm{Obs}^{cl}(U) := \left(\mathcal{O}(\mathcal{E}_U), Q + \{I_{cl}, -\}\right). \tag{4.1}$$

Here I_{cl} is the classical interaction functional and $\mathcal{E}_U = \mathcal{A}_{\Sigma_g}(U) \otimes (\mathfrak{g}_X[1] \oplus \mathfrak{g}_X^\vee)$.

By definition,

$$\mathcal{O}(\mathcal{E}_U) = \widehat{\mathrm{Sym}}\left(\mathcal{E}_U^\vee\right) = \prod_{k\geq 0} \mathrm{Sym}^k\left(\mathcal{E}_U^\vee\right).$$

With the help of the symplectic pairing, we have the following identification:

$$\mathcal{E}_U^\vee \cong \overline{\mathcal{A}}_c(U)[2] \otimes (\mathfrak{g}_X^\vee[-1] \oplus \mathfrak{g}_X),$$

where $\overline{\mathcal{A}}_c(U)$ is the space of compactly supported distribution-valued differential forms on U. Thus we have

$$\begin{aligned} \mathrm{Sym}^n(\mathcal{E}_U^\vee) &= \mathrm{Sym}^n\left(\left(\mathcal{A}(U) \otimes \left(\mathfrak{g}_X[1] \oplus \mathfrak{g}_X^\vee\right)\right)^\vee\right) \\ &\cong \mathrm{Sym}^n\left(\overline{\mathcal{A}}_c(U)[2] \otimes \left(\mathfrak{g}_X^\vee[-1] \oplus \mathfrak{g}_X\right)\right). \end{aligned}$$

We would like to consider local observables in a small disk on Σ_g and define their correlation functions. This can be viewed as the mirror consideration of observables associated to marked points in Gromov-Witten theory. At the classical level, we have

Proposition 4.2. *Let $U \subset \Sigma_g$ be a disk. The cohomology of classical local observables of the B-twisted topological σ-model on U is given by the cohomology of polyvector fields:*

$$H^k(Obs^{cl}(U)) \cong \bigoplus_{p+q=k} H^p(X, \wedge^q T_X).$$

Proof. Recall that $\mathrm{Obs}^{cl}(U)$ is a dg-algebra over $\mathcal{A}_X$. Let $\mathcal{A}_X^k$ denote the smooth k-forms on X. We filter $\mathrm{Obs}^{cl}(U)$ by defining

$$F^k\mathrm{Obs}^{cl}(U) := \mathcal{A}_X^k \mathrm{Obs}^{cl}(U).$$

Since the operator $l_1 + \{I_{cl}, -\}$ increases the degree of differential forms on X by one while d_{Σ_g} preserves it, it is clear that the E_1-page of the spectral sequence is obtained by taking the cohomology with respect to d_{Σ_g}. By Atiyah-Bott's lemma, the chain complex of currents on U is quasi-isomorphic to the chain complex of compactly supported differential forms. Thus we have:

$$E_1 = \left(\widehat{\mathrm{Sym}}\left(H_c^2(U) \otimes \left(\mathfrak{g}_X^\vee[-1] \oplus \mathfrak{g}_X\right)\right), l_1 + \{I_{cl}, -\}\right).$$

The next lemma identifies the E_1-page of the spectral sequence with the de Rham complex of a certain jet bundle on X. It is clear that the spectral sequence degenerates at the E_2-page. Thus we have the quasi-isomorphism

$$\mathrm{Obs}^{cl}(U) \cong \left(\mathcal{A}_X \otimes_{\mathcal{O}_X} \mathrm{Jet}_X^{hol}(\wedge^* T_X), d_{D_X}\right) \cong (\mathcal{A}_X^{0,*} \otimes_{\mathcal{O}_X} \wedge^* T_X, \bar{\partial}).$$

The proposition follows by taking the cohomology of the rightmost cochain complex. q.e.d.

Lemma 4.3. *We have the following isomorphism of cochain complexes over the dga $\mathcal{A}_X$:*

$$\begin{aligned}&\left(\widehat{\mathrm{Sym}}\left(H_c^2(U) \otimes \left(\mathfrak{g}_X[1]^\vee \oplus \mathfrak{g}_X\right)\right), l_1 + \{I_{cl}, -\}\right)\\ \cong &\left(\mathcal{A}_X \otimes_{\mathcal{O}_X} Jet_X^{hol}(\wedge^* T_X), d_{D_X}\right),\end{aligned}$$

where d_{D_X} denotes the differential of the de Rham complex of $Jet_X^{hol}(\wedge^* T_X)$.

Proof. Since U is a disk in Σ_g, we have the canonical isomorphism $H_c^2(U) \cong \mathbb{C}$ induced by the integration of 2-forms. And the following isomorphism is clear:

$$\begin{aligned}&\left(\widehat{\mathrm{Sym}}\left(H_c^2(U) \otimes \left(\mathfrak{g}_X[1]^\vee \oplus \mathfrak{g}_X\right)\right), l_1 + \{I_{cl}, -\}\right)\\ &\qquad\qquad \cong (C^*(\mathfrak{g}_X, \mathrm{Sym}^* \mathfrak{g}_X), d_{CE}).\end{aligned}$$

Thus the lemma follows from Proposition 3.35. q.e.d.

4.2. Quantum observables. Quantum observables are the quantization of classical observables. Let $I[L]$ be a quantization of the classical interaction I_{cl}. The operator $Q_L + \{I[L], -\}_L + \hbar\Delta_L$ squares to zero (Lemma 3.16) and defines a quantization of the classical operator $Q + \{I_{cl}, -\}$.

Definition 4.4. The quantum observables on Σ_g at scale L are defined as the cochain complex

$$\mathrm{Obs}^q(\Sigma_g)[L] := (\mathcal{O}(\mathcal{E})[[\hbar]], Q_L + \{I[L], -\}_L + \hbar\Delta_L).$$

The definition is independent of the scale L since quantum observables at different scales are homotopic equivalent via renormalization group flow (see [5, Chapter 5, Section 9]). Therefore we will also use $\mathrm{Obs}^q(\Sigma_g)$ to denote quantum observables when the scale is not specified.

The quantum observables form a factorization algebra on Σ_g [8]. To define the quantum observables on an arbitrary open subset $U \subset \Sigma_g$, we need the concept of parametrices.

Definition 4.5. A parametrix Φ is a distributional section

$$\Phi \in \mathrm{Sym}^2\left(\overline{\mathcal{E}}\right)$$

with the following properties:

1) Φ is of cohomological degree 1 and $(Q \otimes 1 + 1 \otimes Q)\Phi = 0$;
2) $\frac{1}{2}(H \otimes 1 + 1 \otimes H)\Phi - K_0 \in \mathrm{Sym}^2(\mathcal{E})$ is smooth, where $H = [Q, Q^{GF}]$ is the Laplacian and $K_0 = \lim_{L\to 0} K_L$ is the kernel of the identity operator.

Remark 4.6. We have dropped the "proper" condition as in [8]. This is automatic here since we are working with compact Riemann surface Σ_g. We have also symmetrized $(H \otimes 1)\Phi$ used in [8].

Definition 4.7. We define the propagator $P(\Phi)$ and BV kernel K_Φ associated to a parametrix Φ by

$$P(\Phi) := \frac{1}{2}\left(Q^{GF} \otimes 1 + 1 \otimes Q^{GF}\right)\Phi \in \mathrm{Sym}^2\left(\overline{\mathcal{E}}\right),$$
$$K_\Phi := K_0 - \frac{1}{2}(H \otimes 1 + 1 \otimes H)\Phi.$$

The effective BV operator $\Delta_\Phi := \frac{\partial}{\partial K_\Phi}$ induces a BV bracket $\{-,-\}_\Phi$ on $\mathcal{O}(\mathcal{E})$ in a way similar to the scale L BV bracket $\{-,-\}_L$ is induced by Δ_L.

The following identity describes the relation between the propagator $P(\Phi)$ and BV kernel K_Φ:

$$(Q \otimes 1 + 1 \otimes Q)P(\Phi) = K_0 - K_\Phi,$$

i.e. $P(\Phi)$ gives a homotopy between the singular kernel K_0 and the regularized kernel K_Φ.

Example 4.8. $\Phi = \int_0^L \mathbb{K}_t dt$ is the parametrix we have used to define quantization. There

$$P(\Phi) = \frac{1}{2}\int_0^L \left(Q^{GF} \otimes 1 + 1 \otimes Q^{GF}\right)\mathbb{K}_t dt = \int_0^L \left(Q^{GF} \otimes 1\right)\mathbb{K}_t dt = \mathbb{P}_0^L,$$
$$K_\Phi = \mathbb{K}_L, \quad \Delta_\Phi = \Delta_L.$$

The basic reason we use an arbitrary parametrix here is that the usual renormalization group flow $W\left(\mathbb{P}_\epsilon^L, -\right)$ of observables using length scales does not preserve the property of being supported in an open subset U. Instead, there exist parametrices whose supports are arbitrarily close to the diagonal $\Delta \subset \Sigma_g \times \Sigma_g$ that we can use to achieve this.

Definition 4.9. Let $I[L]$ be a given quantization of I_{cl}, and let Φ be a parametrix. We define the effective quantization $I[\Phi]$ at the parametrix Φ by

$$I[\Phi] := W\left(P(\Phi) - \mathbb{P}_0^L, I[L]\right).$$

Note that $P(\Phi) - \mathbb{P}_0^L \in \mathrm{Sym}^2(\mathcal{E})$ is a smooth kernel since

$$\begin{aligned} &(H \otimes 1 + 1 \otimes H)(P(\Phi) - \mathbb{P}_0^L) \\ =&(Q^{GF} \otimes 1 + 1 \otimes Q^{GF})(\frac{1}{2}(H \otimes 1 + 1 \otimes H)\Phi - \mathbb{K}_0 + \mathbb{K}_L) \end{aligned}$$

is smooth and H is an elliptic operator.

$I[\Phi]$ satisfies a version of the quantum master equation described by the parametrix Φ as in [8] (with a slight modification to include F_{l_1}), and defines the corresponding cochain complex of quantum observables. We leave the details to the readers since we will not use its form for later discussions. Furthermore, different parametrices Φ, Φ' lead to homotopic equivalent cochain complexes which are linked by the renormalization group flow $W(P(\Phi) - P(\Phi'), -)$.

Definition 4.10 ([8])**.** Given a quantum observable $O[L]$ at scale L, we define its value $O[\Phi]$ at the parametrix Φ by requiring that

$$I[\Phi] + \delta O[\Phi] := W\left(P(\Phi) - \mathbb{P}_0^L, I[L] + \delta O[L]\right),$$

where δ is a square-zero parameter. The map $O[L] \mapsto O[\Phi]$ defines a homotopy between the corresponding cochain complexes of observables.

Definition 4.11. Given $O \in \mathcal{O}(\mathcal{E}) = \prod_{k,i\geq 0} \mathrm{Sym}^i\left(\mathcal{E}^\vee\right)\hbar^k$, we will let $O_i^{(k)}$ denote the corresponding component, i.e.

$$O = \sum_{k,i\geq 0} O_i^{(k)}\hbar^k.$$

Definition 4.12 ([8])**.** We say that a quantum observable $O[L]$ has support in U, if for any $k, i \geq 0$, there exists a parametrix Φ such that

$$\mathrm{Supp}\left(O[\Phi]_i^{(k)}\right) \subset U.$$

As shown in [8], the subspace of quantum observables supported in U forms a sub-cochain complex of $\mathrm{Obs}^q(\Sigma_g)$, which will be denoted by $\mathrm{Obs}^q(U)$.

4.2.1. Local quantum observable. Let U be a disk on Σ_g. As shown in [8] with great generality, the cohomology of the local quantum observables

$$H^*\left(\mathrm{Obs}^q(U)\right)$$

defines a deformation of $H^*\left(\mathrm{Obs}^{cl}(U)\right)$:

$$H^*\left(\mathrm{Obs}^q(U)\right) \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C} \cong H^*\left(\mathrm{Obs}^{cl}(U)\right). \tag{4.2}$$

We will construct a splitting map in this subsection, reflecting the vanishing of quantum corrections for observables in our B-model.

Let $\eta \in H^2_c(U)$ be a fixed generator with $\int_U \eta = 1$. By the proof of Proposition 4.2, it induces a quasi-isomorphic embedding

$$\left(\mathcal{A}_X \otimes_{\mathcal{O}_X} \mathrm{Jet}^{hol}_X(\wedge^* T_X), d_{D_X}\right) \hookrightarrow \mathrm{Obs}^{cl}(U),$$

and different choices of η are homotopic equivalent. Let $\mu \in \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathrm{Jet}^{hol}_X(\wedge^* T_X)$, and we will denote by O_μ the corresponding local classical observable. Let $\{O_\mu[L] | L > 0\}$ denote the RG flow of the classical observable O_μ. More explicitly, we define $O_\mu[L]$ by requiring that

$$I[L] + \delta O_\mu[L] = \lim_{\epsilon \to 0} W(\mathbb{P}^L_\epsilon, I_{cl} + \hbar I_{qc} + \delta O_\mu),$$

where $\delta^2 = 0$, and I_{qc} denotes the one-loop quantum correction in equation (3.21). The existence of the limit follows from Lemma/Definition 3.23 and the observation that the distribution O_μ is in fact smooth (tensor products of η's). By construction, $O_\mu[L]$ is a local quantum observable supported in U. We denote the above map by

$$\Psi : \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathrm{Jet}^{hol}_X(\wedge^* T_X) \to \mathrm{Obs}^q(U), \quad \mu \mapsto O_\mu[L].$$

Proposition 4.13. *Ψ is a cochain map.*

Proof. Let $U_L = Q_L + \hbar\Delta_L + \{I[L], -\}_L$ be the differential on quantum observables. By construction,

$$O_\mu[L] e^{I[L]/\hbar} = \lim_{\epsilon \to 0} e^{\hbar \frac{\partial}{\partial \mathbb{P}^L_\epsilon}} \left(O_\mu e^{I_{cl}/\hbar + I_{qc}}\right).$$

By Lemma 3.16,

$$\begin{aligned}
&(U_L(O_\mu[L]) + O_\mu[L] R) e^{I[L]/\hbar} = (Q_L + \hbar\Delta_L + F_{l_1}/\hbar)\left(O_\mu[L] e^{I[L]/\hbar}\right)\\
&= \lim_{\epsilon \to 0} e^{\hbar \frac{\partial}{\partial \mathbb{P}^L_\epsilon}} (Q_\epsilon + \hbar\Delta_\epsilon + F_{l_1}/\hbar)\left(O_\mu e^{I_{cl}/\hbar + I_{qc}}\right)\\
&= \lim_{\epsilon \to 0} e^{\hbar \frac{\partial}{\partial \mathbb{P}^L_\epsilon}} \Big((Q_\epsilon O_\mu + \{I_{cl}, O_\mu\}_\epsilon) e^{I_{cl}/\hbar + I_{qc}}\\
&\qquad + O_\mu (Q_\epsilon + \hbar\Delta_\epsilon + F_{l_1}/\hbar) e^{I_{cl}/\hbar + I_{qc}}\Big)
\end{aligned}$$

where we have used the fact that both O_μ and I_{qc} can only have nontrivial inputs for 0-forms on Σ_g, hence

$$\hbar\Delta_\epsilon O_\mu = \{O_\mu, I_{qc}\}_\epsilon = 0$$

by the type reason. Since the distribution O_μ is in fact smooth, we are safe to take $\epsilon \to 0$ by a similar argument as Lemma/Definition 3.23, Lemma 3.30, and Lemma 3.31. The first term above gives

$$\lim_{\epsilon \to 0} e^{\hbar \frac{\partial}{\partial \mathbb{P}^L_\epsilon}} \left((Q_\epsilon O_\mu + \{I_{cl}, O_\mu\}_\epsilon) e^{I_{cl}/\hbar + I_{qc}}\right) = e^{\hbar \frac{\partial}{\partial \mathbb{P}^L_0}} (O_{d_{D_X}\mu} e^{I_{cl}/\hbar + I_{qc}}).$$

The quantum master equation implies that the second term is

$$\lim_{\epsilon\to 0} e^{\hbar\frac{\partial}{\partial \mathbb{P}_\epsilon^L}}\left(O_\mu(Q_\epsilon+\hbar\Delta_\epsilon+F_{l_1}/\hbar)e^{I_{cl}/\hbar+I_{qc}}\right) = e^{\hbar\frac{\partial}{\partial \mathbb{P}_0^L}}(O_\mu R e^{I_{cl}/\hbar+I_{qc}})$$
$$= O_\mu[L]Re^{I[L]/\hbar}.$$

It follows that

$$U_L(O_\mu[L])e^{I[L]/\hbar} = e^{\hbar\frac{\partial}{\partial \mathbb{P}_0^L}}(O_{d_{D_X}\mu})e^{I_{cl}/\hbar+I_{qc}}$$
$$= O_{d_{D_X}\mu}[L]e^{I[L]/\hbar},$$

i.e. $U_L(\Psi(\mu)) = \Psi(d_{D_X}\mu)$ as desired. q.e.d.

Corollary 4.14. *The cohomology of local quantum observables on a disk U is given by*

$$H^*\left(Obs^q(U)\right) \cong H^*\left(X, \wedge^* T_X\right)[[\hbar]].$$

Proof. In fact, the map Ψ defines a splitting of (4.2).

q.e.d.

This says that the local observables do not receive quantum corrections via perturbative quantization, which is a very special property of the B-model.

4.2.2. Global quantum observable. Now we consider global observables on the Riemann surface Σ_g. The cochain complex of global quantum observables on Σ_g at scale L is defined as

$$\text{Obs}^q(\Sigma_g)[L] := (\mathcal{O}(\mathcal{E})[[\hbar]], Q_L + \{I[L], -\}_L + \hbar\Delta_L). \tag{4.3}$$

Since the complexes of quantum observables are homotopic equivalent for different length scales, we only need to compute the cohomology of global observables at scale $L=\infty$. By considering the d_{Σ_g}-cohomology first, the complex (4.3) at $L=\infty$ is quasi-isomorphic to the following complex:

$$\Big(\mathcal{O}\left(\mathbb{H}^*(\Sigma_g)\otimes\left(\mathfrak{g}_X[1]\oplus\mathfrak{g}_X^\vee\right)\right)[[\hbar]],$$
$$l_1 + \{I^{(0)}[\infty]|_{\mathbb{H}}, -\}_\infty + \hbar(\{I^{(1)}[\infty]|_{\mathbb{H}}, -\}_\infty + \Delta_\infty)\Big),$$

where $\mathbb{H}^*(\Sigma_g)$ denotes the space of harmonic forms on Σ_g. $I^{(0)}[\infty]|_{\mathbb{H}}$ and $I^{(1)}[\infty]|_{\mathbb{H}}$ are the restrictions of the tree-level and one-loop effective interactions to the space of harmonic fields:

$$\mathbb{H} := \mathbb{H}^*(\Sigma_g)\otimes\left(\mathfrak{g}_X[1]\oplus\mathfrak{g}_X^\vee\right).$$

Lemma 4.15. *Restricted to the harmonic fields at scale $L=\infty$, we have*

$$I^{(0)}[\infty]|_{\mathbb{H}} = I_{cl}|_{\mathbb{H}}, \quad I^{(1)}[\infty]|_{\mathbb{H}} = I_{qc}|_{\mathbb{H}} + I^{(1)}_{naive}[\infty]|_{\mathbb{H}}.$$

Proof. We only prove the first identity, and the second one can be proved similarly. Let Γ be a tree diagram with at least two vertices. We show that the Feynman weight $W_\Gamma(\mathbb{P}_0^\infty, I_{cl})$ associated to Γ vanishes when restricted to harmonic fields:

$$W_\Gamma(\mathbb{P}_0^\infty, I_{cl})|_{\mathbb{H}} = 0.$$

We choose an orientation of the internal edges of Γ such that every vertex is connected by a unique oriented path to a vertex $v_\bullet$ in Γ, where $v_\bullet$ has only one edge which is oriented toward $v_\bullet$. The vertex $v_\bullet$ will be called the root. A vertex which has only one edge oriented outward will be called a leaf. By assumption, Γ has at least one leaf which is distinct from the root.

If a leaf has only $\mathbb{H}^0(\Sigma_g)$ and $\mathbb{H}^2(\Sigma_g)$ inputs on its tails, then the propagator $\mathbb{P}_0^\infty$ attached to its edge will annihilate $W_\Gamma(\mathbb{P}_0^\infty, I_{cl})|_{\mathbb{H}}$ since wedge products of harmonic 0-forms and 2-forms are still harmonic, and

$$d^* = 0 \quad \text{on} \quad \mathbb{H}^*(\Sigma_g).$$

Similarly, if a leaf has only one input of type $\mathbb{H}^1(\Sigma_g)$, then $W_\Gamma(\mathbb{P}_0^\infty, I_{cl})|_{\mathbb{H}} = 0$. So we can assume that all leaves have at least two inputs of type $\mathbb{H}^1(\Sigma_g)$ on their tails (possibly other inputs of type $\mathbb{H}^0(\Sigma_g)$). Since $\mathbb{P}_0^\infty$ is a 1-form on $\Sigma_g \times \Sigma_g$, it is easy to see by tracing the path that the incoming edge of the root $v_\bullet$ has to contribute a 1-form to the copy of Σ_g corresponding to $v_\bullet$ which is d^*-exact, and by the type reason there is exactly one extra input of type $\mathbb{H}^1(\Sigma_g)$ on one tail of $v_\bullet$. Since

$$\int_{\Sigma_g} d^*(a) \wedge b = 0, \quad \forall a \in \mathcal{A}(\Sigma_g), b \in \mathbb{H}^*(\Sigma_g).$$

This again implies that $W_\Gamma(\mathbb{P}_0^\infty, I_{cl})|_{\mathbb{H}} = 0$. q.e.d.

For later discussions on correlation functions of observables, we also need some description of the one-loop naive interaction as in the following lemma:

Lemma 4.16. *For Riemann surfaces Σ_g of genus $g = 0$ and $g = 1$, the infinity scale one-loop naive interaction vanishes when restricted to $\mathbb{H}$:*

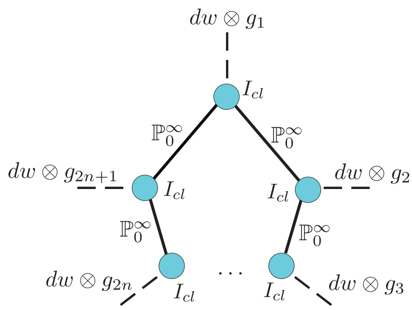
$$I^{(1)}_{naive}[\infty]|_{\mathbb{H}} = 0.$$

Proof. For both genus 0 and genus 1 Riemann surfaces with constant curvature metric, the product of harmonic forms remains harmonic. Thus by the same argument as in the proof of Lemma 4.15, if a one-loop graph γ is a wheel with nontrivial trees attached to it, then

$$W_\gamma(\mathbb{P}_0^\infty, I_{cl})|_{\mathbb{H}} = 0.$$

Hence we only need to deal with wheels. For the genus 0 Riemann surface $\mathbb{P}^1$, since there are no harmonic 1-forms, there must be at least one vertex on the wheel, attached to which all inputs are harmonic 0-forms by the type reason. The corresponding Feynman integral vanishes since the composite of two propagators $\mathbb{P}_0^\infty$ on that vertex is zero by $(d^*)^2 = 0$.

For an elliptic curve $\Sigma_1 = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, if the number of vertices on a wheel is even, then the vanishing of the associated Feynman weight can be proved by the same argument as in Lemma 3.30. For a wheel with an odd number of vertices, a $\mathbb{Z}_2$-symmetry of the analytic propagator P_0^∞ results in the vanishing of the Feynman weights: Let dw be a harmonic 1-form on Σ_1, and we assume without loss of generality that all the vertices of the wheel are trivalent, and all the inputs are of the form $dw \otimes g_i$, where $g_i \in \mathfrak{g}_X$.



Similar to [6, Lemma 17.4.4], the analytic part of the corresponding Feynman weight $W(P_0^\infty, I_{cl})(dw)$ will be a linear combination of

$$\sum_{(a,b)\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{(a\tau+b)^k(a\bar{\tau}+b)^{2n+1-k}},$$

which clearly vanishes. q.e.d.

Now let us compute the cohomology of the global quantum observables.

REMARK 4.17. In the following discussion, the harmonic forms $\mathbb{H}^k(\Sigma_g)$ sit at degree k, and $\Omega_X^1[1] \cong T_X^\vee[1]$ sits at degree -1.

Lemma 4.18. *There is a natural isomorphism of $\mathcal{A}_X$-modules*

(4.4)
$$\mathcal{O}\left(\mathbb{H}^*(\Sigma_g)\otimes\left(\mathfrak{g}_X[1]\oplus\mathfrak{g}_X^\vee\right)\right) \cong \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathrm{Jet}_X^{hol}\left(\widehat{\mathrm{Sym}}\left(T_X\otimes\mathbb{H}^1(\Sigma_g)\right)^\vee \otimes \widehat{\mathrm{Sym}}\left(T_X\otimes\mathbb{H}^2(\Sigma_g)\right)^\vee \otimes \widehat{\mathrm{Sym}}(\Omega_X^1[1]\otimes\mathbb{H}^*(\Sigma_g))^\vee\right).$$

Proof. We have the following isomorphisms:

$$\begin{aligned}&\mathcal{O}(\mathbb{H}^*(\Sigma_g)\otimes(\mathfrak{g}_X[1]\oplus\mathfrak{g}_X^\vee))\\ \cong&\mathcal{O}(\mathbb{H}^0(\Sigma_g)\otimes\mathfrak{g}_X[1])\otimes_{\mathcal{A}_X}\mathcal{O}\left(\bigoplus_{k=1}^2\mathbb{H}^k(\Sigma_g)\otimes\mathfrak{g}_X[1]\oplus\bigoplus_{k=0}^2\mathbb{H}^k(\Sigma_g)\otimes\mathfrak{g}_X^\vee\right)\end{aligned}$$

$$\cong C^*(\mathfrak{g}_X) \otimes_{\mathcal{A}_X} \widehat{\mathrm{Sym}} \left(\bigoplus_{k=1}^{2} \mathbb{H}^k(\Sigma_g) \otimes \mathfrak{g}_X[1] \oplus \bigoplus_{k=0}^{2} \mathbb{H}^k(\Sigma_g) \otimes \mathfrak{g}_X^\vee \right)^\vee .$$

It is clear that the tensor products of the isomorphisms $\tilde{T}$ and $\tilde{K}$ in Propositions D.5 and D.6 respectively give the desired isomorphism.

q.e.d.

Definition 4.19. Let $\pi_2^*\left(\Omega_X^{2g-2}\right)$ denote the canonical flat section of the jet bundle

$$\mathrm{Jet}_X^{hol}\Big(\mathrm{Sym}^{2g\cdot\dim_{\mathbb{C}} X}\left(T_X \otimes \mathbb{H}^1(\Sigma_g)\right)^\vee \otimes \mathrm{Sym}^{\dim_{\mathbb{C}} X}(\Omega_X^1[1] \otimes \mathbb{H}^0(\Sigma_g))^\vee \otimes \mathrm{Sym}^{\dim_{\mathbb{C}} X}(\Omega_X^1[1] \otimes \mathbb{H}^2(\Sigma_g))^\vee\Big)$$

induced by the holomorphic volume form Ω_X. Here we use the notation π_2^* to be consistent with Definition 2.3.

We can view $\pi_2^*\left(\Omega_X^{2g-2}\right)$ as a quantum observable via the identification in Lemma 4.18. The general philosophy in [6] says that the quantization gives rise to a projective volume form, and the next proposition says that the volume form is exactly given by $\pi_2^*\left(\Omega_X^{2g-2}\right)$.

Proposition 4.20. *The following embedding*

$$\iota : \mathcal{A}_X((\hbar)) \hookrightarrow \mathcal{O}\left(\mathbb{H}^*(\Sigma_g) \otimes \left(\mathfrak{g}_X[1] \oplus \mathfrak{g}_X^\vee\right)\right)((\hbar))$$

defined by

$$A \mapsto \iota(A) := \hbar^{-2\dim_{\mathbb{C}} X} A \otimes_{\mathcal{O}_X} \pi_2^*\left(\Omega_X^{2g-2}\right), \quad \forall A \in \mathcal{A}_X$$

is a quasi-isomorphism which is equivariant with respect to the $\mathbb{C}^\times$*-symmetry defined in section* 3.2.3.

Proof. We first show that ι respects the differential. This is equivalent to showing that $\pi_2^*\left(\Omega_X^{2g-2}\right)$ is closed under $l_1 + \left\{I^{(0)}[\infty]|_{\mathbb{H}}, -\right\}_\infty + \hbar(\{I^{(1)}[\infty]|_{\mathbb{H}}, -\}_\infty + \Delta_\infty)$. By Lemma 4.15,

$$\left(l_1 + \left\{I^{(0)}[\infty]|_{\mathbb{H}}, -\right\}_\infty\right)\left(\pi_2^*\left(\Omega_X^{2g-2}\right)\right) = d_{D_X}\left(\pi_2^*\left(\Omega_X^{2g-2}\right)\right) = 0,$$

since $\pi_2^*\left(\Omega_X^{2g-2}\right)$ is flat. Here d_{D_X} is the de Rham differential of the D_X-module.

Claim. $\left\{I_{naive}^{(1)}[\infty], \left(\pi_2^*\left(\Omega_X^{2g-2}\right)\right)\right\}_\infty = 0.$

Proof. It is straightforward to check (similar to the proof of Lemma 4.15) that for any one-loop graph γ, either the Feynman weight $W_\gamma(\mathbb{P}_0^\infty, I_{cl})$ vanishes, or the operator $\{W_\gamma(\mathbb{P}_0^\infty, I_{cl}), -\}_\infty$ applied to

$\pi_2^*\left(\Omega_X^{2g-2}\right)$will generate new terms in $(\mathbb{H}^1(\Sigma_g)\otimes\mathfrak{g}_X[1])^\vee$. These terms force the bracket $\left\{W_\gamma(\mathbb{P}_0^\infty, I_{cl}), \pi_2^*\left(\Omega_X^{2g-2}\right)\right\}_\infty$ to vanish since $\pi_2^*\left(\Omega_X^{2g-2}\right)$ already contains the highest wedge product of $(\mathbb{H}^1(\Sigma_g)\otimes\mathfrak{g}_X[1])^\vee$. q.e.d.

Hence we only need to consider the operator $\Delta_\infty+\{I_{qc},-\}_\infty$. Letting $n=\dim_{\mathbb{C}} X$, the map

$$\wedge^n\left(\partial_{dR}\circ T\right):\Omega_X^n\to C^\infty(X)\otimes_{\mathcal{O}_X}\mathrm{Jet}_X^{hol}(\Omega_X^n)$$

in Proposition 3.37 induces a natural embedding

$$\begin{aligned} T':(\Omega_X^n)^{\otimes(2g-2)}\hookrightarrow\ &\mathrm{Jet}_X^{hol}\Big(\mathrm{Sym}^{2g\cdot n}\left(T_X\otimes\mathbb{H}^1(\Sigma_g)\right)^\vee \\ &\otimes\mathrm{Sym}^n\left(\Omega_X^1[1]\otimes\mathbb{H}^0(\Sigma_g)\right)^\vee\otimes\mathrm{Sym}^n\left(\Omega_X^1[1]\otimes\mathbb{H}^2(\Sigma_g)\right)^\vee\Big). \end{aligned}$$

Let $T'\left(\Omega_X^{2g-2}\right)$ denote the image of the section $(\Omega_X)^{\otimes(2g-2)}$, where Ω_X denotes the volume form and its negative power denotes its dual. By construction,

$$\Delta_\infty T'\left(\Omega_X^{2g-2}\right)=0$$

and by Proposition 3.37, we have

$$\pi_2^*\left(\Omega_X^{2g-2}\right)=e^{-I_{qc}}T'\left(\Omega_X^{2g-2}\right).$$

It follows that

$$\begin{aligned}\Delta_\infty\pi_2^*\left(\Omega_X^{2g-2}\right)=&\Delta_\infty\left(e^{-I_{qc}}T'\left(\Omega_X^{2g-2}\right)\right)\\ =&-e^{-I_{qc}}\left\{I_{qc},T'\left(\Omega_X^{2g-2}\right)\right\}_\infty=-\left\{I_{qc},\pi_2^*\left(\Omega_X^{2g-2}\right)\right\}_\infty,\end{aligned}$$

as desired.

Now we show that ι is a quasi-isomorphism. We consider the filtration on $\mathcal{O}\left(\mathbb{H}^*(\Sigma_g)\otimes(\mathfrak{g}_X[1]\oplus\mathfrak{g}_X^\vee)\right)((\hbar))$ by the degree of the differential forms on X:

$$\begin{aligned}&F^k\mathcal{O}\left(\mathbb{H}^*(\Sigma_g)\otimes\left(\mathfrak{g}_X[1]\oplus\mathfrak{g}_X^\vee\right)\right)((\hbar))\\ &\quad:=\mathcal{A}_X^k\cdot\mathcal{O}\left(\mathbb{H}^*(\Sigma_g)\otimes\left(\mathfrak{g}_X[1]\oplus\mathfrak{g}_X^\vee\right)\right)((\hbar)).\end{aligned}$$

The differential of the graded complex is given by

$$d_1=\hbar\left(\Delta_\infty+\{I_{qc},-\}_\infty\right)=\hbar e^{-I_{qc}}\Delta_\infty e^{I_{qc}}.$$

By the Poincare lemma below, the d_1-cohomology is precisely given by $\mathrm{Im}(\iota)$. It follows that ι is a quasi-isomorphism. q.e.d.

Recall the following Poincare lemma:

Lemma 4.21. *Let $\{x^i\}$ be even elements and let $\{\xi_i\}$ be odd elements; then we have*

$$H^*\left(\mathbb{C}[[x^i,\xi_i]],\Delta=\frac{\partial}{\partial x^i}\frac{\partial}{\partial\xi_i}\right)=\mathbb{C}\xi_1\wedge\cdots\wedge\xi_n.$$

Corollary 4.22. *The top cohomology of* $Obs^q(\Sigma_g)[\hbar^{-1}]$ *is at degree* $(2-2g)\dim_{\mathbb{C}} X$*, given by*

$$H^{(2-2g)\dim_{\mathbb{C}} X}\left(Obs^q(\Sigma_g)[\hbar^{-1}]\right) \cong \mathbb{C}((\hbar)).$$

4.3. Correlation function. Proposition 4.20 implies that a quantization $I[L]$ defines an integrable projective volume form in the sense of [6], which allows us to define correlation functions for quantum observables.

Definition 4.23. Let $O \in \mathrm{Obs}^q(\Sigma_g)$ be a closed element, representing a cohomology class $[O] \in H^k\left(\mathrm{Obs}^q(\Sigma_g)\right)$. We define its correlation function (via Corollary 4.22) by

$$\langle O\rangle_{\Sigma_g} := \begin{cases} 0 & \text{if } k \neq (2-2g)\dim_{\mathbb{C}} X \\ [O] \in \mathbb{C}((\hbar)) & \text{if } k = (2-2g)\dim_{\mathbb{C}} X \end{cases}$$

Recall that the local quantum observables form a factorization algebra on Σ_g. This structure enables us to define correlation functions for local observables, for which let us first introduce some notations.

Definition 4.24. Let $U_1, \cdots, U_n$ be disjoint open subsets of Σ_g. The factorization product

$$\mathrm{Obs}^q(U_1) \times \cdots \times \mathrm{Obs}^q(U_n) \to \mathrm{Obs}^q(\Sigma_g)$$

of local observables $O_i \in \mathrm{Obs}^q(U_i)$ will be denoted by $O_1 \star \cdots \star O_n$. This product descends to cohomologies.

Definition 4.25. Let $U_1, \cdots, U_n$ be disjoint open subsets of Σ_g, and let $O_i \in \mathrm{Obs}^q(U_i)$ be closed local quantum observables supported on U_i. We define their correlation function by

$$\langle O_1, \cdots, O_n\rangle_{\Sigma_g} := \langle O_1 \star \cdots \star O_n\rangle_{\Sigma_g} \in \mathbb{C}((\hbar)).$$

We would like to compute the correlation functions for the B-twisted topological σ-model. By the degree reason, the only nontrivial cases are $g = 0, 1$. We show that they coincide with the prediction from physics.

For later computation, we give an equivalent definition of correlation functions via the BV integration point of view. Let $T^*\widehat{(T^{\Sigma_g}X)}[-1]$ denote the ringed space with underlying topological space X and structure sheaf as that of equation (4.4):

$$T^*\widehat{(T^{\Sigma_g}X)}[-1] = \left(X, \mathcal{O}\left(\mathbb{H}^*(\Sigma_g) \otimes \left(\mathfrak{g}_X[1] \oplus \mathfrak{g}_X^\vee\right)\right)\right).$$

It is clear that the intersection pairing on $\mathbb{H}^*(\Sigma_g)$, together with the canonical pairing between $\mathfrak{g}_X[1]$ and $\mathfrak{g}_X^\vee$, induces an odd symplectic structure on $T^*\widehat{(T^{\Sigma_g}X)}[-1]$. Let $\mathcal{L}$ be the ringed space with underlying topological space X and structure sheaf being generated by the odd generators over $\mathcal{A}_X$ in $\mathcal{O}\left(\mathbb{H}^*(\Sigma_g) \otimes (\mathfrak{g}_X[1] \oplus \mathfrak{g}_X^\vee)\right)$. Thus $\mathcal{L}$ can be viewed as a Lagrangian subspace of $T^*\widehat{(T^{\Sigma_g}X)}[-1]$.

It is clear from the form of the jet bundle in equation (4.4) that there is a canonical projection of functions on $T^*\widehat{(T^{\Sigma_g}X)}[-1]$ to the subspace

$$\mathcal{A}_X \otimes_{\mathcal{O}_X} \mathrm{Jet}_X^{hol}\left(\widehat{\mathrm{Sym}}\left(T_X \otimes \mathbb{H}^2(\Sigma_g)\right)^\vee \otimes \widehat{\mathrm{Sym}}\left(\Omega_X^1[1] \otimes \mathbb{H}^1(\Sigma_g)\right)^\vee\right)\pi_2^*(\Omega_X^{2g-2})$$

generated by $\pi_2^*(\Omega_X^{2g-2})$. We denote by $(f)^{TF}$ the projection of f, where TF is short for "top fermions."

Proposition 4.26. *The map*

$$\begin{aligned} i_L^* : Obs^q(\Sigma_g) &\to \mathcal{A}_X((\hbar))[(2g-2)\dim_{\mathbb{C}} X] \\ O &\mapsto \hbar^{2\dim_{\mathbb{C}} X}\left((e^{I[\infty]/\hbar}\cdot O|_{\mathbb{H}})^{TF}/\pi_2^*(\Omega_X^{2g-2})\right)\Big|_{\mathcal{L}} \end{aligned}$$

is a $\mathbb{C}^\times$-equivariant cochain map. Here the differential on the right hand side is the de Rham differential d_X, and

$$\Big|_{\mathcal{L}} : \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathrm{Jet}_X^{hol}\left(\widehat{\mathrm{Sym}}^*\left(T_X \otimes \mathbb{H}^2(\Sigma_g)\right)^\vee \otimes \widehat{\mathrm{Sym}}^*\left(\Omega_X^1[1] \otimes \mathbb{H}^1(\Sigma_g)\right)^\vee\right) \to \mathcal{A}_X$$

denotes the map which sets all the jet coordinates and that of $T_X \otimes \mathbb{H}^2(\Sigma_g), \Omega_X^1[1] \otimes \mathbb{H}^1(\Sigma_g)$ to be zero.

Proof. Let $O \in \mathrm{Obs}^q(\Sigma_g)$. Recall that from QME, we have

$$(Q_\infty O + \hbar\Delta_\infty O + \{I[\infty], O\}_\infty)\cdot e^{I[\infty]/\hbar} = (Q_\infty + \hbar\Delta_\infty + \frac{F_{l_1}}{\hbar} - R)(e^{I[\infty]/\hbar}\cdot O).$$

We have the following three simple observations:

1) $\left(\hbar\Delta_\infty(e^{I[\infty]/\hbar}O)\right)^{TF} = 0$ since Δ_∞ annihilates one odd generator.
2) $\frac{F_{l_1}}{\hbar}(e^{I[\infty]/\hbar}O)$ vanishes when restricted to the Lagrangian $\mathcal{L}$, since F_{l_1} contains non-trivial bosonic generators.
3) When restricted to $\mathbb{H}$, $Q_\infty = l_1$.

Thus, we only need to show that

$$\begin{aligned} &\left(\left((Q_\infty - R)(e^{I[\infty]/\hbar}\cdot O\big|_{\mathbb{H}})\right)^{TF}/\pi_2^*(\Omega_X^{2g-2})\right)\Big|_{\mathcal{L}} \\ &\qquad = d_X\left((e^{I[\infty]/\hbar}\cdot O\big|_{\mathbb{H}})^{TF}/\pi_2^*(\Omega_X^{2g-2})\right)\Big|_{\mathcal{L}}. \end{aligned}$$

Since l_1 commutes with $\Big|_{\mathcal{L}}$, we can assume that $(e^{I[\infty]/\hbar}O\big|_{\mathbb{H}})^{TF}$ is of the form

$$(e^{I[\infty]/\hbar}O\big|_{\mathbb{H}})^{TF} = B\cdot\pi_2^*(\Omega^{2g-2}) \overset{(1)}{=} B\cdot e^{-I_{qc}}T'(\Omega_X^{2g-2}),$$

where $B \in \mathcal{A}_X$, and identity (1) and the map T' are explained in the proof of Proposition 4.20. Then

$$Q_\infty(B\cdot\pi_2^*(\Omega_X^{2g-2})) = d_X(B)\cdot\pi_2^*(\Omega_X^{2g-2}) + B\cdot l_1(\pi_2^*(\Omega_X^{2g-2})).$$

The fact that $\pi_2^*(\Omega_X^{2g-2})$ is a flat section of the jet bundle is translated to

$$l_1(\pi_2^*(\Omega_X^{2g-2})) + \{I_{cl}, \pi_2^*(\Omega_X^{2g-2})\} = 0,$$

where $I_{cl} = \tilde{l}_0 + \sum_{k\geq 2} \tilde{l}_k$ is the classical interaction functional and $\tilde{l}_k$ is defined in equation (2.2).

The functionals $\{\tilde{l}_k, \pi_2^*(\Omega_X^{2g-2})\}$ for $k \geq 2$ vanish when restricted to the Lagrangian $\mathcal{L}$ since they contain jet coordinates. Thus

$$\begin{aligned}
&\Big(l_1(\pi_2^*(\Omega_X^{2g-2}))\Big)\Big|_{\mathcal{L}} \\
&= -\{\tilde{l}_0, \pi_2^*(\Omega_X^{2g-2})\}\Big|_{\mathcal{L}} \\
&= -\{\tilde{l}_0, e^{-I_{qc}}T'(\Omega_X^{2g-2})\}\Big|_{\mathcal{L}} \\
&= -\Big(-\{\tilde{l}_0, I_{qc}\}\cdot e^{-I_{qc}}T'(\Omega_X^{2g-2}) + e^{-I_{qc}}\cdot\{\tilde{l}_0, T'(\Omega_X^{2g-2})\}\Big)\Big|_{\mathcal{L}} \\
&\overset{(1)}{=} \Big(\{\tilde{l}_0, I_{qc}\}\cdot e^{-I_{qc}}T'(\Omega_X^{2g-2})\Big)\Big|_{\mathcal{L}} \\
&\overset{(2)}{=} R\cdot \pi_2^*(\Omega_X^{2g-2}).
\end{aligned}$$

The identity (1) follows from the fact that $\{\tilde{l}_0, T'(\Omega_X^{2g-2})\} = 0$ by the type reason, and identity (2) follows from Lemma 3.39. Thus we have

$$\begin{aligned}
&\left(\left((Q-R)(e^{I[\infty]/\hbar}\cdot O)\big|_{\mathbb{H}}\right)^{TF}/\pi_2^*(\Omega_X^{2g-2})\right)\Big|_{\mathcal{L}} \\
&= \Big((Q-R)(B\cdot\pi_2^*(\Omega^{2g-2}))/\pi_2^*(\Omega_X^{2g-2})\Big)\Big|_{\mathcal{L}} \\
&= \Big((dB + B\cdot R - B\cdot R)\cdot\pi_2^*(\Omega^{2g-2}))/\pi_2^*(\Omega_X^{2g-2})\Big)\Big|_{\mathcal{L}} \\
&= dB.
\end{aligned}$$

q.e.d.

It is clear that the cochain map i_L^* induces an isomorphism on the degree $(2-2g)\dim_{\mathbb{C}} X$ component. Thus we have the following corollary:

Corollary 4.27. *Let O be a global quantum observable which is closed; then the correlation function of O is the same as the integral of $i_L^*(O)$ on X:*

$$\langle O\rangle_{\Sigma_g} = \hbar^{2\dim_{\mathbb{C}} X}\int_X \Big((e^{I[\infty]/\hbar}\cdot O\big|_{\mathbb{H}})^{TF}/\pi_2^*(\Omega_X^{2g-2})\Big)\Big|_{\mathcal{L}}.$$

We are ready to compute the topological correlation functions on $\mathbb{P}^1$ and elliptic curves.

4.3.1. $g = 0$.

Lemma 4.28. *Let $\{U_i\}$ be a disjoint union of disks contained in a larger disk $U \subset \Sigma_g$, and let $[O_{\mu_i,U_i}] \in H^*(Obs^q(U_i))$ be the local quantum observable associated to $\mu_i \in H^*(X, \wedge^* T_X)$ on U_i. Then*

$$[O_{\mu_1,U_1} \star \cdots \star O_{\mu_m,U_m}] = [O_{\mu_1\cdots\mu_m,U}] \in H^*(Obs^q(U)).$$

Proof. For any parametrix Φ, we have

$$\begin{aligned}(O_{\mu_1,U_1} \star O_{\mu_2,U_2})[\Phi] &\overset{(1)}{=} \lim_{L\to 0} W(P(\Phi) - \mathbb{P}_0^L, I[L], O_{\mu_1,U_1}[L] \star O_{\mu_2,U_2}[L]) \\ &\overset{(2)}{=} W(P(\Phi), I_{cl} + \hbar I_{qc}, O_{\mu_1,U_1} \cdot O_{\mu_2,U_2}) \\ &= O_{\mu_1\mu_2,U}[\Phi].\end{aligned}$$

Here identity (1) is the definition of the factorization product of observables, and identity (2) follows from Proposition 4.13. q.e.d.

Theorem 4.29. *Let $\Sigma_g = \mathbb{P}^1$, and let $\{U_i\}$ be the disjoint union of disks on $\mathbb{P}^1$. Let $O_{\mu_i,U_i} \in H^*(Obs^q(U_i))$ be a local quantum observable associated to $\mu_i \in H^*(X, \wedge^* T_X)$ supported in U_i. Then*

$$\langle O_{\mu_1,U_1}, \cdots, O_{\mu_m,U_m}\rangle_{\mathbb{P}^1} = \hbar^{\dim_{\mathbb{C}} X} \int_X (\mu_1 \cdots \mu_m \vdash \Omega_X) \wedge \Omega_X.$$

Proof. We compute the correlation function at the scale $L = \infty$. By the degree reason and the previous lemma, we can assume $m = 1$, and $\mu = \mu_1 \in H^{\dim_{\mathbb{C}} X}(X, \wedge^{\dim_{\mathbb{C}} X} T_X)$. Let O_μ be the classical observable represented by μ. By Proposition 4.13, the corresponding quantum observable is described by

$$O_\mu[\infty] e^{I[\infty]/\hbar} = \lim_{L\to\infty} \lim_{\epsilon\to 0} e^{\hbar \frac{\partial}{\partial \mathbb{P}_\epsilon^L}} \left(O_\mu e^{I_{cl}/\hbar + I_{qc}} \right).$$

Since $\mathbb{H}^1(\mathbb{P}^1) = 0$, a similar argument as in Lemma 4.15 implies

$$O_\mu[\infty]\big|_{\mathbb{H}} = O_\mu$$

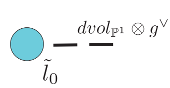
when restricted to harmonic fields. By Corollary 4.27,

$$\langle O_\mu[\infty]\rangle_{\mathbb{P}^1} = \hbar^{2\dim_{\mathbb{C}} X} \int_X \left((e^{I[\infty]/\hbar} \cdot O_\mu\big|_{\mathbb{H}})^{TF} / \pi_2^*(\Omega_X^{-2}) \right)\Big|_{\mathcal{L}}.$$

By Lemma 4.16 and Lemma 4.15,

$$(e^{I[\infty]/\hbar} \cdot O_\mu\big|_{\mathbb{H}})^{TF} = (e^{I_{cl}/\hbar + I_{qc}} \cdot O_\mu\big|_{\mathbb{H}})^{TF}.$$

By the type reason, the only terms in $e^{I_{cl}/\hbar + I_{qc}}$ that will contribute after $\big|_{\mathcal{L}}$ are products of $\tilde{l}_0$:



Let $\{z^i\}$ be holomorphic coordinates on $U \subset X$ such that locally $\Omega_X|_U = dz^1 \wedge \cdots \wedge dz^n$. Let

$$\mu = Ad\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n \otimes \partial_{z^1} \wedge \cdots \wedge \partial_{z^n},$$

where $n = \dim_{\mathbb{C}} X$. We can choose the following element in the jet bundle representing μ:

$$O_\mu = Ad\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n \otimes_{\mathcal{O}_X} \left((\pi_2^*(dz^1) \otimes 1)^\vee \wedge \cdots \wedge (\pi_2^*(dz^n) \otimes 1)^\vee \otimes \mathbf{1} \right),$$

where $\mathbf{1}$ denotes the other component in the jet bundle. On the other hand,

$$e^{\tilde{l}_0/\hbar} = \hbar^{-n} \cdot dz^1 \wedge \cdots \wedge dz^n \otimes_{\mathcal{O}_X} (T(\partial_{z^1}) \wedge \cdots \wedge T(\partial_{z^n})) \\ + \text{lower wedge products}.$$

It follows easily that

$$\left((e^{I[\infty]/\hbar} \cdot O_\mu\big|_{\mathbb{H}})^{TF} / \pi_2^*(\Omega_X^{-2}) \right) \Big|_{\mathcal{L}} = \hbar^{-n} \cdot (\mu \vdash \Omega_X) \wedge \Omega_X.$$

q.e.d.

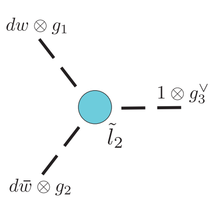
4.3.2. $g = 1$. On elliptic curves, the only nontrivial input is at cohomology degree 0.

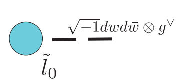
Theorem 4.30. *Let $E = \Sigma_1$ be an elliptic curve. Then $\langle 1 \rangle_E = \chi(X)$ is the Euler characteristic of X.*

Proof. By Corollary 4.27, we only need to look at the term $e^{I[\infty]/\hbar}$. For the case $g = 1$, we have shown in Lemma 4.16 that $I^{(1)}_{naive}[\infty]\big|_{\mathbb{H}}$ vanishes, and the quantum correction I_{qc} also vanishes. Hence $I[\infty] = I^{(0)}[\infty] = I_{cl}$. Let w denote the normalized holomorphic coordinate on the elliptic curve E such that

$$\sqrt{-1} \int_E dw d\bar{w} = 1.$$

It is not difficult to see that by the type reason, the only terms in $e^{I_{cl}/\hbar}|_{\mathbb{H}}$ that can survive under $\big|_{\mathcal{L}}$ are the following:

(4.5) (1) 

, (2) 

Let $\{z^i\}$ be local holomorphic coordinates on X as we chose before; then term (1) in equation (4.5) can be expressed explicitly as:

$$A^k_{ij} \otimes_{\mathcal{O}_X} ((dw)^\vee \otimes \tilde{T}(\widetilde{dz^i})) \otimes ((d\bar{w})^\vee \otimes \tilde{T}(\widetilde{dz^j})) \otimes (1^\vee \otimes \tilde{K}(\widetilde{\partial_{z^k}})). \quad (4.6)$$

And term (2) can be expressed as

$$dz^l \otimes \left((\sqrt{-1} dw d\bar{w})^\vee \otimes \tilde{K}(\widetilde{\partial_{z^l}}) \right). \tag{4.7}$$

By the discussion in [6], the following differential form valued in the bundle $End(T_X) = T_X \otimes_{\mathcal{O}_X} T_X^\vee$

$$(A_{ij}^k dz^i) \otimes_{\mathcal{O}_X} (dz^j \otimes \frac{\partial}{\partial z^k})$$

is exactly the Dolbeault representative of the Atiyah class of the holomorphic tangent bundle T_X. It is straightforward to check that

$$\begin{aligned} \left((\exp(I[\infty]/\hbar)|_{\mathbb{H}})^{TF} / \pi_2^*(\Omega_X^{2-2g}) \right) \Big|_{\mathcal{L}} &= \operatorname{Tr} \left((A_{ij}^k dz^i) \otimes (dz^j \otimes \frac{\partial}{\partial z^k}) \right)^n \\ &= \operatorname{Tr}(At(T_X))^n \\ &= c_n(X). \end{aligned}$$

It then follows easily that

$$\langle 1 \rangle_E = \int_X c_n(X) = \chi(X).$$

q.e.d.

5. Landau-Ginzburg Twisting

In this section we discuss the Landau-Ginzburg twisting of the B-twisted σ-model and establish the topological Landau-Ginzburg B-model via the renormalization method.

Remark 5.1. To avoid confusion, "twisted" and "untwisted" in this section are always concerned with the twist that arises from the superpotential W, instead of the B-twist.

5.1. Classical theory. Let X be a smooth variety with a holomorphic function

$$W : X \to \mathbb{C}.$$

Recall that the B-twisted σ-model describes maps

$$(\Sigma_g)_{dR} \to T^* X_{\bar{\partial}}[1].$$

Let

$$dW \lrcorner : \wedge^* T_X \to \wedge^* T_X$$

be the contraction with the 1-form dW. It induces a differential on $\mathcal{O}\left(T^* X_{\bar{\partial}}[1]\right)$ of degree -1, commuting with d_{D_X}. By abuse of notations, we still denote this differential by $dW \lrcorner$.

Definition 5.2. We define $T^* X_{\bar{\partial}}^W[1]$ to be the L_∞-space with underlying space X, and its sheaf of functions the $\mathbb{Z}_2$-graded complex

$$\mathcal{O}\left(T^* X_{\bar{\partial}}^W[1]\right) := \mathcal{A}_X \otimes_{\mathcal{O}_X} \operatorname{Jet}_X^{hol}\left(\wedge^* T_X\right)$$

with the twisted differential $d_{D_X} + dW \lrcorner$.

Remark 5.3. When $X = \mathbb{C}^n$, and W is a weighted homogeneous polynomial, i.e.

$$W(\lambda^{q_i} z^i) = \lambda W(z^i), \quad \forall \lambda \in \mathbb{C}^*$$

where q_i's are positive rational numbers called the weights, then there emerges a $\mathbb{Q}$-grading by assigning the weights: $wt(z^i) = q_i, wt(\partial_{z^i}) = 1 - q_i, wt(\bar{z}^i) = -q_i, wt(d\bar{z}^i) = -q_i$. However, we will not explore further on this in the current paper.

Note that there is a quasi-isomorphism of $\mathbb{Z}_2$-graded complexes of sheaves

$$\mathcal{O}\left(T^* X^W_{\bar{\partial}}[1]\right) \cong \left(\mathcal{A}^{0,*}_X\left(\wedge^* T_X\right), \bar{\partial}_W\right), \quad \bar{\partial}_W = \bar{\partial} + dW \lrcorner.$$

Therefore $T^* X^W_{\bar{\partial}}[1]$ can be viewed as the derived critical locus of W. The Landau-Ginzburg B-model describes the space of maps

$$(\Sigma_g)_{dR} \to T^* X^W_{\bar{\partial}}[1].$$

As in the untwisted case, we consider those maps in the formal neighborhood of constant maps. This corresponds to the physics statement that path integrals in the B-twisted Landau-Ginzburg model are localized around the neighborhood of constant maps valued in the critical locus of W.

Recall that there exists a Poisson bracket (Schouten-Nijenhuis bracket): $\wedge^* T_X \otimes_{\mathbb{C}} \wedge^* T_X \to \wedge^* T_X$. Viewed as a bi-differential operator, it induces a bracket on the jet bundles, which we denote by

$$\{-,-\}_{T^* X_{\bar{\partial}}[1]} : \mathcal{O}\left(T^* X_{\bar{\partial}}[1]\right) \otimes_{\mathcal{A}_X} \mathcal{O}\left(T^* X_{\bar{\partial}}[1]\right) \to \mathcal{O}\left(T^* X_{\bar{\partial}}[1]\right).$$

Lemma/Definition 5.4. *Let $\widetilde{W} \in \mathcal{O}\left(T^* X_{\bar{\partial}}[1]\right)$ be the image of W under the natural embedding*

$$\mathcal{O}_X \hookrightarrow \mathrm{Jet}^{hol}_X(\mathcal{O}_X) \hookrightarrow \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathrm{Jet}^{hol}_X\left(\wedge^* T_X\right).$$

Then $\{\widetilde{W}, -\}_{T^* X_{\bar{\partial}}[1]} = dW \lrcorner$ as operators on $\mathcal{O}\left(T^* X_{\bar{\partial}}[1]\right)$.

Proof. This follows from the corresponding statement on $\wedge^* T_X$. q.e.d.

Definition 5.5. The space of fields of the topological Landau-Ginzburg B-model is

$$\mathcal{E} := \mathcal{A}_{\Sigma_g} \otimes_{\mathbb{C}} \left(\mathfrak{g}_X[1] \oplus \mathfrak{g}_X^\vee\right),$$

and the classical action functional S^W is defined by

$$S^W = S + I_W,$$

where S is the classical action functional of the Calabi-Yau model, and I_W is the local functional on $\mathcal{A}_{\Sigma_g} \otimes \mathfrak{g}_X[1]$ defined by

$$I_W(\alpha) := \int_{\Sigma_g} \widetilde{W}(\alpha), \quad \alpha \in \mathcal{A}_{\Sigma_g} \otimes \mathfrak{g}_X[1].$$

Here $\widetilde{W}$ is extended linearly in $\mathcal{A}_{\Sigma_g}$ to $\mathcal{A}_{\Sigma_g} \otimes \mathfrak{g}_X[1]$. The LG-twisted interaction is

$$I_{cl}^W = I_{cl} + I_W.$$

REMARK 5.6. The $\mathbb{C}^\times$-symmetry of the untwisted B-model is broken by the term I_W. In particular, the LG-twisted theory is no longer a cotangent theory in the sense of [6].

Lemma 5.7. *The classical interaction I_{cl}^W of the Landau-Ginzburg B-model satisfies the classical master equation $QI_{cl}^W + \frac{1}{2}\left\{I_{cl}^W, I_{cl}^W\right\} + F_{l_1} = 0$.*

Proof.

$$\begin{aligned}
&QI_{cl}^W + \frac{1}{2}\left\{I_{cl}^W, I_{cl}^W\right\} + F_{l_1} \\
=&QI_{cl} + \frac{1}{2}\left\{I_{cl}, I_{cl}\right\} + F_{l_1} + QI_W + \frac{1}{2}\left\{I_W, I_W\right\} + \left\{I_{cl}, I_W\right\} \\
\overset{(1)}{=}&QI_W + \left\{I_{cl}, I_W\right\},
\end{aligned}$$

where (1) follows from the classical master equation of I_{cl} in the untwisted case and the vanishing of $\{I_W, I_W\}$ by the type reason. It is not difficult to see that for $\alpha \in \mathcal{A}_{\Sigma_g}$,

$$(QI_W + \{I_{cl}, I_W\})(\alpha) = \int_{\Sigma_g} d_{D_X}(\widetilde{W})(\alpha) = 0,$$

since $\widetilde{W}$ is flat. q.e.d.

5.2. Quantization. We assume that X is Calabi-Yau with holomorphic volume form Ω_X. Since X is non-compact, the choice of Ω_X will not be unique, and we will always fix one such choice.

Let I_{qc} be the one-loop quantum correction of the B-twisted σ-model associated to (X, Ω_X), with which the quantization of the untwisted theory is defined as

$$I[L] = W(\mathbb{P}_0^L, I_{cl} + \hbar I_{qc}) := \lim_{\epsilon \to 0} W(\mathbb{P}_\epsilon^L, I_{cl} + \hbar I_{qc}).$$

Definition 5.8. We define the Landau-Ginzburg twisting of $I[L]$ by

$$I^W[L] = W(\mathbb{P}_0^L, I_{cl} + I_W + \hbar I_{qc}) := \lim_{\epsilon \to 0} W(\mathbb{P}_\epsilon^L, I_{cl} + I_W + \hbar I_{qc}).$$

Since I_W is only a functional on $\mathcal{A}_{\Sigma_g} \otimes \mathfrak{g}_X[1]$, it is easy to see by the type reason that

$$I^W[L] = I[L] + W_{tree}(\mathbb{P}_0^L, I_{cl}, I_W).$$

Proposition 5.9. *$I^W[L]$ defines a quantization of the B-twisted topological Landau-Ginzburg model S^W in the sense of Definition* 3.14.

Proof. Let $\delta_W[L] = W_{tree}(\mathbb{P}_0^L, I_{cl}, I_W)$. By the type reason, $\Delta_L \delta_W[L] = \{\delta_W[L], \delta_W[L]\}_L = 0$. Since $I[L]$ satisfies the QME, we have

$$\left(Q_L + \hbar\Delta_L + \frac{F_{l_1}}{\hbar} - R\right) e^{I^W[L]/\hbar}$$
$$= \hbar^{-1}\left(Q_L \delta_W[L] + \{I[L], \delta_W[L]\}_L\right) e^{I^W[L]/\hbar}.$$

Since $\delta_W[L]$ is given by sum over trees, it satisfies the classical RG flow equation. The vanishing of $Q_L\delta_W[L] + \{I[L], \delta_W[L]\}_L$ then follows from its vanishing in the classical limit

$$QI_W + \{I_{cl}, I_W\} = 0.$$

q.e.d.

5.3. Observable theory and correlation functions. We would like to explore the correlation functions of local quantum observables of our Landau-Ginzburg theory. One essential difference with the untwisted case is that it is no longer a cotangent theory due to the term I_W; hence the interpretation of quantization as projective volume forms [6] does not work directly in this case. However, the BV integration interpretation in Corollary 4.27 still makes sense in the LG-twisted case, which we will use to define the correlation functions.

For simplicity, we will assume X to be a Stein domain in $\mathbb{C}^n$, and that $Crit(W)$ is finite. We choose the holomorphic volume form $\Omega_X = dz^1 \wedge \cdots \wedge dz^n$, where $\{z^i\}$ are the linear coordinates on $\mathbb{C}^n$.

Definition 5.10. The quantum observables on Σ_g at scale L are defined as the cochain complex

$$\mathrm{Obs}^q(\Sigma_g)[L] := \left(\mathcal{O}(\mathcal{E})[[\hbar]], Q_L + \{I^W[L], -\}_L + \hbar\Delta_L\right).$$

Observables $\mathrm{Obs}^q(U)$ with support in $U \subset \Sigma_g$ are defined in a similar fashion as in section 4.2.

Correlation functions are defined for a proper subspace of quantum observables which are "integrable" in some good sense, since we are working with non-compact space X. We consider the following simplest class:

Definition 5.11. We define the sub-complex $\mathrm{Obs}^q_c(\Sigma_g)[L] \subset \mathrm{Obs}^q(\Sigma_g)[L]$ by

$$\mathrm{Obs}^q_c(\Sigma_g)[L] := \left(\mathcal{O}_c(\mathcal{E})[[\hbar]], Q_L + \{I^W[L], -\}_L + \hbar\Delta_L\right),$$

where $\mathcal{O}_c(\mathcal{E}) := \mathcal{O}(\mathcal{E}) \otimes_{\mathcal{A}(X)} \mathcal{A}_c(X)$ and $\mathcal{A}_c(X)$ is the space of compactly supported differential forms on X. The corresponding local observables supported in $U \subset \Sigma_g$ are denoted by $\mathrm{Obs}^q_c(U)$.

Proposition 5.12. *Let $U \subset \Sigma_g$ be a disk. Then*

$$H^*(Obs^q(U)) \cong H^*(Obs^q_c(U)) \cong \mathrm{Jac}(W)[[\hbar]],$$

where $\mathrm{Jac}(W)$ *is the Jacobian ring of* W.

Proof. The strategy is completely parallel to Corollary 4.14. We just need to observe that the cohomology of classical observables in the twisted case is given by

$$H^*(\mathcal{A}^{0,*}(X, \wedge^* T_X), \bar\partial + dW \lrcorner) \quad \text{or} \quad H^*(\mathcal{A}_c^{0,*}(X, \wedge^* T_X), \bar\partial + dW \lrcorner),$$

both of which are canonically isomorphic to $\mathrm{Jac}(W)$ (see [18]). q.e.d.

Now we define the correlation function of quantum observables. For the Landau-Ginzburg model, notice that the "top fermion" $\pi_2^*(\Omega_X^{2g-2})$ can be defined in a similar way as in the untwisted case. Thus we can define the correlation function of quantum observables via the BV integration in the spirit of Corollary 4.27 (also see there for the notations):

Definition 5.13. Let O be a quantum observable of the Landau-Ginzburg B-model which is closed; then the correlation function of O is defined as

$$\langle O \rangle_{\Sigma_g}^W := \int_X \left((e^{I^W[\infty]/\hbar} O\big|_{\mathbb{H}})^{TF} \Big/ \pi_2^*(\Omega_X^{2g-2}) \right) \Big|_{\mathcal{L}}.$$

As a parallel to Theorem 4.29, we have

Proposition 5.14. *Let $\{U_i\}$ be disjoint disks on Σ_g. Let $O_{f_i,U_i} \in H^*(Obs_c^q(U_i))$ be local observables associated to $f_i \in \mathrm{Jac}(W)$ by Proposition* 5.12*. Then*

$$\langle O_{f_1,U_1} \star \cdots \star O_{f_m,U_m} \rangle_{\Sigma_g}^W = \sum_{p \in Crit(W)} \mathrm{Res}_p \left(\frac{f_1 \cdots f_m \det(\partial_i \partial_j W)^g dz^1 \wedge \cdots \wedge dz^n}{\prod_i \partial_i W} \right),$$

where $\star$ is the local to global factorization product, and Res_p *is the residue at the critical point p* [13].

Proof. As in the non-twisted case, we can assume that $m = 1$ and let $f = f_1 \in \mathrm{Jac}(W)$. Let $O_f[L]$ denote the corresponding quantum observable and $O_f = \lim\limits_{L \to 0} O_f[L]$. By the definition of the correlation function, we have

$$\langle O_f[\infty] \rangle_{\Sigma_g}^W = \int_X \left((e^{I^W[\infty]/\hbar} O_f[\infty]\big|_{\mathbb{H}})^{TF} / \pi_2^*(\Omega_X^{2g-2}) \right) \Big|_{\mathcal{L}}.$$

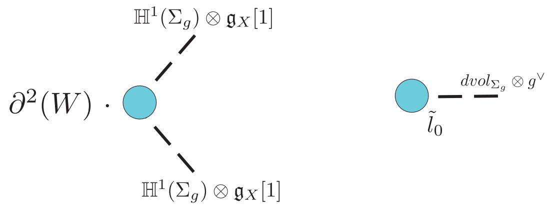
Since $X \subset \mathbb{C}^n$, we can choose the L_∞ structure on $\mathfrak{g}_X$ such that $l_i = 0$ for all $i \geq 2$. It is then clear that the RG flow of the classical interaction of the B-model I_{cl} has only tree parts. Furthermore, when restricted to the subspace of harmonic fields, there is

$$I[\infty]|_{\mathbb{H}} = I_{cl}|_{\mathbb{H}},$$

$$W_{tree}(\mathbb{P}_0^L, I_{cl}, I_W)|_{\mathbb{H}} = I_W|_{\mathbb{H}}.$$

It is then not difficult to see that the only terms in $e^{I^W[\infty]/\hbar}$ that will contribute non-trivially to $\left((e^{I^W[\infty]/\hbar}O_f[\infty]|_{\mathbb{H}})^{TF}/\pi_2^*(\Omega^{2g-2}_{\mathbb{C}^n})\right)\Big|_{\mathcal{L}}$ are the following:

(5.1)



In the first picture, the two harmonic 1-forms on Σ_g attached to the tails must be dual to each other. Since $\dim_{\mathbb{C}}(\mathbb{H}^1(\Sigma_g)) = 2g$, the total contribution of the first terms is

$$\hbar^{-g\cdot n}\,(\det(\partial_i\partial_j W))^g \otimes \pi_2^*(\Omega_X^{2g}).$$

And the contribution of the second terms in equation (5.1) together with the observable O_f is, as in the computation of correlation functions in the non-twisted B-model on $\mathbb{P}^1$, given by

$$\hbar^{-n}((O_f \vdash \Omega_X)\wedge\Omega_X)\otimes\pi_2^*(\Omega_X^{-2}).$$

All together, we have

$$\begin{aligned}\langle O_f[\infty]\rangle^W_{\Sigma_g} =& \hbar^{-(g+1)n}\int_X (\det(\partial_i\partial_j W))^g\,(O_f\vdash\Omega_X)\wedge\Omega_X\\ =&\hbar^{-(g+1)n}\cdot\sum_{p\in Crit(W)}\mathrm{Res}_p\left(\frac{f\det(\partial_i\partial_j W)^g dz^1\wedge\cdots\wedge dz^n}{\prod_i\partial_i W}\right),\end{aligned}$$

where the last equality follows from [18, Proposition 2.5]. q.e.d.

Remark 5.15. This coincides with Vafa's residue formula for topological Landau-Ginzburg models [26].

Appendix A. Proof of Lemma 3.6

The propagator P^L_ϵ on the upper half plane $\mathbb{H}$ with respect to the hyperbolic metric is a 1-form on $\mathbb{H}\times\mathbb{H}$, thus having a decomposition under the isomorphism

$$\mathcal{A}^1(\mathbb{H}\times\mathbb{H})\cong\left(\mathcal{A}^1(\mathbb{H})\otimes\mathcal{A}^0(\mathbb{H})\right)\oplus\left(\mathcal{A}^0(\mathbb{H})\otimes\mathcal{A}^1(\mathbb{H})\right),$$

where $\otimes$ denotes the completed tensor product. Let us call the projection into these two components by the $(1,0)$ and $(0,1)$ parts respectively. There is a similar decomposition of the heat kernel k_t into its $(2,0),(0,2)$, and $(1,1)$ parts. We will use $z_i = x_i + \sqrt{-1}y_i, i = 1,2$ to denote the coordinates on the two copies of $\mathbb{H}$ respectively. The propagator will be denoted by $P^L_\epsilon(z_1,z_2)$, where we have omitted its anti-holomorphic dependence for simplicity, and similarly for the heat kernel $k_t(z_1,z_2)$.

By the fact that $P_\epsilon^L(z_1, z_2)$ is a symmetric tensor in $\mathcal{A}^*(\mathbb{H}) \otimes \mathcal{A}^*(\mathbb{H})$, we only need to compute its $(1,0)$ part. For this, we apply the gauge fixing operator d^* to the $(2,0)$ part of the heat kernel which is given explicitly by:

$$k_t^{scalar}(z_1, z_2)\frac{dx_1 dy_1}{y_1^2} = \frac{\sqrt{2}}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{1}{4}t} \int_\rho^\infty \frac{s e^{-\frac{s^2}{4t}} ds}{(\cosh s - \cosh \rho)^{\frac{1}{2}}} \frac{dx_1 dy_1}{y_1^2}.$$

Here $k_t^{scalar}(z_1, z_2)$ is the heat kernel of the Laplacian on smooth functions, and $\rho(z_1, z_2)$ denotes the geodesic distance between z_1 and z_2 given explicitly by

$$\rho(z_1, z_2) = \operatorname{arcosh}\left(1 + \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{y_1 y_2}\right).$$

In particular, $k_t^{scalar}(z_1, z_2) = k_t^{scalar}(\rho(z_1, z_2))$ is a function of ρ. The $(1,0)$ part of P_ϵ^L is therefore given by (where d_{z_1} is the de Rham differential, $\star_1$ is the Hodge star on the first copy of $\mathbb{H}$)

$$\begin{aligned}
&\int_\epsilon^L dt \left(\star_1 d_{z_1} \star_1 (k_t^{scalar}(z_1, z_2)\frac{dx_1 dy_1}{y_1^2})\right) \\
=&\int_\epsilon^L dt \left(\star_1 d_{z_1} (k_t^{scalar}(\rho(z_1, z_2)))\right) \\
=&\int_\epsilon^L dt f(\rho, t) \left(\star_1 d_{z_1} \cosh(\rho(z_1, z_2))\right) \\
=&\int_\epsilon^L dt f(\rho, t) \star_1 \left(\frac{2(x_1 - x_2)}{y_1 y_2} dx_1 + \frac{(y_1 - y_2)(y_1 + y_2)}{y_1^2 y_2} dy_1\right) \\
=&\int_\epsilon^L dt f(\rho, t) \left(\frac{2(x_1 - x_2)}{y_1 y_2} dy_1 - \frac{(y_1 - y_2)(y_1 + y_2)}{y_1^2 y_2} dx_1\right)
\end{aligned}$$

for some $f(\rho, t)$ clear from the context. By the symmetry property, the full propagator is given by

$$\begin{aligned}
P_\epsilon^L(z_1, z_2) = &\int_\epsilon^L f(\rho, t) dt \\
&\cdot \left(\frac{2(x_1 - x_2)}{y_1 y_2}(dy_1 - dy_2) - \frac{(y_1 - y_2)(y_1 + y_2)}{y_1 y_2}\left(\frac{dx_1}{y_1} - \frac{dx_2}{y_2}\right)\right).
\end{aligned}$$

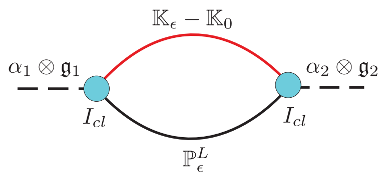
The asymptotic property of $f(\rho, t)$ in equation (3.3) follows from the general property of heat kernels, or an explicit evaluation of $f(\rho, t)$.

Appendix B. Some Feynman graph computations

Proof of Lemma 3.30. It is not difficult to see that the proof of the lemma can be reduced to wheels with two vertices, and we will show that the Feynman weights (3.18) associated to the trivalent wheel vanish. The proof for other wheels with two vertices is similar.

Let $\alpha_1 \otimes g_1$ and $\alpha_2 \otimes g_2$ be the inputs on the tails; the Feynman weight

(B.1)



is the evaluation of

$$\begin{aligned}
&\mathbb{P}_\epsilon^L \otimes (\mathbb{K}_\epsilon - \mathbb{K}_0) \otimes (\alpha_1 \otimes g_1) \otimes (\alpha_2 \otimes g_2) \\
=& \left(P_\epsilon^L \otimes \sum_{i=1}^n (X_i \otimes X^i + X^i \otimes X_i) \right) \\
&\otimes \left((K_\epsilon - K_0) \otimes \sum_{j=1}^n (X_j \otimes X^j + X^j \otimes X_j) \right) \otimes (\alpha_1 \otimes g_1) \otimes (\alpha_2 \otimes g_2)
\end{aligned}$$

under $I_{cl} \otimes I_{cl}$. Here $\{X_i\}$ denotes a basis of $\mathfrak{g}_X$ over $\mathcal{A}_X$ (locally) and $\{X^i\}$ denotes the corresponding dual basis of $\mathfrak{g}_X^\vee$. More explicitly, equation (B.1) is given by

$$\begin{aligned}
&\left(\int_{\Sigma_g \times \Sigma_g} P_\epsilon^L(z_1, z_2)(K_\epsilon(z_1, z_2) - K_0(z_1, z_2))\alpha_1\alpha_2 \right) \left(\langle l_2(-), - \rangle \otimes \langle l_2(-), - \rangle \right) \\
&\left(\sum_{i,j=1}^n (-X_i \otimes g_1 \otimes X^j \otimes X_j \otimes g_2 \otimes X^i + X_j \otimes g_1 \otimes X^i \otimes X_i \otimes g_2 \otimes X^j) \right) \\
=& \, 0.
\end{aligned}$$

Proof of Lemma 3.31. We first prove the lemma for those cases where $n > 3$. As in the proof of Lemma/Definition 3.23, we can replace Σ_g by $\mathbb{H}$ with inputs compactly supported, and assume that γ is a trivalent wheel. We still use the notation K_t for the heat kernel on $\mathbb{H}$ for convenience. Without loss of generality, let us assume that the edge e connects the vertices v_1 and v_n. Let $\alpha_i \otimes \mathfrak{g}_i$ be the input on the vertex v_i. We will show that the following two limits exist and are the same:

$$\lim_{\epsilon \to 0} W_{\gamma,e}(P_\epsilon^L, K_\epsilon, I_{cl}) = \lim_{\epsilon \to 0} W_{\gamma,e}(P_\epsilon^L, K_0, I_{cl}). \tag{B.2}$$

The LHS of equation (B.2) is given explicitly by

$$\begin{aligned}
&W_{\gamma,e}(P_\epsilon^L, K_\epsilon, I_{cl})(\alpha_1, \cdots, \alpha_n) \\
=& \int_{z_1, \cdots, z_n \in \mathbb{H}} P_\epsilon^L(z_1, z_2) \cdots P_\epsilon^L(z_{n-1}, z_n) \\
&\quad \cdot K_\epsilon(z_n, z_1)\alpha_1(z_1, \bar{z}_1) \cdots \alpha_n(z_n, \bar{z}_n) d^2 z_1 \cdots d^2 z_n \\
=& \int_{t_1, \cdots, t_{n-1} = \epsilon}^L dt_1 \cdots dt_{n-1} \int_{z_1, \cdots, z_n \in \mathbb{H}} d^2 z_1 \cdots d^2 z_n
\end{aligned}$$

$$d_{z_1}^* K_{t_1}(z_1, z_2) \cdots d_{z_{n-1}}^* K_{t_{n-1}}(z_{n-1}, z_n) K_\epsilon(z_n, z_1)\alpha_1 \cdots \alpha_n.$$

We claim that the integral

$$\int_{z_1,\cdots,z_n\in\mathbb{H}} d^2z_1 \cdots d^2z_n d^* K_{t_1}(z_1, z_2) \cdots d^* K_{t_{n-1}}(z_{n-1}, z_n) K_\epsilon(z_n, z_1)\alpha_1 \cdots \alpha_n \tag{B.3}$$

is uniformly bounded by a function of $t_1, \cdots, t_{n-1}$ which is integrable on $[0, L]^{n-1}$. Then equation (B.2) follows from the dominated convergence theorem.

Proof of the Claim. By the asymptotic expansion (3.1) (3.3) of K_t and P_ϵ^L respectively, the leading term of the integral (B.3) is given by

$$\begin{aligned}(B.4)\quad &\frac{1}{(4\pi)^n} \int_{z_1,\cdots,z_n\in\mathbb{H}} \prod_{k=1}^{n-1} b_0(\rho(z_k, z_{k+1}))\frac{1}{\epsilon} \cdot a_0(z_n, z_1)e^{-\frac{\rho^2(z_n,z_1)}{4\epsilon}} \alpha_1 \cdots \alpha_n \\ &\left(\prod_{k=1}^{n-1} \frac{1}{t_k^2} e^{-\frac{\rho^2(z_k,z_{k+1})}{4t_k}}\right) \\ &\left(2(x_k - x_{k+1})(dy_k - dy_{k+1}) - \frac{y_k^2 - y_{k+1}^2}{y_k y_{k+1}}\left(\frac{dx_k}{y_k} - \frac{dx_{k+1}}{y_{k+1}}\right)\right).\end{aligned}$$

We provide the estimates for the above leading term, while higher order terms furnish a better convergence property.

We do the same change of coordinates as in the proof of Lemma/Definition 3.23, a procedure after which the integral (B.4) becomes a sum of integrals of the following form:

$$\begin{aligned}\int_{\mathbb{H}} du_0 dv_0 \int_{\mathbb{R}^{2n-2}} du_1 dv_1 \cdots du_{n-1} dv_{n-1} \Phi \cdot \frac{1}{\epsilon}\left(\prod_{k=1}^{n-1} \frac{u_k^{i_k} v_k^{j_k}}{t_k^2}\right) \\ \cdot \exp\left(-\sum_{i=1}^{n-1} \frac{u_i^2 + v_i^2}{4t_i} - \frac{\rho^2(z_n, z_1)}{4\epsilon}\right),\end{aligned} \tag{B.5}$$

where

- for $1 \leqslant k \leqslant n-1$, the functions $u_k^{i_k} v_k^{j_k}$ arise from $x_k - x_{k+1}$ and $y_k - y_{k+1}$, hence $i_k + j_k \geq 1$;
- Φ is a smooth function on $\mathbb{H} \times \mathbb{R}^{2n-2}$ with compact support.

Now we only need to show that for each fixed $(u_0, v_0) \in \mathbb{H}$, the following integral is bounded above in absolute value by an integrable function of

$(t_1, \cdots, t_{n-1})$ on $[0, L]^{n-1}$ independent of ϵ:

(B.6)

$$\int_{\mathbb{R}^{2n-2}} du_1 dv_1 \cdots du_{n-1} dv_{n-1} \frac{1}{\epsilon} \left(\prod_{k=1}^{n-1} \frac{u_k^{i_k} v_k^{j_k}}{t_k^2} \right) \cdot \exp\left(-\sum_{i=1}^{n-1} \frac{u_i^2 + v_i^2}{4t_i} - \frac{\rho^2(z_n, z_1)}{4\epsilon} \right).$$

We show this for the leading term of its Wick expansion. Notice that for each fixed $(u_0, v_0) \in \mathbb{H}$, the function

$$-\sum_{i=1}^{n-1} \frac{u_i^2 + v_i^2}{4t_i} - \frac{\rho^2(z_n, z_1)}{4\epsilon}$$

takes its maximal value 0 at the critical point $(u_1, v_1, \cdots, u_{n-1}, v_{n-1}) = (0, \cdots, 0)$. It is not difficult to see that the Hessian at the critical point is the same as that of the function

$$-\sum_{i=1}^{n-1} \frac{u_i^2 + v_i^2}{4t_i} - \frac{\left(\sum_{i=1}^{n-1} u_i\right)^2 + \left(\sum_{i=1}^{n-1} v_i\right)^2}{4\epsilon}.$$

Thus the leading term in the Wick expansion of equation (B.6) is the same as that of the following integral:

(B.7)

$$\int_{\mathbb{R}^{2n-2}} du_1 dv_1 \cdots du_{n-1} dv_{n-1} \frac{1}{\epsilon} \cdot \left(\prod_{k=1}^{n-1} \frac{u_k^{i_k} v_k^{j_k}}{t_k^2} \right) \cdot \exp\left(-\sum_{i=1}^{n-1} \frac{u_i^2 + v_i^2}{4t_i} - \frac{(\sum_{i=1}^{n-1} u_i)^2 + (\sum_{i=1}^{n-1} v_i)^2}{4\epsilon} \right),$$

which can be evaluated via Gaussian type integral. We rearrange the coordinates on $\mathbb{R}^{2n-2}$ as

$$(u_1, \cdots, u_{n-1}, v_1, \cdots, v_{n-1}),$$

and let $t = (t_1, \cdots, t_{n-1})$. The matrix of the quadratic form in the exponential is given by:

$$M(t, \epsilon) = \frac{1}{4} \begin{pmatrix} A(t, \epsilon) & 0 \\ 0 & A(t, \epsilon) \end{pmatrix},$$

in which

$$A(t, \epsilon) = \begin{pmatrix} \frac{1}{t_1} + \frac{1}{\epsilon} & \frac{1}{\epsilon} & \cdots & \frac{1}{\epsilon} \\ \frac{1}{\epsilon} & \frac{1}{t_2} + \frac{1}{\epsilon} & \cdots & \frac{1}{\epsilon} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\epsilon} & \frac{1}{\epsilon} & \cdots & \frac{1}{t_{n-1}} + \frac{1}{\epsilon} \end{pmatrix}.$$

For convenience, we will also use M for the matrix $M(t,\epsilon)$. It is straightforward to check that

$$\det(M) = \left(\frac{1}{4}\right)^{2(n-1)} \cdot \left(\frac{t_1+\cdots+t_{n-1}+\epsilon}{t_1\cdots t_{n-1}\epsilon}\right)^2. \tag{B.8}$$

The standard trick of the Feynman integral implies that (B.7) equals

$$\begin{aligned} &\frac{1}{\sqrt{\det M}}\left(\prod_{i=1}^{n-1}\frac{1}{t_i^2}\right)\cdot\frac{1}{\epsilon}\sum\left(M^{-1}_{\alpha_1,\beta_1}\cdots M^{-1}_{\alpha_N,\beta_N}\right)\\ =&\frac{4^{n-1}}{t_1\cdots t_{n-1}(t_1+\cdots+t_{n-1}+\epsilon)}\cdot\sum\left(M^{-1}_{\alpha_1,\beta_1}\cdots M^{-1}_{\alpha_N,\beta_N}\right), \end{aligned} \tag{B.9}$$

where the sum is over all pairings of $\prod_{k=1}^{n-1}(u_k^{i_k}v_k^{j_k})$, and $M^{-1}_{\alpha,\beta}$'s are entries of the inverse matrix of M. N is an integer no less than $(n-1)/2$.

We claim that on each region of the form

$$\{(t_1,t_2,\cdots,t_{n-1})\in[0,L]^{n-1}: 0\leqslant t_{\sigma(1)}\leqslant\cdots\leqslant t_{\sigma(n-1)}\leqslant L\},$$

where $\sigma\in S_{n-1}$, equation (B.9) is uniformly bounded above in absolute value by an integrable function. This claim finishes the proof of Lemma 3.31.

We will prove the claim for $\sigma=\mathrm{id}\in S_{n-1}$; the proof for other σ's is similar. The following lemma provides an estimate of the entries of M^{-1}.

Lemma B.1. $|M^{-1}_{i,j}|\leqslant 4\cdot\min\{t_i,t_j\}$.

Proof. There are two possibilities: $i=j$ or $i\neq j$. By symmetry, we only need to consider $M^{-1}_{1,1}$ and $M^{-1}_{1,2}$. We have

$$\begin{aligned} M^{-1}_{1,1}=&\det\begin{pmatrix} \frac{1}{t_2}+\frac{1}{\epsilon} & \frac{1}{\epsilon} & \cdots & \frac{1}{\epsilon}\\ \frac{1}{\epsilon} & \frac{1}{t_3}+\frac{1}{\epsilon} & \cdots & \frac{1}{\epsilon}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{\epsilon} & \frac{1}{\epsilon} & \cdots & \frac{1}{t_{n-1}}+\frac{1}{\epsilon}\end{pmatrix}\cdot\left(\frac{t_1+\cdots+t_{n-1}+\epsilon}{t_1\cdots t_{n-1}\epsilon}\right)^{-1}\cdot 4\\ =&\frac{t_2+\cdots+t_{n-1}+\epsilon}{t_2\cdots t_{n-1}\epsilon}\cdot\left(\frac{t_1+\cdots+t_{n-1}+\epsilon}{t_1\cdots t_{n-1}\epsilon}\right)^{-1}\cdot 4\\ =&t_1\cdot\frac{t_2+\cdots+t_{n-1}+\epsilon}{t_1+\cdots+t_{n-1}+\epsilon}\cdot 4\leqslant 4t_1 \end{aligned}$$

and

$$M^{-1}_{1,2}=\det\begin{pmatrix} \frac{1}{\epsilon} & \frac{1}{\epsilon} & \cdots & \frac{1}{\epsilon}\\ \frac{1}{\epsilon} & \frac{1}{t_3}+\frac{1}{\epsilon} & \cdots & \frac{1}{\epsilon}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{\epsilon} & \frac{1}{\epsilon} & \cdots & \frac{1}{t_{n-1}}+\frac{1}{\epsilon}\end{pmatrix}\cdot\left(\frac{t_1+\cdots+t_{n-1}+\epsilon}{t_1\cdots t_{n-1}\epsilon}\right)^{-1}\cdot 4$$

$$=\frac{1}{t_3\cdots t_{n-1}\epsilon}\cdot\left(\frac{t_1+\cdots+t_{n-1}+\epsilon}{t_1\cdots t_{n-1}\epsilon}\right)^{-1}\cdot 4$$
$$=\frac{t_1t_2}{t_1+\cdots+t_{n-1}+\epsilon}\cdot 4\leqslant 4\cdot\min\{t_1,t_2\}.$$

q.e.d.

With Lemma B.1, we can give an estimate of $\dfrac{M^{-1}_{\alpha_1,\beta_1}\cdots M^{-1}_{\alpha_N,\beta_N}}{t_1\cdots t_{n-1}(t_1+\cdots+t_{n-1}+\epsilon)}$ in (B.9): since $1\in\{\alpha_1,\beta_1,\cdots,\alpha_N,\beta_N\}$, we can always find a subset $\{l_1,l_2,\cdots,l_{\tilde N}\}\subset\{2,3,\cdots,n-1\}$, $\tilde N\leq(n-1)/2$, such that

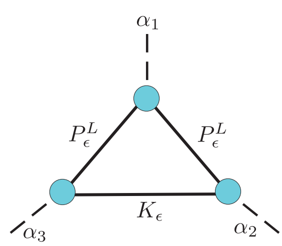
$$\frac{\left|M^{-1}_{\alpha_1,\beta_1}\cdots M^{-1}_{\alpha_N,\beta_N}\right|}{t_1\cdots t_{n-1}(t_1+\cdots+t_{n-1}+\epsilon)}\leqslant\frac{1}{t_{l_1}\cdots t_{l_{\tilde N}}}\frac{1}{t_1+\cdots+t_{n-1}}.$$

It is straightforward to check that the function

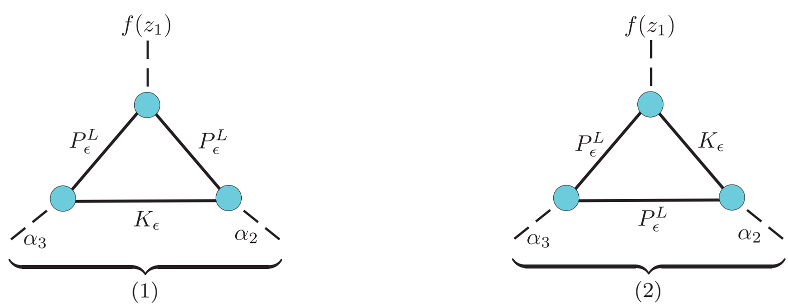
$$\frac{1}{t_{l_1}\cdots t_{l_{\tilde N}}}\frac{1}{t_1+\cdots+t_{n-1}}$$

is integrable on $\{(t_1,\cdots,t_{n-1})\in[0,L]^{n-1}:0\leqslant t_1\leqslant\cdots\leqslant t_{n-1}\leqslant L\}$ if $n\geqslant 4$.

The only case left is when $n=3$. Notice that the following Feynman weight is non-trivial only if at least one α_i is a 0-form.



Let f be a compactly supported function on $\mathbb{H}$. There are the following two possible configurations of the inputs on the graph up to automorphisms:



For configuration (1), we write the corresponding Feynman weight as the sum of

$$\int_{z_1,z_2,z_3\in\mathbb{H}} P_\epsilon^L(z_1,z_2)\ K_\epsilon(z_2,z_3)\ P_\epsilon^L(z_3,z_1)\cdot f(z_2)\ \alpha_2\ \alpha_3 \tag{B.10}$$

and

$$\int_{z_1,z_2,z_3\in\mathbb{H}} P_\epsilon^L(z_1,z_2)\ K_\epsilon(z_2,z_3)\ P_\epsilon^L(z_3,z_1)\cdot(f(z_1)-f(z_2))\ \alpha_2\ \alpha_3. \tag{B.11}$$

Here equation (B.10) actually vanishes since

$$\int_{z_1\in\mathbb{H}} P_\epsilon^L(z_3,z_1)P_\epsilon^L(z_1,z_2)=0,$$

which amounts to $(d^*)^2=0$. The vanishing holds if we replace K_ϵ by K_0. For (B.11), we claim that

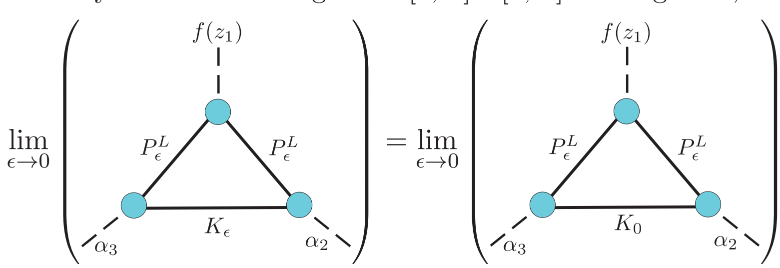
(B.12)

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_{z_1,z_2,z_3\in\mathbb{H}} P_\epsilon^L(z_1,z_2)K_\epsilon(z_2,z_3)P_\epsilon^L(z_3,z_1)\cdot(f(z_1)-f(z_2))\alpha_2\alpha_3\\ &=\lim_{\epsilon\to 0}\int_{z_1,z_2,z_3\in\mathbb{H}} P_\epsilon^L(z_1,z_2)K_0(z_2,z_3)P_\epsilon^L(z_3,z_1)\cdot(f(z_1)-f(z_2))\alpha_2\alpha_3.\end{aligned}$$

To prove the claim, we apply the same argument for the case of $n\geqslant 4$. The leading term of (B.11) is similar to (B.9), except that the function $f(z_1)-f(z_2)$ in (B.11) contributes one more u_i or v_i than in (B.9) (so $N\geq 2$ when $n=3$, hence $\tilde{N}=0$). Thus the leading term is bounded above by a constant times

$$\frac{1}{t_1+t_2},$$

which clearly has a finite integral on $[0,L]\times[0,L]$. All together, we have



For configuration (2), a similar argument as above shows

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_{z_1,z_2,z_3\in\mathbb{H}} K_\epsilon(z_1,z_2)P_\epsilon^L(z_2,z_3)P_\epsilon^L(z_3,z_1)\cdot(f(z_1)-f(z_2))\alpha_2\alpha_3\\ &=\lim_{\epsilon\to 0}\int_{z_1,z_2,z_3\in\mathbb{H}} K_0(z_1,z_2)P_\epsilon^L(z_2,z_3)P_\epsilon^L(z_3,z_1)\cdot(f(z_1)-f(z_2))\alpha_2\alpha_3.\end{aligned}$$

On the other hand,

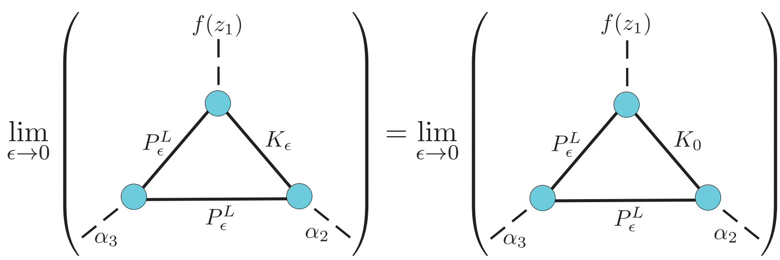
$$\int_{z_1,z_2,z_3\in\mathbb{H}} K_\epsilon(z_1,z_2)P_\epsilon^L(z_2,z_3)P_\epsilon^L(z_3,z_1)\cdot f(z_2)\alpha_2(z_2)\alpha_3(z_3)$$

$$= \pm \int_{z_2, z_3 \in \mathbb{H}} P_\epsilon^L(z_2, z_3) P_{2\epsilon}^{\epsilon+L}(z_3, z_2) f(z_2) \alpha_2(z_2) \alpha_3(z_3).$$

Then similar to Lemma/Definition 3.23, the above limit exists, and

$$\begin{aligned} &\lim_{\epsilon \to 0} \int_{z_2, z_3 \in \mathbb{H}} P_\epsilon^L(z_2, z_3) P_{2\epsilon}^{\epsilon+L}(z_3, z_2) f(z_2) \alpha_2 \alpha_3 \\ =& \lim_{\epsilon \to 0} \int_{z_2, z_3 \in \mathbb{H}} P_\epsilon^L(z_2, z_3) P_\epsilon^L(z_3, z_2) f(z_2) \alpha_2 \alpha_3 \\ =& \lim_{\epsilon \to 0} \int_{z_1, z_2, z_3 \in \mathbb{H}} P_\epsilon^L(z_2, z_3) P_\epsilon^L(z_3, z_1) K_0(z_1, z_2) f(z_2) \alpha_2 \alpha_3. \end{aligned}$$

Altogether we have



Appendix C. One-loop anomaly

In this section, we give a general formula of the one-loop anomaly for perturbative QFT in Costello's formalism. Let $\mathcal{E}$ be the space of fields of a perturbative QFT whose classical interaction is $I \in \mathcal{O}(\mathcal{E})$. Let P_ϵ^L denote the regularized propagator.

Let us first give an explicit description of the one-loop naive quantization $I_{naive}^{(1)}[L]$. Let Γ^{Wheel} denote the set of Feynman diagrams given by wheels (without trees attached), which are the essential part of one-loop diagrams requiring regularization by counter-terms. We fix a renormalization scheme which allows us to decompose any graph integral uniquely into its "smooth part" and "singular part" in the sense of [5]. Let $\gamma \in \Gamma^{Wheel}$; we will write

$$W_\gamma(P_\epsilon^L, I) = W_\gamma(P_\epsilon^L, I)^{sm} + W_\gamma(P_\epsilon^L, I)^{sing} \tag{C.1}$$

for the corresponding decomposition [5, Theorem 9.5.1].

Lemma C.1. *Let $\gamma \in \Gamma^{Wheel}$; then $W_\gamma(P_\epsilon^L, I)^{sing}$ is a local functional on $\mathcal{E}$ independent of L.*

Proof. Since $\frac{\partial}{\partial L} P_\epsilon^L$ is a smooth kernel which does not depend on ϵ, $\frac{\partial}{\partial L} W_\gamma(P_\epsilon^L, I)$ behaves like a tree diagram. Therefore

$$\lim_{\epsilon \to 0} \frac{\partial}{\partial L} W_\gamma(P_\epsilon^L, I) \quad \text{exists.}$$

Hence $W_\gamma(P_\epsilon^L, I)^{sing}$ is independent of the scale L. By [5, Theorem 9.3.1], $W_\gamma(P_\epsilon^L, I)^{sing}$ has a small L asymptotic expansion in terms of local functionals. Since it does not depend on L, it follows that the functional $W_\gamma(P_\epsilon^L, I)^{sing}$ is local. q.e.d.

By the algorithm in [5], $W_\gamma(P_\epsilon^L, I)^{sing}$ is the counter-term associated to γ, and

$$\lim_{\epsilon\to 0} W_\gamma(P_\epsilon^L, I)^{sm} \quad \text{exists.}$$

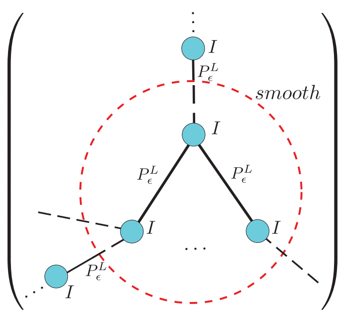
The following proposition now follows easily from the Feynman diagram analysis and the regularization process described in [5].

Proposition C.2. *The one-loop naive quantization is given by*

$$I^{(1)}_{naive}[L] = \lim_{\epsilon\to 0} \sum_{\gamma_1\in trees, v\in V(\gamma_1), \gamma_2\in\Gamma^{Wheel}} W_{\gamma_1, v}(P_\epsilon^L, I, W_{\gamma_2}(P_\epsilon^L, I)^{sm}),$$

where the summation is over all connected tree diagrams γ_1 with a specified vertex v, and a wheel diagram γ_2. $W_{\gamma_1,v}(P_\epsilon^L, I, W_{\gamma_2}(P_\epsilon^L, I)^{sm})$ is the Feynman graph integral on γ_1, where we put I on those vertices not being v, put $W_{\gamma_2}(P_\epsilon^L, I)^{sm}$ on the vertex v, and put P_ϵ^L on all internal edges.

Pictorially,

(C.2) $$I^{(1)}_{naive}[L] =$$ 

Remark C.3. In the above picture, we are taking the sum of weights of all one-loop graphs.

Let

$$(Q + \hbar\Delta_L)\, e^{I^{(0)}_{naive}[L]/\hbar + I^{(1)}_{naive}[L]} = (O_1[L] + O(\hbar)) e^{I^{(0)}_{naive}[L]/\hbar + I^{(1)}_{naive}[L]},$$

where $O_1[L]$ is the leading term in the $\hbar$-expansion. By the construction in [5, Chapter 5], $O_1[L]$ is the anomaly for solving the quantum master equation at the one-loop. Moreover, $O_1[L]$ satisfies a version of classical renormalization group flow, and

$$O_1 := \lim_{L\to 0} O_1[L]$$

exists as a local functional. Our goal is to give a formula for computing O_1 in terms of graphs.

Let

$$I^{CT}(\epsilon) = \sum_{\gamma \in \Gamma^{Wheel}} W_\gamma(P_\epsilon^L, I)^{sing}$$

denote the one-loop counter-terms. Proposition C.2 can be formally written as [5]

$$e^{I^{(0)}_{naive}[L]/\hbar + I^{(1)}_{naive}[L] + O(\hbar)} = \lim_{\epsilon \to 0} e^{\hbar \frac{\partial}{\partial P_\epsilon^L}} e^{I/\hbar - I^{CT}(\epsilon)}.$$

Therefore

$$\begin{aligned}
&(O_1[L] + O(\hbar)) e^{I^{(0)}_{naive}[L]/\hbar + I^{(1)}_{naive}[L] + O(\hbar)} \\
&= (Q + \hbar\Delta_L) \lim_{\epsilon \to 0} e^{\hbar \frac{\partial}{\partial P_\epsilon^L}} e^{I/\hbar - I^{CT}(\epsilon)} = \lim_{\epsilon \to 0} e^{\hbar \frac{\partial}{\partial P_\epsilon^L}} (Q + \hbar\Delta_\epsilon) e^{I/\hbar - I^{CT}(\epsilon)} \\
&= \lim_{\epsilon \to 0} e^{\hbar \frac{\partial}{\partial P_\epsilon^L}} \big(\hbar^{-1}(\{I, I\}_\epsilon - \{I, I\}_0) + \Delta_\epsilon I \\
&\qquad - QI^{CT}(\epsilon) - \{I, I^{CT}(\epsilon)\}_\epsilon + O(\hbar)\big) e^{I/\hbar - I^{CT}(\epsilon)}.
\end{aligned}$$

It follows that

(C.3)

$$\begin{aligned}
O_1[L] = \lim_{\epsilon \to 0} \Bigg(&\sum_{\substack{\gamma:\text{one-loop connected},\\ v \in V(\gamma)}} W_{\gamma, v}(P_\epsilon^L, I, \{I, I\}_\epsilon - \{I, I\}_0) \\
&+ \sum_{\gamma: tree, v \in V(\gamma)} W_{\gamma, v}(P_\epsilon^L, I, \Delta_\epsilon I - QI^{CT}(\epsilon) - \{I, I^{CT}(\epsilon)\}_\epsilon) \Bigg)
\end{aligned}$$

Lemma C.4.

$$\begin{aligned}
QI^{CT}(\epsilon) = &- \{I, I^{CT}(\epsilon)\}_0 + \sum_{\substack{\gamma \in \Gamma^{Wheel}, \sharp E(\gamma) > 1,\\ e \in E(\gamma)}} W_\gamma(P_\epsilon^L, K_\epsilon - K_0, I)^{sing} \\
&+ (\Delta_\epsilon I)^{sing}.
\end{aligned}$$

Proof. It is easy to see that Q preserves the decomposition (C.1); hence

$$QI^{CT}(\epsilon) = Q\left(\sum_{\gamma \in \Gamma^{Wheel}} W_\gamma(P_\epsilon^L, I)^{sing}\right) = \sum_{\gamma \in \Gamma^{Wheel}} (QW_\gamma(P_\epsilon^L, I))^{sing}.$$

By the identity $(Q \otimes 1 + 1 \otimes Q)P_\epsilon^L = K_\epsilon - K_L$ and the classical master equation,

$$\left(\sum_{\gamma \in \Gamma^{Wheel}} QW_\gamma(P_\epsilon^L, I)\right)$$

$$= -\left\{ I, \sum_{\gamma\in\Gamma^{Wheel}} W_\gamma(P_\epsilon^L, I) \right\}_0 - \sum_{\substack{\gamma\in\Gamma^{Wheel}, \sharp E(\gamma)>1,\\ e\in E(\gamma)}} W_\gamma(P_\epsilon^L, K_0, I)$$

$$- \sum_{\gamma\in\Gamma^{Wheel}, e\in E(\gamma)} W_\gamma(P_\epsilon^L, K_L - K_\epsilon, I)$$

$$= -\left\{ I, \sum_{\gamma\in\Gamma^{Wheel}} W_\gamma(P_\epsilon^L, I) \right\}_0 + \sum_{\substack{\gamma\in\Gamma^{Wheel}, \sharp E(\gamma)>1,\\ e\in E(\gamma)}} W_\gamma(P_\epsilon^L, K_\epsilon - K_0, I)$$

$$+ \Delta_\epsilon I - \sum_{\gamma\in\Gamma^{Wheel}, e\in E(\gamma)} W_\gamma(P_\epsilon^L, K_L, I).$$

Since the last term is smooth as $\epsilon \to 0$, it follows that

$$QI^{CT}(\epsilon) = -\{I, I^{CT}(\epsilon)\}_0 + \sum_{\substack{\gamma\in\Gamma^{Wheel}, \sharp E(\gamma)>1,\\ e\in E(\gamma)}} W_\gamma(P_\epsilon^L, K_\epsilon - K_0, I)^{sing} + (\Delta_\epsilon I)^{sing}.$$

q.e.d.

Theorem C.5. *The one-loop anomaly O_1 is given by*

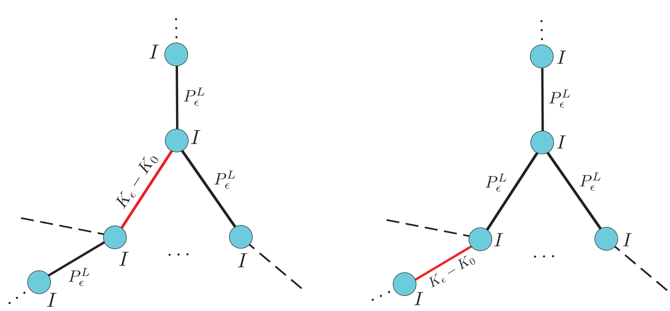
$$O_1 = \lim_{\epsilon\to 0} \sum_{\gamma\in\Gamma^{Wheel}, e\in E(\gamma)} W_\gamma(P_\epsilon^L, K_\epsilon - K_0, I)^{sm} + (\Delta_\epsilon I)^{sm}.$$

Proof. The term

$$\sum_{\substack{\gamma:\text{one-loop connected},\\ v\in V(\gamma)}} W_{\gamma,v}(P_\epsilon^L, I, \{I,I\}_\epsilon - \{I,I\}_0)$$

in equation (C.3) can be expressed as the sum of the following two types of Feynman weights:

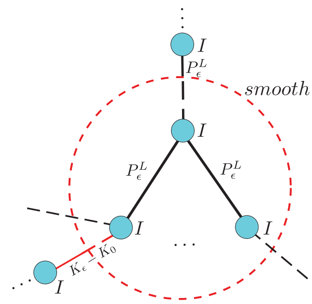
(C.4)



In the left picture $K_\epsilon - K_0$ is labeled on the wheel (the red edge) while in the right picture it is labeled on the external tree. It is not difficult to see that the right picture, together with the term

$$\sum_{\gamma:tree,v\in V(\gamma)} W_{\gamma,v}(P_\epsilon^L, I, \{I, I^{CT}(\epsilon)\}_0 - \{I, I^{CT}(\epsilon)\}_\epsilon)$$

contributes



whose limit vanishes as $\epsilon \to 0$. The theorem then follows easily from equation (C.3), Lemma C.4, and $O_1 = \lim_{L\to 0} O_1[L]$ (which kills all external trees).

q.e.d.

Equation 3.17 is now a graphic expression of Theorem C.5.

Appendix D. Chevalley-Eilenberg complex vs de Rham complex of jet bundles

The main objective of this section is to give an explicit description of the isomorphism in Proposition 3.35. We will also review modules over L_∞ algebras and the corresponding Chevalley-Eilenberg differential for the purpose of our discussion.

L_∞ algebras and their modules. Let us first recall the definition of L_∞ algebras.

Definition D.1. Let A be a commutative differential graded algebra and let $A^\sharp$ denote the underlying graded algebra. A curved L_∞ algebra over A consists of a locally free finitely generated graded $A^\sharp$-module V, together with a cohomological degree 1 and square zero derivation:

$$d : \widehat{\mathrm{Sym}}_{A^\sharp}(V^\vee[-1]) \to \widehat{\mathrm{Sym}}_{A^\sharp}(V^\vee[-1])$$

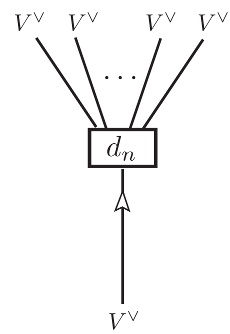
such that the derivation d makes $\widehat{\mathrm{Sym}}_{A^\sharp}(V^\vee[-1])$ into a dga over the dga A. Here $V^\vee$ denotes the $A^\sharp$-linear dual of V. We can decompose the derivation d into components:

$$d_n : V^\vee[-1] \to \mathrm{Sym}^n_{A^\sharp}(V^\vee[-1]), n \geq 0.$$

The structure maps of the curved L_∞ algebra V are defined by dualizing d_n with a degree shift:

$$l_n := d_n^* : \wedge^n V[n-2] \to V.$$

The components d_n of the derivation d can be represented by the following "corollas," which should be read from bottom to top: the bottom line denotes the input of d_n and the top lines denote the outputs.



Modules over L_∞ algebras are defined in a similar fashion:

Definition D.2. Let A and V be the same as in Definition D.1. An $A^\sharp$-module M is called a module over the L_∞ algebra V if there is a differential

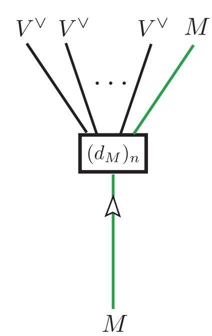
$$d_M : \widehat{\mathrm{Sym}}_{A^\sharp}(V^\vee[-1]) \otimes_{A^\sharp} M \to \widehat{\mathrm{Sym}}_{A^\sharp}(V^\vee[-1]) \otimes_{A^\sharp} M$$

making $\widehat{\mathrm{Sym}}_{A^\sharp}(V^\vee[-1]) \otimes_{A^\sharp} M$ a dg-module over $\widehat{\mathrm{Sym}}_{A^\sharp}(V^\vee[-1])$.

It is clear from the definition that the differential d_M is determined by its components

$$(d_M)_n : M \to \mathrm{Sym}^n_{A^\sharp}(V^\vee[-1]) \otimes_{A^\sharp} M,$$

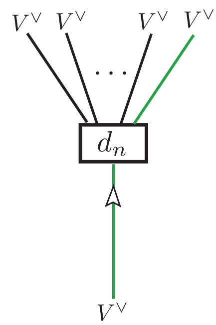
which we represent by the following picture:



Example D.3. $M = V^\vee$ has a naturally induced structure of an L_∞-module over V. We define the map d_M by the following composition:

$$\begin{aligned} M = V^\vee \to V^\vee[-1] &\xrightarrow{d_V} \widehat{\mathrm{Sym}}_{A^\sharp}(V^\vee[-1]) \\ &\xrightarrow{d_{dR}} \widehat{\mathrm{Sym}}_{A^\sharp}(V^\vee[-1]) \otimes V^\vee = \widehat{\mathrm{Sym}}_{A^\sharp}(V^\vee[-1]) \otimes M. \end{aligned} \tag{D.1}$$

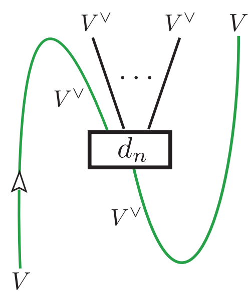
This differential can be represented as follows:



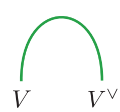
where the green lines denote the module M, and the black lines denote the components in $\widehat{\mathrm{Sym}}_{A^\sharp}(V^\vee[-1])$. Notice that the only difference between d_M and d_V is that there are green lines in the graphical representation of d_M. Thus it is clear that the identity $d_M^2 = 0$ follows from $d_V^2 = 0$, and that the effect of the operator d_{dR} in equation (D.1) is exactly "picking out the green line."

Example D.4. $N = V$. We define the differential d_N by the following graphics:

(D.2)



where the downward "elbow"



in equation (D.2) denotes the evaluation map

$$\langle -, - \rangle : V \otimes V^\vee \to A^\sharp$$

and the reversed "elbow" denotes the coevaluation map.

Again, $d_N^2 = 0$ follows from the identity $d_V^2 = 0$.

Proof of Proposition 3.35. Let X be a complex manifold, and let $\mathfrak{g}_X$ be the curved L_∞ algebra over $A = \mathcal{A}_X$ encoding the complex geometry of X. By the construction of $\mathfrak{g}_X$, there is an isomorphism of cochain complexes

$$\rho^* : \left(\mathcal{A}_X \otimes_{\mathcal{O}_X} \mathrm{Jet}_X^{hol}(\mathcal{O}_X), d_{D_X} \right) \xrightarrow{\sim} \left(C^*(\mathfrak{g}_X), d_{CE} \right).$$

We have the following proposition:

Proposition D.5. *The extension of the map* (3.27) *over* $\mathcal{A}_X$*:*

$$\mathfrak{g}_X^\vee[-1] \cong \mathcal{A}_X \otimes_{\mathcal{O}_X} \Omega_X^1 \xrightarrow{\partial_{dR}\circ T} \mathcal{A}_X \otimes_{\mathcal{O}_X} Jet_X^{hol}(\Omega_X^1)$$

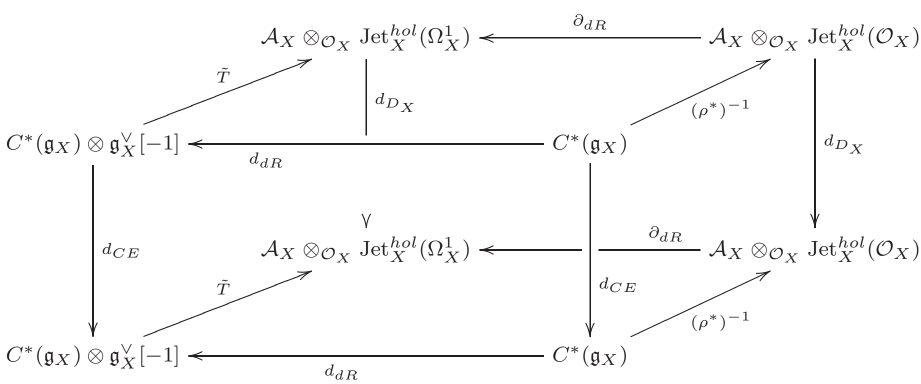
gives rise to an isomorphism of cochain complexes

$$\tilde{T} : \left(C^*(\mathfrak{g}_X) \otimes \mathfrak{g}_X^\vee[-1], d_{CE}\right) \xrightarrow{\sim} \left(\mathcal{A}_X \otimes_{\mathcal{O}_X} \ Jet\ _X^{hol}(\Omega_X^1), d_{D_X}\right).$$

Proof. It is clear from the definition of $\tilde{T}$ that the following diagram commutes:

$$\text{(D.3)} \qquad \begin{array}{ccc} C^*(\mathfrak{g}_X) & \xrightarrow{d_{dR}} & C^*(\mathfrak{g}_X) \otimes \mathfrak{g}_X^\vee[-1] \\ \big\downarrow{\scriptstyle (\rho^*)^{-1}} & & \big\downarrow{\scriptstyle \tilde{T}} \\ \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathrm{Jet}_X^{hol}(\mathcal{O}_X) & \xrightarrow{\partial_{dR}} & \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathrm{Jet}_X^{hol}(\Omega_X^1). \end{array}$$

Here d_{dR} is the de Rham differential of the algebra $C^*(\mathfrak{g}_X)$, and we have identified $C^*(\mathfrak{g}_X) \otimes \mathfrak{g}_X^\vee[-1]$ with 1-forms. Consider the following diagram:



It is straightforward to check that all the squares commute except the left vertical one:

- The commutativity of the top and the bottom squares follows from (D.3).
- The front vertical square commutes by the definition of the Chevalley-Eilenberg differential on $C^*(\mathfrak{g}_X) \otimes \mathfrak{g}_X^\vee[-1]$.
- The commutativity of the back vertical square follows from the fact that ∂_{dR} and d_{D_X} commute with each other.
- The right vertical square commutes by the definition of $\mathfrak{g}_X$.

Since d_{dR} is surjective, a simple diagram chase shows the commutativity of the left vertical square, which implies that the Chevalley-Eilenberg differential d_{CE} on $C^*(\mathfrak{g}_X) \otimes \mathfrak{g}_X^\vee[-1]$ is identified with d_{D_X} on $\mathcal{A}_X \otimes_{\mathcal{O}_X} \mathrm{Jet}_X^{hol}(\Omega_X^1)$ under $\tilde{T}$. q.e.d.

Now we prove the following proposition:

Proposition D.6. *Let K be the smooth homomorphism*

$$K : T_X \to C^\infty(X) \otimes_{\mathcal{O}_X} Jet_X^{hol}(T_X[-1])$$

such that

$$v(\alpha) = \langle K(v), T(\alpha) \rangle, \tag{D.4}$$

for all $\alpha \in \Omega_X^1, v \in T_X$. Then the extension of K over $C^*(\mathfrak{g}_X) \cong \mathcal{A}_X \otimes_{\mathcal{O}_X} Jet_X^{hol}(\mathcal{O}_X)$:

$$\tilde{K} : C^*(\mathfrak{g}_X) \otimes \mathfrak{g}_X \to \mathcal{A}_X \otimes_{\mathcal{O}_X} Jet_X^{hol}(T_X[-1])$$

is an isomorphism of cochain complexes. In particular, $d_{D_X} \circ \tilde{K} = \tilde{K} \circ d_{CE}$.

Proof. It is obvious from equation (D.4) that $\tilde{K}$ is both injective and surjective. We now show that $\tilde{K}$ commutes with differentials. After translating equation (D.2) into homomorphisms, it is clear that the Chevalley-Eilenberg differential d_{CE} on $C^*(\mathfrak{g}_X) \otimes \mathfrak{g}_X$ is given by the following composition:

$$\mathfrak{g}_X \xrightarrow{id \otimes coev} \mathfrak{g}_X \otimes \mathfrak{g}_X^\vee \otimes \mathfrak{g}_X \xrightarrow{id \otimes d_{CE} \otimes id} \mathfrak{g}_X \otimes \mathfrak{g}_X^\vee \otimes C^*(\mathfrak{g}_X) \otimes \mathfrak{g}_X$$
$$\xrightarrow{ev \otimes id \otimes id} C^*(\mathfrak{g}_X) \otimes \mathfrak{g}_X.$$

We pick local holomorphic coordinates $\{z^i\}$ on X. The image of $\{\widetilde{\partial_{z^i}}\}$ under d_{CE} is given by

$$\widetilde{\partial_{z^i}} \mapsto \widetilde{\partial_{z^i}} \otimes \widetilde{dz^j} \otimes \widetilde{\partial_{z^j}} \mapsto \widetilde{\partial_{z^i}} \otimes (\tilde{T}^{-1} \circ d_{D_X} \circ \tilde{T})(\widetilde{dz^j}) \otimes \widetilde{\partial_{z^j}}$$
$$\mapsto \langle \widetilde{\partial_{z^i}}, (\tilde{T}^{-1} \circ d_{D_X} \circ \tilde{T})(\widetilde{dz^j}) \rangle \otimes \widetilde{\partial_{z^j}}.$$

We have the following identities:

$$\begin{aligned}
&\langle \widetilde{\partial_{z^i}}, (\tilde{T}^{-1} \circ d_{D_X} \circ \tilde{T})(\widetilde{dz^j}) \rangle \otimes \widetilde{\partial_{z^j}} \\
\overset{(1)}{=}\ & \langle \tilde{K}(\widetilde{\partial_{z^i}}), (d_{D_X} \circ \tilde{T})(\widetilde{dz^j}) \rangle \otimes \widetilde{\partial_{z^j}} \\
\overset{(2)}{=}\ & \langle (d_{D_X} \circ \tilde{K})(\widetilde{\partial_{z^i}}), \tilde{T}(\widetilde{dz^j}) \rangle \otimes \widetilde{\partial_{z^j}} \\
\overset{(3)}{=}\ & \langle (\tilde{K}^{-1} \circ d_{D_X} \circ \tilde{K})(\widetilde{\partial_{z^i}}), \widetilde{dz^j} \rangle \otimes \widetilde{\partial_{z^j}} \\
=\ & (\tilde{K}^{-1} \circ d_{D_X} \circ \tilde{K})(\widetilde{\partial_{z^i}}),
\end{aligned}$$

where the identities (1) and (3) follow from equation (D.4) and identity (2) follows from the fact that d_{D_X} is a derivation with respect to the pairing $\langle -, - \rangle$. q.e.d.

It is clear that the wedge product of the map $\tilde{K}$ gives the desired isomorphism in Proposition 3.35.

References

[1] M. Alexandrov, A. Schwarz, O. Zaboronsky, and M. Kontsevich, *The geometry of the master equation and topological quantum field theory*, Internat. J. Modern Phys. A **12** (1997), no. 7, 1405–1429. MR1432574, Zbl 1073.81655

[2] S. Barannikov and M. Kontsevich, *Frobenius manifolds and formality of Lie algebras of polyvector fields*, Internat. Math. Res. Notices **4** (1998), 201–215. MR1609624, Zbl 0914.58004

[3] L. Baulieu, A. Losev, and N. Nekrasov, *Target space symmetries in topological theories. I*, J. High Energy Phys. **2** (2002), No. 21, 20. MR1900735

[4] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, Comm. Math. Phys. **165** (1994), no. 2, 311–427. MR1301851, Zbl 0815.53082

[5] K. Costello, *Renormalization and effective field theory*, Mathematical Surveys and Monographs, vol. 170, American Mathematical Society, Providence, RI, 2011. MR2778558, Zbl 1221.81004

[6] ———, *A geometric construction of the Witten genus, II*, available at `arXiv:1111.4234[math.QA]`.

[7] ———, *Notes on supersymmetric and holomorphic field theories in dimensions 2 and 4*, Pure Appl. Math. Q. **9** (2013), no. 1, 73–165, available at `arXiv:1112.0816[math.QA]`. MR3126501, Zbl 1299.14013

[8] K. Costello and O. Gwilliam, *Factorization algebras in quantum field theory*, available at `http://www.math.northwestern.edu/~costello/renormalization`.

[9] K. Costello and S. Li, *Quantum BCOV theory on Calabi-Yau manifolds and the higher genus B-model*, available at `arXiv:1201.4501[math.QA]`.

[10] E. Frenkel and A. Losev, *Mirror symmetry in two steps: A-I-B*, Comm. Math. Phys. **269** (2007), no. 1, 39–86. MR2274462, Zbl 1127.14035

[11] D. Friedan, *Nonlinear models in* $2+\varepsilon$ *dimensions*, Ann. Physics **163** (1985), no. 2, 318–419. MR0811072, Zbl 0583.58010

[12] R. Grady and O. Gwilliam, *One-dimensional Chern-Simons and the* $\hat{A}$*-genus*, Algebr. Geom. Topol. **14** (2014), no. 4, 2299–2377, available at `arXiv:1110.3533[math.QA]`.Zbl 1305.81114

[13] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics. MR1288523, Zbl 0836.14001

[14] O. Gwilliam and Y. Shen, *personal communications*.

[15] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, *Mirror symmetry*, Clay Mathematics Monographs, vol. 1, American Mathematical Society, Providence, RI, 2003. With a preface by Vafa. MR2003030, Zbl 1044.14018

[16] M. Kapranov, *Rozansky-Witten invariants via Atiyah classes*, Compositio Mathematica **115** (1999), no. 01, 71-113. MR1671737, Zbl 0993.53026

[17] A. Kapustin, *Chiral de Rham complex and the half-twisted sigma-model*, available at `arXiv:hep-th/0504074`.

[18] C. Li, S. Li, and K. Saito, *Primitive forms via polyvector fields*, available at `arXiv:1311.1659[math.AG]`.

[19] S. Li, *BCOV theory on the elliptic curve and higher genus mirror symmetry*, available at `arXiv:1112.4063[math.QA]`.

[20] ———, *Renormalization and mirror symmetry*, SIGMA **8** (2012), 101, available at `arXiv:1212.4234[math.AG]`. MR3007258, Zbl 1276.14084

[21] ______, *Variation of Hodge structures, Frobenius manifolds and Gauge theory*, available at arXiv:1303.2782[math.AG].
[22] A. Losev, *personal communications*.
[23] N. Rozenblyum, *personal communications*.
[24] A. Schwarz, *Geometry of Batalin-Vilkovisky quantization*, Comm. Math. Phys. **155** (1993), no. 2, 249–260. MR1230027, Zbl 0786.58017
[25] Y. Shen, *N=2 holomorphic sigma model*, thesis (2012). MR3192890
[26] C. Vafa, *Topological Landau-Ginzburg models*, Modern Phys. Lett. A **6** (1991), no. 4, 337–346. MR1093562
[27] E. Witten, *Mirror manifolds and topological field theory*, Essays on mirror manifolds, Int. Press, Hong Kong, 1992, pp. 120–158. MR1191422, Zbl 0904.58009
[28] ______, *Chern-Simons gauge theory as a string theory*, The Floer memorial volume, Progr. Math., vol. 133, Birkhäuser, Basel, 1995, pp. 637–678. MR1362846

School of Mathematical Sciences
Wu Wen-Tsun Key Laboratory of Mathematics
University of Science and Technology of China
Hefei, China
and
Department of Mathematics
The Chinese University of Hong Kong
Shatin, Hong Kong
E-mail address: qli@math.cuhk.edu.hk

YMSC
Tsinghua university
Beijing, China
E-mail address: sli@math.tsinghua.edu.cn