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TOPOLOGICALLY SLICE KNOTS OF SMOOTH CONCORDANCE ORDER TWO

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Abstract

The existence of topologically slice knots that are of infinite order in the knot concordance group followed from Freedman's work on topological surgery and Donaldson's gauge theoretic approach to four-manifolds. Here, as an application of Ozsváth and Szabó's Heegaard Floer theory, we show the existence of an infinite subgroup of the smooth concordance group generated by topologically slice knots of concordance order two. In addition, no nontrivial element in this subgroup can be represented by a knot with Alexander polynomial one.

1. Introduction.

In [**7**] Fox and Milnor defined the smooth knot concordance group $\mathcal{C}$. Their proof that $\mathcal{C}$ is infinite quickly yields an infinite family of distinct elements of order two. Results of Murasugi [**27**] and Tristram [**41**] demonstrated that $\mathcal{C}$ also contains a free summand of infinite rank. This work culminated in Levine's construction [**22**] of a surjective homomorphism $\phi\colon \mathcal{C} \to \mathcal{G}$, where $\mathcal{G}$ is an algebraically defined group isomorphic to the infinite direct sum $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$.

Classical surgery theory allowed Levine to prove that ϕ is an isomorphism in high (odd) dimensions. The first distinction between classical and high-dimensional concordance was seen in the work of Casson and Gordon [**2**], who showed that the kernel of ϕ is nontrivial; this was followed by a proof by Jiang [**21**] that $\ker(\phi)$ contains a subgroup isomorphic to $\mathbb{Z}^\infty$. In [**23**] it was shown that $\ker(\phi)$ also contains a subgroup isomorphic to $\mathbb{Z}_2^\infty$.

The work of Donaldson [**4**] and Freedman [**8**, **9**] on smooth and topological 4–manifolds, respectively, revealed further subtlety present in low-dimensional concordance. One can define a concordance group $\mathcal{C}^{top}$

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in the topological, locally flat, category. The distinction between the smooth and topological categories is highlighted by considering the kernel of the natural surjection $\mathcal{C} \to \mathcal{C}^{top}$. This kernel is generated by topologically slice knots, and we denote it $\mathcal{C}_{TS}$. To underscore the importance of $\mathcal{C}_{TS}$ it should be mentioned that a single non-trivial element in $\mathcal{C}_{TS}$ implies the existence of a smooth 4–manifold homeomorphic, but not diffeomorphic, to $\mathbb{R}^4$ [**11**, Exercise 9.4.23]. Several people, including Akbulut and Casson, observed that the results of Donaldson and Freedman can be used to produce non-trivial elements in $\mathcal{C}_{TS}$ (see [**3**]), but until recently little was known about the structure of $\mathcal{C}_{TS}$. Using techniques developed by Donaldson [**4**] and later enhanced by Fintushel-Stern [**6**] and Furuta [**10**], Endo [**5**] proved that $\mathcal{C}_{TS}$ contains a subgroup isomorphic to $\mathbb{Z}^\infty$ (see also [**15**, **16**, **17**] for other constructions of infinite rank free subgroups). Techniques derived from Heegaard Floer theory and Khovanov homology (specifically the Rasmussen invariant [**40**]) were used to show that $\mathcal{C}_{TS}$ contains a *summand* isomorphic to $\mathbb{Z}^3$ [**24**, **25**, **26**]. Recently that work has been superseded by work of Hom [**18**] which applies a deep analysis of the structure of Heegaard Floer complexes to construct a summand isomorphic to $\mathbb{Z}^\infty$.

With the abundance of 2–torsion in $\mathcal{C}$, one might expect that $\mathcal{C}_{TS}$ likewise has such torsion. However, producing torsion classes in $\mathcal{C}_{TS}$ is quite difficult since one needs a manifestly smooth invariant to detect them. Many of the known techniques for analyzing $\mathcal{C}_{TS}$, however, fail at detecting torsion classes (for instance, the Ozsváth-Szabó [**34**] or Rasmussen [**40**] concordance invariants). Our main result shows that like the concordance group, $\mathcal{C}_{TS}$ has an abundance of 2–torsion.

Theorem 1. *$\mathcal{C}_{TS}$ contains a subgroup isomorphic to $\mathbb{Z}_2^\infty$.*

We conjecture that, in line with Hom's result, a summand isomorphic to $\mathbb{Z}_2^\infty$ exists, but current tools seem insufficient to prove this.

Freedman's work [**8**, **9**] implied that all knots of Alexander polynomial one are topologically slice, and these knots provided all the early examples of nontrivial elements in $\mathcal{C}_{TS}$. However, in [**16**] it was shown that $\mathcal{C}_{TS}$ in fact contains a subgroup isomorphic to $\mathbb{Z}^\infty$ with no nontrivial element represented by a knot with Alexander polynomial one. Here we extend this to 2–torsion. Let $\mathcal{C}_\Delta$ denote the subgroup of $\mathcal{C}$ generated by knots with Alexander polynomial one.

Theorem 2. *The subgroup from Theorem* 1 *can be chosen so that no nontrivial member is representable by a knot with Alexander polynomial one. In particular, the group $\mathcal{C}_{TS}/\mathcal{C}_\Delta$ contains a subgroup isomorphic to $\mathbb{Z}_2^\infty$.*

This theorem can be strengthened by replacing the subgroup of knots generated by Alexander polynomial one knots with the subgroup generated by knots with determinant one.

To prove these theorems we consider knots $K_{J,n}$ as illustrated in Figure 1. These knots are defined to be the boundaries of surfaces built by adding two bands to a disk as shown: the bands are tied in knots J and $-J$ and have n and $-n$ full twists, where $n > 0$. An important special case occurs when U is the unknot, whereby $K_{U,1}$ is the figure eight knot. We have the following easy proposition:

Proposition 1.1. *$K_{J,n}$ is negative amphicheiral ($K_{J,n} = -K_{J,n}$); in particular, $2K_{J,n} = 0 \in \mathcal{C}$. If J_1 and J_2 are concordant, then $K_{J_1,n}$ and $K_{J_2,n}$ are concordant.*

The amphicheirality of $K_{J,n}$ can be demonstrated just as for the case $J = U$. Indeed, an isotopy to $-K_{J,n}$ is obtained by pulling the bottom band through the rectangular region and then rotating the knot 180° about a vertical axis running down the center of the page. The second part of the lemma follows from the fact that satellite operations descend to concordance, and $K_{J,n}$ is a two-fold satellite operation with companions J and $-J$. The proposition allows for the immediate construction of elements of order at most two in $\mathcal{C}_{TS}$.

Corollary 1.2. *For U the unknot, $2(K_{J,n} \# K_{U,n}) = 0 \in \mathcal{C}$. If J is topologically slice, then the knot $K_{J,n} \# K_{U,n}$ is topologically slice; that is, $K_{J,n} \# K_{U,n} \in \mathcal{C}_{TS}$.*

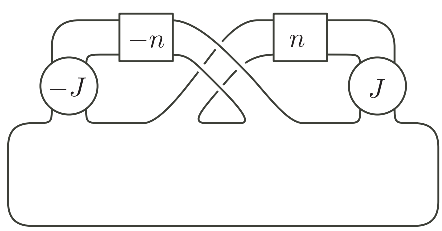

Figure 1

Let D denote the untwisted Whitehead double of the right-handed trefoil knot, $T_{2,3}$, and let D_k denote kD. The knots $K_{D_k,n} \# K_{U,n}$ provide the subgroups appearing in Theorem 1 and Theorem 2.

Theorem 3. *There exists an infinite set of pairs of positive integers $\{(k, n)\}$ with the property that the corresponding set of knots $\{K_{D_k,n} \# K_{U,n}\}$ generates a subgroup of $\mathcal{C}_{TS}$ and of $\mathcal{C}_{TS}/\mathcal{C}_{\Delta}$ as described in Theorems 1 and 2.*

The proof of Theorem 3 is presented in Section 3 after necessary background is given in Section 2. The proof depends on a detailed analysis of the Heegaard Floer d–invariants of the branched cover of S^3

branched over $K_{D_k,n}$. That analysis occupies Sections 4, 5 and 6. Some of the most technical work has been placed in appendices.

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2. Preliminary constructions

2.1. Algebraic slicing obstructions. The proofs of our main results are based on considering two-fold branched covers of S^3 over $K_{J,n}$, which we denote $M(K_{J,n})$. According to [**1**], $M(K_{J,n})$ has a surgery description as illustrated in Figure 2, in which the meridian μ is labeled for later reference. In the diagram, J^r denotes the orientation reverse of J, and the meridian of the surgery curve is oriented consistently with a choice of orientation for that curve. (In general, if a link is formed from the Hopf link by tying a local knot K_1 in one component, K_2 in the second, and then performing n_1 and n_2 surgery on the link, we denote the resulting manifold $S^3_{n_1,n_2}(K_1,K_2)$.) If J is reversible, then $M(K_{J,n})$ has the surgery description $S^3_{-2n,2n}(-2J,2J)$.

From this surgery description, a quick calculation yields a computation of the homology of $M(K_{J,n})$. In particular, $H_1(M(K_{J,n}))$ is a cyclic group of order $4n^2+1$. Notice that given the choice of generator μ of $H_1(M(K_{J,n}))$, the identification with a cyclic group is canonical. In particular, this observation along with Poincaré duality permits us to identify $H^2(M(K_{J,n}))$ with $H^2(M(K_{U,n}))$ for all J. For emphasis and for later reference we state this as a proposition.

Proposition 2.1. *The choice of surgery description of $M(K_{J,n})$ provides a canonical isomorphism $H^2(M(K_{J,n})) \cong H_1(M(K_{U,n})) \cong \mathbb{Z}_{4n^2+1}$.*

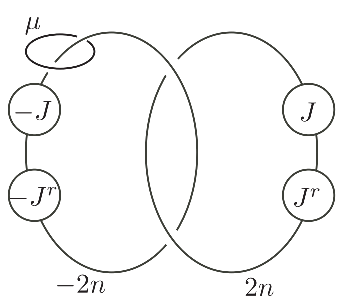


Figure 2

As a special case, we note that $M(K_{U,n})$ is given by $(4n^2+1)/2n$–surgery on the unknot: $M(K_{U,n}) = L(4n^2+1, 2n)$.

If a knot K is slice with slice disk F^2, then $M(K)$ bounds the two-fold branched cover of B^4 branched over the slice disk, $W(F^2)$. In this case we have the following from [**2**].

Proposition 2.2. *The homology groups $H_i(W(F^2), \mathbb{Z}_2) = 0$ for $i \geq 1$. The image I of the restriction map $H^2(W(F^2)) \to H^2(M(K))$ is a subgroup of order satisfying $|I|^2 = |H_1(M(K))|$. Furthermore, I is self-annihilating with respect to the linking form. (Via duality, we can view the linking form, usually defined on $H_1(M(K))$, as a form on $H^2(M(K))$.)*

2.2. Slicing obstructions from Heegaard Floer theory. Heegaard Floer theory associates a (filtered homotopy class of) chain complex $CF^\infty(M, \mathfrak{s})$ to a 3–manifold M with Spin^c structure $\mathfrak{s}$. For a manifold X, the set of Spin^c structures, Spin$^c(X)$, is in bijection with elements in $H^2(X)$, though not canonically so. However, associated to each $\mathfrak{s} \in$ Spin$^c(X)$, there is a first Chern class, $c_1(\mathfrak{s}) \in H^2(X)$, and in the case that $H^2(X, \mathbb{Z}_2) = 0$, the map:

$$c_1 : \mathrm{Spin}^c(X) \to H^2(X)$$

provides a bijection that is natural with respect to the transitive action of $H^2(X)$ on both sides and with respect to pull-back; that is

1) $c_1(\mathfrak{s} + \alpha) = c_1(\mathfrak{s}) + 2\alpha$ for all $\alpha \in H^2(X)$, and
2) $c_1(i^*\mathfrak{s}) = i^*c_1(\mathfrak{s})$ for an embedding $i : Y \to X$ with trivial normal bundle. In particular, for the inclusion of a codimension zero submanifold $Y \subset X$, or for $Y \subset \partial X$, we have $c_1(\mathfrak{s}|_Y) = c_1(\mathfrak{s})|_Y$.

Thus, in cases in which $H^2(X, \mathbb{Z}_2) = 0$, via the Chern class we can denote Spin^c structures by $\mathfrak{s}_\alpha$ for $\alpha \in H^2(X)$. There is an involution on the set of Spin^c structures called conjugation; the conjugate of $\mathfrak{s}$ is denoted $\overline{\mathfrak{s}}$ and one has $\overline{\mathfrak{s}_\alpha} = \mathfrak{s}_{-\alpha}$.

As described in greater detail in Section 4, there is an invariant $d(M, \mathfrak{s})$, called the *correction term*, defined in terms of the filtered homotopy type of $CF^\infty(M, \mathfrak{s})$. It satisfies the following properties.

1) $d(-M, \mathfrak{s}) = -d(M, \mathfrak{s})$.
2) $d(M_1 \# M_2, \mathfrak{s}_1 \# \mathfrak{s}_2) = d(M_1, \mathfrak{s}_1) + d(M_2, \mathfrak{s}_2)$.
3) $d(M, \overline{\mathfrak{s}}) = d(M, \mathfrak{s})$.

The following theorem from [**31**] provides the obstruction we will use to show that knots are not smoothly slice. (The use of d as a slicing obstruction first appeared in [**26**], where it was applied only for the Spin structure. In [**12**, **20**] it was used in conjunction with a careful analysis of Spin^c structures to study concordance.)

Proposition 2.3. *Suppose $(W, \mathfrak{t})$ is a Spin^c four-manifold satisfying $H_i(W, \mathbb{Q}) = 0, i > 0$, and $M = \partial W$. Then $d(M, \mathfrak{t}|_M) = 0$.*

Note. In the case that M^3 is constructed as $-n$–surgery on an oriented knot $K \subset S^3$, there is the following enumeration of Spin^c structures on M, parameterized by integers m with $-n/2 \leq m < n/2$ (see [**33**, Section 4] for details). If W denotes the four-ball with a two-handle added along K with framing $-n < 0$, we let $\mathfrak{t}_m$ denote the Spin^c structure on W satisfying $\langle c_1(\mathfrak{t}_m), [S]\rangle + n = 2m$, where $[S]$ is the generator of $H_2(W)$ represented by an oriented Seifert surface for K, capped off with the core of the two-handle. We denote by $\mathfrak{s}_m$ the restriction of $\mathfrak{t}_m$ to M. This is well-defined whether n is odd or even. The Poincare dual of $c_1(\mathfrak{s}_m)$ satisfies $\mathrm{PD}(c_1(\mathfrak{s}_m)) = 2m[\mu]$, where $[\mu] \in H_1(M)$ is the class represented by the meridian of K.

3. Main theorem.

In Appendix C we use a theorem of Iwaniec to obtain a number theoretic result.

Proposition 3.1. *There exists an infinite set $\mathcal{N}$ of positive integers greater than one such that for all $n \in \mathcal{N}$, $4n^2+1$ is square free and $4n^2+1$ is a product of at most two primes. Furthermore, for each $m, n \in \mathcal{N}$, $4m^2+1$ and $4n^2+1$ are relatively prime.*

The main results of this paper are consequences of the following theorem.

Theorem 3.2. *For each $n \in \mathcal{N}$ there is a positive integer k_n having the following property: If $n \in \mathcal{N}$ and L is any knot with $|H_1(M(L))|$ relatively prime to $4n^2+1$, then $K_{D_{k_n},n} \# K_{U,n} \# L$ is not slice.*

Most important, as an immediate corollary we have the result that implies Theorems 1, 2, and 3 of the introduction.

Corollary 3.3. *For all nonempty finite subsets $\mathcal{N}' \subset \mathcal{N}$,*

$$\sum_{n \in \mathcal{N}'} (K_{D_{k_n},n} \# K_{U,n}) \notin \mathcal{C}_\Delta.$$

In particular, the set of knots $\{K_{D_{k_n},n} \# K_{U,n}\}$ generate a subgroup isomorphic to $\mathbb{Z}_2^\infty$ in $\mathcal{C}_{ts}/\mathcal{C}_\Delta$.

Proof Corollary 3.3. Suppose that $\sum_{n \in \mathcal{N}'} (K_{D_{k_n},n} \# K_{U,n})$ is concordant to a knot K with Alexander polynomial one. Then we have $\sum_{n \in \mathcal{N}'} (K_{D_{k_n},n} \# K_{U,n}) \# -K$ is slice. Let m be the least $n \in \mathcal{N}'$ and let $\mathcal{N}''$ be the set $\mathcal{N}'$ with m removed. We can break up the connected sum of knots as

$$(K_{D_{k_m},m} \# K_{U,m}) \# \left(\sum_{n \in \mathcal{N}''} (K_{D_{k_n},n} \# K_{U,n}) \# -K \right).$$

At this point we can complete the proof by applying Theorem 3.2 with $L = \sum_{n \in \mathcal{N}''} (K_{D_{k_n},n} \# K_{U,n}) \# -K$. q.e.d.

3.1. Proof of Theorem 3.2. The rest of this section presents the proof of Theorem 3.2, calling upon results from later sections as needed. The choice of k_n will be described in the context of the proof.

Abbreviate $K_{D_k,n} \# K_{U,n}$ by $K_{n,k}$. Assuming that $K_{n,k} \# L$ is slice, the manifold $M(K_{n,k}) \# M(L)$ bounds a rational homology ball W. Since the orders of $H_1(M(K_{n,k}))$ and $H_1(M(L))$ are relatively prime, it follows that the image of $H^2(W)$ in $H^2(M(K_{n,k}) \# M(L)) \cong (\mathbb{Z}_{4n^2+1} \oplus \mathbb{Z}_{4n^2+1}) \oplus H^2(M(L))$ contains a subgroup of the form $\mathcal{M} \oplus 0$ where $\mathcal{M} \subset \mathbb{Z}_{4n^2+1} \oplus \mathbb{Z}_{4n^2+1}$ is a metabolizer for the linking form on $H_1(M(K_{n,k}))$. With this we can prove the following.

Lemma 3.4. *If $K_{n,k} \# L$ is slice, then for some metabolizer $\mathcal{M}$ of the linking form on $H_1(M(K_{n,k}))$ and for all $(z_1, z_2) \in \mathcal{M}$,*

$$d(M(K_{D_k,n}), \mathfrak{s}_{z_1}) + d(M(K_{U,n}), \mathfrak{s}_{z_2}) = 0.$$

Proof. It is immediate that $d(M(K_{D_k,n}), \mathfrak{s}_{z_1}) + d(M(K_{U,n}), \mathfrak{s}_{z_2}) + d(M(L), \mathfrak{s}_0) = 0$. Notice that since L is assumed to be concordant to $-K_{n,k}$, which is of order two, L is also of order 2. Because $2L$ is slice, $2M(L)$ bounds a $\mathbb{Z}_2$–homology ball Z. The Spin structure on Z restricts to the Spin structure on $2M(L)$. Thus, the Spin^c structure $\mathfrak{s}_0 \oplus \mathfrak{s}_0$ on $M(L) \# M(L)$ extends to Z. It follows that $2d(M(L), \mathfrak{s}_0) = 0$. q.e.d.

We now must consider metabolizers for the linking form on $(\mathbb{Z}_{4n^2+1})^2$.

Lemma 3.5. *For a fixed non-degenerate linking form on $\mathbb{Z}_N$, with N square-free, each metabolizer for the double of this form on $(\mathbb{Z}_N)^2$ is generated by an element $(1, b)$ where $1 + b^2 \equiv 0 \mod N$.*

Proof. Recall first that a non-degenerate linking form on $\mathbb{Z}_N$ is given by an element $\alpha \in \mathbb{Z}_N$: $lk(x, y) \equiv x\alpha y \mod N$, where α and N are relativity prime.

Since $(\mathbb{Z}_N)^2$ is of rank two, any metabolizer $\mathcal{M}$ is of rank at most two, so is generated by two elements, $\{(a, b), (c, d)\}$. Using Gauss-Jordan elimination, we see it is generated by a pair of elements $\{(a, b), (0, c)\}$. If c is nonzero it would have self-linking 0, which is impossible for a non-degenerate form on $\mathbb{Z}_N$ with N square-free.

Thus $\mathcal{M}$ is generated by a single element (a, b), so (a, b) is of order N. If either a or b were divisible by some prime factor of N, then some multiple of (a, b) would be of the form $(0, c)$ or $(c, 0)$ with c nonzero. But again, the existence of such an element is ruled out by N being square-free and the form being non-degenerate. Since a must be relatively prime to N, some multiple of (a, b) is of the form $(1, b')$, and clearly $b' \neq 0$. In fact, since $(1, b')$ is in the metabolizer $\mathcal{M}$, one has $1 + (b')^2 \equiv 0 \mod N$, as desired. q.e.d.

Combining Lemmas 3.4 and 3.5 yields the following.

Lemma 3.6. *If $K_{n,k} \# L$ is slice, then for some b satisfying $1+b^2 \equiv 0 \mod 4n^2+1$ and for all x, $d(M(K_{D_k,n}), \mathfrak{s}_x) + d(M(K_{U,n}), \mathfrak{s}_{bx}) = 0$.*

Notice that in this statement the subscripts on the Spin^c structures, x and bx, are cohomology classes; the cohomology of the spaces are identified using Proposition 2.1.

For our purposes, a change of signs will be convenient, as follows.

Lemma 3.7. *If $K_{n,k} \# L$ is slice, then there is some b satisfying $b^2 \equiv 1 \mod 4n^2+1$ such that for all x, $d(M(K_{D_k,n}), \mathfrak{s}_x) = d(M(K_{U,n}), \mathfrak{s}_{bx})$.*

Proof. The knot $K_{U,n}$ is of order two: $K_{U,n} \# K_{U,n}$ is slice. Thus, the previous argument shows that there is some b' satisfying $1+b'^2 \equiv 0 \mod 4n^2+1$ such that for all x, $d(M(K_{U,n}), \mathfrak{s}_x) + d(M(K_{U,n}), \mathfrak{s}_{b'x}) = 0$. Replacing x with bx from the previous lemma yields $d(M(K_{U,n}), \mathfrak{s}_{bx}) + d(M(K_{U,n}), \mathfrak{s}_{b'bx}) = 0$. The rest is arithmetic along with a renaming of variables. q.e.d.

Completion of the proof of Theorem 3.2

According to Proposition 6.7, there is a specific Spin^c structure $\mathfrak{s}_\alpha$ such that for all k with $0 \le k < n/2$,

$$d(M(K_{D_k,n}, \mathfrak{s}_\alpha)) - d(M(K_{U,n}, \mathfrak{s}_\alpha)) = -2k.$$

Applying Lemma 3.7, for each k and some b satisfying $1+b^2 \equiv 0 \mod 4n^2+1$, we have

$$d(M(K_{U,n}), \mathfrak{s}_{b\alpha}) - d(M(K_{U,n}), \mathfrak{s}_\alpha) = -2k.$$

Since $4n^2+1$ is the product of at most two primes, there are at most four values of $b \mod 4n^2+1$ for which $b^2 \equiv -1 \mod 4n^2+1$. Thus, the expression on the left of the equality can have at most four distinct values. As long as $n \ge 9$ the set of integers in the interval $0 \le k < n/2$ contains at least five elements, so we can choose k so that the equality is violated. Any such choice can serve as k_n.

4. Heegaard Floer complexes

The computation of the d–invariants of interest depends upon a detailed understanding of related Heegaard Floer complexes. The main result in this section is Theorem 4.2, the *refiltering theorem*, which describes the chain complex associated to the meridian of a knot K within the manifold $S^3_{-N}(K)$ in terms of the chain complex associated to K within S^3.

4.1. Three-manifold complexes. We let $\mathbb{F}$ denote the field with two elements. As mentioned earlier, given a 3–manifold M with Spin^c structure $\mathfrak{s}$, there is an associated $\mathbb{Z}$–filtered $\mathbb{Q}$–graded complex $CF^\infty(M, \mathfrak{s})$. This complex is a free, finitely generated $\mathbb{F}[U, U^{-1}]$–module, which is well-defined up to filtered chain homotopy equivalence. The filtration of $CF^\infty(M,\mathfrak{s})$ by subcomplexes is induced by a natural filtration of $\mathbb{F}[U,U^{-1}]$ by powers of U. More precisely, we can regard $\mathbb{F}[U,U^{-1}]$ as an (infinitely generated) $\mathbb{F}[U]$–module in the obvious way. As such, it has an exhaustive $\mathbb{Z}$-indexed filtration by (free) $\mathbb{F}[U]$–submodules

$$... \subset U^k\mathbb{F}[U] \subset U^{k-1}\mathbb{F}[U] \subset U^{k-2}\mathbb{F}[U] \subset ...,$$

and this filtration induces a $\mathbb{Z}$–filtration of $CF^\infty(M,\mathfrak{s})$ by subcomplexes. Thus the filtration level of a chain in $CF^\infty(M,\mathfrak{s})$, regarded as a sum of Laurent polynomials in the basis elements, is given by the negative of the minimum power of U which appears in this polynomial. The action of U clearly lowers filtration level by one. It lowers grading by two.

Added notation permits the simple representation of subcomplexes; for instance, we denote the subcomplex consisting of elements of filtration level at most n by $CF^\infty(M,\mathfrak{s})_{\{i\leq n\}}$. With this we can define several associated complexes,

$$CF^-(M,\mathfrak{s}) = CF^\infty(M,\mathfrak{s})_{\{i<0\}},$$

$$CF^+(M,\mathfrak{s}) = CF^\infty(M,\mathfrak{s})/CF^\infty(M,\mathfrak{s})_{\{i<0\}},$$

and

$$\widehat{CF}(M,\mathfrak{s}) = CF^\infty(M,\mathfrak{s})_{\{i\leq 0\}}/CF^\infty(M,\mathfrak{s})_{\{i<0\}}.$$

There are corresponding homology groups, $HF^-(M,\mathfrak{s})$, $HF^+(M,\mathfrak{s})$ and $\widehat{HF}(M,\mathfrak{s})$.

There will also be situations in which we must shift the gradings of elements in these chain complexes. For instance, we will write $CF^+(M,\mathfrak{s})[\epsilon]$ for the same complex as $CF^+(M,\mathfrak{s})$, except with the homological grading of any element increased by ϵ; that is,

$$CF^+_*(M,\mathfrak{s})[\epsilon] = CF^+_{*-\epsilon}(M,\mathfrak{s}),$$

for all $*$.

Definition 4.1. The *d–invariant* $d(M,\mathfrak{s})$ is given by

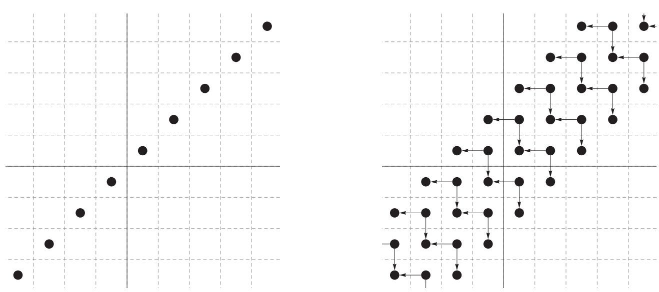
$$\min\{gr(\alpha) \mid \alpha \neq 0 \in HF^+(M,\mathfrak{s}) \text{ and } \alpha \in \text{Image } U^n \text{ for all } n > 0\},$$

where $gr(\alpha)$ is the homological grading.

4.2. Knot complexes. A knot $K \subset M$ induces a second $\mathbb{Z}$–filtration of the complex $CF^\infty(M,\mathfrak{s})$, which thus becomes a $\mathbb{Q}$–graded, $\mathbb{Z}\oplus\mathbb{Z}$–filtered complex. The U action respects the second filtration, lowering this filtration by one as well. This doubly filtered complex is denoted $CFK^\infty(M,K,\mathfrak{s})$, and again there are associated subcomplexes such as $CFK^\infty(M,K,\mathfrak{s})_{\{i\leq m, j\leq n\}}$. As in the 3–manifold case, there are quotient complexes $CFK^+(M,K,\mathfrak{s}) = CFK^\infty(M,K,\mathfrak{s})/CFK^\infty(M,K,\mathfrak{s})_{\{i<0\}}$ and $\widehat{CFK}(M,K,\mathfrak{s}) = CFK^\infty(M,K,\mathfrak{s})_{\{i\leq 0\}}/CFK^\infty(M,K,\mathfrak{s})_{\{i<0\}}$; ignoring the j filtration yields the corresponding complexes for $(M,\mathfrak{s})$.

Figure 3 illustrates the complexes for the unknot and the $(2,5)$–torus knot in S^3. (For alternating knots K, $CFK^\infty(S^3,K)$ is determined simply from the Alexander polynomial [**30**].) The dots represent elements in a filtered $\mathbb{F}$–basis and the line segments indicate components of the boundary operator. Sometimes we will not need to include arrows on the segments; the fact that the boundary map cannot increase either filtration and $\partial^2=0$ will make the direction unambiguous in most of the examples we consider. The gradings are not indicated in the diagram; the coordinates in the diagram correspond to the filtration, as follows: the vertical and horizontal axes in bold separate elements of filtration levels -1 and 0. That is, the dot just above and to the right of the origin has filtration level $(0,0)$. The action of U shifts the diagram down and to the left by one.

Convention. In all the cases we consider, $CFK^\infty(M,K,\mathfrak{s})$ is filtered chain homotopy equivalent to $C\otimes_{\mathbb{F}}\mathbb{F}[U,U^{-1}]$ for some finite $\mathbb{Z}\oplus\mathbb{Z}$–filtered $\mathbb{F}$–complex C. We will simplify our diagrams and illustrate only C, leaving out all of its U translates.



$CFK^\infty(S^3,U)$ $\qquad$ $CFK^\infty(S^3,T_{2,5})$

Figure 3

4.3. Gradings. To this point we have not described how the homological grading is determined. Rather than review this aspect of the theory, we refer the interested reader to [**31**, **35**] for definitions and details. For our purposes, the following elementary observation will be particularly useful: the value of $d(M,\mathfrak{s})$ can be used to determine the gradings of elements in $CFK^\infty(M,K,\mathfrak{s})$. We illustrate this with an important example.

In the special case of S^3 there is only one Spin^c structure, denoted $\mathfrak{s}_0$. We have $HF^+(S^3,\mathfrak{s}_0)=\mathbb{F}[U,U^{-1}]/U\mathbb{F}[U]$ and by definition $d(S^3,\mathfrak{s}_0)=0$. For example, in the complex $CFK^+(S^3,T_{2,5},\mathfrak{s}_0)$ (constructed from the complex illustrated on the right in Figure 3 by quotienting by all elements to the left of the vertical axis), we see that the non-trivial homology class with least grading is represented by the cycles living in filtration levels $(0,2),(1,1)$, and $(2,0)$. Thus, all of these have grading 0.

4.4. Meridians of knots in surgered manifolds. Let $S^3_{-N}(K)$ denote the manifold constructed as $-N$ surgery on $K \subset S^3$ and let μ denote the meridian of K, viewed as a knot in $S^3_{-N}(K)$. The work of [**14**] can be extended to show that for each Spin^c structure $\mathfrak{s}_m$, the complex $CFK^\infty(S^3_{-N}(K),\mu,\mathfrak{s}_m)$ is isomorphic to $CFK^\infty(S^3,K)$, but endowed with a different $\mathbb{Z}\oplus\mathbb{Z}$–filtration and an overall shift in the homological grading. We state the result for a knot in a general 3–manifold.

Notation Notice that until now, Spin^c structures were denoted $\mathfrak{s}_\alpha$, where $\alpha \in H^2(M)$. Here they have been denoted $\mathfrak{s}_m$ with m an integer (viewed, modulo N, in $\mathbb{Z}/N\mathbb{Z}$), according to the convention described in the note at the end of Section 2.2.

Theorem 4.2 (Refiltering Theorem). *Suppose $N \geq 2g(K)$. For m in the interval*

$$\lceil(-N+1)/2\rceil \leq m \leq \lfloor N/2 \rfloor,$$

the complex $CFK^\infty(Y^3_{-N}(K),\mu,\mathfrak{s}_m)$ is isomorphic to $CFK^\infty(Y^3,K)[\epsilon_1]$ as an unfiltered complex, where $[\epsilon_1]$ denotes a grading shift that depends only on m and N. Given a generator $\{[x,i,j]\}$ for $CFK^\infty(Y^3,K)$, the $\mathbb{Z}\oplus\mathbb{Z}$ filtration level of the same generator, viewed as a chain in $CFK^\infty(Y^3_{-N}(K),\mu,\mathfrak{s}_m)$, is given by:

$$\mathcal{F}_m([x,i,j]) = \begin{cases} [i,i] & \text{if } j > i+m, \\ [j-m, j-m-1] & \text{if } j \leq i+m. \end{cases}$$

Before discussing its proof, we illustrate this theorem in Figure 4, which shows for all $N \geq 8$ the complexes $CFK^\infty(S^3_{-N}(K),\mu,\mathfrak{s}_m)$ for $K=U$ and $K=-T_{2,5}$, with $-3 \leq m \leq 4$. We show only the $\mathbb{F}$–subcomplex that generates the full complex over $\mathbb{F}[U,U^{-1}]$.

Proof. The theorem refines [**14**, Theorem 4.1] in two directions:

1) [**14**, Theorem 4.1] determines the $\mathbb{Z}$–filtered chain homotopy type of $\widehat{CFK}(S^3_{-N}(K),\mu,\mathfrak{s}_m)$. Here we seek to understand the $\mathbb{Z}\oplus\mathbb{Z}$–filtered chain homotopy type of $CFK^\infty(Y^3_{-N}(K),\mu,\mathfrak{s}_m)$.
2) [**14**, Theorem 4.1] applies for $N \gg 0$. We wish to show that $N = 2g(K)$ suffices.

The first refinement is an immediate extension of the proof from [**14**], so we do not belabor the details here. To begin, we note that the difference between S^3 and a general 3–manifold is merely notational. The key idea from [**14**] was to observe that with the addition of another basepoint, the natural Heegaard diagram for $-N$–framed surgery on K could be made to represent the knot $\mu \subset Y_{-N}(K)$. The proof of

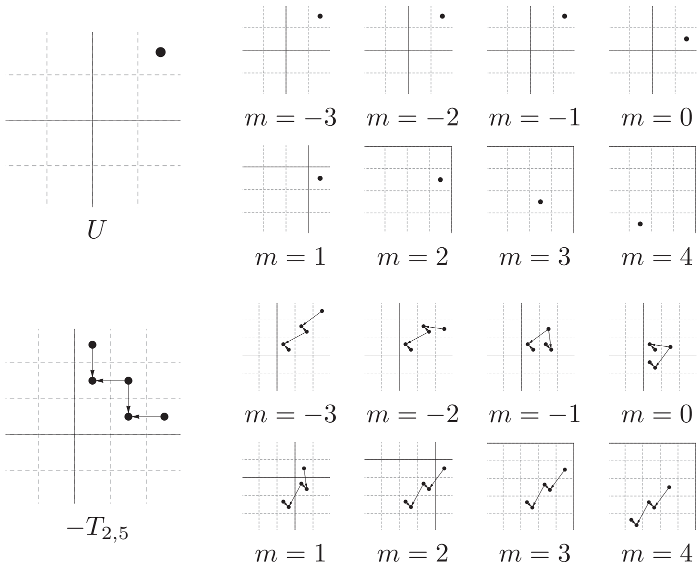


Figure 4

[**33**, Theorem 4.1] shows that the $\mathbb{Z}$–filtered chain homotopy type of $CF^\infty(Y_{-N}(K), \mathfrak{s}_m)$ is determined by that of $CFK^\infty(Y, K)$. This implies that the chain homotopy type of $CF^-(Y_{-N}(K), \mathfrak{s}_m)$, $CF^+(Y_{-N}(K), \mathfrak{s}_m)$, and $\widehat{CF}(Y_{-N}(K), \mathfrak{s}_m)$ are also determined by $CFK^\infty(Y, K)$, as they are sub, quotient, and subquotient complexes of the filtration, respectively. Now the meridian $\mu \subset Y_{-N}(K)$ induces an additional $\mathbb{Z}$–filtration of any of these complexes, and [**14**, Theorem 4.1] determined that in the case of $\widehat{CF}(Y_{-N}(K), \mathfrak{s}_m)$, the additional $\mathbb{Z}$–filtration consists of two steps:

$$0 \subseteq CFK^\infty(Y, K)_{\{i \geq 0, j=m\}} \subseteq CFK^\infty(Y, K)_{\{\min(i, j-m)=0\}},$$

where the subquotient on the right was identified with $\widehat{CF}(Y_{-N}(K), \mathfrak{s}_m)$ by [**33**, Theorem 4.1]. Strictly speaking, the proof of [**14**, Theorem 4.1] only dealt with the case of positive framed surgery explicitly, leaving the case of negative framings to the reader. The analogous proof for negative framings yields the two-step filtration above, and the extension to CFK^∞ follows easily from the same proof. (It is worth noting that the formula from [**14**, Theorem 4.1] was actually for the filtration induced by μ^r, the meridian of K with reversed orientation. The formula above is for the meridian with its standard orientation.) To be more precise, [**33**, Theorem 4.1] identifies $CF^\infty(Y_{-N}(K), \mathfrak{s}_m)$ with $CFK^\infty(Y, K)$ via a chain map which was denoted Φ. This isomorphism of chain complexes respects the $\mathbb{F}[U, U^{-1}]$–module structure of both complexes, and hence one of the $\mathbb{Z}$–filtrations. The additional $\mathbb{Z}$–filtration on $CF^\infty(Y_{-N}(K), \mathfrak{s}_m)$ induced by μ can be determined in exactly the same manner as it was determined for the case of $\widehat{CF}(Y_{-N}(K), \mathfrak{s}_m)$ in [**14**], yielding the statement of the theorem. In both cases, the key lemma is [**14**, Lemma 4.2], which identifies the $\mathbb{Z}$–filtration induced on any given i =constant slice in $CF^\infty(Y_{-N}(K), \mathfrak{s}_m)$ with a two step filtration as above.

For the second refinement, recall that the proof of [**14**, Theorem 4.1] relies on making the surgery parameter large enough so that an entire Spin^c equivalence class of generators for $Y_{-N}(K)$ is *supported in the winding region* (by definition, we say that a generator is supported in the winding region if it is represented by a k–tuple of intersection points which contains a point in the region shown in [**14**, Figure 13]). This is achieved by a pigeonhole argument: there are only finitely many Spin^c equivalence classes that can be represented by the finitely many generators *not* supported in the winding region, and increasing N increases the number of Spin^c structures without bound. Once we have an entire Spin^c equivalence class supported in the winding region, we can appeal to the technique of "moving the basepoint." In the present context this means moving the placement of the meridian and nearby collection of basepoints throughout the winding region; see [**14**, Theorem 4.3]. This technique allows us to use the single Spin^c equivalence class of intersection points which is supported in the winding region to represent all $|H_1(Y)| \cdot N$ different Spin^c structures on $Y_{-N}(K)$ (for a manifold with $b_1(Y) > 0$, $|H_1(Y)|$ should be replaced by the number of Spin^c structures on Y represented by the diagram). Thus the question is reduced to finding a topological interpretation for the number of Spin^c classes represented by generators which are not supported in the winding region. We will henceforth refer to such generators as *exterior*.

To achieve a bound for the number of Spin^c classes represented by exterior generators, we use a particular Heegaard diagram which is adapted to a Seifert surface for K with genus g. A similar Heegaard diagram appears in the proof of the adjunction inequality [**33**, Theorem 5.1]; such a diagram is constructed explicitly in [**32**, Lemma 7.3] and [**28**, Proof of Theorem 2.1]. The diagram consists of a quadruple,

$$(\Sigma_k, \vec{\alpha} = \{\alpha_1, ..., \alpha_k\}, \vec{\beta} = \{\beta_1, ..., \beta_{k-1}, \mu, \lambda\}, \{w \cup z\}),$$

where $(\Sigma, \vec{\alpha}, \vec{\beta} \setminus \mu)$ and $(\Sigma, \vec{\alpha}, \vec{\beta} \setminus \lambda)$ are Heegaard diagrams for $Y_0(K)$ and Y, respectively, and $\{w \cup z\}$ specifies K on the latter diagram. The key features of the diagram are that

- There is a domain $\mathcal{P}$ with

$$\partial\mathcal{P} = \alpha_k \cup \lambda$$

  such that $\mathcal{P} \cup \{\text{Disk bounded by } \alpha_k\}$ is isotopic to the chosen Seifert surface.

- The only α curves which intersect $\mathcal{P}$ are α_k and $\alpha_1, ..., \alpha_{2g}$, where g, as above, is the genus of $K(=$ genus of $\mathcal{P})$.

Now we observe that the diagram

$$(\Sigma_k, \vec{\alpha} = \{\alpha_1, ..., \alpha_k\}, \vec{\beta} = \{\beta_1, ..., \beta_{k-1}, \lambda^{-N}\}, \{w\})$$

specifies $Y_{-N}(K)$, where λ^{-N} is a simple closed curve isotopic to the resolution of N parallel copies of the reversed meridian μ^r and one copy of λ. Furthermore, with an additional point z' the diagram specifies the knot $\mu \subset Y_{-N}(K)$. As above, the generators of $CFK^\infty(Y_{-N}(K), \mu)$ arising from this diagram are split according to whether they are supported in the winding region or are exterior. The exterior generators are characterized by the fact that the point of intersection occurring on λ^{-N} lies outside the winding region (recall that a generator is a k–tuple of intersection points between α and β curves, with each α and β curve appearing exactly once; thus λ^{-N} is used exactly once by any k–tuple comprising a generator). The exterior generators are in bijection with generators for the Heegaard diagram of $Y_0(K)$ (the diagram with λ as the last curve). Our bound of $2g(K)$ in the theorem will be attained if we can argue that the total number of Spin^c equivalence classes represented by the exterior points is less than $|H_1(Y)| \cdot 2g$. This follows from the key properties of our Heegaard diagram. Indeed, recall the first Chern class formula [**32**, Proposition 7.5]:

$$\langle c_1(\mathfrak{s}_w(\mathbf{x})), [\mathcal{P}]\rangle = e(\mathcal{P}) + 2\sum_{x_i \in \mathbf{x}} n_{x_i}(\mathcal{P}). \tag{4.1}$$

Here, $\mathbf{x}$ is a k–tuple generating a Heegaard Floer complex, $[\mathcal{P}] \in H_2$ is the second homology class obtained by capping off the boundary components of a periodic domain, $e(\mathcal{P})$ is the Euler measure of $\mathcal{P}$ (which agrees with the Euler characteristic for periodic domains with all multiplicities zero or one) and $n_{x_i}(\mathcal{P})$ is the average of the local multiplicities of $\mathcal{P}$ in the four regions surrounding an intersection point x_i. For our particular Heegaard diagram for $Y_0(K)$, the right-hand side of 4.1 becomes:

$$-2g + 2\#\{x_i \in \text{interior}(\mathcal{P})\} + 2,$$

where $-2g$ is the Euler characteristic of $\mathcal{P}$. The additional $+2$ term comes from the fact that α_k and λ do not intersect and must each contain an $x_i \in \mathbf{x}$. Since α_k and λ are on the boundary of $\mathcal{P}$, each of these two x_i have $n_{x_i}(\mathcal{P}) = 1/2$. Finally, the fact that there are only $2g$ other α curves which intersect $\mathcal{P}$ and that any k–tuple comprising a generator must use one of these α curves for the intersection point $x_i \subset \lambda$ implies that

$$0 \leq 2\#\{x_i \in \text{interior}(\mathcal{P})\} \leq 2(2g-1),$$

thus showing that

$$-2g + 2 \leq \langle c_1(\mathfrak{s}_w(\mathbf{x})), [\mathcal{P}]\rangle \leq 2g.$$

Now the fact that $\langle c_1(\mathfrak{s}_w(\mathbf{x})), [\mathcal{P}]\rangle$ is an even integer which vanishes if $c_1(\mathfrak{s}_w(\mathbf{x}))$ is torsion implies that there are at most $|H_1(Y)| \cdot 2g$ distinct Spin^c equivalence classes represented on the Heegaard diagram for $Y_0(K)$, and hence the same bound exists for the number of exterior intersection points. This completes the proof. q.e.d.

5. The complex $CFK^\infty(S^3_{-N}(-2D_k), 2D_k)$

In general, the computation of the d–invariant of surgery on a knot $K \subset Y$ from $CFK^\infty(Y, K, \mathfrak{s})$ can be rather challenging; identifying patterns among the values that arise for various values of $\mathfrak{s}$ is even more subtle. If the surgery coefficient is appropriately large, however, there are significant simplifications. This section describes the general theory and demonstrates that in our setting the simplifications that arise from the large surgery assumption do apply.

To be more specific, in [**33**, Theorem 4.1] it was shown that the complex $CFK^\infty(S^3, K)$ determines $CF^+(S^3_N(K), \mathfrak{s}_m)$ for $N \geq 2g(K) - 1$, with a similar result proved for null-homologous knots in arbitrary 3–manifolds. In [**38**, Theorem 4.1] this was generalized to rationally null-homologous knots, in which case $CF^+(Y_N(K), \mathfrak{s}_m)$ depends on the complexes $CFK^\infty(Y^3, K, \mathfrak{s}'_{m'})$ for specified classes $\mathfrak{s}'_{m'}$. However, the generalization of [**38**] did not specify how large the framing parameter had to be in order to apply the result. Rather, it simply showed that for sufficiently large framings such a formula exists, and then a more general formula was proved which holds for arbitrary framings in terms of a mapping cone complex. In our situation we will apply a special case of the results of [**38**], taking advantage of the fact that $Y = S^3_{-2n}(-2D_k)$, and that we are performing $2n$–surgery on a knot formed as the connected sum of a knot in S^3 with the meridian of $-2D_k$. While we utilize the full mapping cone complex, our surgery parameters are chosen so that they will be large enough for the simpler formula to hold. This will manifest itself in a collapse of the mapping cone complex to a single term. In general, "large" should be taken to mean: "large in comparison to the Thurston norm of the complement."

Here is the statement of the result we need. The exact correspondence between the Spin^c structures $\mathfrak{s}_m$ and $\mathfrak{s}'_{m'}$ is implicit in the proof but is not needed in our application of the theorem.

Theorem 5.1. *Let $K_2 \subset Y = S^3_{-N}(K_1)$ be a knot of the form $\mu \# K'_2$ where μ is the meridian of K_1 and K'_2 is a knot in S^3. For any $N \geq \max(2g(K'_2) + 2, 2g(K_1))$, there is an enumeration of Spin^c structures on $Y_N(K_2)$, $\{\mathfrak{s}_m\}_{-N^2/2 \leq m \leq N^2/2}$, such that $CF^+(Y_N(K_2), \mathfrak{s}_m)$ is isomorphic to*

$$CFK^\infty(S^3_{-N}(K_1), K_2, \mathfrak{s}'_{m'})/CFK^\infty(S^3_{-N}(K_1), K_2, \mathfrak{s}'_{m'})_{\{i<0, j<m\}}[\epsilon].$$

Elements in the above quotient with $i = 0, j \leq m$ and $i \leq 0, j = m$ are at filtration level 0 in $CF^+(Y_N(K_2), \mathfrak{s}_m)$; these represent $\widehat{CF}(Y_N(K_2), \mathfrak{s}_m)$. The induced map U lowers filtration level by 1. The grading shift, ϵ, is a function of m and N, and in particular, the grading shift does not depend on K_2'.

Applying this theorem to the relevant manifolds yields the following corollary:

Corollary 5.2. *For any $0 \leq k < n/2$, there is an enumeration of Spin^c structures on $M(K_{D_k,n})$, $\{\mathfrak{s}_m\}_{-2n^2 \leq m \leq 2n^2}$ for which the complex $CF^+(M(K_{D_k,n}), \mathfrak{s}_m)$ is isomorphic to*

$$\frac{CFK^\infty(S^3_{-2n}(-2D_k), \mu\#2D_k, \mathfrak{s}'_{m'})}{CFK^\infty(S^3_{-2n}(-2D_k), \mu\#2D_k, \mathfrak{s}'_{m'})_{\{i<0, j<m\}}}[\epsilon],$$

with filtration, grading shift, and $\mathbb{F}[U]$–module structure as in Theorem 5.1.

Proof. The manifold $M(K_{D_k,n})$ is obtained by performing $2n$–surgery on $\mu\#2D_k \subset S^3_{-2n}(-2D_k)$. Thus we need only verify that

$$2n \geq \max(2g(2D_k) + 2, 2g(-2D_k)),$$

provided that $0 \leq k < n/2$. Both $2D_k$ and $-2D_k$ have genus $2k$, being the connected sum of $2k$ copies of the Whitehead double, a genus one knot. q.e.d.

The rest of this section is devoted to proving Theorem 5.1.

5.1. Heegaard diagrams, Spin^c structures, homology and surgery. Our computation of $HF^+(M, \mathfrak{s})$ relies on results of [**38**], in which the general problem of computing the Heegaard Floer homology of rational surgery on a knot in a rational homology sphere is studied. Although the manifolds we consider are in some respects fairly simple, in order to apply [**38**] it is essential to review some of the foundations.

The manifold M we are considering is formed by surgery on a link $(K', K) \subset S^3$ constructed from the Hopf link by placing local knots in each component. More specifically, M is given by $-N$ surgery on K' followed by N surgery on K. Thus, our approach to computing the Heegaard Floer homology of M is to view it as formed by performing N surgery on knot K, viewed as a knot in $Y = S^3_{-N}(K')$. We begin by considering surgery on the Hopf link, in which case $Y = S^3_{-N}(U) = -L(N, 1)$ and $M = L(N^2 + 1, N)$. We then move to the more general case, encompassing the situation in which the components are knotted.

5.2. Lens space Heegaard diagram. As a starting point, we consider lens spaces $-L(N, 1)$. On the left in Figure 5 is a doubly pointed Heegaard diagram for $Y = -L(2, 1)$, which we use to illustrate the general construction.

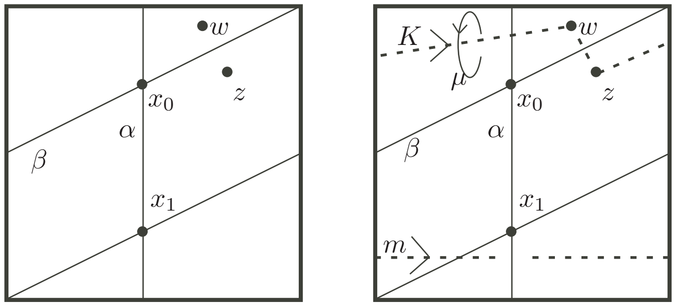


Figure 5. Doubly pointed Heegaard diagram

In the lens space, the surface $\Sigma = T^2$ represented by this diagram bounds solid tori U_α and U_β in which the curves α and β bound embedded disks, respectively. If we let η_α be an arc from w to z on Σ missing α that is pushed into U_α (except at its endpoints) and let η_β be an arc from z to w on Σ missing β pushed into U_β, the union of η_α and η_β forms an oriented knot K in Y. Notice that once isotoped into U_α, K represents the core of U_α.

The meridian to K we denote μ. The complement of K in U_α is homeomorphic to $T^2 \times I$ with $H_1(U_\alpha \setminus K)$ generated by μ and the curve m illustrated on the right in Figure 5. Notice that $H_1(Y \setminus K)$ is generated by μ and m, subject to the relations $Nm - \mu = 0$. This is shown on the left in Figure 6, which illustrates the solid torus U_α. Note that in the figure, K has not yet been isotoped into U_α.

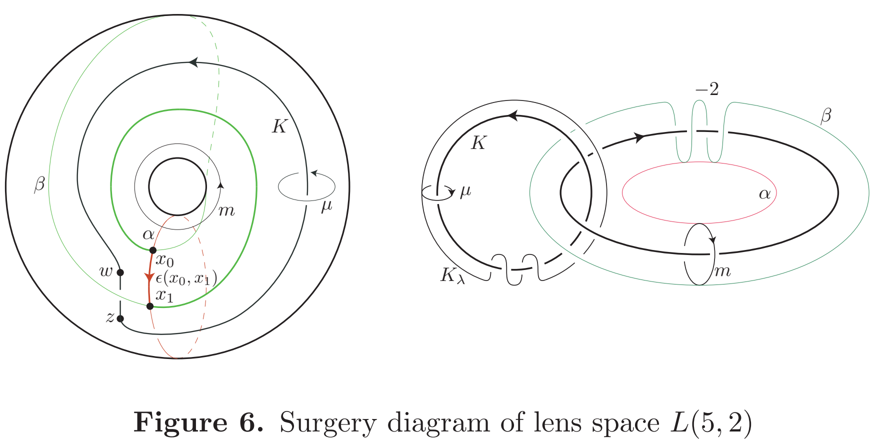


Figure 6. Surgery diagram of lens space $L(5, 2)$

5.3. Relative Spin^c structures. Associated to each intersection point, x_0 or x_1 in the figures and $\{x_0, x_1, \ldots, x_{N-1}\}$ for general $-L(N, 1)$, there is a relative Spin^c structure $\mathfrak{s}_{w,z}(x_i) \in \mathrm{Spin}^c(Y, K)$. The differences between these satisfy

(5.1)
$$\mathfrak{s}_{w,z}(x_{i+1}) - \mathfrak{s}_{w,z}(x_i) = \mathrm{PD}[\epsilon(x_i, x_{i+1})] \in H^2(Y \setminus \nu K, \partial) \cong H^2(Y, K)$$

where
$$\epsilon(x_i, x_{i+1}) \in \frac{H_1(\Sigma \setminus \{z, w\})}{\text{Span } \vec{\alpha} + \text{Span } \vec{\beta}} \cong H_1(Y \setminus \nu K)$$
is the class represented by a path that travels from x_i to x_{i+1} along α and then from x_{i+1} to x_i along β. As seen from the figure, this curve is isotopic to m in $Y \setminus \nu K$. (In all these equations, $i \in \mathbb{Z}/N\mathbb{Z}$.)

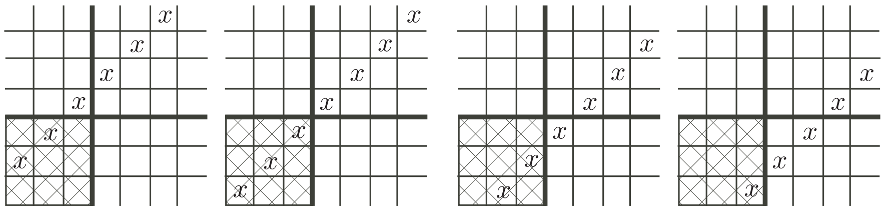
There is a natural map, called the *filling map*, $G_{Y,K}$: $\text{Spin}^c(Y, K) \to \text{Spin}^c(Y)$ which satisfies
$$G_{Y,K}(\xi + k) - G_{Y,K}(\xi) = \iota(k),$$
where $k \in H^2(Y, K)$. If K^r denotes the orientation reverse of K, then
$$G_{Y,K}(\xi) - G_{Y,K^r}(\xi) = -\text{PD}[K].$$

Comment. As described by Turaev [**42**], Spin^c structures on a closed manifold correspond to equivalence classes of nonvanishing vector fields, where two are equivalent if homotopic off a ball. In the case that $K \subset Y$ is an oriented knot, a relative Spin^c structure corresponds to a nonvanishing vector field on $Y \setminus \nu K$ which points outwards on the boundary. The map G is given in terms of a canonical extension of a vector field from $Y \setminus \nu K$ to Y. See [**38**, Section 2.2] for a further discussion.

5.4. $Y_N(K)$**.** We are interested in performing N surgery on K. To be clear about framings, in Figure 6 a push-off of K, K_λ, is illustrated. The surgered manifold, $Y_N(K)$ is built by removing a neighborhood of K and replacing it with a solid torus so that K_λ bounds a meridianal disk in that solid torus. Note that $H_1(Y_N(K))$ is generated by μ and m subject to the relations $Nm - \mu = 0$ and $m + N\mu = 0$. For instance in the illustrated case, with $N = 2$, we get $H_1(Y_N(K)) = \mathbb{Z}/5\mathbb{Z}$. (In fact, $Y_2(K) = L(5, 2)$.) In general, for N surgery on K in $-L(N, 1)$ we end up with $L(N^2 + 1, N) = -L(N^2 + 1, N)$.

5.5. The structure of $CFK^\infty(Y, K)$**.** To each relative Spin^c structure $\xi \in \text{Spin}^c(Y, K)$, there is an associated doubly filtered chain complex $CFK^\infty(Y, K, \xi)$ generated by triples $[\mathbf{x}, i, j]$ satisfying
$$\mathfrak{s}_{w,z}(\mathbf{x}) + (i - j) \cdot \text{PD}[\mu] = \xi. \tag{5.2}$$
Here, $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is an intersection point of the Lagrangian tori in the symmetric product of a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$, and $i, j \in \mathbb{Z}$. For instance, in the case of Y a lens space as above, we have illustrated examples in Figure 7. In the figure, x can denote any of the x_i coming from the Heegaard diagram in Section 5.2. The value of ξ is written beneath each of the complexes. (The shading in these diagrams becomes relevant later.)

Every relative Spin^c structure is of the form $\mathfrak{s}_{w,z}(x_0) + kPD[m]$ for some $k \in \mathbb{Z}$, and this provides a correspondence between $\text{Spin}^c(Y, K)$ and $\mathbb{Z}$. Since $\mu = Nm$, the set of relative Spin^c structures associated to



$\mathfrak{s}_{w,z}(x) - \mathrm{PD}[\mu]$ $\qquad$ $\mathfrak{s}_{w,z}(x)$ $\qquad$ $\mathfrak{s}_{w,z}(x) + \mathrm{PD}[\mu]$ $\qquad$ $\mathfrak{s}_{w,z}(x) + 2\mathrm{PD}[\mu]$

Figure 7. $CFK^\infty(Y, K)$

each x_i is a coset of $N\mathbb{Z} \subset \mathbb{Z}$. Also, since $\mathfrak{s}_{w,z}(x_i) - \mathfrak{s}_{w,z}(x_0) = iPD[m]$, for different x_i the sets are distinct cosets.

5.6. Enumerating relative Spin^c structures for the manifolds at hand. We now move to our particular setting, in which $Y = S^3_{-N}(K')$ is a manifold constructed as surgery on a knot $K' \subset S^3$ and $K \subset Y$ is a knot of the form $\mu \# J$, where μ is the meridian of K' and $J \subset S^3$. Relative Spin^c structures will play a central role in the surgery formula which will be used to compute the Floer homology of $Y_N(K)$. We discuss them now.

The manifold $Y_N(K)$ contains a knot, which we also denote K, induced by the surgery: K is simply the core of the solid torus. There are two surjective filling maps to consider:

$$G_{Y_N(K),K} \colon \mathrm{Spin}^c(Y_N(K), K) \to \mathrm{Spin}^c(Y_N(K))$$

and

$$G_{Y,K} \colon \mathrm{Spin}^c(Y, K) \to \mathrm{Spin}^c(Y).$$

There is a canonical diffeomorphism

$$Y_N(K) \setminus \nu K \to Y \setminus \nu K$$

that provides an identification between $\mathrm{Spin}^c(Y_N(K), K)$ and $\mathrm{Spin}^c(Y, K)$, with which we subsequently conflate elements in the two sets. We will primarily think in terms of $\mathrm{Spin}^c(Y, K)$, and the important point will be to understand the images of this $H^2(Y, K)$–torsor under $G_{Y_N(K),K}$ and $G_{Y,K}$.

Since $H^2(Y, K) \cong \mathbb{Z}$, we can (non-canonically) pick an affine isomorphism which enumerates the relative Spin^c structures on $Y \setminus \nu K$ by integers (or elements in $H^2(Y, K)$). For our purposes, it will be most convenient to pick an enumeration that is compatible with the previously established affine isomorphism $\mathrm{Spin}^c(Y) \cong \mathbb{Z}/N\mathbb{Z} \cong H^2(Y)$ implicit in the statement of the refiltering theorem; that is, we first enumerate elements in $\mathrm{Spin}^c(S^3_{-N}(K'), \mu)$ to be compatible with our enumeration of Spin^c structures on $S^3_{-N}(K')$, and we then use this to induce an enumeration for $\mathrm{Spin}^c(S^3_{-N}(K'), \mu\#J)$.

To make this precise, recall that the refiltering theorem determines the $\mathbb{Z}\oplus\mathbb{Z}$–filtered homotopy type of $CF^\infty(S^3_{-N}(K'), \mu, \mathfrak{s}_m)$, where $\mathfrak{s}_m$ indicates a specific *absolute* Spin^c structure on $S^3_{-N}(K') = Y$ and $m \in \mathbb{Z}/N\mathbb{Z}$. As in the last section, associated to a *relative* Spin^c structure $\xi \in \mathrm{Spin}^c(Y, \mu)$, we obtain a complex $CFK^\infty(Y, \mu, \xi)$ generated by triples satisfying (5.2). We pick an identification of $\mathrm{Spin}^c(Y, \mu)$ with $\mathbb{Z}$ so that the m-th relative Spin^c structure, which we hereafter denote $\underline{\mathfrak{t}}_m \in \mathrm{Spin}^c(Y, \mu)$, or occasionally by $m \in \mathbb{Z}$, has infinity complex given by the refiltering theorem; that is,

$$CFK^\infty(Y, \mu, \underline{\mathfrak{t}}_m) = CFK^\infty(Y, \mu, \mathfrak{s}_m), \quad \text{for } \lceil(-N+1)/2\rceil \le m \le \lfloor N/2\rfloor,$$

where on the left we have the infinity complex associated to a relative Spin^c structure and on the right the filtered complex associated to the absolute Spin^c structure labeled $\mathfrak{s}_m$ by the refiltering theorem. Equation (5.2) then determines the infinity complex for the remaining relative Spin^c structures (outside the interval of the theorem) by the equation:

$$CFK^\infty(Y, \mu, \underline{\mathfrak{t}}_{m+kN}) = CFK^\infty(Y, \mu, \underline{\mathfrak{t}}_m)\{0, -k\},$$

where $\{0, -k\}$ indicates that we have shifted the j–filtration down by k. Finally, we observe that having picked an affine isomorphism

$$\mathrm{Spin}^c(Y, \mu) \cong H^2(Y, \mu) \cong \mathbb{Z},$$

we subsequently obtain an affine isomorphism

$$\mathrm{Spin}^c(Y, \mu\#J) \cong H^2(Y, \mu\#J) \cong \mathbb{Z},$$

via the natural isomorphism $H^2(Y, \mu) \cong H^2(Y, \mu\#J)$.

With our convention in hand, we hereafter regard relative Spin^c structures as integers, or as elements in $H^2(Y, K)$. Similarly, we regard absolute Spin^c structures on Y or $Y_N(K)$ as elements in $H^2(Y) \cong \mathbb{Z}/N\mathbb{Z}$ or $H^2(Y_N(K) \cong \mathbb{Z}/(N^2+1)\mathbb{Z}$, respectively. Our convention is compatible with the filling maps, in the sense that they are now identified with the corresponding restriction maps on cohomology:

$$H^2(Y_N(K), K) \to H^2(Y_N(K))$$

and

$$H^2(Y, K) \to H^2(Y).$$

To illustrate these principles, and for use in the next section, let $\mathfrak{s}$ be some fixed Spin^c structure on $Y_N(K)$. Now define $S(\mathfrak{s}) = G^{-1}_{Y_N(K),K}(\mathfrak{s})$. We have that $S(\mathfrak{s}) = \{\underline{\mathfrak{t}}_{k+(N^2+1)j}\}$ for $j \in \mathbb{Z}$ and some k, $0 \le k \le N^2$. Moreover, for each fixed value of k, there exists an $\mathfrak{s} \in \mathrm{Spin}^c(Y_N(K))$ such that $\underline{\mathfrak{t}}_k \in S(\mathfrak{s})$.

5.7. The mapping cone. $HF^+(Y_N(K), \mathfrak{s})$ can be computed as the homology of a mapping cone complex built from $CFK^\infty(Y, K)$ via a construction of Ozsváth and Szabó which we now recall. We use the notation of [**37**, **38**] and refer the reader there for more details.

Letting $S = S(\mathfrak{s})$ be as above, there are complexes

$$\mathbb{A}^+_{\mathfrak{s}}(Y, K) = \oplus_{\xi \in S} A^+_\xi(Y, K),$$

$$\mathbb{B}^+_{\mathfrak{s}}(Y, K) = \oplus_{\xi \in S} B^+_\xi(Y, K).$$

Here

$$A^+_\xi(Y, K) = CFK^\infty(Y, K, \xi)_{\{\max(i,j) \geq 0\}},$$

and

$$B^+_\xi(Y, K) = CF^+(Y, G_{Y,K}(\xi)).$$

We can write

$$CF^+(Y, G_{Y,K}(\xi)) = CFK^\infty(Y, K, \xi)_{\{i \geq 0\}},$$

where in the term on the right of the equality, K has provided a filtration of $B^+_\xi(Y, K)$.

There are maps:

$$v^+_\xi\colon\ A^+_\xi(Y, K) \to B^+_\xi(Y, K)$$

and

$$h^+_\xi\colon\ A^+_\xi(Y, K) \to B^+_{\xi + \mathrm{PD}[K_\lambda]}(Y, K).$$

The map v is given by the projection map onto the quotient complex of $A^+_\xi(Y, K)$ consisting of triples $[\mathbf{x}, i, j]$ with $i \geq 0$, the so-called vertical complex. The map h is more subtle. Interchanging the roles of i and j replaces K with K^r, its reverse. The associated filling map for K^r is denoted G_{Y,K^r}. Because of the string reversal, $G_{Y,K^r}(\xi) = G_{Y,K}(\xi) + PD(K)$. Thus, if we simply take the quotient corresponding to the horizontal projection, the target of this chain map is a complex homotopy equivalent to $CF^+(Y, G_{Y,K}(\xi) + PD(K))$. The map h^+_ξ is given by horizontal projection, followed by this chain homotopy equivalence.

We now want to consider the set $S(\mathfrak{s})$ in terms of Spin^c structures on Y. To do so we write

$$S(\mathfrak{s}) = \{\underline{\mathfrak{t}}_{k+(N^2+1)j}\}_{j \in \mathbb{Z}}$$

for some fixed k satisfying $0 \leq k \leq N^2$. This set can be partitioned according to its N possible images in $\mathrm{Spin}^c(Y)$ under the filling map $G_{Y,K}$. Let $0 \leq l \leq N-1$. Then S can be written as

$$\bigcup_{0 \leq l \leq N-1} \left(\{\underline{\mathfrak{t}}_{[j(N^2+1)+(l-k)N]N+l}\}_{j \in \mathbb{Z}} \right).$$

Recalling that $\mu = Nm$, this can be rewritten as

$$\bigcup_{0\le l\le N-1} \left(\{\mathfrak{t}_l + [j(N^2+1)+(l-k)N]PD(\mu)\}_{j\in\mathbb{Z}}\right).$$

Deriving the following formula is rather delicate, but its validity is easily checked:

$$l + [j(N^2+1)+(l-k)N]N = l \mod N$$

and

$$l + [j(N^2+1)+(l-k)N]N = k \mod N^2+1.$$

5.8. Reduction to a finite complex. From this discussion it is apparent that, in general, the mapping cone complex is fairly complicated. In this subsection we observe that it always reduces to a complex that is a quotient of a finite dimensional complex over $\mathbb{F}[U,U^{-1}]$. In the next subsection we observe that in our special case the complex reduces to a single A_ξ term.

Consider the complexes $\mathbb{A} = \oplus A_i$ and $\mathbb{B} = \oplus B_i$, joined by the chain map D as illustrated below. We denote the mapping cone complex of D by $\mathbb{C}$. Since CFK^∞ is finitely generated over $\mathbb{F}[U,U^{-1}]$, it follows that $v\colon A_i \to B_i$ is an isomorphism for all large i, and $h\colon A_i \to B_{i+1}$ is an isomorphism as i goes to negative infinity. The diagram below presents a special case.

$$\begin{tikzcd}
\cdots & A_{-3} \arrow[dr, "\cong"] & A_{-2} \arrow[d, "v"] \arrow[dr, "\cong"] & A_{-1} \arrow[d, "v"] \arrow[dr, "h"] & A_0 \arrow[d, "v"] \arrow[dr, "h"] & A_1 \arrow[d, "\cong"] \arrow[dr, "h"] & A_2 \arrow[d, "\cong"] \arrow[dr, "h"] & & \cdots \\
& & B_{-2} & B_{-1} & B_0 & B_1 & B_2 & B_3 &
\end{tikzcd}$$

In this example, we have the following subcomplex, $\mathbb{C}' = \mathbb{A}' \oplus \mathbb{B}'$:

$$\begin{tikzcd}
\cdots & A_{-3} \arrow[dr, "\cong"] & A_{-2} \arrow[d] \arrow[dr, "\cong"] & & & A_1 \arrow[d, "\cong"] \arrow[dr] & A_2 \arrow[d, "\cong"] \arrow[dr] & & \cdots \\
& & B_{-2} & B_{-1} & & B_1 & B_2 & B_3 &
\end{tikzcd}$$

The restriction of D to this subcomplex, which we denote D', induces an isomorphism $D'_*\colon H_*(\mathbb{A}') \to H_*(\mathbb{B}')$. Injectivity is evident; surjectivity follows from the fact that for each x in the right portion of the complex, $(h\circ v^{-1})^k(x) = 0$ for some k. Similarly, for each x in the left portion of the complex, $(v\circ h^{-1})^k(x) = 0$ for some k. There is a long exact sequence

$$\to H_*(\mathbb{B}') \to H_*(\mathbb{C}') \to H_*(\mathbb{A}') \to$$

with connecting homomorphism given by D'. Thus, $H_*(\mathbb{C}') = 0$.

Consider next the short exact sequence $0 \to \mathbb{C}' \to \mathbb{C} \to \mathbb{C}/\mathbb{C}' \to 0$; it leads to a long exact sequence, and we see that $H_*(\mathbb{C}/\mathbb{C}') = H_*(\mathbb{C})$. That is, the homology of $\mathbb{C}$ is the homology of the complex

$$\begin{tikzcd} A_{-1} \arrow[dr] & A_0 \arrow[d] \\ & B_0 \end{tikzcd}$$

Notice that had $h\colon A_{-1} \to B_0$ also been an isomorphism in this example, then the complex would have reduced to a single term, A_0. This occurs in the cases of lens spaces that arise in our work, $L(N^2 + 1, N)$. We will see in the next section that this total collapse also occurs for our manifolds M.

5.9. General complete collapse of the mapping cone complex. In the case of lens spaces constructed as surgery on the unknot, the CFK^∞ complexes which arise are all of the form $(C\otimes_{\mathbb{F}}\mathbb{F}[U, U^{-1}])\{0, k_i\}$, where C is a 1–dimensional doubly filtered $\mathbb{F}$ vector space generated by a single vector x_i at filtration level $(0, 0)$. The shift $\{0, k_i\}$ is a j–filtering shift of k_i for appropriate integers k_i. It thus follows quickly that there is an a such that v_i is an isomorphism for all $i \geq a$ and h_i is an isomorphism for all $i \leq a - 1$. This explains our comment above that for lens spaces there is a complete collapse of the $(\mathbb{A}, \mathbb{B})$ mapping cone complex to a single A_i.

In the more general situation that appears for our M, the CFK^∞ complexes which arise are of the form $(C_{\bar{\imath}} \otimes_{\mathbb{F}} \mathbb{F}[U, U^{-1}])\{0, k_i\}$ for finite dimensional doubly filtered $\mathbb{F}$–chain complexes $C_{\bar{\imath}}$ which are no longer 1–dimensional (here $\bar{\imath} = i \mod n$ for some n). In particular, the CFK^∞ complexes are not restricted to a single diagonal. Instead, they lie in a band; in Figure 8 we illustrate a case in which the band is of height six.

Notice that in the example illustrated in Figure 8, the vertical quotient is not an isomorphism, but the horizontal quotient is. In general, one of the two maps will be an isomorphism unless the origin is contained in the band. Furthermore, if this band is shifted up (by -2 or more) then h continues to be an isomorphism, and if it is shifted down by seven or more, the vertical map becomes an isomorphism.

Recall now that in our decomposition $\mathbb{A}^+_{\mathfrak{s}}(Y, K) = \oplus_{\xi\in S}\mathbb{A}^+_{\xi}(Y, K)$ we have

$$S = \bigcup_{0\leq l\leq N-1} \left(\{\underline{\mathfrak{t}}_l + [j(N^2 + 1) + (l - k)N]PD(\mu)\}_{j\in\mathbb{Z}}\right).$$

In order to state the next result, let the width $\mathrm{w}(C)$ of a doubly filtered complex be defined as: $\mathrm{w}(C) = \max(i-j)-\min(i-j)+1$, where the minimum and maximum are taken over all pairs (i, j) such that there is a nontrivial filtered generator of filtered degree (i, j). Roughly, $\mathrm{w}(C)$

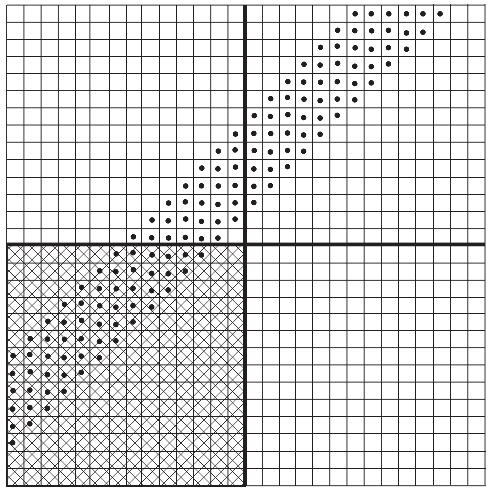


Figure 8

represents the width of the narrowest U–invariant band which contains the full complex. The width determines the Thurston norm of the knot complement [**29**, **36**].

Theorem 5.3 (**Collapse Theorem**). *Suppose that for each* $\underline{\mathfrak{t}}_l \in \mathrm{Spin}^c(Y, K)$ *with* $0 \leq l \leq N-1$, *the complex* $C = CFK^\infty(Y, K, \underline{\mathfrak{t}}_l)$ *satisfies* $\mathrm{w}(C) \leq N$. *Then the mapping cone complex* $\mathbb{A} \to \mathbb{B}$ *that determines* $HF^+(Y_N(K), \mathfrak{s})$ *collapses to a single* A_i *for some* i.

Proof. Recall that as in Section 5.7, k is a specified fixed integer, $0 \leq k \leq N^2$. For simplicity, denote $CFK^\infty(Y, K, \underline{\mathfrak{t}}_l)\{0, s\}_{\{\max(i,j)\geq 0\}}$ by $A'_l(s)$ for $0 \leq l \leq N-1$ where, as above, $\{0, k\}$ indicates that we have shifted the doubly filtered complex up by k. Then the A_i that occur are ordered as follows if we begin with $l = 0$ and $j = 0$:

$$\ldots A'_{N-1}(N+1+kN), A'_0(kN), A'_1((k-1)N), \ldots,$$

$$\ldots A'_{N-1}((k+1-N)N), A'_0(kN-1-N^2), \ldots$$

Notice that the shifts increase by N, or when going from A'_{N-1} to A'_0, by $N+1$. It follows that at most one of A_i is in a band which includes the origin, with all greater A_i being in bands below the origin and all lesser A_i being in bands above the origin. Thus the complex collapses to a single A_i, as desired. q.e.d.

We now have all the pieces necessary to prove Theorem 5.1:

Proof of Theorem 5.1. Given that N is greater than $2g(K_1)$, we are able to apply the refiltering theorem (Theorem 4.2) to prove that the complex $CFK^\infty(S^3_{-N}(K_1), \mu, \mathfrak{s})$ has width at most two for any $\mathfrak{s} \in$

$\mathrm{Spin}^c(S^3_{-N}(K_1))$. The Künneth theorem for the knot Floer homology of connected sums ([**33**, Theorem 7.1] c.f. [**38**, Theorem 5.1]) implies

$$CFK^\infty(S^3_{-N}(K_1), \mu\#K_2', \mathfrak{s})$$
$$\simeq CFK^\infty(S^3_{-N}(K_1), \mu, \mathfrak{s}) \otimes CFK^\infty(S^3, K_2'),$$

for any knot $K_2' \subset S^3$ and any $\mathfrak{s} \in \mathrm{Spin}^c(S^3_{-N}(K_1))$. Now [**36**, Theorem 1.2] implies that the width of $CFK^\infty(S^3, K_2')$ is equal to $2g(K_2') + 1$, and a simple exercise gives the addition formula, $\mathrm{w}(C_1 \otimes C_2) = \mathrm{w}(C_1) + \mathrm{w}(C_2) - 1$. Thus the width of the complex for $K_2 = \mu\#K_2' \subset S^3_{-N}(K_1) = Y$ is at most $2g(K_2') + 2$.

Thus, according to the collapse theorem (Theorem 5.3), for each Spin^c structure $\mathfrak{s}$ on $Y_N(K_2)$, the homology $CF^+(Y_N(K_2), \mathfrak{s})$ is given by a single complex A_i. This complex is of the form $C^\infty/C^\infty_{\{i<0,j<0\}}$ where C^∞ is the complex $CFK^\infty(Y, K_2, \mathfrak{s}')$ shifted down by some parameter m, $-N^2/2 \leq m \leq N^2/2$, and where $\mathfrak{s}'$ is some Spin^c structure on Y. Alternatively, it is the quotient

$$CFK^\infty(Y, K_2, \mathfrak{s}')/CFK^\infty(Y, K_2, \mathfrak{s}')_{\{i<0,j<m\}}.$$

The gradings are shifted, but the shift is independent of the choice of K_1 and K_2'.

The action of U is to shift downward along the diagonal. Thus, the kernel of the U action is precisely the set of elements at filtration level 0 as described in the statement of Theorem 5.1. q.e.d.

6. Computations

6.1. Knot complexes. For a given n and k we have defined $K_{D_k,n}$ to be the knot shown in Figure 1, with the knot J given by kD (where D continues to denote the positive-clasped untwisted Whitehead double of the right-handed trefoil). In this case, $M(K_{D_k,n})$ is given as $(-2n, 2n)$–surgery on the link formed from the Hopf link by replacing the first component with $-2kD$ and the second component by $2kD$; see Figure 2. As mentioned earlier, n will be in the set $\mathcal{N}$ described in Proposition 3.1 and Proposition C.1. For each n we will choose a value for k, denoted k_n, selected to satisfy certain properties. A key result, which follows from the work in Appendices A and B, is the following.

Proposition 6.1.

- *The chain complex $CFK^\infty(S^3, D)$ is filtered chain homotopy equivalent to the chain complex $CFK^\infty(S^3, T_{2,3}) \oplus A$, where A is an acyclic complex. If $[x, i, j]$ is a filtered generator of $CFK^\infty(S^3, D)$, then $|i - j| \leq 1$.*
- *The chain complex $CFK^\infty(S^3, D_k)$ is filtered chain homotopy equivalent to the chain complex $CFK^\infty(S^3, T_{2,2k+1}) \oplus A$, where A is an*

acyclic complex. If $[x, i, j]$ *is a filtered generator of* $CFK^\infty(S^3, D_k)$, *then* $|i - j| \leq k$.

Proof. The first statement of the proposition expands on the computation of $\widehat{CFK}(S^3, D)$ given in [**14**]. Its proof occupies Appendix A. The second statement of the proposition follows from the relationship between $CFK^\infty(S^3, T_{2,3})^{\otimes k}$ and $CFK^\infty(S^3, T_{2,2k+1})$ described in Theorem B.1. q.e.d.

We next compute the knot Floer complex of the meridian of the connected sum of $2k$ copies of the mirror of the doubled trefoil, in the space formed by surgery upon this connected sum.

Theorem 6.2. *For* $2n \geq 4k$ *and* $-n + 1 \leq m \leq n$, *the doubly filtered complex* $CFK^\infty(S^3_{-2n}(-2D_k), \mu, \mathfrak{s}_m)$ *is chain homotopy equivalent to the complex* $C_{2n,k,m} \cong (T \oplus A) \otimes_{\mathbb{F}} \mathbb{F}[U, U^{-1}]$ *where* A *is a finitely generated acyclic complex and* T *has one generator at filtration level* $(0, 0)$ *or* $(0, -1)$. *More precisely, the generator of* T *has filtration level* $(0, 0)$ *if* $m < -2k$ *or* m *odd* $< 2k$, *and has filtration level* $(0, -1)$ *if* $m \geq 2k$ *or* m *even* $\geq -2k$. *For any filtered generator* $[x, i, j]$, $|i - j| \leq 1$.

Proof. The theorem will be a direct application of the refiltering theorem (Theorem 4.2) together with the previous proposition. To begin, note that since the genus of $-2D_k$ is $2k$, we can use the refiltering theorem provided that $2n \geq 4k$ (as assumed). Applying the tensor product to the formula given in Proposition 6.1, we prove in the appendix (Theorem B.1) that there is a ($\mathbb{Z} \oplus \mathbb{Z}$–filtered) chain homotopy equivalence

$$CFK^\infty(S^3, 2kD) \simeq CFK^\infty(S^3, T_{2,4k+1}) \oplus A$$

where A is an acyclic complex. Recalling that $CFK^\infty(S^3, -K) = CFK^\infty(S^3, K)^*$, we obtain a corresponding decomposition for the mirrors:

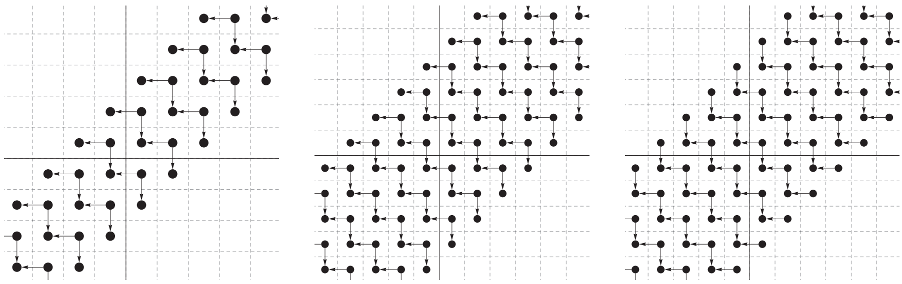
$$CFK^\infty(S^3, -2kD) \simeq CFK^\infty(S^3, -T_{2,4k+1}) \oplus A^*.$$

Applying the refiltering theorem then gives a decomposition

$$CFK^\infty(S^3_{-2n}(-2D_k), \mu, \mathfrak{s}_m) \simeq CFK^\infty(S^3_{-2n}(-T_{2,4k+1}), \mu, \mathfrak{s}_m) \oplus A',$$

where A' is an acyclic complex concentrated on one or both of the diagonals mentioned in the theorem (note that, by an abuse of notation, μ is the meridian to $-2D_k$ and $-T_{2,4k+1}$ on the left and right sides of the equivalence, respectively). Precisely, A' is the $\mathbb{Z} \oplus \mathbb{Z}$–filtered chain complex which results from the refiltration of A^*.

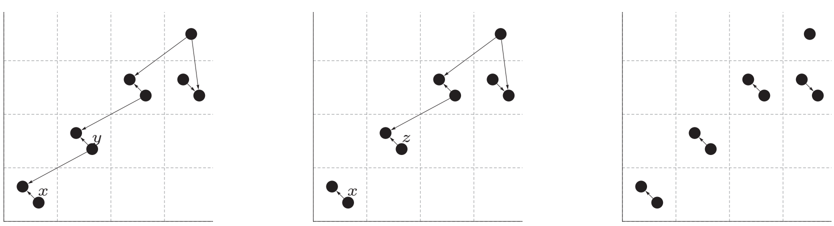
Thus it remains to understand the result of applying the refiltering theorem to the $CFK^\infty(S^3, -T_{2,4k+1})$. For reference, the figure illustrates the complexes associated to $T_{2,5}, T_{2,9}$, and $-T_{2,9}$. Applying Theorem 4.2, one sees that for each Spin^c structure $\mathfrak{s}_m$, the complex $CFK^\infty(S^3_{-2n}(-T_{2,4k+1}), \mu, \mathfrak{s}_m)$ is given by a complex concentrated on the diagonal and one below the diagonal. We wish to understand this complex better.



$CFK^\infty(S^3, T_{2,5})$ $CFK^\infty(S^3, T_{2,9})$ $CFK^\infty(S^3, -T_{2,9})$

Figure 9

For example, Figure 10(a) provides an illustration of the complex $CFK(S^3_{-2n}(-T_{2,9}), \mu, \mathfrak{s}_{-3})$, in which we have labeled two of the generators x and y. Replacing y with $x + y$ gives a filtered change of basis, and the new complex is as shown in Figure 10(b). Notice that this has introduced an acyclic piece. Repeating the process yields the complex illustrated in Figure 10(c). Applying this simplification in general shows that for each m, the complex $CFK^\infty(S^3_{-2n}(-T_{2,4k+1}), \mu, \mathfrak{s}_m)$ splits as a direct sum of an acyclic complex (necessarily on the two stated diagonals) plus a complex of the form $T \otimes \mathbb{F}[U, U^{-1}]$, where T is a single generator of the stated filtration (in fact the complex is filtered homotopy equivalent to $T \otimes \mathbb{F}[U, U^{-1}]$). This completes the proof of Theorem 6.2.



(a) (b) $z = x + y$ (c) $T \oplus A$

Figure 10

q.e.d.

We next want to consider the second component of the link. This is obtained from the meridian of the first component by forming the connected sum with a knot whose CFK^∞ is identical, modulo acyclic summands, to $T_{2,4k+1}$. Moreover, the complex for the meridian, modulo acyclic summands, is simply that of the unknot with a filtration shift. Given these observations, the following result is immediate.

Theorem 6.3. *For $2n \geq 4k$ and for $-n+1 \leq m \leq n$, we have*

$$CFK^\infty(S^3_{-2n}(-2D_k), \mu\#2D_k, \mathfrak{s}_m)_{i,j} = CFK^\infty(S^3, T_{2,4k+1})_{i,j-\delta} \oplus A,$$

with A an acyclic complex. Here, $\delta = 0$ if $m < -2k$ or m odd $< 2k$; $\delta = -1$ if $m \geq 2k$ or m even $\geq -2k$. Cycles representing nontrivial classes of grading 0 are located at filtration levels $i + j = 2k + \delta$. For any filtered generator $[x, i, j]$, $|i - j| \leq 2k + 1$.

We will need to compare this with the case of J the unknot, for which the computation is simpler. The result is as follows.

Theorem 6.4. *For $2n \geq 3$ and for $-n+1 \leq m \leq n$, we have*

$$CFK^\infty(S^3_{-2n}(-U), \mu\#U, \mathfrak{s}_m)_{i,j} = CFK^\infty(S^3, U)_{i,j-\delta}.$$

Here, $\delta = 0$ if $m < 0$ and $\delta = -1$ if $m \geq 0$. The cycle representing a nontrivial homology class is at filtration level $(0, \delta)$.

6.2. d–invariants and acyclic summands. As already seen, many of the complexes that arise have included acyclic summands. We will need to see that these summands do not affect the computations of the relevant d–invariants. Rather than present the most general theorem concerning acyclic summands, we will restrict ourselves to a simpler setting for which the proof is more straightforward.

Let $\mathcal{D}$ be a free, finitely generated $\mathbb{F}[U, U^{-1}]$–chain complex that is $\mathbb{Q}$–graded. Moreover, suppose that $\mathcal{D}$ has a distinguished basis, and is $\mathbb{Z}$–filtered (by subcomplexes) by the corresponding distinguished $\mathbb{F}[U]$–submodules

$$... \subset U^k\mathbb{F}[U] \subset U^{k-1}\mathbb{F}[U] \subset U^{k-2}\mathbb{F}[U] \subset ...$$

Thus the action of U lowers filtration level by one. Assume that it lowers grading by two. We let $d(\mathcal{D})$ denote the least grading of a nontrivial homology class $z \in H(\mathcal{D}/\mathcal{D}_{i<0})$ where z is in the image of U^k for all k; here U^k is viewed as an endomorphism of $H(\mathcal{D}/\mathcal{D}_{i<0})$. (If such an element does not exist, then $d(\mathcal{D}) = -\infty$.)

A particular example can be built from a finitely generated acyclic $\mathbb{F}$–chain complex A which is filtered and graded: regard a filtered generator $\mathbf{x}$ of A as a monomial $\mathbf{x} \otimes U^{-(\text{Filtration of } \mathbf{x})}$ and form the $\mathbb{F}[U, U^{-1}]$–complex $\mathcal{A} = A \otimes \mathbb{F}[U, U^{-1}]$, so that $A \otimes 1$ has the same filtration and grading as A, and U acts on the right, decreasing filtration level by one and grading by two. Write $A_k = A \otimes U^k$. Thus an element in A_k has filtration level equal to the filtration level of the corresponding element in A, shifted down by k.

Proposition 6.5. *If $\mathcal{D}' \cong \mathcal{D} \oplus \mathcal{A}$ with $\mathcal{D}$ and $\mathcal{A}$ as above, then $d(\mathcal{D}') = d(\mathcal{D})$.*

Proof. Forming the quotient complex of $\mathcal{A}$ with the subcomplex $\mathcal{A}^- = \mathcal{A}_{i<0}$ of elements with filtration level less than 0 yields a complex $\mathcal{A}^+ = \mathcal{A}/\mathcal{A}^-$. This complex decomposes over $\mathbb{F}$ as $\oplus_k A_k/(A_k \cap \mathcal{A}^-)$.

Since A is finitely generated, there is an N such that: (1) if $k > N$, then $(A_k \cap \mathcal{A}^-) = A_k$; and (2) if $k < -N$, then $(A_k \cap A^- = 0)$. Recalling that A is acyclic, we see that for all k with $|k| > N$, the homology group $H(A_k/(A_k \cap \mathcal{A}^-)) = 0$.

The action of U maps $H(A_k/(A_k \cap \mathcal{A}^-))$ to $H(A_{k+1}/(A_{k+1} \cap \mathcal{A}^-))$. The only possible nontrivial elements in the homology of $\mathcal{A}/\mathcal{A}^-$ are sums of elements in $H(A_{k+1}/(A_{k+1} \cap \mathcal{A}^-))$ for $|k| \leq N$. But no such element can be in the image of U^{2N} since it would then be in the image of an element in $H(A_k/(A_k \cap \mathcal{A}^-))$, for some $k < -N$, and we have seen these groups are trivial.

Given this, we see each nontrivial elements of $H(\mathcal{D}'/\mathcal{D}'_{<0})$ that is in the image of U^k for arbitrarily large k is also in the image of an element of $H(\mathcal{D}/\mathcal{D}_{<0})$. q.e.d.

6.3. Computations of d–invariants. We need to compute the difference of d–invariants, $d(M(K_{D_k,n}), \mathfrak{s}_n) - d(M(K_{U,n}), \mathfrak{s}_n)$, for any k with $0 \leq k < n/2$. Recall that the manifold $M(K_{D_k,n})$ is also denoted by $S^3_{-2n,2n}(-2D_k, 2D_k)$.

In the following theorem we use ϵ_i to denote a grading shift. As stated in the theorem, these are homological invariants that depend on the value of n, but are independent of the particular knots chosen. Thus, the values of ϵ in the first two equations and in the last two equations are equal and their particular values irrelevant. For this reason we denote them simply by ϵ_1 and ϵ_2. We also include in the statement the number $d(S^3, \mathfrak{s}_0)$ despite it equalling 0; this highlights the role of the d–invariant of the base space in which the knot lies.

Theorem 6.6. *For any* $0 \leq k < n/2$

$$0 = d(S^3_{-2n}(-2D_k), \mathfrak{s}_{-n}) - d(S^3, \mathfrak{s}_0) - \epsilon_1 \tag{6.1}$$

$$0 = d(S^3_{-2n}(-U), \mathfrak{s}_{-n}) - d(S^3, \mathfrak{s}_0) - \epsilon_1 \tag{6.2}$$

$$-2k = d(S^3_{-2n,2n}(-2D_k, 2D_k), \mathfrak{s}_n) - d(S^3_{-2n}(-2D_k), \mathfrak{s}_{-n}) - \epsilon_2 \tag{6.3}$$

$$0 = d(S^3_{-2n,2n}(-U, U), \mathfrak{s}_n) - d(S^3_{-2n}(-U), \mathfrak{s}_{-n}) - \epsilon_2, \tag{6.4}$$

where $\mathfrak{s}_0$ is the unique spin structure on S^3 and ϵ_i are grading shifts. The grading shifts ϵ_i are homological invariants [**31**] *and hence* (6.1) *and* (6.3)*, respectively, have the same grading shifts ϵ_1 and ϵ_2 as* (6.2) *and* (6.4)*, respectively.*

Proof. Ozsváth and Szabó [**33**, Corollary 4.2] showed that for a knot K in S^3 and $|-2n| \geq 2g(K) - 1$, the complex $CF^+(S^3_{-2n}(K), \mathfrak{s}_{-n})$ is filtered chain homotopic to

$$CFK^\infty(S^3,K)/CFK^\infty(S^3,K)_{\{i<0\cup j<-n\}}[\epsilon],$$

where the grading shift ϵ is $d(S^3,\mathfrak{s}_0)+\epsilon_1$ according to [**31**]. (The value of ϵ_1 can be computed explicitly, but we do not need its exact value in our computations.)

Proposition 6.1 along with Proposition 6.5 allows us to replace $-2D_k$ with $-T_{2,4k+1}$. We see that in the complex $CFK^\infty(S^3,-T_{2,4k+1})$ the cycle at filtration level $(0,-2k)$ is the cycle of grading zero having the least j–filtration among all grading zero cycles, and all cycles of grading less than zero have i–filtration less than zero. Since $-2k>-n$, the cycle at filtration level $(0,-2k)$ lives and all the cycles of grading less than zero vanish in the quotient. See Figure 11 for the case $-n=-4$ and $k=1$.

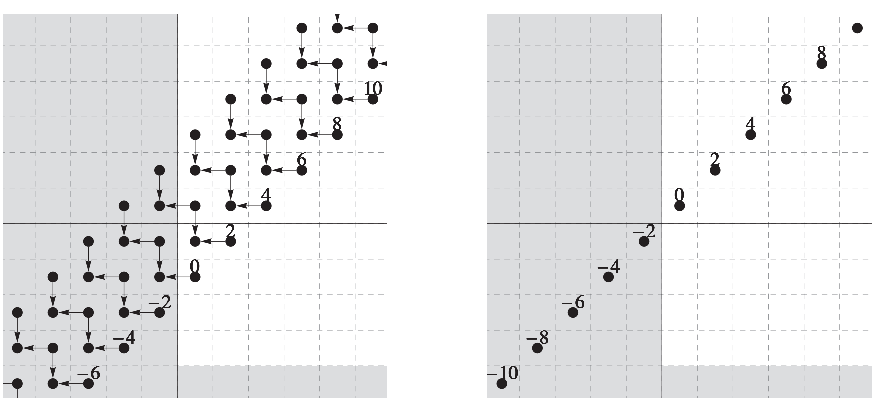


Figure 11

This shows the identity $d(S^3_{-2n}(-2D_k),\mathfrak{s}_{-n})-d(S^3,\mathfrak{s}_0)-\epsilon_1=0$. A similar argument shows $d(S^3_{-2n}(-U),\mathfrak{s}_{-n})-d(S^3,\mathfrak{s}_0)-\epsilon_1=0$. These two identities give rise to Equation 6.1. Equation 6.2 is similar.

By Theorem 6.3, noting $n>2k$, we can identify

$$CFK^\infty(S^3_{-2n}(-2D_k),\mu\#2D_k,\mathfrak{s}_n)_{i,j}=CFK^\infty(S^3,T_{2,4k+1})_{i,j+1}\oplus A,$$

where the complex A is acyclic complex. Stated otherwise, the complex $CFK^\infty(S^3_{-2n}(-2D_k),2D_k,\mathfrak{s}_n)$ is filtered chain homotopic to the complex $CFK^\infty(S^3,T_{2,4k+1})$ with j–filtration shifted downward by one plus an acyclic complex A. Combining this with Theorem 5.1 and using Proposition 6.5 to eliminate the acyclic summand from the computation, we have that the d–invariant associated to $CF^+(S^3_{-2n,2n}(-2D_k,2D_k),\mathfrak{s}_n)$ is equal to that of

$$CFK^\infty(S^3,T_{2,4k+1})_{i,j+1}/CFK^\infty(S^3,T_{2,4k+1})_{i,j+1\{i<0,\,j<0\}}[\epsilon],$$

where $\epsilon = d(S^3_{-2n}(-2D_k), \mathfrak{s}_n) + \epsilon_2$ for some ϵ_2 independent of $T_{2,4k+1}$. The cycles x at filtration level $(i, 2k-i-1)$, $0 \leq i \leq 2k$, are all of the grading zero cycles in $CFK^\infty(S^3, T_{2,4k+1})_{i,j+1}$. It is easy to see that the cycles $U^k x$ at filtration level $(i', -i'-1)$, $-k \leq i' \leq k$, have grading $-2k$ and none of them vanish in the quotient, while at least one of $U^{k'} x$ vanishes in the quotient if $k' > k$. See Figure 12 for the case $n = 4$ and $k = 1$.

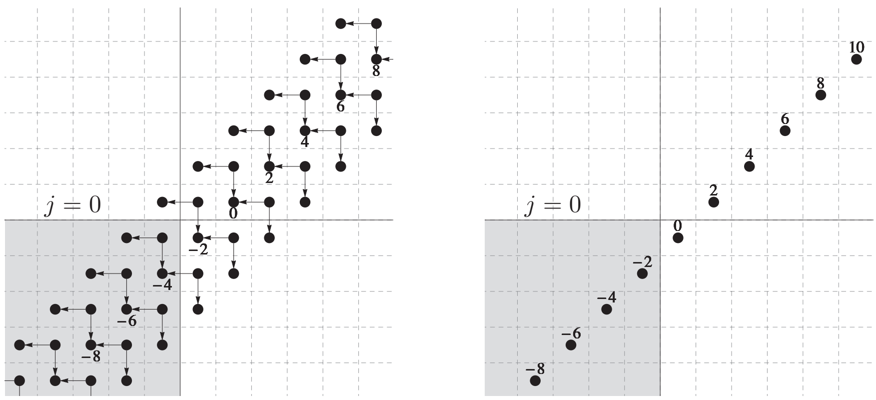


Figure 12

This implies Equation 6.3. Combining Theorems 6.4 and 5.1, a similar argument as done above provides the proof of Equation 6.4. q.e.d.

Combining the equations in Theorem 6.6 immediately yields the following proposition, which was the key step in the completion of the proof of Theorem 3.2 at the end of Section 3.

Proposition 6.7. $d(M(K_{D_k,n}), \mathfrak{s}_n) - d(M(K_{U,n}), \mathfrak{s}_n) = -2k.$

Appendix A. The infinity complex of the Whitehead doubled trefoil

Let D denote the positive-clasped untwisted Whitehead double of the right-handed trefoil. In this appendix we prove:

Proposition 6.1. *The chain complex $CFK^\infty(S^3, D)$ is chain homotopy equivalent to the chain complex $CFK^\infty(S^3, T_{2,3}) \oplus A$, where A is an acyclic complex. The presence of the acyclic summand does not change the width:*

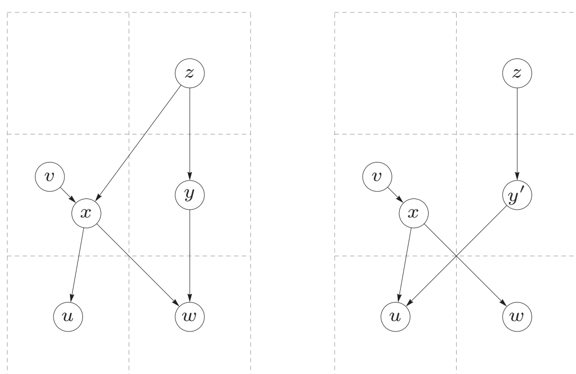
$$\mathrm{w}(CFK^\infty(S^3, D)) = \mathrm{w}(CFK^\infty(S^3, T_{2,3})).$$

In order to prove this proposition, we need the following well-known lemma about how a basis change affects the two-dimensional diagram of a knot Floer complex. See [**14**, Lemma 6.1] for instance.

Lemma A.1. *Let C_* be a knot Floer complex with a two-dimensional arrow diagram D given by an $\mathbb{F}$–basis. Suppose that x, y are two basis elements of the same grading such that each of the i and j filtrations of x is not greater than that of y. Then the basis change given by $y' = y + x$ gives rise to a diagram D' of C_* which differs from D only at y and x as follows:*

- *Every arrow from some z to y in D adds an arrow from z to x in D'.*
- *Every arrow from x to some w in D adds an arrow from y' to w in D'.*

Proof. First note that this basis change does not alter the grading or double filtrations. If $\partial z = y + \alpha$ for $z, \alpha \in C_*$, then $\partial z = y' + x + \alpha$, which shows that every arrow from z to y should add an arrow from z to x. Since $\partial y' = \partial y + \partial x$, every arrow from x should add an arrow from y'. See Figure 13 for an example. q.e.d.



(a) (b) $y' = y + x$

Figure 13. The figure represents the effect of a filtered basis change to a portion of a $\mathbb{Z} \oplus \mathbb{Z}$–filtered chain complex over $\mathbb{F}$.

Proof of Proposition 6.1. Theorem 1.2 of [**14**] shows that

$$\widehat{HFK}_*(D, j) = \begin{cases} \mathbb{F}^2_{(-1)} \oplus \mathbb{F}^2_{(0)}, & j = 1 \\ \mathbb{F}^4_{(-2)} \oplus \mathbb{F}^3_{(-1)}, & j = 0 \\ \mathbb{F}^2_{(-3)} \oplus \mathbb{F}^2_{(-2)}, & j = -1 \\ 0, & \text{otherwise.} \end{cases}$$

We assign $\mathbb{F}$–bases to each summand in the direct sum decomposition as follows:

$$\widehat{HFK}_*(D,j) = \begin{cases} \langle u_1, u_2\rangle \oplus \langle x_1, x_2\rangle, & j=1 \\ \langle v_1, v_2, v_3, v_4\rangle \oplus \langle y_1, y_2, y_3\rangle, & j=0 \\ \langle w_1, w_2\rangle \oplus \langle z_1, z_2\rangle, & j=-1. \end{cases}$$

Following Rasmussen [**39**, Lemma 4.5] (or [**14**, Lemma 5.3]), $\widehat{HFK}_*(D)$ is chain homotopy equivalent to $\widehat{CFK}(D)$. Thus we can assume that $CFK^\infty(D)_{0,j} = \widehat{HFK}_*(D,j)$ and $CFK^\infty(D)_{i,j} \cong U^{-i}\, CFK^\infty(D)_{0,j-i} = \widehat{HFK}_{*-2i}(D,j-i)$. If necessary, we put the grading in the superscript of the generator; for instance, x_1^2 denotes the grading 2 generator among $U^i x_1$ for $i \in \mathbb{Z}$. See Figure 14 for an example.

First note that there are no components of boundary maps between generators of the same (i,j)–filtration since they would be reduced in $\widehat{HFK}_*(D,j)$. If we denote the vertical, horizontal, and diagonal components of the boundary map ∂ of $CFK^\infty(D)$ by ∂_V, ∂_H, and ∂_D, respectively, then $\partial = \partial_V + \partial_H + \partial_D$. We will determine ∂ by first determining ∂_V, then ∂_H, and lastly ∂_D.

Note that $\mathbb{F}^2_{(0)} \overset{\partial_V}{\to} \mathbb{F}^3_{(-1)} \overset{\partial_V}{\to} \mathbb{F}^2_{(-2)}$, or, $\langle x_1, x_2\rangle \overset{\partial_V}{\to} \langle y_1, y_2, y_3\rangle \overset{\partial_V}{\to} \langle z_1, z_2\rangle$ is a chain subcomplex of $\widehat{CFK}(D)$ since ∂ lowers the grading by one. Since $\widehat{HF}(S^3) = \mathbb{F}_{(0)}$, by changing basis we may assume that $\partial_V(x_1) = \partial_V(y_1) = \partial_V(z_1) = \partial_V(z_2) = 0$, $\partial_V(x_2) = y_1$, $\partial_V(y_2) = z_1$, and $\partial_V(y_3) = z_2$. See Figure 14(b).

We will find $\partial_V(u_1)$, which must lie in $\langle v_1, v_2, v_3, v_4, z_1, z_2\rangle$. If $\partial_V(u_1) = az_1 + bz_2 \in \langle z_1, z_2\rangle$ for $a, b \in \mathbb{F}$, then $u_1 + ay_1 + by_2$ represents a nontrivial element of grading -1 in $\widehat{HF}(S^3)$, which is impossible. Thus $\partial_V(u_1)$ must have a nontrivial component in $\langle v_1, v_2, v_3, v_4\rangle$, which may be assumed to be v_1 by changing the basis for $\langle v_1, v_2, v_3, v_4\rangle$. If $\partial_V(u_1) = v_1 + az_1 + bz_2$, then the change of basis $v_1' = v_1 + az_1 + bz_2$ gives rise to $\partial_V(u_1) = v_1'$ and $\partial_V v_1' = \partial_V v_1$, as in Lemma A.1. So we may assume that $\partial_V u_1 = v_1$ and similarly that $\partial_V u_2 = v_2$. The image of $\langle v_3, v_4\rangle$ under ∂_V should be equal to $\langle w_1, w_2\rangle$ since $\widehat{HF}(S^3) = \mathbb{Z}$ in which v_3, v_4, w_1 and w_2 should vanish. So $\{\partial_V(v_3), \partial_V(v_4)\}$ is a basis for $\langle w_1, w_2\rangle$ and we may assume that $w_1 = \partial_V(v_3)$ and $w_2 = \partial_V(v_4)$. The vertical components of the boundary maps are all determined as shown in Figure 14(c).

Next, we will determine the horizontal components of the boundary map of $CFK^\infty(D)$, whose columns look like those illustrated in Figure 14. We will argue that the complex will have a two-dimensional illustration described in Figure 15. By analogy with the vertical case, note that $\langle z_1, z_2\rangle \overset{\partial_H}{\to} \langle y_1, y_2, y_3\rangle \overset{\partial_H}{\to} \langle x_1, x_2\rangle$ is a chain subcomplex S of

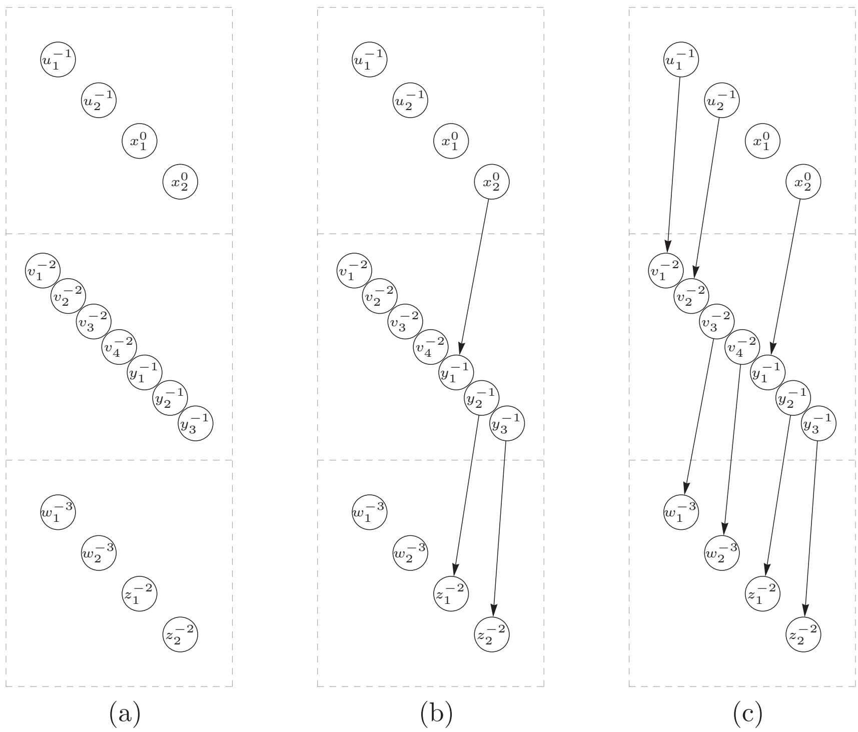


Figure 14

$CFK^\infty(D)_{\{j\le 0\}}/\, CFK^\infty(D)_{\{j<0\}}$, since ∂ lowers the degree by one. Observe as well that for any $s\in S$, elements with grading one lower than s are either to the left or below and hence $\partial s=\partial_V s+\partial_H s$. In particular there are no diagonal components of the boundary maps restricted to S. This implies that $\partial x_1=\partial_V x_1=0$ and $\partial x_2=\partial_V x_2=y_1$.

Since $\widehat{HF}(S^3)\cong \mathbb{F}_{(0)}$ is isomorphic to

$$H_*\left(CFK^\infty(D)_{\{j\le 0\}}/\, CFK^\infty(D)_{\{j<0\}}\right),$$

we may choose an $\mathbb{F}$–basis $\{z_1,z_2\}$ so that $\partial_H(z_1)=0$. To keep the same vertical description as in Figure 14(c), we adjust the basis for $\langle y_2,y_3\rangle$ accordingly. Observe that ∂z_2 is the source of no diagonal arrows, since elements with grading one lower are located only to the left. So we have $\partial z_2\in\langle y_1,y_2,y_3\rangle$. If ∂z_2 is of the form $y_2+\beta$ for $\beta\in\langle y_1,y_3\rangle$, then $0=\partial^2 z_2=\partial y_2+\partial\beta=z_1+\partial_H y_2+\partial\beta$, which, on the other hand, can never be zero since $\partial_H y_2\in\langle x_1,x_2\rangle$, $\partial\beta\in\langle\partial y_1,\partial y_3\rangle\in\langle z_2,x_1,x_2\rangle$, and z_1 does not belong to $\langle x_1,x_2,z_2\rangle$. Thus y_2 does not appear in ∂z_2. Similarly, y_3 does not appear in ∂z_2, and thus ∂z_2 must be y_1.

Similarly, grading considerations and the fact that the homology of the quotient $CFK^\infty(D)_{\{j\le m\}}/\, CFK^\infty(D)_{\{j<m\}}$ is $\mathbb{F}_{(2m)}$ implies that $\partial_H\langle y_2,y_3\rangle=\langle x_1,x_2\rangle$. If $\partial_H y_2$ is of the form x_2+ax_1 for $a\in\mathbb{F}$, then

$\partial y_2 = (\partial_H + \partial_V)y_2 = x_2 + ax_1 + z_1$ and $0 = \partial^2 y_2 = \partial(x_2 + ax_1 + z_1) = y_1 \neq 0$, which is impossible. Thus we have $\partial_H y_2 = x_1$. Then $\partial_H y_3$ should be of the form $x_2 + ax_1$. By the change of basis $x_2' = x_2 + ax_1$ we may assume $\partial_H y_3 = x_2$.

To complete the analysis of ∂_H we must consider the w, v and u generators. The argument is similar to what we have done already. First notice that these elements might not generate a subcomplex of the horizontal complex; $\partial_H(w_i^j)$ could contain terms of the form x_k^{j-1} (which are at the same j–filtration level but at i–filtration two lower). A change of basis, adding some of the elements x_k^{j-1} to some of the v_i^{j-1}, eliminates this possibility, at the expense of perhaps adding diagonal maps. Since the change of basis combines elements at different i–filtration levels, the vertical map is unchanged. Thus, we can assume that the w, v and u generate a subcomplex of the horizontal complex which is complementary to the subcomplex generated by the x, y, and z generators. Using the fact that $\partial^2(u_k^{i-1}) = 0$ we conclude that ∂_H must vanish on the v_1^i and v_2^i.

Using the known homology of the horizontal complex (in particular, that the horizontal homology at j–filtration level 0 is generated by a single element at grading 0, and thus a z_i) we can conclude that ∂_H maps the subgroup generated by v_3^0 and v_4^0 isomorphically to the subgroup generated by u_1^{-1} and u_2^{-1}, and similarly for their U translates.

A change of basis among the u_1 and u_2 generators ensures each v_3 maps to the corresponding u_1 and each v_4 maps to a corresponding u_2. A change of basis among the v_1 and v_2 elements reestablish that ∂_V maps each u_1^i and u_2^i to a v_1^{i-1} and v_2^{i-1}, respectively. That $\partial^2(u_k^i) = 0$ implies that w_i^j maps horizontally to a corresponding v_i^{j-1}.

At this point we have a diagram for $CFK^\infty(D)$ as in Figure 15 with only vertical and horizontal components of the boundary maps shown.

Finally, we will deal with the diagonal components of the boundary maps. As mentioned earlier, due to grading constraints there are no diagonal maps coming from the x, y, or z generators, while there may be diagonals going in. On the other hand, there are no diagonal maps going into the u, v, or w generators. All possible cases of diagonal maps are: (1) from u's to x's, (2) from v's to y's, and (3) from w's to z's. This implies that the complex T generated by x_1, y_2 and z_1 is indeed a subcomplex of $CFK^\infty(D)$.

We will show that filtered basis changes can eliminate all the diagonal arrows going into T. Then $CFK^\infty(D)$ splits into $\mathbb{F}[U, U^{-1}] \otimes T$ and a subcomplex A. Note that $\mathbb{F}[U, U^{-1}] \otimes T$ is isomorphic to $CFK^\infty(T(2,3))$ and A is acyclic (that is, $H_*(A) = 0$). This follows from [**32**, Section 10], which showed $HF^\infty(S^3) \cong \mathbb{F}[U, U^{-1}]$ and $HF^\infty(S^3) \cong H_*(CFK^\infty(D)) \cong H_*(\mathbb{F}[U, U^{-1}] \otimes T)$ as $\mathbb{F}[U, U^{-1}]$–modules.

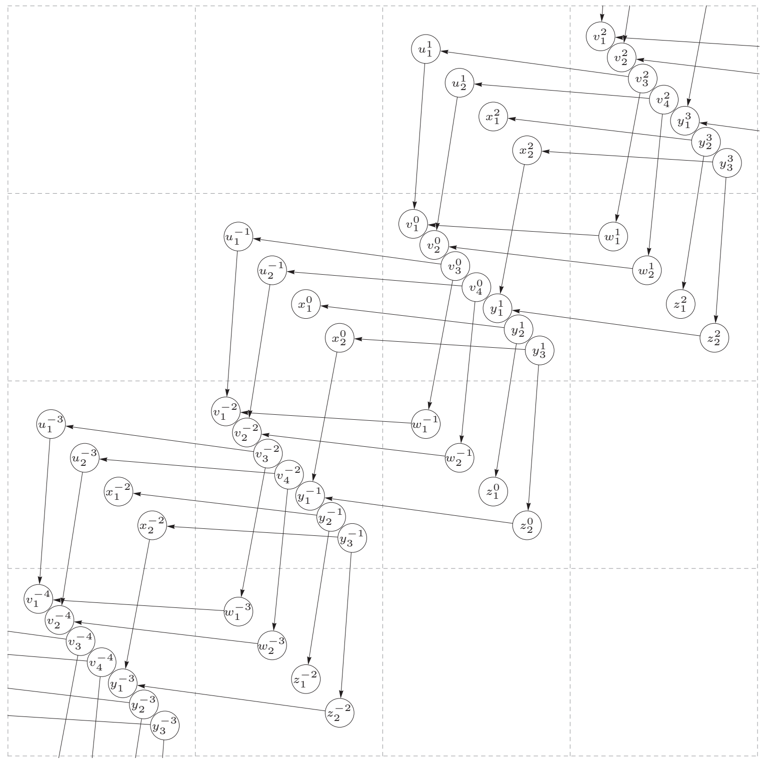


Figure 15

First, we show that ∂v_1 and ∂v_2 cannot include y_2. Note that they have zero vertical and horizontal components. If $\partial v_1 = y_2 + ay_1 + by_3$ for $a, b \in \mathbb{F}$, then $0 = \partial^2 v_1 = x_1 + z_1 + bx_2 + bz_2$ which cannot be zero for any a, b. So there are no arrows from v_1 or v_2 to y_2.

We claim that, for any $a, b \in \mathbb{F}$ and $i = 1, 2$, the following are equivalent:

1) $\partial_D u_i = ax_1 + bx_2$
2) $\partial_D v_{i+2} = ay_2 + by_3 + cy_1$ for some $c \in \mathbb{F}$
3) $\partial_D w_i = az_1 + bz_2$.

We prove the claim only for $i = 1$; almost the same argument applies to $i = 2$. Let $\partial_D u_1 = ax_1 + bx_2$, $\partial_D v_3 = c_1 y_1 + c_2 y_2 + c_3 y_3$, and $\partial_D w_1 = d_1 z_1 + d_2 z_2$ for $a, b, c_*, d_* \in \mathbb{F}$. The constraint $\partial^2 = 0$ gives rise to the equalities

$$\begin{aligned} 0 = \partial^2 v_3 &= \partial(u_1 + w_1 + c_1 y_1 + c_2 y_2 + c_3 y_3) \\ &= (v_1 + ax_1 + bx_2) + (v_1 + d_1 z_1 + d_2 z_2) + c_2(x_1 + z_1) + c_3(x_2 + z_2) \\ &= (a + c_2)x_1 + (b + c_3)x_2 + (d_1 + c_2)z_1 + (d_2 + c_3)z_2. \end{aligned}$$

Thus $a = c_2 = d_1$ and $b = c_3 = d_2$.

Suppose $a = 1$ for $i = 1$. Let $v_1' = v_1 + x_1$ and $w_1' = w_1 + y_2$. Then all arrows going into v_1 or w_1 come from u_1, w_1 or v_3 and hence we need to check the boundaries of u_1, v_3, w_1' and v_1': $\partial u_1 = v_1' + bx_2$, $\partial v_3 = u_1 + w_1' + by_3 + cy_1$, $\partial w_1' = \partial w_1 + \partial y_2 = (v_1 + z_1 + bz_2) + (x_1 + z_1) = v_1 + bz_2$, and $\partial v_1' = \partial v_1 + \partial x_1 = 0$. With these new basis elements v_1' and w_1' there are no diagonal components from u_1, v_3, w_1 to x_1, y_2, z_1. A similar argument works for u_2, v_4, w_2. Thus T can be assumed to be a direct summand as desired. q.e.d.

We remark that a similar process of changing bases as in the previous proof can be used to prove that $CFK^\infty(S^3, D)$ is isomorphic to the complex in Figure 15. Since this result is unnecessary for our purposes or any foreseeable applications to concordance we leave it as an exercise for the interested reader.

Appendix B. $CFK^\infty(S^3, T_{2,2k+1})$

Theorem B.1. *$CFK^\infty(S^3, T_{2,3})^{\otimes k} = CFK^\infty(S^3, T_{2,2k+1}) \oplus A$ where A is acyclic. The presence of the acyclic summand does not change the width:*

$$\mathrm{w}(CFK^\infty(S^3, T_{2,2k+1})) = \mathrm{w}(CFK^\infty(S^3, T_{2,2k+1})).$$

Proof. The proof is by induction. We show that

$$CFK^\infty(S^3, T_{2,2k+1}) \otimes CFK^\infty(S^3, T_{2,3}) = CFK^\infty(S^3, T_{2,2k+3}) \oplus A.$$

The complex $CFK^\infty(S^3, T_{2,2k+1})$ has filtered generators at grading 0: $[x, i, j]$ where $i \geq 0$, $j \geq 0$ and $i + j = k$. There are also generators at grading level 1, $[y, i, j]$ with $i \geq 1, j \geq 1$ and $i + j = k + 1$. The boundary map is given by $\partial[y, i, j] = [x, i - 1, j] + [x, i, j - 1]$. (Notice that the symbols x and y do not correspond to intersection points in a Heegaard diagram. The i and j denote the filtration levels.)

In order to distinguish the complex for $T_{2,3}$, we replace x and y with z and w, so that the complex is generated by $[z, 0, 1], [z, 1, 0]$, and $[w, 1, 1]$.

The tensor product $CFK^\infty(S^3, T_{2,2k+1}) \otimes CFK^\infty(S^3, T_{2,3})$ has generators of type $x \otimes z$ at grading level 0, $x \otimes w$ and $y \otimes z$ at grading level 1, and $y \otimes w$ at grading level 2. In total there are $3(2k + 1)$ generators.

We now make a basis change, replacing certain generators with their sums with other generators, relabeled as indicated:

- $[x, i, j] \otimes [w, 1, 1] \to [x, i, j] \otimes [w, 1, 1] + [y, i + 1, j] \otimes [z, 0, 1] = \alpha_i$, for all $0 \leq i < k$.
- $[x, i, j] \otimes [z, 1, 0] \to [x, i, j] \otimes [z, 1, 0] + [x, i + 1, j - 1] \otimes [z, 0, 1] = \beta_i$, for all $0 \leq i < k$.
- $[y, i, j] \otimes [z, 1, 0] \to [y, i, j] \otimes [z, 1, 0] + [x, i, j - 1] \otimes [w, 1, 1] = \gamma_i$, for all $0 < i \leq k$.

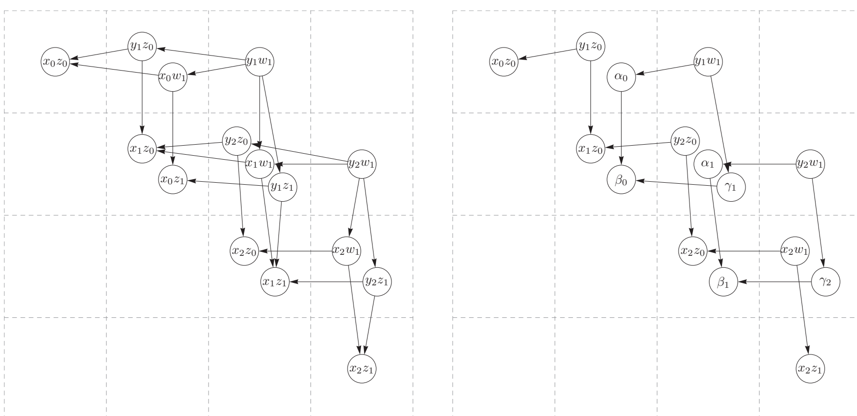
Now we isolate out acyclic pieces, using the following four observations.

- $\partial([y,i,j]\otimes[w,1,1]) = [x,i-1,j]\otimes[w,1,1]+[x,i,j-1]\otimes[w,1,1]+[y,i,j]\otimes[z,0,1]+[y,i,j]\otimes[z,1,0] = \alpha_{i-1}+\gamma_i$.
- $\partial\alpha_{i-1} = \partial([x,i-1,j]\otimes[w,1,1]+[y,i,j]\otimes[z,0,1]) = [x,i-1,j]\otimes[z,0,1]+[x,i-1,j]\otimes[z,1,0]+[x,i-1,j]\otimes[z,0,1]+[x,i,j-1]\otimes[z,0,1] = [x,i-1,j]\otimes[z,1,0]+[x,i,j-1]\otimes[z,0,1] = \beta_{i-1}$.
- $\partial\gamma_i = [x,i-1,j]\otimes[z,1,0]+[x,i,j-1]\otimes[z,1,0]+[x,i,j-1]\otimes[z,0,1]+[x,i,j-1]\otimes[z,1,0] = [x,i-1,j]\otimes[z,1,0]+[x,i,j-1]\otimes[z,0,1] = \beta_{i-1}$.
- $\partial\beta_{i-1} = 0$.

From this we see that there is an acyclic *summand*

$$\langle[y,i,j]\otimes[w,1,1]\rangle \xrightarrow{\partial} \langle\alpha_{i-1},\gamma_i\rangle \xrightarrow{\partial} \langle\beta_i\rangle\,.$$

For instance, see Figure 16 for the case $k=2$.



$CFK^\infty(S^3,(T_{2,5})\otimes CFK^\infty(S^3,T_{2,3})$ $\qquad$ $CFK^\infty(S^3,T_{2,7})\oplus A$

Figure 16. Notation: $x_iz_{i'} = [x,i,k-i]\otimes[z,i',1-i']$, $x_iw_1 = [x,i,k-i]\otimes[w,1,1]$, $y_iz_{i'} = [y,i,k+1-i]\otimes[z,i',1-i']$, $y_iw_1 = [y,i,k+1-i]\otimes[w,1,1]$, $\alpha_i = x_iw_1 + y_{i+1}z_0$, $\beta_i = x_iz_1 + x_{i+1}z_0$, and $\gamma_i = y_iz_1 + x_iw_1$.

There are k such summands, with a total rank of $4k$. The original complex had rank $3(2k+1) = 6k+3$. Thus, splitting off the acyclic summands leaves a complex of rank $2k+3$. Generators for a complement to the acyclic summand are given by the set $\{[x,i,j]\otimes[z,0,1],[y,i,j]\otimes[z,0,1]\}$ and two more elements, $[x,k,0]\otimes[w,1,1]$ and $[x,k,0]\otimes[z,1,0]$. Finally, we note that this is a subcomplex of the desired isomorphism type, as follows from three simple observations: $\partial([x,i,j]\otimes[z,0,1]) = 0$,

$\partial([y,i,j]\otimes[z,0,1]) = [x,i-1,j]\otimes[z,0,1]+[x,i,j-1]\otimes[z,0,1]$ and $\partial([x,k,0]\otimes[w,1,1]) = [x,k,0]\otimes[z,0,1]+[x,k,0]\otimes[z,1,0]$. q.e.d.

Note that similar computations have recently appeared in [**13**].

Appendix C. Number theoretic results

Theorem C.1. *There is an infinite set $\mathcal{N}$ of natural numbers $\{n_i\}$ satisfying:*

1) *For all n_i, $4n_i^2+1\geq 9$ and is either prime or the product of two distinct primes; thus $4n_i^2+1$ is square free;*
2) *The values $\{4n_i^2+1\}$ are pairwise relatively prime.*

Proof. A theorem of Iwaniec [**19**] states that if $G(n) = an^2+bn+c$ is an irreducible integer polynomial with $a>0$ and $c\equiv 1 \mod 2$, then there exist infinitely many n such that $G(n)$ has at most two prime factors, counted with multiplicity. We will apply this for $G(n)$ of the form $4A^2n^2+1$, for appropriate values of $A>0$. Notice that any $G(n)$ of this form is never a perfect square (for any $n>0$). Thus, by Iwaniec's theorem we have that for an infinite set of positive n, $G(n)$ is either prime or a product of two distinct primes. In particular, it is square free.

The n_i are defined inductively, starting with $n_1=2$, so $4n_1^2+1=17$ is prime. Suppose that for all $i<k$, values of n_i have been selected so as to satisfy the conditions of the theorem. Let A denote the product of all $4n_i^2+1$, $i<k$. Apply Iwaniec's theorem to choose an N so that $4A^2N^2+1$ is the product of at most two prime factors. No prime factor of the $4n_i^2+1$, $i<k$, can divide this number, so $4A^2N^2+1$ is relatively prime to $4n_i^2+1$ for all $i<k$. Let $n_k=AN$. q.e.d.

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