J. DIFFERENTIAL GEOMETRY
102 (2016) 173-178

ON REGULAR ALGEBRAIC SURFACES OF $I\!R^3$ WITH CONSTANT MEAN CURVATURE

J. Lucas M. Barbosa & Manfredo P. do Carmo

Abstract

We consider regular surfaces M that are given as the zeros of a polynomial function $p : I\!R^3 \to I\!R$, where the gradient of p vanishes nowhere. We assume that M has non-zero constant mean curvature and prove that there exist only two examples of such surfaces, namely the sphere and the circular cylinder.

1. Introduction

An algebraic set in $I\!R^3$ will be the set

$$M = \{(x, y, z) \in I\!R^3;\ p(x, y, z) = 0\}$$

of zeros of a polynomial function $p : I\!R^3 \to R$. An algebraic set is regular if the gradient vector $\nabla p = (p_x, p_y, p_z)$ vanishes nowhere in M; here p_x, p_y, and p_z denote the derivative of p with respect to x, y, or z respectively.

The condition of regularity is essential in our case. It allows us to parametrize the set M locally by differentiable functions $x(u, v)$, $y(u, v)$, $z(u, v)$ (not necessarily polynomials), so that M becomes a regular surface in the sense of differential geometry (see [**3**], chapter 2, section 2.2, in particular Proposition 2); here (u, v) are coordinates in an open set of $I\!R^2$.

Since M is a closed set in $I\!R^3$, it is a complete surface. In addition, being a regular surface, it is properly embedded, i.e., the limit set of M (if any) does not belong to M (cf. [**16**], chapter IV, A.1 p. 113). In particular, regular algebraic surfaces are locally graphs over their tangent planes.

From now on, M will denote a regular algebraic surface in $I\!R^3$. Due to the regularity condition, one can define on M the basic objects of differential geometry of surfaces and pose some differential-algebraic questions within this algebraic category.

For instance, in the last 60 years (namely after the seminal work [**5**] of Heinz Hopf in 1951), many questions have been worked out on differentiable surfaces of non-zero constant mean curvature H. See also [**6**].

Received 5/1/2014.

In our case, we have two examples of algebraic regular surfaces that have non-zero constant mean curvature, namely,

(1) spheres, $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = r^2$ with center $(x_0, y_0, z_0) \in \mathbb{R}^3$ and radius $r = 1/H$;
(2) circular right cylinders, $(x - x_0)^2 + (y - y_0)^2 = r^2$, whose basis is a circle in the plane xy with center (x_0, y_0) and whose axis is a straight line passing through the center and parallel to the z axis.

A first natural question is: Are there further examples?

The first time we heard about this question was in a preprint of Oscar Perdomo (recently published in [**14**]) where he proves that for polynomials of degree three there are no such surfaces.

In this note, we prove the following general result:

Theorem 1.1. *Let M be a regular algebraic surface in $\mathbb{R}^3$. Assume that it has constant mean curvature $H \neq 0$. Then M is a sphere or a right circular cylinder.*

Acknowledgments. We want to thank Fernando Codá Marques for a crucial observation, Oscar Perdomo for having written [**14**], and Karl Otto Stöhr for his help in our first attempts to solve the problem. We also want to thank the referee for useful comments.

Both authors are partially supported by CNPq, Brazil.

2. Preliminaries

We first observe that, in the compact case, this theorem follows immediately from Alexandrov's well-known result: *An embedded compact surface in $\mathbb{R}^3$ with constant mean curvature is isometric to a sphere.*

The second observation is that the total curvature of an algebraic surface is finite. This was first proved by Osserman [**12**] in the case that the surface is an immersion parametrized by polynomials in two variables. In this case, it follows from a theorem by Huber [**7**] that a complete parametrized algebraic surface has finite topology, i.e., it is a compact surface with a finite number of ends.

It is likely that a similar proof can be given to our (implicitly defined) regular algebraic surface M. The proof by Osserman uses Bezout's theorem and the same will occur in the implicit case. Since we had difficulties in finding a reference for the appropriate version of Bezout's theorem, we followed another way.

The fact that algebraic surfaces in $\mathbb{R}^3$ have finite topology is just a particular case of a more general theorem which states that all algebraic subsets of $\mathbb{R}^n$ defined by any number of real polynomials with bounded degree belong to a finite number of topological types. This is proved in [**2**], chapter 9, Theorem 9.3.5. Applied to surfaces, this proves that our M is a compact surface with finitely many ends.

The proof of our theorem uses in a crucial way the structure theory for embedded, complete finitely connected surfaces with non-zero constant mean curvature developed by Korevaar, Kusner, and Solomon in [**8**] after some preliminary work by Meeks [**10**]. The statement that we need from these papers is as follows:

Theorem A ([**10**] and [**8**]) Let M be a complete, non-compact, properly embedded surface in $I\!R^3$ with non-zero constant mean curvature. Assume that M is finitely connected. Then, the ends of M are cylindrically bounded. Furthermore, for each end E of M, there exists a Delaunay surface $\Sigma \subset I\!R^3$ such that E and Σ can be expressed as cylindrical graphs ρ_E and ρ_Σ so that, near infinity, $|\rho_E - \rho_\Sigma| < Ce^{-\lambda x}$ where $C \geq 0$ and $\lambda > 0$ are constants.

Remark 2.1. *The first assertion in Theorem A comes from* [**10**]. *The final assertion is from* [**8**], *Theorem 5.18.*

3. Proof of the Theorem

We can assume that M is complete and non-compact; otherwise it is a sphere. Thus, M has finite topology, that is, M is compact with finitely many ends. By Theorem A, each end E of M converges exponentially to a Delaunay surface Σ. Since M is embedded, the Delaunay surface Σ to which an end E converges has to be an onduloid or a right circular cylinder.

We first claim that the Delaunay surface Σ toward which E converges is actually a cylinder.

Suppose it is not. By a rigid motion, we can assume that the axis of Σ is parallel to the y axis and meets the z axis. Then, there is a value z_0 of z such that the line $y \to (0, y, z_0)$ intersects Σ infinitely often. Since E approaches Σ at infinity, the algebraic equation $p(0, y, z_0) = 0$ has infinitely many solutions. This is impossible. So Σ is a cylinder as we claimed.

We claim now that E contains an open set of the cylinder Σ.

To see this, we take a rigid motion so that one of the straight lines of the cylinder Σ agrees with the coordinate y-axis. Thus, one of the intersection curves of E with the plane $x = 0$ is a curve β that converges to the y-axis. If y is large enough, β is given by

$$\beta(y) = (0, y, z(y)),$$

where $z(y)$ is a function that satisfies

$$\lim_{y\to\infty} z(y) = 0.$$

Since the curve β belongs to the end E, we have

$$p(0, y, z(y)) = 0.$$

Observe that the polynomial p can be written as

$$p(x,y,z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

where $a_k = a_k(x,y)$ is a polynomial in x and y of degree $\leq n$. By Theorem A, we have that

$$\lim_{y\to\infty} z(y) = \lim_{y\to\infty} Ce^{-\lambda y} = 0.$$

By a known result in calculus, we have, for any integer k,

$$\lim_{y\to\infty} y^k e^{-\lambda y} = 0$$

for any integer k.

Thus, by computing the limit in the equation $p(0,y,z(y)) = 0$ as $y \to \infty$ along the curve β, we obtain that a_0 does not depend on y, and $a_0 = 0$. This means that, for any y, the equation $p(0,y,z) = 0$ has $z = 0$ as a root, i.e., the straight line $y \to (0,y,0)$ is contained in E.

The above argument applies to an arbitrary straight line of Σ. It follows that an open set in E is a cylinder. This proves our claim.

Thus, there exists an open set U in M with the property that the Gaussian curvature K vanishes in M. Since M is analytic, K vanishes identically in M. It is then well known (see e.g. [**9**]) that M is a cylinder. Since H is constant, this is a circular cylinder. This proves the theorem.

Remark 3.1. *A crucial point in the proof is that the convergence in* [**8**] *is exponential. It allows us to prove that not only an arbitrary straight line in the cylinder* Σ *converges to* E *but that actually it is contained in* E.

4. Final Remarks

The case $H = 0$. There are many algebraic minimal surfaces in $I\!R^3$ (see p. 161 of the English translation of Nitsche's book [**11**]). However, the examples we are most familiar with, namely, the Enneper surface and the Hennenberg surface, are not embedded; thus they are not regular algebraic surfaces.

In fact it is simple to prove the following proposition.

Proposition 4.1. *There are no regular algebraic minimal surfaces in* $I\!R^3$ *except the plane.*

Proof. Let M be an algebraic minimal surface in $I\!R^3$. As we have seen, such a surface is finitely connected, i.e., it is a compact surface with a finite number of ends. We also know that M is properly embedded.

Let E be one of its ends. Parametrically E can be described by a map $x : D - \{O\} \to I\!R^3$, where D is an open disk of $I\!R^2$ centered at the origin and O is the origin.

We may assume, after a rotation if necessary, that the Gauss map, which extends to O (see Osserman [**13**]), takes on the value $(0,0,1)$ at

O. The two simplest examples of such ends are the plane and (either end of) the catenoid.

Now we use a result proved by R. Schoen [**15**]. He showed that such an end is the graph of the function x_3 defined over the (x_1, x_2)-plane and

$$x_3(x_1, x_2) = a \log \rho + \beta + \rho^{-2}(\gamma_1 x_1 + \gamma_2 x_2) + O(\rho^{-2}). \tag{1}$$

When $a \neq 0$ the end is of catenoid type. When $a = 0$ the end is of the planar type. In fact, if $a \neq 0$ the function x_3 will be asymptotic to the graph of the function $\log \rho$; if $a = 0$ it will be asymptotic to the graph of a constant function (equal to β).

Let's assume that E is of the catenoid type. Consider the curve α intersection of the E with the plane $x_2 = 0$ in the region $x_1 > 0$. Since M is given by the equation $p(x_1, x_2, x_3) = 0$, the curve α is algebraic, given by $p(x_1, 0, x_3) = 0$. This curve must be asymptotic to the graph of the function $x_3 = a \log x_1$. But this is impossible. Hence, M can not have an end of the catenoid type.

Thus, all the ends of M are of the planar type. But they are in finite number. Since M is embedded, the planes asymptotic to M must be parallel. It follows that there are two parallel planes such that M is contained in the region bounded by them. It follows by the halfspace theorem for minimal surfaces [**4**] that M must be a plane. q.e.d.

Hypersurfaces in $\mathbb{R}^{n+1}$, $n \geq 3$. In this case we consider the zeros of a polynomial function $p(x_0, x_1, \ldots, x_n)$, $n \geq 3$, with $\nabla p \neq 0$ everywhere, and call them *regular algebraic hypersurfaces* M^n of $\mathbb{R}^{n+1}$. Similar to the case $n = 2$, the only compact examples of such hypersurfaces are spheres. This follows immediately from Alexandrov theorem. So, we are left to consider the complete non-compact case. A generalized cylinder C^k in $\mathbb{R}^{n+1}$ is a product $B^k \times \mathbb{R}^{n-k}$, where the basis $B^k \subset \mathbb{R}^{k+1} \subset \mathbb{R}^{n+1}$ is a hypersurface of $\mathbb{R}^{k+1}$ and the product is embedded in $\mathbb{R}^{n+1}$ in the canonical way, i.e., $B^k \times \mathbb{R}^{n-k} \subset \mathbb{R}^{k+1} \times \mathbb{R}^{n-k}$. It is easily checked that when B is a k-sphere, C^k has nonzero constant mean curvature. The following lemma is again a consequence of Alexandrov's theorem.

Lemma 4.2. *Let C^k be an algebraic regular generalized cylinder in $\mathbb{R}^{n+1}$ whose basis B is a compact hypersurface. If C^k has constant mean curvature then the base B^k is a k-sphere.*

We do not know any further examples of a regular algebraic hypersurface in $\mathbb{R}^{n+1}$, $n > 2$, with nonzero constant mean curvature. We can ask a question similar to the one we answered for $n = 2$. The possible extension of our proof, however, needs new ideas. Although the topology is again finite, the proof of the structure theorem of [**8**] does not work for hypersurfaces in $\mathbb{R}^{n+1}$, $n > 2$.

References

[1] A.D. Alexandrov, *A characteristic property of spheres*, Ann. Math. Pura App. **58** (1962), 303–315, MR 0143162, Zbl 0107.15603.

[2] J. Bochnak, M. Coste & M-F Roy, *Real algebraic geometry*, A series of modern surveys in mathematics, vol. **36**, Springer, Berlin, Heidelberg, 1998, MR 1659509, Zbl 0912.14023.

[3] M. do Carmo, *Differential geometry of curves and surfaces*, Prentice Hall, 1976, MR 0394451, Zbl 0326.53001.

[4] D. Hoffman & W.H. Meeks, III, *The strong halfspace theorem for minimal surfaces* Inventiones Math. **101** (1990), 373–377, MR 1062966, Zbl 0722.53054.

[5] H. Hopf, *Über Flächen mit einer relation zwischen hauptkrümungen*, Math. Nachr. **4** (1951), 232–249, MR 0040042, Zbl 0042.15703.

[6] H. Hopf, *Differential geometry in the large*, Springer Lecture Notes in Mathematics **1000**, 2^{nd} edition, 1989, MR 1013786, Zbl 0669.53001.

[7] A. Huber, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv. **32** (1957), 13–72, MR 0094452, Zbl 0080.15001.

[8] N.J. Korevaar, R. Kusner & B. Solomon, *The structure of complete embedded surfaces with constant mean curvature*, J. Diff. Geometry **30** (1989), 465–503, MR 1010168, Zbl 0726.53007.

[9] W.S. Massey, *Surfaces of Gaussian curvature zero in Euclidean* 3*-space*, Tohoku Math. J. **14** (1962), 73–79, MR 0139088, Zbl 0114.36903.

[10] W.H. Meeks, III, *The topology and geometry of embedded surfaces of constant mean curvature*, J. Diff. Geometry **27** (1988), 539–552, MR 0940118, Zbl 0617.53007.

[11] J.C.C. Nitsche, *Lectures on minimal surfaces*, vol. 1, Cambridge University Press, Cambridge, 1989, MR 1015936, Zbl 1209.53002.

[12] R. Osserman, *The total curvature of algebraic surfaces*, Contributions to Analysis and Geometry, 249–257, Johns Hopkins University Press, Baltimore, 1981, MR 0648469, Zbl 0548.53048.

[13] R. Osserman, *Global properties of minimal surfaces in* E^3 *and* E^n, Annals of Mathematics **80** (1964), 340–364, MR 0179701, Zbl 0134.38502.

[14] O.M. Perdomo, *Algebraic constant mean curvature surfaces in Euclidean space*, Houston Journal of Mathematics **39** (2013), 127–136, MR 3056432, Zbl 1271.53006.

[15] R. Schoen, *Uniqueness, symmetry, and embeddedness of minimal surfaces*, Journal of Differential Geometry **18** (1983), 791–809, MR 0730928, Zbl 0575.53037.

[16] H. Whitney, *Geometric integration theory*, Princeton University Press, Princeton, N.J., 1957, MR 0087148, Zbl 0083.28204.

Rua Carolina Sucupira 723 Ap 2002
60140-120 Fortaleza - Ce, Brazil
E-mail address: joaolucasbarbosa@terra.com.br

Instituto Nacional de Matemática Pura e Aplicada - IMPA
Estrada Dona Castorina 110
22460-320 Rio de Janeiro - RJ, Brazil
E-mail address: manfredo@impa.br