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Pin$^-$(2)-MONOPOLE INVARIANTS

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Abstract

We introduce a diffeomorphism invariant of 4-manifolds, the Pin$^-$(2)-monopole invariant, defined by using the Pin$^-$(2)-monopole equations. We compute the invariants of several 4-manifolds, and prove gluing formulae. By using the invariants, we construct exotic smooth structures on the connected sum of an elliptic surface $E(n)$ with arbitrary number of the 4-manifolds of the form of $S^2 \times \Sigma$ or $S^1 \times Y$ where Σ is a compact Riemann surface with positive genus and Y is a closed 3-manifold. As another application, we give an estimate of the genus of surfaces embedded in a 4-manifold X representing a class $\alpha \in H_2(X; l)$, where l is a local coefficient on X.

1. Introduction

In the paper [**16**], we introduced the Pin$^-$(2)-monopole equations which are a twisted or a real version of the Seiberg-Witten equations, and obtained several constraints on the intersection forms with local coefficients of 4-manifolds by analyzing the moduli spaces. In this article, we investigate diffeomorphism invariants defined by using the Pin$^-$(2)-monopole equations, which we will call Pin$^-$(2)-*monopole invariants*. We compute the invariants of several 4-manifolds, and prove connected-sum formulae. We give two applications. The first application is to construct exotic smooth structures on $E(n)\#(\#_{i=1}^k(S^2 \times \Sigma_i))\#(\#_{j=1}^l(S^1 \times Y_j))$ where Σ_i are compact Riemann surfaces with positive genus and Y_j are closed 3-manifolds. The second application is an estimate of the genus of surfaces embedded in a 4-manifold X representing a class $\alpha \in H_2(X; l)$, where l is a local coefficient on X, which can be considered as a local coefficient analogue of the adjunction inequalities in the Seiberg-Witten theory [**11**, **5**, **14**, **19**].

1.1. Exotic smooth structures. We state the first application:

Theorem 1.1. *For any positive integer n, there exists a set $\mathcal{S}_n$ of infinitely many distinct smooth structures on the elliptic surface $E(n)$ which have the following significance: For $\sigma \in \mathcal{S}_n$, let $E(n)_\sigma$ be the*

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manifold with the smooth structure σ homeomorphic to $E(n)$. Let Z be a connected sum of arbitrary positive number of 4-manifolds, each of which is $S^2 \times \Sigma$ or $S^1 \times Y$ where Σ is a compact Riemann surface with positive genus and Y is a closed 3-manifold. Then, $E(n)_\sigma \# Z$ for different σ are mutually non-diffeomorphic.

Remark 1.2. A famous result due to C. T. C. Wall tells us that any pair of simply-connected smooth 4-manifolds M_1 and M_2 which have isomorphic intersection forms are *stably diffeomorphic* for stabilization by taking connected sums with $k(S^2 \times S^2)$ for sufficiently large k. (See e.g. [**9**].) Theorem 1.1 says that there exist infinitely many exotic structures on $E(n)$ which can not be stabilized by $S^2 \times \Sigma$ with positive $g(\Sigma)$ or $S^1 \times Y^3$.

1.2. $\mathrm{Pin}^-(2)$**-monopole invariants.** To prove the theorem above, the $\mathrm{Pin}^-(2)$-monopole invariant will be defined and used. We remark that the $\mathrm{Pin}^-(2)$-monopole equations are defined on a Spin^{c-}*-structure* (§2.1 and [**16**], Section 3), which is a $\mathrm{Pin}^-(2)$-analogue of Spin^c-structure. One of the special features of the $\mathrm{Pin}^-(2)$-monopole theory is that the moduli spaces may be nonorientable. Hence, in general, $\mathbb{Z}_2$-valued invariants will be defined. Only when the moduli space is orientable, $\mathbb{Z}$-valued invariants can be defined. Here, we state several non-vanishing results on the $\mathrm{Pin}^-(2)$-monopole invariants.

A Spin^{c-}-structure is an object on a double covering $\tilde{X} \to X$ of a 4-manifold X rather than on X itself. For a Spin^{c-}-structure on $\tilde{X} \to X$, an $\mathrm{O}(2)$-bundle E called the characteristic $\mathrm{O}(2)$-bundle is associated (§2.1). Let l be the $\mathbb{Z}$-bundle associated to the double covering $\tilde{X} \to X$, i.e., $l = \tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$. The l-coefficient Euler class of E in $H^2(X;l)$ is denoted by $\tilde{c}_1(E)$. More precisely, we need to fix an l-coefficient orientation of E to define the Euler class $\tilde{c}_1(E)$. (See §2.1.)

An Enriques surface N_0 has a double covering $\pi\colon K_0 \to N_0$ with K_0 a $K3$ surface. More generally, a smooth 4-manifold N which is homotopy equivalent to an Enriques surface is known to be homeomorphic to the standard Enriques surface [**18**], and has a double covering $\pi\colon K \to N$ such that K is a homotopy $K3$ surface. Let $l_K = K \times_{\{\pm 1\}} \mathbb{Z}$.

Theorem 1.3. *There exists a* Spin^{c-}*-structure c on $\pi\colon K \to N$ which satisfies the following:*

- $\pi^* \tilde{c}_1(E) = 0$*, where E is the characteristic* $\mathrm{O}(2)$*-bundle and* $\pi^*\colon H^2(N;l_K) \to H^2(K;\mathbb{Z})$ *is the induced homomorphism,*
- *the* $\mathbb{Z}_2$*-valued* $\mathrm{Pin}^-(2)$*-monopole invariant of* (N,c) *is nontrivial.*

Remark 1.4. The virtual dimension of the moduli space of (N,c) is 0.

Remark 1.5. Theorem 1.3 is proved by Theorem 2.22 which relates the Pin$^-$(2)-monopole invariants of N with the Seiberg-Witten invariants of the double covering K, together with the non-vanishing result due to J. Morgan and Z. Szabó [**13**] for homotopy $K3$ surfaces.

Next we state a connected-sum formula for Pin$^-$(2)-monopole invariants. Before that, we note the following remarks. In general, an ordinary Spin^c-structure can be seen as a reduction of an *untwisted* Spin^{c-}-structure defined on a *trivial* double cover $\tilde{X} \to X$ (§2.1). Furthermore, the Seiberg-Witten (U(1)-monopole) equations on a Spin^c-structure can be identified with the Pin$^-$(2)-monopole equations on the corresponding untwisted Spin^{c-}-structure (§2.4). Often, we will not distinguish an untwisted Spin^{c-}-structure and the Spin^c-structure which is its reduction, and use the same symbol. In the following, we consider the gluing of Pin$^-$(2)-monopoles and ordinary Seiberg-Witten U(1)-monopoles.

Let X_1 be a 4-manifold with an ordinary Spin^c-(or untwisted Spin^{c-}-) structure c_1. Let X_2 be the manifold Z in Theorem 1.1 whose connected-summands are of the form of $S^2 \times \Sigma$ or $S^1 \times Y$. To define a $\mathbb{Z}$-bundle on X_2, consider a 2-torus T^2 with a nontrivial $\mathbb{Z}$-bundle l_T. An oriented Riemann surface Σ with positive genus g can be considered as a connected sum of g tori: $\Sigma = T^2 \# \cdots \# T^2$. Let l_Σ be the $\mathbb{Z}$-bundle over Σ which is given by the connected sum of l_T: $l_\Sigma = l_T \# \cdots \# l_T$. For a Riemann surface Σ with positive genus, consider the product $S^2 \times \Sigma$ with the $\mathbb{Z}$-bundle l which is the pull-back

$$l = \pi^* l_\Sigma,$$

where $\pi\colon S^2 \times \Sigma \to \Sigma$ is the projection. We also consider $S^1 \times Y$ with the $\mathbb{Z}$-bundle l' which is the pullback of a nontrivial $\mathbb{Z}$-bundle l_{S^1} over S^1.

Remark 1.6. For $(X;l)$, let $b_k^l = b_k(X;l) = \dim H^k(X; l \otimes \mathbb{Q})$. For $(X,l) = (S^2 \times \Sigma_g; l)$, $b_0^l = b_2^l = b_4^l = 0$ and $b_1^l = b_3^l = 2g-2$. For $(X,l) = (S^1 \times Y; l)$, $b_k^l = 0$ for all k.

Recall X_2 is a connected-sum of 4-manifolds of the form of $S^2 \times \Sigma$ or $S^1 \times Y$. Equip each component of the form of $S^2 \times \Sigma$ (resp. $S^1 \times Y$) with the $\mathbb{Z}$-bundles l (resp. l') as above, and define the $\mathbb{Z}$-bundle l_{X_2} on X_2 as their connected sum. If we write the cardinality of $H^2(X_2; l_{X_2})$ as n, there are n distinct isomorphism classes of Spin^{c-}-structures for $\tilde{X}_2 \to X_2$, where $\tilde{X}_2$ is the double covering associated to l_{X_2}. (See Proposition 2.3.) Each of these Spin^{c-}-structures has a characteristic O(2)-bundle E with torsion $\tilde{c}_1(E)$. Let c_2 be such a Spin^{c-}-structure on X_2. We consider the connected sum $X_1 \# X_2$ with the Spin^{c-}-structure $c_1 \# c_2$ which is the connected sum of the Spin^{c-}-structures c_1 and c_2. (Here we assume c_1 is an untwisted Spin^{c-}-structure.) Then, the following holds:

Theorem 1.7. *Let X_1 be a closed oriented connected 4-manifolds with a* Spin^c*(untwisted* Spin^{c-}*)-structure such that*

- $b_+(X_1) \geq 2$,
- *the virtual dimension of the Seiberg-Witten moduli space for (X_1, c_1) is zero,*
- *the Seiberg-Witten invariant for (X_1, c_1) is odd.*

Let X_2 and l_{X_2} be as above. Then, for any Spin^{c-}*-structure c_2 on $\tilde{X}_2 \to X_2$, the* $\mathrm{Pin}^-(2)$*-monopole invariant of $(X_1\#X_2, c_1\#c_2)$ is nonzero.*

Remark 1.8. The virtual dimension d of the moduli space of $(X_1\#X_2, c_1\#c_2)$ is positive: For instance, if

$$X_2 = \#_{i=1}^k (S^2 \times \Sigma_i) \# \#_{j=1}^m (S^1 \times Y_j),$$

then

$$d = \sum_{i=1}^k (2g(\Sigma_i) - 2) + (k+m) = 2\sum_{i=1}^k g(\Sigma_i) - k + m \geq k + m.$$

Remark 1.9. This non-vanishing result would be interesting because of the following two points: First, although the dimension of the moduli space is positive, the (co)homological (not cohomotopical) invariant is nontrivial. Second, if X_2 contains a component of the form of $S^2 \times \Sigma$, all of the Seiberg-Witten invariants and the cohomotopy refinement [**2**] of $X_1\#X_2$ are 0 because $S^2 \times \Sigma$ admits a positive scalar curvature metric and $b_+(S^2 \times \Sigma) > 0$.

Remark 1.10. It is worth to notice that $b_+(X_2; l) = 0$. In fact, Theorem 1.7 can be considered as a $\mathrm{Pin}^-(2)$-monopole analogue of the Seiberg-Witten gluing formulae for connected sums $X_1\#X_2$ when X_1 is a 4-manifold with positive $b_+(X_1)$ and X_2 is one of the following:

(1) X_2 is a 4-manifold with $b_1(X_2) = b_+(X_2) = 0$, (Froyshov [**7**], Chapter 14 for general cases; Fintushel-Stern [**5**], Theorem 1.4 and Nicolaescu [**17**], §4.6.2 for $\overline{\mathbb{CP}}^2$; Kotschick-Morgan-Taubes [**10**], Proposition 2 for rational homology 4-spheres),
(2) $X_2 = S^1 \times S^3$, (Ozsváth-Szabó [**20**]) or
(3) X_2 is a connected sum of several manifolds in (1) or (2) above.

Remark 1.11. Theorem 1.7 is a special case of Theorem 3.8.

As mentioned above, the $\mathrm{Pin}^-(2)$-monopole invariants are defined as $\mathbb{Z}_2$-valued invariants. But in some exceptional cases, we can define $\mathbb{Z}$-valued invariants. For instance, the non-vanishing result for homotopy Enriques surfaces (Theorem 1.3) is refined as follows:

Theorem 1.12. *The $\mathbb{Z}$-valued* $\mathrm{Pin}^-(2)$*-monopole invariant for (N, c) in Theorem 1.3 is odd.*

Furthermore, the following holds for connected sums of homotopy Enriques surfaces.

Theorem 1.13. *For any integer $n \geq 2$, let $X_n = N_1 \# N_2 \# \cdots \# N_n$ where each N_i is a homotopy Enriques surface. Then X_n has a Spin^{c-}-structure c_n such that*

- *the $\mathbb{Z}_2$-valued $\mathrm{Pin}^-(2)$-monopole invariant is 0, but*
- *the $\mathbb{Z}$-valued invariant is nontrivial.*

Remark 1.14. Since $b_+(N_i) \geq 1$, the Seiberg-Witten invariants and Donaldson invariants of X_n are 0.

1.3. The genus of embedded surfaces. We state the second application of the $\mathrm{Pin}^-(2)$-monopole invariants, which is an estimate of the genus of embedded surfaces representing a local-coefficient class. Let X be a closed oriented connected 4-manifold and suppose a nontrivial double covering $\tilde{X} \to X$ is given, and let $l = \tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$. Then a homology class $\alpha \in H_2(X; l)$ is represented by an embedded surface Σ as follows:

- Σ is a connected surface embedded in X. Let $i \colon \Sigma \to X$ be the embedding map.
- The orientation system of Σ is identified with the pull-back $i^* l$ of l by i.
- If $i_* \colon H_2(\Sigma; i^* l) \to H_2(X; l)$ is the induced homomorphism and $[\Sigma] \in H_2(\Sigma; i^* l)$ is the fundamental class, then $\alpha = i_*[\Sigma]$.

Conversely, a connected embedded surface Σ whose orientation system is the restriction of l has its fundamental class $[\Sigma]$ in $H_2(X; l)$.

For such embedded surfaces, the following adjunction inequality holds.

Theorem 1.15. *Let c be a Spin^{c-}-structure on $\tilde{X} \to X$, and $\tilde{c}$ be the Spin^c-structure on $\tilde{X}$ induced from c (see §2). Suppose at least one of the following occurs*:

- *$b_+(X; l) \geq 2$ and the $\mathrm{Pin}^-(2)$-monopole invariant of (X, c) is nontrivial.*
- *$b_+(\tilde{X}) \geq 2$ and the ordinary Seiberg-Witten invariant of $(\tilde{X}, \tilde{c})$ is nontrivial.*

Suppose a class $\alpha \in H_2(X; l)$ is represented by a connected embedded surface as above. If α has infinite order and $\alpha \cdot \alpha \geq 0$, then

$$-\chi(\Sigma) \geq |\tilde{c}_1(E) \cdot \alpha| + \alpha \cdot \alpha,$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ.

Combining Theorem 1.15 with the non-vanishing results in §1.2, we obtain the following estimates for several concrete 4-manifolds.

Theorem 1.16. *Suppose a pair (X, l) of 4-manifold X, and a $\mathbb{Z}$-bundle l over X is one of the following:*

- $(N_1\#N_2\#\cdots\#N_n, l_1\#\cdots\#l_n)$, *where each* N_i *is a homotopy Enriques surface, and* l_i *is a nontrivial* $\mathbb{Z}$*-bundle, or*
- $(E(2)\#Z, l)$ *as in Theorem* 1.1.

Let Σ *be a connected embedded surface as above representing a class* $\alpha \in H_2(X; l)$. *If* α *has infinite order and* $\alpha\cdot\alpha \geq 0$, *then*

$$-\chi(\Sigma) \geq \alpha\cdot\alpha.$$

Remark 1.17. The number $\alpha\cdot\alpha$ is the normal Euler number of the embedding $\Sigma \subset X$.

From this, we can also obtain some kind of equivariant adjunction inequality on the double coverings:

Corollary 1.18. *Let* $\tilde{X} \to X$ *be the double covering associated with* (X, l) *in Theorem* 1.16, *and* $\iota\colon \tilde{X} \to \tilde{X}$ *be the covering transformation. Suppose an oriented connected surface* Σ *embedded in* $\tilde{X}$ *satisfies the property that* $[\Sigma] - \iota_*[\Sigma]$ *has infinite order in* $H_2(\tilde{X};\mathbb{Z})$ *and* $[\Sigma]\cdot[\Sigma] \geq 0$. *If* $\Sigma \cap \iota(\Sigma) = \emptyset$, *then*

$$-\chi(\Sigma) \geq [\Sigma]\cdot[\Sigma]. \tag{1.19}$$

Example 1.20. Let us examine Corollary 1.18 for a simple example. Let $X = K3\#(T^2\times S^2)$. Consider the double cover $\tilde{X} \to X$ which is associated to a nontrivial double cover $T^2\times S^2 \to T^2\times S^2$. Then $\tilde{X} = K_1\#(T^2\times S^2)\#K_2$, where K_i are copies of $K3$. Let $\sigma = [pt\times S^2]$ and $\tau = [T^2\times pt]$ in $H_2(T^2\times S^2;\mathbb{Z})$. Take a 2-sphere S representing σ embedded in the $T^2\times S^2$-component, and oriented connected surfaces Σ_i $(i = 1, 2)$ embedded in the K_i-components so that $[\Sigma_i] \neq 0$, $[\Sigma_i]^2 \geq 0$, $\iota(\Sigma_1)\cap\Sigma_2 = \emptyset$ and $\iota_*[\Sigma_1] \neq [\Sigma_2]$. Then we can arrange to take a connected sum $\Sigma = \Sigma_1\#S\#\Sigma_2$ in $\tilde{X}$ such that $\Sigma\cap\iota(\Sigma) = \emptyset$. Such a Σ certainly satisfies (1.19) because of the adjunction inequality for $K3$. On the other hand, we can construct oriented connected surfaces Σ embedded in $\tilde{X}$ with $\Sigma\cap\iota(\Sigma) \neq \emptyset$ which violate (1.19) as follows. Let g_1 be the genus of Σ_1 above. We can take an embedded 2-torus T representing $\tau + n\sigma$ so that $2n > 2g_1 - [\Sigma_1]^2$. Then take a connected sum $\Sigma = \Sigma_1\#T$ in $\tilde{X}$. Since $[\Sigma]\cdot\iota_*[\Sigma] = (\sigma + n\tau)^2 = 2n > 0$, we have $\Sigma\cap\iota(\Sigma) \neq \emptyset$.

The organization of the paper is as follows. In Section 2, we introduce $\mathrm{Pin}^-(2)$-monopole invariants, and discuss the relation with the Seiberg-Witten invariants on the double covering, and prove Theorem 1.3 and Theorem 1.12. In Section 3, several versions of gluing formulae are stated, and assuming these, we prove Theorem 1.1 and Theorem 1.13. Sections 4-6 are devoted to the proof of the gluing theorems stated in §2. Section 4 describes the $\mathrm{Pin}^-(2)$-monopole theory on 3-manifolds. Section 5 deals with finite energy $\mathrm{Pin}^-(2)$-monopoles on 4-manifolds with tubular ends. In Section 6, we give the proofs of the gluing theorems. In Section 7, the proof of the genus estimate (Theorem 1.15) is given.

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2. $\mathrm{Pin}^-(2)$-monopole invariants

2.1. Spin^{c-}**-structures.** The $\mathrm{Pin}^-(2)$-monopole equations are defined on Spin^{c-}-structures, which are a $\mathrm{Pin}^-(2)$-version of the Spin^c-structures. While a Spin^c-structure is given as a $\mathrm{Spin}^c(4) = \mathrm{Spin}(4) \times_{\{\pm 1\}} U(1)$-lift of the frame bundle, a Spin^{c-}-structure is given by a $\mathrm{Spin}(4)\times_{\{\pm 1\}}\mathrm{Pin}^-(2)$-lift of it. The precise definition is given as follows. (See also [**16**], Section 3.) The group $\mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{Pin}^-(2)$ is denoted by $\mathrm{Spin}^{c-}(4)$. Let X be a closed oriented connected Riemannian 4-manifold with double covering $\tilde{X} \to X$. The SO(4)-frame bundle on X is denoted by $Fr(X)$. Since $\mathrm{Pin}^-(2) = U(1) \cup j\,U(1)$, $\mathrm{Spin}^c(4)$ is the identity component of $\mathrm{Spin}^{c-}(4)$, and $\mathrm{Spin}^{c-}(4)/\mathrm{Spin}^c(4) = \{\pm 1\}$. Also we have $\mathrm{Spin}^{c-}(4)/\mathrm{Pin}^-(2) = \mathrm{SO}(4)$ and $\mathrm{Spin}^{c-}(4)/\mathrm{Spin}(4) = \mathrm{O}(2)$.

Definition 2.1. A Spin^{c-}-structure on $\tilde{X} \to X$ is a triple (P, σ, τ) where

- P is a $\mathrm{Spin}^{c-}(4)$-bundle over X,
- σ is an isomorphism between the $\mathbb{Z}/2$-bundles $P/\mathrm{Spin}^c(4)$ and $\tilde{X}$,
- τ is an isomorphism between the SO(4)-bundles $P/\mathrm{Pin}^-(2)$ and $Fr(X)$.

Instead of the determinant U(1)-bundle for a Spin^c-structure, an O(2)-bundle $E = P/\mathrm{Spin}(4)$ is associated to a Spin^{c-}-structure. We call this E the *characteristic* O(2)*-bundle*. Let l be the $\mathbb{Z}$-bundle $\tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$ over X. Then l is related to E by $\det E = l \otimes \mathbb{R}$. The l-coefficient orientation of E (and hence $\tilde{c}_1(E) \in H^2(X; l)$) is determined via the isomorphism $\sigma\colon P/\mathrm{Spin}^c(4) \overset{\cong}{\to} \tilde{X}$ as follows. As described in [**16**], §3.3, the $\mathrm{Spin}^c(4)$-bundle $P \to P/\mathrm{Spin}^c(4) \cong \tilde{X}$ defines a Spin^c-structure on $\tilde{X}$. Let L be its determinant line bundle, and $D(L)$, $S(L)$ be its disk and sphere bundles. Let $E_{\mathbb{R}}$ be the $\mathbb{R}^2$-bundle associated to E, and $D(E_{\mathbb{R}})$, $S(E_{\mathbb{R}})$ be similar objects. Then choose the l-coefficient orientation of E so that the Thom classes $\tilde{u} \in H^2(D(L), S(L); \mathbb{Z})$ of L and $u \in H^2(D(E_{\mathbb{R}}), S(E_{\mathbb{R}}); l)$ of $E_{\mathbb{R}}$ satisfy the relation

$$\pi^* u = \tilde{u}, \tag{2.2}$$

where π^* is the homomorphism induced from the projection $\pi\colon \tilde{X} \to X$. Then we also have the relation $\pi^*\tilde{c}_1(E) = c_1(L)$.

The basic fact on Spin^{c-}-structures on $\tilde{X} \to X$ is as follows:

Proposition 2.3. (1) For an O(2)-bundle E over X with $\det E = l \otimes \mathbb{R}$ as above, there exists a Spin^{c-}-structure on $\tilde{X} \to X$ whose characteristic bundle is isomorphic to E if and only if $w_2(X) = w_2(E) + w_1(l \otimes \mathbb{R})^2$. (2) If a Spin^{c-}-structure on $\tilde{X} \to X$ is given, there is a bijective correspondence between the set of isomorphism classes of Spin^{c-}-structures on $\tilde{X} \to X$ and $H^2(X; l)$.

Proof. The assertion (1) is proved in [**16**]. To prove the assertion (2), let us consider the exact sequence,

$$1 \to S^1 \to \mathrm{Spin}^{c-}(4) \to \mathrm{SO}(4) \times \{\pm 1\} \to 1. \tag{2.4}$$

From this, we have a fibration,

$$BS^1 \to B\,\mathrm{Spin}^{c-}(4) \to B(\mathrm{SO}(4) \times \{\pm 1\}). \tag{2.5}$$

In (2.4), $\{\pm 1\}$ gives rise to an automorphism of S^1 of complex conjugation. If we identify BS^1 with $\mathbb{C}\mathrm{P}^\infty$, the action of $\pi_1(B(\{\pm 1\})) \cong \mathbb{Z}_2$ on a fiber of (2.5) can be homotopically identified with complex conjugation on $\mathbb{C}\mathrm{P}^\infty$. Then Spin^{c-}-structures on $\tilde{X} \to X$ are classified by

$$H^2(X; \tilde{\pi}_2(BS^1)) \cong H^2(X; l),$$

where $\tilde{\pi}_2$ is the local coefficient with respect to the $\pi_1(B(\{\pm 1\}))$-action on fibers. q.e.d.

Usually, we will assume the covering $\tilde{X} \to X$ is nontrivial. But in the case when $\tilde{X} \to X$ is trivial, the $\mathrm{Spin}^{c-}(4)$-bundle of a Spin^{c-}-structure on X has a $\mathrm{Spin}^{c}(4)$-reduction, and in fact, this reduction induces a Spin^{c}-structure on X. We will refer to a Spin^{c-}-structure with trivial $\tilde{X}$ as an *untwisted* Spin^{c-}*-structure*.

2.2. Definition of $\mathrm{Pin}^-(2)$-monopole invariants. In this subsection, we introduce $\mathrm{Pin}^-(2)$-monopole invariants. Let X be an oriented closed connected 4-manifold with double covering $\tilde{X} \to X$, and suppose a Spin^{c-}-structure c on $\tilde{X} \to X$ is given. Let $l = \tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$, $\lambda = l \otimes \mathbb{R}$, and E be the characteristic O(2)-bundle. Then we have $\lambda = \det E$. Let $\mathcal{A}$ be the space of O(2)-connections on E, $\mathcal{C}$ the configuration space $\mathcal{C} = \mathcal{A} \times \Gamma(S^+)$, and $\mathcal{C}^*$ the space of irreducible configurations, $\mathcal{C}^* = \mathcal{A} \times (\Gamma(S^+) \setminus 0)$. Fix $k \geq 3$ and take L^2_k-completion of $\mathcal{C}$ and $\mathcal{C}^*$. The gauge transformation group $\mathcal{G}$ is the L^2_{k+1}-completion of $\Gamma(\tilde{X} \times_{\{\pm 1\}} \mathrm{U}(1))$, where $\{\pm 1\}$ acts on U(1) by complex conjugation. We use the same symbols for the completed spaces. Let $\mathcal{B}^* = \mathcal{C}^*/\mathcal{G}$.

The (perturbed) Pin$^-$(2)-monopole equations for $(A, \Phi) \in \mathcal{C}$ are given as follows:

$$\begin{cases} D_A\Phi = 0, \\ \dfrac{1}{2}F_A^+ = q(\Phi) + \mu, \end{cases} \tag{2.6}$$

where D_A is the Dirac operator, q is a quadratic form and $\mu \in \Omega^+(i\lambda)$. (See Section 4 of [**16**] for the precise meaning and definition of each term of the equations.)

Remark 2.7. Here we adopt the convention according to [**12**], slightly different from [**16**], with $\frac{1}{2}$ on the curvature term F_A^+. Of course, this set of the equations is essentially same with that in [**16**], because they coincide after an appropriate rescaling.

The moduli space $\mathcal{M}(X, c) = \mathcal{M}_{\mathrm{Pin}^-(2)}(X, c)$ is defined as the space of solutions modulo gauge transformations. (The perturbed moduli space is usually denoted by the same symbol.)

Remark 2.8. When the Spin^{c-}-structure is untwisted, since $\tilde{X} \to X$ is trivial, we have $\mathcal{G} = \Gamma(\tilde{X} \times_{\{\pm 1\}} \mathrm{U}(1)) \cong \mathrm{Map}(X, \mathrm{U}(1))$. While the stabilizer of the Pin$^-$(2)-monopole reducible on a twisted Spin^{c-}-structure is $\{\pm 1\}$, that in the untwisted case is U(1). (See also §2.4.)

For the time being, we suppose the Spin^{c-}-structure is twisted. Suppose $b_+(X; l) \geq 1$. Then, as in the case of the ordinary Seiberg-Witten theory, by a generic choice of μ, the moduli space $\mathcal{M}(X, c)$ has no reducible and is a compact manifold whose dimension is given by

$$d(c) = \frac{1}{4}(\tilde{c}_1(E)^2 - \mathrm{sign}(X)) - (b_0(X; l) - b_1(X; l) + b_+(X; l)). \tag{2.9}$$

Note that the index of the Dirac operator D_A is given by $\frac{1}{4}(\tilde{c}_1(E)^2 - \mathrm{sign}(X))$ and $b_0(X; l) = 0$ if l is nontrivial.

In a sense, the Pin$^-$(2)-monopole invariant of (X, c) is defined as the fundamental class of the moduli space $[\mathcal{M}(X, c)] \in H_{d(c)}(\mathcal{B}^*)$. We can obtain a numerical invariant by evaluating $[\mathcal{M}(X, c)]$ by a cohomology class in $H^{d(c)}(\mathcal{B}^*)$. If $\tilde{X} \to X$ is nontrivial, $\mathcal{B}^*$ has the homotopy type of the classifying space of the group $\mathbb{Z}/2 \times \mathbb{Z}^{b_1(X;l)}$. This fact is stated in [**16**], Proposition 25. However, the proof of Lemma 27 in [**16**] which is used in the proof of Proposition 25 is incomplete in that it is not proved there that the identity component of $\mathcal{G}$ is contractible. Here we complement it.

Lemma 2.10. *The gauge transformation group $\mathcal{G}$ is homotopy equivalent to* $(\mathbb{Z}/2) \times \mathbb{Z}^{b_1(X;l)}$.

Proof. Let $\tilde{\mathcal{G}} = \mathrm{Map}(\tilde{X}, \mathrm{U}(1))$. Define the involution I on $\tilde{\mathcal{G}}$ by $u \mapsto \overline{\iota^* u}$ where $\iota\colon \tilde{X} \to \tilde{X}$ is the covering transformation and " $\bar{\cdot}$ " means the complex conjugation. Then $\mathcal{G}$ is identified with the I-fixed point set $\tilde{\mathcal{G}}^I$. Let $h\colon \tilde{\mathcal{G}} \to [\tilde{X}, S^1] \cong H^1(\tilde{X};\mathbb{Z}) \cong \mathbb{Z}^{b_1(\tilde{X})}$ be the map which sends each element of $\tilde{\mathcal{G}}$ to its homotopy class. Put $\tilde{\mathcal{K}} = \ker h$. Consider the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \tilde{\mathcal{K}} & \longrightarrow & \tilde{\mathcal{G}} & \longrightarrow & [X, S^1] & \xrightarrow{h} & 1 \\ & & \uparrow & & \uparrow & & \uparrow j & & \\ 1 & \longrightarrow & \tilde{\mathcal{K}} \cap \tilde{\mathcal{G}}^I & \longrightarrow & \tilde{\mathcal{G}}^I & \longrightarrow & h(\tilde{\mathcal{G}}^I) & \longrightarrow & 1 \end{array}$$

The vertical map j is injective since the first and second vertical maps are inclusions. It is proved that $\pi_0\tilde{\mathcal{G}}^I = \pi_0\mathcal{G} = \mathbb{Z}_2 \oplus \mathbb{Z}^{b_1(X;l)}$ in the proof of Lemma 27 in [**16**]. Now it suffices to see that $\tilde{\mathcal{K}} \cap \tilde{\mathcal{G}}^I$ is homotopy equivalent to $\{\pm 1\}$. Each element $u \in \tilde{\mathcal{K}}$ can be written as $u = \exp(2\pi\sqrt{-1}f)$ for some function $f\colon \tilde{X} \to \mathbb{R}$. If $u = \exp(2\pi\sqrt{-1}f)$ is in $\tilde{\mathcal{K}} \cap \tilde{\mathcal{G}}^I$, then there is an integer m so that $f(\iota x) = m - f(x)$ for every $x \in \tilde{X}$. If we fix a base point $x_0 \in \tilde{X}$ and choose f so that $f(x_0) \in [0, 1)$, then such an m is uniquely determined. Then the homotopy $f_t = tf + (1-t)m/2$ gives the homotopy between u and ± 1.

q.e.d.

In contrast to the ordinary Seiberg-Witten theory, the moduli space $\mathcal{M}(X, c)$ may be non-orientable. (A necessary condition for $\mathcal{M}(X, c)$ to be orientable will be given in §2.3.) In general, we can define the following $\mathbb{Z}/2$-valued version of the $\mathrm{Pin}^-(2)$-monopole invariants.

Definition 2.11. The $\mathrm{Pin}^-(2)$-monopole invariant of (X, c) is defined as a map

$$\mathrm{SW}^{\mathrm{Pin}}(X, c)\colon H^{d(c)}(\mathcal{B}^*;\mathbb{Z}/2) \to \mathbb{Z}/2,$$

given by

$$\mathrm{SW}^{\mathrm{Pin}}(X, c)(\xi) := \langle \xi, [\mathcal{M}(X, c)] \rangle.$$

If $b_+(X;l) \geq 2$, then $\mathrm{SW}^{\mathrm{Pin}}(X, c)$ is a diffeomorphism invariant. If $b_+(X;l) = 1$, then $\mathrm{SW}^{\mathrm{Pin}}(X, c)$ depends on the chamber structure of the space of metrics and perturbations.

Remark 2.12. We give a geometric description of the cohomology classes of $\mathcal{B}^*$ in §3.1 and §3.2.

Remark 2.13. The compactness of $\mathcal{M}(X, c)$ enables us to develop the Bauer-Furuta theory [**2**] for the $\mathrm{Pin}^-(2)$-monopole equations. In fact, we can define a stable cohomotopy refinement of the $\mathrm{Pin}^-(2)$-monopole invariants. This will be discussed elsewhere.

2.3. Orientability of the moduli spaces. The purpose of this subsection is to discuss the orientability of the moduli spaces. Let us consider the family of Dirac operators $\tilde{\delta}_{Dirac} = \{D_A\}_{A\in\mathcal{A}}$. In [**16**], §4, we introduced a subgroup $\mathcal{K}_\gamma$ in $\mathcal{G}$, which has the properties:

- $\mathcal{G}/\mathcal{K}_\gamma = \{\pm 1\}$.
- $\mathcal{K}_\gamma$ acts on $\mathcal{A}$ freely, and $\mathcal{A}/\mathcal{K}_\gamma$ has the same homotopy type of $H^1(X;\lambda)/H^1(X;l)$.

Remark 2.14. Here γ is a circle embedded in X on which l is nontrivial. The subgroup $\mathcal{K}_\gamma$ is defined as the set of gauge transformations whose restrictions to γ are homotopic to 1.

Dividing $\tilde{\delta}_{Dirac}$ by $\mathcal{K}_\gamma$, we obtain the family $\delta_{Dirac} = \tilde{\delta}_{Dirac}/\mathcal{K}_\gamma$ over $\mathcal{A}/\mathcal{K}_\gamma$.

Proposition 2.15. *If the index of the Dirac operator is even and* $\det\operatorname{ind}\delta_{Dirac}$ *is trivial, then the moduli space is orientable.*

Proof. For a configuration (A,Φ), let us consider the sequence,

$$0 \longrightarrow \Omega^0(i\lambda) \xrightarrow{\mathcal{I}_\Phi} \Omega^1(i\lambda)\oplus\Gamma(S^+)$$
$$\xrightarrow{\mathcal{D}_{(A,\Phi)}} \Omega^+(i\lambda)\oplus\Gamma(S^-) \longrightarrow 0,$$

where $\mathcal{I}_\Phi(f) = (-2df, f\Phi)$ and $\mathcal{D}_{(A,\Phi)}(a,\phi) = d^+a - Dq_\Phi(\phi), D_A\phi + \frac{1}{2}\rho(a)\Phi)$, which are the linearizations of the gauge group action and the monopole map. Let $V = \Omega^1(i\lambda)\oplus\Gamma(S^+)$, and $W = (\Omega^0\oplus\Omega^+)(i\lambda)\oplus\Gamma(S^-)$ and define $\delta_{(A,\Phi)}\colon V\to W$ by,

$$\delta_{(A,\Phi)} = \mathcal{I}_\Phi^*\oplus\mathcal{D}_{(A,\Phi)}.$$

Then the family $\tilde{\delta} = \{\delta_{(A,\Phi)}\}_{(A,\Phi)\in\mathcal{C}}$ defines a bundle homomorphism between the bundles over $\mathcal{C}$,

$$\tilde{\delta}\colon \mathcal{C}\times V\to\mathcal{C}\times W.$$

Restricting $\tilde{\delta}$ to $\mathcal{C}^*$ and dividing by $\mathcal{G}$, we obtain a bundle homomorphism over $\mathcal{B}^* = \mathcal{C}^*/\mathcal{G}$,

$$\delta\colon \mathcal{C}^*\times_{\mathcal{G}} V\to\mathcal{C}^*\times_{\mathcal{G}} W.$$

The moduli space is orientable if $\det\operatorname{ind}\delta$ is trivial. By deforming $\delta_{(A,\Phi)}$ by $\delta_{(A,t\phi)}$ $(0\le t\le 1)$, we may assume $\tilde{\delta} = \{(d^*\oplus d^+)\oplus D_A\}_{(A,\Phi)\in\mathcal{C}}$. Since $(d^*\oplus d^+)$ does not depend on (A,Φ), $\det\operatorname{ind}(d^*\oplus d^+)$ is trivial. Therefore it suffices to consider the Dirac family

$$\tilde{\delta}' = \{D_A\}_{(A,\Phi)\in\mathcal{C}}\colon \mathcal{C}\times\Gamma(S^+)\to\mathcal{C}\times\Gamma(S^-). \tag{2.16}$$

Then (2.16) can be identified with the pull-back of $\tilde{\delta}_{Dirac}$, via the projection $p\colon\mathcal{C}\to\mathcal{A}$ with $p(A,\Phi) = A$. Dividing (2.16) by $\mathcal{K}_\gamma$, we obtain $\tilde{\delta}'/\mathcal{K}\colon \mathcal{C}\times_{\mathcal{K}_\gamma}\Gamma(S^+)\to\mathcal{C}\times_{\mathcal{K}_\gamma}\Gamma(S^-)$. Note that $\mathcal{C}/\mathcal{K}_\gamma$ is homotopic to $\mathcal{A}/\mathcal{K}_\gamma$. Thus $\operatorname{ind}(\tilde{\delta}'/\mathcal{K})$ is identified with $p^*\operatorname{ind}(\delta_{Dirac})$, which is trivial by the assumption. Hence $\det \operatorname{ind} \delta$ is trivial if and only if $\det((p^* \operatorname{ind}(\delta_{Dirac}))|_{\mathcal{C}^*})/\{\pm 1\})$ over $\mathcal{C}^*/\mathcal{G}$ is trivial. Note that $\mathcal{C}^*/\mathcal{G} \simeq \mathbb{R}\mathrm{P}^\infty \times T^{b_1(X;l)}$. Let $\eta \to \mathcal{C}^*/\mathcal{G}$ be the nontrivial real line bundle which represents the generator of $H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2)$. Then by the assumptions, we see that $\det((p^* \operatorname{ind}(\delta_{Dirac})|_{\mathcal{C}^*})/\{\pm 1\}) \cong \eta^{\otimes \operatorname{ind} D}$. Thus the proposition is proved. q.e.d.

Remark 2.17. For instance, if $b_1(X;l) = 0$ and the Dirac index is even, then the moduli space is orientable.

Note that $H^*(\mathcal{B}^*; \mathbb{Z})/\mathrm{Tor} \cong H^*(T^{b_1(X;l)}; \mathbb{Z})$. Suppose the moduli space $\mathcal{M}(X)$ is orientable. Fixing an orientation, we can define $\mathbb{Z}$-valued $\mathrm{Pin}^-(2)$-monopole invariants $\mathrm{SW}^{\mathrm{Pin}}_{\mathbb{Z}}$ by evaluating the fundamental class $[\mathcal{M}(X)]$ by infinite-order classes ξ in $H^*(\mathcal{B}^*; \mathbb{Z})$:

$$\mathrm{SW}^{\mathrm{Pin}}_{\mathbb{Z}}(X, c)(\xi) = \langle \xi, [\mathcal{M}(X)] \rangle.$$

2.4. $\mathrm{Pin}^-(2)$-monopoles on untwisted Spin^{c-}-structures. Let us consider an untwisted Spin^{c-}-structure $c = (P, \sigma, \tau)$ on a (trivial) double covering $\tilde{X} \to X$. The two connected components of $\tilde{X}$ will be denoted by X_+ and X_- according to the rule described below. Consider the Spin^c-structure on $\tilde{X}$ which is defined by the projection $P \to P/\mathrm{Spin}^c(4) \cong \tilde{X}$. Its restrictions to the components X_+ and X_- of $\tilde{X}$ are mutually complex conjugate Spin^c-structures c_+ and c_- (see [**16**], §2(iii)). Let $i_\pm \colon X_\pm \to \tilde{X}$ be the inclusion maps. Let $L_\pm$ be the determinant line bundles of $c_\pm$, and their Thom classes be $u_\pm$. Then X_+ is chosen to satisfy

$$u_+ = i_+^*(\tilde{u}) = i_+^* \circ \pi^*(u),$$

where u and $\tilde{u}$ are the Thom classes as in (2.2). We call the Spin^c-structure c_+ the *canonical reduction*.

Remark 2.18. When a Spin^c-structure c_0 with $\mathrm{Spin}^c(4)$-bundle $P_c \to X$ is given, the $\mathrm{Spin}^{c-}(4)$-bundle $P = P_c \times_{\mathrm{Spin}^c(4)} \mathrm{Spin}^{c-}(4)$ defines an untwisted Spin^{c-}-structure c on $\tilde{X} = P/\mathrm{Spin}^c(4) \to X$. Then c_0 is the canonical reduction of c.

As *real* vector bundles, we have identifications among spinor bundles for c, c_+ and c_-,

$$S^\pm_c \cong S^\pm_{c_+} \cong S^\pm_{c_-}.$$

Also as *real* vector bundles, we have identifications among the $\mathbb{R}^2$-vector bundle associated to the characteristic $\mathrm{O}(2)$-bundle E of c and the determinant line bundles $L_\pm$. If an $\mathrm{O}(2)$-connection A on E is given, we have $\mathrm{U}(1)$-connections $A_\pm$ on $L_\pm$ induced from A by reduction. As *real* operators, the covariant derivatives of A and $A_\pm$ can be identified, and therefore the Dirac operators induced from A and $A_\pm$ can also be identified as real operators. Furthermore, it can be seen that the $\mathrm{Pin}^-(2)$-monopole solutions on c can be identified with the Seiberg-Witten solutions on $c_\pm$ via the identifications above:

Proposition 2.19. *Let c be an untwisted Spin^{c-}-structure, and $c_\pm$ the Spin^c-structures which are its reductions as above. Then there are identifications among the set of $\mathrm{Pin}^-(2)$-monopole solutions on c and the sets of Seiberg-Witten solutions on $c_\pm$. Moreover, at the level of moduli spaces, we have*

$$\mathcal{M}_{\mathrm{Pin}^-(2)}(X,c) \cong \mathcal{M}_{\mathrm{U}(1)}(X,c_+) \cong \mathcal{M}_{\mathrm{U}(1)}(X,c_-),$$

where $\mathcal{M}_{\mathrm{U}(1)}$ means the ordinary Seiberg-Witten ($\mathrm{U}(1)$*-monopole*) *moduli spaces.*

In what follows, when we use a phrase like "a Spin^c(untwisted Spin^{c-})-structure c", it means an untwisted Spin^{c-}-structure and its canonical reduction. We consider them to be an equivalent object, and use them alternatively according to situations.

2.5. Relation with the Seiberg-Witten invariants of the double coverings. Let us consider a twisted Spin^{c-}-structure c on a (nontrivial) covering $\pi\colon \tilde{X} \to X$. If we pull-back the Spin^{c-}-structure c to $\tilde{X}$, the pulled-back Spin^{c-}-structure $\tilde{c}$ on $\tilde{X}$ is *untwisted*. If P is the $\mathrm{Spin}^{c-}(4)$-bundle for c, the projection $P \to P/\mathrm{Spin}^c \cong \tilde{X}$ can be considered as a $\mathrm{Spin}^c(4)$-bundle over $\tilde{X}$ which defines a Spin^c-structure $\tilde{c}_+$ over $\tilde{X}$ which is, in fact, the canonical reduction of $\tilde{c}$. Then π^*P is identified with $P \times_{\mathrm{Spin}^c(4)} \mathrm{Spin}^{c-}(4)$. The covering transformation $\iota\colon \tilde{X} \to \tilde{X}$ has a natural lift $\tilde{\iota}$ on $\tilde{c}$ which is given by a $\mathrm{Spin}^{c-}(4)$-bundle morphism of $P \times_{\mathrm{Spin}^c(4)} \mathrm{Spin}^{c-}(4)$ defined by $\tilde{\iota}([p,g]) = [pJ, J^{-1}g]$ for $[p,g] \in P \times_{\mathrm{Spin}^c(4)} \mathrm{Spin}^{c-}(4)$, where $J = [1, j^{-1}] \in \mathrm{Spin}^{c-}(4) = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{Pin}^-(2)$. Then there is a bijective correspondence between the configuration space of c and the space of $\tilde{\iota}$-invariant configurations on $\tilde{c}$. If we interpret the objects on $\tilde{c}$ in terms of the Spin^c-structure $\tilde{c}_+$, the $\tilde{\iota}$-action is identified with the antilinear involution I defined in [**16**], §4(v). Thus we can identify configurations on (X,c) with I-invariant configurations on $(\tilde{X}, \tilde{c}_+)$. In particular, we have,

Proposition 2.20 ([**16**], Proposition 4.11)**.** *There is a bijective correspondence between the set of* $\mathrm{Pin}^-(2)$*-monopole solutions on* (X,c) *and the set of I-invariant Seiberg-Witten solutions on* $(\tilde{X}, \tilde{c}_+)$*. Moreover we have*

$$\mathcal{M}_{\mathrm{Pin}^-(2)}(X,c) \cong \mathcal{M}_{\mathrm{U}(1)}(\tilde{X}, \tilde{c}_+)^I. \tag{2.21}$$

Let us discuss the relation of the $\mathrm{Pin}^-(2)$-monopole invariants of X and the Seiberg-Witten invariants of $\tilde{X}$. Mimicking the arguments in [**21**] or [**15**], we can prove a formula which relates the $\mathrm{Pin}^-(2)$-monopole invariants of (X, c) with the Seiberg-Witten invariants of $(\tilde{X}, \tilde{c}_+)$ as follows.

Theorem 2.22. *If* $d(c) = 0$ *and* $b_1(\tilde{X}) = 0$*, then*

$$\mathrm{SW}^{\mathrm{U}(1)}(\tilde{X}, \tilde{c}_+) \equiv \sum_{c_\sigma} \mathrm{SW}^{\mathrm{Pin}}(X, c_\sigma) \mod 2 \tag{2.23}$$

where $\mathrm{SW}^{\mathrm{U}(1)}(\tilde{X}, \tilde{c}_+)$ *is the Seiberg-Witten invariant of* $(\tilde{X}, \tilde{c}_+)$*, and* c_σ *runs through all* Spin^{c-}*-structures on* X *whose pull-back on* $\tilde{X}$ *are isomorphic to* $\tilde{c}_+$*.*

Remark 2.24. Since the I-action is free and $d(c) = 0$, the virtual dimension of the Seiberg-Witten moduli for $(\tilde{X}, \tilde{c}_+)$ is also zero.

Remark 2.25. The set of c_σ's as above is identified with

$$\{c + a \mid a \in \ker(\pi^*\colon H^*(X; l) \to H^*(\tilde{X}; \pi^* l))\}.$$

Proof of Theorem 2.22*.* In the I-equivariant setting, the moduli space $\mathcal{M}_{\mathrm{U}(1)}(\tilde{X}, \tilde{c}_+)$ is decomposed into the I-invariant part and the free part. The I-invariant part is identified with $\mathcal{M}_{\mathrm{Pin}^-(2)}(X, c)$ as in (2.21). On the other hand, if the free part is a 0-dimensional manifold, then the number of elements in the free part is even, because $\mathbb{Z}/2$ acts freely. Now, the theorem follows if the equivariant transversality can be achieved by an equivariant perturbation. This issue is discussed in [**15**]. (*Cf.* [**21**].) It is easy to achieve the transversality on the free part. For the I-invariant part, on each point $\xi \in \mathcal{M}_{\mathrm{U}(1)}(\tilde{X}, \tilde{c}_+)^I$, consider the Kuranishi model $f_\xi\colon H_1 \to H_2$, where H_1 and H_2 are finite dimensional I-linear vector spaces. Since the I-action on the base space $\tilde{X}$ is free, the Lefschetz formula tells us that H_1 and H_2 are isomorphic as the I-spaces. Then fixing an I-linear isomorphism $L_\xi\colon H_1 \to H_2$, we can perturb the equations I-equivariantly by using L_ξ to achieve the transversality around ξ. q.e.d.

Now, we can prove Theorem 1.3 and Theorem 1.12.

Proof of Theorem 1.3 *and Theorem* 1.12*.* There exists a Spin^{c-}-structure c on N whose associated $\mathrm{O}(2)$-bundle is isomorphic to $\underline{\mathbb{R}} \oplus (l_K \otimes \mathbb{R})$. Then the associated Spin^c-structure $\tilde{c}$ on the double cover K has a trivial determinant line bundle. Then $\mathrm{SW}^{\mathrm{U}(1)}(K, \tilde{c})$ is congruent to one modulo 2 by Morgan-Szabó [**13**]. On the other hand, since $b_1(N; l) = 0$, the Dirac index is even and $d(c) = 0$ for the Spin^{c-}-structure c, the moduli space is orientable, and by fixing an orientation, the $\mathbb{Z}$-valued invariant is defined. Then, by Theorem 2.22, there is a Spin^{c-}-structure c' such that $\mathrm{SW}^{\mathrm{Pin}}_{\mathbb{Z}}(N, c')$ is odd. q.e.d.

Remark 2.26. At present, the author does not know the exact value of $\mathrm{SW}^{\mathrm{Pin}}_{\mathbb{Z}}(N, c')$ for any homotopy Enriques surface N.

3. Gluing formulae

In this section, we state several versions of gluing formulae for the Pin$^-$(2)-monopole invariants, and prove Theorem 1.1 and Theorem 1.13. Before that, we introduce two kinds of μ-maps in order to represent various cohomology classes of $\mathcal{B}^*$.

3.1. μ-map (1). In this subsection, we define the first μ-map, $\mu_{\mathcal{E}}$. The isomorphism class of a double cover $\tilde{X} \to X$ is determined by a homomorphism $\rho\colon \pi_1(X) \to \{\pm 1\}$. Let $H = \pi_1(\tilde{X})$. When the double cover $\tilde{X} \to X$ is nontrivial, we have the exact sequence

$$1 \to H \to \pi_1(X) \xrightarrow{\rho} \{\pm 1\} \to 1.$$

Let ι_* be the involution on the rational cohomology group $H_1(\tilde{X};\mathbb{Q})$ induced from the covering transformation $\iota\colon \tilde{X} \to \tilde{X}$. If we write its $(+1)$(resp. (-1))-eigenspace as H_1^+ (resp. H_1^-), we have the identifications $H_1^+ \cong H_1(X;\mathbb{Q})$ and $H_1^- \cong H_1(X; l \otimes \mathbb{Q})$, where $l = \tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$. On the other hand, $H_1(\tilde{X};\mathbb{Q})$ is identified with $(H/[H,H]) \otimes \mathbb{Q}$. Then we can choose loops $\gamma_1, \dots, \gamma_b$ in X, where $b = b_1(X;l)$, such that

(C1) the homotopy class of each γ_i is in $\ker\rho$, and
(C2) the homology classes of $\gamma_1, \dots, \gamma_b$ generate $H_1(X;l)/\mathrm{Tor}$.

Note that the restriction of l to γ_i is a trivial $\mathbb{Z}$-bundle and the restriction E_{γ_i} of E to γ_i has a unique U(1)-reduction according to the l-orientation of E.

Let E be the characteristic O(2)-bundle of a Spin^{c-}-structure on a nontrivial double covering $\tilde{X} \to X$, and $\pi\colon X \times \mathcal{C}^* \to X$ be the projection. We define the universal characteristic O(2)-bundle $\mathcal{E}$ over $X \times \mathcal{B}^*$ as $\mathcal{E} = \pi^*E/\mathcal{G}$. Then we have its characteristic classes

$$\tilde{c}_1(\mathcal{E}) \in H^2(X \times \mathcal{B}^*; l\hat{\otimes}\mathbb{Z}), \quad w_2(\mathcal{E}) \in H^2(X \times \mathcal{B}^*; \mathbb{Z}_2),$$

where $\hat{\otimes}$ denotes the exterior tensor product of local coefficients. Now let us define the μ-maps

$$\hat{\mu}_{\mathcal{E}}\colon H_1(X;l) \to H^1(\mathcal{B}^*;\mathbb{Z}), \quad \mu_{\mathcal{E}}\colon H_1(X;\mathbb{Z}_2) \to H^1(\mathcal{B}^*;\mathbb{Z}_2),$$

by the formula

$$\hat{\mu}_{\mathcal{E}}(\alpha) = \tilde{c}_1(\mathcal{E})/\alpha, \quad \mu_{\mathcal{E}}(\alpha) = w_2(\mathcal{E})/\alpha.$$

Since the restriction of $\det E = l \otimes \mathbb{R}$ to γ_i is a trivial $\mathbb{R}$-bundle over γ_i, for any O(2)-connection A on E, the holonomy $\mathrm{Hol}_{\gamma_i}(A)$ around γ_i is contained in SO(2) $\subset$ O(2). Let $\hat{\theta} \in H^1(\mathrm{SO}(2);\mathbb{Z})$ and $\theta \in H^1(\mathrm{SO}(2);\mathbb{Z}_2)$ be the generators.

Proposition 3.1. *$\hat{\mu}_{\mathcal{E}}(\gamma_i) = \mathrm{Hol}^*_{\gamma_i}\hat{\theta}$, $\mu_{\mathcal{E}}(\gamma_i) = \mathrm{Hol}^*_{\gamma_i}\theta$.*

Remark 3.2. As in the proposition above, we sometimes abuse the symbol for a loop to denote its homotopy class or homology class.

Proof. (The proof is parallel to the ordinary Seiberg-Witten case. *Cf.* [**20**], §9.) For a loop $\beta\colon S^1 \to \mathcal{B}^*$, the restriction $\mathcal{E}|_{\gamma_i\times\beta}$ has a U(1)-reduction associated to the U(1)-reduction of $E|_{\gamma_i}$. Then

$$\langle \tilde{c}_1(\mathcal{E})/\gamma_i, \beta\rangle = \langle c_1(\mathcal{E}|_{\gamma_i\times\beta}), \gamma_i\times\beta\rangle = \deg(\mathrm{Hol}_{\gamma_i}\circ\beta).$$

q.e.d.

Since $\mathcal{B}^* \simeq \mathbb{R}\mathrm{P}^\infty \times T^b$, $H_1(\mathcal{B}^*;\mathbb{Z}_2)$ and $H_1(\mathcal{B}^*;\mathbb{Z})$ have decompositions

$$H_1(\mathcal{B}^*;\mathbb{Z}_2) = H_P\oplus H_T, \quad H_1(\mathcal{B}^*;\mathbb{Z}) = \hat{H}_P\oplus\hat{H}_T,$$

where H_P is a subgroup isomorphic to $H_1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2)\cong\mathbb{Z}_2$, $\hat{H}_P\cong H_1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z})\cong\mathbb{Z}_2$, $H_T\cong H_1(T^b;\mathbb{Z}_2)\cong\mathbb{Z}_2^b$ and $\hat{H}_T\cong H_1(T^b;\mathbb{Z})\cong\mathbb{Z}^b$. Let η_1 (resp. $\hat{\eta}_1$) be the generator of H_P (resp. $\hat{H}_P$).

Corollary 3.3. *There exist basis $\tau_1,\dots,\tau_b$ for H_T and $\hat{\tau}_1,\dots,\hat{\tau}_b$ for $\hat{H}_T$ such that*

- $\langle\mu_{\mathcal{E}}(\gamma_i),\tau_j\rangle = \delta_{ij}$, $\quad\langle\mu_{\mathcal{E}}(\gamma_i),\eta_1\rangle = 0$,
- $\langle\hat{\mu}_{\mathcal{E}}(\gamma_i),\hat{\tau}_j\rangle = \delta_{ij}$, $\quad\langle\hat{\mu}_{\mathcal{E}}(\gamma_i),\hat{\eta}_1\rangle = 0$.

Proof. The assertions for τ_i and $\hat{\tau}_i$ are obvious from Proposition 3.1. On the other hand, the class η_1 is represented by a path $\tilde{\eta}_1 = \{(A_t,\Phi_t)\}_{t\in[0,1]}$ in $\mathcal{C}^*$ such that $A_t = A_0$ and $\Phi_1 = -\Phi_0$, and therefore (A_1,Φ_1) is gauge equivalent to (A_0,Φ_0) by the constant gauge transformation -1. q.e.d.

Remark 3.4. For each γ_i as above, the holonomy map $\mathrm{Hol}_{\gamma_i}\colon \mathcal{A}/\mathcal{G}\to S^1$ represents a cohomology class $\bar{\gamma}_i$ in $H^1(\mathcal{A}/\mathcal{G};\mathbb{Z})\cong[\mathcal{A}/\mathcal{G},S^1]$. In fact, $(\bar{\gamma}_1,\dots,\bar{\gamma}_b)$ gives a basis for $H^1(\mathcal{A}/\mathcal{G};\mathbb{Z})$.

3.2. μ-map (2). We define the second μ-map $\mu_{\mathcal{F}}$. When we define the involution I on $\tilde{X}\times\mathbb{C}$ by $I(x,v) = (\iota x,\bar{v})$, we have an $\mathbb{R}^2$-bundle $E_0 = (\tilde{X}\times\mathbb{C})/I$ over X which is identified with $\underline{\mathbb{R}}\oplus(l\otimes\sqrt{-1}\mathbb{R})$. Then $\mathcal{G} = \Gamma(\tilde{X}\times_{\{\pm1\}}\mathrm{U}(1))$ naturally acts on E_0 by $(x,v)\mapsto(x,u(x)v)$. Let $\pi\colon X\times\mathcal{C}^*\to X$ be the projection, and define the $\mathbb{R}^2$-bundle $\mathcal{F}$ over $X\times\mathcal{B}^*$ by $\mathcal{F} = \pi^*E_0/\mathcal{G}$. By using the Stiefel-Whitney class $w_2(\mathcal{F})\in H^2(X\times\mathcal{B}^*;\mathbb{Z}_2)$, define the μ-map $\mu_{\mathcal{F}}$ for $k=0,1$ as follows:

$$\mu_{\mathcal{F}}\colon H_k(X;\mathbb{Z}_2)\to H^{2-k}(\mathcal{B}^*;\mathbb{Z}_2), \quad \mu_{\mathcal{F}}(\alpha) = w_2(\mathcal{F})/\alpha.$$

Let us consider the case when $\alpha\in H_1(X;\mathbb{Z}_2)$. By the universal coefficient theorem, we have a split exact sequence

$$0\to H_1(X;l)\otimes\mathbb{Z}_2\to H_1(X;\mathbb{Z}_2)\to\mathrm{Tor}(H_0(X;l),\mathbb{Z}_2)\to 0.$$

Then there is a loop ν in X such that

(N) the homology class of ν in $H_1(X;\mathbb{Z})$ corresponds to the generator of $\mathrm{Tor}(H_0(X;l),\mathbb{Z}_2)\cong\mathbb{Z}_2$.

Let η_1 and $\tau_1,\dots,\tau_b$ be the basis for $H_1(\mathcal{B}^*;\mathbb{Z}_2) = H_P\oplus H_T$ as in §3.1.

Proposition 3.5. *$\langle \mu_{\mathcal{F}}(\nu), \eta_1\rangle = 1$, and $\langle \mu_{\mathcal{F}}(\nu), \tau_i\rangle = 0$ for any i.*

Proof. As in the proof of Corollary 3.3, the class η_1 is represented by a path $\tilde{\eta}_1 = \{(A_t, \Phi_t)\}_{t\in[0,1]}$ in $\mathcal{C}^*$ such that $A_t = A_0$ and $(A_1, \Phi_1) = (-1)(A_0, \Phi_0)$. Then $\mathcal{F}|_{\nu\times\eta_1}$ is identified with $[0,1]\times[0,1]\times\mathbb{C}/\sim$, where

$$(0, y, v) \sim (1, y, \bar{v}), \quad (x, 0, v) \sim (x, 1, -v).$$

In other words, when $\pi_i\colon S^1\times S^1 \to S^1$ is the i-th projection and $\varepsilon \to S^1$ is a nontrivial $\mathbb{R}$-bundle over S^1,

$$\mathcal{F}|_{\nu\times\eta_1} \cong \pi_2^*\varepsilon \oplus (\pi_1^*\varepsilon \otimes \pi_2^*\varepsilon).$$

Then the first assertion follows because $w_2(\mathcal{F}|_{\nu\times\eta_1}) = w_1(\pi_2^*\varepsilon)w_1(\pi_1^*\varepsilon \otimes \pi_2^*\varepsilon)$ is the generator of $H^2(\nu\times\eta;\mathbb{Z}_2)$.

Recall that $\pi_0\mathcal{G} \cong H^1(X;l)\oplus\mathbb{Z}_2$. For the dual basis $\check{\gamma}_i \in H^1(X;l)$ of $\gamma_i \in H_1(X;l)$, we can take $u_i \in \mathcal{G}$ representing $\check{\gamma}_i$. Then $u_i|_\nu \simeq 1$, and we may assume $u_i|_\nu = 1$. The homology class $\tau_i \in H_1(X;\mathbb{Z}_2)$ is represented by a path $\tilde{\tau}_i = \{(A_t, \Phi_t)\}_{t\in[0,1]}$ such that $\Phi_1 = u_i\Phi_0$ and $A_t = A_0 + t(2u_i^{-1}du_i)$. Then $\mathcal{F}|_{\nu\times\tau_i}$ can be identified with $[0,1]\times[0,1]\times\mathbb{C}/\sim$, where

$$(0, y, v) \sim (1, y, \bar{v}), \quad (x, 0, v) \sim (x, 1, v).$$

Hence $w_2(\mathcal{F}|_{\nu\times\tau_i})$ is 0. q.e.d.

Corollary 3.6. *$H^*(\mathcal{B}^*;\mathbb{Z}_2)$ is generated by $\mu_{\mathcal{F}}(\nu)$ and $\mu_{\mathcal{E}}(\gamma_i)$ for $i = 1,\dots,b$.*

Next we consider $\mu_{\mathcal{F}}(x_0)$ for a generator x_0 of $H_0(X;\mathbb{Z}_2)$.

Proposition 3.7. *$\mu_{\mathcal{F}}(x_0) = \mu_{\mathcal{F}}(\nu)\cup\mu_{\mathcal{F}}(\nu)$.*

Proof. Since $\mathcal{B}^* \simeq \mathbb{R}\mathrm{P}^\infty \times T^b$, $H_2(\mathcal{B}^*;\mathbb{Z}_2)$ is generated by

- η_2 corresponding to the generator of $H_2(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2)$,
- $\eta_1\otimes\tau_j$, where η_1 and τ_j are as in §3.1, and
- $\tau_i\times\tau_j$ $(i\neq j)$.

First we prove that $\langle\mu_{\mathcal{F}}(x), \eta_2\rangle \neq 0$. Fix an O(2)-connection A_0 on E, and choose $\phi_0, \phi_1, \phi_2 \in \Gamma(S^+)$ which are linearly independent. Let S be the 2-sphere in $\mathcal{C}^*$ defined as

$$S = \{A_0\}\times\{p\phi_0 + q\phi_1 + r\phi_2 \,|\, p, q, r\in\mathbb{R}, \quad p^2+q^2+r^2 = 1\}.$$

Then the class η_2 is represented by $[S/\{\pm1\}]$. Let $\varepsilon \to \mathbb{R}\mathrm{P}^2$ be the canonical line bundle. We see that $\mathcal{F}|_{\{x_0\}\times S/\{\pm1\}}$ is isomorphic to $\varepsilon\oplus\varepsilon$.

Next we prove that $\langle\mu_{\mathcal{F}}(x), \eta_1\otimes\tau_i\rangle = \langle\mu_{\mathcal{F}}(x), \tau_i\times\tau_j\rangle = 0$. As in the proof of Proposition 3.5, we can choose $u_i \in \mathcal{G}$ representing $\check{\gamma}_i$. We may assume $u_i(x_0) = 1$. The homology class $\tau_i \in H_1(X;\mathbb{Z}_2)$ is represented by a path $\tilde{\tau}_i = \{(A_t, \Phi_t)\}_{t\in[0,1]}$ such that $\Phi_1 = u_i\Phi_0$ and $A_t = A_0 + t(2u_i^{-1}du_i)$. Then we can see that

$$w_2(\mathcal{F}|_{\{x_0\}\times(\eta_1\times\tau_i)}) = w_2(\mathcal{F}|_{\{x_0\}\times(\tau_i\times\tau_j)}) = 0.$$

q.e.d.

For cohomology classes of $\mathcal{B}^*$, let

$$\nu^* = \mu_{\mathcal{F}}(\nu), \quad \gamma_i^* = \mu_{\mathcal{E}}(\gamma_i), \quad \hat{\gamma}_i^* = \hat{\mu}_{\mathcal{E}}(\gamma_i).$$

Then, for example, $H^*(\mathcal{B};\mathbb{Z}_2)$ can be written as

$$H^*(\mathcal{B};\mathbb{Z}_2) = \mathbb{Z}_2[\nu^*] \otimes \bigwedge(\mathbb{Z}_2\gamma_1^* \oplus \cdots \oplus \mathbb{Z}_2\gamma_b^*),$$

and a cohomology class $\xi \in H^*(\mathcal{B};\mathbb{Z}_2)$ can be written as

$$\xi = (\nu^*)^a \prod_{i\in I} \gamma_i^*,$$

where a is a non-negative integer and I is a subset of $\{1,\dots,b\}$.

For a Spin^c(untwisted Spin^{c-})-structure, we have the μ-map of ordinary Seiberg-Witten theory ([**20**], §9):

$$\mu_0 \colon H_k(X;\mathbb{Z}) \to H^{2-k}(\mathcal{B}^*;\mathbb{Z}) \quad (k = 0, 1).$$

For $x \in H_0(X;\mathbb{Z})$ and $\gamma \in H_1(X;\mathbb{Z})$, let $x^* = \mu_0(x)$, $\gamma^* = \mu_0(\gamma)$.

3.3. Cutting down the moduli spaces. The purpose of this subsection is to construct the submanifolds in the moduli spaces which are dual to the classes $\mu_{\mathcal{F}}(\nu)$, $\mu_{\mathcal{F}}(x_0)$ and $\mu_{\mathcal{E}}(\gamma_i)$. (*Cf.* [**4**], §5.2 and [**19**], §9.) For a loop ν in X as in §3.2, fix a tubular neighborhood $n(\nu)$ of ν which is a smooth open submanifold with smooth boundary in X. Let $\mathcal{C}^*_{n(\nu)}$ be the space of irreducible configurations on $n(\nu)$, $\mathcal{G}_{n(\nu)}$ be the gauge transformation group and $\mathcal{B}^*_{n(\nu)} = \mathcal{C}^*_{n(\nu)}/\mathcal{G}_{n(\nu)}$. Note that $\pi_0\mathcal{G}_{n(\nu)} = \{\pm 1\}$. Let $\mathcal{G}_{n(\nu)}$ act on $\mathbb{R}$ via the projection $\mathcal{G}_{n(\nu)} \to \pi_0\mathcal{G}_{n(\nu)} = \{\pm 1\}$ and the multiplication of $\{\pm 1\}$. Dividing by the diagonal action, we obtain a real line bundle

$$\mathcal{L}_\nu = \mathcal{C}^*_{n(\nu)} \times_{\mathcal{G}_{n(\nu)}} \mathbb{R} \to \mathcal{B}^*_{n(\nu)}.$$

Suppose thet the moduli space $\mathcal{M}(X)$ contains no reducibles and is perturbed to be a smooth manifold. Let M be $\mathcal{M}(X)$ itself or its smooth submanifold. Since the restriction of an irreducible solution on X to an open subset of X is also irreducible by the unique continuation property of the Dirac operator, we have a well-defined restriction map

$$r_\nu \colon M \to \mathcal{B}^*_{n(\nu)}.$$

We can choose a section s of $\mathcal{L}_\nu$ so that the pull-back $r_\nu^* s$ is transverse to the zero-section of $r_\nu^*\mathcal{L}_\nu$ ([**4**], 5.2.2). Then the zero-set of $r_\nu^* s$ is a codimension-one submanifold of M which is dual to the class $\mu_{\mathcal{F}}(\nu)$ in M, and is denoted by

$$M \cap V_\nu.$$

Similarly, for the class $\mu_{\mathcal{F}}(x_0)$, we can construct a codimension-two submanifold of M which is dual to $\mu_{\mathcal{F}}(x_0)$ in M, and is denoted by

$$M \cap V_{x_0}.$$

For the loops γ_i chosen in §3.1, let $\mathrm{Hol}_{\gamma_i}\colon M \to S^1$ be the smooth map defined by the holonomy around γ_i. When we take a regular value $\theta \in S^1$ of Hol_{γ_i}, the inverse image $\mathrm{Hol}_{\gamma_i}^{-1}(\theta)$ is a codimension-one submanifold of M which is dual to $\mu_{\mathcal{E}}(\gamma_i)$ in M, and is denoted by

$$M \cap V_{\gamma_i}.$$

3.4. Gluing theorems. In this subsection, we state several gluing formulae for Pin$^-$(2)-monopole invariants, which will be proved in later sections. The formulae have different forms depending on whether the Spin^{c-}-structures are twisted or untwisted, and the moduli spaces contain reducibles or not. For local coefficients l_1 and l_2 over X_1 and X_2, if both of l_i are nontrivial, then we have $b_1(X_1\#X_2;l_1\#l_2) = b_1(X_1;l_1) + b_1(X_2;l_2) + 1$ by the Meyer-Vietoris sequence. Hence there is an extra generator of $H_1(X_1\#X_2)$ which does not come from X_1 or X_2. On the other hand, if one of l_i is trivial, then $b_1(X_1\#X_2;l_1\#l_2) = b_1(X_1;l_1) + b_1(X_2;l_2)$. Choose loops $\alpha_1,\dots,\alpha_{b_1(l_1)}$ in X_1, and $\beta_1,\dots,\beta_{b_1(l_2)}$ in X_2, where $b_1(l_i) = b_1(X_i;l_i)$ for $i = 1,2$, and δ in $X_1\#X_2$ representing an extra generator if both of l_1 and l_2 are nontrivial, such that

- $\alpha_1,\dots,\alpha_{b_1(l_1)}$ and $\beta_1,\dots,\beta_{b_1(l_2)}$ satisfy the conditions (C1) and (C2) in §3.1 for (X_1,l_1) and (X_2,l_2), respectively, and
- $\alpha_1,\dots,\alpha_{b_1(l_1)},\beta_1,\dots,\beta_{b_1(l_2)}$ and δ (if exists) satisfy the conditions (C1) and (C2) for $(X_1\#X_2,l_1\#l_2)$. (We assume α_i and β_j are also contained in $X_1\#X_2$.)

For each $i = 1,2$, if l_i is nontrivial, then choose another loop ν_i in X_i satisfying the condition (N) before Proposition 3.5. We also assume that ν_i is contained in $X_1\#X_2$.

The first gluing formula is on the gluing of U(1)-irreducible monopoles and Pin$^-$(2)-reducible monopoles.

Theorem 3.8. *Let X_1 be a closed oriented connected 4-manifold with $b_+(X_1) \geq 2$ and a* Spin^c*(untwisted* Spin^{c-}*)-structure c_1. Let X_2 be a closed oriented connected 4-manifold which satisfies the following:*

- *There exists a nontrivial double covering $\tilde{X}_2 \to X_2$ such that $b_+(X_2;l_2) = 0$ where $l_2 = \tilde{X}_2 \times_{\{\pm 1\}} \mathbb{Z}$.*
- *There exists a* Spin^{c-}*-structure c_2 on $\tilde{X}_2 \to X_2$ such that $\tilde{c}_1(E)^2 = \mathrm{sign}(X_2)$ (and hence the Dirac index is 0 and $d(c_2) = b_1(X_2;l_2)$).*

For a cohomology class $\xi \in H^*(\mathcal{B}^*(X_1,c_1);\mathbb{Z}_2)$ of the form $\xi = \prod_{i\in I}\mu_0(\alpha_i)$ where $I \subset \{1,\dots,b_1(l_1)\}$, let

$$\xi' = \prod_{i\in I}\mu_{\mathcal{E}}(\alpha_i) \in H^*(\mathcal{B}^*(X_1\#X_2,c_1\#c_2);\mathbb{Z}_2).$$

Then we have

$$\mathrm{SW}^{\mathrm{Pin}}(X_1\#X_2, c_1\#c_2)(\xi'(\nu_2^*)^{2a+1}\beta_1^*\cdots\beta_{b_1(l_2)}^*) \equiv \mathrm{SW}^{\mathrm{U}(1)}(X_1, c_1)(\xi(x^*)^a) \mod 2.$$

Theorem 1.7 is a corollary of Theorem 3.8.

The second one is a generalized blow-up formula by the gluing of $\mathrm{Pin}^-(2)$-irreducibles and $\mathrm{U}(1)$-reducibles.

Theorem 3.9 (*Cf.* [**5**, **17**, **7**])**.** *Let X_1 be a closed oriented connected 4-manifold with a Spin^{c-}-structure c_1 with $b_+(X_1; l_1) \geq 2$. Let X_2 be a closed oriented connected 4-manifold with a Spin^c(untwisted Spin^{c-})-structure c_2 such that $b_1(X_2) = b_+(X_2) = 0$ and $d(c_2) = -1$. For any $\xi = (\nu_1^*)^a \prod_{i\in I}\alpha_i^*$ where $I \subset \{1, \ldots, b_1(l_1)\}$,*

$$\mathrm{SW}^{\mathrm{Pin}}(X_1\#X_2, c_1\#c_2)(\xi) = \mathrm{SW}^{\mathrm{Pin}}(X_1, c_1)(\xi).$$

Remark 3.10. In Theorem 3.9, ξ is assumed to represent both of the cohomology classes of $\mathcal{B}^*(X_1, c_1)$ and $\mathcal{B}^*(X_1\#X_2, c_1\#c_2)$. The similar remark is valid for the following theorems.

The third one is on the gluing of $\mathrm{Pin}^-(2)$-irreducibles and $\mathrm{Pin}^-(2)$-reducibles.

Theorem 3.11. *Let X_1 be a closed oriented connected 4-manifold with a twisted Spin^{c-}-structure c_1 with $b_+(X; l_1) \geq 2$, and X_2 be a manifold with a Spin^{c-}-structure c_2 as in Theorem 3.8. Then, for any $\xi = (\nu_1^*)^a \prod_{i\in I}\alpha_i^*$ where $I \subset \{1, \ldots, b_1(l_1)\}$,*

$$\mathrm{SW}^{\mathrm{Pin}}(X_1\#X_2, c_1\#c_2)(\xi\delta^*\beta_1^*\cdots\beta_{b_1(l_2)}^*) = \mathrm{SW}^{\mathrm{Pin}}(X_1, c_1)(\xi).$$

If 4-manifolds X_1 and X_2 have positive b_+, then the Seiberg-Witten invariants of $X_1\#X_2$ are always 0. Likewise, the $\mathbb{Z}_2$-valued $\mathrm{Pin}^-(2)$-monopole invariants have a similar property.

Theorem 3.12. *Let X_1 be a closed oriented connected 4-manifold with a twisted Spin^{c-}-structure c_1 with $b_+(X_1; l_1) \geq 1$. Let X_2 be a closed oriented connected 4-manifold with a (twisted or untwisted) Spin^{c-}-structure c_2, and suppose one of the following:*

(i) *$b_+(X_2) \geq 1$ and c_2 is an untwisted Spin^{c-}-structure on X_2.*
(ii) *c_2 is a twisted Spin^{c-}-structure on X_2 with $b_+(X_2; l_2) \geq 1$.*

Then $\mathrm{SW}^{\mathrm{Pin}}(X_1\#X_2, c_1\#c_2)(\xi) = 0$ for any class $\xi \in H^*(\mathcal{B}; \mathbb{Z}_2)$.

On the other hand, the $\mathbb{Z}$-valued invariants can be nontrivial for a connected sum $X_1\#X_2$ even when both of $b_+(X_1; l_1)$ and $b_+(X_2; l_2)$ are positive. Consider (X_i, l_i) $(i = 0, 1, \ldots, n)$ with nontrivial l_i. We assume $b_1(X_i, l_i) = 0$ for every i. Let $X = X_0\#\cdots\#X_n$ and $l = l_0\#\cdots\#l_n$. As noticed above, each time we take a connected sum, we have an extra generator in the first homology of the connected sum. Choose loops $\delta_1, \dots, \delta_n$ in X representing such extra generators in $H_1(X; l)$ satisfying the conditions (C1), (C2).

Theorem 3.13. *Let n be any positive integer. For $i = 0, 1, \dots, n$, let X_i be a closed oriented connected 4-manifold with a twisted Spin^{c-}-structure c_i satisfying*

- $b_1(X_i; l_i) = 0$, $b_+(X_i; l_i) \geq 2$.
- $d(c_i) = 0$, *and*
- *the index of the Dirac operator is positive and even.*

Note that in this situation, the moduli space $\mathcal{M}(X_i, c_i)$ is orientable, and the $\mathbb{Z}$-valued invariant $\mathrm{SW}^{\mathrm{Pin}}_{\mathbb{Z}}(X_i, c_i)(1)$ is defined for a choice of orientation. Let $X = X_0 \# \cdots \# X_n$ and $c = c_0 \# \cdots \# c_n$. Then the glued moduli space $\mathcal{M}(X, c)$ is orientable, and

$$\mathrm{SW}^{\mathrm{Pin}}_{\mathbb{Z}}(X, c)(\hat{\delta}_1^* \cdots \hat{\delta}_n^*) = 2^n \prod_{i=0}^{n} \mathrm{SW}^{\mathrm{Pin}}_{\mathbb{Z}}(X_i, c_i)(1),$$

for a choice of orientation.

3.5. Proofs of Theorem 1.1 and Theorem 1.13. In this subsection, we prove Theorem 1.1 and Theorem 1.13 by assuming Theorem 3.8 and Theorem 3.13.

Proof of Theorem 1.1. Let (X_2, l_{X_2}) be as in Theorem 1.7. Then this satisfies the conditions for X_2 in Theorem 3.8.

For given n, required exotic structures on $E(n)$ can be constructed by either logarithmic transformation (see e.g., [**8**]) or Fintushel-Stern's knot surgery [**6**].

First, we discuss on the case of logarithmic transformation. Let $E(n)_{p.q}$ be the log transformed $E(n)$ with two multiple fibers of multiplicities p and q. For odd n, all of $E(n)_{p.q}$ with $\gcd(p, q) = 1$ is homeomorphic to $E(n)$. On the other hand, for even n, $E(n)_{p.q}$ is homeomorphic to $E(n)$ if and only if $\gcd(p, q) = 1$ and pq is odd. Let $f \in H^2(E(n)_{p,q})$ be the Poincaré dual of the homology class of a regular fiber. Then there is a primitive class f_0 with $f = pqf_0$, and the Poincaré duals f_p and f_q of the multiple fibers of p and q are given by $f_p = qf_0$ and $f_q = pf_0$. If we put

$$D(a, b, c) = af + bf_p + cf_q,$$

then, for $n \geq 2$, the canonical class K is given as $K = D(n-2, p-1, q-1)$. The Seiberg-Witten basic classes are given by $K - 2D(a, b, c)$, where $0 \leq a \leq n-2$, $0 \leq b \leq p-1$, $0 \leq c \leq q-1$, and the value the Seiberg-Witten invariant for the class $K - 2D(a, b, c)$ is

$$\mathrm{SW}^{\mathrm{U}(1)}(E(n)_{p,q}, K - 2D(a, b, c)) = (-1)^a \binom{n-2}{a},$$

which is independent of b and c. Similar facts hold for the case when $n = 1$. In general, the number of basic classes whose Seiberg-Witten invariants are odd is changed if p and q are varied. By using these facts together with Theorem 3.8, we can find infinitely many $\{p, q\}$ such that $E(n)_{p,q}\#X_2$ have different numbers of basic classes for $\mathrm{Pin}^-(2)$-monopole invariants.

For a knot K, let $E(n)_K$ be the manifold obtained by the knot surgery on a regular fiber T with K. If we consider the Seiberg-Witten invariant as a symmetric Laurent polynomial as in [**6**], the invariant of $E(n)$ is related to that of $E(n)_K$ by

$$\mathrm{SW}^{\mathrm{U}(1)}_{E(n)_K} = \mathrm{SW}^{\mathrm{U}(1)}_{E(n)} \cdot \Delta_K(t),$$

where $t = \exp(2[T])$ and $\Delta_K(t)$ is the (symmetrized) Alexander polynomial of K. Now, let $X_K = E(n)_K$, and let us fix a Spin^{c-}-structure c_2 on X_2 as in Theorem 3.8, and consider a function of $\mathrm{Pin}^-(2)$-monopole invariants of $X_K\#X_2$,

$$\mathrm{SW}^{\mathrm{Pin}}_{X_K\#(X_2,c_2)} \colon \{h \in H^2(X_K;\mathbb{Z}) \,|\, h \equiv w_2(X) \mod 2\} \to \mathbb{Z}_2,$$

which is defined as

$$\mathrm{SW}^{\mathrm{Pin}}_{X_K\#(X_2,c_2)}(h) = \mathrm{SW}^{\mathrm{Pin}}(X_K\#X_2, c(h)\#c_2)(\nu_2^*\beta_1^*\cdots\beta^*_{b_1(l_2)}),$$

where $c(h)$ is the Spin^c-structure on X_K with $c_1 = h$. If we assume $\mathrm{SW}^{\mathrm{Pin}}_{X_K\#(X_2,c_2)}$ as a $\mathbb{Z}_2$-coefficient polynomial, then Theorem 1.7 implies that $\mathrm{SW}^{\mathrm{Pin}}_{X_K\#(X_2,c_2)}$ is the $\mathbb{Z}_2$-reduction of the $\mathbb{Z}$-coefficient polynomial $\mathrm{SW}^{\mathrm{U}(1)}_{E(n)_K}$. Then we can find infinitely many K so that $\mathrm{SW}^{\mathrm{Pin}}_{X_K\#(X_2,c_2)}$ are different. q.e.d.

Proof of Theorem 1.13. For each (N_i, l_i), we have $b_1(X_i; l_i) = 0$ and $b_+(X_i; l_i) = 2$. By Theorem 1.12, there is a twisted Spin^{c-}-structure c_i such that $d(c_i) = 0$, the Dirac index is 2 and $\mathrm{SW}^{\mathrm{Pin}}_{\mathbb{Z}}(X_i, c_i)$ is odd. Then the theorem follows from Theorem 3.13. q.e.d.

4. $\mathrm{Pin}^-(2)$-monopole theory on 3-manifolds

Sections 4-6 are devoted to the proof of the gluing theorems in §3.4, and this preparatory section is on the $\mathrm{Pin}^-(2)$-monopole theory on 3-manifolds. We refer to [**12**, **7**] for the Seiberg-Witten counterpart of the topics in this section.

4.1. Spin^{c-}-structures on 3-manifolds. Define the group $\mathrm{Spin}^{c-}(3)$ by

$$\mathrm{Spin}^{c-}(3) = \mathrm{Spin}(3) \times_{\{\pm 1\}} \mathrm{Pin}^-(2) = \mathrm{Sp}(1) \times_{\{\pm 1\}} \mathrm{Pin}^-(2).$$

Let Y be an oriented closed connected Riemannian 3-manifold, and $Fr(Y)$ its SO(3)-frame bundle. Suppose a double covering $\tilde{Y} \to Y$ is given. A Spin^{c-}-structure on $\tilde{Y} \to Y$ consists of a principal Spin$^{c-}(3)$-bundle P and isomorphisms $\sigma\colon P/\operatorname{Spin}^c(3) \to \tilde{Y}$ and $\tau\colon P/\operatorname{Pin}^-(2) \to Fr(Y)$. The characteristic O(2)-bundle E is defined as $E = P/\operatorname{Spin}(3)$.

Remark 4.1. As in the 4-dimensional case, if $\tilde{Y} \to Y$ is trivial, then a Spin^{c-}-structure on $\tilde{Y} \to Y$ can be reduced to a Spin^c-structure on Y, and is called *untwisted*.

Define the action of Spin$^{c-}(3)$ on $\operatorname{Im}\mathbb{H}$ by

$$[q,u]\cdot v = qvq^{-1},$$

for $[q,u] \in \operatorname{Spin}^{c-}(3)$ and $v \in \operatorname{Im}\mathbb{H}$. Then the associated bundle $P \times_{\operatorname{Spin}^{c-}(3)} \operatorname{Im}\mathbb{H}$ is identified with the tangent bundle TY. Define the Spin$^{c-}(3)$-action on $\mathbb{H}$ by

$$[q,u]\cdot \psi = q\psi u^{-1},$$

for $[q,u] \in \operatorname{Spin}^{c-}(3)$ and $\psi \in \mathbb{H}$. Then we obtain the associated bundle $S = P \times_{\operatorname{Spin}^{c-}(3)} \mathbb{H}$ which is the spinor bundle for the Spin^{c-}-structure.

The Clifford multiplication is defined as follows. The identity component of Spin$^{c-}(3)$ is Spin$^c(3)$, and the quotient group $\operatorname{Spin}^{c-}(3)/\operatorname{Spin}^c(3)$ is isomorphic to $\{\pm 1\}$. Let $\mathbb{C}_-$ be a copy of $\mathbb{C}$ with the $\{\pm1\}$-action by complex conjugation. Then Spin$^{c-}(3)$ acts on $\mathbb{C}_-$ via the projection $\operatorname{Spin}^{c-}(3) \to \operatorname{Spin}^{c-}(3)/\operatorname{Spin}^c(3) = \{\pm 1\}$. If we define

$$\rho_0\colon (\operatorname{Im}\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C}_- \times \mathbb{H} \to \mathbb{H}$$

by $\rho_0(v \otimes a, \psi) = \bar{v}\psi\bar{a}$, then ρ_0 is Spin$^{c-}(3)$-equivariant. Let $K = \tilde{Y} \times_{\{\pm 1\}} \mathbb{C}_-$. Then we can define the Clifford multiplication

$$\rho\colon T^*Y \otimes_{\mathbb{R}} K \to \operatorname{Hom}(S,S),$$

which induces

$$\rho\colon \Omega^1(Y;K) \times \Gamma(S) \to \Gamma(S).$$

Note that $K = \underline{\mathbb{R}} \oplus i\lambda$, and so $\Omega^1(Y;K) = \Omega^1(Y;\underline{\mathbb{R}}) \oplus \Omega^1(Y;i\lambda)$. Although the spinor bundle S does not have an ordinary hermitian inner product, the pointwise *twisted* hermitian product

$$\langle\cdot,\cdot\rangle_{K,x}\colon S_x \times S_x \to K_x \tag{4.2}$$

is defined. For $\alpha \otimes 1 \in T^*Y \otimes K$, the image $\rho(\alpha\otimes 1)$ is a traceless endomorphism which is *skew-adjoint* with respect to the inner product (4.2). The whole image of T^*Y by ρ forms the subbundle of $\operatorname{Hom}(S,S)$, which we write as $\tilde{\mathfrak{su}}(S)$, equipped with the inner product $\frac{1}{2}\operatorname{tr}(a^*b)$. When $\{e_1, e_2, e_3\}$ is an oriented frame on $\Lambda^1(Y)$, we assume the orientation convention

$$\rho(e_1)\rho(e_2)\rho(e_3) = 1.$$

We extends ρ to forms by the rule,

$$\rho(\alpha\wedge\beta) = \frac{1}{2}\left(\rho(\alpha)\rho(\beta) + (-1)^{\deg\alpha\deg\beta}\rho(\beta)\rho(\alpha)\right).$$

The orientation convention implies $\rho(*\alpha) = -\rho(\alpha)$ for 1-forms.

4.2. $\mathrm{Pin}^-(2)$**-monopole equations on 3-manifolds.** An $\mathrm{O}(2)$-connection B on E together with the Levi-Civita connection defines a $\mathrm{Spin}^{c-}(3)$-connection on P. Then we have the Dirac operator $D_B\colon \Gamma(S) \to \Gamma(S)$ associated to B.

The bundle $\Lambda^1(Y) \otimes_{\mathbb{R}} i\lambda$ is also associated with P as follows. Let $\varepsilon\colon \mathrm{Pin}^-(2) \to \mathrm{Pin}^-(2)/\mathrm{U}(1) \cong \{\pm 1\}$ be the projection, and let $\mathrm{Spin}^{c-}(3)$ act on $\operatorname{Im}\mathbb{H}$ by

$$v \in \operatorname{Im}\mathbb{H} \to \varepsilon(u) q v q^{-1} \quad \text{for } [q,u] \in \mathrm{Spin}^{c-}(3).$$

Then $\Lambda^1(Y) \otimes_{\mathbb{R}} i\lambda$ is identified with $P \times_{\mathrm{Spin}^{c-}(3)} \operatorname{Im}\mathbb{H}$. For $\psi \in \mathbb{H}$, $\psi i \bar{\psi}$ is in $\operatorname{Im}\mathbb{H}$. Then the map $\psi \in \mathbb{H} \to \psi i \bar{\psi} \in \operatorname{Im}\mathbb{H}$ is $\mathrm{Spin}^{c-}(3)$-equivariant, and induces a quadratic map

$$q\colon \Gamma(S) \to \Omega^1(Y; i\lambda).$$

For a closed 2-form $\eta \in \Omega^2(i\lambda)$, the perturbed $\mathrm{Pin}^-(2)$-monopole equations on Y are defined as

$$\begin{cases} D_B\Psi = 0, \\ -\dfrac{1}{2}(*(F_B + \eta)) = q(\Psi), \end{cases} \tag{4.3}$$

for $\mathrm{O}(2)$-connections B on E and $\Psi \in \Gamma(S)$. The gauge transformation group is given by

$$\mathcal{G}_Y = \Gamma(\tilde{Y} \times_{\{\pm 1\}} \mathrm{U}(1)),$$

where $\{\pm 1\}$ acts on $\mathrm{U}(1)$ by complex conjugation.

Remark 4.4. If the Spin^{c-}-structure is untwisted, then the 3-dimensional $\mathrm{Pin}^-(2)$-monopole equations are also identified with the 3-dimensional Seiberg-Witten equations.

4.3. $\mathrm{Pin}^-(2)$**-Chern-Simons-Dirac functional.** Choose a reference $\mathrm{O}(2)$-connection B_0 on E. Let $\mathcal{A}(E)$ be the space of $\mathrm{O}(2)$-connections on E, and $\mathcal{C} = \mathcal{A}(E) \times \Gamma(S)$.

Definition 4.5. Let η be a closed 2-form in $\Omega^2(\lambda)$. The (perturbed) $\mathrm{Pin}^-(2)$-Chern-Simons-Dirac functional $\vartheta\colon \mathcal{C} \to \mathbb{R}$ is defined by

$$\vartheta(B,\Psi) = -\frac{1}{8}\int_Y (B - B_0) \wedge (F_B + F_{B_0} + i\eta) + \frac{1}{2}\int_Y \langle D_B\Psi, \Psi\rangle_{\mathbb{R}} \mathrm{dvol}_Y. \tag{4.6}$$

A few comments on the definition. For $\alpha \in \Omega^1(i\lambda)$ and $\beta \in \Omega^2(i\lambda)$, $\alpha \wedge \beta$ is in $\Omega^3(Y;\mathbb{R})$ since $\lambda^{\otimes 2}$ is trivial. The inner product $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ is the real part of (4.2).

The tangent space of $\mathcal{C}$ at (B,Ψ) is $T_{(B,\Psi)}\mathcal{C} = \Omega^1(i\lambda)\oplus\Gamma(S)$. We equip the tangent space with an L^2 metric. Then the gradient of ϑ with respect to the L^2-metric is given by

$$\nabla\vartheta = \left(\frac{1}{2}(*(F_B + i\eta)) + q(\Psi), D_B\Psi\right).$$

Hence the critical points of ϑ are the solutions of the Pin$^-$(2)-monopole equations on Y.

For a critical point (B,Ψ) of ϑ, let $\mathcal{H}_{(B,\Psi)}\colon \Omega^1(i\lambda)\oplus\Gamma(S)\to\Omega^1(i\lambda)\oplus\Gamma(S)$ be the derivative of $\nabla\vartheta$ at (B,Ψ) given as

$$\mathcal{H}_{(B,\Psi)}(b,\psi) = \left(\frac{1}{2}*db - Dq_\Psi(\psi), -D_B\psi - \frac{1}{2}b\Psi\right),$$

where Dq_Ψ is the linearization of q. A critical point (B,Ψ) is called *non-degenerate* if the middle cohomology group of the following complex is 0:

$$\Omega^0(i\lambda) \xrightarrow{\mathcal{I}_\Psi} \Omega^1(i\lambda)\oplus\Gamma(S) \xrightarrow{\mathcal{H}_{(B,\Psi)}} \Omega^1(i\lambda)\oplus\Gamma(S),$$

where $\mathcal{I}_\Psi$ is defined by $\mathcal{I}_\Psi(f) = (-2df, f\Psi)$.

For $g\in\mathcal{G}_Y$, $g^{-1}dg$ is an $i\lambda$-valued 1-form, and the λ-valued 1-form $\frac{1}{2\pi i}g^{-1}dg$ represents an integral class $[g]\in H^1(Y;l)/\mathrm{Tor}$.

Proposition 4.7. *For $(B,\Psi)\in\mathcal{C}$ and $g\in\mathcal{G}_Y$,*

$$\vartheta(g(B,\Psi)) - \vartheta(B,\Psi) = 2\pi([g]\cup(\pi\tilde{c}_1(E) - [\eta]))[Y],$$

where $[\eta]\in H^2(Y;\lambda)$ is the de Rham cohomology class of η.

4.4. Non-degenerate critical point on S^3. Here, we suppose $Y = S^3$ with a positive scalar curvature metric. Since S^3 is simply-connected, every Spin^{c-}-structure is untwisted. This is unique up to isomorphism and identified with a unique Spin^c-structure. For a positive scalar curvature metric, every monopole solution is a reducible one, say $(\theta,0)$, which is unique up to gauge. Furthermore, the kernel of the Dirac operator D_θ is trivial. Since the index of D_θ is 0, the cokernel is also trivial, and this implies $(\theta,0)$ is nondegerate. The stabilizer of $(\theta,0)$ of the gauge group action is denoted by Γ_θ:

$$\Gamma_\theta = \{g\in\mathrm{Map}(S^3;\mathrm{U}(1))\,|\,g(\theta,0) = (\theta,0)\}.$$

Note that $\Gamma_\theta\cong S^1$.

5. Pin$^-$(2)-monopoles on a 4-manifold with a tubular end

In this section, we continue the preparation for gluing, and discuss on finite energy Pin$^-$(2)-monopoles on 4-manifolds with tubular ends. We refer to [**3**] as well as [**12**, **7**].

5.1. Setting. Let X be a Riemannian 4-manifold with a Spin^{c-}-structure containing a tubular end $[-1,\infty)\times Y$, where Y is a closed, connected, Riemannian 3-manifold with a Spin^{c-}-structure. More precisely, suppose we are given

(1) an orientation preserving isometric embedding $i\colon [-1,\infty)\times Y\to X$ such that
$$X^t = X\setminus i((t,\infty)\times Y)$$
is compact for any $t\geq -1$,

(2) an isomorphism between Spin^{c-}-structure on $[-1,\infty)\times Y$ induced from Y and the one inherited from X via the embedding i.

Remark 5.1. If the Spin^{c-}-structure on X is twisted but its restriction on the tube $[-1,\infty)\times Y$ is untwisted, then the double cover $\tilde{X}$ has two tubular ends.

In order to define weighted Sobolev norms on various sections over X, take a C^∞-function $w\colon X\to\mathbb{R}$ such that

$$w(t)=\begin{cases}1 & \text{on } X^{-1}\\ e^{\alpha t} & \text{for } (t,y)\in[0,\infty)\times Y\end{cases}\tag{5.2}$$

where α is a small positive number which will been chosen later to be suitable for our purpose. For a nonnegative integer k, we will use the weighted Sobolev norm of a section f (e.g., a form or a spinor) on X given by
$$\|f\|_{L^{2,w}_k}=\|wf\|_{L^2_k}.$$

Let X_1 and X_2 be 4-manifolds with tubular ends as above with isometric embeddings
$$i_1\colon[-1,\infty)\times Y\to X_1,\quad i_2\colon[-1,\infty)\times\bar{Y}\to X_2,$$
where $\bar{Y}$ is Y with opposite orientation. For $T\geq 0$, let $X^{\#T}$ be the manifold obtained by gluing X_1^{2T} and X_2^{2T} via the identification
$$i_1(t,y)\sim i_2(2T-t,y).$$
Then we naturally have an isometric embedding of a neck $i_T\colon[-T,T]\times Y\to X^{\#T}$. (Here, the negative side is connected to X_1^0 and the positive side to X_2^0.) When we take functions w_1, w_2 as (5.2), a continuous function $w_T\colon X^{\#T}\to\mathbb{R}$ is induced by gluing w_1 and w_2 such that

$$w_T(t)=e^{\alpha(T-|t|)}\tag{5.3}$$

for $(t,y)\in[-T,T]\times Y$. For the sections over $X^{\#T}$, we will use the weighted norm
$$\|f\|_{L^{2,w_T}_k}=\|w_Tf\|_{L^2_k}.$$

5.2. Exponential decay. The purpose of this subsection is to give exponential decay estimates for Pin$^-$(2)-monopoles on a cylinder $[0,\infty)\times Y$ and a band $(-T,T)\times Y$. Since a Pin$^-$(2)-monopole on an untwisted Spin^{c-}-structure is identified with an ordinary Seiberg-Witten monopole, the estimates for Seiberg-Witten monopoles on a cylinder $[0,\infty)\times Y$ hold for Pin$^-$(2)-monopoles on an untwisted Spin^{c-}-structure. On the other hand, we can also obtain an estimate for Pin$^-$(2)-monopoles on a twisted Spin^{c-}-structure by lifting everything to the double cover $[0,\infty)\times\tilde{Y}$ on which the corresponding Spin^{c-}-structure is untwisted and applying the estimate for the Seiberg-Witten monopole. Thus, invoking the results due to Froyshov [**7**] for the Seiberg-Witten monopoles, we obtain the estimates for Pin$^-$(2)-monopoles as follows.

Let β be a nondegenerate monopole over Y, and $U\subset\mathcal{B}_Y$ is an L^2-closed subset which contains no monopoles except perhaps $[\beta]$. Define $B_t=[t-1,t+1]\times Y$.

Theorem 5.4 ([**7**], Theorem 6.3.1.)**.** *There exists a constant λ_+ which has the following significance. For any $C>0$, there exist constants ϵ and C_k for nonnegative integer k such that the following holds. Let $x=(A,\Phi)$ be a* Pin$^-$(2)*-monopole in temporal gauge over $(-2,\infty)\times Y$ such that $x(t)\in U$ for some $t\geq 0$. Set*

$$\bar{\nu}=\|\nabla\vartheta\|_{L^2((-2,\infty)\times Y)},\quad \nu(t)=\|\nabla\vartheta\|_{L^2(B_t)}.$$

If $\|\Phi\|_\infty\leq C$ and $\bar{\nu}\leq\epsilon$ then there is a smooth Pin$^-$(2)*-monopole α over Y, gauge equivalent to β, such that if B is the connection part of $\pi^*\alpha$ then for every $t\geq 1$ and nonnegative integer k one has*

$$\sup_{y\in Y}|\nabla_B^k(x-\pi^*\alpha)|_{(t,y)}\leq C_k\sqrt{\nu(0)}e^{-\lambda^+t}.$$

Theorem 5.5 ([**7**], Theorem 6.3.2.)**.** *There exists a constant λ_+ which has the following significance. For any $C>0$, there exist constants ϵ and C_k for nonnegative integer k such that the following holds for every $T>1$. Let $x=(A,\Phi)$ be a* Pin$^-$(2)*-monopole in temporal gauge over the band $[-T-2,T+2]\times Y$ such that $x(t)\in U$ for some $t\in[-T-2,T+2]$. Set*

$$\bar{\nu}=\|\nabla\vartheta\|_{L^2([-T-2,T+2]\times Y)},\quad \nu(t)=\|\nabla\vartheta\|_{L^2(B_t)}.$$

If $\|\Phi\|_\infty\leq C$ and $\bar{\nu}\leq\epsilon$ then there is a smooth Pin$^-$(2)*-monopole α over Y, gauge equivalent to β, such that if B is the connection part of $\pi^*\alpha$ then for every $t\leq T-1$ and nonnegative integer k one has*

$$\sup_{y\in Y}|\nabla_B^k(x-\pi^*\alpha)|_{(t,y)}\leq C_k(\nu(-T)+\nu(T))^{1/2}e^{-\lambda^+(T-|t|)}.$$

5.3. Energy. Let Z be a Riemannian Spin^{c-}-4-manifold possibly non-compact or with boundaries, such as X with a tubular end, or its compact submanifolds X^t or a compact tube $[a,b]\times Y$. Let μ be a closed 2-form in $\Omega^2(i\lambda)$, and assume μ is the pull-back of η on the tube. For configurations (A,Φ), we define the energy by

$$\mathcal{E}(A,\Phi)=\frac{1}{4}\int_Z|F_A-\mu|^2+\int_Z|\nabla_A\Phi|^2 + \frac{1}{4}\int_Z\Big(|\Phi|^2+\frac{s}{2}\Big)^2-\int_Z\frac{s^2}{16}+2\int_Z\langle\Phi,\rho(\mu)\Phi\rangle,$$

where s is the scalar curvature.

Proposition 5.6 ([**12**], Chapter II and Chapter VIII)**.** (1) *If* (A,Φ) *is a* $\mathrm{Pin}^-(2)$*-monopole on* $Z=X^T$ *with a finite cylinder* $(-1,T]\times Y$ *near the boundary* Y*, then*

$$\mathcal{E}(A,\Phi)=\frac{1}{4}\int_Z(F_A-\mu)\wedge(F_A-\mu)-\int_Y\langle\Phi|_Y,D_B(\Phi|_Y)\rangle,$$

where B *is the boundary connection induced from* A*.*
(2) *If* (A,Φ) *is a* $\mathrm{Pin}^-(2)$*-monopole on* $[t_0,t_1]\times Y$ *in temporal gauge, then*

$$\frac{1}{2}\mathcal{E}(A,\Phi)=\vartheta(A(t_1),\Phi(t_1))-\vartheta(A(t_0),\Phi(t_0)).$$

5.4. Compactness. We invoke a compactness result due to Kronheimer and Mrowka.

Proposition 5.7 ([**12**], Theorem 5.1.1)**.** *Let* Z *be a compact Riemannian* Spin^{c-}*-4-manifold with boundary. Suppose there exists a constant* C *so that a sequence* (A_n,Φ_n) *of smooth solutions to* $\mathrm{Pin}^-(2)$*-monopole equations satisfies the bound* $\mathcal{E}(A_n,\Phi_n)\le C$*. Then there exists a sequence* g_n *of (smooth) gauge transformations with the following properties: after passing to a subsequence, the transformed solutions* $g_n(A_n,\Phi_n)$ *converges weakly in* L^2_1 *to a* L^2_1*-configuration* (A,Φ) *on* Z*, and converges strongly in* C^∞ *on every interior domain* $Z'\subset Z$*.*

Corollary 5.8. *Let* $x(t)=(A(t),\Phi(t))$ *be a smooth monopole on* $[-1,\infty)\times Y$ *in temporal gauge. If* $\mathcal{E}(A,\Phi)$ *is finite, then* $[x(t)]$ *converges in* $\mathcal{B}_Y$ *to some critical point as* $t\to\infty$*.*

Proof. By translation, $(A_T,\Phi_T)=(A,\Phi)|_{[T-1,T+1]\times Y}$ can be considered as a monopole on $[-1,1]\times Y$. Let T_n be any sequence with $T_n\to\infty$ as $n\to\infty$. Since $\mathcal{E}(A,\Phi)$ is finite, $\mathcal{E}(A_{T_n},\Phi_{T_n})\to 0$ as $n\to\infty$. Then, after some gauge transformations, we may assume (A_{T_n},Φ_{T_n}) converges in C^∞ on $(-1,1)\times Y$ to the pull-back of some critical point. From this, the corollary is proved. q.e.d.

Proposition 5.9. *Let X be a* Spin^{c-}*-4-manifold X with an end $[-1,\infty)\times Y$. If a smooth monopole (A,Φ) over X has a finite energy $\mathcal{E}(A,\Phi)$, then we have either*

$$\Phi = 0, \qquad \textit{or} \qquad \|\Phi\|_{C^0} \le -\frac{1}{2}\inf_{x\in X} s(x) + 4\|\mu\|_{C^0},$$

where s is the scalar curvature of X.

Proof. By Corollary 5.8, we may assume (A,Φ) converges to a monopole (B,Ψ) on Y. If $|\Phi|$ takes its maximum on X, then the argument in [**11**], Lemma 2, implies the proposition. Otherwise we have $\|\Phi\|_{C^0} = \|\Psi\|_{C^0}$. Since (B,Ψ) is a 3-dimensional monopole, Ψ also satisfies

$$\Psi = 0 \quad \text{ or } \quad \|\Psi\|_{C^0} \le -\frac{1}{2}\inf_{y\in Y} s(y) + 4\|\eta\|_{C^0}.$$

q.e.d.

5.5. Weighted moduli spaces. Throughout this subsection, we assume X is a Spin^{c-}-4-manifold with the end $[-1,\infty)\times S^3$. Let us fix a smooth reference connection A^0 which is the pull-back of θ on the tube $[0,\infty)\times S^3$. For later purpose, we choose an integer k so that $k\ge 3$. We consider the space of configurations

$$\mathcal{C}^w = \{(A^0+a,\Phi)\,|\,a\in L^{2,w}_k(\Lambda^1(i\lambda)),\Phi\in L^{2,w}_k(S^+)\}.$$

Let us consider the set of gauge transformations

$$\mathcal{G}^w = \{g\in L^2_{k+1,\mathrm{loc}}(\Gamma(\tilde{X}\times_{\{\pm1\}}\mathrm{U}(1)))\,|\,\nabla_0 g\in L^{2,w}_k\},$$

where ∇_0 denotes the covariant derivative of A^0. We can prove,

Proposition 5.10 ([**22**], Section 7, *Cf.* [**3**], §4.3, [**7**] Chapter 2)**.** (1) *Let $L\mathcal{G}^w$ be the set defined by*

$$L\mathcal{G}^w = \{\xi\in L^2_{k+1,\mathrm{loc}}(\Lambda^0(i\lambda))\,|\,\nabla_0\xi\in L^{2,w}_{k+1}\}.$$

Then each element $\xi\in L\mathcal{G}^w$ tends to a limit in $\mathrm{Lie}\,\Gamma_\theta\cong i\mathbb{R}$ *at infinity, and therefore the evaluation map is defined*:

$$r\colon L\mathcal{G}^w\to \mathrm{Lie}\,\Gamma_\theta.$$

When we define the inner product on $L\mathcal{G}^w$ by

$$\langle\xi,\eta\rangle = \langle\nabla_0\xi,\nabla_0\eta\rangle_{L^{2,w}_{k+1}} + \langle r(\xi),r(\eta)\rangle_{i\mathbb{R}},\quad \xi,\eta\in L\mathcal{G}^w,$$

$L\mathcal{G}^w$ is a Hilbert space.
(2) *$\mathcal{G}^w$ is a Hilbert Lie group which is modeled on the Lie algebra $L\mathcal{G}^w$. Each element $g\in\mathcal{G}^w$ tends to a limit in Γ_θ at infinity, and the evaluation map is defined*:

$$R\colon \mathcal{G}^w\to\Gamma_\theta.$$

Let $\mathcal{G}_0^w$ be the kernel of R. Then $\mathcal{G}^w/\mathcal{G}_0^w \cong \Gamma_\theta$. Now the Lie algebra of $\mathcal{G}_0^w$ is given by

$$L\mathcal{G}_0^w = L^{2,w}_{k+1}(\Lambda^0(i\lambda)).$$

For a configuration $(A,\Phi) \in \mathcal{C}^w$, the infinitesimal $\mathcal{G}_0^w$-action is given by the map

$$\mathcal{I}_\Phi \colon L^{2,w}_{k+1}(\Lambda^0(i\lambda)) \to L^{2,w}_k(\Lambda^1(i\lambda) \oplus S^+)$$

defined by $\mathcal{I}_\Phi(f) = (-2df, f\Phi)$. When $\mathcal{I}_\Phi^*$ is the formal adjoint of $\mathcal{I}_\Phi$, the adjoint of $\mathcal{I}_\Phi$ with respect to the *weighted* norm is given by

$$\mathcal{I}_\Phi^{*,w}(\alpha) = w^{-2}\mathcal{I}_\Phi^*(w^2\alpha).$$

This gives the decomposition(*Cf.* [**7**]):

$$L^{2,w}_k(\Lambda^1(i\lambda) \oplus S^+) = (\ker \mathcal{I}_\Phi^{*,w} \subset L^{2,w}_k) \oplus \mathcal{I}_\Phi(L^{2,w}_{k+1}).$$

Since the $\mathcal{G}_0^w$-action on $\mathcal{C}^w$ is free, the quotient space $\tilde{\mathcal{B}}^w = \mathcal{C}^w/\mathcal{G}_0^w$ is a Hilbert manifold, with a local model

$$T_{[(A,\Phi)]}\tilde{\mathcal{B}}^w = \ker \mathcal{I}_\Phi^{*,w} \cap L^{2,w}_k.$$

The $\mathrm{Pin}^-(2)$-monopole map is defined as

$$\Theta = \Theta_\mu \colon \mathcal{C}^w \to L^{2,w}_{k-1}(\Lambda^+(i\lambda) \oplus S^-),$$
$$\Theta_\mu(A,\Phi) = \left(\frac{1}{2}F_A^+ - q(\Phi) - \mu, D_A\Phi\right),$$

where μ is a (compact-supported) $i\lambda$-valued self-dual 2-form. The moduli space is defined by $\mathcal{M} = \Theta^{-1}(0)/\mathcal{G}^w$.

Proposition 5.11. *The moduli space $\mathcal{M}$ is compact.*

Proof. Let $[(A_n,\Phi_n)]$ be any sequence in $\mathcal{M}$. In general, one can prove that the sequence has a chain convergent subsequence. ([**3**], Chapter 5 and [**7**], Chapter 7.) Since there is only one critical point on $Y = S^3$, the subsequence converges in $\mathcal{M}$. q.e.d.

The differential of Θ at $x = (A,\Phi)$ is given by

$$\mathcal{D}_{(A,\Phi)} = D\Theta \colon L^{2,w}_k(\Lambda^1(i\lambda) \oplus S^+) \to L^{2,w}_{k-1}(\Lambda^+(i\lambda) \oplus S^-),$$
$$\mathcal{D}_{(A,\Phi)}(a,\phi) = \left(\frac{1}{2}d^+a - Dq_\Phi(\phi), D_A\phi + \frac{1}{2}\rho(b)\Phi\right),$$

where Dq_Φ is the differential of q. Then

$$\mathcal{D}_{(A,\Phi)} \circ \mathcal{I}_\Phi(f) = (0, fD_A\Phi). \tag{5.12}$$

Therefore, if (A,Φ) is a $\mathrm{Pin}^-(2)$-monopole solution, then $\mathcal{D}_{(A,\Phi)} \circ \mathcal{I}_\Phi(f) = 0$, which forms the deformation complex:

$$0 \longrightarrow L^{2,w}_{k+1}(\Lambda^0(i\lambda)) \xrightarrow{\mathcal{I}_\Phi} L^{2,w}_k(\Lambda^1(i\lambda) \oplus S^+)$$
$$\xrightarrow{\mathcal{D}_{(A,\Phi)}} L^{2,w}_{k-1}(\Lambda^+(i\lambda) \oplus S^-) \longrightarrow 0.$$

The cohomology groups are denoted by $H^i_{(A,\Phi)}$.

The monopole map Θ defines a Γ_θ-invariant section of a bundle over $\tilde{\mathcal{B}}^w$ whose linearization is given by $\mathcal{I}^{*;w}_\Phi \oplus \mathcal{D}_{(A,\Phi)}$. When Y is the standard S^3, the virtual dimension of the moduli space "framed at infinity" $\tilde{\mathcal{M}} = \Theta^{-1}(0)/\mathcal{G}^w_0 \subset \tilde{\mathcal{B}}^w$ is given by

$$\operatorname{ind}^+(\mathcal{I}^*_\Phi \oplus \mathcal{D}_{(A,\Phi)}) + \dim \Gamma_\theta = d(c) + 1,$$

where $d(c)$ is in (2.9). The genuine moduli space is $\mathcal{M} = \tilde{\mathcal{M}}/\Gamma_\theta$ whose virtual dimension is $d(c)$. In general, $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are not smooth manifolds, and we need to perturb the equations. Before that, we introduce a term.

Definition 5.13. The moduli space $\mathcal{M}$ is said to be *regular* if all of elements $[(A,\Phi)]$ of $\mathcal{M}$ have $H^2_{(A,\Phi)} = 0$.

Remark 5.14. If $\mathcal{M}$ contains no reducibles, then $H^0_{(A,\Phi)} = 0$ for all $[(A,\Phi)] \in \mathcal{M}$. But the converse is not necessarily true, because the stabilizer of a Pin$^-$(2)-monopole reducible $[(A,0)]$ on a twisted Spin^{c-}-structure is $\{\pm 1\}$, and then $H^0_{(A,0)} = 0$.

If $b_+(X;l) \geq 1$, by perturbing the equation by adding a *compactly-supported* self-dual 2-form as in (2.6), we obtain a smooth $\tilde{\mathcal{M}}$:

Theorem 5.15 ([**7**], Proposition 8.2.1)**.** *Suppose $b_+(X;l) \geq 1$. For generic compactly-supported self-dual 2-forms, by perturbing the equations as in (2.6), the perturbed moduli space $\tilde{\mathcal{M}}$ is regular and contains no reducibles, and therefore is a smooth manifold of dimension $d(c)+1$. Then $\mathcal{M}$ is a smooth manifold of dimension $d(c)$.*

When $\mathcal{M}(X)$ has no reducibles, the cutting-down method described in §3.3 works well for $\mathcal{M}(X)$ in this section. However, if $\mathcal{M}(X)$ contains a reducible, we need a little care for it as follows. Choose loops $\gamma_1, \dots, \gamma_b$, where $b = b_1(X;l)$, satisfying the conditions (C1) and (C2) in §3.1. Define the map $\mathfrak{h}\colon \mathcal{M}(X) \to T^b$ by $\mathfrak{h} = \mathrm{Hol}_{\gamma_1} \times \cdots \times \mathrm{Hol}_{\gamma_b}$.

Theorem 5.16. *Suppose $b_+(X;l) = 0$, $\tilde{c}_1(E)^2 = \operatorname{sign}(X)$ (and hence the Dirac index is 0 and $d(c) = b_1(X;l)$). For a generic choice of $\alpha \in T^b$ and a compactly-supported self-dual 2-form, the cut-down moduli space $\mathcal{M} \cap \mathfrak{h}^{-1}(\alpha)$ is regular, and therefore consists of one reducible point and a finite number of irreducible points.*

Proof. The proof is similar to that in [**16**], Subsection 4.8. Due to the noncompactness of X, we need to modify the following point: The space $L^{2,w}_k(\Lambda^1(i\lambda))$ is decomposed into the direct sum of $\ker d^+$ and its complement $(\ker d^+)^\perp$. Furthermore, since $b_+(X;l) = 0$, $d^+\colon (\ker d^+)^\perp \to L^{2,w}_{k-1}(\Lambda^+(i\lambda))$ is an isomorphism. Mimicking the argument in the proof of Lemma 14.2.1 of [**7**], we can prove the following.

Claim. Fix a compact codimension-0 submanifold $K \subset X$ and let $\Omega^+_{X,K}(i\lambda)$ be the space of smooth self-dual 2-forms on X supported on K with C^∞-topology. For $(b,\mu) \in \ker d^+ \oplus \Omega^+_{X,K}(i\lambda)$, let $A(b,\mu)$ be the connection $A^0 + b + (d^+)^{-1}(\mu)$. Let $\mathcal{R}$ be the set of $(b,\mu) \in \ker d^+ \oplus \Omega^+_{X,K}(i\lambda)$ such that $D_{A(b,\mu)}$ is surjective, (and hence, of course, also injective). Then $\mathcal{R}$ is open-dense.

Claim. There exists a gauge invariant open-dense subset $\mathcal{R}' \subset \mathcal{C}^w \times \Omega^+_{X,K}(i\lambda)$ such that the restriction of the $\mathrm{Pin}^-(2)$-monopole map Θ_μ to $\mathcal{R}'$ has 0 as regular value.

With these understood,

$$\mathcal{Z} = \{(A,\Phi,\mu) \in \mathcal{R}' \mid \Theta_\mu(A,\Phi) = 0\,\}$$

is a submanifold in $\mathcal{R}'$. Then it suffices to apply the Sard-Smale theorem to the map

$$\mathfrak{h} \times \pi \colon \mathcal{Z} \to T^b \times \Omega^+_{X,K}(i\lambda),$$

where π is the projection. q.e.d.

6. Proofs of gluing formulae

The purpose of this section is to give proofs of the gluing formulae in §3.4.

6.1. Gluing monopoles. Let X_1 and X_2 be Spin^{c-}-4-manifolds with ends $[-1,\infty) \times Y_1$ and $[-1,\infty) \times Y_2$, where $Y_1 = \bar{Y}_2 = S^3$. Fix a reducible solution $(\theta, 0)$ on S^3, and choose a C^∞ reference connection A_i^0 on each X_i which is the pull-back of θ on the tube. Let $x_i = (A_i, \Phi_i)$ be finite energy monopole solutions on X_i $(i = 1, 2)$. Furthermore, we also suppose $H^2_{x_1} = H^2_{x_2} = 0$. We assume each A_i is in temporal gauge on the tube, and if necessary, consider it as a one-parameter family of connections $\theta + a_i(t)$ on the tube. The spinors Φ_i are also considered as one-parameter families $\Phi_i(t)$ on the tube.

Now, we construct an approximated solution on $X^{\#T}$ from (A_1, Φ_1) and (A_2, Φ_2) by splicing construction. Choose a smooth cut-off function γ, with $\gamma(t) = 1$ for $t \leq 0$ and $\gamma(t) = 0$ for $t \geq 1$. Define $x_1' = (A_1', \Phi_1')$ over X_1 by

$$\begin{aligned} A_1' &= \theta + \gamma(t - T + 3) a_1(t), \\ \Phi_1' &= \gamma(t - T + 3)\Phi_1(t). \end{aligned} \tag{6.1}$$

Define $x_2' = (A_2', \Phi_2')$ over X_2 in a similar fashion.

Fix an identification of the Spin^{c-}-structures on $[0, 2T] \times Y_1$ and $[0, 2T] \times Y_2$ with respect to θ. Note that the Spin^{c-}-structures on the tubes are untwisted, which are identified with ordinary Spin^c-structures. The all possibilities of such identifications are parameterized by Γ_θ, which are called the *gluing parameters*. If we fix an identification σ_0, then the other identifications are indicated as $\sigma = \exp(v)\sigma_0$ for $v \in \operatorname{Lie}\Gamma_\theta \cong i\mathbb{R}$. For an identification σ, we can glue x_1' and x_2' via σ to give a configuration over $X^{\#T}$. The glued configuration is denoted by

$$x'(\sigma) = (A'(\sigma), \Phi'(\sigma)).$$

Then it is easy to see the following

Proposition 6.2. *For each $i = 1, 2$, let Γ_i be the stabilizer of the monopole x_i. Then $x'(\sigma_1)$ and $x'(\sigma_2)$ are gauge equivalent if and only if $[\sigma_1] = [\sigma_2]$ in $\Gamma_\theta/(\Gamma_1 \times \Gamma_2)$, where Γ_i are the stabilizers of x_i.*

Let $\mathrm{Gl} = \Gamma_\theta/(\Gamma_1 \times \Gamma_2)$. Define the map $\mathfrak{F}' \colon \mathrm{Gl} \to \mathcal{B}(X^{\#T})$ by the splicing construction above: $[\sigma] \mapsto [x'(\sigma)]$. If $H^2_{x_1} = H^2_{x_2} = 0$ and T is sufficiently large, then we can find in a unique way a monopole solution $x(\sigma)$ on $X^{\#T}$ near the spliced configuration $x'(\sigma)$. (This is standard in gluing theory. See [**4**, **3**, **7**, **17**].) Then we have a smooth map

$$\mathfrak{F} \colon \mathrm{Gl} \to \mathcal{M}(X^{\#T}), \quad [\sigma] \mapsto [x(\sigma)]. \tag{6.3}$$

Before proceeding, we give another description of the spliced family $\{[x'(\sigma)]\}$ for gluing parameters $\sigma \in \Gamma_\theta$. According to the definition of $x'(\sigma)$, for different σ, $x'(\sigma)$ are objects on different bundles parameterized by σ. It is convenient if we can represent all $[x'(\sigma)]$ as objects on a fixed identification, say σ_0, of bundles. This is also done in [**4**], §7.2.4, in the ASD case.

Recall $X^{\#T} = X_1^0 \cup ([-T,T] \times Y) \cup X_2^0$, and X_1^{2T} and X_2^{2T} are assumed to be embedded in $X^{\#T}$. Choose a smooth function λ_1 on $X^{\#T}$ such that $\lambda_1 = 1$ on X_1^0, $\lambda_1 = 0$ on X_2^0 and

$$\lambda_1(t,y) = \begin{cases} 1 & -T \le t \le -1, y \in Y, \\ 0 & 1 \le t \le T, y \in Y, \end{cases}$$

and satisfies $|\nabla\lambda| = O(1)$. Define another function λ_2 on $X^{\#T}$ by $\lambda_2 = 1 - \lambda_1$. Let $v \in \operatorname{Lie}\Gamma_\theta = i\mathbb{R}$, and $\sigma = \sigma_0 \exp(v)$. Define gauge transformations h_1 and h_2 on $X^{\#T}$ by

$$\begin{aligned} h_1 &= \exp(\lambda_2 v) \\ h_2 &= \exp(-\lambda_1 v). \end{aligned} \tag{6.4}$$

Note that $h_1 h_2^{-1} = \exp(\lambda_1 + \lambda_2)v = \exp v$. Then $h_1 x_1' = h_2 x_2'$ over $[-2,2] \times Y$ on which x_1' and x_2' are flat, and therefore we can glue them. The glued configuration is denoted by $x'(\sigma_0, v)$. Then, by definition, it can be seen that $x'(\sigma)$ and $x'(\sigma_0, v)$ are gauge equivalent. Often, we will not distinguish these two, and use the same symbol $x'(\sigma)$.

6.2. Gluing maps between the moduli spaces. The gluing construction (6.3) can be globalized to whole moduli spaces. In fact, we can define the map

$$\Xi \colon \tilde{\mathcal{M}}(X_1) \times_{\Gamma_\theta} \tilde{\mathcal{M}}(X_2) \to \mathcal{M}(X^{\#T}),$$

for sufficiently large T.

Theorem 6.5. *Let X_1 and X_2 be Spin^{c-}-4-manifolds with ends $[-1,\infty)\times Y_1$ and $[-1,\infty)\times Y_2$, where $Y_1=\bar{Y}_2=S^3$. Suppose the following.*

- *The Spin^{c-}-structure on X_1 may be twisted or untwisted, and $\mathcal{M}(X_1)$ contains no reducibles.*
- *The Spin^{c-}-structure on X_2 is twisted, and $\mathcal{M}(X_2)$ may contain a reducibles.*
- *Both of $\mathcal{M}(X_1)$ and $\mathcal{M}(X_2)$ are regular and 0-dimensional.*

Then Ξ is a diffeomorphism between 1-dimensional compact manifolds.

Theorem 6.6. *Suppose X_1 is a Spin^{c-}-4-manifold with the end $[-1,\infty)\times S^3$ whose moduli space $\mathcal{M}(X_1)$ is regular and contains no reducibles. Suppose X_2 is a Spin^c(untwisted Spin^{c-})-4-manifold with the end $[-1,\infty)\times S^3$ such that $b_1(X_2)=b_+(X_2)=0$ and $\dim\mathcal{M}(X_2)=-1$. Then Ξ induces a diffeomorphism*

$$\mathcal{M}(X_1)\to\mathcal{M}(X^{\#T}).$$

With the results in the previous subsections understood, we can prove these theorems by a similar way to those of the corresponding theorems in the Seiberg-Witten and Donaldson theory. (See [**4**, **3**, **7**, **17**]).

6.3. The images of the map $\mathfrak{F}$. To prove the gluing formulae, we want to know what is the homology class of the image of $\mathfrak{F}$ in $H_*(\mathcal{B})$. The homology class depends on whether each of the Spin^{c-}-structures on X_1 and X_2 is twisted or untwisted, and whether each of monopoles x_1 and x_2 is irreducible or not. We call an irreducible/reducible monopole on a twisted Spin^{c-}-structure $\mathrm{Pin}^-(2)$-*irreducible/reducible*, and an irreducible/reducible monopole on an untwisted Spin^{c-}-structure U(1)-*irreducible/reducible*. We assume that the Spin^{c-}-structures of x_2 is twisted. Then $\mathcal{B}(X^{\#T})$ is homotopy equivalent to $\mathbb{R}\mathrm{P}^\infty\times T^{b_1(X^{\#T};l)}$. Let $\nu_2^*=\mu_{\mathcal{F}}(\nu_2)$ and $\hat{\delta}^*=\hat{\mu}_{\mathcal{E}}(\delta)$ for the loops ν_2 and δ in $X^{\#T}$ chosen as in §3.4. For monopoles x_1 and x_2 on X_1 and X_2, let C be the image of $\mathfrak{F}$. Suppose x_1 and x_2 are not U(1)-reducible. Then C is a circle.

Theorem 6.7. *For the homology classes $[C]\in H_1(\mathcal{B};\mathbb{Z})$ and $[C]_2\in H_1(\mathcal{B};\mathbb{Z}_2)$ of C, we have the following*:

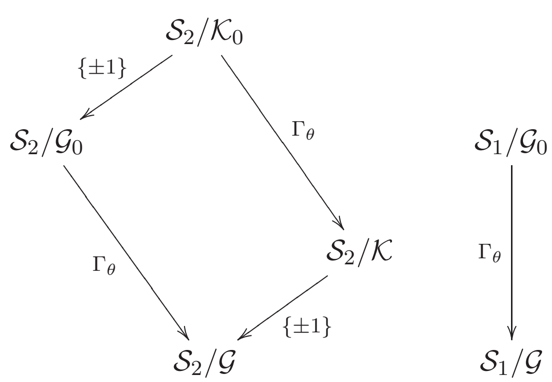
1. *If x_1 is a U(1)-irreducible and x_2 is a $\mathrm{Pin}^-(2)$-reducible, then $\langle\nu_2^*,[C]_2\rangle\neq 0$.*
2. *If x_1 is a U(1)-irreducible and x_2 is a $\mathrm{Pin}^-(2)$-irreducible, then $[C]=[C]_2=0$.*
3. *If x_1 is a $\mathrm{Pin}^-(2)$-irreducible and x_2 is a $\mathrm{Pin}^-(2)$-reducible, then $\langle\hat{\delta}^*,[C]\rangle=\pm1$.*
4. *If both of x_1 and x_2 are $\mathrm{Pin}^-(2)$-irreducibles, then $\langle\hat{\delta}^*,[C]\rangle=\pm2$.*

Before proving the theorem, we give some preliminaries. In the following, we simplify the notation as $\mathcal{G} = \mathcal{G}^w$, $\mathcal{G}_0 = \mathcal{G}_0^w$ and $\mathcal{K} = \mathcal{K}_\gamma$ which is in Remark 2.14. Let $\mathcal{K}_0 = \mathcal{K} \cap \mathcal{G}_0$. For each $i = 1, 2$, let $\mathcal{S}_i$ be the set of solutions which are $\mathcal{G}$-equivalent to x_i. Now, we prove the assertions (1) and (2).

Proof of (1) *and* (2). We have a commutative diagram whose vertical and horizontal arrows are exact:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \{\pm 1\} & = & \{\pm 1\} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
1 & \longrightarrow & \mathcal{G}_0 & \longrightarrow & \mathcal{G} & \longrightarrow & \Gamma_\theta & \longrightarrow & 1, \\
 & & \uparrow & & \uparrow & & \| & & \\
1 & \longrightarrow & \mathcal{K}_0 & \longrightarrow & \mathcal{K} & \longrightarrow & \Gamma_\theta & \longrightarrow & 1, \\
 & & \uparrow & & \uparrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

We also have the following diagrams of various quotient maps:



By definition, $\mathcal{S}_1/\mathcal{G}$ and $\mathcal{S}_2/\mathcal{G}$ are one-point sets. Then $\mathcal{S}_1/\mathcal{G}_0$ is a circle on which Γ_θ acts freely. Hence, $C = \operatorname{Im} \mathfrak{F}$ can be written as

$$C = (\mathcal{S}_1/\mathcal{G}_0) \times_{\Gamma_\theta} (\mathcal{S}_2/\mathcal{G}_0) = \frac{(\mathcal{S}_1/\mathcal{G}_0) \times_{\Gamma_\theta} (\mathcal{S}_2/\mathcal{K}_0)}{\{\pm 1\}} = (\mathcal{S}_2/\mathcal{K}_0)/\{\pm 1\}.$$

First, let us consider the case of (2). In this case, $\mathcal{G}$ acts on $\mathcal{S}_2$ freely. Therefore $\mathcal{S}_2/\mathcal{K}_0 \cong \Gamma_\theta \times \{\pm 1\}$, and we can see that the homology class of C is zero. In the case of (1), each element of $\mathcal{S}_2$ has the stabilizer $\{\pm 1\} \subset \mathcal{G}$. Since $\mathcal{G}_0 \cap \{\pm 1\} = \{1\}$, we see that $\mathcal{S}_2/\mathcal{G}_0 \cong \Gamma_\theta/\{\pm 1\}$ and $[C]$ is the generator of $H_1(\mathbb{RP}^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2$. q.e.d.

We give an alternative proof which gives more intuitive understanding of the homology class of $[C]$.

Alternative proof of (1) *and* (2). For a section Φ_i of the spinor bundle S_i^+ of c_i $(i=1,2)$, let Φ_i' be the cut-off section as in (6.1). Define

$$\Gamma(S_i^+)' := \{\Phi_i' \,|\, \Phi_i \in \Gamma(S_i^+)\,\}.$$

Let $S_{\sigma_0}^+ = S_1^+ \#_{\sigma_0} S_2^+$ be the glued spinor bundle over $X^{\#T}$ via the gluing parameter σ_0. Then we can assume $\Gamma(S_1^+)' \oplus \Gamma(S_2^+)'$ is a subspace of $\Gamma(S_{\sigma_0}^+)$ via the splicing construction. For the monopoles $x_i = (A_i, \Phi_i)$ and $\sigma = \sigma_0 \exp(v)$ $(v \in \operatorname{Lie}\Gamma_\theta)$, define the configuration $y(\sigma)$ on $X^{\#T}$ by

$$y(\sigma) = (A'(\sigma_0), (\exp(v)\Phi_1', \Phi_2'))$$

where $(\exp(v)\Phi_1', \Phi_2') \in \Gamma(S_1^+)' \oplus \Gamma(S_2^+)'$ is assumed to be an element of $\Gamma(S_{\sigma_0}^+)$ as above. Then the homology class $[C]$ is represented by

$$\{y(\sigma)\}_{\sigma\in\Gamma_\theta} = \{A'(\sigma_0)\} \times C_1 \times \{\Phi_2'\},$$

where $C_1 = \{\exp(v)\Phi_1'\}_v$. Note that C_1 is a circle in the complex line generated by Φ_1'. Let $\mathcal{P} = (\Gamma(S_{\sigma_0}^+) \setminus \{0\})/\{\pm 1\}$. Then $\mathcal{P}$ is homotopy equivalent to $\mathbb{R}\mathrm{P}^\infty$. Consider the following map

$$q\colon \mathcal{P} \to \mathcal{B}^* = \mathcal{C}^*/\mathcal{G}, \quad [\Phi] \mapsto [(A'(\sigma_0), \Phi)].$$

Then the map q induces an injective homomorphism

$$q_*\colon H_*(\mathcal{P}) \to H_*(\mathcal{B}^*).$$

If x_2 is a $\mathrm{Pin}^-(2)$-reducible, then $\Phi_2' \equiv 0$ and $[C_1 \times \{\Phi_2'\}]$ is a generator of $H_1(\mathcal{P}; \mathbb{Z}_2)$. On the other hand, if x_2 is a $\mathrm{Pin}^-(2)$-irreducible, then $\Phi_2' \neq 0$ and $[C_1 \times \{\Phi_2'\}]$ is null-homologous. q.e.d.

In order to prove the assertions (3) and (4), we first consider the gluing of connections. For each $i = 1, 2$, let A_i be a connection on the characteristic bundle E_i for c_i. For $\sigma \in \Gamma_\theta$, let $A_1 \#_\sigma A_2$ be the spliced connection on $E = E_1 \#_\sigma E_2$ as in §6.1. Note that $A_1 \#_\sigma A_2$ is gauge equivalent to $A_1 \#_{-\sigma} A_2$, where $-\sigma = \sigma \exp \pi i$.

Lemma 6.8. *Let* $S = \{[A_1 \#_\sigma A_2]\}_{\sigma\in\Gamma_\theta/\{\pm1\}} \subset \mathcal{A}(E)/\mathcal{G}$ *be the set of gauge equivalence classes of the family* $\{A_1 \#_\sigma A_2\}_{\sigma\in\Gamma_\theta}$. *Then its homology class* $[S] \in H_1(\mathcal{A}(E)/\mathcal{G}; \mathbb{Z})$ *satisfies the following:*

(1) $\langle \bar\alpha_i, [S]\rangle = \langle \bar\beta_j, [S]\rangle = 0$ *for* $i = 1, \ldots, b_1(l_1)$, $j = 1, \ldots, b_1(l_2))$,
(2) $\langle \bar\delta, [S]\rangle = \pm 1$,

where $\bar\alpha_i$, $\bar\beta_j$, $\delta \in H^1(\mathcal{A}/\mathcal{G}; \mathbb{Z})$ *as in Remark* 3.4.

Proof. The assertion (1) is obvious. We prove the assertion (2). Fix $\sigma_0 \in \Gamma_\theta$ as based point, and the spliced connections $A_1 \#_\sigma A_2$ for other

σ are constructed by using (6.4) as in §6.1. For $\sigma \in \Gamma_\theta$, $A_1 \#_\sigma A_2$ and $A_1 \#_{-\sigma} A_2$ are gauge equivalent by the gauge transformation $\check{g}$ such that

$$\check{g} = \begin{cases} 1 & \text{on } X_1^0 \\ -1 & \text{on } X_2^0 \\ \exp(\lambda_2 \pi i) & \text{on } [-T, T] \times Y \end{cases}$$

where λ_2 is the function defined around (6.4). On the other hand, for any w with $0 < w < \pi$, if we put $\sigma_w = \sigma \exp(iw)$, then $A_1 \#_\sigma A_2$ and $A_1 \#_{\sigma_w} A_2$ are not gauge equivalent. Therefore S is a circle embedded in $\mathcal{A}(E)/\mathcal{G}$. By taking homotopy class and projection, we have a surjection $\rho\colon \mathcal{G} \to H^1(X; l)/\mathrm{Tor}$ (see [**16**], Lemma 4.22). Then it suffices to prove $\langle \rho(\check{g}), [\delta] \rangle = \pm 1$. To see this, consider the following commutative diagram:

$$\begin{array}{ccc} \tilde{\mathcal{G}} = \mathrm{Map}(\tilde{X}; \mathrm{U}(1)) & \xrightarrow{\tilde{\rho}} & H^1(\tilde{X}; \mathbb{Z}) \\ \varpi \Big\uparrow & & \varpi' \Big\uparrow \\ \mathcal{G} = \Gamma(\tilde{X} \times_{\{\pm 1\}} \mathrm{U}(1)) & \xrightarrow{\rho} & H^1(X; l)/\mathrm{Tor}, \end{array}$$

where the maps ϖ and ϖ' are the pull-back maps to the double covering $\tilde{X}$. Note the following:

- The image of ϖ is the fixed point set $\tilde{\mathcal{G}}^I$, where the I-action is given by $I\tilde{g} = \overline{\iota^* \tilde{g}}$.
- Let $\tilde{X}_i$ $(i = 1, 2)$ be the double coverings of X_i. Then $\tilde{X}$ is the connected sum "at two points" of $\tilde{X}_1$ and $\tilde{X}_2$. That is, this is obtained as follows: For each $i = 1, 2$, removing two 4-balls from each of $\tilde{X}_i$, we obtain a manifold $\tilde{X}_i'$ whose boundary $\tilde{Y}_i$ is a disjoint union of two S^3. Then $\tilde{X} = \tilde{X}_1' \cup_{\tilde{Y}_1 = \tilde{Y}_2} \tilde{X}_2'$.

Consider a circle $\tilde{\delta}$ embedded in $\tilde{X}$ starting from a point x_1 in $\tilde{X}_1'$ and entering $\tilde{X}_2'$ via a component of $\tilde{Y}_1 = \tilde{Y}_2$ and returning to x_1 via another component of $\tilde{Y}_1 = \tilde{Y}_2$. Then the restriction of $\varpi(\check{g})$ to $\tilde{\delta}$ gives a degree one map from $\tilde{\delta}$ to U(1). q.e.d.

Proof of (3) *and* (4). Let us consider the projection

$$\pi\colon C = (\mathcal{S}_1/\mathcal{G}_0) \times_{\Gamma_\theta} (\mathcal{S}_2/\mathcal{G}_0) \to S,$$

which is defined by $\pi([x_1], [x_2]) = [A_1 \#_\sigma A_2]$, where each A_i is the connection part of x_i. Note that π is a map between two S^1. Then, π has degree 1 in the case of (3), and degree 2 in the case (4). q.e.d.

6.4. Proofs of the gluing theorems.

Proof of Theorem 3.8. First assume $d(c_i) = \dim \mathcal{M}(X_i) = 0$ for $i = 1, 2$. For each $i = 1, 2$, let X_i' be the manifold with cylindrical end obtained from removing a 4-ball from X_i. By perturbing the equations with a compactly-supported 2-form, Theorem 6.6 implies that

$\tilde{\mathcal{M}}(X_i) \cong \tilde{\mathcal{M}}(X_i')$ for a metric on X_i with long neck. By the assumption, $\mathcal{M}(X_1')$ consists of odd numbers, say k, of U(1)-irreducible points. The assumption that $\dim \mathcal{M}(X_2) = 0$ implies $b_1^l = b_1(X_2; l_2) = 0$, and then $\mathcal{M}(X_2)$ consists of one $\mathrm{Pin}^-(2)$-reducible point and maybe several $\mathrm{Pin}^-(2)$-irreducible points. By Theorem 6.5, $\mathcal{M}(X^{\#T})$ is a disjoint union of several circles:

$$\mathcal{M}(X^{\#T}) = \bigcup_{i=1}^{k} C_i \cup \bigcup_j C_j',$$

where C_i are obtained by gluing U(1)-irreducibles and a $\mathrm{Pin}^-(2)$-reducible, and C_j' are made from U(1)-irreducibles and $\mathrm{Pin}^-(2)$-irreducibles. Then Theorem 6.7(1)(2) implies that $\langle h, [\mathcal{M}(X^{\#T})]\rangle = k \bmod 2$, and this implies the theorem.

In the case when $d(c_1)$ or $d(c_2)$ is positive, Theorem 6.5 can be generalized to give the diffeomorphism between 1-dimensional cut-down moduli spaces:

$$\Xi \colon \tilde{M}_1 \times_{\Gamma_\theta} \tilde{M}_2 \to M_T,$$

where

$$\begin{aligned} \tilde{M}_1 &= \tilde{\mathcal{M}}(X_1) \cap \bigcap_{i\in I} V_{\alpha_i} \cap \bigcap_{k=1}^{a} V_{x_k}, \\ \tilde{M}_2 &= \tilde{\mathcal{M}}(X_2) \cap \mathfrak{h}^{-1}(\alpha), \\ M_T &= \mathcal{M}(X^{\#T}) \cap \bigcap_{i\in I} V_{\alpha_i} \cap \bigcap_{k=1}^{a} V_{x_k} \cap \mathfrak{h}^{-1}(\alpha), \end{aligned} \tag{6.9}$$

and $\tilde{M}_1$, $\tilde{M}_2$ and M_T are assumed to be smooth and 1-dimensional. When N is a closed submanifold of $\mathcal{M}(X^{\#T})$, as a homology class,

$$[N \cap V_{x_0}] = \mu_{\mathcal{F}}(x_0) \cap [N] = (\mu_{\mathcal{F}}(\nu) \cup \mu_{\mathcal{F}}(\nu)) \cap [N].$$

From these, the theorem follows. q.e.d.

Proof of Theorem 3.9. This is a corollary of Theorem 6.6. q.e.d.

Proof of Theorem 3.11. The proof is similar to that of Theorem 3.8, by using Theorem 6.7 (3)(4). q.e.d.

Proof of Theorem 3.12. For each $i = 1, 2$, $\mathcal{M}(X_i)$ is perturbed to have no reducibles since $b_+(X_i; l_i) \geq 1$. The cut-down moduli space M_T as in (6.9) is a disjoint union of circles C_i. In the case (i), each C_i is null-homologous by Theorem 6.7(2). In the case (ii), $\langle \gamma_0^*, [C_i]\rangle = \pm 2$ by Theorem 6.7(4). Therefore the $\mathbb{Z}_2$-valued invariant is zero. q.e.d.

By the proof of the case (ii) of Theorem 3.12, Theorem 3.13 is true if the glued moduli space is orientable. The orientability of the glued moduli space follows from the next lemma:

Lemma 6.10. *For $i = 1, 2$, let X_i be a closed oriented connected 4-manifold with a twisted Spin^{c-}-structure c_i whose Dirac index is positive and even, and A_i be a connection on the characteristic bundle E_i. Then for S in Lemma 6.8, the restriction of $\operatorname{ind}\delta_{Dirac}$ to S, $\operatorname{ind}(\delta_{Dirac}|_S)$, is orientable.*

Proof. We construct a framing of the index bundle $\operatorname{ind}(\delta_{Dirac}|_S)$. For simplicity, we assume $\operatorname{ind} D_{A_2} = 2$, and the general case will be clear. Let us consider the family $\{D_{A_1\#_\sigma A_2}\}_{\sigma\in\Gamma_\theta}$. By Proposition 2.2 in [**1**], we may assume $\operatorname{Coker} D_{A_1\#_\sigma A_2} = 0$ for any σ. Since $\ker D_\theta = 0$ on S^3, we can construct an isomorphism for each σ ([**3**], §3.3):

$$\alpha_\sigma\colon \ker D_{A_1} \oplus \ker D_{A_2} \to \ker D_{A_1\#_\sigma A_2}.$$

In the proof of Lemma 6.8, we have seen that $A_1\#_\sigma A_2$ is gauge equivalent to $A_1\#_{-\sigma} A_2$ by a gauge transformation g. Now we can see that, for $\psi \in \operatorname{Ker} D_{A_1}$ and $\phi \in \operatorname{Ker} D_{A_2}$,

$$\alpha_\sigma(\psi, \phi) = g\alpha_{-\sigma}(\psi, -\phi).$$

Let $\{\psi^j\}$ be a basis for $\ker D_{A_1}$, and $\{\phi^1, \phi^2\}$ be a basis for $\ker D_{A_2}$. Fix $\sigma_0 \in \Gamma_\theta$ and let $\sigma_w = \sigma_0 \exp(iw)$ for $0 \le w \le \pi$, and

$$\begin{pmatrix}\phi^1_w\\ \phi^2_w\end{pmatrix} = \begin{pmatrix}\cos w & -\sin w\\ \sin w & \cos w\end{pmatrix}\begin{pmatrix}\phi^1\\ \phi^2\end{pmatrix}.$$

Then the following gives a framing for $\operatorname{ind}(\delta_{Dirac}|_S)$:

$$\{\alpha_{\sigma_w}(\psi^j, \phi^1_w), \alpha_{\sigma_w}(\psi^j, \phi^2_w)\}.$$

q.e.d.

Corollary 6.11. *For each $i = 1, 2$, let X_i be a closed oriented connected 4-manifold with a twisted Spin^{c-}-structure which has the following properties*:

- *the index of the Dirac operator is positive and even, and*
- *the moduli space $\mathcal{M}(X_i)$ is orientable.*

Then the glued moduli space $\mathcal{M}(X_1\#X_2)$ is also orientable.

Proof of Theorem 3.13. Since each of $\mathcal{M}(X_i)$ is orientable, Corollary 6.11 implies the moduli space of $X_0\#\cdots\#X_n$ is also orientable. The statement for the invariant is proved by Theorem 6.7. q.e.d.

7. Proofs of Theorem 1.15 and Corollary 1.18

We begin with the proof of Theorem 1.15. Our proof of Theorem 1.15 is similar to the proof of Thom conjecture due to Kronheimer and Mrowka [**11**]. (*Cf.* [**17**].)

7.1. Reduction to the case when $\alpha\cdot\alpha = 0$. Suppose $n := \alpha\cdot\alpha > 0$. Let $X' = X\#n\overline{\mathbb{CP}}^2$, and E_i $(i = 1,\dots,n)$ be the (-1)-sphere in the i-th $\overline{\mathbb{CP}}^2$. Take the connected sum in X',

$$\Sigma' = \Sigma\#E_1\#\cdots\#E_n.$$

Then $[\Sigma']\cdot[\Sigma'] = 0$.

Even if we replace X by X', the $\mathrm{Pin}^-(2)$-monopole invariant is unchanged by Theorem 3.9. Furthermore, even if we replace $\tilde{X}$ by $\tilde{X}'$, the Seiberg-Witten invariant is also unchanged by the ordinary blow-up formulae [**5**, **17**]. The quantity $-\chi(\Sigma)$ and $\alpha\cdot\alpha + |\tilde{c}_1(E)\cdot\alpha|$ are also unchanged. Thus, we may assume $\alpha\cdot\alpha = 0$.

In the remainder of this section, we suppose (X,α,Σ) satisfies the assumption of the beginning of §1.3, and

- $\alpha = [\Sigma] \in H_2(X;l)$ has infinite order, and
- $\alpha\cdot\alpha = 0$.

7.2. The case when $\chi(\Sigma) > 0$. Here, we prove that, under the assumption of Theorem 1.15, the Euler characteristic of Σ cannot be positive:

Proposition 7.1. *If $\chi(\Sigma) > 0$, then the $\mathrm{Pin}^-(2)$-monopole invariants of (X,c) and the Seiberg-Witten invariants of $(\tilde{X},\tilde{c})$ are trivial.*

Proof. The Seiberg-Witten case is proved by Theorem 1.1.1 in [**7**] or Proposition 4.6.5 in [**17**]. The $\mathrm{Pin}^-(2)$-monopole case is similar. Take a tubular neighborhood N of Σ, and let $Y = \partial N$ and $X_0 = \overline{X\setminus N}$. Then Y admits a positive scalar curvature metric g_Y. Decompose X as $X = X_0\cup_Y N$. For a positive real number T, let us insert a cylinder between X_0 and N as:

$$X_T = X_0\cup([-T,T]\times Y)\cup N.$$

Fix a metric on X_T which is product on the cylinder: $dt^2 + g_Y$. Let $\alpha_{\mathbb{R}}$ be the class in $H_2(X_T;\lambda) = H_2(X_T;l\otimes\mathbb{R})$ corresponding to $\alpha \in H_2(X_T;l)$. Since α is suuposed to have infinite order, $\alpha_{\mathbb{R}}$ is a nonzero class in $H_2(X_T;\lambda)$. Let $a \in H^2(X_T;\lambda)$ be the Kronecker dual of $\alpha_{\mathbb{R}}$ such that $\langle a,\alpha_{\mathbb{R}}\rangle = 1$. Then the image of a by the restriction map $r\colon H^2(X_T;\lambda)\to H^2(Y;i^*\lambda)$ is also a nonzero class. Choose a 2-form $\eta\in\Omega^2(Y;i^*\lambda)$ representing $r(a)$. Let us perturb the $\mathrm{Pin}^-(2)$-monopole equations on Y by η as in (4.3). Since every $\mathrm{Pin}^-(2)$-monopole solution for a positive scalar curvature metric g_Y is reducible, a generic small choice of η makes the perturbed Chern-Simons-Dirac functional (4.6) have no critical point. Choose a 2-form $\mu\in i\Omega^2(X;\lambda)$ whose restriction to the cylinder is the pull-back of $i\eta$.

Now suppose the $\mathrm{Pin}^-(2)$-monopole invariants of (X,c) is nontrivial. Then the moduli space $\mathcal{M}(X_T)$ is nonempty for all T. Taking the limit $T\to\infty$, we can obtain a finite energy solution on the manifold with cylindrical end, $X_0 \cup [-1, \infty) \times Y$. Since a finite energy solution should converge to a critical point at infinity (Corollary 5.8), this is a contradiction. q.e.d.

7.3. The case when Σ is nonorientable. Take a tubular neighborhood N of Σ, and let $Y = \partial N$ and $X_0 = \overline{X \setminus N}$. Decompose X as $X = X_0 \cup_Y N$. For a large $T > 0$, insert a long cylinder between X_0 and N as:

$$X_T = X_0 \cup ([-T, T] \times Y) \cup N.$$

Fix a metric on X_T which is product on the cylinder: $dt^2 + g_Y$. (Below, we will take a special metric g_Y on Y.) Let $\tilde{X}_T$ be the associated double covering. Then

$$\tilde{X}_T = \tilde{X}_0 \cup ([-T, T] \times \tilde{Y}) \cup \tilde{N},$$

where $\tilde{Y} = S^1 \times \tilde{\Sigma}$ and $\tilde{N} = D^2 \times \tilde{\Sigma}$. (The object with $\tilde{}$ is the associated double covering.) Take the metric g_Y on Y so that its pull-back metric on $\tilde{Y} = S^1 \times \tilde{\Sigma}$ is of the form

$$d\theta^2 + g_{\tilde{\Sigma}},$$

where $g_{\tilde{\Sigma}}$ is the metric with constant scalar curvature $-2\pi(4g(\tilde{\Sigma}) - 4)$. Then the volume of $\tilde{\Sigma}$ is 1.

Now, consider the limit $T \to \infty$. For $\tilde{X}_T$, the following is known.

Proposition 7.2 ([**11**], Proposition 8)**.** *If the Seiberg-Witten invariant of $(\tilde{X}, \tilde{c})$ is nontrivial, then there is a translation invariant Seiberg-Witten solution on $\mathbb{R} \times \tilde{Y}$.*

The same method of proof as in [**11**] yields the following:

Proposition 7.3. *If the $\mathrm{Pin}^-(2)$-monopole invariant of (X, c) is nontrivial, then there exists a translation invariant $\mathrm{Pin}^-(2)$-monopole solution on $\mathbb{R} \times Y$.*

Under the situation of Proposition 7.3, by pulling back the $\mathrm{Pin}^-(2)$-monopole solution on $\mathbb{R} \times Y$ to $\mathbb{R} \times \tilde{Y}$, we also have a translation invariant Seiberg-Witten solution on $\mathbb{R} \times \tilde{Y}$.

By the argument in [**11**], the existence of a translation invariant solution on $\mathbb{R} \times \tilde{Y}$ implies

$$-\chi(\tilde{\Sigma}) \geq |c_1(L)[\tilde{\Sigma}]|,$$

where L is the determinant line bundle of the Spin^c-structure $\tilde{c}$. This immediately implies

$$-\chi(\Sigma) \geq |\tilde{c}_1(E)[\Sigma]|.$$

7.4. The case when Σ is orientable. Since the restriction of the local system l to Σ is trivial for orientable Σ, the restrictions of the Spin^{c-}-structure to Y and N are untwisted. This reduces the argument to the Seiberg-Witten case [**11**]. Let us consider the case when the Seiberg-Witten invariant of $(\tilde{X}, \tilde{c})$ is nontrivial. Since Σ is orientable, $\tilde{\Sigma}$ has two components: $\tilde{\Sigma} = \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2$. Then take a tubular neighborhood $\tilde{N}_1$ of $\tilde{\Sigma}_1$, and let $\tilde{Y}_1 = \partial \tilde{N}_1$ and $\tilde{X}_0 = \overline{\tilde{X} \setminus \tilde{N}_1}$. Let us consider

$$\tilde{X}'_T = \tilde{X}_0 \cup ([-T, T] \times \tilde{Y}_1) \cup \tilde{N}_1,$$

for large T. This also reduces the argument to the Seiberg-Witten case [**11**].

Proof of Corollary 1.18. Since $(\iota_*)^2 = \mathrm{id}$, $H_2(\tilde{X}; \mathbb{Q})$ splits into (± 1)-eigenspaces. Then (-1)-eigenspace is identified with $H_2(X; l \otimes \mathbb{Q})$. Let $\pi\colon \tilde{X} \to X$ be the projection. Then $\pi_*\colon H_2(\tilde{X}; \mathbb{Q}) \to H_2(X; l \otimes \mathbb{Q})$ can be identified with $\alpha \mapsto \frac{1}{2}(\alpha - \iota_* \alpha)$. It follows from these and the assumption that $\Sigma \cap \iota\Sigma = \emptyset$ that $\pi(\Sigma)$ satisfies the conditions in Theorem 1.16. q.e.d.

References

[1] M. F. Atiyah & I. M. Singer, *The index of elliptic operators: IV*, Ann. of Math. **93**, 119–138.

[2] S. Bauer & M. Furuta, *A stable cohomotopy refinement of Seiberg-Witten invariants. I*, Invent. Math. **155** (2004), no. 1, 1–19.

[3] S. K. Donaldson, Floer homology groups in Yang-Mills theory. With the assistance of M. Furuta and D. Kotschick. Cambridge Tracts in Mathematics, 147. Cambridge University Press, Cambridge, 2002.

[4] S. K. Donaldson & P. B. Kronheimer, The geometry of four-manifolds. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1990.

[5] R. Fintushel & R. Stern, *Immersed spheres in 4-manifolds and the immersed Thom conjecture*, Turkish J. Math. **1**9 (1995), no. 2, 145–157.

[6] R. Fintushel & R. Stern, *Knots, links, and 4-manifolds*, Invent. Math. **1**34 (1998), no. 2, 363–400.

[7] K. A. Froyshov, *Compactness and gluing theory for monopoles,* Geometry & Topology Monographs, 15. Geometry & Topology Publications, Coventry, 2008.

[8] R. Gompf & A. Stipsicz, 4-manifolds and Kirby calculus. Graduate Studies in Mathematics, 20. American Mathematical Society, Providence, RI, 1999.

[9] R. Kirby, Topology of 4-manifolds, Lecture Notes in Mathematics, 1374. Springer-Verlag, Berlin, 1989.

[10] D. Kotschick, J. W. Morgan & C. H. Taubes, *Four-manifolds without symplectic structures but with nontrivial Seiberg-Witten invariants*, Math. Res. Lett. **2** (1995), no. 2, 119–124.

[11] P. B. Kronheimer & T. Mrowka, *The genus of embedded surfaces in the projective plane,* Math. Res. Lett. **1** (1994), no. 6, 797–808.

[12] P. B. Kronheimer & T. Mrowka, Monopoles and three-manifolds. New Mathematical Monographs, 10. Cambridge University Press, Cambridge, 2007.

[13] J. W. Morgan, Z. Szabó, *Homotopy* $K3$ *surfaces and mod* 2 *Seiberg-Witten invariants*, Math. Res. Lett. **4** (1997), 17–21.

[14] J. W. Morgan, Z. Szabó, C. H. Taubes, *A product formula for the Seiberg-Witten invariants and the generalized Thom conjecture,* J. Differential Geom. **44** (1996), no. 4, 706–788.

[15] N. Nakamura, *A free* $\mathbb{Z}_p$*-action and the Seiberg-Witten invariants*, J. Korean Math. Soc. **39** (2002), no. 1, 103–117.

[16] N. Nakamura, Pin$^-$(2)*-monopole equations and intersection forms with local coefficients of* 4*-manifolds*, Math. Ann. **357** (2013), pp 915–939

[17] L. I. Nicolaescu, Notes on Seiberg-Witten theory, Graduate Studies in Mathematics, 28. American Mathematical Society, Providence, RI, 2000.

[18] C. Okonek, *Fake Enriques surfaces*, Topology **27** (1988), no. 4, 415–427.

[19] P. Ozsváth & Z. Szabó, *The symplectic Thom conjecture*, Ann. of Math. (2) **1**51 (2000), no. 1, 93–124.

[20] P. Ozsváth & Z. Szabó, *Higher type adjunction inequalities in Seiberg-Witten theory*, J. Differential Geom. **5**5 (2000), no. 3, 385–440.

[21] Y. Ruan & S. Wang, *Seiberg-Witten invariants and double covers of* 4*-manifolds,* Comm. Anal. Geom. **8** (2000), no. 3, 477–515.

[22] C. H. Taubes, *Gauge theory on asymptotically periodic* 4*-manifolds*, J. Differential Geom. **25** (1987), no. 3, 363–430.

[23] C. T. C. Wall, *Diffeomorphism of* 4*-manifolds*. J. London Math. Soc. (1964) s1-39 (1): 131–140.

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