J. DIFFERENTIAL GEOMETRY
101 (2015) 423-465

FAMILIES OF CONTACT 3-MANIFOLDS WITH ARBITRARILY LARGE STEIN FILLINGS

R. İnanç Baykur, Jeremy Van Horn-Morris,
Appendix: Samuel Lisi & Chris Wendl

Abstract

We show that there are vast families of contact 3-manifolds each member of which admits infinitely many Stein fillings with arbitrarily large Euler characteristics and arbitrarily small signature—disproving a conjecture of Stipsicz and Ozbagci. To produce our examples, we use a framework which generalizes the construction of Stein structures on allowable Lefschetz fibrations over the 2-disk to those over any orientable base surface, along with the construction of contact structures via open books on 3-manifolds to spinal open books introduced in [**24**].

1. Introduction

Understanding the topology of possible Stein fillings of a fixed contact 3-manifold has been an active line of research in the past couple of decades. By now it is known that there are contact 3-manifolds which admit no Stein filling, as well as a unique Stein filling, or many, and even infinitely many ones, up to diffeomorphisms. However, all examples of Stein fillings of a fixed contact 3-manifold known up to date bore the same curious aspect: their characteristic numbers constitute a finite set. Andras Stipsicz conjectured that the set of signatures and Euler characteristics of all possible Stein fillings of a closed contact 3-manifold is finite [**30**, Conjecture 1.2]. The same conjecture was also formulated by Burak Ozbagci and Andras Stipsicz for the Euler characteristics alone [**29**, Conjecture 1.2]; [**28**, Conjecture 12.3.16], and more specifically as the Euler characteristics being bounded above [**28**, Conjecture 1.3.9]. There are many examples of Stein fillable contact structures for which the finiteness of both characteristic numbers is seen to hold true: those on 3-manifolds which are non-flat circle bundles over orientable surfaces [**30**], or those which admit compatible planar open books [**22**] are a few. Our main theorem, however, disproves this conjecture, for all of its flavors:

Received 4/12/2013.

Theorem 1.1. *There are infinite families of contact* 3*-manifolds, where each contact* 3*-manifold admits a Stein filling whose Euler characteristic is larger and signature is smaller than any two given numbers.*

Let us call a Lefschetz fibration on a 4-manifold "allowable" if its base and regular fibers are connected, compact surfaces with *non-empty* boundaries, and if each vanishing cycle is homologically non-trivial in the fiber. Following the works of Eliashberg and Gompf on handle decompositions of compact Stein manifolds, Loi and Piergallini proved that any Stein domain admits a Lefschetz fibration structure [**25**] (and an alternative proof was later given by Akbulut and Ozbagci [**1**]). Moreover, the Stein structure on an *allowable Lefschetz fibration* can be chosen so that the contact structure it induces on the boundary agrees with the one that the Thurston-Winkelnkemper construction would produce when applied to the natural open book induced by the Lefschetz fibration on the boundary. We will use an extension of this result to Lefschetz fibrations over *arbitrary compact surfaces* (that is, orientable surfaces with any number of boundary components and of any genera) filling the same contact structure on the 3-manifold boundary induced by a *generalized* open book structure: roughly speaking, we will use a decomposition of a 3-manifold as a certain "plumbing" of a surface bundle over a disjoint union of circles and circle bundles over arbitrary surfaces, where the surfaces in the former and latter collections have the same topology, respectively. These generalized open books are introduced and studied in [**24**] under the name *spinal open books*, which we will adopt here. Note that when we have a surface bundle over a circle and a circle bundle over a 2-disk, this is the usual notion of an open book decomposition of a 3-manifold, and thus exists on all 3-manifolds. We prove the following theorem using handle decompositions and convex surface theory:

Theorem 1.2. *If $f : X \to \Sigma$ is an allowable Lefschetz fibration with bounded fiber (where Σ is any compact surface with non-empty boundary), then X admits a Stein structure. Moreover, the Stein structures on any two allowable Lefschetz fibrations filling the same spinal open book can be chosen so that they induce the same contact structure on the boundary.*

This result was known to Sam Lisi and Chris Wendl, who provide their proof, a variation of a technique of Gompf and Thurston, in the appendix to this article. Combined with Loi and Piergallini's stronger result on the existence of allowable Lefschetz fibrations (over the 2-disk) on compact Stein manifolds, this theorem generalizes, in the obvious way, the characterization of Stein manifolds in terms of the Lefschetz fibrations they can be equipped with. (See Corollary 3.9.)

The organization of our article is as follows:

We discuss spinal open books and the natural contact structures we associate to them in Section 3. These parallel the descriptions in [**24**] and in the appendix. For completeness, we show, using convex surface theory, that there is a unique choice of a compatible contact structure on a given spinal open book (Propositions 3.2 and 3.5). Discussing the handle decompositions and induced Stein cobordisms for building an allowable Lefschetz fibration over an arbitrary compact surface with non-empty boundary, we prove Theorem 1.2 in the same section using a cut-and-paste operation we call *folding* (or a *spinal tap* in the case of a spinal open book). Our techniques have the same flavor as those used in [**5**] and mimic the construction in [**4**].

Section 4 is where we present our families of examples for Theorem 1.1. Our main examples will be the graph manifolds $Y(g,h,n)$, which are plumbings of circle bundles with Euler numbers 0 and n over surfaces of genera g and h. A surgery diagram of $Y(g,h,n)$ is given in Figure 8 in Section 4. We will define a distinguished contact structure $\xi_{Y(g,h,n)}$ on each $Y(g,h,n)$ via the framed spinal open book on it. Here, for each triple of integers $g \geq 2, h \geq 1, n \leq 2h-2$, we produce infinite families of Stein fillings of contact 3-manifolds $(Y(g,h,n), \xi_{g,h,n})$, by constructing infinite families of Lefschetz fibrations, whose Euler characteristics can be chosen to be arbitrarily large. We will also show that the Stein fillings of $(Y(2,h,n), \xi_{2,h,n})$ can be chosen so that they have arbitrarily small (negative) signatures. All these examples are derived from special families of Lefschetz fibrations on *closed* 4-manifolds (Theorem 4.1), which are built using relations in the mapping class groups of surfaces with boundaries after [**6**]. Lastly, we outline how to get similar families of Stein fillings of a fixed contact structure on more general 3-manifolds, so as to illustrate that the contact 3-manifolds $(Y(g,h,n), \xi_{g,h,n})$ above demonstrate a general phenomenon and are nowhere close to being special in this sense.

Acknowledgments. The first author was partially supported by the NSF grant DMS-0906912. The authors would like to thank the organizers of the 2012 Georgia Topology Conference for the wonderfully stimulating atmosphere they created, where this work was shaped. We thank John Etnyre, Burak Ozbagci, and Chris Wendl for their comments on the first version of this paper. We would also like to thank Sam Lisi and Chris Wendl for the appendix.

2. Preliminaries

Here we review the background material we will use and generalize in the later sections. All manifolds in this article are assumed to be compact, smooth, and oriented, whereas the maps between them are always smooth.

Guide to notation. We will generally follow the following conventions for notation:

- X, X^4: a connected, compact 4-manifold, possibly with boundary
- Y, Y^3: closed, connected 3-manifold
- F, Σ: compact, connected surfaces, possibly with boundary
- S: a convex surface with positive and negative regions S_+ and S_- and dividing set Γ
- $\widehat{F}$, $\widehat{\Sigma}$: compact, (possibly) disconnected surfaces with (each component having) boundary
- Γ_g^s: the mapping class group of a genus g surface with s boundary components (fixing the boundary pointwise)
- ϕ: is a mapping class element of a connected surface
- $\widehat{\phi}$: is a mapping class element of a disconnected surface

2.1. Lefschetz fibrations and mapping class groups. A *Lefschetz fibration* is a surjective map $f\colon X \to \Sigma$, where X and Σ are 4- and 2-dimensional compact manifolds, respectively, such that f fails to be a submersion along a finite set C, and around each *critical point* in C it conforms to the local model $f(z_1, z_2) = z_1 z_2$, compatible with orientations. If the regular fiber F has genus g and Σ has genus h, we say that (X, f) is a genus g Lefschetz fibration over a genus h surface. The critical points arise from attaching 2-handles to regular fibers with framing -1 with respect to the framing induced by the fiber. We will refer to these 2-handles as *Lefschetz handles*. We will assume that each *singular fiber* contains only one critical point, which can be achieved after a small perturbation of any given Lefschetz fibration. When there are no critical points, $f : X \to \Sigma$ is nothing but a surface bundle over a surface, so f always restricts to a surface bundle over $\Sigma \setminus f(C)$ on $X \setminus f^{-1}(f(C))$ and, in particular, over $\partial\Sigma$ on ∂X. The reader is advised to turn to [**18**] for a detailed treatment of Lefschetz fibrations via handlebody decompositions.

We will call a Lefschetz fibration *allowable*, if both the base and the regular fiber have non-empty boundaries, and if no fiber contains a closed embedded surface. In the literature, allowable Lefschetz fibrations over the 2-disk are called *PALFs*, "positive allowable Lefschetz fibrations," where positivity emphasizes the orientation-preserving local model we prescribed for the Lefschetz singularities.

Let $\Sigma^s_{g,r}$ denote a compact oriented surface of genus g with s boundary components and r marked points in the interior. The *mapping class group*, $\Gamma^s_{g,r}$, of $\Sigma^s_{g,r}$ is the group of isotopy classes of orientation-preserving self-diffeomorphisms of $\Sigma^s_{g,r}$, which are compactly supported in the interior of $\Sigma^s_{g,r}$, fixing r marked points and the points on the boundary. For simplicity, we write $\Sigma_{g,r} = \Sigma^0_{g,r}$, $\Sigma^s_g = \Sigma^s_{g,0}$, and $\Sigma_g = \Sigma^0_{g,0}$. We also use the similar simplified notation for the corresponding mapping class groups. It is well-known that $\Gamma^r_{g,m}$ is generated by positive (right-handed) Dehn twists along non-separating curves.

For a smooth surface bundle $f\colon E \to \Sigma$ with fibers Σ^s_g, the *monodromy representation* of f is defined to be the map $\Psi\colon \pi_1(\Sigma) \to \Gamma^s_g$ relative to a fixed identification φ of F with the fiber over the base point of Σ: For each loop $\gamma\colon I \to \Sigma$ the bundle $f_\gamma\colon \gamma^*(E) \to I$ is canonically trivial, inducing a diffeomorphism $f_\gamma^{-1}(0) \to f_\gamma^{-1}(1)$ up to isotopy. Using φ to identify $f_\gamma^{-1}(0)$ and $f_\gamma^{-1}(1)$ with F, we get the element $\Psi(\gamma) \in \Gamma_g$. Changing the identification φ changes Ψ by a conjugation with an element of Γ_g. We will use the functional notation for the mapping class group: i.e., for $f_1, f_2 \in \Gamma_g$, the product $f_1 f_2$ means that we first apply f_2 and then f_1—thus the map $\Psi\colon \pi_1(\Sigma) \to \Gamma_g$ is an anti-homomorphism.

A genus g Lefschetz fibration $f\colon X \to \Sigma$ with a regular fiber $F \cong \Sigma_g$ can be defined combinatorially using the *monodromy representation* $\Psi\colon \pi_1(B \setminus f(C)) \to \Gamma^r_g$, which determines f up to isomorphism (and X up to diffeomorphism), provided $g \geq 2$. (This is due to the fact that for $g \geq 2$ the space of self-diffeomorphisms of F isotopic to the identity is contractible.) Importantly, isotopy type of a surface bundle over S^1 with fiber F is determined by the return map of a flow transverse to the fibers, which can be identified with an element $\mu \in \Gamma_g$, called *monodromy* of this fibration over S^1.

It turns out that the monodromy of a Lefschetz fibration $f\colon X \to D^2$ over the disk with a single critical point is a right Dehn twist along the vanishing cycle creating the singular fiber. Therefore, the monodromy of a Lefschetz fibration $f\colon X \to \Sigma_h$ with n critical points is given by a factorization of the identity element $1 \in \Gamma_g$ as

$$1 = \prod_{i=1}^{n} t_{v_i} \prod_{j=1}^{h} [\alpha_j, \beta_j]\,, \tag{1}$$

where v_i are the vanishing cycles of the singular fibers and t_{v_i} is the positive Dehn twist about v_i. This factorization of the identity is called the *monodromy factorization*. Here the mapping classes a_i and b_i specify the monodromies along a free generating system $\langle \alpha_1, \beta_1, \ldots, \alpha_h, \beta_h \rangle$ of $\pi_1(\Sigma^1_h)$ such that $\prod_{i=1}^h [\alpha_i, \beta_i]$ is parallel to the boundary component of Σ^1_h. In particular, when there are no t_{v_i} in the factorization, this prescribes a surface bundle. Conversely, a word

$$w = \prod_{i=1}^{n} t_{v_i} \prod_{j=1}^{h} [\alpha_j, \beta_j]$$

prescribes a Lefschetz fibration over Σ^1_h, and if $w = 1$ in Γ_g we get a Lefschetz fibration $X \to \Sigma_h$.

For a Lefschetz fibration $f : X \to \Sigma$, a map $\sigma\colon \Sigma \to X$ is called a *section* if $f \circ \sigma = id_\Sigma$. Suppose that a fibration $f\colon X \to \Sigma$ admits a section σ. Set $S = \sigma(\Sigma) \subset X$. This section S provides a lift of the representation $\Psi : \pi_1(\Sigma \setminus f(C)) \to \Gamma_g$ to the mapping class group $\Gamma_{g,1}$. One can then fix a disk neighborhood of this section preserved under the monodromy, and get a lift to Γ_g^1. Conversely, every such representation with a lift determines a fibration with a section: Gluing a disk with a marked point to a surface with one boundary component along the boundary, and extending self-diffeomorphisms of the surface by the identity on the disk, we obtain a surjective homomorphism $\Gamma_g^1 \to \Gamma_{g,1}$, whose kernel is freely generated by the right Dehn twist t_δ along a simple closed curve δ parallel to the boundary. If the factorization

$$1 = \prod_i t_{v_i} \prod_j [\alpha_j, \beta_j]$$

lifts from Γ_g to a similar factorization in $\Gamma_{g,1}$, then the corresponding fibration has a section. Moreover, if we lift this product to Γ_g^1 we get

$$t_\delta^m = \prod_i t_{v_i'} \prod_j [\alpha_j', \beta_j']$$

for some m. Here, $t_{v_i'}$ is a Dehn twist mapped to t_{v_i} under $\Gamma_g^1 \to \Gamma_g$. Similarly, α_j' and β_j' are mapped to α_j and β_j, respectively. An elementary observation is that the power m of t_δ in the above factorization in Γ_g^1 is the negative of the self-intersection number of the section S that we obtain.

These observations generalize in a straightforward fashion to the case when we have r *disjoint* sections $S_1, \dots, S_r$, corresponding to r marked points captured in the mapping class group $\Gamma_{g,r}^s$.

2.2. Open book decompositions. An *open book decomposition* $\mathcal{B}$ of a 3-manifold Y^3 is a pair (K, f) where K is an oriented link in Y, called the *binding*, and $f \colon Y \setminus K \to S^1$ is a fibration such that $f^{-1}(t)$ is the interior of a compact oriented surface $F_t \subset Y$ and $\partial F_t = K$ for all $t \in S^1$. The surface $F = F_t$, for any t, is called the *page* of the open book. The *monodromy* of an open book is given by the return map of a flow which is transverse to the pages and meridional near the binding. We consider this as an element $\phi \in \Gamma_g^s$, where g is the genus of the page F, and s is the number of components of $K = \partial F$. Equivalently, and more fitting with our later definitions, we can think of an open book decomposition as a decomposition of $Y = Y_P \cup Y_\Sigma$, where Y_P is the fiber bundle $f : Y_P \to S^1$ (as before) with compact fibers, Y_Σ is a union of solid tori $S^1 \times \{D_1^2, \dots, D_s^2\}$ (the neighborhoods of the binding components K) and each meridional disk $p \times D^2$ intersects the boundary of the fibers of f in a single point. (And hence that, up to isotopy, D^2 is determined by the topology of Y_Σ.)

Suppose we have a Lefschetz fibration $f \colon X \to D^2$ with bounded regular fiber F, and let p be a regular value in the interior of the base D^2.

Composing f with the radial projection $D^2 \setminus \{p\} \to \partial D^2$ we obtain an open book decomposition on $Y = \partial X$ with binding $\partial f^{-1}(p)$. Identifying $f^{-1}(p) \cong F$, we can write

$$Y = (\partial F \times D^2) \cup f^{-1}(\partial D^2)\,.$$

Thus we view $\partial F \times D^2$ as the tubular neighborhood of the binding $K = \partial f^{-1}(p)$, and the fibers over ∂D^2 as its *truncated pages*. The monodromy of this open book is prescribed by that of the fibration. In this case, we say that the open book $(K, f|_{\partial X\setminus K})$ is *filled by*, or *is the boundary of*, the Lefschetz fibration (X, f). Any open book whose monodromy can be written as a product of positive Dehn twists can be filled by a Lefschetz fibration over the 2-disk.

We can think of the second definition of an open book in this language as well. As a Lefschetz fibration, the boundary of X inherits a Künneth-like decomposition consisting of vertical and horizontal boundaries (as viewed by f). In that case the fibered region Y_P is the vertical boundary of f, $f^{-1}(\partial D^2)$, and Y_Σ is the horizontal boundary, which is the (trivial) bundle of boundary circles ∂F_t over D^2. As a bundle, we think of this as $f|_{\partial F}$. Each component of Y_Σ is topologically $S^1 \times D^2$ and there is a unique isotopy class of section which trivializes the bundle.

2.3. Contact structures and compatibility. A 1-form $\alpha \in \Omega^1(Y)$ on a $(2n-1)$-dimensional oriented manifold Y is called a *contact form* if it satisfies $\alpha \wedge (d\alpha)^{n-1} \neq 0$. A *co-oriented contact structure* on Y is then a hyperplane field ξ which is globally written as the kernel of a contact 1-form α. In dimension three, this is equivalent to asking $d\alpha$ to be nondegenerate on the plane field ξ.

A contact structure ξ on a 3-manifold Y is said to be *supported by an open book* $\mathcal{B} = (K, f)$ if ξ is isotopic to a contact structure given by a 1-form α satisfying $\alpha > 0$ on positively oriented tangents to K and $d\alpha$ is a positive volume form on every page. When this holds, we say that the open book $\mathcal{B}$ is *compatible with the contact structure* ξ on Y. It is a classical result of Thurston and Winkelnkemper [**31**] that any open book admits such a contact structure (where the notion of "compatibility" is due to Giroux).

Considering contact 3-manifolds as boundaries of certain 4-manifolds together with various compatibility conditions has been an active research topic in low dimensional topology. From the contact topology point of view, it is the study of different types of *fillings* of a fixed contact manifold. In dimension four, there are essentially two considerations. Let (X, ω) be a symplectic 4-manifold with cooriented nonempty boundary $Y = \partial X$. If there exists a *Liouville vector field* ν defined on a neighborhood of ∂X pointing out along ∂X, then we obtain a positive contact structure ξ on ∂X, which can be written as the kernel of contact 1-form $\alpha = \iota_\nu \omega|_{\partial X}$. When this holds, we say (Y, ξ) is the *ω-convex* *boundary* or *strongly convex boundary* of (X, ω). (When ν points inside, we say (Y, ξ) is the *ω-concave boundary of* (X, ω).)

Now if (X, J) is almost-complex, then the complex tangencies on $Y = \partial X$ give a uniquely defined oriented hyperplane field. It follows that there is a 1-form α on Y such that $\xi = \mathrm{Ker}\alpha$. We define the *Levi form* on Y as $d\alpha|_\xi(\cdot, J\cdot)$. If this form is positive definite then (Y, ξ) is said to be a *strictly J-convex boundary of* (X, J), and if it is J-convex for an unspecified J (for instance when J is tamed by a given symplectic form), we say (Y, ξ) is a *strictly pseudoconvex boundary*. If (X, ω, J) is an almost-Kähler manifold, i.e., a manifold equipped with a symplectic form ω and a compatible almost-complex structure J, then it can be shown that strict pseudoconvexity of the boundary is equivalent to the condition that $\omega|_\xi > 0$.

For detailed and comparative discussions of these concepts, as well as proofs of some facts mentioned in the next subsection, the reader can turn to [**10**] and [**12**]. For further basic notions from contact topology of 3-manifolds such as Legendrian knots, Thurston-Bennequin framing (which appears below), the text of Ozbagci–Stipsicz [**29**], or that of Geiges [**14**] would be valuable sources.

2.4. Stein manifolds. A smooth function $\psi\colon X \to \mathbb{R}$ on a complex manifold X of real dimension $2n$ is called *strictly plurisubharmonic* if ψ is strictly subharmonic on every holomorphic curve in X. We call a complex manifold X *Stein*, if it admits a proper strictly plurisubharmonic function $\psi\colon X \to [0, \infty)$ (after Grauert [**19**]). Thus a compact manifold X with boundary which is equipped with a complex structure in its interior is called *compact Stein* if it admits a proper strictly plurisubharmonic function which is constant on the boundary.

Given a function $\psi\colon X \to \mathbb{R}$ on a Stein manifold, we can define a 2-form $\omega_\psi = -dJ^* d\psi$. It turns out that ψ is a strictly plurisubharmonic function if and only if the symmetric form $g_\psi(\cdot, \cdot) = \omega_\psi(\cdot, J\cdot)$ is positive definite. So every Stein manifold X admits a Kähler structure ω_ψ, for any strictly plurisubharmonic function $\psi\colon X \to [0, \infty)$. It is easy to see that the restriction of ω_ψ to each level set $\psi^{-1}(t)$ gives a Levi form on $\psi^{-1}(t)$, implying that all nonsingular level sets of ψ are strictly pseudoconvex hypersurfaces. Thus one can equivalently call a Stein manifold a *strictly pseudoconvex manifold*. Moreover, it was observed in [**10**] that the gradient vector field of ψ defines a (global) Liouville vector field $\nu = \nabla_\psi$, making all nonsingular level sets ω_ψ-convex. Hence, Stein manifolds exhibit strongest filling properties for a contact manifold which can be realized as their boundary. Given contact 3-manifold (Y, ξ), we will call Stein surface (X, J) a *Stein filling* of (Y, ξ) if $\partial X = Y$ and $J|_Y$ induces the contact structure ξ.

In this article, we are mainly interested in compact Stein surfaces with convex boundaries, up to diffeomorphisms. A topologist's characterization of these manifolds in terms of Weinstein structures ([**34**], [**7**]) follows from the work of Eliashberg and Gompf:

Theorem 2.1 (Eliashberg [**8**], Gompf [**17**]). *A smooth oriented compact 4-manifold with boundary is a Stein surface, up to orientation preserving diffeomorphisms, if and only if it has a handle decomposition* $X_0 \cup h_1 \cup \ldots \cup h_m$*, where* X_0 *consists of* 0- *and* 1-*handles and each* h_i*,* $1 \leq i \leq m$*, is a* 2-*handle attached to*

$$X_i = X_0 \cup h_1 \cup \ldots \cup h_i$$

along a Legendrian circle L_i *with framing* $tb(L_i) - 1$.

Theorem 2.2 (Loi–Piergallini [**25**]; also see Akbulut–Ozbagci [**1**]). *An oriented compact 4-manifold with boundary is a Stein surface, up to orientation preserving diffeomorphisms, if and only if it admits an allowable Lefschetz fibration over the* 2-*disk, a.k.a "PALF." Moreover, any two allowable Lefschetz fibrations over the* 2-*disk filling the same open book carry Stein structures which fill the same contact structure (induced by the open book).*

2.5. Convex surfaces. In this article, we will make extensive use of convex surface theory, which we review briefly here. For details and proofs, see [**21**]. A surface S in a contact 3-manifold Y is *convex with Legendrian boundary* if any boundary component of S is tangent to the contact planes and there is a vector field X defined in a neighborhood of S that is positively transverse to S and which preserves the contact planes. In that case, we assume that X is transverse to ξ and let S_+ denote the set of points for which X is positively transverse to ξ, S_- the set of points where X is negatively transverse to ξ, and Γ the set where X is tangent to ξ. Γ is then a collection of properly embedded, simple closed curves which separate S_+ and S_- called the *dividing set*.

Theorem 2.3 (Giroux [**15**], Honda [**21**]). *For a convex surface* S *with Legendrian boundary, the subsets* S_+ *and* S_- *are embedded submanifolds whose boundary constitutes a collection of properly embedded circles* Γ*. Further, the isotopy class of* ξ *in a neighborhood of* S *is determined by* Γ.

The *standard convex* S^2 has a single circle as its dividing set. The *standard* 3-*ball* is the contact manifold which is tight on B^3 and with boundary the standard convex S^2. A *bypass* is a convex bigon with Legendrian boundary and whose dividing set consists of a single arc with both boundary points on the same boundary arc.

A contact 3-manifold Y admits a *decomposing disk* if there is a proper, non-boundary parallel convex disk D with Legendrian boundary whose dividing set consists of a single arc. We say Y is *disk decomposable* if there is a collection of disjoint decomposing disks so that cutting and rounding gives a collection of standard contact 3-balls. A *product contact manifold* is a contact manifold, diffeomorphic to $S \times I$ for some compact, convex surface S with Legendrian boundary. The notions of a disk decomposable and product contact manifold are equivalent up to smoothing the boundary.

An S^1*-invariant contact structure* is a contact structure on a surface bundle over S^1 with convex torus (or empty) boundary, whose fibers are all convex surfaces. Equivalently, an S^1-invariant contact structure is made by taking a product contact manifold and gluing the top to the bottom by a diffeomorphism preserving the dividing set.

3. Contact structures on spinal open books and Stein structures on allowable Lefschetz fibrations over arbitrary surfaces

3.1. Spinal open books. The notion of a spinal open book was introduced by Lisi, Wendl, and the second author in [**24**] and used to classify fillings of certain contact manifolds. It is (roughly speaking) the right kind of structure to study contact structures arising as the boundaries of Lefschetz fibrations over non-disk bases. For completeness, we give a set of proofs and constructions here based on convex surface theory. In the appendix, Lisi and Wendl give what should be considered the standard characterization of compatibility, existence, and uniqueness of contact structures in terms of Reeb fields and Giroux forms. The following definitions are equivalent but have been altered to accommodate the spinal tap construction of Section 3.3. A *spinal open book decomposition* $\mathcal{B}$ of a 3-manifold Y is a decomposition of Y into regions $Y_P \cup_T Y_\Sigma$, where

- Y_P is a compact, embedded, codimension-0 submanifold with torus boundary components, equipped with the structure of a fiber bundle $\widehat{F} \hookrightarrow Y_P \xrightarrow{\pi_P} S^1$ for some possibly disconnected surface $\widehat{F}$ with boundary.
- Y_Σ is a compact, embedded, codimension-0 submanifold with torus boundary (the same boundary as ∂Y_P), equipped with the structure of a circle bundle $S^1 \hookrightarrow Y_\Sigma \xrightarrow{\pi_\Sigma} \widehat{\Sigma}$, over a (possibly disconnected) surface with non-empty boundary.
- The (oriented) boundary components of a fiber $\widehat{F}_t$ in Y_P are S^1 fibers of π_Σ (equipped with the same orientation).

We call the fibered region Y_P the *paper*, and the fibers $\widehat{F}_t$ the *pages* (though we may also use *page* to refer to a component of $\widehat{F}_t$). The product region Y_Σ we call the *spine*. For any section of π_Σ, we call a connected component, Σ, a *vertebra* (plural *vertebrae*), though we will also use this terminology for the connected component of $\widehat{\Sigma}$ that it lies over. The tori boundaries T between the two we call *interface tori*.

Note: the specific vertebrae can change depending on which section is chosen. However, we will always have a fixed section in mind (or possibly be altering one section to get another).

For the purposes of this paper, all spinal open books will be *symmetric, uniform, and simple* (in the terminology of [**24**]). By this we mean every component of $\widehat{F}$ is homeomorphic, every component of $\widehat{\Sigma}$ is homeomorphic, and every component of Y_Σ is adjacent to every component of Y_P along a single interface torus.

An *abstract spinal open book* is a 4-tuple $(\widehat{F}, \widehat{\phi}, \widehat{\Sigma}, G)$ where:

- $\widehat{F}$ is a possibly disconnected surface with non-empty boundary, where we write the connected components as $\widehat{F} = F_1 \cup \cdots \cup F_n$
- $\widehat{\phi}$ is an orientation preserving self-diffeomorphism of $\widehat{F}$ fixing the boundary pointwise
- $\widehat{\Sigma}$ is a disjoint union of connected surfaces with non-empty boundaries: $\widehat{\Sigma} = \Sigma_1 \cup \cdots \cup \Sigma_m$
- G is a bijection $G : |\partial\widehat{F}| \to |\partial\widehat{\Sigma}|$ between the boundary components of $\widehat{F}$ and $\widehat{\Sigma}$

To construct an isomorphism class of 3-manifolds with embedded spinal open books from this, we form the surface bundle Y_P over S^1 with fiber $\widehat{F}$ and monodromy $\widehat{\phi}$. The spine Y_Σ is the trivial bundle $S^1 \times \widehat{\Sigma}$. We glue the resulting boundaries together using G to identify components and following the Thurston-Winkelnkemper/Giroux conventions:

- each component of the oriented boundary of a fiber $\widehat{F}$ is an S^1 fiber in the component of $S^1 \times \partial\widehat{\Sigma}$ determined by G and
- each component of the boundary of $\{pt.\} \times \widehat{\Sigma}$ is an orbit of a point in $\partial\widehat{F}$ in Y_P.

3.2. Framed spinal open books. For our purposes an equivalent definition of compatibility between spinal open books and contact structures will be useful, for which we first introduce the following: A *framed spinal open book decomposition* is a spinal open book decomposition along with a section of the spine, up to isotopy. Equivalently (at least up to isotopy), it is a spinal open book decomposition along with an identification of Y_Σ with $\widehat{\Sigma} \times S^1$. (Note that the monodromy condition on an abstract open book means we are automatically constructing a *framed* spinal open book decomposition when we form the closed manifold.)

A *framed* spinal open book decomposition *supports* (or *is compatible with*) a contact structure ξ if the following conditions are satisfied:

- The interface tori are convex with dividing set two parallel curves of negative slope (i.e., in the $(\partial\Sigma, \partial F)$-basis, the dividing set is of

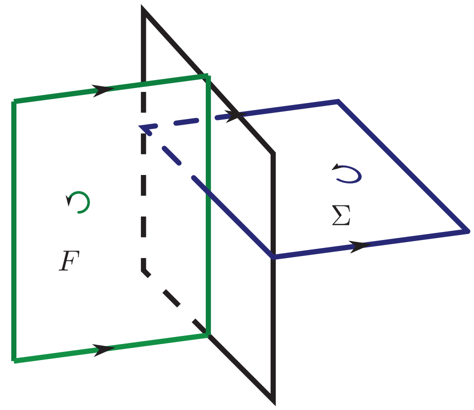

Figure 1. Orientations of the fiber and vertebra at an interface torus T.

the form $\pm(-p, q)$ for $p, q > 0$). We can thicken the interface tori to a convex, i-invariant $T^2 \times I$ region so that each boundary torus has a Legendrian ruling whose slope is given by the slope of the pages or of the vertebrae.

- On each component of the paper Y_P, we can isotope a page F to be convex with Legendrian boundary on $\mathcal{T}$ and with a dividing set consisting of boundary parallel arcs so that the negative regions F_- are boundary parallel bigons, and so that after cutting along F, $Y_P \setminus F$ is a product contact manifold. (Equivalently, we could ask that the complement of F in Y_P be *disk decomposable* after rounding.)
- On each component of the spine Y_Σ, we can make a vertebra Σ convex with Legendrian boundary on $\mathcal{T}$ and with dividing set consisting of boundary parallel bigons, so that after cutting along Σ, $Y_\Sigma \setminus \Sigma$ is a product contact manifold. (Equivalently, we could ask that the complement of Σ in Y_Σ be *disk decomposable* after rounding.)

Intuitively, one should think of the contact structure associated to a framed spinal open book as being a deformation of the tangent planes to the fibers and vertebrae, and rotating a quarter-turn between them in a small neighborhood of the interface torus, just as the contact structure we associate to a standard open book is a deformation of the tangent planes to the fibers and to a small disk neighborhood of the binding, with a quarter-turn rotation in between.

Remark 3.1. The requirement that the slope of the interface torus be negative is necessary for a good definition of compatibility. While there are indeed tight contact structures with a positively sloped dividing set along the interface torus, this does not fit into our definition. Because of our choice of orientation and prescribed positive and negative regions, any compatible contact structure constructed as above with a positively sloped dividing set on T is overtwisted with an overtwisted disk located in a small neighborhood of the interface torus (see Figure 2). This would also be true for a standard open book (whose spinal components are all $S^1 \times D^2$) if the slope is not of the form $(1, q)$.

Proposition 3.2. *Every framed spinal open book decomposition admits a unique isotopy class of compatible contact structures.*

Proof. From the description of compatibility via convex surfaces, given the dividing set on the interface tori T, such a contact structure both exists and is unique up to isotopy. To see this, we can make the given interface tori convex. After thickening, we can make the fiber and the vertebra simultaneously convex with Legendrian boundary and with the specified dividing set. A neighborhood of this union has a unique contact structure and its complement is disk decomposable.

To show that the definition is well-defined, though, we need to see that it is independent of the slope of the dividing set on the interface torus. This is guaranteed by the orientations chosen and described in Figure 1. In particular, we could have chosen the slope to be -1. Here are the details:

By switching to a framed spinal open book, we can make both Y_Σ and Y_P contact bundles with convex fibers. Let $\widehat{\Sigma}$ be the fiber of Y_Σ and let Σ be a connected component of $\widehat{\Sigma}$. Similarly, $\widehat{F}$ is the fiber of Y_P and F is a connected component of $\widehat{F}$. Since the interface torus has dividing sets of slope $(-p, q)$, the fiber Σ has q components in the dividing set parallel to T, and F has p components. We want to show that we can decrease each of q and p to 1 while still keeping Y_Σ and Y_P contact bundles whose convex fibers have boundary parallel dividing sets. To do this, observe that each dividing curve on F determines a bypass for Y_Σ and each dividing curve for Σ determines a bypass for Y_P. Forgetting the contact structure on the complement of T, if we attach $p-1$ bypasses from F and $q-1$ bypasses from Σ, the resulting torus has slope -1. It suffices to show, then, that after sliding along one of these bypasses, the resulting spinal open book remains compatible. Thus the next lemma completes the proof. q.e.d.

Lemma 3.3. *If we slide T over a bypass from F, the resulting spinal open book remains compatible with the contact structure, and similarly for a bypass from Σ.*

Proof. We know the contact structure on Y_P is tight and the complement of F is disk decomposable. Let Y'_P be the result of cutting out the bypass layer from Y_P and let F' be the subsurface of Y'_P consisting of F with the bypass removed. Then the contact structure on $Y_{\mathcal{P}'}$ is tight

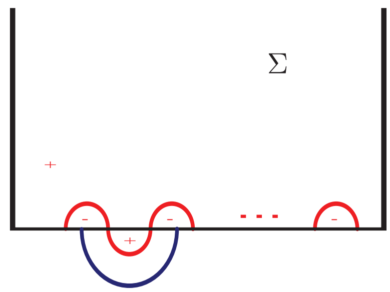

Figure 2. Attaching a bypass from the pages to the spine (or vice versa).

and the complement of F' is disk decomposable. Thus ξ is compatible along $Y_{\mathcal{P}'}$.

Now suppose we attach this bypass to Y_Σ. Coming from F, this bypass is being attached along a vertical Legendrian arc straddling three adjacent arcs of the dividing set. The bypass arc can be slid down the T so it is parallel with the given section Σ of the spine. Because the dividing set on T was chosen with the appropriate slope, attaching this bypass merges two adjacent disks in Σ_- (as opposed to capping a negative bypass to form an overtwisted disk). The dividing sets of Σ and this added bypass are shown in Figure 2. As before, after cutting along Σ, now extended by the bypass, the result is disk decomposable. q.e.d.

The next lemma shows how the framings on a given spinal open book relate to each other, indicating how we get an equivalent definition of a compatible contact structure on an unframed *spinal open book*, up to isotopy.

Lemma 3.4. *Let $\mathcal{B}$ and $\mathcal{B}'$ be two different* framed *spinal open book decompositions which represent the same spinal open book. (In particular, $\mathcal{B}$ and $\mathcal{B}'$ correspond to two different choices of sections of Y_Σ.) Then the two contact structures ξ and ξ' supported by $\mathcal{B}$ and $\mathcal{B}'$ are isotopic.*

Proof. The general idea is this: on an S^1-invariant contact structure, any section is convex with dividing curves coming from the projection to the base. It is slightly trickier (though not much) to deal with contact structures with convex boundary which are S^1-invariant except for possible holonomy along the boundary. We analyze this directly and produce an appropriate convex surface.

Start with the product $Y_\Sigma = S^1 \times \Sigma$ with the vertebra $\{pt\} \times \Sigma$ convex. Changing a section of Y_Σ is equivalent to choosing a function from $\widehat{\Sigma}$ to S^1 and looking at its graph in $S^1 \times \widehat{\Sigma}$. Such a map is determined by its degree on an embedded basis of $\pi_1(\widehat{\Sigma})$. Changing the degree by

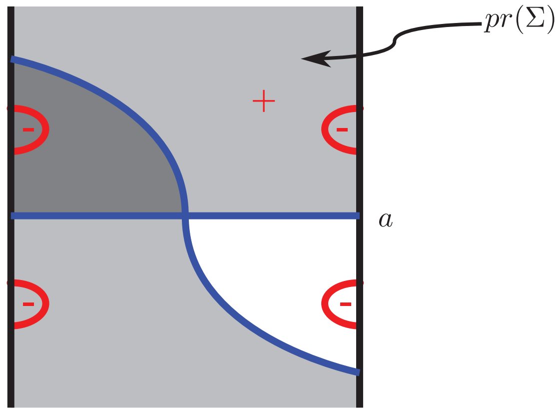

Figure 3. Spinning a vertebra Σ along an arc: the figure shows a neighborhood of an arc in a vertebra of the spine, along with its image after spinning. We want to form a smooth, convex surface from Σ and $S^1 \times a$. To do this, start by cutting the surface Σ along a and dragging one side of it around the S^1 direction. (In the figure, we drag the lower edge of $\Sigma \setminus a$.) This slides each boundary point of a to the right across some number bigons determined by the slope of the dividing set. The figure shows the projection onto Σ of its image after being slid along a, where the light gray regions have multiplicity 1, the dark gray has multiplicity 2, and the white region is missed entirely. To form a closed surface, then, we need to remove one sheet above the dark gray region and add in a copy of the white region. This has the effect of moving some number of boundary parallel bypasses from one boundary component of Σ to another.

1 on a single generator is equivalent to taking a properly embedded dual arc a in a component $\widehat{\Sigma}$ and "spinning" the section Σ around the S^1 direction above a. (Equivalently, if we look at the product annulus $S^1 \times a$ sitting over this arc, we can form the resolution with the section to get the spun surface.) To prove the lemma, we need to show that, by adding an annulus to Σ in $\mathcal{B}$, we get a new framed spinal open book compatible with the same (isotopy class of) contact structure. Since we know that the contact structure on Y_Σ is tight, it's enough to find a convex representative of the new section with boundary parallel dividing curves.

As in the proof of Proposition 3.2 (Lemma 3.3), we can choose any slope for the dividing set on the interface tori which is negative (in the $(\partial\Sigma, \partial F)$ basis). For ease, we assume that we have arranged the contact structure so that the interface tori at the boundaries of Y_Σ touching ∂a have dividing sets of slope $\pm(-1, 2)$. Choose a representative of a on Σ which is disjoint from the dividing set. If we slide a in the S^1 direction so that its feet follow the dividing curves, it will return to Σ having moved to the right by jumping over one disk component of Σ_- on each boundary. We can spin Σ in the vertical S^1 direction in a small neighborhood of a, keeping it convex. We start with the partial annulus given by flowing a along the contact vector field. Once we return to the vertebra Σ, in order to glue to get a closed surface, we need to remove a small triangle of Σ on one boundary, and wrap by an additional triangle on the other as shown in Figure 3. (This has the effect of removing one bigon component of Σ_- on one boundary and adding a bigon component on the other boundary.) This gives a new section with boundary parallel dividing curves, as required. q.e.d.

Combining Lemma 3.4 and Proposition 3.2 gives a new proof of the following result from [**24**]:

Proposition 3.5. *Every spinal open book decomposition is compatible with a unique isotopy class of contact structure.*

3.3. Spinal tap on a spinal open book. We will now define an operation on embedded spinal open books, which comes with a natural Stein cobordism, as we will discuss shortly. This operation (in both directions) has been studied already by Baldwin [**5**]. Avdek [**4**] gives the inverse operation. For ease, we restrict to symmetric, uniform, simple open books, though the operation works in much more generality.

As a motivation, we outline the plan to prove Theorem 1.2. Suppose we start with a Lefschetz fibration $F \hookrightarrow X \to \Sigma$, where the base surface Σ has nonempty boundary. To construct a Stein structure on (X, f), we start with a Lefschetz fibration over the disk and extend it as a bundle over the 1-handles of Σ. To produce a Stein structure, we invoke [**16**, **25**, **1**] to get a Stein structure on the Lefschetz fibration over the disk. The only thing to prove, then, is that we can extend the Stein structure over the 1-handles of Σ. We do this a handle at a time, showing that at every stage we get a spinal open book at the boundary of the Lefschetz fibration which is compatible with the contact structure at the boundary of the Stein structure.

The act of cutting the Lefschetz fibration open along a single arc in Σ (that is, removing a 1-handle and the bundle above it) has a corresponding operation on the spinal open book at the boundary of (X, f) which we call a *spinal tap*. The spinal tap has an analogue in the setting of contact manifolds and convex surfaces. We'll call a closed convex (or sutured) surface *foldable* if it admits an orientation reversing self-diffeomorphism, fixing the dividing set (or sutures) and sending R_+

to R_- (and vice versa). Equivalently, a foldable convex surface is one that arises as the boundary of an I-invariant neighborhood of a convex surface with boundary parallel dividing set (or as the boundary of a product contact or sutured manifold).

Given a contact 3-manifold Y with convex boundary and a foldable boundary component $S = R_+ \cup R_-$, we call the operation of gluing in $R_+ \times I$ by its boundary to S, respecting their dividing sets (and using any identification of R_+ and R_-), *folding* Y along S. For a non-boundary surface S, we first cut open Y along S and then fold $Y \backslash S$ along each of the new boundary components, adding two copies of $R_+ \times I$.

A spinal tap is a folding along a convex surface which sits nicely with respect to a spinal open book, and which uses the same (or, rather, inverse) identifications to glue in the two copies of $R_+ \times I$.

The following describes this motivating cut-and-paste operation in the topological category.

For an embedded spinal open book $\mathcal{B} = Y_P \cup Y_\Sigma$, let S be a surface consisting of two connected components F_1 and F_2 of a single fiber in Y_P along with some annuli made up of some S^1 fibers in Y_Σ connecting them. There is one such annulus in each component of Y_Σ. Since the orientations of F_1 and F_2 don't agree along the annuli, we orient S as $F_1 \cup -F_2$. Call such a surface a *spinal tap surface.* A spinal tap along S is the following operation:

- Cut $\mathcal{B}$ along S. The resulting manifold has two boundary components $S_+ = F_{1+} \cup -F_{2+}$ and $S_- = F_{1+} \cup -F_{2+}$.
- Fold S_+ by gluing F_{1+} to F_{2+} by a diffeomorphism $h : F_1 \to F_2$.
- Fold S_- by gluing F_{2-} to F_{1-} by the inverse diffeomorphism h^{-1} : $F_2 \to F_1$.

The resulting open book $\mathcal{B}' = Y'_P \cup Y'_\Sigma$ has the following:

- The new spine Y'_Σ: Y'_Σ is obtained from Y_Σ by cutting along the connecting annuli of S.
- Y'_P is the bundle made by removing the two products $F_1 \times [0, 1]$ and $F_2 \times [0, 1]$ from Y_P and identifying $F_1 \times \{0\}$ and $F_2 \times \{1\}$ by h and $F_2 \times \{0\}$ and $F_1 \times \{1\}$ by h^{-1}.

It might be helpful to see the effect of a spinal tap on an abstract open book.

Let $\mathcal{B} = (\widehat{F}, \widehat{\phi}, \widehat{\Sigma}, G)$ be an abstract spinal open book. One can choose a foldable surface (up to isotopy) by choosing a collection of arcs $\{a_i\}$, one in each Σ_i, which are all required to connect to the same two boundary components adjacent to the paper over F_j and F_k. If the arc has both endpoints on the same boundary component, then F_j must be equal to F_k. Given a diffeomorphism $h : F_j \to F_k$, we can form a new spinal open book $\mathcal{B}' = (\widehat{F}', \widehat{\phi}', \widehat{\Sigma}', G')$ as follows: There are two cases. If

$j \neq k$, then the spinal tap merges F_j and F_k and, using h, composes their monodromies:

- $\widehat{\Sigma}' = \widehat{\Sigma} \setminus \{a_i\}$, $i = 1, \ldots, n$
- $\widehat{F}' = \widehat{F} - F_k$
- $\widehat{\phi}' = \begin{cases} \text{if } i \neq j \text{ or } k, & \widehat{\phi}'|_{F_i} = \widehat{\phi}|_{F_i} \\ \text{if } i = j, & \widehat{\phi}'|_{F_j} = h^{-1}\widehat{\phi}|_{F_k} h\, \widehat{\phi}|_{F_j} \end{cases}$

If $j = k$, then the spinal tap splits the component of Y_P with fiber F_j into two bundles by adding a new component F_{n+1} to $\widehat{F}$, and decomposing the monodromy:

- $\widehat{\Sigma}' = \widehat{\Sigma} \setminus \{a_i\}$, $i = 1, \ldots, n$
- $\widehat{F}' = \widehat{F} \cup F_{n+1}$
- $\widehat{\phi}' = \begin{cases} \text{if } i \neq j \text{ or } n+1, & \widehat{\phi}'|_{F_i} = \widehat{\phi}'|_{F_i} \\ \text{if } i = j, & \widehat{\phi}'|_{F_j} = h^{-1}\widehat{\phi}|_{F_j} \\ \text{if } i = n+1, & \widehat{\phi}'|_{F_{n+1}} = h \end{cases}$

Proposition 3.6. *Suppose $(Y, \mathcal{B}, \xi)$ is a contact spinal open book and suppose that $(Y', \mathcal{B}', \xi')$ is obtained from $\mathcal{B}$ by a spinal tap. Then there is a Stein cobordism from (Y', ξ') to (Y, ξ). Moreover, if $\mathcal{B}'$ is the boundary of a Lefschetz fibration (X', f'), then this Lefschetz fibration can be extended, along this Stein cobordism, to a Lefschetz fibration (X, f) with boundary $\mathcal{B}$.*

The proof of this proposition is broken down into two parts. First we construct the desired Stein cobordism, verifying that the two contact structures at either end of the cobordism are compatible with the specified spinal open book. Then we show that this cobordism behaves nicely with respect to a Lefschetz fibration with boundary $\mathcal{B}'$.

The following proposition shows that there are nice surfaces in spinal open books (which generalize the idea of a union of a pair of fibers in an open book as in [**33**], for example), for which the *spinal tap* operation on spinal open books described above is equivalent to the convex cut-and-paste operation of *cutting* and *folding* along a convex surface.

Proposition 3.7. *Let $S = F_1 \cup F_2$ be a spinal tap surface in a spinal open book $\mathcal{B}$ and let $\mathcal{B}'$ be the result of a spinal tap along S. Denote by ξ the contact structure supported by $\mathcal{B}$, and denote by ξ' the structure supported by $\mathcal{B}'$. Then we can make S into a foldable convex surface in ξ so that ξ' is the result of cutting and folding ξ along S.*

Proof. Our goal is to construct a suitable, foldable, convex surface isotopic to the spinal tap surface $S = F_1 \cup F_2$, where the positive and negative regions will be F_+ and F_-, respectively. Let Σ be a vertebra in a component of the spine and let a be the isotopy class of the arc in Σ used to construct S. Denote by F_1 and F_2 the two fibers of Y_P with boundary isotopic to the circle in Y_Σ above ∂a. We want to construct a nice convex representative of the annulus A over a so that the boundary circles are Legendrian isotopic in a product neighborhood of T to the boundaries of F_1 and F_2. We can Legendrian realize a on Σ so that it misses the dividing set. After sliding around the S^1 direction, the holonomy of the dividing set on the boundary will force the endpoints of a to jump some number of boundary parallel bigons of Σ_- to the right on each boundary (see Figure 3). To form a closed annulus, A, we first move to the right along one boundary component, then move to the right on the other. We can think of this strip as curved back over itself (as seen from above in the S^1-direction), with a fold that runs from one boundary arc to the other. Everywhere except that fold remains transverse to the contact vector field, giving a convex annulus with dividing set shown in Figure 4.

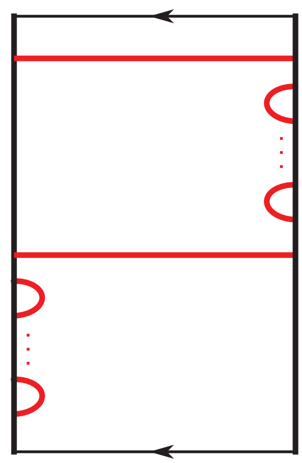

Figure 4. The dividing set on the convex annulus A.

The new spine Y'_Σ will be formed by cutting Y_Σ along A. In particular, we want to add a neighborhood of A to Y_P and round the resulting boundary to get our new interface tori T', with Y'_Σ being the portion of Y_Σ on the inside of T' and taking $\Sigma \backslash a$, rounded, to be our new section Σ'. A schematic is shown in Figure 5. Since Y'_Σ is tight (as a subset of Y_Σ), cutting Y'_Σ along Σ' and rounding gives a disk decomposable handlebody, and so ξ_0 and Y'_Σ remain compatible.

Now let's look at the complement of Y'_Σ, which is $Y_P \cup \nu(A)$ with the boundary rounded. To construct ξ', we will cut along S and fold the two boundaries back upon themselves. However, it will be easier to actually "crease" the boundary of $Y \backslash S$ (that is, to undo the edge-rounding) so that it has a convex boundary with corners, and to glue in the product contact manifold $F_1 \times [0, 1]$, where F_1 is convex with boundary parallel dividing curves, keeping A and its standard neighborhood isolated. A schematic of the cut, crease, and gluing is shown in Figure 6.

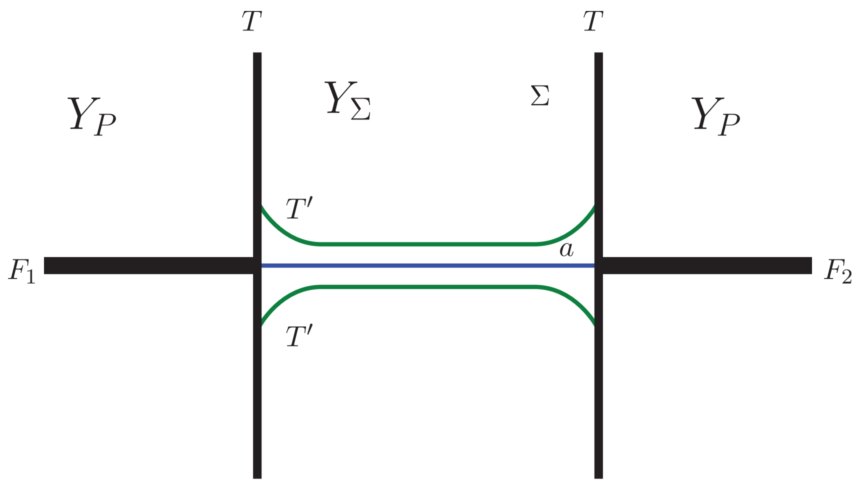

Figure 5. A look at the spine and paper projected to an $I \times I$ subsurface of Σ in a neighborhood of the spinal tap surface. The spinal tap surface is $F_1 \cup S^1 \times a \cup -F_2$. A vertebra Σ lies in the plane of the diagram. Both the old (T) and new (T') interface tori are shown.

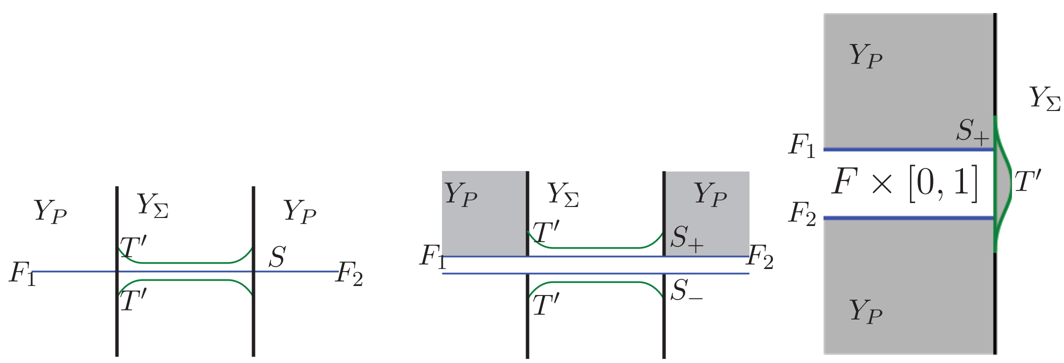

Figure 6. Preparing to fold $Y_P - S$: first we cut along S, then wrap the complement around itself. This manifold has a smooth convex boundary, so we "crease" the boundary near the edges of R_+, undoing edge-rounding, so that we may glue in the product contact manifold with corners, $F \times [0,1]$, where F is convex with dividing set consisting of boundary parallel bigons.

We can then glue in $F_1 \times [0,1]$ as prescribed by the spinal tap. Moreover, since we folded S so as to preserve the dividing set on A, after gluing in $F_1 \times [0,1]$ the contact structure is isotopic to an S^1-invariant contact structure—i.e., the "bump" we added rounding near A simply extends the contact structure by a standard product neighborhood. This constructs the contact structure on Y'_P, and since it is S^1-invariant, it remains compatible with Y'_P. q.e.d.

We can now give the proof of Proposition 3.6.

Proof of Proposition 3.6. To prove Proposition 3.6, we equate convex folding with a sequence of contact (+1)-surgeries and 1-handle removal. We will prove the following: Let S be a spinal tap surface in a contact 3-manifold $(Y, \mathcal{B}, \xi)$ and let $(Y', \mathcal{B}', \xi')$ be the result of folding along S. There is a Stein cobordism from (Y', ξ') to (Y, ξ). Moreover, if $\mathcal{B}'$ is the boundary of a Lefschetz fibration (X', f'), this fibration can be extended along the Stein cobordism to a Lefschetz fibration (X, f) with boundary $\mathcal{B}$.

Let S be a convex surface in (Y, ξ) and suppose S_+ and S_- are homeomorphic surfaces. Folding Y along S gives a new contact manifold—first we cut Y along S and then we glue in two copies of $S_+ \times [0, 1]$, one to each boundary component S_1 and S_2 of $Y \backslash S$. As in the definition of the spinal tap, we choose an orientation-reversing diffeomorphism $h : S_+ \to S_-$, which preserves the identification of ∂S_+ with ∂S_- given by S, and glue by the following identifications:

$$\begin{aligned} S_+ \times \{0\} &\xrightarrow{id} S_+ \\ S_+ \times \{1\} &\xrightarrow{h} S_- \end{aligned}$$

on $S \times \{1\}$ and

$$\begin{aligned} -S_+ \times \{0\} &\xrightarrow{\bar{h}} -S_- \\ -S_+ \times \{1\} &\xrightarrow{id} -S_+ \end{aligned}$$

on $S \times \{0\}$.

Since $S_+ \times [0, 1]$ is disk decomposable, we can take a collection of decomposing arcs for S_+ and extend them to a collection of decomposing disks for $S_+ \times [0, 1]$. Gluing in $S_+ \times [0, 1]$, then, is the same as gluing in these decomposing disks and then filling in the remaining S^2 boundaries (or boundary, if the collection is minimal) with standard contact 3-balls.

We want to compare this with the following surgery construction. Let $\{a_i\}$ be an arc decomposition of S_+ and extend each arc into S_- on S by $h(a_i)$. This will give a Legendrian link $\mathcal{L}$ on S, each component of which has Thurston-Bennequin number one less than the framing induced by S, i.e., $tb = pf - 1$. Contact (+1)-surgery is then topological 0-surgery.

Let l_i be a component of $\mathcal{L}$. A standard neighborhood of l_i, $\nu(l_i)$, framed by S is $l_i \times [0, 1] \times [0, 1]$, where the first I factor is the neighborhood in S and the second gives the vertical direction. The boundary of this neighborhood consists of four annuli: two horizontal and two vertical. Cutting along S is the same as cutting out all of these neighborhoods, plus removing a neighborhood of the resulting complementary punctured disk $S \backslash \mathcal{L}$. Topological 0-surgery along $\mathcal{L}$ glues in a solid torus so that meridional disks get attached to the longitudinal fibers of $\partial\nu(l_i)$. These are the same longitudes as the fibers in the four annuli which make up $\partial\nu(l_i)$. Attaching disks along the horizontal annuli folds the skeleta of S_1 and S_2 together. Attaching disks along the horizontal annuli caps off the punctures of the complementary disk $S - \mathcal{L}$, yielding an $S^2 \times I$ region. If we further cut along this S^2 and glue in two 3-balls, this gives the result of folding along S.

On the contact side, then, we can think of $(+1)$-surgery on $\mathcal{L}$ as follows. First we cut along S. Then we attach a pair of thickened standard decomposing disks $D^2 \times [0,1]$ along each component l_i of $\mathcal{L}$, one sitting on each boundary S_1 and S_2. The new boundary is then a pair of standard convex 2-spheres, which get glued together (cf. Baldwin [**5**]).

To finish the construction and end with the folded manifold Y', we need to cut along this convex S^2 and fill in with two standard contact 3-balls—i.e., we need to surger out a standard contact $S^2 \times I$.

Thus Y' is built from Y by a sequence of contact $(+1)$-surgeries and a 4-dimensional 1-handle removal. The reverse cobordism from Y' to Y consists of a single Weinstein 1-handle, and $b_1(S_+)$ Weinstein 2-handles, which gives a Stein cobordism from Y' to Y.

We can understand the upside-down cobordism as well. After folding, Y' has two surfaces $\tilde{S}_1$ and $\tilde{S}_2$ which are naturally convex and have transverse boundary and trivial dividing set. In particular, they are (subsets of) pages of the spinal open book $\mathcal{B}'$. Unfolding consists of removing neighborhoods of these two surfaces and gluing together the resulting convex boundaries by attaching a $DS \times I$, where DS is the (convex) double of $\tilde{S}_i$. This product has a nice handle decomposition, and adding the 1-handle is easy. The 2-handles are attached along the dual link to $\mathcal{L}$ in $Y' \# S^1 \times S^2$. Each component of the dual link consists of four arcs, a_1, a_2, a_3, a_4, each dual to the $D^2 \times I$ subset of the surgery solid torus as described above. The arc a_1 lies on $\tilde{S}_1$, a_3 lies on $\tilde{S}_2$, and the arcs a_2 and a_4 run between them across the 1-handle (cf. Avdek [**4**]).

If $\mathcal{B}'$ is already the boundary of a Lefschetz fibration L', then the surfaces $\tilde{S}_1$ and $\tilde{S}_2$ are each fibers of $L' = F \hookrightarrow X' \xrightarrow{\pi} \Sigma'$ and this handle decomposition is precisely the handle decomposition used to extend L' as an F-bundle over an additional 1-handle attached to Σ'. Moreover, the gluing map used to extend L' identifies the arcs a_1 on $\tilde{S}_1$ with a_3 on $\tilde{S}_2$. q.e.d.

3.4. Stein structures on Lefschetz fibrations over arbitrary surfaces. With the machinery we have developed in the previous subsections in hand, we can now prove:

Theorem 3.8. *Suppose $(Y, \mathcal{B})$ is the boundary of an allowable Lefschetz fibration (X, f), and let ξ be the contact structure supported by*

$\mathcal{B}$. Then X admits a Stein structure J whose convex, contact boundary is ξ, i.e., (X, J) is a Stein filling of (Y, ξ).

Proof. Let $F \hookrightarrow X \xrightarrow{f} \Sigma$ be a Lefschetz fibration with boundary $\mathcal{B}$. Take a properly embedded arc in Σ which is disjoint from the critical values of f. Then $S = f^{-1}(a)\big|_{\partial X}$ is a spinal tap surface in $\mathcal{B}$. If we cut X along $f^{-1}(a)$ we get a new Lefschetz fibration $F \hookrightarrow X' \xrightarrow{f'} \Sigma'$ with boundary $\mathcal{B}'$, the spinal open book formed by the spinal tap along S. By Proposition 3.6, there is a Stein cobordism which extends f' on X' to f on X. If we then take a set of decomposing arcs for Σ and cut along their $F \times I$ preimages in L, we are left with a Lefschetz fibration over the disk whose boundary is the result of the successive spinal taps on $\mathcal{B}$. Since the resulting Lefschetz fibration over a disk admits a Stein structure filling the boundary open book, repeatedly applying Proposition 3.6 proves that X also admits a Stein structure filling its boundary spinal open book $\mathcal{B}$. q.e.d.

Combining the above theorem with the stronger result of Loi and Piergallini (see Theorem 2.2), we get:

Corollary 3.9. *An oriented compact 4-manifold with boundary is a Stein surface, up to orientation-preserving diffeomorphisms, if and only if it admits an allowable Lefschetz fibration over a compact surface with non-empty boundary. Moreover, any two allowable Lefschetz fibrations filling the same spinal open book carry Stein structures which fill the same contact structure induced by the spinal open book.*

In our constructions to follow we make repeated use of the following corollary, which generalizes a result of Akbulut and Ozbagci [**2**]:

Corollary 3.10. *Let X be a 4-manifold, closed or with boundary, and $f\colon X \to \Sigma$ be an allowable Lefschetz fibration over any compact surface Σ, closed or bounded, F a regular fiber and $S_1, \dots, S_m \subset Int(X) \setminus Crit(f)$ a non-empty collection of* disjoint *sections of this fibration. Let X_0 be the 4-manifold we obtain from X by excising fibered tubular neighborhoods of $F, S_1, \dots, S_m$. Then X_0 admits a Stein structure. In particular, this holds when $f\colon X \to \Sigma$ is a Lefschetz fibration on a closed 4-manifold X and none of the Lefschetz vanishing cycles are separating. Moreover, if $f\colon X \to \Sigma$ and $f'\colon X' \to \Sigma$ are any two allowable Lefschetz fibrations over a closed surface Σ with regular fibers $F \cong F'$, and with disjoint sections $S_1, \dots, S_m$ and $S'_1, \dots, S'_m$ of matching self-intersection numbers, then $X_0 = X \setminus (F \cup S_1 \cup \dots S_n)$ and $X'_0 = X' \setminus (F \cup S'_1 \cup \dots S'_n)$ admit Stein structures inducing the same contact structure on their identified boundaries.*

Proof. When we remove a fiber and a collection of disjoint sections from an allowable Lefschetz fibration (and in particular from a Lefschetz fibration with no separating vanishing cycles), we are left with another allowable Lefschetz fibration. The boundaries of the induced Lefschetz fibrations on X_0 and X_0' are isomorphic spinal open books with spine $\widehat{\Sigma} = (\Sigma - D^2)_1 \cup \cdots \cup (\Sigma - D^2)_m$ and page $F - (D_1^2 \cup \cdots \cup D_m^2)$. So the statements follow from the previous theorems. q.e.d.

4. Contact 3-manifolds admitting arbitrarily large Stein fillings

4.1. Main construction. We are going to produce the families of Stein fillings promised in Theorem 1.1, by first engineering certain families of Lefschetz fibrations with distinguished sections.

Theorem 4.1. *Let $g \geq 2$, $h \geq 1$, and $n \leq 2h-2$ be fixed integers. For any positive m, there is a genus g Lefschetz fibration $(X(m), f(m)) = (X_{g,h,n}(m), f_{g,h,n}(m))$ over a genus h surface, such that*

1) *$(X(m), f(m))$ has only non-separating Lefschetz vanishing cycles, and the number of critical points of $\kappa(m) = \kappa_{g,h,n}(m)$ is strictly increasing in m.*
2) *$(X(m), f(m))$ admits a section $S_n = S_{g,h,n}(m)$ of self-intersection n.*

Moreover, when $g = 2$, for any fixed $h \geq 1$, the signature of $X(m) = X_{2,h,n}(m)$ is strictly decreasing in m.

For any section of a genus g Lefschetz fibration over a genus $h \geq 1$ surface, its self-intersection number is determined by the number of critical points when $g = 1$, and is bounded above by $2h-2$, when $g \geq 2$ and $h \geq 1$, as shown in [**6**]. So the triples (g, h, n) realized in the theorem above are all one can possibly get.

Proof. We will construct the families of Lefschetz fibrations and sections prescribed in the statement using factorizations in the mapping class groups of surfaces. As outlined in Section 2, we need to obtain relations in Γ_g^1 of the form

t_δ^{-n} = Product of $\kappa(m)$ positive Dehn twists along non-separating curves and of h commutators

where n is the self-intersection of a section S_n and $\kappa(m)$ is a multivariable function depending on g, h, n, m, which is strictly increasing in $m > 0$.

Let $g \geq 2$ and $h \geq 1$ be fixed integers. All the relations below should be understood to take place in Γ_g^1. Our key input is the following family of relations obtained in [**6**] (see proof of Theorem 21; relations 12–20). See Figure 7 for the curves that appear below.

When $h = 1$, the following relation holds for any positive integer m:

$$t_\delta^0 = 1 = C(m)\, T^m \,, \tag{2}$$

where $C(m)$ is a single commutator that depends on m, namely

$$C(m) = t_{c_1}^m t_{d_1}^{-m} t_{c_3}^m t_{d_2}^{-m} = [\psi^{-1}, t_{c_1}^{-m} t_{d_1}^m]\,,$$

with d_1, d_2 the bounding curves for the chain c_1, c_2, c_3 in the figure, and the right hand side a commutator since there is a self-diffeomorphism ψ of the surface mapping (c_1, d_1) to (d_2, c_2), and

$$T = t_{c_2} t_{c_1} (t_{c_1} t_{c_2} t_{c_3})^2 t_{c_1} t_{c_2}$$

is a product of positive Dehn twists. (See relation 20 of [**6**]; here we chose $l = 2$.) Note that $c_2 = b$ for $g = 2$.

In contrast, for $h > 1$, for any positive integer m we have

$$t_\delta^{2-2h} \quad = \quad C_{h-1} \cdots C_1\, C(m) T_1^m\, T_2^m\,, \tag{3}$$

where $C_1, \ldots, C_{h-1}$ are fixed commutators, and $C(m)$ is a single commutator that depends on m, defined as follows:

$$C_{h-1} = t_{a_3}^{h-1} t_{x_4}^{1-h} t_{a_4}^{h-1} t_{x_3}^{1-h} \text{ and } C_i = t_{x_3}^i C t_{x_3}^{-i}, 1 \leq i \leq h-2 \text{ for}$$
$$C = t_{a_1} t_{x_2}^{-1} t_{a_2} t_{x_1}^{-1}.$$

Here T_1 and T_2 are products of positive Dehn twists, where

$$T_1 = t_r t_{a_1} t_b t_r (t_{a_1} t_r t_b)^2\,,$$

and

$$T_2 = (t_{c_1} t_{c_2} \cdots t_{c_{2g-3}} t_{c_{2g-2}} t_b)^{2g} (t_{c_1} t_{c_2} \cdots t_{c_{2g-3}})^{-2g-2}\,,$$

and thus

$$T_2 = (t_{c_1} t_{c_2} \cdots t_{c_{2g-3}})^{2g-2} P (t_{c_1} t_{c_2} \cdots t_{c_{2g-3}})^{-2g-2}\,,$$

where P is a product of $8g - 6$ positive Dehn twists along nonseparating curves one obtains from the previous product after applying the braid relation repeatedly. (These relations are found in the paragraph following Equation 17 in [**6**].)

On the other hand, we have the (one boundary) chain relation

$$t_\delta \quad = \quad (t_{c_1} t_{c_2} \cdots t_{c_{2g-3}} t_{c_{2g-2}} t_b t_r)^{4g+2}\,.$$

Let R denote the product of positive Dehn twists appearing on the right hand side of this relation, so it contains $8g^2 + 4g$ Dehn twists. We can multiply the two sides of equations (2) and (3) by t_δ^k and R^k to get

$$t_\delta^k = C(m)\, T^m\, R^k\,, \text{when } h = 1, \text{ and} \tag{4}$$

$$t_\delta^{2-2h+k} \quad = \quad C_1 \cdots C_{h-1}\, C(m)\, T_1^m\, T_2^m\, R^k\,, \text{when } h > 1. \tag{5}$$

So both relations prescribe genus g Lefschetz fibrations over genus h surfaces with sections of self-intersection $n = 2h - 2 - k$ and with only non-separating vanishing cycles.

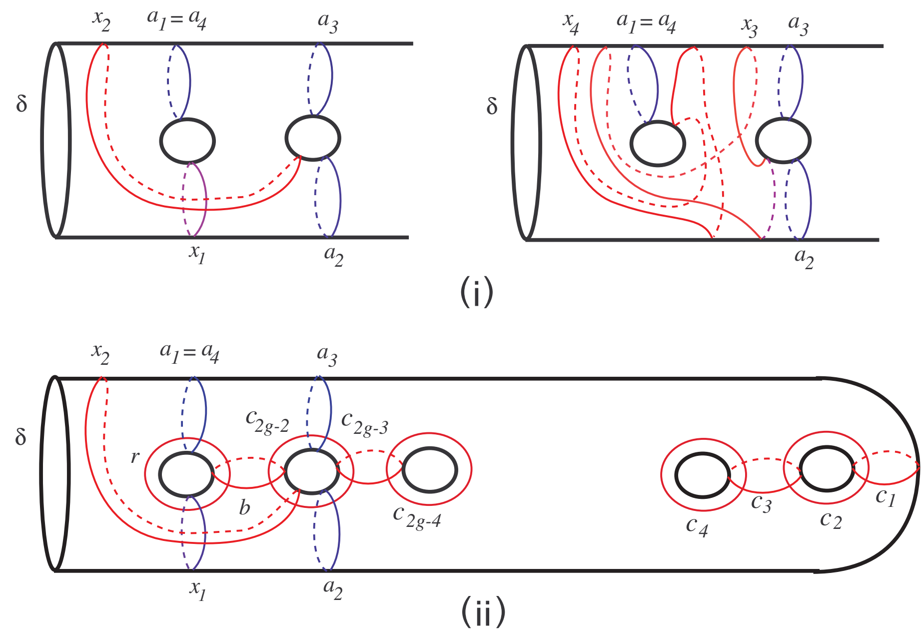

Figure 7. The curves of the mapping class group relations. When $g = 2$, we have $b = c_{2g-1}$ and $r = c_{2g}$.

The number of Lefschetz critical points $\kappa(m) = \kappa_{g,h,n}(m)$ can be calculated as

$$\kappa(m) = \begin{cases} 10m + (8g^2+4g)k = 10m - (8g^2+4g)n & \text{for } h = 1 \\ (8g+4)m + (8g^2+4g)k = (8g+4)m + (8g^2+4g)(2h-2-n) & \text{for } h > 1 \end{cases}$$

which is strictly increasing in m for any g, h, n. Thus, the same holds for

$$\mathrm{e}(X(m)) = 4(g-1)(h-1) + \kappa(m)\,.$$

The signatures of the 4-manifolds $X(m) = X_{g,h,n}(m)$ can be calculated from the explicit monodromy factorizations (4) and (5) above, by looking at their images under the boundary capping homomorphism $\Gamma_g^1 \to \Gamma_g$. We will carry out this calculation in a simpler case, when the fibration is hyperelliptic, in which case the following signature formula of Endo's [**11**] comes in handy:

$$\sigma(X) = -\frac{g+1}{2g+1}N + \sum_{j=1}^{[\frac{g}{2}]}\Big(\frac{4j(g-j)}{2g+1} - 1\Big)s_j.$$

Here X is the total space of the hyperelliptic fibration, and N and $s = \sum_{j=1}^{[\frac{g}{2}]} s_j$ are the numbers of nonseparating and separating vanishing cycles, respectively, whereas s_j denotes the number of separating vanishing cycles which separate the surface into two subsurfaces of genera j and $g - j$.

When $g = 2$, the mapping class group Γ_2 is hyperelliptic, and thus the genus two fibrations are guaranteed to be hyperelliptic. (Indeed, the reader can check that, in this case, all the curves on the closed surface isotope to curves which are symmetric under the hyperelliptic involution, whereas for $g > 2$ they do not.) So we calculate the signature of $X(m) = X_{2,h,n}(m)$ as

$$\begin{cases} -\frac{3}{5}(10m - 40n) + 0 = -6m + 24n & \text{for } h = 1 \\ -\frac{3}{5}(20m + 40(2h - 2 - n)) + 0 = -12m - 24(2h - 2 - n) & \text{for } h > 1 \end{cases}$$

which for any $h \geq 1$ is seen to be strictly decreasing in m. q.e.d.

Now let $Y_{g,h,n}$ be the graph 3-manifold described in Figure 8. The next theorem provides the promised families of contact 3-manifolds and their Stein fillings.

Theorem 4.2. *Let $g \geq 2$, $h \geq 1$, and $n \leq 2h - 2$ be fixed integers. Then $Y_{g,h,n}$ admits a contact structure $\xi_{g,h,n}$, which admits an infinite sequence of Stein fillings $(X(m), J(m)) = (X_{g,h,n}(m), J_{g,h,n}(m))$, for $m = 0, 1, \ldots$, such that the Euler characteristic of $X(m)$ is increasing in m. Moreover, when $g = 2$, for any fixed $h \geq 1$ and n, the signature of $X(m)$ is decreasing in m.*

Proof. From the above theorem, we have a family of Lefschetz fibrations

$$(X(m), f(m)) = (X_{g,h,n}(m), f_{g,h,n}(m))$$

with distinguished sections $S = S_{g,h,n}(m)$ of self-intersection n. Removing fibered neighborhoods of a regular fiber and the section S of $(X(m), f(m))$ hands us an allowable Lefschetz fibration $(\check{X}(m), \check{f}(m))$ which induces a framed spinal book $\mathcal{B}_{g,h,n}$ on its boundary $Y_{g,h,n}$, which is fixed for any $m = 0, 1, \ldots$. By Proposition 3.5, $Y_{g,h,n}$ admits a unique contact structure $\xi_{g,h,n}$ compatible with the spinal open book $\mathcal{B}_{g,h,n}$. On the other hand, by Corollary 3.10, $\check{X}(m)$ admits a Stein structure $J(m)$ filling the contact structure $\xi_{g,h,n}$ on $Y = Y_{g,h,n}$.

The Euler characteristics and signatures of $X(m)$ and $\check{X}(m)$ are related by the formulae

$$\mathrm{e}(X(m)) = \mathrm{e}(\check{X}(m)) + 3 - 2(g + h), \text{and}$$
$$\sigma(X(m)) = \sigma(\check{X}(m)) + 0\,,$$

where the latter follows from Novikov additivity. Therefore we see that $\mathrm{e}(\check{X}(m))$ is strictly increasing in m, and for $g = 2$, the $\sigma(\check{X}(m))$ is strictly decreasing. This completes the proof. q.e.d.

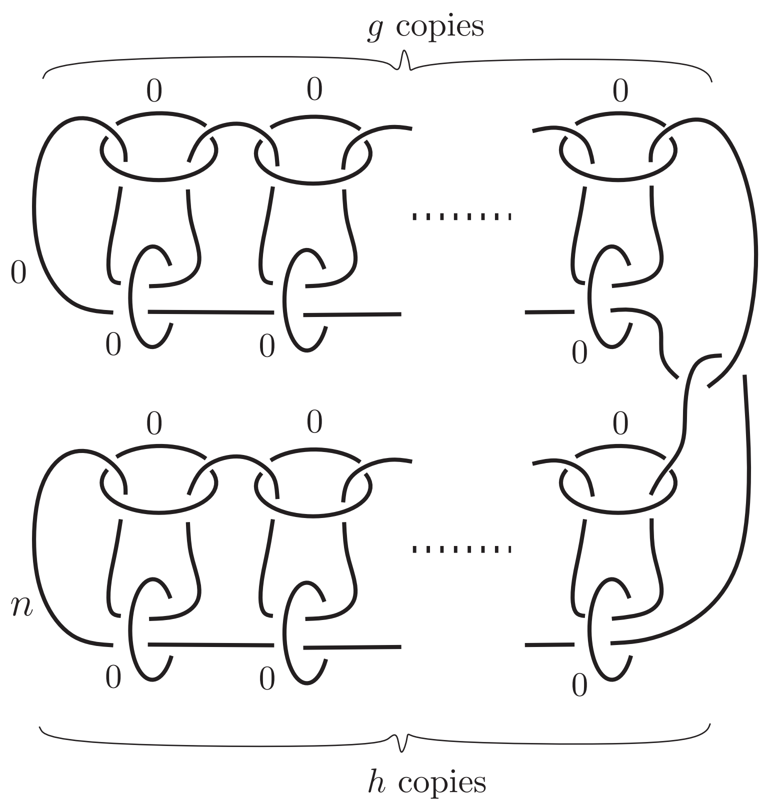

Figure 8. A surgery description of the graph manifold $Y_{g,h,n}$ as a plumbing of a circle bundle over Σ_g with Euler number 0 and a circle bundle over Σ_h with Euler number n. The linking patterns are repeated g times on the top and h times on the bottom.

The proof of Theorem 1.1 immediately follows:

Proof of Theorem 1.1. For $g = 2$, $h \geq 1$, and $n \leq 2h - 2$, $(Y_{2,h,n}, \xi_{2,h,n})$ admits Stein fillings $(\check{X}(m), J(m)) = (\check{X}_{2,h,n}(m), J_{2,h,n}(m))$ such that $\{\mathrm{e}(\check{X}(m))\}$ is a strictly increasing sequence, and $\{\sigma(\check{X}(m))\}$ is a strictly decreasing sequence, for $m = 0, 1, \ldots$. So for any given pair of integers E, S, there exists a positive integer P such that the infinite subsequence $\{(\check{X}(m), J(m))\}_{m \geq P}$ consists of members whose Euler characteristics are greater than E and signatures are smaller than S. q.e.d.

Remark 4.3. We shall note that when discussing the signatures, we restricted ourselves to families with $g = 2$ above for brevity. Otherwise, it is possible to see that the signature of $X_{g,h,n}(m)$ is decreasing in m for any fixed $g > 2$, $h \geq 1$, and $n \leq 2h - 2$ as well—which, however, requires a significantly more tedious calculation, since the fibrations we obtain in this case are not hyperelliptic.

We also note that

Corollary 4.4. *There are infinite families of contact 3-manifolds, where each contact 3-manifold admits Stein filling with infinitely many different chern numbers c_1^2 and c_2.*

Proof. We calculate $c_1^2(X) = 2\mathrm{e}(X) + 3\sigma(X)$ of the Stein fillings of $Y_{2,h,n}$ given in the proof of Theorem 4.2 as $2m - 8n$ for $h = 1$ and $4m + 8(2h - 2 - n)$ for $h \geq 2$, which constitute an infinite family for varying $m \geq 0$. Since $c_2(X) = \mathrm{e}(X)$, the latter claim is already proved above. q.e.d.

Remark 4.5. Filling the fiber component in our allowable Lefschetz fibrations above with $n < 2-2h$, we get new 4-manifolds whose boundaries are non-flat circle bundles over a closed surface Σ_h of genus $h \geq 1$. However, in [**30**], Stipsicz showed that *any* contact structure on a non-flat circle bundle over a surface Σ_h admits at most finitely many Stein fillings, which implies that the cobordisms we get this way can never be Stein.

4.2. Further constructions. We will now outline how to obtain similar families of contact structures on more general 3-manifolds, admitting Stein fillings which have arbitrarily large Euler characteristics and arbitrarily small signatures,

More general graph manifolds. We can generalize the above construction to many more graph manifolds, by removing more than one fiber and/or using Lefschetz fibrations with many disjoint sections, and following the same steps as above. The former is straightforward: We can simply remove fibered tubular neighborhoods of l disjoint fibers for $l \geq 2$ to obtain more general graph manifolds, which can be described by a surgery diagram similar to the one given in Figure 8, where we will instead have l copies of the top part of the diagram, each one of which links the bottom part once. Therefore, the same families of Lefschetz fibrations $X_{g,h,n}(m)$ in Theorem 4.1 can be employed to obtain arbitrarily large Stein fillings of the contact structures given by the spinal open books on these more general graph manifolds.

We can also consider graph manifolds which can be described by a surgery diagram similar to the one given in Figure 8, where this time we would have l copies of the *bottom* part of the diagram, each one of which links the top part once. However, we are now in need a sequence of Lefschetz fibrations with increasing Euler characteristics (and decreasing signatures) *which have l disjoint sections*. Such families can be deduced from the ones we presented in Theorem 4.1 as follows: Consider the family $(X_{g,1,0}(m), f_{g,1,0}(m))$, for any fixed $g \geq 2$. Each one of these Lefschetz fibrations has a section S of self-intersection 0. By taking l disjoint push-offs of S, we get l disjoint sections of this Lefschetz fibration. We can then take the fiber sum of $(X(m), f(m)) = (X_{g,1,0}(m), f_{g,1,0}(m))$ with any genus g Lefschetz fibration over the 2-sphere with l disjoint sections $S_1, \dots, S_l$ of self-intersections $r_1, \dots, r_l$, with only non-separating vanishing cycles. Possibly after an isotopy, we can patch the disjoint sections coming from both summands so as to get a new family of Lefschetz fibrations $(X'(m), f'(m))$ with l disjoint sections $S'_1, \dots, S'_l$ of self-intersections $r_1, \dots, r_l$. As before, we see that the Euler characteristic of $X'(m)$ is strictly increasing in m (and its signature for $g = 2$ is strictly decreasing). Hence, excising fibered neighborhoods of a regular fiber and these l sections, we obtain the desired Stein fillings of the 3-manifold on the boundary, equipped with the natural contact structure induced by the spinal book. It is worth noting that there are many examples of Lefschetz fibrations with disjoint sections of *different* self-intersections. Thus we can obtain graph manifolds where the framings $r_1, \dots, r_l$ on the l copies mentioned above are not necessarily the same. Lastly, we can push for even more general families of graph manifolds by taking out more than one fiber in these Lefschetz fibrations as before.

Non-graph manifolds. It is also possible to generalize our constructions to the case of Stein fillable contact 3-manifolds supported by spinal open books whose page monodromies are non-trivial—which typically will hand us non-graph manifolds.

Let $f : X \to \Sigma$ be a (not necessarily allowable) Lefschetz fibration with regular fiber F and base Σ compact surfaces with non-empty boundary and $Y = \partial X$. Similar to the description of a standard open book, $f|_Y$ gives a spinal open book $\mathcal{B}$. The paper of this spinal open book is the vertical boundary of X, $Y_P = f^{-1}(\partial D)$. The spine is the complementary region $Y \setminus Y_P$, and is the horizontal boundary of X. The Lefschetz fibration then equips Y_Σ with the structure of a circle bundle with fibers consisting of the boundaries of all fibers of f, ∂F, as a bundle over the base Σ. If ∂F is disconnected, then the vertebra $\widehat{\Sigma}$ consists of $\#|\partial F|$ copies of Σ. As the boundary of a Lefschetz fibration, $\mathcal{B}$ is a symmetric, uniform, simple spinal open book: every fiber is isotopic within X and so every component of the fiber $\widehat{F}$ of $\mathcal{B}$ is isomorphic. The spine consists of the bundle of the disconnected union of circle boundaries of the fibers, and as such circles are isotopic within each horizontal boundary to the boundary of a single fiber, there is a single boundary component of each component of the total fiber $\widehat{F}$ which gets glued to a component of $\widehat{\Sigma}$.

For a standard open book $\mathcal{B}$, there exists a Lefschetz fibration with boundary $\mathcal{B}$ if we can find a factorization of the monodromy of $\mathcal{B}$ into positive Dehn twists. Any such factorization gives a Lefschetz fibration filling of $\mathcal{B}$. For spinal open books, the picture is slightly more complicated:

Given a spinal open book $\mathcal{B} = (\widehat{F}, \widehat{\phi}, \widehat{\Sigma}, G)$, there exists a Lefschetz fibration with boundary $\mathcal{B}$ if and only if we can find identifications $\hat{i} : \widehat{F} \to F$ and a factorization of the *total monodromy* $\Phi_{\hat{i}} = \phi_1^{i_1} \circ \cdots \circ \phi_n^{i_n}$ (where $\phi^i = i \circ \phi \circ i^{-1}$) as

$$\Phi_{\hat{i}} = \prod_{i=1}^{m} t_i \prod_{j=1}^{h} [\alpha_j, \beta_j]$$

in the mapping class group of F, where h is the genus of Σ, α_j, β_j are isotopy classes of diffeomorphisms of F for $j = 1, \ldots, h$, and t_i, $i = 1, \ldots, m$ are Dehn twists on F. In particular, such a factorization corresponds to the monodromy presentation of the bundle of non-singular fibers in a Lefschetz fibration with this boundary. The commutators correspond to the genus of Σ and the identification maps i used to conjugate ϕ to ϕ^i correspond to 1-handles producing the additional boundary components of Σ.

Given any mapping class element Φ, we define the *positive coset commutator length* of Φ to be the smallest h so that we can write Φ as the product of a length h commutator and positive Dehn twists as above.

Theorem 4.6. *Let $\mathcal{B}$ be a symmetric, uniform, simple spinal open book with page F of genus greater than two, spine Σ of genus h. If there is a set of identifications $\hat{i}$ so that the positive coset commutator length of the total monodromy $\Phi_{\hat{i}}$ is strictly less than h, then $\xi_{\mathcal{B}}$ admits Stein fillings of arbitrarily large Euler characteristic.*

Proof. If there is such a total monodromy $\Phi_{\hat{i}}$ with commutator length strictly less than h, then in the monodromy presentation of the associated Lefschetz fibration, we can choose a single commutator to be that of the identity maps. We can extend this factorization to new Lefschetz fibrations by making a monodromy substitution using the relations given in (2) above so as to produce arbitrarily large allowable Lefschetz fibrations filling $\mathcal{B}$ as before. Thus, any contact 3-manifold satisfying these properties will admit arbitrarily large Stein fillings. q.e.d.

REMARK 4.7. Note that we can produce further families of contact 3-manifolds with arbitrarily large Stein fillings simply by building Stein cobordisms from our examples above. In this way, one can get contact 3-manifolds with various other topological properties, including hyperbolic manifolds. (For example, every closed 3-manifold contains a hyperbolic knot ([**27**]) and this plus Thurston's hyperbolic Dehn surgery theorem implies that we can always find a Legendrian surgery from our examples to some (and indeed many) hyperbolic $3-$manifold(s).)

Remark 4.8. A curious point that arises in our work is as follows: There are Stein fillable contact 3-manifolds which admit (1) a unique Stein filling, (2) more than one but finitely many Stein fillings, and (3) infinitely many Stein fillings, up to diffeomorphisms. That is, there is an intrinsic property one can associate to Stein fillable contact 3-manifolds in terms of the number of Stein fillings they admit. There are examples of contact 3-manifolds which carry only one of these properties. For example, Gromov [**20**] proved that there is a unique minimal symplectic filling of S^3, whereas McDuff [**26**] proved that there are exactly two minimal fillings of the standard contact structure on $L(4,1)$. Ozbagci and Stipsicz in [**28**] showed that the Seifert manifolds with a single Seifert fiber of order 2 and with base a surface of genus $g \geq 2$ admit a contact structure with infinitely many Stein fillings; also see [**3**]. In this article, we have shown that there is a fourth class of Stein fillable contact 3-manifolds, namely those which admit (4) infinitely many Stein fillings with arbitrarily large Euler characteristics. It is therefore worth asking whether or not there are Stein fillable contact 3-manifolds which belong to the class (3) but not (4).

Appendix: Stein structures on Lefschetz fibrations and their contact boundaries

(by Samuel Lisi and Chris Wendl)

In this appendix we explain a special case of a theorem from [**24**] which implies that an allowable Lefschetz fibration over an arbitrary oriented surface with boundary can always be viewed in a canonical way as a Stein filling of a contact structure determined by the spinal open book at the boundary (see Theorem 1.2). Our proof is a variation on the technique of Thurston [**32**] and Gompf [**18**] for constructing symplectic structures on Lefschetz fibrations.

Guide to notation. Throughout the appendix, we will generally use the following notation:

- E will be a 4-manifold with boundary and the total space of a Lefschetz fibration
- M will be a contact 3-manifold
- g, f, F will be functions to $\mathbb{R}$
- Σ will be a surface with boundary which will be either connected or disconnected, depending on the setting
- ϕ, θ will be the coordinate on S^1

Let E be a smooth, compact, oriented and connected 4-manifold with boundary and corners such that ∂E is the union of two smooth faces

$$\partial E = \partial_h E \cup \partial_v E$$

intersecting at a 2-dimensional corner. Let Σ denote a compact, oriented and connected surface with nonempty boundary. We consider a Lefschetz fibration $\Pi \colon E \to \Sigma$ with the following properties:

1) The sets of critical points E^{crit} and critical values Σ^{crit} lie in the interiors of E and Σ respectively.
2) $\Pi^{-1}(\partial\Sigma) = \partial_v E$ and $\Pi|_{\partial_v E} \colon \partial_v E \to \partial\Sigma$ is a smooth fiber bundle.
3) $\Pi|_{\partial_h E} \colon \partial_h E \to \Sigma$ is also a smooth fiber bundle.
4) All fibers $E_z := \Pi^{-1}(z)$ for $z \in \Sigma$ are connected and have nonempty boundary in $\partial_h E$.

As we will review in §A.2 below, any Lefschetz fibration of this type induces a spinal open book at its boundary. We say that Π is **allowable** if all the irreducible components of its fibers have nonempty boundary, i.e., none of its vanishing cycles are homologically trivial.

Theorem A.1. *If the Lefschetz fibration $\Pi \colon E \to \Sigma$ is allowable, then after smoothing corners on ∂E, E admits (canonically up to Stein homotopy) the structure of a Stein domain, and the filled contact structure at the boundary is uniquely determined up to isotopy by the induced spinal open book.*

In the background of this theorem is the corresponding existence and uniqueness result (also a special case of a theorem in [**24**]) for contact structures supported by spinal open books. We shall state and prove this in §A.1, and then prove Theorem A.1 in §A.2.

Remark A.2. A version of Theorem A.1 also holds without the allowability assumption, but in that case E generally becomes a strong symplectic filling instead of a Stein filling. See [**24**] for details.

Acknowledgments. We thank Kai Cieliebak for helpful conversations. SL was partially supported by the ERC Starting Grant of Frédéric Bourgeois StG-239781-ContactMath. CW was partially supported by a Royal Society University Research Fellowship.

A.1. Spinal open books and contact structures. To establish notation, we begin by reviewing some essential definitions (see Section 3).

Definition A.3. A **spinal open book decomposition** on a closed oriented 3-manifold M is a decomposition $M = M_\Sigma \cup M_P$, where the pieces M_Σ and M_P (called the **spine** and **paper** respectively) are smooth compact 3-dimensional submanifolds with disjoint interiors such that $\partial M_\Sigma = \partial M_P$, carrying the following additional structure:

1) A smooth fiber bundle $\pi_\Sigma \colon M_\Sigma \to \Sigma$ with fiber S^1, such that each fiber is either disjoint from ∂M_Σ or contained in it. Here, Σ is a compact oriented surface whose connected components (called **vertebrae** (singular, **vertebra**) all have nonempty boundary.

2) A smooth fiber bundle $\pi_P\colon M_P \to S^1$ such that the connected components (called **pages**) of fibers are all compact surfaces with nonempty boundary, where they meet ∂M_P transversely. Moreover, the boundary components of each page are fibers of π_Σ.

We shall denote by

$$\boldsymbol{\pi} := \left(\pi_\Sigma\colon M_\Sigma \to \Sigma, \pi_P\colon M_P \to S^1\right)$$

the collection of information encoded in a spinal open book. We will say additionally that $\boldsymbol{\pi}$ admits a **smooth overlap** if the fibration $\pi_P\colon M_P \to S^1$ can be extended over an open neighborhood $M_P' \subset M$ containing M_P such that all fibers of π_Σ intersecting M_P' are contained in fibers of the extended π_P. Note that while an arbitrary spinal open book does not always admit a smooth overlap, it can always be deformed continuously to one that does, and the result is unique up to smooth isotopy.

Definition A.4. Given a spinal open book $\boldsymbol{\pi}$ on M, a positive contact form α on M will be called a **Giroux form** for $\boldsymbol{\pi}$ if the following conditions hold:

1) The 2-form $d\alpha$ is positive on the interior of every page.
2) The Reeb vector field R_α determined by α is positively tangent to every oriented fiber of $\pi_\Sigma\colon M_\Sigma \to \Sigma$.

A contact structure ξ on M is **supported** by $\boldsymbol{\pi}$ whenever it admits a contact form which is a Giroux form.

Theorem A.5. *If $\boldsymbol{\pi}$ is a spinal open book on M which admits a smooth overlap, then the space of Giroux forms for $\boldsymbol{\pi}$ is nonempty and contractible. In particular, any isotopy class of spinal open books gives rise to a canonical isotopy class of supported contact structures.*

Remark A.6. One can also formulate the above definitions and prove a generalization of Theorem A.5 for compact manifolds with boundary, which allows for a useful alternative characterization of certain “local” filling obstructions such as Giroux torsion and planar torsion; see [**24**] for details.

The proof of Theorem A.5 will occupy the remainder of this subsection. As a first step, we define a **fiberwise Giroux form** for $\boldsymbol{\pi}$ to be any smooth 1-form α on M for which the following conditions hold:

- $d\alpha$ is positive on the interior of every page,
- α is positive on the fibers of $\pi_\Sigma\colon M_\Sigma \to \Sigma$, and the tangent spaces to these fibers are contained in $\ker d\alpha$.

A fiberwise Giroux form is a Giroux form if and only if it is contact, but since we have not required the latter in the above definition, the space of fiberwise Giroux forms is convex.

Choose for each connected component of $\partial\Sigma$ a collar neighborhood $(-1,0]\times S^1$ with coordinates (s,ϕ), and enlarge Σ by attaching $[0,1)\times S^1$ in the obvious way to each of these collars, denoting the resulting surface by $\overrightarrow{\Sigma}$. If $\boldsymbol{\pi}$ admits a smooth overlap, then this can be done so that there is also an open neighborhood $\mathcal{U}_\Sigma$ of ∂M_Σ in M which we can identify with $(-1,1)\times\partial M_\Sigma$ such that the fibration

$$\mathcal{U}_\Sigma = (-1,1)\times\partial M_\Sigma \to (-1,1)\times S^1 \colon (s,x)\mapsto (s,\pi_P(x))$$

matches π_Σ on $\mathcal{U}_\Sigma \cap M_\Sigma$, which is the region $\{s\le 0\}$. In fact, this defines an extended fibration

$$\overline{\pi}_\Sigma \colon \overline{M}_\Sigma \to \overline{\Sigma},$$

where $\overline{M}_\Sigma := M_\Sigma \cup \mathcal{U}_\Sigma$. We shall continue to denote the coordinates on the collars $(-1,1)\times S^1 \subset \overline{\Sigma}$ by (s,ϕ) and, in light of the compatibility of the two fibrations, also use $\phi\in S^1$ to denote the coordinate on the base of $\pi_P \colon M_P \to S^1$.

Choose a Liouville form σ on $\overline{\Sigma}$ that matches $e^s\,d\phi$ on the collars $(-1,1)\times S^1$. For convenience, we can also fix an identification of the (necessarily trivial) bundle $\overline{M}_\Sigma \to \overline{\Sigma}$ with $\overline{\Sigma}\times S^1$ such that $\vec{\pi}_\Sigma(z,\theta)=z$. This identifies each connected component of $\mathcal{U}_\Sigma$ with $(-1,1)\times S^1\times S^1$, carrying coordinates (s,ϕ,θ). In these coordinates on the collar $M_P \cap \mathcal{U}_\Sigma \cong [0,1)\times S^1\times S^1$ we have $\pi_P(s,\phi,\theta)=\phi$.

To keep orientations straight, it will also be convenient to define an alternative coordinate system on $M_P\cap\mathcal{U}_\Sigma$ by

$$(t,\phi,\theta) := (-s,\phi,\theta)\in(-1,0]\times S^1\times S^1 \subset M_P\cap\mathcal{U}_\Sigma.$$

This has the advantage that $(t,\theta)\in(-1,0]\times S^1$ now defines a set of positively oriented collar neighborhoods of the boundary of each page. Note that the monodromy of the bundle $\pi_P\colon M_P\to S^1$ cannot be assumed trivial near the boundary, but up to isotopy we can still assume that it takes the form $(t,\theta)\mapsto(t,\theta)$ in the above collars while also permuting boundary components. With this understood, the following lemma is proved by a standard argument (see, for example, [**13**]).

Lemma A.7. *On M_P there exists a 1-form η such that $d\eta$ is positive on each fiber of $\pi_P\colon M_P\to S^1$ and, in the collar neighborhoods of ∂M_P with coordinates (t,ϕ,θ) as defined above, $\eta = e^t\,d\theta$.* *q.e.d.*

We can now construct a fiberwise Giroux form. Let $F\colon M_P\to(0,1]$ denote a smooth function which is identically 1 outside of $\mathcal{U}_\Sigma$ and takes the form $e^s f(s)$ in the collar coordinates $(s,\phi,\theta)\in\mathcal{U}_\Sigma$, where $f\colon(-1,1)\to(0,1]$ is a smooth function satisfying the conditions

- $f(s)=1$ for $s\le 0$,
- $f'(s)<0$ for $s>0$,
- $f(s)=e^{-s}$ for s near 1.

Now if η is given by Lemma A.7, the expression

$$\alpha = \begin{cases} d\theta & \text{on } M_\Sigma, \\ F\eta & \text{on } M_P \end{cases}$$

defines a fiberwise Giroux form on M.

We will use a version of the Thurston trick to turn fiberwise Giroux forms into Giroux forms. Given a constant $\delta \in (0, 1]$, choose a smooth function $g_\delta \colon [0, \infty) \to [0, 2]$ with

- $g_\delta(s) = e^s$ for s near 0,
- $g'_\delta(s) \geq 0$ for all s,
- $g_\delta(s) = 2$ for all $s \geq \delta$,

and define from this a smooth function $G_\delta \colon M_P \to [0, 2]$ by

$$G_\delta = \begin{cases} 2 & \text{on } M_P \setminus \mathcal{U}_\Sigma, \\ g_\delta(s) & \text{for } (s, \phi, \theta) \in \mathcal{U}_\Sigma. \end{cases}$$

Then identifying the Liouville form σ on Σ with its pullback $\pi_\Sigma^*\sigma$ on M_Σ, we define for any $\delta \in (0, 1]$ another smooth 1-form on M by

$$\beta_\delta = \begin{cases} \sigma & \text{on } M_\Sigma, \\ G_\delta \, d\phi & \text{on } M_P. \end{cases}$$

Lemma A.8. *For any fiberwise Giroux form α, there exist constants $\delta_0 \in (0, 1]$ and $K_0 \geq 0$ such that for all constants $\delta \in (0, \delta_0]$ and $K \geq K_0$,*

$$\alpha_{K,\delta} := \alpha + K\beta_\delta$$

is a Giroux form. Whenever α itself is a Giroux form, one can take $K_0 = 0$.

Proof. Observe that $\alpha_{K,\delta}$ is automatically a *fiberwise* Giroux form for all $K \geq 0$, $\delta \in (0, 1]$, so we only need to show that $\alpha_{K,\delta}$ is contact for the right choices of these constants. Since $\beta_\delta \wedge d\beta_\delta \equiv 0$, we have

$$\alpha_{K,\delta} \wedge d\alpha_{K,\delta} = K \left(\alpha \wedge d\beta_\delta + \beta_\delta \wedge d\alpha \right) + \alpha \wedge d\alpha;$$

thus it suffices to show that whenever $\delta > 0$ is sufficiently small,

$$\alpha \wedge d\beta_\delta + \beta_\delta \wedge d\alpha > 0. \tag{6}$$

The conditions on fiberwise Giroux forms imply that $\alpha(\partial_\theta) > 0$ at ∂M_P, so this is also true on collars of the form $\{s \leq \delta_0\} \subset \mathcal{U}_\Sigma$ for sufficiently small $\delta_0 > 0$. Assuming $0 < \delta \leq \delta_0$, we shall now show that (6) holds everywhere on M.

On M_Σ, $\beta_\delta \wedge d\alpha = \sigma \wedge d\alpha = 0$ since $\sigma(\partial_\theta) = d\alpha(\partial_\theta, \cdot) = 0$, but $\alpha \wedge d\beta_\delta > 0$ since $\alpha(\partial_\theta) > 0$ and $d\beta_\delta = d\sigma$ is positive on Σ.

On M_P outside of the collars $\{s \leq \delta\}$, we have $\beta_\delta = 2\, d\phi$ and thus $d\beta_\delta = 0$, while $\beta_\delta \wedge d\alpha = 2\, d\phi \wedge d\alpha > 0$ due to the assumption that $d\alpha$ is positive on the fibers of π_P.

On the collars $\{s \le \delta\}$, we have $\beta_\delta = G_\delta \, d\phi$, with $G_\delta > 0$ on the interior of M_P; hence $\beta_\delta \wedge d\alpha = G_\delta \, d\phi \wedge d\alpha > 0$ again except at ∂M_P. It thus remains only to show that $\alpha \wedge d\beta_\delta \ge 0$, with strict positivity at ∂M_P. This follows from the fact that $\alpha(\partial_\theta) > 0$ on this region, since $\alpha \wedge d\beta_\delta = g_\delta'(s) \, \alpha \wedge ds \wedge d\phi$, where $g_\delta'(s)$ was assumed to be nonnegative and strictly positive at $s = 0$. q.e.d.

The above implies Theorem A.5: indeed, since the space of fiberwise Giroux forms is nonempty and convex, Lemma A.8 shows that Giroux forms exist, and for any integer $n \ge 0$, a continuous S^n-parametrized family of Giroux forms can be contracted through Giroux forms. It follows by Whitehead's theorem that the space of Giroux forms is contractible.

A.2. Lefschetz fibrations and Stein structures. In this section, we take $\Pi \colon E \to \Sigma$ to be a Lefschetz fibration as described in the discussion preceding Theorem A.1. This naturally gives rise to a spinal open book on ∂E, with spine $M_\Sigma := \partial_h E$ and paper $M_P := \partial_v E$. The fibration $\pi_P \colon \partial_v E \to S^1$ is defined as the restriction $\Pi|_{\partial_v E} \colon \partial_v E \to \partial\Sigma$ after choosing an orientation-preserving identification of each connected component of $\partial\Sigma$ with S^1. Likewise, $\Pi|_{\partial_h E} \colon \partial_h E \to \Sigma$ defines a smooth fibration whose fibers are disjoint unions of finitely many circles; hence it can be factored as

$$\partial_h E \xrightarrow{\pi_\Sigma} \widetilde{\Sigma} \xrightarrow{p} \Sigma,$$

where $\pi_\Sigma \colon \partial_h E \to \widetilde{\Sigma}$ is a fiber bundle with connected fibers over another compact oriented surface $\widetilde{\Sigma}$ with boundary, and $p \colon \widetilde{\Sigma} \to \Sigma$ is a smooth finite covering map. After smoothing the corner at $\partial_h E \cup \partial_v E$, this construction gives rise to a unique isotopy class of spinal open books admitting smooth overlaps.

To construct Stein structures on E, we will consider a special class of almost complex structures that always admit plurisubharmonic functions, thus giving rise to a distinguished deformation class of Weinstein structures. This in turn yields a canonical deformation class of Stein structures due to a theorem of Eliashberg [**7**]. Recall that a function $f \colon W \to \mathbb{R}$ on an almost complex manifold (W, J) is called ***J*-convex** if the 1-form $\lambda := -df \circ J$ is the primitive of a symplectic form that tames J. We will make repeated use of the standard fact that *every* complex structure J on a compact and connected surface with nonempty boundary admits a J-convex function which has the boundary as a regular level set. Indeed, such a function can be found by starting with a Morse function that is J-convex near its critical points and postcomposing with a positive function with large second derivative (see, e.g., [**23**, Lemma 4.1]); in this way, one can also choose the function's value and normal derivative at the boundary to be arbitrarily large.

Denote by $\mathcal{J}(\Pi)$ the space of smooth almost complex structures J on E that are compatible with its orientation and satisfy the following conditions:

1) There exists a smooth complex structure j on Σ, compatible with the given orientation, such that $\Pi\colon (E, J) \to (\Sigma, j)$ is pseudoholomorphic.
2) J is integrable on some neighborhood of E^{crit}.
3) The maximal J-complex subbundle in $T(\partial_h E)$ is preserved by some smooth S^1-action on $\partial_h E$ which restricts to a free and transitive action on each boundary component of each fiber X_z.

Observe that any $J \in \mathcal{J}(\Pi)$ makes the fibers into J-complex curves, with the induced orientation matching their natural orientation. An element of $\mathcal{J}(\Pi)$ can be constructed by picking complex Morse coordinates near E^{crit}, then choosing a suitable horizontal subbundle outside this neighborhood which is S^1-invariant at $\partial_h E$, and extending the resulting complex structures on the vertical and horizontal subbundles globally. Since both are oriented bundles of real rank 2, the space $\mathcal{J}(\Pi)$ is contractible.

Given $J \in \mathcal{J}(\Pi)$, we will say that a J-convex function $f\colon E \to \mathbb{R}$ is **admissible** if the Liouville form $\lambda := -df \circ J$ restricts to a contact form on both of the smooth boundary faces $\partial_h E$ and $\partial_v E$, such that for all $z \in \Sigma$, $\partial X_z \subset \partial_h E$ is a union of closed Reeb orbits. Observe that since J is tamed by the symplectic form $d\lambda$, this construction automatically makes the fibers symplectic, including the pages in $\partial_v E$ of the induced spinal open book at the boundary, and in this sense one can reasonably say that λ *restricts to a Giroux form* on ∂E. The contact condition implies that the induced Liouville vector field at ∂E is outwardly transverse to both smooth faces; hence one can smooth the corner so that the Liouville vector field is also transverse to the smoothened boundary, and in so doing one can arrange for λ to be a Giroux form for the resulting spinal open book with smooth overlap. Moreover, the Liouville vector field is gradient-like with respect to f, and one can then homotope f near the smoothened boundary through Lyapunov functions to make the smoothened boundary a regular level set, producing a Weinstein structure uniquely up to Weinstein homotopy. In this way, any choice of admissible J-convex function f determines a homotopy class of Weinstein structures which fill the contact structure supported by the spinal open book at the boundary.

The above discussion reduces the proof of Theorem A.1 to the following:

Proposition A.9. *If $\Pi\colon E \to \Sigma$ is allowable, then for every $J \in \mathcal{J}(\Pi)$, the space of admissible J-convex functions is nonempty and contractible.*

Proof. We proceed in three steps.

Step 1: Existence of a fiberwise J-convex function. Given $J \in \mathcal{J}(\Pi)$, let us call a smooth function $f\colon E \to \mathbb{R}$ **admissibly fiberwise J-convex** if the 1-form $\lambda := -df \circ J$ has the following properties:

1) At E^{crit}, $d\lambda$ is symplectic and tames J.
2) On $E \setminus E^{\mathrm{crit}}$, $d\lambda$ is symplectic on every fiber.
3) For all $z \in \Sigma$, the tangent spaces to $\partial X_z \subset \partial_h E$ are positive for λ but in the kernel of $d\lambda|_{T(\partial_h E)}$.

The space of admissibly fiberwise J-convex functions is convex and thus contractible. Such a function is admissibly J-convex if and only if $d\lambda$ is a symplectic form taming J and λ defines contact forms on $\partial_h E$ and $\partial_v E$.

Our first task is to construct an admissibly fiberwise J-convex function $f\colon E \to \mathbb{R}$. By our assumptions on J, there is a uniquely determined complex structure j on Σ such that $\Pi\colon (E, J) \to (\Sigma, j)$ is pseudoholomorphic. There is also a vertical vector field ∂_θ on $\partial_h E$ whose flow generates an S^1-action that preserves the maximal J-complex subbundle

$$\xi_h := \{v \in T(\partial_h E) \mid Jv \in T(\partial_h E)\} \subset T(\partial_h E).$$

Note that Π is J–j holomorphic so $J|_{\xi_h} = \Pi^* j$; hence it is automatic that the flow of ∂_θ also preserves $J|_{\xi_h}$. We assume ∂_θ is positive with respect to the boundary orientation of each fiber, so $-J\partial_\theta$ points transversely outward.

To construct the desired function $f\colon E \to \mathbb{R}$, we begin by choosing for each $z \in \Sigma \setminus \Sigma^{\mathrm{crit}}$ a J-convex function $f_z\colon X_z \to \mathbb{R}$ which at ∂X_z satisfies $f_z \equiv c_z$ and $df_z(-J\partial_\theta) = \nu_z$ for some constants $c_z, \nu_z > 0$. We can then find a neighborhood $\mathcal{U}_z \subset \Sigma \setminus \Sigma^{\mathrm{crit}}$ containing z such that f_z admits an extension to a smooth function $f_z\colon E|_{\mathcal{U}_z} \to \mathbb{R}$ having these same properties on every fiber in $E|_{\mathcal{U}_z}$. Observe that the constants c_z and ν_z can always be made larger without changing the choice of neighborhood $\mathcal{U}_z$. The 1-form $\lambda_z := -df_z \circ J$ on $E|_{\mathcal{U}_z}$ now satisfies $d\lambda_z|_{TX_z} > 0$ for each $z \in \mathcal{U}_z$, and its restriction to the horizontal boundary $\alpha_z^h := \lambda_z|_{T(\partial_h E)}$ satisfies $\alpha_z^h(\partial_\theta) = \nu_z$, $\alpha_z^h|_{\xi_h} = 0$.

We next construct similar functions near the singular fibers. For $z \in \Sigma^{\mathrm{crit}}$, let E_z^{crit} denote the finite set of critical points in the fiber X_z. For each $p \in E_z^{\mathrm{crit}}$, fix a neighborhood $\mathcal{U}_p \subset E$ containing p on which J is integrable, and choose holomorphic coordinates (z_1, z_2) identifying $\mathcal{U}_p$ with a neighborhood of 0 in $\mathbb{C}^2$ such that $\Pi(z_1, z_2) = z_1^2 + z_2^2$ for a suitable choice of holomorphic coordinate near $\Pi(p) \in \Sigma$. We use these coordinates to define a function $f_z\colon \mathcal{U}_p \to \mathbb{R}$ by

$$f_z(z_1, z_2) = \frac{1}{2}\left(|z_1|^2 + |z_2|^2\right).$$

Then $-df_z \circ J$ is the primitive of a positive symplectic form in $\mathcal{U}_p$ which tames J and restricts symplectically to the vertical subspaces. Since Π

is allowable, the connected components of $X_z \setminus E_z^{\mathrm{crit}}$ are all compact oriented surfaces with nonempty boundary and finitely many punctures. It follows that f_z can be extended so that it is J-convex on X_z and satisfies $f_z \equiv c_z$, $df_z(-J\partial_\theta) \equiv \nu_z$ at ∂X_z for some large constants $c_z, \nu_z > 0$. Since the J-convexity condition is open, we can then extend f_z over $E|_{\mathcal{U}_z}$ for some neighborhood $z \in \mathcal{U}_z \subset \Sigma$ so that it has these same properties on each fiber, and the constants c_z, ν_z can be made larger if desired without changing $\mathcal{U}_z$.

Since Σ is compact, there is a finite subset $I \subset \Sigma$ such that the neighborhoods $\mathcal{U}_z$ constructed above for $z \in I$ cover Σ. By making the functions f_z more convex near $\partial_h E$, we can then increase the constants $c_z > 0$ for all $z \in I$ so that they match a single constant $c > 0$, and likewise increase ν_z for $z \in I$ to match some large number $\nu > 0$. Choose a partition of unity $\{\rho_z \colon \mathcal{U}_z \to [0,1]\}_{z\in I}$ subordinate to the covering $\{\mathcal{U}_z\}_{z\in I}$, and define $f \colon E \to \mathbb{R}$ by

$$f = \sum_{z\in I} (\rho_z \circ \Pi) f_z.$$

If $\lambda = -df \circ J$, we now have $d\lambda$ positive on all fibers, while $d\lambda$ is symplectic and tames J near E^{crit}, and the restriction $\alpha^h := \lambda|_{T(\partial_h E)}$ to the horizontal boundary satisfies

$$\alpha^h(\partial_\theta) \equiv \nu > 0, \qquad \alpha^h|_{\xi_h} \equiv 0.$$

It follows that α^h is invariant under the flow of ∂_θ; thus

$$0 \equiv \mathcal{L}_{\partial_\theta} \alpha^h \equiv d\alpha^h(\partial_\theta, \cdot).$$

Step 2: The Thurston trick. Suppose $f \colon E \to \mathbb{R}$ is any admissibly fiberwise J-convex function and denote $\lambda = -df \circ J$. Choose a j-convex function $\varphi \colon \Sigma \to \mathbb{R}$ which has $\partial\Sigma$ as a regular level set. Let $\sigma := -d\varphi \circ j$ denote the resulting Liouville form on Σ. For any constant $K \geq 0$, consider the function

$$F_K := f + K(\varphi \circ \Pi) \colon E \to \mathbb{R}.$$

We claim that this is admissibly J-convex whenever K is sufficiently large, and that this is also true for all $K \geq 0$ if f itself is admissibly J-convex. Indeed, since $\Pi \colon (E, J) \to (\Sigma, j)$ is pseudoholomorphic, we find

$$\Lambda_K := -dF_K \circ J = \lambda + K\Pi^*\sigma.$$

Choose a neighborhood $\mathcal{U}^{\mathrm{crit}} \subset E$ of E^{crit} on which J is integrable and $d\lambda$ is a symplectic form taming J. Then for any nonzero vector $X \in T\mathcal{U}^{\mathrm{crit}}$,

$$d\Lambda_K(X, JX) = d\lambda(X, JX) + K\, d\sigma(\Pi_* X, j\Pi_* X) \tag{7}$$

is positive; here we've used the fact that Π is J-j-holomorphic and $d\sigma$ tames j, implying that the second term is nonnegative.

To see that $d\Lambda_K$ also tames J on $E \setminus \mathcal{U}^{\mathrm{crit}}$, observe that the second term in (7) is always nonnegative, and is positive for $K > 0$ if and only if the vector χ is not vertical. Likewise, the first term in (7) is positive for nonzero vertical vectors V and therefore also for all nonzero vectors in some open neighborhood of the vertical subbundle. It follows that the sum can always be made positive if K is sufficiently large. Moreover, if f is J-convex then the first term is positive for any $X \neq 0$, and the sum is then positive for all $K \geq 0$.

It remains to check that the restrictions

$$\alpha_K^v := \Lambda_K|_{T(\partial_v E)}, \qquad \alpha_K^h := \Lambda_K|_{T(\partial_h E)}$$

are both contact for suitable choices of $K \geq 0$. Let $\alpha^v := \lambda|_{T(\partial_v E)}$. Then since $d\sigma$ vanishes on $T(\partial\Sigma)$, $\Pi^* d\sigma$ vanishes on $\partial_v E$, implying

$$\alpha_K^v \wedge d\alpha_K^v = (\alpha^v + K\Pi^*\sigma) \wedge (d\alpha^v + K\Pi^* d\sigma) = \alpha^v \wedge d\alpha^v + K(\Pi^*\sigma \wedge d\alpha^v).$$

Here, the second term is positive since $d\alpha^v > 0$ on fibers; thus α_K^v is contact for all K sufficiently large, and for all $K \geq 0$ if α^v is contact; the latter is the case if f is admissibly J-convex. Likewise on $\partial_h E$, we write $\alpha^h := \lambda|_{T(\partial_h E)}$ and observe that $\Pi^*\sigma \wedge \Pi^* d\sigma$ vanishes for dimensional reasons, while $\Pi^*\sigma \wedge d\alpha^h = 0$ since the vertical direction is in $\ker d\alpha^h$. Thus

$$\alpha_K^h \wedge d\alpha_K^h = (\alpha^h + K\Pi^*\sigma) \wedge (d\alpha^h + K\Pi^* d\sigma) = \alpha^h \wedge d\alpha^h + K(\alpha^h \wedge \Pi^* d\sigma).$$

Once again the second term is positive, as $\alpha^h > 0$ in the vertical direction, and the contact condition for α_K^h follows.

Step 3: Contractibility. The existence of an admissible J-convex function follows immediately by combining steps 1 and 2. Moreover, since the space of admissibly fiberwise J-convex functions is convex, step 2 implies that any continuous S^n-parametrized family of admissible J-convex functions is contractible, so the result follows via Whitehead's theorem. q.e.d.

References

[1] S. Akbulut & B. Ozbagci, *Lefschetz fibrations on compact Stein surfaces*, Geom. Topol. **5** (2001) 319–334, MR 1825664, Zbl 1002.57062; `http://dx.doi.org/10.2140/gt.2001.5.319`.

[2] S. Akbulut & B. Ozbagci, *On the topology of compact Stein surfaces*, Int. Math. Res. Not. 2002, no. 15, 769–782, MR 1891172, Zbl 1007.57023.

[3] A. Akhmedov, J.B. Etnyre, T. Mark & I. Smith, *A note on Stein fillings of contact manifolds*, Math. Res. Lett. **15** (2008), no. 6, 1127–1132, MR 2470389, Zbl 1156.57019.

[4] R. Avdek, *Liouville hypersurfaces and connect sum cobordisms*, preprint; arXiv:1204.3145.

[5] J. Baldwin, *Contact monoids and Stein cobordisms*, Math. Res. Lett. **19** (2012), no. 1, 31–40, MR 2923174, Zbl 06194633.

[6] R.I. Baykur, M. Korkmaz & N. Monden, *Sections of surface bundles and Lefschetz fibrations*, Trans. Amer. Math. Soc. **365** (2013), no. 11, 5999–6016, MR 3091273, Zbl 06208247.

[7] K. Cieliebak & Y. Eliashberg, *From Stein to Weinstein and back, Symplectic geometry of affine complex manifolds*, American Mathematical Society Colloquium Publications. American Math Society 59 Providence, RI: American Mathematical Society (AMS) (2012), MR 3012475, Zbl 1262.32026.

[8] Y. Eliashberg, *Topological characterization of Stein manifolds of dimension* > 2, Internat. J. Math. **1** (1990) 29–46, MR 1044658, Zbl 0699.58002; `http://dx.doi.org/10.1142/S0129167X90000034`.

[9] Y. Eliashberg, *A few remarks about symplectic filling*, Geom. Topol. **8** (2004) 277–293, MR 2023279, Zbl1067.53070, `http://dx.doi.org/10.2140/gt.2004.8.277`.

[10] Y. Eliashberg & M. Gromov, *Convex symplectic manifolds*, from: "Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989)," Proc. Sympos. Pure Math. **52**, Amer. Math. Soc., Providence, RI (1991) 135–162, MR 1128541, Zbl 0742.53010.

[11] H. Endo, *Meyer's signature cocycle and hyperelliptic fibrations*, Math. Ann. **316** (2000), no. 2, 237–257, MR 1741270, Zbl 0948.57013.

[12] J.B. Etnyre, *Symplectic convexity in low-dimensional topology*, Topology Appl. **88** (1998) 3–25, MR 1634561, Zbl 0930.53049, `http://dx.doi.org/10.1016/S0166-8641(97)00196-X`.

[13] J.B. Etnyre, *Lectures on open book decompositions and contact structures*, Floer homology, gauge theory, and low-dimensional topology, Clay Math. Proc. **5**, Amer. Math. Soc., Providence, RI (2006) 103–141, MR 2249250, Zbl 1108.53050.

[14] H. Geiges, *An introduction to contact topology,* Cambridge Studies in Advanced Mathematics **109**, Cambridge University Press, Cambridge, 2008, xvi+440 pp, MR 2397738, Zbl 1153.53002.

[15] E. Giroux, *Structures de contact en dimension trois et bifurcations des feuilletages de surfaces*, Invent. Math. **141** (2000) 615–689, MR 1779622, Zbl 1186.53097, `http://dx.doi.org/10.1007/s002220000082`.

[16] E. Giroux, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, from: "Proceedings of the International Congress of Mathematicians, Vol. II" (Beijing, 2002), Higher Ed. Press, Beijing (2002) 405–414, MR 1957051, Zbl 1015.53049.

[17] R.E. Gompf, *Handlebody construction of Stein surfaces*, Ann. of Math. (2) **148** (1998) 619–693, MR 1668563, Zbl 0948.57021.

[18] R.E. Gompf & A. I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics 20, American Mathematical Society, Providence, RI (1999), MR 1707327, Zbl 0933.57020.

[19] H. Grauert, *On Levi's problem and the imbedding of real-analytic manifolds*, Ann. of Math. (2) **68** (1958), 460–472, MR 0098847, Zbl 0108.07804.

[20] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), no. 2, 307–347, MR 809718, Zbl 0592.53025.

[21] K. Honda, *On the classification of tight contact structures. I*, Geom. Topol. **4** (2000), 309–368, MR 1786111, Zbl 0980.57010, `http://dx.doi.org/10.2140/gt.2000.4.309`.

[22] A. Kaloti, *Stein fillings of planar open books*, preprint, `http://arxiv.org/abs/1311.0208`.

[23] J. Latschev & C. Wendl, *Algebraic torsion in contact manifolds,* with an appendix by Michael Hutchings, Geom. Funct. Anal. **21** (2011), 5, 1144–1195, MR 2846386, Zbl 1248.53072.

[24] S. Lisi, J. Van Horn-Morris & C. Wendl, *On symplectic fillings of spinal open book decompositions*, preprint in preparation.

[25] A. Loi & R. Piergallini, *Compact Stein surfaces with boundary as branched covers of* B^4, Invent. Math. **143** (2001), 325–348, MR 1835390, Zbl 0983.32027, `http://dx.doi.org/10.1007/s002220000106`.

[26] D. McDuff, *The structure of rational and ruled symplectic 4-manifolds*, J. Amer. Math. Soc. **3** (1990), no. 3, 679–712, MR 1049697, Zbl 0723.53019.

[27] R. Myers, *Simple knots in compact, orientable 3-manifolds*, Trans. of the Amer. Math. Soc. **273**, no. 1, 75–91 (1982), MR 0664030, Zbl 0508.57008.

[28] B. Ozbagci & A.I. Stipsicz, *Contact* 3*-manifolds with infinitely many Stein fillings,* Proc. Amer. Math. Soc. **132** (2004), no. 5, 1549–1558, MR 2053364, Zbl 1045.57014.

[29] B. Ozbagci & A.I. Stipsicz, *Surgery on contact 3-manifolds and Stein surfaces*, Bolyai Society Mathematical Studies **13**, Springer, Berlin (2004), MR 2114165, Zbl 1067.57024.

[30] A.I. Stipsicz,*On the geography of Stein fillings of certain* 3*-manifolds*, Michigan Math. J. **51** (2003), no. 2, 327–337, MR 1992949, Zbl 1043.53066.

[31] W.P. Thurston & H. E. Winkelnkemper, *On the existence of contact forms*, Proc. Amer. Math. Soc. **52** (1975) 345–347, MR 0375366, Zbl 0312.53028.

[32] W. P. Thurston, *Some simple examples of symplectic manifolds*, Proc. Amer. Math. Soc. **55** (1976) 467–468, MR 0402764, Zbl 0324.53031.

[33] I. Torisu, *Convex contact structures and fibered links in 3-manifolds*, Internat. Math. Res. Notices, (2000), no. 9, 441-454, MR 1756943, Zbl 0978.53133.

[34] A. Weinstein, *Contact surgery and symplectic handlebodies*, Hokkaido Math. J. **20** (1991), no. 2, 241–251, MR 1114405, Zbl 0737.57012.

Department of Mathematics and Statistics
University of Massachusetts
Amherst, MA 01003-9305
E-mail address: baykur@math.umass.edu

Department of Mathematical Sciences
University of Arkansas
Fayetteville, AR 72701
E-mail address: jvhm@uark.edu

Université de Nantes
Laboratoire de Mathématiques Jean Leray
F-44322 Nantes Cedex 3, France
E-mail address: samuel.lisi@univ-nantes.fr

Department of Mathematics
University College London
Gower Street
London WC1E 6BT, United Kingdom
E-mail address: c.wendl@ucl.ac.uk