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COMPLETE WILLMORE SURFACES IN $\mathbb{H}^3$ WITH BOUNDED ENERGY: BOUNDARY REGULARITY AND BUBBLING

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Abstract

We study various aspects related to boundary regularity of complete properly embedded Willmore surfaces in $\mathbb{H}^3$, particularly those related to assumptions on boundedness or smallness of a certain weighted version of the Willmore energy. We prove, in particular, that small energy controls $\mathcal{C}^1$ boundary regularity. We examine the possible lack of $\mathcal{C}^1$ convergence for sequences of surfaces with bounded Willmore energy and find that the mechanism responsible for this is a bubbling phenomenon, where energy escapes to infinity.

1. Introduction

In our previous paper [**1**] we studied the renormalized area, $\mathrm{RenA}(Y)$, as a functional on the space of all properly embedded minimal surfaces Y in $\mathbb{H}^3$ with a sufficiently smooth boundary curve at infinity. Area or volume renormalization of a properly embedded *minimal* submanifold of arbitrary dimension or codimension in hyperbolic space was introduced by Graham and Witten [**9**]; the renormalization is accomplished by a Hadamard regularization of the asymptotic expansion of areas (or volumes) of a family of compact truncations of the submanifold. The renormalized area of such a minimal surface in $\mathbb{H}^3$ turns out to be a classical quantity. The first result in [**1**] is that

$$\mathrm{RenA}(Y) = -2\pi\chi(Y) - \frac{1}{2}\int_Y |\mathring{A}|^2\, d\mu, \tag{1.1}$$

where $\chi(Y)$ and $\mathring{A}$ are the Euler characteristic and trace-free second fundamental form of Y, respectively. Since Y is minimal, $\mathring{A}$ equals the full second fundamental form A, so $\int_Y |\mathring{A}|^2\, d\mu$ is the same as the total curvature $\int_Y |A|^2\, d\mu$ of the surface Y. In other words, $\mathrm{RenA}(Y)$ differs

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from $\mathcal{E}[Y] := \int_Y |A|^2\, d\mu$ by a purely topological term. In general, even if Y is not minimal, then a short computation shows that

$$|A|^2 = 2|\mathring{A}|^2 + 2K_Y - 2K_Y^{\mathbb{H}^3} = 2\left(|\mathring{A}|^2 + K_Y + 1\right), \tag{1.2}$$

where K_Y is the Gauss curvature of Y and the third term on the right is the contribution from the sectional curvature of the tangent space of Y with respect to the hyperbolic metric on $\mathbb{H}^3$.

We can also relate $\mathcal{E}[Y]$ to the total curvature of $\overline{Y}$, regarded as a compact surface with boundary in the upper half-space $\mathbb{R}^3_+$ with the Euclidean metric. Indeed, the density $|\mathring{A}|^2\, d\mu$ is invariant with respect to conformal changes of the ambient space. Decorating all quantities computed with respect to the Euclidean metric with bars, and using the Gauss-Bonnet theorem and the analogue of (1.2) for the Euclidean curvatures, one obtains

$$\int_Y |\bar{A}|^2\, d\bar{\mu} = 2\int_Y |\mathring{\bar{A}}|^2\, d\bar{\mu} + 4\pi\chi(Y) = 2\int_Y |\mathring{A}|^2\, d\mu + 4\pi\chi(Y). \tag{1.3}$$

There is no boundary term here because if $\partial_\infty \overline{Y}$ is smooth, then $\overline{Y}$ is smooth up to its boundary and meets $\partial\mathbb{R}^3_+$ orthogonally, so the geodesic curvature of this boundary vanishes.

We shall generalize our class of surfaces slightly and regard $\mathring{\mathcal{E}}[Y] := \int_Y |\mathring{A}|^2 d\mu$ as an energy on the space of complete, properly embedded surfaces in $\mathbb{H}^3$ with asymptotic boundary of some fixed regularity which meet $\partial_\infty\mathbb{H}^3$ orthogonally. We show later that this orthogonality is a consequence of the finiteness of the total curvature $\mathcal{E}[Y]$ (computed with respect to the hyperbolic metric!). The critical points of $\mathring{\mathcal{E}}$ in this extended class are the so-called Willmore surfaces. To set this into context, recall that the Willmore energy of a closed surface Y in a Riemannian manifold (M, g) is $\int_Y \left(\frac{1}{4}|H|^2 + K_Y^M\right) d\mu = \int_Y \left(\frac{1}{2}|\mathring{A}|^2 + K_Y\, d\mu\right) = \frac{1}{2}\int_Y |\mathring{A}|^2\, d\mu + 2\pi\chi(Y)$ (see [**17**]); here K_Y is the Gauss curvature of Y and K_Y^M the sectional curvature with respect to M of the tangent space of Y. Willmore surfaces are the critical points of this energy. For the complete surfaces in $\mathbb{H}^3$ considered here, the integral $\int_Y K_Y\, d\mu$ always diverges, so we regard $\int_Y |\mathring{A}|^2 d\mu$ as the Willmore energy instead. Critical points (with respect to compactly supported variations) are the usual Willmore surfaces in $\mathbb{H}^3$. The Euler-Lagrange equation for $\int_Y |\mathring{A}|^2\, d\mu$, with respect to compactly supported variations, is conformally invariant; hence Willmore surfaces are the same for either the hyperbolic or the Euclidean metric on the upper half-space.

Stated more plainly, the objects studied in this paper are the critical points of $\mathring{\mathcal{E}}$, which we call Willmore surfaces, for which the total curvature $\mathcal{E}[Y] := \int_Y |A|^2 d\mu$ is finite (and hence $Y \perp \partial\mathbb{H}^3$). Our aim is to study sequences Y_j of Willmore surfaces with fixed genus and number of ends; we wish, in particular, to examine how boundedness of $\mathcal{E}[Y_j]$ controls regularity of these surfaces at their boundaries. Unfortunately, $\mathcal{E}$ itself does not seem to provide adequate control of this boundary regularity—we comment on this further below, where we describe an analogue of our results for harmonic functions, so we instead consider a weighted energy

$$\mathcal{E}_p(Y) := \int_Y |\mathring{A}|^2 f^{2p}\, d\mu.$$

Here f is the intrinsic distance function in Y to a given finite collection of points in Y, which we call *poles*, and $p > 1$ is fixed. These poles are in the interior of Y so that near $\partial_\infty \mathbb{H}^3 = \{x = 0\}$ (in the upper half-space model), $f \sim |\log x|$. For brevity we refer to $\mathcal{E}_p(Y)$ as the *weighted energy* of Y. We still require the finiteness of $\int |A|^2\, d\mu$, instead of some weighted version of it.

We shall study the following problem: If Y_j is a sequence of Willmore surfaces with $\mathcal{E}_p(Y_j) \leq C < \infty$, then does some subsequence of the Y_j converge in $\mathcal{C}^1$ up to the boundary? In fact, we show that $\mathcal{C}^1$ convergence may fail at a finite set of points at the boundary, but we are able to understand this phenomenon via the loss of energy in the limit. Since convergence of Willmore surfaces in any compact set of $\mathbb{H}^3$ is well understood, we focus almost entirely on the behavior of these surfaces near and at their asymptotic boundaries.

Before stating our results, we put this into a broader context. The study of failure of compactness for variational problems goes back at least to [**30**] and has now been explored in a wide variety of settings; we refer to [**28**] for a good overview of results and methods. Particularly relevant to our problem are the many deep advances in understanding the analytic aspects of the Willmore functional; we refer in particular to the fundamental paper of L. Simon [**31**], the more recent work by Kuwert and Schätzle [**16**], and the powerful new approach developed by Rivière [**27**]; see also [**24**]. Regularity at a free boundary for submanifolds with prescribed mean curvature has been studied in [**14**], and related problems have been studied in many other settings. However, the analysis of bubbling phenomena at the boundary seems to be far less well investigated. Often this failure of compactness at the boundary is excluded by imposing a priori bounds on boundary regularity. Our particular geometric problem presents a natural situation where it is unnatural to impose such boundary control, and where this bubbling phenomenon is then unavoidable.

The second context in which to view our work is slightly more tenuous. To explain it we first recall the computation from [**1**] which gives the first variation of $\mathcal{E}$ at a minimal surface Y. If $\gamma = \partial_\infty Y$ is the boundary curve at infinity, then there is function u_3 associated to Y such that

$$D\,\mathcal{E}|_Y\,(\psi) = 6\int_\gamma u_3\psi_0\,ds. \tag{1.4}$$

Here ψ is a Jacobi field along Y, i.e. an infinitesimal variation of Y amongst minimal surfaces and ψ_0 its boundary value at γ, and s is the arclength parameter along γ. The pair (γ, u_3) can be regarded as the Cauchy data of Y. It follows from the basic regularity theory for such surfaces, due to Tonegawa [**32**], that if the 'Dirichlet data' γ is $\mathcal{C}^\infty$, then $\overline{Y}$ is $\mathcal{C}^\infty$ up to the boundary. By analogy with classical elliptic theory, one might also expect that control on the Neumann data, u_3, should also control regularity of Y near its boundary. In particular, if Y_j is a Palais-Smale sequence for $\mathcal{E}$ (or $\mathcal{E}_p$), then the functions $u_3^{(j)}$ converge to zero in some weak sense, and the question then becomes whether quantitative measures of smallness on these functions yield greater control on the boundary curves γ_j. We do not emphasize this point of view, however, since it has been difficult to make precise.

Results. Our first theorem is an ϵ-regularity result: if the weighted energy of a Willmore surface in a Euclidean half-ball in the upper half-space model around some point $P \in \partial_\infty Y$ is small, then the $\mathcal{C}^1$ norm of the surface is controlled uniformly up to the boundary. To explain this, regard Y as a horizontal graph over a vertical half-plane (its tangent plane at some boundary point). Finiteness of the weighted energy is slightly weaker than bounding the (weighted) $W^{2,2}$ norm of the graph function, with the same logarithmic weight. The conclusion that Y is $\mathcal{C}^1$ shows that this graph function exhibits *better* regularity near the boundary than would follow from the Sobolev embedding theorem. (Indeed, there are $\mathcal{C}^1$ functions with compact support for which the weighted $W^{2,2}$ norm is finite and the $W^{1,\infty}$ norm is arbitrarily large. It is not hard to construct an infinite sum of these, with disjoint supports, so that the weighted $W^{2,2}$ norm is finite and the $W^{1,\infty}$ norm is infinite.)

This $\mathcal{C}^1$ regularity is nearly optimal. Indeed, since dilations are hyperbolic isometries, the energy $\mathcal{E}_p$ is dilation-invariant provided we let the dilations act on the set of poles as well. However, if we take a blow-down limit of a given surface, then the $\mathcal{C}^{1,\alpha}$ norm of the boundary curve diverges, so we cannot expect that norm to be controlled only by the weighted energy. It is not clear how to characterize the optimal regularity associated with finiteness of weighted energy, nor is it obvious whether there is an optimal weight function that guarantees $\mathcal{C}^1$ regularity.

One application of this first result is that if Y_j is a sequence of Willmore surfaces with $\mathcal{E}_p$ bounded, and with well-separated boundary components, then some subsequence converges to a Willmore surface Y_*, the boundary at infinity of which is a priori Lipschitz except at a finite number of bad points. We then show that except possibly at these exceptional points, the limit curve is $\mathcal{C}^1$. This is a gain of regularity compared to Sobolev embedding. We note that the convergence of $\gamma_j = \partial_\infty Y_j$ to $\gamma_* = \partial_\infty Y_*$ need not be $\mathcal{C}^1$; in fact we construct counterexamples to this at the end of this paper: Using fairly simple gluing arguments, we obtain a sequence Y_j with energy $\mathcal{E}_p(Y_j) \leq C$ which converges to a totally geodesic hemisphere, but where the convergence is not $\mathcal{C}^1$ at a finite number of boundary points. At each of those points, one sees a sequence of increasingly strong blow-downs of a fixed Willmore surface, which carries a fixed positive amount of energy, shrink to a point; we regard this as a type of bubbling. However, unlike the various 'interior' bubbling phenomena mentioned earlier which only occur when the energy is above a certain threshold, in this setting *arbitrarily small* amounts of energy can disappear in these limits.

Our final result is that the phenomenon exhibited by these examples above is the only mechanism through which the convergence $Y_j \to Y_*$ can fail to be $\mathcal{C}^1$ near the boundary, at least in regions of small energy. In such regions we show that if $P_j \in \gamma_j$, $P_j \to P_* \in \gamma_*$, but the tangent lines $T_{P_j}\gamma_j$ fail to converge to $T_{P_*}\gamma_*$, then there exist a sequence of hyperbolic isometries φ_j which dilate away from P_j and are such that $\varphi_j(Y_j) \to \tilde{Y}_*$, where $\mathcal{E}(\tilde{Y}_*) > 0$. Finally, we show that such a bubble of energy (which is already receding to infinity before applying the dilations φ_j) carries with it one of the poles used to define the weight function f. The investigation of regularity gain and bubbling in regions of large energy presents various technical difficulties (some of which are already apparent in [**19**]) which are beyond the scope of this paper. We intend to return to this in the future.

A model problem. We conclude this introductory material by discussing a model linear problem where one already sees that it is impossible to obtain adequate control of boundary regularity from the unweighted energy alone. This is, of course, only a hint that the same should be true for our nonlinear problem.

Set $D := \{x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$ and consider the Bergman-type space of harmonic functions

$$\mathcal{B} := \big\{u \in W^{1,2}(D) : \Delta u = 0,\ \int_{D(1)} |\nabla u|^2 dxdy < \infty\big\}.$$

The standard Sobolev trace theorem shows that $\mathcal{B} \ni u \mapsto u|_{\partial D} \in H^{1/2}(\partial D)$ is continuous, and conversely, the Poisson map $H^{1/2}(\partial D) \ni f \to \mathcal{P}f \in \mathcal{B}$, where u is the harmonic extension of f, is also bounded. For any function $f \in H^{1/2} \setminus L^\infty$, the extension $u = \mathcal{P}f$ has bounded energy $\int |\nabla u|^2 < \infty$. In particular, the harmonicity of u does not guarantee any extra control at the boundary.

On the other hand, for any $p > 1$, consider

$$\mathcal{B}_p := \big\{u \in W^{1,2}(D) : \Delta u = 0,\ \int_D |\nabla u|^2[1 + \log(1 - r^2)]^{2p}\, dxdy < \infty\big\}.$$

We claim that any $u \in \mathcal{B}_p$ restricts to a bounded function on $\partial D(1)$. The argument is a simpler version of some part of one of the main arguments in this paper, so we present it now.

Proof. To educe the analogy with section 5 below, and since the argument is local, we consider the equivalent problem of showing that if u is any harmonic function on $Q = \{0 \le x \le 1, |y| \le 1\}$ satisfying

$$\mathcal{E}_p(u) := \int |\nabla u|^2(1 - \log x)^{2p}\, dxdy < \infty,$$

then its boundary value $u(0, y)$ is continuous (note that $1 - \log x$ is comparable to the hyperbolic distance between (x, y) and $(0, 1)$ in Q (at least for $x \le 1 - \epsilon$). More specifically, we show that $u(x, y)$ converges uniformly as x tends to 0, and that the limiting continuous function $u(0, y)$ is bounded by $u(1, y)$ and $\mathcal{E}_p(u)$.

The first step is to show that the function $v := (\partial_x u)^2(1 - \log x)^{2p}$ is subharmonic. This is true with respect to either the Euclidean or the hyperbolic Laplacians. Computing with the former, after some arithmetic we see that

(1.5)
$$\begin{aligned}\Delta v &= 2|\nabla\partial_x u|^2(1 - \log x)^{2p} + 2\nabla(\partial_x u)^2 \cdot \nabla(1 - \log x)^{2p} \\ &\quad + (\partial_x u)^2\Delta(1 - \log x)^{2p} \ge 2(1 - \log x)^{2p-2}\big(|\partial_x^2 u|^2(1 - \log x)^2 \\ &\qquad -4px^{-1}\partial_x u\, \partial_x^2 u(1 - \log x) + 4p^2|\partial_x u|^2 x^{-2}\big) \ge 0,\end{aligned}$$

as claimed.

Now apply the mean value formula for subharmonic functions on the (Euclidean) ball $B_{x_0}((x_0, y))$. This gives that for any $0 < x_0 < 1$,

$$\begin{aligned}&|\partial_x u(x_0, y)|^2(1 - \log x_0)^{2p} \\ &\qquad \le \frac{1}{4\pi x_0^2}\int_{B_{x_0}((x_0,y))} u(x)^2(1 - \log x)^{2p}\, dxdy \le \frac{1}{4\pi x_0^2}\mathcal{E}_p(u),\end{aligned}$$

or equivalently,

$$|\partial_x u(x_0, y)| \le \frac{1}{2\sqrt{\pi}x_0(1 - \log x_0)^p}\mathcal{E}_p(u)^{1/2}.$$

The right hand side is integrable on $0 < x_0 < 1$, which proves that

$$|u(1, y) - u(0, y)| \le C\mathcal{E}_p(u)^{1/2},$$

and also shows that $u(x_0, y)$ converges in the uniform norm as $x_0 \searrow 0$, so that $u(0, y)$ is continuous. q.e.d.

Outline. We now provide a brief outline of some of the key ideas and arguments in this paper. The preamble to each section contains more extensive discussion of the main idea in that section.

The argument commences in §2, where we prove two 'soft' results about boundary regularity for Willmore surfaces with finite energy. Together, these show that any such surface must meet $\partial_\infty \mathbb{H}^3$ orthogonally and have a good local graphical representation over a vertical plane provided the boundary curve has a corresponding graphical representation over a line. This relies only on interior regularity results for Willmore surfaces and simple Morse-theoretic arguments.

ϵ-regularity. Our first 'hard' result is that for (local) Willmore surfaces with sufficiently small weighted energy, the boundary curve is controlled in $\mathcal{C}^1$. Indeed, if this were to fail, then we could construct a sequence of Willmore surfaces, the energies of which vanish in the limit, but such that there is a jump in the tangent lines in the limit. To reach a contradiction, we wish to relate the slope of the tangent line at the boundary to information on a parallel curve in the interior of the surface and then use the known $\mathcal{C}^\infty$ convergence in the interior.

The relationship between derivative information in the interior and at the boundary, i.e. the difference between the 'horizontal' derivatives at height 0 and 1, say, is given by integrating the mixed second derivative of the graph function along a vertical line and showing that this is controlled by the energy. To do this we must use a choice of 'gauge', which is a special isothermal coordinate system for which we have explicit pointwise control of the conformal factor. We do so adapting some deep results of Müller and Sverak [**26**] (see also Hélein [**11**]) to our setting. Throughout this adaptation we must be careful that the jump in first derivatives at the origin still holds in these new coordinates. Note that this requires the finiteness of the unweighted energy only. This boundedness of the conformal factor, along with the boundedness of the weighted energy, allows us to obtain pointwise control of the mixed component of the 2nd derivative of the graph function and bounding of its line integral. This pointwise bound relies on a realization of Willmore surfaces as harmonic maps into the deSitter space; a mean value inequality for this map yields a bound on a specific component of the second fundamental form, which in turn implies our desired pointwise bounds. It is at this point that the finiteness of the weighted Willmore energy is essential. In this argument there is a second line integral which it is necessary to control in terms of the energy of Y_j in a half-ball. This second line integral (which can be controlled by the regular rather than the weighted energy) plays a crucial role in the later analysis of bubbling.

These arguments occupy §3–6. In §7 we use the techniques developed up to that point to derive the regularity gain for the limit surface Y_* in regions of small energy.

Bubbling. Section 8 contains the argument that if the convergence $Y_j \to Y_*$ is not $\mathcal{C}^1$ at some sequence of boundary points $P_j \to P_* \in \partial_\infty Y_*$, then we can perform a sequence of blow-ups near those boundary points to produce a sequence of Willmore surfaces $\tilde{Y}_j$ which converge to a limit surface $\tilde{Y}_*$ of non-zero energy; prior to the blow-up the surfaces $\tilde{Y}_j$ are disappearing in the limit toward P_*. In other words, the $\mathcal{C}^1$ loss of compactness is due to portions of Y_j with fixed (but arbitrary) nonzero energy disappearing at infinity. Unlike similar arguments for bubbling in the interior, since our surfaces have infinite area, it is not initially clear that we can find points $Y_j \ni Q_j \to \partial_\infty \mathbb{H}^3$ on which $|\mathring{A}|_g$ is bounded below; the rescalings we wish to perform should be centered at such points. Their existence is proved indirectly, by arguing that it is impossible for all possible blow-ups near the points $P_j \in Y_j$ to converge to surfaces of zero energy. This argument makes essential use of the second line integral mentioned above. The key point is to show that this line integral can be controlled by the energy of Y_j in a conical region emanating from (rather than a half-ball containing) P_j.

Further questions. There are several questions and problems which are closely related to the themes in this paper and which seem of particular interest. We hope to return to some of these soon.

Despite the fact that the problems which led us to the current investigations involve Willmore surfaces of finite weighted energy in $\mathbb{H}^3$, one could equally study Willmore surfaces in the Euclidean ball, with boundaries lying on the boundary $\mathbb{S}^2$. In fact, the present work makes clear that only a weighted version of the traceless part $\int_Y |\mathring{A}|^2 f^{2p} d\mu$ of the total curvature is needed for our results; in view of the conformal invariance of the form $|\mathring{A}|^2 d\mu$, this suggests that the results here may also hold in a Euclidean ball, assuming an upper bound on the weighted energy $\int_Y |\mathring{\overline{A}}|^2 f^{2p} d\overline{\mu}$ and imposing bounds on the angle of intersection between Y and $\mathbb{S}^2 = \partial B^3$. Indeed, many of the methods developed here transfer to that more general setting with no difficulty.

Another question, which was a motivation for this work but not studied explicitly here, concerns the analysis of sequences of Willmore surfaces Y_j which are Palais-Smale for the functional $\mathcal{E}$. Recall that this means that $\mathcal{E}(Y_j)$ tends to a critical value and $D\mathcal{E}|_{Y_j}$ converges to 0. The goal would be to find critical points for $\mathcal{E}$. Our results show that critical sequences may converge to surfaces with strictly lower genus, and this convergence often occurs only in a weak norm at the boundary, but it may still be possible to produce $\mathcal{E}$-critical surfaces this way.

Finally, one other set of problems we wish to mention involve an analogous though more complicated problem of studying sequences of Poincaré-Einstein metrics in four dimensions. Recall that (M, g) is said to be Poincaré-Einstein if M is a compact manifold with boundary, and g is conformally compact (hence is complete on the interior of M) and

Einstein; see [**22**, **3**] for more details and further references. These objects can be studied in any dimension, but it is known that dimension 4 is critical in the same way that dimension 2 is critical for Willmore surfaces. This is reflected in two formulæ due to Anderson [**2**]: the first is an explicit local integral expression for the renormalized volume of a four-dimensional Poincaré-Einstein space as the sum of an Euler characteristic and the squared L^2-norm of the Weyl curvature, while the second describes the differential of renormalized volume with respect to Poincaré-Einstein deformations. These are entirely analogous to (and indeed were the motivatations for) the corresponding formulæ here. It is therefore reasonable to ask whether results like the ones here can be proved in that Poincaré-Einstein setting. Slightly more generally, reflecting the passage from minimal to Willmore, one should study these questions in the setting of Bach-flat metrics. More specifically, suppose that (M^4, g_j) is a sequence of Poincaré-Einstein (or Bach-flat) metrics such that $\int |W_j|^2 \, dV_{g_j} \leq C < \infty$, where W_j is the Weyl tensor of g_j. The specific issue is to determine how this uniform energy bound (or some suitably weighted version of such a bound) controls the regularity of the sequence of conformal infinities of g_j. This is related to the questions studied by Anderson [**3**] and more recently by Chang-Qing-Yang [**4**].

1.1. Notation and terminology. Almost all of the results below are local, so we always work in the upper half-space model $\mathbb{R}^3_+$ of $\mathbb{H}^3$, with vertical (height) coordinate x, and with linear coordinates (y, z) on $\mathbb{R}^2 = \{x = 0\}$.

All of the surfaces studied here are assumed to be smooth and Willmore (or minimal, if noted explicitly). We always assume that any such Y is connected and has closure $\overline{Y} \subset \overline{\mathbb{H}^3}$, a compact surface with boundary curve $\gamma = \partial_\infty Y \subset \mathbb{R}^2$ which is embedded and closed, but possibly disconnected. We assume that $\overline{Y}$ is at least $\mathcal{C}^2$ unless explicitly stated otherwise. Since $\mathbb{H}^3$ has many isometries, including dilation and horizontal $(\mathbb{R}^2)$ translation, it is convenient to fix a normalization and scale. We say that Y is normalized if the length of its boundary curve (measured with respect to the Euclidean metric on $\mathbb{R}^2$) satisfies $|\gamma| = 100\pi$ and if the center of mass of γ in $\mathbb{R}^2$ is 0. The class of all Willmore surfaces with k ends and genus g, normalized in this way, and which meet $\partial\mathbb{R}^3_+$ orthogonally, is denoted $\mathcal{M}_{k,g}$, and $\mathcal{M} = \bigcup_{k,g} \mathcal{M}_{k,g}$. For each $Y \in \mathcal{M}$, $\overline{Y}$ is the closure of Y in $\mathbb{R}^3_+$. We prove later that any complete Y with finite energy necessarily meets $\partial\mathbb{H}^3$ orthogonally, so we can omit this condition from the definition of elements of $\mathcal{M}$.

Many of the arguments below use the interplay between the metrics g and $\bar{g}$ on a surface Y induced from the ambient hyperbolic and Euclidean metrics, respectively. We denote by A and $\bar{A}$, and $\mathring{A}$ and $\mathring{\bar{A}}$, the corresponding second fundamental form and trace-free second fundamental forms of Y, and by $d\mu, d\overline{\mu}$ the area elements. As noted earlier,

$$|\mathring{A}|_g^2 \, d\mu = |\mathring{\bar{A}}|_{\bar{g}}^2 \, d\bar{\mu}. \tag{1.6}$$

If Y is minimal (rather than just Willmore) with respect to g, then $A = \mathring{A}$ and

$$|A|_g^2 \, d\mu = |\mathring{A}|_g^2 \, d\mu = |\mathring{\bar{A}}|_{\bar{g}}^2 \, d\bar{\mu}. \tag{1.7}$$

For brevity, the subscripts g and $\bar{g}$ are often omitted when the meaning is clear.

Definition 1.1. If $Y \in \mathcal{M}$, then we define the Willmore energy of Y to equal $\mathcal{E}(Y) := \int_Y |A|^2 \, d\mu$.

Key Assumption: In the entirety of this paper, we restrict to the subset of surfaces $Y \in \mathcal{M}_{k,g}$ for which $\mathcal{E}(Y) \leq M$ for some fixed $M < \infty$. We remark that in view of (1.3) and (1.6), this restriction ensures a uniform bound on $\int_Y |\bar{A}|^2 \, d\overline{\mu}$, which we shall use frequently, and without further mention.

1.2. Results. As explained earlier, we shall need to consider surfaces which satisfy a slightly stronger condition than finiteness of Willmore energy. This involves a weighted version of the Willmore energy which we now define.

Definition 1.2. Fix $N \in \mathbb{N}$. Given any finite set of points $\mathcal{O} = \{O_1, \ldots, O_N\}$, where each $O_j \in Y$, let $f_{\mathcal{O}}(P) := \mathrm{dist}(P, \mathcal{O}) + 5$. We call the points O_k the poles of $f_{\mathcal{O}}$. If $P \in Y$ and O_k is one of the poles nearest to P, we write $P \sim O_k$. We frequently write f instead of $f_{\mathcal{O}}$ for brevity; thus f is the distance function from *some* set of N points which may be anywhere on Y.

Now define the weighted energy

$$\mathcal{E}_p(Y, \mathcal{O}) := \int_Y |\mathring{A}|^2 f_{\mathcal{O}}^{2p} \, d\mu;$$

we sometimes write this simply as $\mathcal{E}_p(Y)$.

Definition 1.3. Fix $Y \in \mathcal{M}$ and $\gamma = \partial_\infty \overline{Y}$. Writing $B(P,R)$ as the open Euclidean half-ball centered at P of radius R, for any $P \in \gamma$ and $R > 0$, denote by $Y'_{B(P,R)}$ the path component of $\overline{Y} \cap B(P,R)$ which contains P in its closure and $\gamma'_{B(P,R)} = \overline{Y'_{B(P,R)}} \cap \partial\mathbb{R}^3_+$. The weighted and unweighted localized energies of $Y'_{B(P,R)}$ are given by

$$\mathcal{E}_p^{B(P,R)}(Y, \mathcal{O}) := \int_{Y'_{B(P,R)}} |\mathring{A}|^2 f_{\mathcal{O}}^{2p} \, d\mu, \quad \mathcal{E}^{B(P,R)}(Y) := \int_{Y'_{B(P,R)}} |A|^2 \, d\mu. \tag{1.8}$$

Definition 1.4. The ζ-*Lipschitz radius* of a normalized, closed embedded $\mathcal{C}^1$ curve $\gamma \subset \mathbb{R}^2$ is defined as follows. If $P \in \gamma$ and $\ell_\gamma(P) = T_P\gamma$, then let $\gamma_P \subset \gamma$ be the largest open connected arc containing P which is a graph over $\ell_\gamma(P)$. Thus if $P = 0$ and $\ell_P = \{(0, y, 0)\}$, then $\gamma_P = \{(y, f(y)) : a < y < b\}$ for some maximal $a < 0 < b$. We then define $\mathrm{LipRad}^\zeta_\gamma(P)$ to be the largest number M such that $(-M, M) \subset (a, b)$ and $\frac{|f(y)-f(y')|}{|y-y'|} < \zeta$ for every $y, y' \in (-M, M)$. Finally, we set

$$\mathrm{LipRad}^\zeta(\gamma) = \inf_{P\in\gamma} \mathrm{LipRad}^\zeta_\gamma(P). \tag{1.9}$$

Since γ is compact, the easily verified lower semicontinuity of LipRad^ζ implies that the infimum is attained at some point and $\mathrm{LipRad}^\zeta(\gamma) > 0$.

Theorem 1.1. *There is a $\zeta_0 \in (0, 1/20)$ with the property that for any $\zeta \in (0, \zeta_0)$, there exists an $\epsilon(\zeta, p) > 0$ such that if $Y \in \mathcal{M}_{k,g}$ and $\mathcal{E}_p^{B(P,R)}(Y) < \epsilon$ for some $P \in \gamma = \partial_\infty Y$ and $R \leq 1$, then*

$$\mathrm{LipRad}^\zeta_\gamma(Q) \geq \zeta \cdot \frac{R - |PQ|}{10}$$

for all $Q \in \gamma'_{B(P,R)}$.

Let us note that this lower bound on $\mathrm{LipRad}^\zeta_\gamma(Q)$ is completely independent of the set of poles $\mathcal{O}$ in Y.

From this and Lemma 2.4 below, we can deduce the

Corollary 1.1. *In the setting of Theorem 1.1, there exists $\epsilon'(\zeta, p) \leq \epsilon(\zeta, p)$ such that if $\mathcal{E}_p^{B(P,R)}[Y] < \epsilon'(\zeta, p)$, then $Y'_{B(P,R/2)}$ is a horizontal graph $z = u(x, y)$ over the half-disc $D_+(P, R/2)$ in the vertical half-plane $\mathbb{R}_+ \times \ell_P$, and $|\nabla u| \leq 2\zeta$ in $D_+(P, R/2)$.*

The Lipschitz radius is a reasonable measure of regularity on the space of normalized embedded curves γ. Note that if γ_j is a sequence of such curves with $\mathrm{LipRad}^\zeta(\gamma_j) \geq C > 0$, then there are uniform Lipschitz parametrizations around each point of every γ_j; hence in particular some subsequence of the γ_j converge in $\mathcal{C}^{0,\alpha}$ for any $\alpha < 1$ to a limit curve γ which is itself Lipschitz.

For future use, we state a slightly modified version of this result. Let $\mathcal{M}'$ be the space of properly embedded Willmore surfaces $Y \subset \mathbb{H}^3$ for which $\int_Y |A|^2\, d\mu < \infty$ and with $\mathcal{C}^1$ boundary curves $\gamma = \partial_\infty Y$. Note that $Y'_{B(P,R)}$ and $\mathcal{E}_p^{B(P,R)}(Y)$ are still defined when $Y \in \mathcal{M}'$. The modification deals with surfaces $Y \in \mathcal{M}'$ for which $\gamma'_{B(P,R)}$ intersects $\partial B(P, R)$.

Theorem 1.2. *For some $\zeta_0 > 0$ and every $\zeta \in (0, \zeta_0)$ there exists an $\epsilon(\zeta, p) > 0$ such that if $Y \in \mathcal{M}'$, $\mathcal{E}_p^{B(P,R)} \leq \epsilon(\zeta)$ (with $R \leq 1$) and*

$\gamma'_{B(P,R)}$ intersects $\partial B(P,R)$ for some $P \in \gamma = \partial_\infty Y$ and if $\gamma'_{B(P,R)}$ is $\mathcal{C}^1$ up to its endpoints, then

$$\mathrm{LipRad}^{\zeta}_{\gamma}(Q) \geq \zeta \cdot \frac{R - |PQ|}{10}$$

for all $Q \in \gamma'_{B(P,R)}$.

Theorem 1.1 leads to the following characterization of the possible limits of sequences of Willmore surfaces for which there is a uniform upper bound on the weighted energy.

Theorem 1.3. *Let $Y_j \in \mathcal{M}_{k,g}$ and suppose that $\mathcal{E}_p(Y_j) \leq M$ for some $M > 0$. Suppose too that the distance between the various components of $\gamma_j = \partial_\infty Y_j$ is uniformly bounded away from 0. Then if $0 < \zeta \leq \zeta_0$, there is a subsequence, again relabelled as Y_j, which converges to a finite multiplicity (but possibly disconnected) Willmore surface Y_* with boundary curve γ_*. The convergence $Y_j \to Y_*$ is smooth away from $\{x = 0\}$, except at a finite number of interior points, where Y_* may fail to be smooth. In this limit, the set of poles $\mathcal{O}^{(j)} \subset Y_j$ converges to a set of poles $\mathcal{O}^* \subset \overline{Y_*}$.*

Furthermore, there exist points $P_1, \ldots, P_\Lambda \in \gamma_*$, $\Lambda = \Lambda(\zeta)$, and corresponding sequences $P_i^{(j)} \in \gamma_j$, $i = 1, \ldots, \Lambda$, with $P_i^{(j)} \to P_i$ for all i, such that the convergence of γ_j to γ_* is $\mathcal{C}^{0,\alpha}$ for every $\alpha < 1$ away from the points $P_i^{(j)}$. Finally, if $P \in \gamma \setminus \{P_1, \ldots, P_\Lambda\}$, then there is a line ℓ_P such that Y_* is the graph of a Lipschitz function with Lipschitz constant 2ζ over some disc in the half-plane $\mathbb{R}^+ \times \ell_P$.

There are two distinctly separate aspects to the proof of this result. One establishes convergence in the interior, i.e., for the truncated surfaces $Y_j^\eta := Y_j \bigcap \{x \geq \eta\}$ for any fixed $\eta > 0$, while the second focuses on convergence near the boundary. Interior convergence (in the sense allowed in the theorem) follows from standard arguments once we show that there are uniform energy and area bounds for the Y_j^η. These arguments involve ϵ-regularity results for smooth Willmore surfaces, which can be deduced from [**15**, Theorem 2.1] (note that the method there can be applied when the ambient metric is hyperbolic since the Willmore equation still leads to an elliptic equation for $\Delta|A|^2$), along with a covering argument and a further diagonalization.

The assumptions of Theorem 1.3 include the uniform bounds for $\int_{Y_j \bigcap \{x \geq \eta\}} |A|^2 \, d\mu_j$. In fact, these energy bounds also imply area bounds for the Y_j^η; this is done in Proposition 2.1. This relies on the fact from our earlier paper [**1**] that the total curvature of each Y_j, together with a bound on the geodesic curvature of $\partial Y_j^{\theta_j}$ for some $\theta_j \in (\eta/2, \eta)$, bounds the *renormalized area* of each Y_j (see [**1**] for the definition of this concept), which in turn bounds the area of each Y_j^η. We defer details of this argument to the next section.

Convergence at the boundary, on the other hand, follows directly from Theorem 1.1 and Corollary 1.1. Indeed, let $\delta(Q)$ be the largest radius such that the half-ball $B(Q, \delta(Q))$ centered at $Q \in \gamma_j$ satisfies $\mathcal{E}_p^{B(Q,\delta(Q))}(Y_j') \leq \epsilon'(\zeta)$. If t denotes the arclength parameter along any component of γ_i, then the upper bound on $\mathcal{E}_p(Y_j)$ implies that for all but those finitely many values of t corresponding to the points $Q_i^{(j)}$, $\liminf_j \delta(\gamma_j(t)) > 0$. From this and a diagonalization argument, we deduce the asserted convergence near the boundary.

We next show that the limit curve γ_* is $\mathcal{C}^1$, rather than just Lipschitz, away from a finite set of points.

Theorem 1.4. *In the setting of Theorem* 1.3*, the curve* $\gamma_* = \partial_\infty Y_*$ *is piecewise* $\mathcal{C}^1$*, with singularities occurring (at most) at the set* $\{P_1, \ldots, P_\Lambda\} \subset \gamma_*$.

Remark 1.1. A modification of the proof of Lemma 2.4 below shows that $\overline{Y}_*$ is then $\mathcal{C}^1$ up to $\gamma_* \setminus \{P_1, \ldots, P_\Lambda\}$.

We also describe bubbling in this setting by showing that away from points where the convergence $\gamma_j \to \gamma_*$ is not $\mathcal{C}^1$, the loss of compactness is due to some portion of the Willmore surfaces *with non-zero energy* escaping to infinity:

Theorem 1.5. *Let* Y_j *be a sequence in* $\mathcal{M}_{k,g}$, $\mathcal{E}_p(Y_j) \leq M < \infty$, *with* $Y_j \to Y_*$ *where* Y_* *is* $\mathcal{C}^1$ *up to* $\gamma_* \setminus \{P_1, \ldots, P_\Lambda\}$. *After rotating and translating, we write each* Y_j *as a horizontal graph* $z = u_j(x, y)$ *over the half-disc* $\{x^2 + y^2 \leq \delta^2, z = 0\}$, *with* $|\nabla u_j| \leq 2\zeta$ *and* $u_j \to u_*$ *in* $\mathcal{C}^\infty$ *away from* $\{x = 0\}$ *and in* $\mathcal{C}^{0,\alpha}$ *up to* $\{x = 0\}$. *Suppose too that for some* $y_0 \in (-\delta, \delta)$, $\lim_{j\to\infty} \partial_y u_j(y_0, 0) \neq \partial_y u_*(y_0, 0)$. *Then there exists a sequence of interior points* $Q_j \in Y_j' \bigcap B(0, \delta)$ *with* $Q_j \to A_j := (0, y_0, u_j(y_0, 0))$ *and a sequence of hyperbolic isometries* ψ_j *mapping* Q_j *to* $(1, 0, 0)$ *so that* $\psi_j(Y_j) \to Y_*'$ *for some complete Willmore surface* Y_*' *with* $\mathcal{E}(Y_*') > 0$.

At most N *non-isometric blow-ups can be obtained in this way, and there exists a number* $M' > 0$ *such that for each sequence* Q_j *there is a sequence of poles* $\mathcal{O}^{(j)} \in Y_j$ *such that* $\mathrm{dist}_{Y_j}(Q_j, O^{(j)}) \leq M'$.

This theorem actually proves that the convergence $\partial Y_j \to \partial Y_*$ is $\mathcal{C}^1$ near all points $P \in \partial Y_* \setminus (\partial Y_* \bigcap \{P_1, \ldots, P_\Lambda\})$ *except* at those points on $\partial_\infty Y_*$ which are limits of the poles $O_i^{(j)}$. Thus, contrary to Theorem 1.1, the possibilities for bubbling depend very strongly on the positions of the poles in each Y_j.

Corollary 1.2. *Assume that for some point* $P \in \partial Y_* \setminus (\partial Y_* \bigcap \{P_1, \ldots, P_\Lambda\})$ *there exists a relatively open set* $\Omega \subset \overline{\mathbb{R}^3_+}$ *such that* $O_i \notin \Omega$ *for all poles* $O_i \in Y_j$ *and for all* j *large enough. Then the curves* ∂Y_j *converge to* ∂Y_* *in the* $\mathcal{C}^1$ *norm in the domain* Ω.

Indeed, if this were not the case at some point $Q \in \partial Y_* \bigcap \Omega$, then for $p' \in (1,p)$ and any $\epsilon > 0$ there exists an open half-ball $B(P,\delta)$ such that $\mathcal{E}_{p'}^{B(P,\delta)}(Y_j) \leq \epsilon$ for all j large enough. If ϵ is small enough, this implies a lower bound on $\mathrm{LipRad}(\gamma_j)$ in the half-balls $B(P,\delta/2)$.

Applying Theorem 1.5, we obtain a blow-up limit $\varphi_j(Y_j) \to Y'_*$ with $\mathcal{E}(Y'_*) > 0$, where the φ_j are hyperbolic isometries centered at $Q \in \partial_\infty \mathbb{H}^3$. The $\mathcal{C}^\infty$ convergence away from the boundary of $\varphi_j(Y_j)$ implies that there exist balls $B(P_j, 1) \subset Y_j$ of (intrinsic) radius 1 in Y_j with $P_j \to Q$ and with $\mathcal{E}^{B(P_j,1)}(Y_j) \geq \epsilon_0 > 0$. (Indeed, it suffices to let $\epsilon_0 < \mathcal{E}^{B(P,1)}(Y'_*)$ where $B(P,1) \subset Y'_*$ is any intrinsic ball where the energy is non-zero.) Now, $P_j \to Q$ readily implies that $f_{j,\mathcal{O}}(Q_j) \to \infty$ ($f_{j,\mathcal{O}}$ is the weight function for the surface Y_j), so that $\mathcal{E}_{p'}^{B(P,\delta)}(Y_j) \to \infty$. This is a contradiction.

In the last section of this paper, we construct examples where bubbling to infinity does occur. These are sequences of minimal (and thus Willmore) surfaces $Y_j \in \mathcal{M}$ with $\mathcal{E}(Y_j) \leq M < \infty$ and with $\overline{Y}_j$ converging smoothly away from a finite number of points on the boundary. At these points, the convergence fails to be $\mathcal{C}^1$, despite the fact that the curves γ_j and γ_* are all $\mathcal{C}^\infty$.

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2. Some geometric lemmas

2.1. Graphicality near the boundary. We begin with some geometric results, pertaining primarily to *fixed* complete Willmore surfaces Y with $\mathcal{E}(Y) < \infty$ and with $\gamma = \partial_\infty Y$ a finite union of compact embedded Lipschitz curves. We prove first that any such Y meets $\partial_\infty \mathbb{H}^3$ orthogonally, which is the well-known behavior, when Y is $\mathcal{C}^2$ up to the boundary. We then show that if some segment of γ is graphical with a bounded Lipschitz constant, then a portion of the Willmore surface directly above this segment is also graphical, with bounded gradient. The proofs are almost entirely geometric, involving blow-up arguments, though we rely on one analytic fact which is the ϵ-regularity theorem for (interior) Willmore discs. The finiteness of energy is used crucially at several places.

Lemma 2.1. *Let Y be a complete properly embedded Willmore surface such that $\gamma = \partial_\infty Y$ is a finite collection of embedded Lipschitz curves. Let P_j be a sequence of points in Y converging to a point on $\partial_\infty Y$. If $\bar{\nu}$ denotes the Euclidean unit normal to Y, then $\langle \partial_x, \bar{\nu}\rangle_{\overline{g}}(P_j) \to 0$ as $j \to \infty$.*

Proof. Since Y has finite energy, then $\int_{Y\cap\{x\leq Cx(P_j)\}} |A|^2\, d\mu \to 0$ for any $C > 0$. Now suppose that the assertion is false. Thus, passing to a subsequence if necessary, $\langle \partial_x, \bar{\nu}\rangle_{\overline{g}}(P_j) \to \beta \neq 0$. Let $B_1(P_j)$ be the ball of radius 1 around P_j with respect to the metric g. Passing to a further subsequence, we may assume that $B_1(P_j) \cap B_1(P_k) = \emptyset$ for $j \neq k$. Then $\int_{B_1(P_j)} |A|^2\, d\mu \to 0$, since otherwise $\mathcal{E}(Y)$ would be infinite.

Now translate Y horizontally and dilate by the factor $1/x(P_j)$ so that P_j is mapped to $(1, 0, 0)$ and denote by Y_j the resulting sequence of surfaces. Since each Y_j passes through the fixed point $(1, 0, 0)$ and $\int_{Y_j\cap\{x\leq M\}} |A|^2\, d\mu \to 0$ for any $M > 0$, we can invoke the a priori pointwise bounds for $|\nabla^p A_j|$ on a ball of any fixed radius around this fixed point using [**27**, Theorem I.5]; see also [**18**]. These show that yet a further subsequence of the Y_j converges in the $\mathcal{C}^\infty$ topology on compact sets to a complete Willmore surface Y_*.

Since $\mathcal{E}^{B_R(P_j)}(Y_j) \to 0$ for any $R > 0$, we see that Y_* is totally geodesic, and hence is either a vertical plane or a hemisphere; its slope at $(1, 0, 0)$ equals $\beta \neq 0$, so we must be in the latter case. This shows that there is a fixed constant $R = R(\beta) > 0$ such that if j is large, then the ball $B_R(P_j)$ in Y contains a point Q_j where $T_{Q_j}Y$ is horizontal, i.e. parallel to $\{x = 0\}$.

We can assume (passing again to a further subsequence) that $x(Q_j)$ is strictly monotone decreasing, so a standard minimax argument shows that we may choose a sequence of points $Q_j' \in Y_j$ which are critical points of index one for the function x. In other words, writing Y_j as a graph $x = v(y, z)$ near Q_j', v has a saddle at Q_j'. We can therefore translate horizontally and dilate by the factor $1/x(Q_j')$ to obtain a sequence Y_j' of Willmore surfaces which converge locally in $\mathcal{C}^\infty$ to a complete Willmore surface Y_*' passing through the point $(1, 0, 0)$. By construction, for any $M > 0$,

$$\int_{Y_j'\cap\{x\leq M\}} |A_j|^2\, d\mu_j \to 0. \tag{2.1}$$

Using the interior curvature estimates that follow from the ϵ-regularity in [**27**] again, we see that the convergence of Y'_j to Y'_* is $\mathcal{C}^\infty$ near the point $(1,0,0)$; hence Y'_* has a horizontal tangent plane at this point. Furthermore, the two principal curvatures at Q'_j relative to the ambient Euclidean metric are $\kappa_1 \geq 0$ and $\kappa_2 \leq 0$, and these inequalities must persist in the limit. This means that Y'_* cannot be a hemisphere. However, (2.1) implies that $\mathcal{E}(Y'_*) = 0$, which yields a contradiction. q.e.d.

An almost identical argument proves

Lemma 2.2. *Let Y be a fixed Willmore surface in $\mathbb{H}^3$ with $\mathcal{E}(Y) < \infty$. Let P_j be a sequence of points in $\partial_\infty Y$ and choose $\delta_j \searrow 0$. Denote by $B^+_{\delta_j}(P_j)$ the Euclidean half-ball centered at P_j and with radius δ_j. Assume that the sequence of dilated translates $\delta_j^{-1}(Y \cap B^+_1(P_j) - P_j)$ converges to a surface $\tilde{Y}$ (which is necessarily Willmore). Then $\tilde{Y}$ must be a vertical half-plane.*

(We note that the two lemmas above do not require the boundary curves to be locally Lipschitz.)

We next turn to proving local graphicality of any Willmore surface of finite energy near points where the boundary curve is Lipschitz.

Lemma 2.3. *Let Y be a complete properly embedded Willmore surface in $\mathbb{H}^3$ with finite energy and such that $\partial_\infty Y = \gamma$ is a finite union of closed embedded rectifiable loops. Define $\mathcal{S}_B$ to be the set of all points $P \in \gamma$ for which there exists a connected subarc $\gamma_P \subset \gamma$ which is a graph over a straight line $\ell_P \subset \mathbb{R}^2$ containing P which, if we rotate and translate so that ℓ_P is the y-axis, has graph function $z = f(y)$, $|y| \leq \delta(P)$, satisfying* $\mathrm{Lip}(f) \leq B$.

Then there exists an $h > 0$, independent of $P \in \mathcal{S}_B$, such that the portion $Y'_{B(P,h\,\delta(P))}$ of the surface Y is graphical over the half-disc $\{\sqrt{x^2+y^2} \leq h\,\delta(P),\ z = 0\}$ with graph function $z = u(x,y)$, where u satisfies $|\nabla u| \leq 2B$.

Proof. If this were false, then there would exist a sequence $P_j \in \gamma$, lines ℓ_j and graph functions $f_j : [-\delta_j, \delta_j] \to \mathbb{R}$ for γ with Lipschitz constant B, and sequences of numbers $h_j \to 0$ and points $Q_j \in Y'_{B(P_j,h_j\delta_j)}$ with coordinates (x_j, y_j, z_j) (using coordinates (x,y,z) where P_j is the origin and ℓ_{P_j} is the y-axis), such that the angle between the unit (Euclidean) normal $\bar{\nu}(Q_j)$ to Y at Q_j and ∂_z is greater than $\arctan(2B)$.

Since Y has finite energy, we have that $\mathcal{E}^{B(P_j,h_j\delta_j)}(Y) \to 0$, so a contradiction can be drawn by a blow-up argument. Translate so that $y_j = 0$, then dilate by the factor $\frac{1}{x_j}$ to obtain a sequence of surfaces $\tilde{Y}_j$. By construction, $\partial_\infty \tilde{Y}_j$ is graphical over the y-axis at least over the interval $|y| \leq \frac{1}{h_j}$, with Lipschitz constant B. Furthermore, the angle between $\bar{\nu}$ and ∂_z at $(1,0,0)$ is greater than $\arctan(2B)$. However, Y_j converges to a vertical half-plane Y_*, and since the convergence is $\mathcal{C}^\infty$ away from the boundary by [**6**], the angle condition at $(1,0,0)$ is preserved in the limit. However, by Lemma 2.2, from the Lipschitz bound on the graph function f_j, we see that $Y_* = \{z = \alpha y + \beta, x > 0\}$ for some α with $|\alpha| \leq B$. This contradicts the angle condition at $(1,0,0)$. q.e.d.

We also need a slight variant of this.

Lemma 2.4. *Consider a sequence of complete Willmore surfaces Y_j, the closures of which pass through the $(0,0,0)$. Assume that the subdomains $Y'_{j,B(0,3)}$ satisfy $\mathcal{E}^{B(0,3)}(Y_j) \to 0$, and that $\gamma_j = \partial_\infty Y'_{j,B(0,3)}$ is a graph $z = v_j(y)$ over the interval $|y| \leq 2$ with $v_j \in \mathcal{C}^1$ and $|v'_j(y)| \leq \delta$ for some $\delta > 0$. Then there exists an $\epsilon_0(\delta) > 0$ such that $Y'_{j,B(0,3)}$ is a graph $z = u_j(x,y)$ over the rectangle $\mathcal{R} := \{0 \leq x \leq \epsilon_0(\delta), |y| \leq 2\}$, and $|\nabla u_j| \leq 2\delta$ on $\mathcal{R}$.*

Proof. This is proved essentially as before. We pick $\epsilon_0(\delta)$ small enough so that all functions $u_j(x,y)$ with $(x,y) \in [0,\epsilon_0(\delta)] \times [-2,2]$ whose graphs are portions of upper half-spheres and satisfy $|\partial_y u_j|_{x=0}| \leq \delta$ also satisfy $|\nabla u_j(x,y)| \leq \frac{3\delta}{2}$ for $(x,y) \in [0,\epsilon_0(\delta)] \times [-2,2]$. If the claim were to fail for this $\epsilon_0(\delta)$, we could choose points $P_j \in Y_j$ contained in the portion of Y_j which is graphical over this rectangle where $|\nabla u_j(P_j)| > 2\delta$. Since $Y_j \to Y_*$ smoothly away from $x = 0$ and Y_* must be a portion of a hemisphere, we must have $x(P_j) \to 0$. Now dilate by the factor $x(P_j)^{-1}$; this produces a sequence of Willmore surfaces $\tilde{Y}_j$ which converge to a vertical half-plane Y_* which meets the xy-plane at a small angle bounded above by $|\arctan(\delta)|$, but such that the corresponding graph functions $\tilde{u}_j$ satisfy $|\nabla \tilde{u}_j| > 2\delta$ at a fixed point $(0,0,1)$. However, convergence in this dilated setting is still smooth away from $x = 0$ by [**27**], so this is a contradiction. q.e.d.

2.2. Interior area bounds. We now explain how to derive the interior area bounds required for Theorem 1.3.

Proposition 2.1. *Consider any sequence $Y_j \in \mathcal{M}_{k,g}$ of normalized surfaces with $\int_{Y_j} |A|^2 \, d\mu_j \leq M < \infty$. Then for any $\eta > 0$, there is a uniform bound on the areas of the truncations $Y_j^\eta = Y_j \cap \{x \geq \eta\}$, at least for some subsequence of the Y_j.*

Proof. The first step is a lemma which shows that the Y_j remain outside a fixed half-space $B_+ \subset \mathbb{H}^3$. Note that this would be trivial by virtue of the maximum principle if the Y_j were minimal. In this more general setting, however, this follows from the energy bounds. Recall too that we are assuming that each $\gamma_j = \partial_\infty Y_j$ has $|\gamma_j| = 100\pi$.

Lemma 2.5. *Suppose that $Y_j \in \mathcal{M}_{k,g}$ is a sequence with $\int_{Y_j} |A|^2\, d\mu_j \leq M < \infty$. Then there exists a closed half-ball $B_* = \{(x,y,z) : x^2 + (y-y_0)^2 + (z-z_0)^2 \leq R^2\}$ so that $Y_j \cap B_* = \emptyset$ (at least for some subsequence of the Y_j).*

Proof. We first note that because of the uniform bound on the length and number of components of each γ_j, there must exist a disk $D'_* = \{x = 0, (y-y'_0)^2 + (z-z'_0)^2 \leq (R')^2\} \subset \partial_\infty \mathbb{H}^3$ such that $\gamma_j \cap D_* = \emptyset$ for infinitely many j.

Now, if the assertion were false, then there would exist points $P_j \in Y_j$ for infinitely many j where $x(P_j) \to 0$ and such that $T_{P_j} Y_j$ is horizontal and Y_j locally lies above this tangent plane. By virtue of the uniform energy bound, the energies of $Y_j \cap B(P_j, 1)$ (i.e., hyperbolic radius 1) must converge to 0. Use a dilation and horizontal translation to transform P_j to $(1,0,0) \in \mathbb{H}^3$. The Y_j must then converge, locally in $\mathcal{C}^\infty$ around $(1,0,0)$, to the half-sphere $\{x^2+y^2+z^2 = 1, x \geq 0\}$, which contradicts the fact that Y_j locally lies above P_j. q.e.d.

By virtue of this lemma, we may as well now assume that each Y_j lies in the half-ball $\{x^2 + y^2 + z^2 \leq 1, x \geq 0\}$. The key idea in the rest of the proof is that there exists a sequence of good truncations $Y_j^\sharp := Y_j^{\theta_j} = Y_j \cap \{x \geq \theta_j\}$, with $\theta_j \geq c > 0$, determined by the requirement that $\left| \int_{\gamma_j^\sharp} \kappa\, ds \right| \leq C$, where $\gamma_j^\sharp = \partial Y_j^\sharp$ and κ is the geodesic curvature of this boundary in $Y_j^\sharp$. Indeed, suppose that we have determined these truncations, and suppose too that we have shown that the Euler characteristics of the $Y_j^\sharp$ remain bounded. Recalling from (1.2) that the Gauss curvature of $Y_j^\sharp$ satisfies

$$K = \frac{1}{2}|A|^2 - |\mathring{A}|^2 - 1,$$

then the Gauss-Bonnet formula

$$\int_{Y_j^\sharp} \frac{1}{2}|A|^2 - |\mathring{A}|^2\, d\mu_j - \mathrm{Area}(Y_j^\sharp) + \int_{\gamma_j^\sharp} \kappa\, ds_j = 2\pi\chi(Y_j^\sharp),$$

combined with the bounds on the energy, the boundary term, and the Euler characteristic, shows that the areas of the $Y_j^\sharp$ remain uniformly bounded.

Let us first explain how to find truncations with bounded Euler characteristic. The only way this could fail is if, for each $\eta > 0$, the surfaces Y_j have increasingly large numbers of boundary components as $j \to \infty$. This, in turn, could happen only if there were to exist a sequence of points $P_j \in Y_j$ with horizontal tangent and where Y_j lies above the tangent plane at P_j. (The idea is simply that each new boundary component of the truncation would bound a compact portion of the surface, on which the height function attains a minimum.) However, we have already showed in the lemma that this is impossible.

Now let us show that there are truncations for which the integral of geodesic curvature on the boundary remains bounded. For any constants $0 < a < b$, let $Y^{a,b} = Y \cap \{a \le x \le b\}$. We shall use the argument in Lemma 2.1 and the ϵ-regularity result [**15**, Theorem 2.1], which together imply that there exist small constants $\delta, \delta' > 0$ so that for any $0 < \delta'' < \delta'$, if $Y \in \mathcal{M}$ satisfies

$$\int_{Y^{\delta''/4,\delta''}} |A|^2 \, d\mu \le \delta, \tag{2.2}$$

then for any point $P \in Y^{\delta''/3,\delta''/2}$, Y is a graph over a disc of radius $\delta''/10$ lying in a vertical half-plane, where the $\mathcal{C}^k$ norm of the graph function is controlled by δ and $(\delta'')^{-1}$, and so that Y is almost vertical in the sense that $|\overline{g}(\overline{\nu}, \partial_x)| \le 1/10$, where $\overline{\nu}$ is the Euclidean unit normal vector to Y at P.

Having fixed the constants δ, δ', we must show that (2.2) holds for infinitely many Y_j, for some δ'' which is uniformly bounded below. We have already chosen one value of η which is suitable for the topological bound. Now fix $N \ge M/\delta + 1$ (where M is the energy bound), and divide the interval $[2^{-2N}\eta, \eta]$ into the union of $N-1$ subintervals $I_\ell = [2^{-2\ell}\eta, 2^{-2\ell+2}\eta]$. The sum of the energies of Y_j on each of these pieces is no larger than M, so for each j, the energy of Y_j on at least one of the I_ℓ is less than δ. Finally, choose ℓ so that infinitely many of the Y_j have energy less than δ on I_ℓ and relabel that subsequence as the whole sequence.

The $\mathcal{C}^2$ bounds of the vertical graph functions show that the geodesic curvature of any cut $Y_j \bigcap \{x = \theta\}$ in Y_j^θ, $\theta \in I_\ell$, is bounded; this fact, recalling that $Y_j \bigcap \{x = \theta\}$ are contained in a disc of fixed radius R yields a length bound for these curves. Thus we conclude at last that

$$\left| \int_{Y_j^\theta} \kappa \, ds \right| \le C$$

for any such θ, independently of j. q.e.d.

3. The ϵ-regularity results: Small energy controls boundary regularity

3.1. Vanishing energy implies $\mathcal{C}^1$ boundary convergence. We first state a key proposition, and then deduce Theorems 1.1 and 1.2 from it.

For any $\zeta \in (0, 1]$, consider the (unique) circle C_*^ζ in the yz-plane which is tangent to the y-axis at the origin and whose graph function $f_*^\zeta(y)$ over the interval $[-1, 1]$ satisfies $(f_*^\zeta)'(1) = \zeta$. Pick ζ_0 small enough so that for each $\zeta \in (0, \zeta_0]$ the circle C_*^ζ is contained in the open ball $B(0, \frac{5}{\zeta}) \subset \mathbb{R}^2$.

Proposition 3.1. *Let ζ and ζ_0 be as above. Suppose that Y_j is a sequence of connected Willmore surfaces in $\mathbb{H}^3 \bigcap B(0,2)$ with boundaries at infinity $\partial_\infty Y_j = \gamma_j$, and the remaining boundary components on the outer boundary of this half-ball. Assume $\int_{Y_j} |\bar{A}|^2 d\overline{\mu} \leq M < \infty$. We assume furthermore that:*

a) *Each γ_j is the graph of a function f_j over $[-1, 1]$, which satisfies $|f_j(y) - f_j(y')| \leq \zeta |y - y'|$ for all $y, y' \in [-1, 1]$ and $f_j(0) = 0$, $f_j'(0) = 0$, $f_j'(1) = \zeta$;*
b) $\mathrm{LipRad}^\zeta_{\gamma_j}(P) \geq \frac{2-|P|}{A}$, *for some fixed $A > 0$;*
c) $\mathcal{E}_p^{B(0,2)}(Y_j) \to 0$ *as $j \to \infty$.*

Then $f_j \to f_*^\zeta$ in $\mathcal{C}^1([-1, 1])$.

In other words, if the weighted energies of a sequence of Willmore surfaces converge to zero in some fixed half-ball, and if the boundaries at infinity of these Willmore surfaces are uniformly Lipschitz in the qualitative sense above, then these boundaries must converge to a particular circular arc defined by the normalization, and the convergence is actually in $\mathcal{C}^1$.

For future reference, we state another proposition which guarantees $\mathcal{C}^1$ convergence of boundary curves under slightly different assumptions. This will be used in the proof of Theorem 1.4 above.

Proposition 3.2. *Assume that Y_j is a sequence of connected Willmore surfaces in $\mathbb{H}^3 \bigcap B(0,2)$, with boundaries at infinity $\partial_\infty Y_j = \gamma_j$, and with all other boundaries contained in the outer boundary of the half-ball $B(0,2)$. Assume $\int_{Y_j} |\bar{A}|^2 d\overline{\mu} \leq M < \infty$. Assume also that*

a) *Y_j is the graph of a function $z = u_j(x,y)$ over the half-disc $\{x^2 + y^2 \leq 2, z = 0\}$.*
b) *$|\nabla u_j| \leq 2\zeta \leq 1/10$ for $x > 0$ and $f_j(y) := u_j(0,y)$ is a Lipschitz function with Lipschitz constant ζ.*
c) *$\mathcal{E}_p(Y_j) \to 0$, and Y_j converges (in the $\mathcal{C}^\alpha$ norm, $\alpha < 1$) to the upper half-disc $\{z = 0, x^2 + y^2 < 2\}$.*
d) *All f_j are differentiable at $y = 0$.*

Then $\lim_{j\to\infty} f_j'(0) = 0$.

3.2. Proposition 3.1 implies ϵ-regularity. We now show that Theorems 1.1 and 1.2 can be deduced from Proposition 3.1.

The argument is by contradiction. Assume that for every $j \geq 1$ there exist surfaces $Y_j \in \mathcal{M}$ and points $P_j \in \gamma_j := \partial Y_j$ (and radii $R_j \leq 1$ in the context of Theorem 1.1) such that $\mathcal{E}_p^{B(P_j,R_j)}(Y_j) < \frac{1}{j}$, yet

$\mathrm{LipRad}^\zeta_{\gamma_j}(Q_j) < \zeta\frac{R_j - |P_jQ_j|}{10}$ for some $Q_j \in \gamma_j \cap B(P_j, R_j)$. Observe that the points Q_j must lie in the *open* ball $B(P_j, R_j)$ since $\mathrm{LipRad}^\zeta(\gamma_j) > 0$.

Select a point $Z_j \in \gamma_j$ in the open ball $B(P_j, R_j)$ so that

$$\inf_Q \frac{\mathrm{LipRad}^\zeta_{\gamma_j}(Q)}{(R_j - |P_jQ|)} = \frac{\mathrm{LipRad}^\zeta_{\gamma_j}(Z_j)}{(R_j - |P_jZ_j|)},$$

and note that this ratio is less than $\zeta/10$. Let $\delta_j := \mathrm{LipRad}^\zeta_{\gamma_j}(Z_j)$. By translation and rotation, assume that $Z_j = 0$ and $T_{Z_j}\gamma_j$ is the y-axis. Now dilate by δ_j^{-1}. Denote the rescaled surface by $\tilde{Y}_j$ and the rescaled boundary curve by $\tilde{\gamma}_j$; note that $|\tilde{\gamma}_j| = 100\pi\delta_j^{-1}$. Thus $\tilde{\gamma}_j$ is a graph $z = f_j(y)$ over $[-1, 1]$, with $f_j(0) = 0$, $f_j'(0) = 0$, and $|f_j(y) - f_j(y')| \le \zeta|y - y'|$. Moreover, because $[-1, 1]$ is the maximal interval on which the Lipschitz norm of f_j is bounded by ζ, we must have either $|f_j'(-1)| = \zeta$ or $|f_j'(1)| = \zeta$, and to be definite we suppose that $f_j'(1) = \zeta$ for each j.

The translated and rescaled ball $\tilde{B}_j$ contains $B(0, \frac{5}{\zeta})$. Furthermore, by the choice of Z_j and the dilation, we see that there exists an $\eta > 0$ such that for each $P \in \tilde{\gamma}_j \bigcap B(0, \frac{5}{\zeta})$, $\mathrm{LipRad}^\zeta_{\tilde{\gamma}_j}(P) \ge \eta$.

We claim that $\tilde{\gamma}_j \to C^\zeta_*$ in $\mathcal{C}^1$. Assuming this for the moment, we show that this leads to a contradiction in Theorems 1.1 and 1.2.

For Theorem 1.2, the contradiction is immediate. Indeed, the curves $\tilde{\gamma}_j$ intersect the circle $\partial\tilde{B}_j$, which contradicts the fact that $\tilde{\gamma}_j \to C^\zeta_*$, which lies strictly in the interior of $\tilde{B}_j$.

As for Theorem 1.1, let $g_j : [-50\pi\delta_j^{-1}, 50\pi\delta_j^{-1}] \to \mathbb{R}^2$ parametrize $\tilde{\gamma}_j$ by arclength, so the length along the curve between $g_j(0)$ and $g_j(s)$ is $|s|$; similarly, let $g_* : [-50\pi\delta_j^{-1}, 50\pi\delta_j^{-1}] \to C^\zeta_*$ be a (multi-covering) arclength parametrization of C^ζ_*.

Observe that $|C^\zeta_*| = 2\pi R_\zeta$, with $R_\zeta = \sqrt{1 + \frac{1}{\zeta^2}}$. Our claim gives that $g_j(s) \to g_*(s)$ in $\mathcal{C}^1([-\pi R_\zeta, \pi R_\zeta])$, so in particular,

$$\lim_{j\to\infty} g_j(-\pi R_\zeta) = \lim_{j\to\infty} g_j(\pi R_\zeta), \quad \lim_{j\to\infty} g_j'(-\pi R_\zeta) = -\lim_{j\to\infty} g_j'(\pi R_\zeta).$$

This shows that $\lim_{j\to\infty} |\tilde{\gamma}_j| = |C^\zeta_*|$. On the other hand, we know that $|\tilde{\gamma}_j| = 100\pi\delta_j^{-1}$, which is impossible since $|C^\zeta_*| = 2\pi R_\zeta$.

Proof that Proposition 3.1 *implies* $\gamma_j \to C^\zeta_*$ *in* $\mathcal{C}^1$*:* Let $2\zeta'$ be the length of the arc in C^ζ_* which is a graph over the interval $y \in [-1, 1]$. By Proposition 3.1, $g_j(s) \to g_*(s)$ for $s \in [-\zeta', \zeta']$. (Note that the required bound on $\int_{Y_j} |\bar{A}|^2 d\overline{\mu}$ follows from the key assumption at the end of subsection 1.1.) Let $M > 0$ be the largest number in $[0, \pi R_\zeta]$ such that $g_j(s) \to g_*(s)$ in $\mathcal{C}^1([-M, M])$. We must prove that $M = \pi R_\zeta$, and moreover, for any small $\epsilon > 0$ and j sufficiently large, that there exists an $\epsilon_j > 0$ with $\lim_{j\to\infty} \epsilon_j = \epsilon$ and $g_j(-\pi R_\zeta - \epsilon_j) = g_j(\pi R_\zeta - \epsilon)$. The first claim ensures that $\gamma_j([-\pi R_\zeta, \pi R_\zeta]) \to C_*^\zeta$, while the second implies that $\gamma_j(s)$ closes up on a small extension of the interval $[-\pi R_\zeta, \pi R_\zeta]$.

The first part is proved by contradiction: Assume $M < \pi R_\zeta$, and consider the pairs $(g_j(M), g_j'(M))$. These converge to $(g_*(M), g_*'(M))$, so for j large they lie in the open set $\mathcal{U}$ where LipRad is bounded below by some $\eta > 0$. Now let ℓ_j be the tangent line to $\tilde{\gamma}_j$ at $g_j(M)$. Consider the intervals of length η centered at $g_j(M)$ on each ℓ_j. After translation, rotation, and dilation by the factor η^{-1}, the rescaled $\tilde{\gamma}_j$ can be written as the graphs of functions ϕ_j on $[-1, 1]$. Applying Proposition 3.1 to these functions, we see that $\phi_j \to f_*$ in $\mathcal{C}^1$. Hence $g_j \to g_*$ on a larger interval $[-M', M']$, which contradicts the maximality of M.

As for the second part of the claim, note that the argument above shows that for $|\tau| \leq \eta$ we have

$$\lim_{j\to\infty} (g_j(-\pi R_\zeta - \tau), g_j'(-\pi R_\zeta - \tau)) = \lim_{j\to\infty} (g_j(\pi R_\zeta - \tau), -g_j'(\pi R_\zeta - \tau)),$$

because of the lower bound $\mathrm{LipRad}^\zeta \geq \eta$ and the $\mathcal{C}^1$ convergence of the g_j on $[-\pi R_\zeta - \tau, -\pi R_\zeta + \tau]$ to an arc of C_*^ζ.

Now, assume that for some fixed $\epsilon > 0$, there exists a subsequence in j such that $g_j(-\pi R_\zeta - \epsilon) \neq g_j(\pi R_\zeta - s)$ for any $s \in (0, 2\epsilon)$. In particular this says that $g_j(t)$ does not 'close up' for $t \leq -\pi R_\zeta$ and $t \geq \pi R_\zeta$.

This gives a sequence of values $\tau_j \in [-50\pi\delta_j^{-1}, -\pi R_\zeta] \bigcup [\pi R_\zeta, 50\pi\delta_j^{-1}]$ such that $\tau_j \to \tau_*$, $g_j(-\tau_j) \to P$, with $P \in C_*^\zeta$, yet $g_j'(\tau_j) \to T_*$ for some vector T_* which is transverse to the tangent vector T of C_*^ζ at P. However, if j is large enough, then $g_j(\tau_j) \in \mathcal{U}$, and hence $\mathrm{LipRad}^\zeta_{\gamma_j}(g_j(\tau_j)) \geq \eta > 0$. But this implies that γ_j must *self-intersect* near P, which contradicts that the boundary curves are embedded. □

3.3. An overview of the strategy. In the next two sections, we prove Propositions 3.1 and 3.2. In a nutshell, both results show, in slightly different settings, that if the weighted energies of portions of the Willmore surfaces $Y_j \subset \mathbb{H}^3$ converge to zero, then $\gamma_j = \partial_\infty Y_j$ must converge *in the* $\mathcal{C}^1$ *norm* to the boundary curve γ_* of a totally geodesic surface Y_*. We stress that the convergence of the graphical portions of the surfaces Y_j to Y_* is $\mathcal{C}^\infty$ away from $\{x = 0\}$; the novelty here is the $\mathcal{C}^1$ convergence at the boundary.

Since the argument has several steps, we now provide a moderately detailed outline of the strategy. If the results were false, we could find a sequence of Willmore surfaces Y_j satisfying the hypotheses but for which the $\mathcal{C}^1$ convergence fails at some boundary point. Thus, having written the boundary curves graphically, we assume that there exists $y_0 \in [0, 1]$ such that $\lim_{j\to\infty} f_j'(y_0) = b_1 \neq b_2 = f_*'(y_0)$. Because the local energy converges to zero, the limit Y_* is totally geodesic, and the convergence is $\mathcal{C}^\infty$ away from $\{x = 0\}$. Furthermore, at $\{x = 0\}$, $f_j \to f_*$ in $\mathcal{C}^\rho, \rho < 1$ where the graph of f_* is a circular arc.

Compose with a suitable sequence of rotations, reflections, and inversions so that we can assume that $(y_0, f_j(y_0)) = (0, 0)$ and (maintaining the names of all surfaces and curves) that Y_* is a portion of the vertical plane $\{z = 0\}$. By assumption b) of Proposition 3.1, each γ_j is the graph of a function f_j defined on a fixed interval $[-1, 1]$, and the limiting curve γ_* is the graph of $f_* = 0$ on this same interval. The hypothesis is that $\lim_{j\to\infty} f_j'(0) = \alpha > 0$, although $f_*'(0) = 0$.

The argument proceeds in two steps. We first show that there exists a sequence of hyperbolic isometries φ_j such that the surfaces $\varphi_j(Y_j)$ satisfy all the assumptions of Propositions 3.1 and 3.2 (including the jump in the limit of the first derivatives), but so that some fixed portion of $\varphi_j(Y_j)$ is covered by isothermal coordinates, the associated conformal factor of which is uniformly bounded. This construction relies crucially on ideas in [**8**], many of which go back to the influential paper [**26**]. The work here will involve modifying some arguments in [**8**], which is possible because of some special features of our setting, to ensure that the jump in the first derivative has a fixed size $\alpha - \beta > \frac{\alpha}{2}$.

However, we then use particular properties of these isothermal coordinate systems to prove that no such jump in the limit of the first derivatives can occur. Writing $\varphi_j(Y_j)$ as the graphs of functions u_j, and denoting the isothermal coordinates by (q_j, w_j), the idea is to control $\partial_{w_j} u_j|_{(0,0)}$ using that $\partial_{w_j} u_j \to 0$ as $j \to \infty$ uniformly along $\{x = 1\}$. The relationship between these derivatives at $x = 0$ and $x = 1$ is obtained using two integrals, the first of the mixed component of the second fundamental form of $\varphi_j(Y_j)$ with respect to the Euclidean metric, and the second depending on a derivative of the conformal factor. We show that these integrals are bounded in terms of $\mathcal{E}_p(\varphi_j(Y_j))$ and hence converge to 0. The estimate for the first integral uses a realization of Willmore surfaces as harmonic maps into the (3+1)-dimensional deSitter space. In Euclidean coordinates, the energy integrand for this map turns out precisely to be the traceless second fundamental form $|\mathring{A}|^2$. This, together with the harmonic map equation and the strong subharmonicity of the distance function on our surfaces, yields bounds on $|\mathring{A}|$ which are *integrable* in x. It is at this point exactly that the boundedness of *weighted* Willmore energy (as opposed to the regular Willmore energy) is used. The control of the second integral follows from interpreting it as one term in a flux formula whose interior term is controlled by $\mathcal{E}(\varphi_j(Y_j))$.

Remark 3.1. The jump of the first derivative of γ_j can also be described in terms of the Euclidean coordinate function z restricted to the surface Y_j. Indeed, the jump condition is the same as

$$|\lim_{j\to\infty} \bar{\nu}_j(z) - \bar{\nu}_*(z)| = \alpha > 0, \tag{3.1}$$

where $\bar{\nu}_j$ and $\bar{\nu}_*$ are the Euclidean unit tangent vectors to $\partial_\infty Y_j$ and $\partial_\infty Y_*$ at $(0,0,0)$.

4. Uniform isothermal parametrizations

We now choose a sequence of hyperbolic isometries φ_j which map the surfaces Y_j to a new sequence of surfaces which satisfy the assumptions of our propositions (in particular they converge to a vertical half-plane) but such that some fixed portions of these rescaled surfaces admit isothermal coordinates (q_j, w_j), the conformal factors of which are uniformly bounded in $\mathcal{C}^0$, $W^{2,1}$, and $W^{1,2}$. We must also ensure that the transformed surfaces still exhibit a jump in first derivative at the origin. The main tool we employ is the work of Müller and Sverak [**26**] that provides the desired isothermal coordinates for complete graphs of finite total curvature.

This construction is used in the proofs of Propositions 3.1 and 3.2 in slightly different settings, so we prove the present result in two different settings as well. These involve different hypotheses on the boundary curves $\gamma_j = \partial_\infty Y_j$ (assumed as always to pass through the origin). The γ_j are $\mathcal{C}^1$ or Lipschitz, respectively, with uniform control on the norms, and in the second setting, we assume that γ_j is differentiable at the origin. Let us now describe these more carefully. In the following, and throughout the rest of this section, we write

$$D_+(a) = \{(x, y, 0) : x^2 + y^2 \leq a^2,\ x \geq 0\},$$
$$D(a) = \{(x, y, 0) : x^2 + y^2 \leq a^2\}$$

for the half-disk or disk of radius a in the vertical plane $\{z = 0\}$.

Setting 1: Y_j is a sequence of incomplete Willmore surfaces, where each Y_j is a horizontal graph $z = u_j(x, y)$ over $D_+(3)$ with $u_j \in \mathcal{C}^2$, $||u_j||_{W^{2,2}} \leq M < \infty$, $u_j(0,0) = 0$, and $\partial_y u_j(0,0) = \alpha > 0$. We assume that $|\nabla u_j|_{\overline{g}} \leq \zeta \leq 1/20$, and finally that $\mathcal{E}(Y_j) \leq \mu < 2\pi$ and $Y_j \to Y_* = D_+(3)$.

Setting 2: Y_j is a sequence of incomplete Willmore surfaces which are again horizontal graphs $z = u_j(x, y)$ over $D_+(3)$ with $u_j(0,0) = 0$ and $u_j \in W^{2,2}$, $||u_j||_{W^{2,2}} \leq M < \infty$, and $u_j \in \mathcal{C}^2$ away from $\{x = 0\}$. We assume that $(0,0)$ is a point of differentiability for u_j and $\partial_y u_j(0,0) = \alpha > 0$. We also assume that $y \mapsto u_j(0, y)$ is Lipschitz with constant $\zeta \leq 1/20$, and furthermore, $|\nabla u_j| \leq 2\zeta$ for $x > 0$. Finally, suppose that $\mathcal{E}(Y_j) \leq 2\pi$ and $Y_j \to Y_* = D_+(3)$.

Recalling that α is the jump in the derivative, choose any number β with $0 < \beta \ll \alpha$. Consider the straight line $\ell_\beta = \{z = \beta y\}$ in the horizontal plane $\{x = 0\}$. Since the curves γ_j converge to a segment in the y-axis containing the subinterval $[-1, 1]$, then for j large, there must exist values $-1 < y_j^- < 0 < y_j^+ < 1$ such that the two points $F_j^\pm = (0, y_j^\pm, u_j(y_j^\pm))$ both lie on the line ℓ_β. We assume that y_j^+ is chosen as large as possible in the interval $(0,1)$, and similarly for y_j^-. Since $\gamma_j = \mathrm{Graph}(u_j|_{x=0})$ converges to the line $\ell_0 = \{z = 0\}$, it is necessarily the case that $|F_j^\pm| \to 0$. Let $R_{-\beta}$ denote the rotation of the yz-plane by the small negative angle which sends ℓ_β to ℓ_0; thus $R_{-\beta}(F_j^\pm) = (\pm|F_j^\pm|, 0)$.

Suppose, to be definite, that $|F_j^+| \geq |F_j^-|$. Dilating the entire surface by the factor $|F_j^+|^{-1}$ pushes the point F_j^+ to $(1,0)$. The key observation is that this dilation of $R_{-\beta}Y_j$ converges to a vertical plane (since it must be totally geodesic and graphical over $\{z = 0\}$), and since this plane contains the two points $(0,0)$ and $(1,0)$, it *must* be $\{z = 0, x \geq 0\}$. This holds even though, before dilating, the sequence $R_{-\beta}Y_j$ converges to the vertical plane $\{y = -\beta z, x \geq 0\}$. Denote this dilated, rotated surface by $\tilde{R}_{-\beta}(Y_j)$. Note also that our assumed $W^{2,2}$ bound implies that $\int_{\tilde{R}_{-\beta}(Y_j)} |\bar{A}|^2 d\overline{\mu} \leq M$, and hence given $\gamma > 0$ and fixing $0 < \beta_- < \beta_+ \ll \alpha$, then for j large enough there exists some $\beta_j \in (\beta_-, \beta_+)$ such that

$$\int_{\tilde{R}_{-\beta_j}(Y_j)\bigcap\{1/4\leq x^2+y^2+z^2\leq 9\}} |\bar{A}|^2 d\overline{\mu} \leq \gamma. \tag{4.1}$$

Remark 4.1. By this observation, we can pick a sequence $\beta_j, 0 < \beta_- \leq \beta_j < \beta_+ \ll \alpha$ such that:

$$\int_{\tilde{R}_{-\beta_j}(Y_j)\bigcap\{1/4\leq x^2+y^2+z^2\leq 9\}} |\bar{A}|^2 d\overline{\mu} = o(1). \tag{4.2}$$

We make this choice hereafter.

For simplicity, now reset the notation and write the rotated dilated surfaces as Y_j, with boundary curves γ_j, graph functions u_j, etc.

Lemma 4.1. *Consider a sequence of incomplete Willmore surfaces Y_j which are graphs $z = u_j(x,y)$ over $D_+(3)$ with $|\nabla u_j| \leq 2\zeta$, $\mathrm{Lip}(u_j|_{x=0}) \leq \zeta$, $8\mathcal{E}(Y_j) \leq \pi$, $\int_{Y_j\cap\{1/4\leq x^2+y^2+z^2\leq 9\}} |\bar{A}|^2 d\overline{\mu} \to 0$, $u_j(0,0) = 0, u_j(0,1) = 0$, and $u_j \to 0$, where the convergence is in $\mathcal{C}^\infty$ away from $\{x = 0\}$ and in $\mathcal{C}^{0,\alpha}$ up to $x = 0$. Assume further that there is a jump in the first derivative at the origin:*

$$\lim_{j\to\infty} \partial_y u_j(0,0) - \partial_y u_*(0,0) \geq \alpha - 2\beta_j > \frac{1}{2}\alpha. \tag{4.3}$$

Then, setting $\mathcal{U}_j = \mathrm{Graph}(u_j)|_{D_+(2)}$, we claim that there exist relatively open domains $\mathcal{D}_j \subset \overline{\mathbb{R}^2_+}$ and conformal maps $\Psi_j : \mathcal{D}_j \to \mathcal{U}_j$, with $\Psi_j : \mathcal{D}_j \bigcap\{x = 0\} \to \mathcal{U}_j \bigcap\{x = 0\}$, and so that the conformal factor ϕ_j in

$$(\Psi_j)^*\overline{g}_j = e^{2\phi_j}\delta_{ij} \qquad (\delta_{ij} \text{ is the Euclidean metric}),$$

which is defined on $\mathcal{D}_j \subset \mathbb{R}^2_+$, *satisfies*

$$(4.4)\quad ||\phi_j||_{\mathcal{C}^0(\mathcal{D}_j)} + ||\phi_j||_{W^{1,2}(\mathcal{D}_j)} + ||\phi_j||_{W^{2,1}(\mathcal{D}_j)} \leq C \int_{\mathcal{D}_j} |\mathring{A}_j|^2 \, d\overline{\mu} + o(1).$$

Remark 4.2. Since Ψ_j is conformal, $|D\Psi_j(P)| = e^{\phi_j(P)}$ at any point P, which implies that the upper and lower bounds on $||\phi_j||_{\mathcal{C}^0}$ automatically bound from above and below the distortion of distance with respect to Ψ_j. In particular, the distance between $\Psi_j^{-1}(0,0,0)$ and $\Psi_j^{-1}(0,1,0)$ is uniformly bounded above and below by positive constants, and in addition, there exist $c, C > 0$ so that $D_+(0,c) \subset \mathcal{D}_j \subset D_+(0,C)$.

Remark 4.3. Note, for future reference, that we actually prove that the surfaces $\mathcal{U}_j \subset \mathrm{Graph}(u_j)|_{D_+(2)}$ are subregions of complete, smooth graphical surfaces $Y_j^\flat$ in $\mathbb{R}^3$ which are reflection-symmetric across $\{x = 0\}$. If $u_j^\flat(x,y)$ is the graph function of $Y_j^\flat$, then the (distorted) annular regions $\{(x, y, u_j^\flat(x,y)), 2 \leq \sqrt{x^2+y^2} \leq 4\}$ of these larger surfaces are *not* Willmore with respect to the hyperbolic metric. On the other hand, $u_j^\flat(x,y) = 0$ for $\sqrt{x^2+y^2} \geq 5$; we denote this portion of $Y_j^\flat$ by $Y_j^\sharp$. The isothermal coordinates (q_j, w_j) cover the entire surface $Y_j^\flat$, and the associated conformal factor ϕ_j satisfies (4.4) on all of $Y_j^\flat$ and $\phi_j \to 0$ as $\sqrt{x^2+y^2} \to \infty$. In the first setting above, $\mathcal{E}(Y_j^\flat) \to 0$.

Remark 4.4. We let q_j, w_j be the isothermal coordinates induced onto $\mathcal{U}_j$ by the map Ψ_j. In other words, letting x, y be the standard coordinates on $\mathbb{R}^2_+ := \{x \geq 0, y \in \mathbb{R}\}$, we write

$$(4.5)\qquad (q_j, w_j) = (\Psi_j)_*(x,y).$$

We also note that the pointwise bound on ϕ_j yields bounds on $|\nabla q_j|_{\overline{g}}$, $|\nabla w_j|_{\overline{g}}$. Indeed, dropping the subscript j momentarily, we have

$$(4.6)\quad \overline{g} = e^{2\phi}(dq^2 + dw^2) = (1+(u_x)^2)dx^2 + 2u_x u_y dx dy + (1+(u_y)^2)dy^2,$$

so in particular

$$\overline{g}(\partial_x, \partial_x) + \overline{g}(\partial_y, \partial_y) = 2 + u_x^2 + u_y^2 = e^{2\phi}(|\partial_x q|^2 + |\partial_y q|^2 + |\partial_x w|^2 + |\partial_y w|^2).$$

Using $|u_x|, |u_y| \leq 1/10$ and $|dq|_{\overline{g}} = |dw|_{\overline{g}}$, $dq \perp dw$, the equivalence of pointwise bounds on ϕ_j and $|\nabla q_j|_{\overline{g}}$ follows directly.

Proof. As described earlier, we apply a result of Müller and Sverak [**26**], which yields a *special* isothermal parametrization of a complete graphical surface of bounded total curvature. To reduce to this setting, consider the graph functions u_j in Lemma 4.1. Reflect these across the plane $\{x = 0\}$ and then modify them outside a disk of radius 2 to equal 0. The resulting surface is not Willmore outside the disk of radius 2, but the Müller-Sverak theorem yields a good isothermal parametrization, the restriction of which to the small disk is the one required in Lemma 4.1.

In any case, we obtain isothermal coordinates (q, w) which still detect the jump in the first derivative, and with $0 < C_1 \leq |\nabla q|, |\nabla w| \leq C_2$.

Reflection. We first reflect Y_j across the horizontal plane to obtain a surface Y_j' in $\mathbb{R}^3$ invariant with respect to the vertical reflection $x \mapsto -x$. The doubled surface is graphical over $D(3)$, and has graph function $\tilde{u}_j \in W^{2,2}(D(3))$. This is straightforward to check using Lemma 2.1. We change notation, denoting the doubled surface Y_j' by Y_j again.

Extension. We now claim that the doubled incomplete surface Y_j can be extended to a complete surface Y_j^{ext} which is a graph over the entire vertical plane $\{z = 0\}$ with graph function u_j^{ext} which vanishes when $x^2 + y^2 \geq 25$ and also satisfies

$$\mathcal{E}(Y_j^{\text{ext}}) \leq 2\mathcal{E}(Y_j) + o(1). \tag{4.7}$$

Notice that Y_j^{ext} is no longer Willmore in the transition annulus $4 \leq x^2 + y^2 \leq 9$.

This construction is a bit lengthy, so we defer it to the subsection below, but let us grant that for the moment.

Now invoke [**26**, Theorem 5.2] to obtain a conformal map

$$\Psi_j : \mathbb{R}^2 \longrightarrow Y_j^{\text{ext}}.$$

If $\overline{g}_0$ is the standard flat metric on $\mathbb{R}^2$ and $\overline{g}_j$ is the metric on Y_j^{ext} induced from the Euclidean metric in $\mathbb{R}^3$, then define ϕ_j by

$$\Psi_j^* \overline{g}_j = e^{2\phi_j} \overline{g}_0.$$

Then [**26**, Theorem 5.2] and the $W^{2,1} \mapsto W^{1,2}$ estimates immediately before it show that

$$\begin{aligned} ||\phi_j||_{\mathcal{C}^0(\mathbb{R}^2)} + ||\phi_j||_{W^{1,2}(\mathbb{R}^2)} + ||\phi_j||_{W^{2,1}(\mathbb{R}^2)} &\leq C \int_{Y_j^{\text{ext}}} |\mathring{\bar{A}}_j|^2 \, d\bar{\mu}_j \\ &\leq 2C\mathcal{E}(Y_j) + o(1). \end{aligned} \tag{4.8}$$

The second inequality here follows from (4.7) and (4.1). The restriction of Ψ_j to $\mathcal{U}_j := \Psi_j^{-1}(\text{Graph}(u_j)|_{D_+(2)})$ gives our claim.

As a brief hint of the idea of the proof of this fact (but see [**26**] for further details), ϕ_j is a solution of the semilinear elliptic PDE, $\Delta_{\overline{g}_0} \phi_j = \hat{K}_j e^{2\phi_j}$, where $\hat{K}_j$ is the Gauss curvature function on $M_{t_j}(\hat{Y}_j)$. The main term $\hat{K}_j e^{2\phi_j}$ on the right has a 'determinant structure', since it can be expressed via the pullback of the area form on S^2 by the Gauss map. This allows one to conclude that the right hand side lies in the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$, and from there the estimates follow from some important and well-known theorems in harmonic analysis. q.e.d.

Construction of the extension. We now prove the fact claimed in the proof of Lemma 4.1 that the reflected surface Y_j' can be extended to a graph over the entire plane $\{z=0\}$ in such a way that the increase of energy is controlled. This is straightforward using mollification. The point is that each of our surfaces is graphical with bounded tilt, so the total curvature is equivalent to the L^2-norm of the Hessian of its graph function. In particular, if $Y = \mathrm{Graph}(u)$ for $u \in \mathcal{C}^2(D')$, $D' = \{1/4 \leq x^2+y^2 \leq 9\}$, with $|\nabla u| \leq 2\zeta$, then

(4.9)
$$\frac{1}{(1+4\zeta^2)}\int_{D'} |\partial^2 u|^2 dxdy \leq \int_{Y'} |\bar{A}|^2 d\overline{\mu} \leq (1+4\zeta^2)\int_{D'} |\partial^2 u|^2 dxdy.$$

Lemma 4.2. *Let u be a $W^{2,2}$ function defined on the half-disc $D_+(3) := \{\sqrt{x^2+y^2} \leq 3, x > 0\}$. If $Y = \mathrm{Graph}(u)$, then write*

$$\int_Y |\mathring{\bar{A}}|^2 d\bar{\mu} := \mathcal{E}, \quad \int_{Y\cap 1/2\leq\sqrt{x^2+y^2}\leq 3} |\bar{A}|^2 d\bar{\mu} := \mathcal{E}'$$

and assume that $|\nabla u| \leq 1$, and in addition

1) *there exist $\epsilon, \delta > 0$ such that $|\nabla u(P)| \leq \delta$ for all $P \in D' \bigcap \{x \geq \epsilon\}$;*
2) *for any $P \in \overline{D'} \cap \{x = 0\}$, and any sequence $P_j \in D'$ with $P_j \to P$, we have $\lim_{j\to\infty} \partial_x u(P_j) = 0$.*

Let U be the even extension of u to $D = \{\sqrt{x^2+y^2} \leq 3\}$. Then there exists a function $\overline{u}$ such that $\overline{u} = U$ on $\{\sqrt{x^2+y^2} \leq 1\}$, $\overline{u} = 0$ on $\{\sqrt{x^2+y^2} \geq 5\}$, and if we let $\overline{Y} := \mathrm{Graph}(\overline{u})$ then $\int_{\overline{Y}} |\mathring{\bar{A}}_{\overline{Y}}|^2 \leq 2\mathcal{E} + 1000(\delta+\epsilon) + 10\mathcal{E}'$.

By Remark 4.1 and the fact that the Y_j converge locally in $\mathcal{C}^\infty$ to a vertical half-plane away from $\{x = 0\}$, this lemma then implies the claim on extension from above.

Proof. First note that if $u \in W^{2,2}$, then using the fact that $\partial_x u = 0$ on $\{x = 0\}$, we have $U \in W^{2,2}$. Furthermore, using the formula for the second fundamental form of a graph $z = U(x,y)$, we have

$$\int_{Y'} |\mathring{\bar{A}}|^2 d\overline{\mu} = 2\mathcal{E}.$$

To construct the extension, fix a smooth cutoff function $\chi(x,y) \in \mathcal{C}_0^\infty(B_1(0))$ with $\int_{\mathbb{R}^2} \chi dxdy = 1$, and such that $|\partial\chi| \leq 10, |\partial^2\chi| \leq 100$. Given any $\rho > 0$, we let $\chi_\rho := \rho^{-2}\chi(\frac{x}{\rho}, \frac{y}{\rho})$. We work in polar coordinates $r := \sqrt{x^2+y^2}$, $\theta := \arctan(y/x)$.

Define a function $u^\sharp(r,\theta)$ which equals $u(r,\theta)$ for $r \leq 5/2$, and which vanishes for $r > 5/2$. In addition, let $\psi(r)$ be a $\mathcal{C}^\infty$ function which vanishes when $r \leq 1$, equals 2 for $r \geq 3$, is strictly monotone increasing in the interval $[1,3]$, and satisfies $|\psi'(r)| \leq 10, |\psi''(r)| \leq 100$. Then define the function

$$\overline{u}(r,\theta) := (u^\sharp * \chi_{\psi(r)})(r,\theta),$$

where χ_0 is understood as the δ function. It is straightforward that $\overline{u}$ is $\mathcal{C}^2$ away from $\{x=0\}$, and it is also obvious that $\overline{u}(r,\theta)=0$ for $r\geq 5$ and that $|\nabla\overline{u}|\leq 1$ throughout $\mathbb{R}^2$. What remains is to show that the surface $Y=\mathrm{Graph}(\overline{u})$ satisfies the claims of our lemma.

To do this, we recall some facts about the Hardy-Littlewood maximal functions. For $\chi\in\mathcal{C}_0^\infty$ and $f\in L^1_{loc}(\mathbb{R}^2)$, define

$$M(f)(x) := \sup_{\rho>0}|(f*\chi_\rho)(x)|.$$

Then (for an appropriate choice of cutoff function χ),

$$||M(f)||_{L^2}\leq 10||f||_{L^2}. \tag{4.10}$$

Using (4.10) and (4.9) we derive:

$$\int_{Y\bigcap\{1\leq r\leq 2\}}|\bar{A}|^2 d\overline{\mu}\leq 20\mathcal{E}'.$$

This implies immediately that

$$\int_{Y\bigcap\{r\leq 2\}}|\mathring{\bar{A}}|^2 d\overline{\mu}\leq 2\mathcal{E}+20\mathcal{E}'.$$

Thus matters are reduced to estimating $\int_{Y_j\bigcap\{r\geq 2\}}|\mathring{\bar{A}}|^2 d\overline{\mu}$. Given (4.6), it suffices to control:

$$\int_{\{r\geq 2\}}|\partial^2\overline{u}|^2 dxdy.$$

We use the formula $\partial^2\overline{u}(P)=[\partial^2(\chi_{\psi(r(P))})*u^\sharp](P)$. Using the pointwise bounds on $\partial^j\chi_r$, $\partial^j\psi$, $j=0,1,2$ and on $|u|$, we directly derive that:

$$\int_{Y\bigcap\{2\leq r\leq 5\}}|\bar{A}|^2 d\overline{\mu}\leq 1000\int_{\mathbb{R}^2\bigcap\{2\leq r\leq 5\}}|\overline{u}|^2 dxdy\leq 1000(\epsilon+\delta).$$

Finally, using the definition of $u^\sharp$, we deduce that $\int_{Y\bigcap\{5\leq r\}}|\bar{A}|^2 d\overline{\mu}=0$.

q.e.d.

5. The key estimates: Small weighted energy in a half-ball controls $\mathcal{C}^1$ regularity

We now prove Propositions 3.1 and 3.2. In both cases, consider the sequence of (incomplete, graphical) Willmore surfaces furnished by Lemma 4.1, but dilate these further so that they are all graphical over $D_+(3)$. We denote by $\hat{Y}_j$ the image of the original surfaces under the Möbius transformation in the previous subsection.

Recall also the connection between the bounds on the conformal factor ϕ_j and on $|dq_j|_{\overline{g}}$ and $|dw_j|_{\overline{g}}$, as explained in Remark 4.3. Recall equation (4.4) and choose j large enough so that $||\phi_j||_{\mathcal{C}^0} \leq 10^{-2}$; this implies that the image of the entire rectangle $0 \leq q_j \leq 1, |w_j| \leq 1$ lies in Y_j. Furthermore, since $Y_j^\flat$ converges to a vertical plane Y_* (see Remark 4.3), and since q_j has gradient bounded above and below and vanishes along $\{x = 0\}$, and $\Delta_{\overline{g}_j} q_j = 0$, there is a subsequence of the q_j converging to a harmonic function of linear growth which vanishes at $x = 0$. The only possible limit is λx for some $\lambda \in [C^{-1}, C]$. As before, the convergence is $\mathcal{C}^\infty$ away from $\{x = 0\}$.

Again, using the $\mathcal{C}^0$ and $W^{1,2}$ bounds for ϕ_j in (4.4), we may replace coordinate derivatives by covariant derivatives in these bounds, at worst only increasing the constant.

We can now prove the main analytic estimate, which shows that the energy of Y_j controls the jump in the first derivative of the boundary curve of Y_j at the origin.

In the following, we often write ∂_1 and ∂_2 for the coordinate vector fields ∂_{q_j}, ∂_{w_j} on Y_j, and also set $\bar{A}_j(\partial_1, \partial_2) = (\bar{A}_j)_{12}$, or simply $\bar{A}_{12}$. Since the coordinate function z equals u_j on Y_j, letting ν be the (Euclidean) unit normal to Y_j we have

$$(\bar{A}_j)_{12}\nabla_\nu z = \nabla_{12} u_j = \partial_{12} u_j - \partial_1 u_j\, \partial_2\phi_j - \partial_1\phi_j\, \partial_2 u_j. \tag{5.1}$$

The two equalities are just specializations of basic definitions to this situation. Now drop the subscript j for simplicity. Multiply the equation by $e^{-\phi}$. Noting that $e^{-\phi}(\partial_{12}u - \partial_1\phi\partial_2 u) = \partial_1(e^{-\phi}\partial_2 u)$, then integrating along the line segment $0 \leq q \leq 1$, $w = 0$ gives

$$\begin{aligned}(e^{-\phi}\partial_2 u)(1,0) - (e^{-\phi}\partial_2 u)(0,0) &= \int_{(0,0)}^{(1,0)} e^{-\phi}\bar{A}_{12}\nabla_\nu u\, dq \\ &\quad + \int_{(0,0)}^{(1,0)} e^{-\phi}\partial_2\phi\partial_1 u\, dq.\end{aligned} \tag{5.2}$$

We now state our main estimate.

Proposition 5.1. *Set $\mathcal{E}_j := \mathcal{E}(Y_j)$ and $\mathcal{E}_{j,p} = \mathcal{E}_p(Y_j)$. Then there exist constants C, C' such that*

$$\int_{(0,0)}^{(1,0)} \left|(\bar{A}_j)_{12}e^{-\phi_j}\right| dq_j + \int_{(0,0)}^{(1,0)} e^{-\phi_j}\partial_2\phi_j\partial_1 u_j\, dq_j \leq C\sqrt{\mathcal{E}_{j,p}} + C'\sqrt{\mathcal{E}_j} + o(1). \tag{5.3}$$

Before proving this, let us explain how it leads to a contradiction, thus establishing Propositions 3.1 and 3.2. In view of the $\mathcal{C}^\infty$ convergence of $u_j \to u_*$ and $(q_j, w_j) \to (q_*, w_*)$ (both for $x > 0$),

$$|e^{-\phi_j}\partial_{w_j} u_j(1,0) - e^{-\phi_*}\partial_{w_*} u_*(1,0)| \leq \alpha/10 \tag{5.4}$$

for j sufficiently large. Furthermore,

$$e^{-\phi_j}\partial_{w_j} u_j(1,0) - e^{-\phi_j}\partial_{w_j} u_j(0,0) = \int_0^1 \partial_1(e^{-\phi_j}\partial_2 u_j)\, dq_j, \tag{5.5}$$

and since Y_* lies in the plane $\{z = 0\}$, we also have

$$e^{-\phi_*}\partial_{w_*}u_*(1,0) - e^{-\phi_*}\partial_{w_*}u_*(0,0) = 0. \tag{5.6}$$

Proposition 5.1 then yields that for large enough j,

$$\int_{(0,0)}^{(1,0)} \partial_{q_j}(e^{-\phi_j}\partial_{w_j}u_j)\, dq_j \le \alpha/10. \tag{5.7}$$

Combining these facts along with $|e^{-\phi_j}\partial_{w_j}u_j - \partial_y u_j| \le \frac{1}{10}|\partial_y u_j|$, which holds since $|\nabla u_j| < 2\zeta \le 1/10$ and $|\partial_{w_j} - \partial_y|_{\overline{g}}$ is also small, we conclude that

$$|\partial_y u_j(0,0) - \partial_y u_*(0,0)| \le \alpha/3 \tag{5.8}$$

when j is large, which contradicts (4.3).

5.1. Regularity from the interior: the two line integrals. Proposition 5.1 is a consequence of the following two results:

Proposition 5.2. *With all notation as above, suppose that $||\phi_j||_{\mathcal{C}^0(Y_j)} \le K$ for all j. Then there exists a constant $C(K) > 0$ such that for each point $P \in Y_j$ with $q_j(P) \in [0,1]$ and $w_j(P) = 0$ we have*

$$|(\mathring{\bar{A}}_j)_{12}|(P) \le C(K)\frac{\sqrt{\int_{B^2(P)} |\mathring{\bar{A}}_j|^2 f_j^{2p}\, d\bar{\mu}}}{U_j(q_j)}, \tag{5.9}$$

where $B^2(P)$ is the (intrinsic) ball of radius 2 centered at P and the functions $U_j(q_j)$ satisfy $\int_0^1 \frac{dq_j}{U_j(q_j)} \le M' < \infty$ for some uniform constant M'.

Proposition 5.3. *For some constants C, C' independent of j, we have*

$$\int_{(0,0)}^{(1,0)} e^{-\phi_j}\partial_2\phi_j\partial_1 u_j\, dq_j \le C\int_{Y_j} |\mathring{\bar{A}}_j|^2\, d\bar{\mu} + C'\sqrt{\int_{Y_j} |\mathring{\bar{A}}_j|^2\, d\bar{\mu}} + o(1). \tag{5.10}$$

These are proved in the remaining subsections of §5.

Remark 5.1. In our setting, the ϵ-regularity result [**27**, Theorem I.5] applied to intrinsic discs of radius 1 in Y_j (with respect to g_j) yields that $|\mathring{A}_j| \le C\sqrt{\mathcal{E}_j}$, which implies that $x|\mathring{\bar{A}}_j| \le C\sqrt{\mathcal{E}_j}$. Using that $0 < C_1 \le q_j/x < C_2$, which we have already noted follows from the upper and lower bounds on $||\phi_h||_{\mathcal{C}^0}$, we see that $q_j|\mathring{\bar{A}}_j| \le C'\sqrt{\mathcal{E}_j}$. Therefore, Proposition 5.2 actually shows that assuming bounded *weighted* energy yields a stronger pointwise decay estimate for $|\mathring{\bar{A}}_j|$.

5.2. Proof of Proposition 5.2. The argument relies on obtaining pointwise control on $|\mathring{A}|$ at each point on the segment $\{0 \leq q_j \leq 1, w_j = 0\}$ using the weighted energy of Y_j on a ball of (hyperbolic) radius 1 around that point.

To this end, we use a well-known realization of Willmore surfaces as harmonic maps into the $(3+1)$-dimensional deSitter space $(dS_{1,3}, h)$, which we regard as a hypersurface in the $(4+1)$-dimensional Minkowski space $\mathbb{R}^{1,4}$. This is useful since the norm of this map, $|d\Phi|^2$, is precisely equal to $|\mathring{A}|^2$.

Willmore surfaces as harmonic maps. Consider the (incomplete) Willmore surfaces $Y_j \subset \mathbb{R}^3_+ \subset \mathbb{R}^3$, equipped with the isothermal coordinates $q_j \in [0,1], w_j \in [-1,1]$. For simplicity, denote Y_j as Y for the moment. Let $\overline{g}$, $\overline{\nabla}$, and $\overline{\Delta}$ be the induced Euclidean metric, connection, and corresponding Laplacian. The Willmore surface in $\mathbb{R}^3$ determines a unique conformal harmonic map

$$\Phi : Y \to (dS_{1,3}, h) \subset \mathbb{R}^{1,4}$$

(see [**12**] for details). Using coordinates (t, x^1, x^2, x^3, x^4) so that the Minkowski metric on $\mathbb{R}^{1,4}$ is $g_{\mathrm{Mink}} := -dt^2 + \sum (dx^j)^2$, then $dS_{1,3} = \{-t^2 + \sum (x^j)^2 = 1\}$. We recall first that

$$\frac{1}{2}|\mathring{A}|^2 = (d\Phi)^i_\alpha (d\Phi)^j_\beta \overline{g}^{\alpha\beta} h_{ij} := |d\Phi|^2. \tag{5.11}$$

Since harmonic maps from 2-dimensional domains are conformally invariant, we may as well use the flat metric $g_{\mathbb{E}^2} := dq^2 + dw^2$ on Y rather than $e^{2\phi}(dq^2 + dw^2)$. Observe that $|d\Phi|^2_{\mathbb{E}^2} = e^{-2\phi}|d\Phi|^2$, so if ϕ is bounded above and below, then $|T|_{\mathbb{E}^2}$ and $|T|$ are comparable; in particular, if $e^{|\phi|} \leq \sqrt{2}$, then

$$\frac{1}{2}|\mathring{A}|^2 \leq |d\Phi|^2_{\mathbb{E}^2} \leq 2|\mathring{A}|^2. \tag{5.12}$$

Now recall that

(5.13)

$$\Delta_{\mathbb{E}^2}|d\Phi|^2_{\mathbb{E}^2} = 2|\nabla d\Phi|^2_{\mathbb{E}^2} + (\mathrm{Riem}_{dS})_{ijkl}(d\Phi)^i_\alpha (d\Phi)^j_\beta (d\Phi)^k_\gamma (d\Phi)^l_\delta (g_{\mathbb{E}^2})^{\alpha\gamma}(g_{\mathbb{E}^2})^{\beta\delta},$$

which is the special case of the Bochner-type formula for any harmonic map ([**13**, Eqn. (8.7.13)]). We recall the Riemann curvature tensor of deSitter space:

$$(\mathrm{Riem}_{dS})_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk}).$$

Also, since Φ is conformal (and the metric induced by h on $\Phi(Y)$ is Riemannian), $d\Phi(\partial_q)$, $d\Phi(\partial_w)$ are orthogonal and have the same length; hence

$$\begin{aligned} &h_{ik}h_{jl}(d\Phi)^i_\alpha (d\Phi)^j_\beta (d\Phi)^k_\gamma (d\Phi)^l_\delta (g_{\mathbb{E}^2})^{\alpha\gamma}(g_{\mathbb{E}^2})^{\beta\delta} \\ &= 2h_{il}h_{jk}(d\Phi)^i_\alpha (d\Phi)^j_\beta (d\Phi)^k_\gamma (d\Phi)^l_\delta (g_{\mathbb{E}^2})^{\alpha\gamma}(g_{\mathbb{E}^2})^{\beta\delta}. \end{aligned}$$

In particular, (5.13) implies that:

$$\Delta_{\mathbb{E}^2}|d\Phi|^2_{\mathbb{E}^2} = 2|\nabla d\Phi|^2_{\mathbb{E}^2} - (|d\Phi|^2_{\mathbb{E}^2})^2. \tag{5.14}$$

Remark 5.2. The sign of the second term on the right here is what makes it necessary to assume finiteness of the *weighted* energy. Indeed, this formula seems to imply that finiteness of some sort of weighted energy is actually necessary. As in the introduction, even for scalar harmonic functions Φ in $\mathbb{R}^2_+$, $\int_{\mathbb{R}^2_+} |d\Phi|^2\, dxdy < \infty$ does not imply that $||\Phi||_{L^\infty(\mathbb{R}^2_+)} < \infty$. The sign of $|d\Phi|^4_{\mathbb{E}^2}$ suggests that finiteness of $\int |d\Phi|^2$ is even less effective in controlling $\sup|\Phi|$ than in the linear case. In particular, the theorems above would fail if we only assume finiteness of the (unweighted) energy.

For future reference, note that by Remark 5.1 we have

$$q_j^2|d\Phi|^2_{\mathbb{E}^2} \leq 1. \tag{5.15}$$

Modified weight function and the isothermal parametrization: Now recall the weight function f. We need to replace f by a new function $\tilde{f}_j$ (one for each j) which is better suited for computations; the weighted energy with respect to the new weight function is still bounded. After isolating the part Y_j in the original Willmore surface $\hat{Y}_j$ which admits good isothermal coordinates, some poles lie in Y_j and others lie in $\hat{Y}_j\backslash Y_j$. We shall modify the weight function f to omit those poles which do not lie in Y_j, but the justification for this requires some preliminary estimates.

Lemma 5.1. *Consider $Y \subset \mathbb{H}^3$, and assume that the connected component of $Y \bigcap B_+(0,3)$ containing the origin is a graph $z = u(x,y)$ over the half-disc $D_+(3)$, with $|\nabla u| \leq 1$. Then for points $B \in Y \setminus \mathrm{Graph}(u)$ and $A_x := (x, 0, u(x,0)) \in \mathrm{Graph}(u)$, with $x \leq 1 + 10^{-1}$, we have*

$$d_g(B, A_x) \geq |\log x|.$$

Proof. The proof is elementary: if $B = (x_0, y_0, z_0)$, then set $\tilde{B} = (x_0, y_0, 0)$ and $\tilde{A}_x = (x, 0, 0)$. It suffices to check that $d_{\mathbb{H}^3}(\tilde{B}, \tilde{A}_x) \geq |\log x|$. The geodesic $\gamma(\tilde{B}, \tilde{A}_x)$ joining $\tilde{B}, \tilde{A}_x$ is a circular arc. If $x_0 \geq 1 + 10^{-1}$, the claim is obvious since $d_{\mathbb{H}^3}(\tilde{B}, \tilde{A}_x) \geq d_{\mathbb{H}^3}((x_0, 0, 0), \tilde{A}_x) \geq |\log x|$. If $x_0 \leq 1 + 10^{-1}$, however, then since $(x_0)^2 + (y_0)^2 \geq 4$, this circular arc must intersect the line $\{x = 1 + 10^{-1}, y = 0\}$; we can apply the previous argument. q.e.d.

Now consider the *hyperbolic* metric $\mathbf{g}_{\mathbb{H}} := q_j^{-2}((dq_j)^2 + (dw_j)^2)$ on Y_j. This is conformal to the metric g_j induced by the embedding $Y_j \subset \mathbb{H}^3$; indeed,

$$\frac{x^2 e^{-2\phi_j}}{(q_j)^2} g_j = g_{\mathbb{H}}.$$

Quantities computed with respect to this metric will be labelled with an $\mathbb{H}$. In particular, with $\mathcal{B}_j = \{0 \le q_j \le 1, -1 \le w_j \le 1\} \subset Y_j$, then for any $P \in \mathcal{B}_j$, we write $B^R(P)$ and $B^R_{\mathbb{H}}(P)$ for the balls around P of radius R with respect to d_{g_j} and $d_{\mathbb{H}}$.

The bounds that $\sup|\phi_j|, \sup|\nabla q_j| \le 1 + 10^{-1}$, from the requirement at the beginning of this section, give upper and lower bounds on q_j/x (in particular $x \le q_j(1+10^{-1})$), which imply that

$$|d_{\mathbb{H}}(P,Q) - d_{g_j}(P,Q)| \le 1 \qquad \forall\ P,Q \in Y_j. \tag{5.16}$$

Lemma 5.1 implies that for each point $P \in \{0 \le q_j \le 1, w_j = 0\}$ and each pole $O_k \in \hat{Y}_j \setminus Y_j$, we have

$$d_{g_j}(O_k, P) \ge |\log x(P)|, \tag{5.17}$$

and hence also

$$d_{g_j}(O_k, P) + 1 \ge |\log q_j(P)|, \tag{5.18}$$

using the upper and lower bounds on $\frac{q_j}{x}$ in $\mathcal{B}_j$.

These considerations make it natural to modify the weight function f slightly. Thus, define the new function $\tilde{f}_j$ on $\mathcal{B}_j$ by setting $\tilde{f}_j(Q) = f_j(Q)$ if $Q \sim O_k$, provided $O_k \in \mathcal{B}_j$, and $\tilde{f}_j(Q) := |\log(q_j(Q))| + 5$ otherwise. Denote the weighted energy associated to $\tilde{f}$ by $\tilde{\mathcal{E}}_p$, and observe that by (5.17) and (5.18),

$$\tilde{\mathcal{E}}_p[Y_j] = \int |\mathring{A}_j|^2 \tilde{f}_j^2 d\mu_j \le 2\mathcal{E}_p[Y_j].$$

Proposition 5.4. *On the segment $\{0 \le q_j \le 1, w_j = 0\}$, we have*

$$|\mathring{A}_j|_{\mathbb{E}^2}(P) \le 4\frac{\sqrt{\tilde{\mathcal{E}}_p^{B^1_{\mathbb{H}}(P)}[Y_j]}}{\tilde{f}_j^p(P) q_j(P)}.$$

This will be proved below.

Proposition 5.4 *implies Proposition 5.2:* Assume (passing to a subsequence) that there are K poles O_i in $\mathcal{B}_j$. There is an obvious bound

$$\begin{aligned} 1/\tilde{f}_j^p(P) &\le \frac{N-K}{(-\log q_j(P)+5)^p} + \frac{1}{(\min_{i\le K} \mathrm{d}_{g_j}(P,O_i)+5)^p} \\ &\le \frac{N-K}{(-\log q_j(P)+5)^p} + \sum_{i=1}^K \frac{1}{(\mathrm{d}_{g_j}(P,O_i)+5)^p}. \end{aligned} \tag{5.19}$$

Observe that

$$\int_0^1 \frac{1}{q(|\log q|+5)^p} dq = \frac{5^{1-p}}{p-1}.$$

Hence it suffices to obtain uniform bounds for the individual integrals

$$\int_0^1 \frac{1}{q_j[\mathrm{d}_{g_j}(\cdot, O_i)+5]^p}\, dq_j,\ i \le K.$$

Consider the poles $O_i \in \mathcal{B}_j$ and set $q_j(O_i) := \delta_{j,i} \in (0,1]$. Equation (5.16) implies that $d_{g_j}(P, O_i) + 5 \geq |\log q_j(P) - \log \delta_{j,i}| + 4$, so we finish the proof by noting that

$$\int_0^1 \frac{1}{q_j(|\log q_j - \log \delta_{j,i}| + 4)^p}\, dq_j \leq \frac{2}{p-1}.$$

Proof of Proposition 5.4. We begin by recalling a mean value inequality:

Lemma 5.2 ([**25**])**.** *There exists a constant $C > 0$ such that if F is any positive $\mathcal{C}^2$ function on the ball $B^1_{\mathbb{H}}(P) \subset \mathbb{H}$ satisfying $\Delta_{\mathbb{H}} F \geq -2F$, then*

$$F(P) \leq C \int_{B^1_{\mathbb{H}}(P)} F\, d\mu_{\mathbb{H}}.$$

To use this, we modify $\tilde{f}_j$ slightly further: suppose that $P \in Y_j$ and $P \sim O_k$; if $O_k \in \mathcal{B}_j$ we define $\overline{f}_j(Q) := d_{\mathbb{H}}(Q, O_k) + 5$ for $Q \in B^1_{\mathbb{H}}(P)$, while if $O_k \notin \mathcal{B}_j$ (so $\tilde{f}_j(P) = -\log(q_j(P)) + 5$) then $\overline{f}_j(Q) = -\log(q_j(Q)) + 5$ for $Q \in B^1_{\mathbb{H}}(P)$. In other words, if $Q \in B^1_{\mathbb{H}}(P)$, then $\overline{f}_j(Q)$ either equals the $\mathbb{H}$-distance to the pole closest to P, or else, if the nearest pole does not lie in $\mathcal{B}_j$, it equals $-\log q_j + 5$.

Observe that $\tilde{f} \leq \overline{f} + 1 \leq 2\overline{f}$ on $B^1_{\mathbb{H}}(P)$. To compare these functions in the other direction, suppose $Q \in B^1_{\mathbb{H}}(P)$, with $P \sim O_k$ and $Q \sim O_r$. Using (5.16) and the triangle inequality, we find

$$\begin{aligned}
&\overline{f}(Q) - 5 = d_{\mathbb{H}}(Q, O_k) \leq d_{\mathbb{H}}(Q, P) + d_{\mathbb{H}}(P, O_k) \leq 2 + d_{g_j}(P, O_k) \\
&\leq 2 + d_{g_j}(P, O_r) \leq 2 + d_{g_j}(P, Q) + d_{g_j}(Q, O_r) \leq 3 + d_{g_j}(Q, O_r) \leq \tilde{f}(Q),
\end{aligned}$$

and hence

$$\overline{f}_j(Q) \leq 2\tilde{f}_j(Q). \tag{5.20}$$

One consequence is that

$$\int_{B^1_{\mathbb{H}}(P)} |A_j|^2 \overline{f}_j^{2p}\, d\mu_{\mathbb{H}} \leq 4^p \int_{B^1_{\mathbb{H}}(P)} |A_j|^2 \tilde{f}_j^{2p}\, d\mu_{\mathbb{H}}. \tag{5.21}$$

In any case, we have proved that $1/2 \leq |\overline{f}_j/\tilde{f}_j| \leq 2$ in $B^1_{\mathbb{H}}(P)$, and so it suffices to prove Proposition 5.4 with $\overline{f}_j$ replacing $\tilde{f}_j$.

We now prove this proposition. We first claim that $(\Delta_{\mathbb{H}} \overline{f}_j)\overline{f}_j - 5|\nabla \overline{f}_j|^2_{\mathbb{H}} \geq 0$. Indeed, in the region where $\overline{f}_j = d_{\mathbb{H}}(O_k, \cdot) + 5$, then $|\nabla \overline{f}_j|_{\mathbb{H}} = 1$, and the differential inequality follows from the standard formula $\Delta_{\mathbb{H}} \overline{f}_j = \coth(\overline{f}_j - 5)$. On the other hand, when $\overline{f}_j = -\log q_j + 5$, it follows by calculating that

$$(\Delta_{\mathbb{H}} \overline{f}_j)\overline{f}_j - 5|\nabla \overline{f}_j|^2_{\mathbb{H}} = (-\log q_j + 5 - 5) \geq 0, \tag{5.22}$$

since $q_j \in (0,1]$. In any case, using (5.14), (5.22), as well as the Cauchy-Schwarz inequality to handle the cross term in the derivative, we derive

(5.23)
$$\begin{aligned}
&\Delta_{\mathbb{E}^2}(|d\Phi_j|^2_{\mathbb{E}^2}\overline{f}_j^{2p}) \\
&= \overline{f}_j^{2p-2}\Big(\Delta_{\mathbb{E}^2}(|d\Phi_j|^2_{\mathbb{E}^2})\overline{f}_j^2 + |d\Phi_j|^2_{\mathbb{E}^2}(2p(\Delta_{\mathbb{E}^2}\overline{f}_j)\overline{f}_j \\
&\quad + 2p(2p-1)|d\Phi_j|^2_{\mathbb{E}^2}|\nabla\overline{f}_j|^2_{\mathbb{E}^2}) + 8p\langle\nabla d\Phi, d\Phi\rangle\cdot\overline{f}_j\nabla\overline{f}_j\Big) \\
&\geq \overline{f}_j^{2p-2}\Big(-|d\Phi|^4_{\mathbb{E}^2}\overline{f}_j^2 + (10p + 2p(2p-1) - 8p^2)|d\Phi_j|^2_{\mathbb{E}^2}|\nabla\overline{f}_j|^2_{\mathbb{E}^2}\Big) \\
&\geq -|d\Phi_j|^4_{\mathbb{E}^2}\overline{f}_j^{2p}
\end{aligned}$$

because $10p + 2p(2p-1) - 8p^2 = -4p^2 + 8p \geq 0$, which holds since $p \in (1,2]$. The reader should compare this with the model version of this calculation (1.5). Using the conformal covariance of the Laplacian and (5.15) this implies:

$$\Delta_{\mathbb{H}}(|d\Phi_j|^2_{\mathbb{E}^2}\overline{f}_j^{2p}) \geq -(|d\Phi_j|^2_{\mathbb{E}^2}\overline{f}_j^{2p}). \tag{5.24}$$

Using Lemma 5.2 and (5.12), there exists a universal constant $C > 0$ such that

(5.25)
$$\begin{aligned}
\frac{1}{4}|\mathring{A}_j(P)|^2_{\mathbb{E}^2}\overline{f}_j^{2p}(P) &\leq |d\Phi_j(P)|^2_{\mathbb{E}^2}\overline{f}_j^{2p}(P) \\
&\leq C\int_{B^1_{\mathbb{H}}(P)}|d\Phi_j|^2_{\mathbb{E}^2}\overline{f}_j^{2p}\,d\mu_{\mathbb{H}} \leq 4C\int_{B^1_{\mathbb{H}}(P)}|\mathring{A}_j|^2_{\mathbb{E}^2}\overline{f}_j^{2p}\,d\mu_{\mathbb{H}}.
\end{aligned}$$

Now, since $e^{-2} \leq \frac{q_j(Q)}{q_j(P)} \leq e^2$ for $Q \in B^1_{\mathbb{H}}(P)$ and $|\phi|$ is uniformly bounded,

$$|\mathring{A}_j|^2_{\mathbb{E}^2} = \frac{e^{-2\phi_j}}{q_j^2}|\mathring{A}_j|_{g_j}, \quad \text{and} \quad d\mu_{\mathbb{H}} = e^{-2\phi_j}\frac{x^2}{q_j^2}\,d\mu_j$$

implies that

(5.26)
$$|\mathring{A}_j(P)|_{\mathbb{E}^2} \leq C\frac{\sqrt{\int_{B^1_{\mathbb{H}}(P)}|\mathring{A}_j|^2_{\mathbb{H}}\overline{f}_j^{2p}d\mu_{\mathbb{H}}}}{q_j(P)\overline{f}_j^p} \leq C\frac{\sqrt{\int_{B^2(P)}|\mathring{A}_j|^2_{\mathbb{H}}\overline{f}_j^{2p}d\mu_{\mathbb{H}}}}{q_j(P)\overline{f}_j^p}.$$

This finishes the proof of Proposition 5.4 and hence the proof of Proposition 5.2 as well.

5.3. Proof of Proposition 5.3: the second line integral. The analysis of the second line integral in (5.3) differs from that of the first one. In particular, rather than deriving pointwise control for the integrand (which appears hopeless), we express the entire integral as a flux of a suitable vector field across the line $\{0 \leq q_j \leq 1, w_j = 0\}$ using Stokes' theorem. The bound is then obtained by controlling the integral of the divergence over boxes adjacent to this line.

Since $e^{2\phi_j} = \overline{g}_j(\partial_1, \partial_1)$, we obtain that for j large enough, $|\phi_j| \leq 1/10$, and hence

$$e^{|\phi_j|} \leq 2, \quad \text{and} \qquad 1/2 \leq |\partial_1|_{\overline{g}_j} \leq 2.$$

From these bounds we also obtain

$$|\nabla_1 u_j| \leq 4\zeta \Longrightarrow -\frac{1}{2} \leq \partial_1 u_j \leq 2. \tag{5.27}$$

Recall also the basic equation, which follows from the Codazzi formulæ,

$$-\Delta_{\overline{g}}\,\phi = \frac{4\overline{H}^2 - |\bar{A}|^2}{4} = \frac{4\overline{H}^2 - |\mathring{\bar{A}}|^2}{8}, \tag{5.28}$$

as well as the identity

$$\Delta_{\overline{g}}\, e^{-\phi} = -\Delta_{\overline{g}}\phi e^{-\phi} + |\nabla\phi|^2_{\overline{g}}\, e^{-\phi}. \tag{5.29}$$

The key for proving (5.10) is to express the integral I on the left in that formula as one of the boundary flux terms of an integration of the divergences of two vector fields over the two rectangles $\mathcal{D}_1 := \{0 \leq q_j \leq 1, 0 \leq w_j \leq 1\}$ and $\mathcal{D}_2 := \{0 \leq q_j \leq 1, -1 \leq w_j \leq 0\}$. To do this, introduce a cutoff function $\chi(w)$ such that $\chi \in \mathcal{C}^2$ with $0 \leq \chi \leq 1$, $\chi(0) = 1, \chi(-1) = \chi(1) = 0$, and such that $|\chi'| \leq 4$.

By Stokes' formula, and with summation over s implied,

(5.30)

$$\begin{aligned} I = &\int_{\mathcal{D}_1} \Delta_{\overline{g}_j} e^{-\phi_j}(\partial_1 u_j + 1)\chi\, d\overline{\mu}_j + \int_{\mathcal{D}_1} \partial^s e^{-\phi_j}\partial_s(\partial_1 u_j + 1)\chi\, dq_j dw_j \\ &+ \int_{\mathcal{D}_1} \partial_s(e^{-\phi_j})(\partial_1 u_j + 1)\partial^s\chi\, d\overline{\mu}_j - \int \partial_1 e^{-\phi_j}(\partial_1 u_j + 1)\chi\, dw_j\Big|_{q_j=0}^{q_j=1} \\ &+ \int_{\mathcal{D}_2} \Delta_{\overline{g}_j} e^{-\phi_j}\chi\, d\overline{\mu}_j + \int_{\mathcal{D}_2} \partial_s e^{-\phi_j} e^{\phi_j}\partial_s\chi\, dq_j dw_j - \int \partial_1 e^{-\phi_j} e^{\phi}\chi\, dw_j\Big|_{q_j=0}^{q_j=1}. \end{aligned}$$

Some of these integrals are expressed with respect to the $d\bar{\mu}_j$, others with respect to the volume form $dq_j dw_j$, but the difference is not large since $d\bar{\mu}_j = e^{2\phi_j} dq_j dw_j$.

Since $Y_j \to Y_*$ smoothly away from $\{x = 0\}$, then for *any* $\eta \in (0, 1]$, $\partial_{q_j}|_{x=\eta} \to \partial_{q_*}|_{x=\eta}$ and $\phi_j|_{x=\eta} \to \phi_*|_{x=\eta}$ smoothly. The bounds on $|\partial_1|_{\overline{g}}$ and the fact that $\phi_* = \text{const}$ show that

$$\int_{-1}^{1} |\partial_1 \phi_j|\, dw_j\Big|_{q_j=1} = o(1). \tag{5.31}$$

On the other hand, by (4.4),

$$\int_0^1 \int_{-1}^1 |\partial_{12}\phi_j| dw_j dq_j \le \mathcal{E}_j + o(1). \tag{5.32}$$

Combining these last two equations, we see that if $\eta \in (0,1]$, then for $-1 \le a \le w \le b \le 1$,

$$\left| \int_a^b \partial_1\phi_j \, dw_j \right|_{q_j=\eta} \le \left| \int_a^b \partial_1\phi_j \, dw_j \right|_{q_j=1} + \int_\eta^1 \int_a^b |\partial_{12}\phi_j| \, dw_j dq_j. \tag{5.33}$$

Since this is true for all subintervals $[a,b]$, then for each η we can divide the integral on the left into subintervals $[a,b]$ where $\partial_1\phi_j|_{q_j=\eta}$ has constant sign, and then add these subintervals, to obtain that

$$\int_{-1}^1 |\partial_1\phi_j| \, dw_j \Big|_{q_j=\eta} \le \mathcal{E}_j + o(1), \tag{5.34}$$

where the error term is independent of η. Letting $\eta \to 0$ gives

$$\int_{-1}^1 |\partial_1\phi_j| \, dw_j \Big|_{q_j=0} \le \mathcal{E}_j + o(1). \tag{5.35}$$

Now consider the interior integral terms in (5.30). By (5.29), the first interior integral can be written as

$$\int_{\mathcal{D}_1} -\Delta_{\overline{g}_j}\phi_j(e^{-\phi_j})(\partial_1 u_j + 1)\chi \, d\bar{\mu} + \int_{\mathcal{D}_1} |\nabla\phi_j|^2_{\overline{g}_j} \, e^{-\phi_j}(\partial_1 u_j + 1)\chi \, d\bar{\mu}.$$

The first term on the right here is controlled using (5.28):

(5.36)

$$\begin{aligned} -\int_{\mathcal{D}_1} \Delta_{\overline{g}_j}\phi_j e^{-\phi_j}(\partial_1 u_j + 1)\chi \, d\bar{\mu} &= \int_{\mathcal{D}_1} \frac{|\mathring{\bar{A}}_j|^2 - 4\overline{H}_j^2}{8}(\partial_1 u_j + 1)\chi e^{-\phi_j} \, d\bar{\mu} \\ &\le 4 \int_{\mathcal{D}_1} \left(2|\mathring{\bar{A}}_j|^2 - \frac{\overline{H}_j^2}{64} \right) \chi e^{-\phi_j} \, d\bar{\mu}. \end{aligned}$$

Next, define

$$\mathcal{T} := \int_{\mathcal{D}_1} |\nabla\phi_j|^2_{\overline{g}_j} e^{-\phi_j}(\partial_1 u_j + 1)\chi d\bar{\mu} + \int_{\mathcal{D}_1} \partial^s e^{-\phi_j}\partial_{s1}u_j \chi \, d\bar{\mu}.$$

Replace $\partial_{1s}u_j$ by $\nabla_{1s}u_j$ using $\nabla_{ab}u_j = \partial_{ab}u_j - \Gamma^t_{ab}\partial_t u_j$, where

(5.37)

$$\Gamma^1_{21} = \partial_1\phi_j, \Gamma^2_{22} = \partial_2\phi_j, \Gamma^2_{21} = -\partial_2\phi_j, \Gamma^1_{22} = \partial_1\phi_j, \Gamma^1_{12} = \partial_2\phi_j, \Gamma^2_{12} = \partial_1\phi_j,$$

to get that $\mathcal{T}$ equals

(5.38)

$$\int_{\mathcal{D}_1} |\nabla\phi_j|^2_{\overline{g}_j} e^{-\phi_j}(\nabla_1 u_j + 1)\chi\, d\bar{\mu} + \int_{\mathcal{D}_1} \partial^s e^{-\phi_j}\nabla_{s1}u_j\chi\, d\bar{\mu}$$

$$+ \int_{\mathcal{D}_1} \partial^s e^{-\phi_j}\Gamma^t_{s1}\partial_t u_j\chi\, d\bar{\mu} = \int_{\mathcal{D}_1}\left(\partial^s e^{-\phi_j}\nabla_{s1}u_j\chi + |\nabla\phi_j|^2_{\overline{g}_j} e^{-\phi_j}\chi\right) d\bar{\mu}.$$

Applying Cauchy-Schwarz, (4.4), and $|\nabla_{ab}u_j|_{\overline{g}_j} \leq 2|(\bar{A}_j)_{ab}|_{\overline{g}_j}$ we derive:

(5.39)

$$\mathcal{T} \leq 100\int_{\mathcal{D}} |\nabla\phi_j|^2\chi\, d\bar{\mu} + \frac{1}{100}\int_{\mathcal{D}} |\bar{A}_j|^2\chi\, d\bar{\mu} \leq C200\,\mathcal{E}_j + \frac{1}{50}\int_{\mathcal{D}} |\overline{H}_j|^2 d\bar{\mu} + o(1).$$

As for the third bulk term, using (4.4) again, we derive

(5.40)

$$\mathcal{Z} := \int_{\mathcal{D}_1} e^{-2\phi_j}\partial_2 e^{-\phi_j}(\partial_1 u_j + 1)\partial_2\chi\, d\bar{\mu} = \int_{\mathcal{D}_1} \partial_2 e^{-\phi_j}(\partial_1 u_j + 1)\partial_2\chi\, dqdw$$

$$\leq 4\sqrt{2\int_{\mathcal{D}_1} |\nabla\phi_j|^2_{\overline{g}_j} d\bar{\mu}}\cdot\sqrt{\int_{\mathcal{D}_1} dqdw} \leq 10C\sqrt{\mathcal{E}_j + o(1)}.$$

We control the last two bulk terms by

$$\begin{aligned} &\int_{\mathcal{D}_2} (-\Delta_{\overline{g}_j}\phi e^{-\phi_j} + |\nabla\phi_j|^2_{\overline{g}_j} e^{-\phi_j})\chi\, d\bar{\mu} \\ &\leq \int_{\mathcal{D}_2} \frac{|\mathring{\bar{A}}_j|^2 - \overline{H}_j^2}{4}|\chi|\, d\bar{\mu} + 2C\mathcal{E}_j + o(1). \end{aligned} \tag{5.41}$$

Finally, the Cauchy-Schwarz inequality together with (4.4) one last time gives

(5.42)

$$\int_{\mathcal{D}_2} e_j^{-2\phi}\partial_2\phi_j\partial_2\chi\, d\overline{\mu} \leq 2\sqrt{\int_{\mathcal{D}} |\nabla e^{-\phi_j}|^2\, d\bar{\mu}}\sqrt{\int_{\mathcal{D}_2} dqdw} \leq 2\sqrt{\int_{\mathcal{D}_2} |\mathring{\bar{A}}_j|^2\, d\bar{\mu}}.$$

Taken together, these estimates complete the proof. The only thing to observe is that the terms $\int_{\mathcal{D}} \overline{H}_j^2\, d\bar{\mu}$ appear with a *negative* coefficient in the end, and so can be discarded, since our proposition only claims an upper bound on I. □

6. Regularity gain for the limit surface in the small energy regions

We now turn to a closer look at the relationship between finiteness of the weighted energy and the regularity of the boundary curve at infinity, and prove Theorem 1.4. In fact, we prove the $\mathcal{C}^1$ regularity for all Willmore surfaces with finite weighted total curvature near points where the boundary curve is locally graphical and Lipschitz.

Definition 6.1. Consider a rectifiable, closed, embedded loop $\gamma \subset \mathbb{R}^2$, with arclength parametrization $t \to (y(t), z(t)) = \gamma(t)$. We say that γ is locally Lipschitz at $P = \gamma(t_0)$ if there exists a $\delta(t_0) > 0$ and a constant $M(t_0) < \infty$ such that (after a rotation) the portion of γ parametrized by $(t_0 - \delta, t_0 + \delta)$ coincides with the graph $z = f(y)$ over an interval of length $\eta(t_0)$ centered at $\gamma(t_0)$. Thus $\gamma((t_0 - \delta, t_0 + \delta)) = \mathrm{Graph}(f)$ and $|f(y_1) - f(y_2)| \leq M(t_0)|y_1 - y_2|$.

Our main result in this section is the

Theorem 6.1. *Let $Y \subset \mathbb{H}^3$ be a complete Willmore surface with $\gamma = \partial_\infty Y$ a possibly disconnected embedded rectifiable curve. Suppose that there exists a set of poles $\mathcal{O} = \{O_1, \dots, O_K\} \subset Y$ such that $\mathcal{E}_p(Y) < \infty$ (the weight function f relative to $\mathcal{O}$ is implicit) and that γ is locally graphical and Lipschitz except at a finite number of points $\{P_1, \dots, P_\Lambda\}$. Assume finally that if $\gamma(t_0) \neq P_j$ for any j, $M(t_0) = \zeta$ and $\mathcal{E}_p^{B(\gamma(t_0),\delta(t_0))}(Y) \leq \epsilon'(\zeta)$. Then $\gamma \setminus \{P_1, \dots, P_\Lambda\}$ is a $\mathcal{C}^1$ curve.*

In the setting in Theorem 1.4, the assumption that the boundary curve is locally Lipschitz away from $\{P_1 \dots, P_\Lambda\}$ holds for the curve $\partial_\infty Y_*$ which is the limit of the $\partial_\infty Y_j$. Indeed, Corollary 1.1 ensures the graphicality and Lipschitz bound away from the bad points $P_1, \dots P_\Lambda$. We distinguish two further cases. Either there exists a sequence of poles $O_k^{(j)}$ converging to an interior point $O_* \in Y_*$, or else any sequence of poles $O_k^{(j)}$ diverges to infinity in the limit. In the first case, without precluding that some poles disappear to infinity, suppose that the limits of the poles occur at $O_{*,1}, \dots, O_{*,K} \in Y_*$. We can also assume that the poles $O_k^{(j)} \in Y_j$ converge to $O_{*,k} \in Y_*$. Also, using the weight function f_* on Y_* corresponding to the poles $\{O_{*,1}, \dots, O_{*,K}\}$, the weighted Willmore energy is finite. To see this, note that for all $\epsilon > 0$

$$\mathcal{E}_p(Y_j \cap \{x \geq \epsilon\}) \to \mathcal{E}_p(Y_* \cap \{x \geq \epsilon\}).$$

This follows readily since all poles other than $O_1, \dots, O_K$ disappear toward infinity, thus $f_j \to f_*$ over the portion of the surfaces contained in $\{x \geq \epsilon\}$. So consider the second case, where $O_k^{(j)} \to \partial_\infty \mathbb{H}^3$ for all k. We claim that Y_* must then be a finite union of half-spheres; this implies Theorem 1.4 immediately. To prove this assertion, just note that if all poles disappear toward infinity then $|\mathring{A}| = 0$ on all of Y_*: If this were false, then there would exist an interior point $P \in Y_*$ and a ball $B^1(P) \subset Y_*$ such that $\int_{B^1(P)} |\mathring{A}|^2 d\mu = \epsilon > 0$. But then, since all the $O_k^{(j)}$ diverge to infinity, $f|_{B^1(P)} \to \infty$ on $B^1(P)$, which implies that $\mathcal{E}_p[Y_j] \to \infty$ as well. This is a contradiction.

Therefore we have reduced matters to proving Theorem 6.1. This, in turn, is a consequence of the following result.

Proposition 6.1. *Let $\gamma_k(t)$, $0 < t < M_k$, be an arclength parametrization of the k^{th} connected component of γ. Suppose that $\gamma(t_*) \notin \{P_1, \dots, P_\Lambda\}$. Choose any Cauchy sequence $t_j \in (0, M_k)$ where γ_k is differentiable at t_j, with $t_j \to t_* \in (0, M_k)$. Then $\dot\gamma(t_j)$ is a Cauchy sequence.*

Proof. Since $\gamma(t_*)$ is not equal to one of the bad points P_j, there exists a line ℓ through $\gamma_k(t_*)$ and a number δ such that $\gamma|_{(t_*-\delta,t_*+\delta)}$ is a graph over the interval of length δ in ℓ centered at $\gamma(t_*)$ with graph function $z = f(y)$ having Lipschitz constant ζ. Lemma 2.3 guarantees graphicality of $Y'_{B(\gamma(t_0),h\,\delta)}$ over the region $\sqrt{x^2+y^2} \le h\delta$ in the vertical half-plane $\ell \times \mathbb{R}^+$ (where we take ℓ as the y-axis), with graph function $z = u(x, y)$, where $|\nabla u| \le 2\zeta$.

Since t_j is Cauchy, it lies in $(t_* - h\,\delta, t_* + h\,\delta)$ for j large, so if we write $\gamma(t_j) = (y_j, u(0, y_j))$, then $y_j \to 0$.

Now, argue by contradiction and assume that $\dot\gamma(t_j)$ is not Cauchy. Then there exists $\theta > 0$ and a subsequence j_k such that $|f'(y_{j_{k-1}}) - f'(y_{j_k})| \ge \theta$. Reset notation so that the index is simply j again. Translate and rotate so that $(y_{j-1}, f(y_{j-1})) = (0, 0)$ and $(y_j, f(y_j))$ lies on the y-axis, then dilate by the factor $\lambda_j := |y_j - y_{j-1}|^{-1}$. Denote the resulting Willmore surface by $\tilde Y_j$ and write $\partial\tilde Y_j = \tilde\gamma_j$.

This surface is still graphical with Lipschitz norm no larger than ζ, and furthermore, $\mathcal{E}^{B(0,\lambda_j h\delta)}(\tilde Y_j) \le \epsilon'(\zeta)$. By Lemma 2.2, Y_j must converge to a vertical half-plane Y_*, and since $\partial_\infty Y_*$ passes through the origin and $(0, 1, 0)$, necessarily $Y_* = \{z = 0\}$. Thus $Y_* \cap \{x = 1\}$ must converge to the line $\{z = 0, x = 1\}$ for some α with $|\alpha| \le 2\zeta$. But now, since $|f_j'(0) - f_j'(1)| \ge \theta$, it follows that for at least one of the two values $y = 0, y = 1$ there is a jump in the derivative of size at least $\theta/2$ between the heights $x = 0$ and 1. We can assume that this jump occurs at $y = 0$.

However, this contradicts Proposition 3.2. The graphicality and Lipschitz bound in that proposition still hold by virtue of the assumption and Lemma 2.3. The fact that the weighted energy goes to zero follows from the dilation invariance of $\mathcal{E}_p$, and the fact that after dilation, the graphs satisfy $\mathcal{E}_p(\mathrm{Graph}(u_j)) \le \mathcal{E}_p(Y \cap B(P, 2h\lambda_j^{-1})]$, and $\lambda_j^{-1} = |y_j - y_{j-1}| \to 0$. This proves the proposition and Theorem 6.1 as well. q.e.d.

7. Bubbling in the small energy regions

We now turn to a closer examination of how bubbling occurs, aiming toward the proof of Theorem 1.5.

The argument leading to the fact that bubbling occurs is indirect. We first construct a sequence of Möbius transformations φ_j to obtain uniform isothermal parametrizations for the surfaces $\varphi_j(Y_j)$. If the surfaces $\varphi_j(Y_j)$ converge to a non-trivial surface, we are done. Otherwise, we must prove that one can take a further sequence of dilations to obtain a nontrivial limit.

The idea is to use the jump in the first derivative coupled with the bounds (5.1) to argue that one of the two line integrals on the right side of that equation must be bounded below. In particular, with $4\epsilon_0 := lim_{j\to\infty}\partial_y u_j(0,0) - \partial_y u_*(0,0)$, then by (5.1), either

$$\int_0^1 |\mathring{\bar{A}}_{12}|e^{-\phi_j}\,dq_j \geq \epsilon_0 \text{ or} \tag{7.1}$$

$$\left|\int_0^1 \partial_2 e^{-\phi_j}\partial_q u_j\,dq_j\right| \geq \epsilon_0. \tag{7.2}$$

These cases are treated separately in the next two subsections. We also show, in §7.3, that each bubble remains at finite distance from one of the poles.

7.1. The integral $\int_0^1 |\mathring{\bar{A}}_{12}|e^{-\phi_j}dq_j$ bounded below implies bubbling. Assuming (7.1), from Proposition 5.2, we derive that

$$\sup_j \sup_{P\in\{0\leq q_j\leq 1, w_j=0\}} \mathcal{E}_p(B^1_{\mathbb{H}}(P)) \geq \frac{(p-1)}{C(K)M'}\epsilon_0.$$

Consider the set of points $P \in \{0 \leq q_j \leq 1, w_j = 0\}$ where $\mathcal{E}_p(B^1_{\mathbb{H}}(P)) \geq \frac{(p-1)\epsilon_0}{10C(K)M'}$. We know that such points exist when j is large. We then ask whether there exist a constant M and points $P_j \in \{0 \leq q_j \leq 1, w_j = 0\}$ with $\mathcal{E}_p(B^1_{\mathbb{H}}(P_j)) \geq \frac{(p-1)\epsilon_0}{10C(K)M'}$ such that $f_j(P_j) \leq M$. (The requirement $f_j(P_j) \leq M$ is equivalent to the existence of $M < \infty$ such that $\mathrm{dist}(P_j, O^{(j)}_k) \leq M$, where $P_j \sim O^{(j)}_k$.)

If such a sequence P_j exists, then consider the isometry φ_j of $\mathbb{H}^3$ which maps P_j to $P_* = (1,0,0)$. The surfaces $\varphi_j(Y_j)$ converge (in a large closed ball around P_*) to a Willmore surface Y_* with $\mathcal{E}(B^1_{\mathbb{H}}(P_*), Y_*) \geq \epsilon_0/10M^2$, and this would be the desired 'bubble'.

It suffices then to show that (7.1) must fail if no such sequence P_j exists. Indeed, observe that if $1 < p' < p$, then Proposition 5.4 gives the new bound

$$|\mathring{\bar{A}}_j|_{\mathbb{E}^2}(P) \leq \frac{C_{p'}\sqrt{\mathcal{E}_{p'}^{B^1_{\mathbb{H}}(P)}(Y_j)}}{q_j(P)(\tilde{f}_j^{p'}(P))}, \quad \text{whence}$$

$$\int_0^1 |\mathring{\bar{A}}_j|_{\mathbb{E}^2}\,dq \leq C'_{p'} \sup_{P\in l_j} \sqrt{\mathcal{E}_{p'}^{B^1_{\mathbb{H}}(P)}[Y_j]}.$$

However, observe that $\limsup_{j\to\infty} \sup_{P\in l_j} \mathcal{E}_{p'}(B^1_{\mathbb{H}}(P)) = 0$; this holds because by definition $\mathcal{E}_{p'}^{B^1_{\mathbb{H}}(P)}(Y_j) \leq 2f_j^{p'-p}(P)\mathcal{E}_p^{B^1_{\mathbb{H}}(P)}(Y_j)$, and $\mathcal{E}_p^{B^1_{\mathbb{H}}(P_j)}$

$(Y_j) \leq M''$, while $f_j(P_j) \to \infty$. Taken together, this all shows that

$$\limsup_{j\to\infty} \int_0^1 |\mathring{\bar{A}}_j|_{\mathbb{E}^2}\, dq_j = 0,$$

contradicting (7.1), as claimed.

7.2. A lower bound on the flux (7.2) **implies bubbling.** Our goal is to show that such a lower bound (7.2) implies the existence of further blow-ups $\varphi_j : \mathbb{H}^3 \to \mathbb{H}^3$ such that $\varphi_j(Y_j) \to Y_*$ with $\mathcal{E}[Y_*] > 0$. Unlike in the final subsection of §5, it is not enough to bound the line integral $|\int_0^1 \partial_2 e^{-\phi_j} \partial_q u_j dq_j|$ by the energy in a box. Fortunately, we can bound it in terms of the energy in a sector $|\frac{w}{q}| \leq 1$ emanating from the distinguished boundary point in the isothermal coordinates (q, w). This bound can then be used to show the existence of a sequence of points where either $|x\bar{A}_j|$ or $|\nabla \phi_j|_g$ are bounded away from zero. Either alternative provides the points around which we can recenter the rescalings. In the first case, we obtain a limit surface with non-zero curvature at one interior point, which must therefore be nontrivial. In the second we obtain a complete Willmore surface for which the canonical isothermal coordinates have non-constant conformal factor, and therefore the surface must be nontrivial. The key difficulty in bounding the left side of (7.2) in terms of the energy in a sector is that the cutoff function depends on w/q, so a derivative of this cutoff function produces a power of $1/x$. The resulting integral is controlled by using the specific algebraic form of the integrand on the left in (7.2). This somewhat remarkable fact is further evidence of the delicate nature of the blow-up procedure.

Proof. First, by translating and dilating, assume that $y_0 = 0$, and that the Y_j and Y_* are graphical over the vertical half-disc $\{x^2 + y^2 \leq 1000, z = 0\}$, with graph functions u_j and u_*, where $|\nabla u_j|, |\nabla u_*| \leq 2\zeta \ll 1$. We can also assume that $\partial_y u_j(0,0) = \alpha > 0$ and $\partial_y u_*(0,0) = 0$. Now, consider any subinterval $[\beta_-, \beta_+] \subset [0, \alpha]$ with $\beta_+ \ll \alpha$. For any sequence $\beta_j \in [\beta_-, \beta_+]$, $\beta_j \to \beta_*$, the line $z = \beta_j y$ intersects $\gamma_j = \partial_\infty Y_j$ at a point $(y_j, u_j(0, y_j))$, $y_j > 0$, and just as in §4, we have $\lim_{j\to\infty} y_j = 0$.

Now dilate Y_j by $\rho_j := \frac{1}{y_j}$ to obtain a new surface $\tilde{Y}_j$ which converges to Y', where Y' is graphical over the entire vertical half-plane $\{z = 0\}$ and passes through the fixed point $(1, \beta_*)$. If, for *any* such sequence β_j, $\mathcal{E}(Y') \neq 0$, then the proof is complete.

Otherwise, $\mathcal{E}(Y') = 0$ so Y' is totally geodesic and graphical over a half-plane, and hence is the half-plane $\{z = \beta_* y, x > 0\}$. Rotating again to make this the xy-plane, the original graph function u_j must satisfy $\partial_y u_j(0,0) = 0$ while $\partial_y u_*(0,0) \sim \alpha - \beta_* > \frac{\alpha}{2} > 0$. All of this is true for any $\beta_* \in [\beta_-, \beta_+]$. Using Remark 4.1, there exists a sequence β_j such that $\int_{\tilde{Y}_j \cap \{1/4 \leq \sqrt{x^2+y^2+z^2}\} \leq 4} |\bar{A}^j|^2 d\bar{\mu} \to 0$.

By Lemma 4.1, there exists a sequence of hyperbolic isometries φ_j such that $\varphi_j(\tilde{Y}_j)$ have all the properties listed there, and in particular admit isothermal coordinates (q_j, w_j) for which the conformal factor ϕ_j satisfies

$$\begin{aligned} ||\nabla^2\phi_j||_{L^1(\varphi_n(\tilde{Y}_j))} + ||\nabla\phi_j||_{L^2(\varphi_j(\tilde{Y}_j))} + ||\phi_j||_{\mathcal{C}^0(\varphi_j(\tilde{Y}_j))} \\ \leq \mathcal{E}(\varphi_j(\tilde{Y}_j)) + o(1) < 2\epsilon'(\zeta) + o(1). \end{aligned} \tag{7.3}$$

Moreover, there is still a jump of $\alpha - \beta_*$ in the first derivative at the origin in these coordinates. The $\varphi_j(\tilde{Y}_j)$ are graphical over the disc $\{x^2 + y^2 \leq 10, z = 0\}$ (for simplicity, we denote the graph function by u_j) with $|\nabla u_j| \leq 4\zeta$, and the image of the rectangle $0 \leq q_j \leq 1, |w_j| \leq 1$ is entirely contained in Graph(u_j). Recall from Remark 4.3 that the surfaces $\varphi_j(\tilde{Y}_j)$ admit an extension $Y_j^\flat$ which is a graph over the xy-plane with graph function u_j, where $u_j = 0$ for $\sqrt{x^2+y^2} \geq 50$. The bounds (7.3) continue to hold for this extended surface.

Using Remark 4.4 and the smallness of the energy we derive that $|y|/x \leq 10$ and $x^2 + y^2 \leq 10$ at all points in the sector

$$\mathcal{S}_j := \{(q_j, w_j) \in Y_j,\ |w_j|/q_j \leq 1,\ q_j^2 + w_j^2 \leq 4\}.$$

We now claim that one of the following must be true:

a) Either $\varphi_j(\tilde{Y}_j)$ converge to a *nontrivial* limit $\tilde{Y}_*$, or else
b) there exists a sequence $\omega_j \to \infty$ such that the dilates $\omega_j \cdot \varphi_j(\tilde{Y}_j)$ converge to a non-trivial limit $\tilde{Y}_*$.

The theorem will be proved once we show that these are the only possibilities.

As many times before, write $\varphi_j(\tilde{Y}_j)$ as just Y_j. If alternative a) does not occur, then Y_j converges to a vertical half-plane Y_*. We claim that for some $\mu > 0$, there exists a sequence $P_j \in \mathcal{S}_j$ such that either $|\nabla\phi_j|_g(P_j) \geq \mu$ or else $x\,|\bar{A}|_{\overline{g}}(P_j) \geq \mu$.

Observe that either of these two possibilities implies our theorem. Indeed, suppose the former of these is true and consider the dilated surfaces $\frac{1}{x(P_j)}\,Y_j$. The images $\tilde{P}_j$ of the points P_j have height $x_j = 1$ and $|y_j| \leq 10$. Setting $\lambda_j := \frac{1}{x(P_j)}$, consider the isothermal coordinates

$$\tilde{q}_j(\lambda_j x, \lambda_j y) = \lambda_j q_j(x,y),\ \tilde{w}_j(\lambda_j x, \lambda_j y) = \lambda_j w_j(x,y),$$

and the corresponding conformal factor $\tilde{\phi}_j(\lambda_j x, \lambda_j y) = \phi_j(x,y)$ on $\lambda_j Y_j$. Clearly, $|\nabla\tilde{\phi}_j|_{\overline{g}}(\tilde{P}_j) \geq \mu$. Also, passing to a subsequence, $\tilde{P}_j \to \tilde{P}_*$ where $x(\tilde{P}_*) = 1$, $|y(\tilde{P}_*)| \leq 10$, and $|z(\tilde{P}_*)| \leq 10\zeta$.

Using the estimates in [**27**, Thm. I.5], some subsequence of the surfaces $\lambda_j Y_j$ must converge, smoothly in the interior and in $\mathcal{C}^{0,\alpha}$ up to the boundary, to a surface $\tilde{Y}_*$, and this limit surface admits isothermal coordinates $(\tilde{q}_*, \tilde{w}_*)$ where $\tilde{q}_* = 0$ on $\{x = 0\}$ and $1/10 \leq |\tilde{q}|/|x| \leq 10$. The convergence $(\tilde{q}_j, \tilde{w}_j, \tilde{\phi}_j) \to (\tilde{q}_*, \tilde{w}_*, \tilde{\phi})$ is smooth away from $\{x = 0\}$.

We claim that $\tilde{Y}_*$ cannot be a vertical half-plane. Indeed, if it were, then following the same argument as in the second paragraph of §5, $\tilde{q}_* = Cx$ for some $1/10 \leq C \leq 10$, and in that case, the corresponding conformal factor $\tilde{\phi}_*$ would be constant. This contradicts the smooth convergence and the fact that $|\nabla\tilde{\phi}_j(\tilde{P}_j)|_{\overline{g}} \geq \mu$.

The proof that $x|\bar{A}|_{\overline{g}}(P_j) \geq \mu$ implies the result is even simpler. Indeed, the same sequence of dilations of Y_j converges to a Willmore surface with $|\bar{A}|_{\overline{g}}(\tilde{P}_*) \geq \mu$, and this must be nontrivial since we know that it is graphical over the half-plane $\{z = 0\}$ and hence cannot be a sphere.

We have therefore reduced the proof to showing that conditions i) to vi) below lead to a contradiction.

i) Each Y_j is a graphical Willmore surface, with graph function u_j, over $\{x^2 + y^2 \leq 10, x > 0, z = 0\}$, with $\mathcal{E}(Y_j) \leq \epsilon'(\zeta)$ and $\int_{Y_j} |\bar{A}_j|^2 d\bar{\mu} \leq M$ for some fixed $M < \infty$. The surface Y_j extends to a (non-Willmore) graphical surface $Y_j^\flat$. The region $30 \leq \sqrt{x^2+y^2}$ is denoted $Y_j^\sharp$ and $u_j = 0$ there.

ii) Each Y_j, and its extension $Y_j^\flat$ too, admits an isothermal coordinate chart (q_j, w_j) with conformal factor ϕ_j satisfying (7.3).

iii) $Y_j \to Y_* := \{x^2 + y^2 \leq 10, z = 0\}$.

iv) The conformal factors ϕ_j satisfy $|\nabla\phi_j|_g \to 0$ uniformly in $\mathcal{S}_j$.

v) $x \cdot |\bar{A}_j|_{\overline{g}} \to 0$ uniformly in $\mathcal{S}_j$.

vi) $|\int_0^1 \partial_2 e^{-\phi_j} \partial_q u_j dq_j| \geq \epsilon_0 > 0$.

To reach the contradiction it suffices to prove that conditions i)–v) contradict condition vi). In other words, we need to show that i)–v) imply:

$$\lim_{j\to\infty} \left| \int_{(0,0)}^{(1,0)} \partial_2 e^{-\phi_j} \partial_1 u_j \, dq_j \right| = 0. \tag{7.4}$$

Proof of (7.4)*:* Recall that since the conformal factor ϕ_j is bounded, the quantities $|\partial_q|$, $|\partial_x|$ and $|\partial_y|$, $|\partial_w|$ are comparable. In the following, $|\cdot|$ denotes the norm with respect to $dq^2 + dw^2$. In many expressions below, we suppress the subscripts j for simplicity.

The strategy is to express the second integral $\int_{(0,0)}^{(1,0)} \partial_2 e^{-\phi_j} \partial_1 u \, dq$ as the flux of the integral of a divergence over some part of the circular sector $\mathcal{S}_j$. Introduce polar coordinates $r_j = \sqrt{q_j^2 + w_j^2}$ and θ_j with $\tan(\theta_j + \frac{\pi}{2}) = \frac{w_j}{q_j}$, so that $\mathcal{S}_j := \{0 \leq r_j \leq 1, \pi/4 \leq \theta_j \leq 3\pi/4\} \subset Y_j$. Let $\mathcal{S}^l$ denote the region where $\pi/2 \leq \theta \leq 3\pi/4$, and define $\chi^l(\theta) = (3 - \frac{4\theta}{\pi})$ in $\mathcal{S}^l$.

By the divergence theorem,

(7.5)
$$\int_{(0,0)}^{(1,0)} \partial_2 e^{-\phi_j}\partial_1 u\, dq = \int_{\mathcal{S}^l} \Delta_{\overline{g}} e^{-\phi_j}\partial_1 u \chi^l\, d\overline{\mu} + \int_{\mathcal{S}^l} \nabla_s e^{-\phi_j}\nabla^s(\partial_1 u)\chi^l\, d\overline{\mu}$$
$$+ \frac{4}{\pi}\int_{\mathcal{S}^l} e^{-2\phi_j}\partial_\theta e^{-\phi_j}\frac{1}{r^2}\partial_1 u\, d\overline{\mu} + \int_{\pi/2}^{3\pi/4}(\partial_1 u)\partial_1 e^{-\phi_j}(1,\theta)\, d\theta.$$

(The coefficient $\frac{4}{\pi}$ arises from $\partial_\theta\chi^l$.) The final boundary term tends to zero since Y_j converges to a vertical half-plane, so in particular $|\partial u_j|, |\partial\phi_j| \to 0$ away from $\{x = 0\}$.

Now consider the bulk terms. First, observe that the pointwise bounds on ϕ_j and on $|\nabla u|_{\overline{g}}$ imply that $|\partial_1 u|_{\overline{g}} \le 3\zeta$. This uses the formula for the second fundamental form for a graph in $\mathbb{R}^3$ and implies that:

$$\int_{\mathcal{S}^l} |\partial^2 u|^2\, d\overline{\mu} \le 10\int_{\mathcal{S}^l} |\bar{A}|^2\, d\mu \le 10M. \tag{7.6}$$

Using (5.28) and $|\partial_r u|_{\overline{g}} \le 3\zeta$, we have

$$\begin{aligned} &\left|\int_{\mathcal{S}^l} \Delta_{\overline{g}} e^{-\phi}\partial_1 u\chi^l\, d\bar{\mu}\right| \\ &\le 4\int_{\mathcal{S}^l} e^{-\phi}\overline{H}^2|\partial_1 u|\, d\bar{\mu} + \int_{\mathcal{S}^l} |\mathring{\bar{A}}|^2 e^{-\phi}|\partial_1 u|\, d\bar{\mu} + \int_{\mathcal{S}^l} |\nabla\phi|^2 e^{-\phi}\, d\bar{\mu}. \end{aligned} \tag{7.7}$$

Clearly,

$$\int_{\mathcal{S}^l} \overline{H}^2|\partial_1 u|\, d\bar{\mu} + 4\int_{\mathcal{S}^l} |\mathring{\bar{A}}|^2 e^{-\phi}|\partial_1 u|\, d\bar{\mu} \le 10\int_{\mathcal{S}^l} |\bar{A}|^2|\partial_1 u|\, d\bar{\mu}. \tag{7.8}$$

In addition, using (7.3) and (7.6),

(7.9)
$$\left|\int_{\mathcal{S}^l} \nabla_s e^{-\phi}\nabla^s(\partial_1 u)\chi^l\, d\bar{\mu}\right| = \left|\int_{\mathcal{S}^l} e^{-2\phi}\partial_s e^{-\phi}\partial^s(\partial_1 u)\chi^l\, d\bar{\mu}\right|$$
$$\le 4\left(\int_{\mathcal{S}^l} |\nabla\phi|^2\, d\bar{\mu}\right)^{\frac{1}{2}}\left(\int_{\mathcal{S}^l} |\partial^2 u|^2\, d\bar{\mu}\right)^{\frac{1}{2}} \le 100\sqrt{M}\left(\int_{\mathcal{S}^l} |\nabla\phi|^2\, d\bar{\mu}\right)^{\frac{1}{2}}.$$

The main issue is to control the term

$$T_2 := \left|\int_{\mathcal{S}^l} \frac{1}{r^2} e^{-2\phi}\partial_\theta e^{-\phi}\partial_1 u\, d\bar{\mu}\right|.$$

Recall that by Lemma 2.1, $\partial_1 u = 0$ on $\{q = 0\} = \{x = 0\}$, and also $\frac{|\partial_1 u|^2}{r^2} \le \frac{|\partial_1 u|^2}{q^2}$. The Hardy inequality now gives

$$\begin{aligned} \int_{\mathcal{S}^l} \frac{|\partial_1 u|^2}{r^2} dq dw &\le 10\int_{\mathcal{B}} \frac{|\partial_1 u|^2}{q^2} dq dw \le \int_D |\partial^2 u|^2 dq dw \\ &\le 100\int_{Y_j} |\bar{A}|^2_{\overline{g}} d\overline{\mu} \le 100M, \end{aligned} \tag{7.10}$$

where $\mathcal{B} := \{0 \le w \le 1, 0 \le q \le 1\}$. Thus

(7.11)

$$T_2 \le 10 \int_{\mathcal{S}^l} r^{-2} |\partial_\theta \phi \partial_1 u| \, d\bar{\mu} \le 10 \left(\int_{\mathcal{S}^l} |\nabla \phi|^2_{\bar{g}} \, d\bar{\mu} \right)^{\frac{1}{2}} \left(\int_{\mathcal{S}^l} r^{-2} |\partial_1 u|^2 \, d\bar{\mu} \right)^{\frac{1}{2}}$$

$$\le 100 \left(\int_{\mathcal{S}^l} |\nabla \phi|^2_{\bar{g}} \, d\bar{\mu} \right)^{\frac{1}{2}} \left(\int_{\mathcal{B}} |\partial^2 u|^2 \, d\bar{\mu} \right)^{\frac{1}{2}} \le 100\sqrt{M} \left(\int_{\mathcal{S}^l} |\nabla \phi|^2_{\bar{g}} \, d\bar{\mu} \right)^{\frac{1}{2}}.$$

We then claim that

$$\lim_{j \to \infty} \int_{\mathcal{S}^l} |\nabla \phi_j|^2_{\bar{g}} \, d\bar{\mu} = 0, \tag{7.12}$$

and

$$\lim_{j \to \infty} \int_{\mathcal{S}^l} |\bar{A}_j|^2 |\partial_1 u_j| \, d\bar{\mu} = 0. \tag{7.13}$$

These estimates will prove (7.4), and thus our theorem.

Proof of (7.12)*:* We assert first that on the family of lines $\ell_{\theta_0} := \{0 \le r \le 1, \theta = \theta_0\}$, $\frac{\pi}{2} \le \theta \le \frac{3\pi}{4}$, there is a uniform bound

$$\int_{\ell_{\theta_0}} |\partial \phi_j| dr \le \epsilon'(\zeta) + M'. \tag{7.14}$$

Before proving (7.14), let us see how it proves the estimate.

Since $1/10 \le q/x \le 10$ in $\mathcal{S}^l$, we have

$$r |\nabla \phi_j|_{\bar{g}} \le 100 |\nabla \phi_j|_g$$

in this sector, so that

$$\begin{aligned}\int_{\mathcal{S}^l} |\nabla \phi_j|^2_{\bar{g}} \, d\bar{\mu} &= \int_{\pi/2}^{3\pi/4} \int_0^1 |\partial \phi_j| |\partial \phi_j| r \, dr d\theta \\ &\le 100 \sup_{\mathcal{S}^l} |\nabla \phi_j|_g \sup_{\theta \in [\pi/2, 3\pi/4]} \int_{\ell_\theta} |\partial \phi_j| \, dr.\end{aligned}$$

Since the first factor tends to 0 by the assumption v) above and the second one is bounded, we obtain (7.12).

Thus matters are reduced to showing (7.14). Recall from (7.3) that $||\partial^2 \phi_j||_{L^1(\mathbb{R}^2)} \le \epsilon'(\zeta) + o(1)$, where ∂ is differentiation with respect to (q_j, w_j). Given any ray ℓ_{θ_0}, consider the right-angle rectangle $R_{\theta_0} \subset Y_j^\flat$ which is defined by four straight (with respect to the coordinates q_j, w_j) line segments: ℓ_{θ_0} is one segment, then s^1, s^2 are two line segments of length 50, emanating from the endpoints $(0,0)$ and $(\cos\theta_0, \sin\theta_0)$ of ℓ_{θ_0} and normal to it; finally ℓ'_{θ_0} joins the other endpoints of s^1, s^2. Thus ℓ'_{θ_0} is parallel to ℓ_{θ_0} (with respect to the flat coordinates q_j, w_j) and lies in the portion $Y_j^\sharp$ of $Y_j^\flat$ defined in Remark 4.3. Let n be the unit vector field (with respect to $dq^2 + dw^2$) normal to ℓ_{θ_0}, so n is also normal to ℓ'_{θ_0} and tangent to the lines s^1, s^2.

Integrating $\partial_n(\partial\phi_j)$ over the rectangle R_{θ_0} and decomposing ℓ_{θ_0} into sets where a given component of $\partial\phi_j$ has constant sign, we obtain that

$$\int_{\ell_{\theta_0}} |\partial\phi_j| \leq \int_{R_{\theta_0}} |\partial^2\phi_j|\, dqdw + \int_{\ell'_{\theta_0}} |\partial\phi_j|.$$

The first integral on the right is bounded above by (7.3). We obtain a uniform upper bound on $\int_{\ell'_{\theta_0}} |\partial\phi_j|$ by proving that $\partial\phi_j$ is bounded above pointwise over ℓ'_{θ_0}, which is true because in $Y_j^\sharp$ we have $\Delta\phi_j = 0$ and $|\phi_j|$ is uniformly bounded. This proves (7.14). $\square$

Proof of (7.13)*:* Recall that condition vi) implies that $\lim_{j\to\infty} \sup_{\mathcal{S}^l} r\,|\bar{A}_j| = 0$; using Cauchy-Schwarz, the Hardy inequality, and (7.6), we get

$$\begin{aligned} &\int_{\mathcal{S}^l} |\bar{A}_j|^2 |\partial_1 u_j|\, d\bar{\mu} = \int_{\mathcal{S}^l} r|\bar{A}_j| \cdot |\bar{A}_j| \cdot \frac{1}{r} |\partial_1 u_j|\, d\bar{\mu} \\ &\leq \left(\sup_{\mathcal{S}^l} r|\bar{A}_j|\right) \left(\int_{\mathcal{S}^l} |\bar{A}_j|^2\, d\bar{\mu}\right)^{\frac{1}{2}} \left(\int_{\mathcal{S}^l} r^{-2} |\partial_1 u_j|^2\, d\bar{\mu}\right)^{\frac{1}{2}} \\ &\leq 10 \left(\sup_{\mathcal{S}^l} r|\bar{A}_j|\right) M \to 0. \end{aligned} \tag{7.15}$$

This proves (7.4), and hence completes the proof of our theorem. q.e.d.

7.3. Finitely many bubbles. We conclude by showing that there can exist at most N nontrivial nonisometric limits in the sequence Y_j, which completes the proof of Theorem 1.5. In other words, given Y_j and any sequence of isometries φ_j such that $\varphi_j(Y_j)$ converges to a limiting Willmore surface Y_* with $\mathcal{E}(Y_*) > 0$, there can be at most N distinct possible limits Y_*.

Arguing as usual by contradiction, assume there exist $N+1$ nonisometric limits, $Y_{*,1}, \dots, Y_{*,N+1}$, with induced metrics $g_{*,k}$, $k \leq N+1$. Since the limits $Y_{*,k}$ are non-trivial, there exist points A_k on each $Y_{*,k}$ such that the intrinsic balls $B^1_{g_{*,k}}(A_k)$ have non-zero energy.

The fact that the $Y_{*,k}$ arise as limits of $\varphi_j(Y_j)$ (with $\mathcal{C}^\infty$ convergence on compact sets) gives balls $B^1_{g_j}(C_{j,k}) \subset Y_j$ with $\varphi_k(B^1_{g_j}(C_{j,k})) \to B^1_{g_{*,k}}(A_k)$. In particular we may assume that there is an $\epsilon > 0$ such that $\mathcal{E}[B^1_{g_j}(C_{j,k})] > \epsilon$ for all j, k. We then claim that for any pair of distinct values $k \neq l$ the points $C_{j,k}, C_{j,l} \in Y_j$ drift infinitely far apart, i.e.

$$\mathrm{dist}_{g_j}(C_{j,k}, C_{j,l}) \to \infty. \tag{7.16}$$

If we can prove this, then using the triangle inquality, there is a sequence of points $C_{j,k(j)} \in Y_j$ with $\min_r \mathrm{dist}_{g_j}(O_r, C_{j,k(j)}) \to \infty$. But this cannot hold since then

$$\begin{aligned}\mathcal{E}_p(Y_j) &\geq \int_{B^1_{g_j}(C_{j,k})} [\min_r \operatorname{dist}_{g_j}(O_r, C_{j,k(j)})]^{2p} |A_j|^2 \, d\mu_j \\ &\geq [\min_r \operatorname{dist}_{g_j}(O_r, C_{j,k(j)})]^{2p} \epsilon^2 \to \infty.\end{aligned} \tag{7.17}$$

We have reduced matters to proving (7.16). But if this were not true, then a large enough ball centered at $C_{j,k}$ must contain $C_{j,l}$, which would imply that the limit surfaces $Y_{*,k}$ and $Y_{*,l}$ coincide up to a hyperbolic isometry; this contradicts our assumption.

8. Examples

In this final section we show that the putative modes of convergence described above actually occur. Namely, we exhibit sequences Y_j of complete, properly embedded minimal (and therefore Willmore) surfaces in $\mathbb{H}^3$ with fixed genus which lose (unweighted) energy in the limit because some portions separate and disappear toward infinity. These Y_j have (unweighted) energy tending to zero and converge smoothly away from a finite number of points on the boundary curve at infinity. The limit is another complete, properly embedded surface Y_*, and we find such sequences where the genus of Y_* is strictly less than that of each of the Y_j. In other words, there can be a loss of genus in the limit. The construction of these surfaces proceeds by a fairly standard gluing result. There are many very similar ways to prove such theorems, and we follow a method used in the papers [**23**, **21**, **22**]. Since this method is well documented in these papers, we provide only a sketch of the argument.

Theorem 8.1. *Choose a finite number, $Y_1, \dots, Y_k$, of complete, properly embedded minimal surfaces, each with finite energy. Suppose that each $\gamma_r = \partial_\infty Y_r$, $r = 1, \dots, k$ is a $\mathcal{C}^2$ curve, and assume also that each $Y_r, r = 1, \dots, k$ is nondegenerate in the sense that it admits no Jacobi fields which decay at γ_r. Then there is a family of complete, properly embedded minimal surfaces Y_t with boundary curves $\partial_\infty Y_t = \gamma_t$, a small perturbation of the unit circle. These boundary curves converge in $\mathcal{C}^2$ to the unit circle away from k distinct points $q_1, \dots, q_k$. Furthermore, there exist rescalings of Y_t at q_j which converge to an isometric copy of Y_r. Finally,*

$$\mathcal{E}(Y_t) = \sum_{r=1}^{k} \mathcal{E}(Y_r) + o(1)$$

as $t \to \infty$.

Proof. There are three steps: we first construct a family of approximate solutions Y_t' which are approximately minimal and have the stated concentration properties; we next analyze the mapping properties of the

Jacobi operators on these surfaces, focusing on estimates which are uniform in the parameter t; the final step is to perturb Y_t' to a minimal surface Y_t when t is sufficiently large.

Approximate solutions: First, choose two separate collections of points $p_1, \ldots, p_k$ and $q_1, \ldots, q_k$ on the unit circle S^1 in the boundary at infinity $\{x = 0\}$, such that no two of these points coincide. For simplicity of notation, assume that $p_r = -q_r$ below. Next, fix points $p_r', q_r' \in \gamma_r$, $r = 1, \ldots, k$, and choose a hyperbolic isometry F_r which carries p_r' to p_r and q_r' to q_r, and set $Y_r' = F_r(Y_r)$. Finally, let $M_{r,t}$ be the family of hyperbolic dilations with source p_r and sink q_r, and set $Y_{r,t} = M_{r,t}(Y_r')$.

As $t \to +\infty$, the surfaces $Y_{r,t}$ converge locally uniformly in $\mathcal{C}^2$ in the region $\overline{\mathbb{H}^3} \setminus \{q_r\}$ to the totally geodesic hemisphere H bounded by the unit circle, and this convergence is $\mathcal{C}^\infty$ away from $\{x = 0\}$. In particular, $\gamma_{r,t} := \partial_\infty Y_{r,t}$ converges in $\mathcal{C}^2$ away from the point q_r. Applying the inverse dilations $M_{r,-t}$, we see that rescalings of Y_t' converge to Y_r', which is an isometric copy of Y_r.

For each r, choose a closed spherical cap A_r (intersected with the half-space $x \geq 0$) centered at q_r in the unit hemisphere H. We can do this so that these caps are disjoint from one another, and we then let $B_r = H \setminus A_r$. Choose a slightly larger spherical cap $B_r' \supset B_r$, so $B_r' \cap A_r$ is diffeomorphic to a rectangle. Let A_r' be the complement of B_r' in H. By the convergence explained in the last paragraph, some portion $B_{r,t}' \subset Y_{r,t}$ is a normal graph over B_r' with graph function $u_{r,t}$ converging to 0 in $\mathcal{C}^2(B_r') \cap \mathcal{C}^\infty(B_r' \setminus (B_r' \cap \{x = 0\})$. Finally, choose a smooth nonnegative cutoff function χ_r which has support in $A_r \setminus (A_r \cap B_r')$ and which equals 1 in A_r'. Let $Y_{r,t}'$ be the surface which agrees with $Y_{r,t}$ over A_r' and which has graph function $\chi_r u_{r,t}$ over B_r'.

By construction, each $Y_{r,t}'$ coincides with the totally geodesic hemisphere in the region B_r, and this region is disjoint from all of the other regions A_i, $i \neq r$. This means that we may define the surface Y_t' to be the superposition of these k separate surfaces, since they all agree on the complement of the union of the A_r in the hemisphere H.

Observe that these surfaces are minimal in $H \setminus (A_1 \cup \ldots \cup A_k)$ and in $A_1' \cup \ldots \cup A_k'$, and the discrepancy from being minimal in the overlap regions tends to 0 as $t \to \infty$.

Analysis of the Jacobi operator. Consider the Jacobi operator

$$L_r = \Delta_{Y_r} + |A_r|^2 - 2$$

on the surface Y_r. This operator has continuous spectrum filling out the half-line $(-\infty, -9/4]$ and a finite number of L^2 eigenvalues above that ray. The assumption that Y_r is nondegenerate means that $L_r : H^2(Y_r) \to L^2(Y_r)$ is an isomorphism, i.e. 0 is not an L^2 eigenvalue. It is also the case, cf. [**1**], that under this condition, L_r is an isomorphism on other function spaces better suited for the gluing argument. In particular, let $x^\delta \mathcal{C}^{k,\alpha}$ denote the intrinsic Hölder space (relative to the metric on Y_r induced from the hyperbolic metric) weighted by the function x^δ, where x is the upper half-space coordinate restricted to Y_r. As described carefully in [**1**], if $0 < \delta < 3$, then

$$L_r : x^\delta \mathcal{C}^{2,\delta}(Y_r) \longrightarrow x^\delta \mathcal{C}^{0,\alpha}(Y_r)$$

is an isomorphism. Denote its inverse by G_r. It is very important that we do not just know the existence of this operator abstractly, but realize that it is a pseudodifferential operator for which we have a rather explicit description of the asymptotic behavior of its Schwartz kernel.

Let us now define a family of weighted Hölder spaces on the surfaces Y_t'. We have already defined the cutoff functions χ_r, $r = 1, \ldots, k$, and it is clearly possible to add one extra smooth nonnegative function χ_0 which equals 1 on $H \setminus (A_1 \cup \ldots \cup A_k)$ and is supported away from $A_1' \cup \ldots, A_k'$, such that $\{\chi_0, \ldots, \chi_k\}$ is a partition of unity on Y_t'. (We suppress the dependence on t in the χ_r.) Now define

$$\mathcal{C}^{\ell,\alpha}_{\delta,t}(Y_t') = \{u = \sum_{r=0}^{k} \chi_r u_r, \text{ where } u_r = (M_t \circ F_r)^* v_r, \quad v_r \in x^\delta \mathcal{C}^{\ell,\delta}(Y_r),$$
$$r = 1, \ldots, k, \text{and } u_0 \equiv v_0 \in x^\delta \mathcal{C}^{\ell,\alpha}(H)\},$$

endowed with the norm

$$||u||_{\delta,t} = \sum_{r=0}^{k} ||v_r||_{\ell,\alpha,\delta}.$$

Notice that the elements of $\mathcal{C}^{\ell,\alpha}_{\delta,t}(Y_t')$ coincide with those in $x^\delta \mathcal{C}^{\ell,\alpha}(Y_t')$, but the norm in which there is a hidden extra t dependence, so in particular this norm is not uniformly equivalent as $t \nearrow \infty$ to the standard norm on $x^\delta \mathcal{C}^{\ell,\alpha}(Y_t')$, which is given by an expression similar to the one above, but using the summands u_r instead of v_r.

Next, we can transfer the inverse G_r on Y_r using the mapping $M_t \circ F_r$ to an operator $G_{r,t}$ on $Y_{r,t}'$, and then define

$$\widetilde{G}_t = \sum_{r=0}^{k} \tilde{\chi}_r G_{r,t} \chi_r.$$

Here each $\tilde{\chi}_r$ is a nonnegative smooth cutoff function which is equal to 1 on the support of χ_r and vanishes outside a larger neighbourhood. We compute that if L_t denotes the Jacobi operator on Y_t', then

$$L_t \widetilde{G}_t = \mathrm{Id} - \sum_{r=0}^{k} [L_t \tilde{\chi}_r] G_{r,t} \chi_r := \mathrm{Id} - K_t.$$

The operator K_t is a smoothing operator; this is because the supports of $[L_t, \tilde{\chi}_r]$ and χ_r are disjoint from one another, and because G_r is a pseudodifferential operator, the Schwartz kernel of which is necessarily singular only along the diagonal. Moreover, it is possible to choose the supports of these two functions, $[L_t, \tilde{\chi}_r]$ and χ_r, very far from one another. On the other hand, the Schwartz kernel of $G_{r,t}$ has a decay profile equivalent to the one of G_r; namely, $G_r(z,z') \leq C\exp(-3\,d_{Y_r}(z,z'))$. Taking these facts together, and arguing exactly as in [**26**], we conclude that the norm of K_t as a mapping on $\mathcal{C}^{\ell,\alpha}_{\delta,t}$ for any fixed $\delta \in (0,3)$ can be made as small as desired, uniformly in t, by choosing the supports of these cutoff functions appropriately. We conclude from this that

$$L_t : \mathcal{C}^{2,\alpha}_{\delta,t}(Y_t') \longrightarrow \mathcal{C}^{0,\alpha}_{\delta,t}$$

is an isomorphism for all $t > 0$ whenever $0 < \delta < 3$, and the norm of its inverse is uniformly bounded in t as $t \to \infty$.

The gluing construction. If ν is the (hyperbolic) unit normal to Y_t' and ϕ is any function on Y_t', then define the normal graph

$$Y_{t,\phi} = \{\exp_z(\phi(z)\nu(z)) : z \in Y_t'\}.$$

Let $\mathcal{M}$ denote the minimal surface operators on Y_t', i.e. $\mathcal{M}(\phi)$ is the (hyperbolic) mean curvature function of $Y_{t,\phi}$, viewed as a graph over Y_t'. This is a second order quasilinear operator which can be written as a small perturbation of the minimal surface operators for normal graphs on $Y_{r,t}$ and H, but the main thing we need to know about it is that its linearization at $\phi = 0$ is simply the Jacobi operator L_t.

The perturbation argument is standard. Set $\mathcal{M}(0) = f$. It is not hard to see that $||f||_{0,\alpha,\delta} \to 0$ as $t \to \infty$. Expand $\mathcal{M}(\phi) = 0$ as

$$f + L_t\phi + Q_t(\phi) = 0 \Longrightarrow L_t\phi = -f - Q_t(\phi);$$

here Q_t is a quadratic remainder term involving the terms ϕ, $\nabla\phi$, and $\nabla^2\phi$ which satisfies

$$||Q_t(\phi)||_{0,\alpha,\delta} \leq C||\phi||^2_{2,\alpha,\delta}$$

and

$$||Q_t(\phi) - Q_t(\psi)||_{0,\alpha,\delta} \leq C(||\phi||_{2,\alpha,\delta} + ||\psi||_{2,\alpha,\delta})||\phi - \psi||_{2,\alpha,\delta}.$$

The equation

$$\phi = -G_t(f + Q_t(\phi))$$

can then be solved using the estimates above by a straightforward contraction mapping argument.

It is easy from the construction to see that if t is quite large, then $||\phi||_{2,\alpha,\delta}$ is small and the surface $Y_t := Y_{t,\phi}$ is embedded. Since $\phi \to 0$ at $\partial_\infty Y_t'$, we see that the new surface has the same boundary curve at infinity. The fact that Y_t converges in $\mathcal{C}^2$ away from the points $q_1, \ldots, q_k$ follows directly from the construction. q.e.d.

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