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VIRTUAL HOMOLOGICAL TORSION OF CLOSED HYPERBOLIC 3-MANIFOLDS

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Abstract

In this paper, we will use Kahn and Markovic's immersed almost totally geodesic surfaces ([**KM1**]) to construct certain immersed π_1-injective 2-complexes in closed hyperbolic 3-manifolds. Such 2-complexes are locally almost totally geodesic except along a 1-dimensional subcomplex. By using Agol 's result that the fundamental groups of closed hyperbolic 3-manifolds are vitually compact special ([**Ag**], [**Wi**]) and other works on geometric group theory, we will show that any closed hyperbolic 3-manifold virtually contains any prescribed subgroup in the homological torsion. More precisely, our main result is, for any finite abelian group A, and any closed hyperbolic 3-manifold M, M admits a finite cover N, such that A is a direct summand of $Tor(H_1(N;\mathbb{Z}))$.

1. Introduction

1.1. Background. In [**Lü1**], Lück showed that the L^2-betti numbers of a CW-complex with residually finite fundamental group can be approximated by the betti numbers of a cofinal tower of finite regular cover. For the definitions of various L^2-invariants, see [**Lü2**].

Theorem 1.1 ([**Lü1**])**.** *Let X be a finite, connected CW-complex with residually finite fundamental group Γ. Let $\Gamma \supset \Gamma_1 \supset \cdots \supset \Gamma_n \supset \cdots$ be a nested sequence of finite index normal subgroups of Γ with $\cap\Gamma_n = \{1\}$, and let X_n be the finite cover of X associated with $\Gamma_n \subset \Gamma$, then*

$$\lim_{n\to\infty} \frac{b_p(X_n)}{[\Gamma : \Gamma_n]} = b_p^{(2)}(X).$$

Since finite volume hyperbolic 3-manifolds have vanishing L^2-betti numbers ([**LL**]), by applying the above result to hyperbolic 3-manifolds, we have the following immediate corollary.

Corollary 1.2. *For any hyperbolic 3-manifold M with finite volume, and any tower of finite regular covers $\cdots \to M_n \to \cdots \to M_1 \to M$ with*

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$\cap\pi_1(M_i) = \{1\}$,

$$\lim_{n\to\infty} \frac{b_1(M_n)}{[\pi_1(M):\pi_1(M_n)]} = 0.$$

Along with Agol's virtually infinite first betti number theorem ([**Ag**]), these results imply that the first betti numbers of finite covers of a fixed hyperbolic 3-manifold can go to infinity, but this trend does not grow very fast, which is a very interesting phenomenon.

On the other hand, a natural question is, whether the above approximation of the L^2-betti number can be generalized to some approximation of the L^2-torsion.

In particular, in [**LS**], Lück and Schick showed that, for a finite volume hyperbolic 3-manifold M, its L^2-torsion is related with its hyperbolic volume by the following equality:

$$\rho^{(2)}(\widetilde{M}) = -\frac{Vol(M)}{6\pi}.$$

So there arises the following natural question (see [**Lü2**] Question 13.73, [**Lü3**] Question 1.12 and [**BV**]).

Question 1.3. *Let M be a hyperbolic 3-manifold with finite volume, does there exist a cofinal tower of finite regular covers $\cdots \to M_n \to \cdots \to M_1 \to M$, such that*

$$\lim_{n\to\infty} \frac{\ln|Tor(H_1(M_n;\mathbb{Z}))|}{[\pi_1(M):\pi_1(M_n)]} = \frac{Vol(M)}{6\pi}?$$

In [**Le**], Le claim that

$$\lim_{n\to\infty} \frac{\ln|Tor(H_1(M_n;\mathbb{Z}))|}{[\pi_1(M):\pi_1(M_n)]} \leq \frac{Vol(M)}{6\pi}$$

holds for any cofinal tower of finite regular covers.

If the answer of Question 1.3 is yes, any hyperbolic 3-manifold admits a certain cofinal tower of finite regular covers, with exponential growth on their homological torsion. However, a much weaker question, whether any hyperbolic 3-manifold virtually has nontrivial homological torsion, was still unknown. In the survey paper [**AFW**], Aschenbrenner, Friedl and Wilton asked the following question.

Question 1.4. *Let M be a hyperbolic 3-manifold with finite volume, does M admit a finite cover N such that $Tor(H_1(N;\mathbb{Z})) \neq 0$?*

This paper is devoted to answer Question 1.4 for closed hyperbolic 3-manifolds. Actually, we will prove that any closed hyperbolic 3-manifold virtually contains any prescribed subgroup in its homological torsion.

Theorem 1.5. *For any finite abelian group A, and any closed hyperbolic 3-manifold M, M admits a finite cover N, such that A is a direct summand of $Tor(H_1(N;\mathbb{Z}))$.*

Since Agol showed that hyperbolic 3-manifolds have virtually infinite first betti number ([**Ag**]) and the first betti number does not decrease under taking finite cover, we have the following immediate corollary.

Corollary 1.6. *For any finitely generated abelian group A, and any closed hyperbolic 3-manifold M, M admits a finite cover N, such that A is a direct summand of $H_1(N;\mathbb{Z})$.*

Remark 1.7. In a previous draft of this paper, the author used the result that $\pi_1(M)$ is LERF ([**Ag**],[**Wi**]), and only showed that A embeds into $H_1(N;\mathbb{Z})$. Then Agol and Friedl informed the author about the virtual retract property of quasi-convex subgroups in $\pi_1(M)$ ([**HW**]), then we could promote the result to make A to be a direct summand.

In the proof of Theorem 1.5, we will use Kahn and Markovic's construction of immersed almost totally geodesic surfaces in closed hyperbolic 3-manifolds ([**KM1**]). Since Kahn and Markovic's construction requires the manifold has a positive injectivity radius, it does not work for hyperbolic 3-manifolds with cusps. So we can not show the same result for cusped 3-manifolds, and we have the following natural question.

Question 1.8. *Whether Theorem* 1.5 *holds for finite volume hyperbolic* 3*-manifolds with cusps?*

1.2. Sketch of the Proof. In this paper, we will always use the symbols $\mathbf{l}$, $\mathbf{d}$ to denote the complex length and complex distance. The definitions of $\mathbf{l}$, $\mathbf{d}$, $\mathbf{hl}_{\Pi}$ and s are given in Section 2, which are standard notations in [**KM1**].

We will use Kahn and Markovic's construction of immersed almost totally geodesic surfaces in closed hyperbolic 3-manifolds ([**KM1**]) to do the following construction. For any closed hyperbolic 3-manifold M and any positive integer $p \geq 2$, we will construct an immersed π_1-injective 2-complex $f: X_p \looparrowright M$, which provides us the virtual homological torsion.

More precisely, suppose $f: S \looparrowright M$ is a Kahn-Markovic surface, and S is equipped with a pants decomposition $\mathcal{C}$. Then Kahn and Markovic's theorem implies that, there exist some small number $\epsilon > 0$ and some large number $R > 0$, such that for any simple closed curve $C \in \mathcal{C}$, $|\mathbf{hl}(C) - \frac{R}{2}| < \epsilon$ and $|s(C) - 1| < \frac{\epsilon}{R}$ holds. The exponential mixing property of the frame flow ([**Mo**],[**Po**]) implies that there exists a closed geodesic γ in M, such that $|\mathbf{l}(\gamma) - \frac{R+2\pi i}{p}| < \frac{\epsilon}{p}$. Moreover, we can choose S such that $f(C)$ goes along γ for p times for some $C \in \mathcal{C}$.

By passing to a two-fold cover of S if necessary, we can cut S along C to get a connected surface S' with two oriented boundary components C_1 and C_2, with $[C_1] - [C_2] = 0$ in $H_1(S';\mathbb{Z})$. Then X_p is defined to be the quotient space of S' under the $\frac{2\pi}{p}$-rotations on C_1 and C_2 respectively. Let c_1 and c_2 denote the image of C_1 and C_2 in X_p respectively (with induced orientations), and we still use f to denote the map $f : X_p \looparrowright M$ induced by the immersion $S \looparrowright M$.

Geometrically, in the closed hyperbolic 3-manifold M, away from points in $c_1 \cup c_2$, $f(X_p)$ locally looks like an almost totally geodesic surface. On a neighborhood of $f(c_i)$, $f(X_p)$ is almost a (p-prong) $\times I$ with the top and bottom identified by the $\frac{2\pi}{p}$-rotation. Here the p-prong satisfies that any two adjacent edges have angle $\frac{2\pi}{p}$.

By doing cut-and-paste surgeries on X_p with other Kahn-Markovic surfaces, we can assume that any essential arc in X_p with end points in $c_1 \cup c_2$ is very long. In this case we will show that $f : X_p \looparrowright M$ is π_1-injective.

Now we give two strategies to construct virtual homological torsions for closed hyperbolic 3-manifolds. One strategy uses LERF and the other one uses the virtual retract property. The strategy using LERF can only give an embedding of the finite abelian group A into $Tor(H_1(N, \mathbb{Z}))$ for some finite cover N; while the second strategy can show that A is actually a virtual direct summand, which is stronger. However, the first strategy gives us an interesting codimension-0 submanifold in some finite cover N, which might be useful in some further research, so we give both strategies here.

Strategy I: Let $\widetilde{M}$ be the infinite cover of M associate to $f_*(\pi_1(X_p)) \subset \pi_1(M)$, then $\widetilde{M}$ is a geometric finite hyperbolic 3-manifold. Let $\hat{M}$ be a compact core of $\widetilde{M}$, then the boundary of $\hat{M}$ is incompressible, and there exists an order-p element $\alpha = [c_1] - [c_2] \in H_1(\hat{M}; \mathbb{Z})$. It is also easy to show that, in the order-p subgroup of $H_1(\hat{M}; \mathbb{Z})$ generated by α, only 0 and $\frac{p}{2}\alpha$ can be carried by $H_1(\partial\hat{M}; \mathbb{Z})$ when p is even, and only 0 is carried by $H_1(\partial\hat{M}; \mathbb{Z})$ when p is odd.

Since fundamental groups of hyperbolic 3-manifolds are LERF ([**Ag**], [**Wi**]), by Scott's criterion of LERF ([**Sc**]), there exists an intermediate finite cover $N \to M$ of $\widetilde{M} \to M$ such that $\hat{M}$ embeds into N. By an M-V sequence argument, $\hat{M}$ gives a $\mathbb{Z}_{\sigma(p)}$ subgroup in $Tor(H_1(N; \mathbb{Z}))$. Here $\sigma(p) = p$ when p is an odd number, and $\sigma(p) = p/2$ if p is even.

For two such geometrically finite subgroups $G_1 = (f_1)_*(\pi_1(X_{p_1}))$ and $G_2 = (f_2)_*(\pi_1(X_{p_2}))$, we can find $g \in \pi_1(M)$ such that both of the limit points of g in S^2_∞ do not lie in the limit sets $\Lambda(G_1)$ and $\Lambda(G_2)$. Then for a large enough positive integer n, the same argument as above shows that the geometric finite subgroup $G_1 * g^n G_2 g^{-n} \subset \pi_1(M)$ gives a $\mathbb{Z}_{\sigma(p_1)} \oplus \mathbb{Z}_{\sigma(p_2)}$ subgroup in the homology of some finite cover N. The result for a general finite abelian group A can be shown by induction as the $\mathbb{Z}_{\sigma(p_1)} \oplus \mathbb{Z}_{\sigma(p_2)}$ case.

Strategy II: Agol and Wise showed that $\pi_1(M)$ is virtually special ([**Ag**],[**Wi**]), so we can suppose that $\pi_1(M)$ is already the group of a special cube complex. Since quasi-convex subgroups of special groups are virtual retract ([**HW**]), there exists a finite cover N of M, such that the following conditions hold.

1) $\pi_1(X_p) \subset \pi_1(N)$.

2) For the inclusion map $i : \pi_1(X_p) \to \pi_1(N)$, there exists a retract homomorphism $r : \pi_1(N) \to \pi_1(X_p)$ such that $r \circ i = id_{\pi_1(X_p)}$.

The maps on fundamental groups induce maps on the first homology:

$$H_1(X_p; \mathbb{Z}) \xrightarrow{i_*} H_1(N; \mathbb{Z}) \xrightarrow{r_*} H_1(X_p; \mathbb{Z}).$$

Since $r_* \circ i_* = id$, we know that $H_1(X_p; \mathbb{Z})$ is a direct summand of $H_1(N; \mathbb{Z})$.

It is easy to compute that $H_1(X_p; \mathbb{Z}) \cong \mathbb{Z}^{2g+1} \oplus \mathbb{Z}_p$, so $\mathbb{Z}_p$ is a direct summand of $H_1(N; \mathbb{Z})$.

For a general finite abelian group A, we can do the induction as in Strategy I and use the virtual retract property to construct our desired finite cover N.

This paper is organized as the following. In Section 2, we will give a quick review of Kahn and Markovic's result on constructing immersed almost totally geodesic surfaces in closed hyperbolic 3-manifolds ([**KM1**]), and prove some related lemmas. In Section 3, we will carry out the above discussion more concretely and rigorously, modulo the π_1-injectivity result (Theorem 3.4). The π_1-injectivity property of $f : X_p \looparrowright M$ is a technical result and the proof will be deferred to Section 4.

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2. A Review of Kahn and Markovic's Works and Further Results

In this section, we give a quick review of Kahn and Markovic's works on constructing immersed almost totally geodesic surfaces in closed hyperbolic 3-manifolds (see [**KM1**]). After introducing their works, we will develop a few related lemmas.

In [**KM1**], Kahn and Markovic proved the following Surface Subgroup Theorem, which is the first step to prove Thurston's Virtual Haken and Virtual Fibered Conjectures. (The conjectures were raised in [**Th2**], and settled in [**Ag**]).

Theorem 2.1 ([**KM1**]). *For any closed hyperbolic 3-manifold M, there exists an immersed closed hyperbolic surface $f : S \looparrowright M$, such that $f_* : \pi_1(S) \to \pi_1(M)$ is an injective map.*

Actually, the surfaces constructed in Theorem 2.1 are almost totally geodesic surfaces, which are constructed by pasting oriented *good pants* together along oriented *good curves* in an almost totally geodesic way. In the following, we will describe Kahn and Markovic's construction with more details.

At first, we need to give some geometric definitions.

Let α be an oriented geodesic arc in a closed hyperbolic 3-manifold with initial point p and terminal point q. For two unit normal vectors $\vec{v}$ and $\vec{w}$ of α at p and q respectively, we define $\mathbf{d}_\alpha(\vec{v},\vec{w})$ by the following way. Let $\vec{v}'$ be the parallel transportation of $\vec{v}$ to q along α, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be the oriented angle between $\vec{v}'$ and $\vec{w}$ (with respect to the orientation of α), and the length of α be $l > 0$. Then the complex distance between $\vec{v}$ and $\vec{w}$ along α is defined to be $\mathbf{d}_\alpha(\vec{v},\vec{w}) = l + \theta i$.

For an oriented closed geodesic γ in a hyperbolic 3-manifold, we define its complex length in a similar way. Choose an arbitrary point p on γ and a unit normal vector $\vec{v}$ of γ at p, then we can consider γ as an oriented geodesic arc from p to p. Then the complex length of γ is defined to be $\mathbf{l}(\gamma) = \mathbf{d}_\gamma(\vec{v},\vec{v})$. This complex length not only measures the length of γ in the usual sense, but also measures the rotation angle of the corresponding hyperbolic isometry. Note that the complex length of a closed geodesic does not depend on the orientation and the choices we made.

In the following, we will use Π^0 to denote the oriented pair of pants.

Definition 2.2. For a closed hyperbolic 3-manifold M, a map $f : \Pi^0 \to M$ is called a *skew pair of pants* if $f_* : \pi_1(\Pi^0) \to \pi_1(M)$ is injective, and $f(\partial\Pi^0)$ is a union of three closed geodesics.

We will always think about homotopic skew pair of pants as the same object, and we will use Π to denote a skew pair of pants $f : \Pi^0 \to M$ when it does not cause any confusion.

Let C_1, C_2 and C_3 be the three oriented boundary components of Π^0, then let γ_1, γ_2 and γ_3 be the three oriented closed geodesics $f(C_1)$, $f(C_2)$ and $f(C_3)$ respectively. Let a_i be the simple arc on Π^0 which connects C_{i-1} and C_{i+1}, such that a_1, a_2, and a_3 are disjoint with each other (they are called seams of Π_0). Then we can assume that $f(a_i)$ is a geodesic arc perpendicular with both γ_{i-1} and γ_{i+1} for $i = 1, 2, 3$, and denote $f(a_i)$ by η_i.

Now we fix one γ_i, and give orientations for η_{i-1} and η_{i+1} such that they are both pointing away from γ_i. Then η_{i-1} and η_{i+1} divide γ_i to two oriented geodesic arcs γ_i^1 and γ_i^2, such that the orientation on γ_i^1 goes from $\eta_{i-1} \cap \gamma_i$ to $\eta_{i+1} \cap \gamma_i$. Let $\vec{v}_{i-1}$ and $\vec{v}_{i+1}$ be the unit tangent vectors of η_{i-1} and η_{i+1} at $\eta_{i-1} \cap \gamma_i$ and $\eta_{i+1} \cap \gamma_i$ respectively, then we have a pair of vectors $(\vec{v}_{i-1}, \vec{v}_{i+1})$ on the unit normal bundle $N^1(\gamma_i)$, which is called the pair of feet of Π on γ_i. The hyperbolic geometry of right-angled hexagons in $\mathbb{H}^3$ implies that $\mathbf{d}_{\gamma_i^1}(\vec{v}_{i-1},\vec{v}_{i+1}) = \mathbf{d}_{\gamma_i^2}(\vec{v}_{i+1},\vec{v}_{i-1})$. So we can define the half length of γ_i with respect to Π by

$$\mathbf{hl}_\Pi(C_i) = \mathbf{d}_{\gamma_i^1}(\vec{v}_{i-1},\vec{v}_{i+1}) = \mathbf{d}_{\gamma_i^2}(\vec{v}_{i+1},\vec{v}_{i-1}).$$

Now we are ready to define *good curves* and *good pants*.

Definition 2.3. Fix a small number $\epsilon > 0$ and a large number $R > 0$. For a closed oriented geodesic γ in M, we say γ is an (R,ϵ)-*good curve* if $|\mathbf{l}(\gamma) - R| < 2\epsilon$. The set of (R,ϵ)-good curves is denoted by $\mathbf{\Gamma}_{R,\epsilon}$.

For a skew pair of pants $f : \Pi^0 \to M$, we say it is an (R,ϵ)-*good pants* if $|\mathbf{hl}_\Pi(C) - \frac{R}{2}| < \epsilon$ holds for all the three cuffs (boundary components) of Π^0. The set of (R,ϵ)-good pants is denoted by $\mathbf{\Pi}_{R,\epsilon}$.

In the following, we will work with a very small number $\epsilon > 0$ and a very large number $R > 0$, and the precise value of ϵ and R will be determined later. When R and ϵ have been fixed, we will only talk about good curves and good pants, instead of (R,ϵ)-good curves and (R,ϵ)-good pants, when it does not cause any confusion. Note that oriented boundary components of (R,ϵ)-good pants are (R,ϵ)-good curves.

For a good curve $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$, the normal bundle $N^1(\gamma)$ of γ is naturally identified with $\mathbb{C}/\mathbf{l}(\gamma)\mathbb{Z}+2\pi i\mathbb{Z}$. If we have a skew pair of pants Π which has γ as one of its oriented boundary component, we can define the half normal bundle of γ by $N^1(\sqrt{\gamma}) = \mathbb{C}/\mathbf{hl}_\Pi(\gamma)\mathbb{Z} + 2\pi i\mathbb{Z}$. Then the pair of feet of Π on γ are identified to one point in $N^1(\sqrt{\gamma})$, which is called the *foot* of Π on γ, and denoted by $foot_\gamma(\Pi)$.

Now we are ready to talk about maps from surfaces to closed hyperbolic 3-manifolds. In the following, we will fix a closed hyperbolic 3-manifold and work on it.

Suppose S is a compact oriented closed surface with negative Euler characteristic, equipped with a pants decomposition $\mathcal{C}$. Then the closure of each component of $S\setminus\mathcal{C}$ is an oriented pair of pants, and we call such a component a pants in S.

Definition 2.4. A map $f : S \to M$ is called *viable* if the following conditions hold.

- For each pants Π in S, $f|_\Pi : \Pi \to M$ is a skew pair of pants.
- For any two pants Π and Π' in S sharing a curve $C \in \mathcal{C}$, $\mathbf{hl}_\Pi(C) = \mathbf{hl}_{\Pi'}(C)$ holds.

So for a viable map $f : S \to M$, we will use $\mathbf{hl}(C)$ to denote $\mathbf{hl}_\Pi(C)$ for each $C \in \mathcal{C}$. For two pants in S sharing a curve $C \in \mathcal{C}$, we give C an arbitrary orientation. Let Π be the pants lies to the left of C on S, and Π' lies to the right. Let $\gamma = f(C)$, and $\bar{\gamma}$ be the same closed geodesic with the opposite orientation, then we can compare the feet of Π and Π' on $N^1(\sqrt{\gamma})$ by the following shearing parameter (here $N^1(\sqrt{\gamma})$ and

$N^1(\sqrt{\bar{\gamma}})$ are naturally identified with each other):

$s(C) = foot_\gamma(f|_\Pi) - foot_{\bar{\gamma}}(f|_{\Pi'}) - \pi i \in N^1(\sqrt{\gamma}) = \mathbb{C}/\mathbf{hl}(C)\mathbb{Z} + 2\pi i\mathbb{Z}.$

Now we can precisely describe the immersed almost totally geodesic surfaces constructed in [**KM1**].

Theorem 2.5 ([**KM1**]). *For any closed hyperbolic 3-manifold M, there exists constants $q > 0$ and $K > 0$, such that for every small enough $\epsilon > 0$ and every large enough $R > 0$, the following statement holds. There exists a closed surface S equipped a pants decomposition $\mathcal{C}$, and a viable map $f : S \to M$ such that for any $C \in \mathcal{C}$, we have*

$$\text{(1)} \qquad \begin{cases} |\mathbf{hl}(C) - \frac{R}{2}| < \epsilon, \\ |s(C) - 1| < KRe^{-qR} < \frac{\epsilon}{R}. \end{cases}$$

Moreover, $f_* : \pi_1(S) \to \pi_1(M)$ is injective.

We will call a viable map $f : S \to M$ an (R,ϵ)-*almost totally geodesic surface*, if the inequality (1) holds for each $C \in \mathcal{C}$.

The existence of such (R,ϵ)-almost totally geodesic closed surfaces is proved by the following strategy in [**KM1**]. For any good curve γ, one can consider all the good pants in M with γ as one of its oriented boundary component, then consider all the feet $foot_\gamma(\Pi)$ on $N^1(\sqrt{\gamma})$. Kahn and Markovic showed that these feet on $N^1(\sqrt{\gamma})$ are very equidistributed, so they can paste all the good pants together in a proper way such that $|s(C) - 1| < \frac{\epsilon}{R}$ holds.

More precisely, Kahn and Markovic constructed an integer valued measure μ_0 on $\mathbf{\Pi}_{R,\epsilon}$, with the following nice property.

Proposition 2.6 ([**KM1**]). *There exists an integer valued measure μ_0 on $\mathbf{\Pi}_{R,\epsilon}$ with the following properties. Let $\hat{\partial}\mu_0$ be the counting measure on*

$$N^1(\sqrt{\mathbf{\Gamma}_{R,\epsilon}}) = \bigcup_{\gamma \in \mathbf{\Gamma}_{R,\epsilon}} N^1(\sqrt{\gamma})$$

given by the feet of pants in $\mathbf{\Pi}_{R,\epsilon}$ and weighted by μ_0. Then for any $\gamma \in \mathbf{\Gamma}_{R,\epsilon}$, there exists a constant $K_\gamma \geq 0$, such that $\hat{\partial}\mu_0|_{N^1(\sqrt{\gamma})}$ is KRe^{-qR}-equivalent to $K_\gamma\lambda$ for some universal constant $K > 0$. Here λ is the standard Lebesgue measure on $N^1(\sqrt{\gamma}) \cong \mathbb{C}/\mathbf{hl}(\gamma)\mathbb{Z} + 2\pi i\mathbb{Z}$.

For two Borel measures μ and ν on a compact metric space X, we say μ and ν are δ-*equivalent* for some $\delta > 0$ if the following conditions hold.

- $\mu(X) = \nu(X)$.
- For any Borel measurable subset $A \subset X$, $\mu(A) < \nu(N_\delta(A))$ holds. Here $N_\delta(A)$ is the δ-neighborhood of A in X.

In the proof of the existence of good pants (curves) and the equidistribution result, the following exponential mixing property of the frame flow played a crucial role.

Theorem 2.7 ([**Mo**],[**Po**]). *Let M be a closed hyperbolic 3-manifold, $\mathcal{F}(M)$ be the frame bundle of M, Λ be the Liouville measure on $\mathcal{F}(M)$ which is invariant under the frame flow $g_t : \mathcal{F}(M) \to \mathcal{F}(M)$.*

Then there exists a constant $q > 0$ that depends only on M, such that the following statement holds. Let $\psi, \phi : \mathcal{F}(M) \to \mathbb{R}$ be two C^1 functions, then for any $r \in \mathbb{R}$,

$$\left|\Lambda\left(\mathcal{F}(M)\right) \int_{\mathcal{F}(M)} (g_r^* \psi) \cdot \phi \, d\Lambda - \int_{\mathcal{F}(M)} \psi \, d\Lambda \cdot \int_{\mathcal{F}(M)} \phi \, d\Lambda\right| \leq C e^{-q|r|}.$$

Here $C > 0$ only depends on the C^1-norms of ψ and ϕ.

For technical reasons, we need a slightly stronger condition than (1) in this paper, so we need the following proposition. In [**Sa**], Šarić has shown that we can strengthen $|\mathbf{hl}(C) - \frac{R}{2}| < \epsilon$ to $|\mathbf{hl}(C) - \frac{R}{2}| < \frac{\epsilon}{R}$, and it is the essential part of the proof, so we only give a very brief proof here.

Proposition 2.8. *For any closed hyperbolic 3-manifold M, there exists a constant $q > 0$ and a polynomial $P(\cdot)$, such that for every small enough $\epsilon > 0$ and large enough $R > 0$, the following statement holds. There exists a closed surface S equipped with a pants decomposition $\mathcal{C}$, and a viable map $f : S \to M$, such that for any $C \in \mathcal{C}$, we have*

$$\begin{cases} |\mathbf{hl}(C) - \frac{R}{2}| < \frac{\epsilon}{R}, \\ |s(C) - 1| < P(R) e^{-qR} < \frac{\epsilon}{R^2}. \end{cases} \tag{2}$$

Moreover, $f_* : \pi_1(S) \to \pi_1(M)$ is injective.

Proof. In the introduction of [**Sa**], Šarić pointed out that $|\mathbf{hl}(C) - \frac{R}{2}| < \epsilon$ can be replaced by $|\mathbf{hl}(C) - \frac{R}{2}| < \frac{\epsilon}{R}$. The main reason that such a refinement is applicable is, the exponential mixing property of frame flow ([**Mo**], [**Po**]) gives the exponential rate, which beats any polynomial rate.

More precisely, in Kahn and Markovic's construction in [**KM1**], the following function is a crucial ingredient. For an arbitrary point F_0 in the frame bundle $\mathcal{F}(\mathbb{H}^3)$, we can choose a C^1 bump function $f_\epsilon^{F_0} : \mathcal{F}(\mathbb{H}^3) \to \mathbb{R}_{\geq 0}$ supporting on the ϵ-neighborhood of F_0 in $\mathcal{F}(\mathbb{H}^3)$ (here $\epsilon > 0$ is smaller than the injectivity radius of M), such that

$$\int_{\mathcal{F}(\mathbb{H}^3)} f_\epsilon^{F_0}(x) d\Lambda(x) = 1.$$

By pulling back $f_\epsilon^{F_0}$ by $Isom_+(\mathbb{H}^3)$ and projecting to $\mathcal{F}(M)$, we get a function $f_\epsilon^F : \mathcal{F}(M) \to \mathbb{R}_{\geq 0}$ centered at F for each $F \in \mathcal{F}(M)$. Then Kahn and Markovic's constructions of (R, ϵ)-good pants and immersed almost totally geodesic surfaces start from the function f_ϵ.

In [**Sa**], Šarić gave the following observation. For the time t frame flow, we consider an alternative bump function $f_{\frac{\epsilon}{t}}$. By taking $f_{\frac{\epsilon}{t}}(x)$ to be $t^6 \cdot f_\epsilon(xt)$ up to a constant close to 1, we can suppose

$$\int_{\mathcal{F}(\mathbb{H}^3)} f_{\frac{\epsilon}{t}}(x) d\Lambda(x) = 1.$$

Since the frame flow has exponential mixing rate in term of t, while the constant C in Theorem 2.7 can be estimated by the H_2^2-Sobolev norm of $f_{\frac{\epsilon}{t}}$, which grows in a polynomially rate, so we have

$$(3) \quad \left| \Lambda\left(\mathcal{F}(M)\right) \int_{\mathcal{F}(M)} (g_t^* f_{\frac{\epsilon}{t}}^{F_1})(x) f_{\frac{\epsilon}{t}}^{F_2}(x) d\Lambda(x) - 1 \right| \leq P(t) e^{-q|t|} \to 0$$

when t goes to ∞. Then all the works in [**KM1**] are still available under the inequality (3), and $|\mathbf{hl}(C) - \frac{R}{2}| < \frac{\epsilon}{R}$ holds.

For the second inequality in (1), under the new bump function $f_{\frac{\epsilon}{R}}$ and inequality (3), the same argument as in [**KM1**] gives an integer valued measure μ_0 on $\mathbf{\Pi}_{R,\frac{\epsilon}{R}}$, such that $\hat{\partial}\mu_0|_{N^1(\sqrt{\gamma})}$ is $P'(R)e^{-qR}$-equivalent to $K_\gamma \lambda$ for each $\gamma \in \mathbf{\Gamma}_{R,\frac{\epsilon}{R}}$, with P' being a polynomial and $K_\gamma \geq 0$.

So we can get a gluing of good pants with $|s(C) - 1| < P'(R)e^{-qR} < \frac{\epsilon}{R^2}$, and the proof of this proposition is done. q.e.d.

We also need the following two lemmas which are basic for our construction, and their proofs are closely related with Kahn and Markovic's works in [**KM1**].

Lemma 2.9. *For any positive integers $p \geq 2$ and $q \geq 1$, then for small enough $\epsilon > 0$ and large enough $R > 0$, there exists a closed geodesic γ' in M, such that $|\mathbf{l}(\gamma') - \frac{R+2\pi i}{p}| < \frac{\epsilon}{qR}$.*

Proof. Take an arbitrary frame $F = (p, \vec{v}, \vec{n})$ in $\mathcal{F}(M)$, with $p \in M$, $\vec{v}, \vec{n} \in T_p^1(M)$ with $\vec{v} \perp \vec{n}$. Let $\vec{n}'$ be the $\frac{2\pi}{p}$ rotation of $\vec{n}$ along $\vec{v}$, and let $F' = (p, \vec{v}, \vec{n}')$.

Now we consider functions $f^{F}_{\frac{\epsilon}{4qR}}$ and $f^{F'}_{\frac{\epsilon}{4qR}}$ as in the proof of Proposition 2.8. By applying equation (3), with t replaced by $\frac{R}{p}$ and ϵ replaced by $\frac{\epsilon}{4pq}$, there exists an oriented geodesic arc α in M, with two frames $\hat{F}$ and $\hat{F}'$ at its initial and terminal points respectively, such that the following conditions hold.

1) The parallel transportation of $\hat{F}$ along α to its terminal point equals $\hat{F}'$.
2) The first vector component of $\hat{F}$ is tangent to α.
3) The distance between F and $\hat{F}$, and the distance between F' and $\hat{F}'$ in $\mathcal{F}(M)$ are both smaller than $\frac{\epsilon}{4qR}$.

Then by connecting the initial and terminal points of α by an $\frac{\epsilon}{2qR}$-short geodesic in M, we get a closed path which is homotopic to a closed geodesic γ'. Then this γ' satisfies $|\mathbf{l}(\gamma') - \frac{R+2\pi i}{p}| < \frac{\epsilon}{qR}$, by elementary estimations in the hyperbolic geometry. q.e.d.

Lemma 2.10. *There exists a universal constant $D > 0$, such that for any small enough $\epsilon > 0$ and large enough $R > 0$, the following statement holds. For any closed geodesic $\gamma \in \mathbf{\Gamma}_{R,\frac{\epsilon}{DR}}$, there exists a closed surface S with a pants decomposition $\mathcal{C}$, such that there exists an $(R, \frac{\epsilon}{R})$-almost totally geodesic immersion $f : S \looparrowright M$, and $f(C) = \gamma$ for some $C \in \mathcal{C}$.*

Proof. In [**KM1**], the integer valued measure μ_0 in Proposition 2.6 is given by a real valued measure μ on $\mathbf{\Pi}_{R,\epsilon}$ ($\mathbf{\Pi}_{R,\frac{\epsilon}{R}}$ in our case), by first perturbing μ to a rational valued measure, then take an integer multiple.

So we need only to show that $\hat{\partial}\mu(N^1(\sqrt{\gamma})) > 0$. In this case, we can take a small enough perturbation of μ so that $\hat{\partial}\mu_0(N^1(\sqrt{\gamma})) > 0$ holds. Then in the construction of almost totally geodesic surfaces instructed by μ_0, we must use some good pants with one of its cuff being γ. Then we take the component of the Kahn-Markovic surface which contains this good pants.

By the proof in Section 4.8 of [**KM1**], $\hat{\partial}\mu(N^1(\sqrt{\gamma})) > 0$ if and only if there exist two frames $F_1 = (p_1, \vec{v}_1, \vec{n}_1)$ and $F_2 = (p_2, \vec{v}_2, \vec{n}_2)$, and two geodesic arcs α_1 and α_2 in M, such that the following conditions hold.

1) α_1 has initial point p_1 and terminal point p_2, while α_2 has initial point p_2 and terminal point p_1.
2) $\alpha_1\alpha_2$ is homotopic to γ.
3) Let $\omega(F_1) = (p_1, \omega(\vec{v}_1), \vec{n}_1)$ be the $\frac{2\pi}{3}$-rotation of F_1 with respect to $\vec{n}_1$, and $\bar{\omega}(F_2) = (p_2, \bar{\omega}(\vec{v}_2), \vec{n}_2)$ be the $\frac{4\pi}{3}$-rotation of F_2 with respect to $\vec{n}_2$, then both $a_{\alpha_1}(F_1, F_2)$ and $a_{\alpha_2}(\omega(F_1), \bar{\omega}(F_2))$ are positive.

Under the modified Kahn-Markovic condition (2), for two frames F_1 and F_2 in M with a geodesic arc α connecting their base points, $a_\alpha(F_1, F_2)$ is defined by the following way. Take two frames $\hat{F}_1$ and $\hat{F}_2$ in $\mathcal{F}(\mathbb{H}^3)$ projecting to F_1 and F_2 respectively, such that the geodesic arc in $\mathbb{H}^3$ connecting the base points of $\hat{F}_1$ and $\hat{F}_2$ projects to α. Let $r = \frac{R}{2} + \ln\frac{4}{3}$, and let $g_{\frac{r}{4}}(\hat{F}_1) = (p_1', \vec{v}_1', \vec{n}_1')$, $g_{\frac{r}{4}}(\hat{F}_2) = (p_2', \vec{v}_2', \vec{n}_2')$, then $a_\alpha(F_1, F_2)$ is defined by:

$$a_\alpha(F_1, F_2) = \int_{\mathcal{F}(M^3)} (g_{\frac{r}{2}}^* f_{\frac{\epsilon}{DR}}^{(p_1', \vec{v}_1', \vec{n}_1')})(x) f_{\frac{\epsilon}{DR}}^{(p_2', -\vec{v}_2', \vec{n}_2')}(x) d\Lambda(x).$$

Here $D > 0$ is some universal constant.

Then it is easy to check that if $\gamma \in \mathbf{\Gamma}_{R,\frac{\epsilon}{DR}}$, frames F_1 and F_2 satisfying the above conditions do exist. q.e.d.

3. Geometric Constructions

In this section, for each closed hyperbolic 3-manifold M, we will construct an immersed π_1-injective 2-complex $X_p \looparrowright M$ which has good pants as its building blocks. Here X_p is a local model of the homological $\mathbb{Z}_p$-torsion. Then we will show that the immersion $X_p \looparrowright M$ provides homological torsion in some finite cover of M.

3.1. Construction of a 2-complex. We first give a brief sketch of our construction $X_p \looparrowright M$.

At first, there exists a Kahn-Markovic surface $f : S \looparrowright M$, such that for some $C \in \mathcal{C}$, $f(C)$ goes along some closed geodesic γ' for p times with $\mathbf{l}(\gamma')$ close to $\frac{R+2\pi i}{p}$. Then we cut S along C, and quotient the two boundary components by $\frac{2\pi}{p}$-rotations, to get an immersed 2-complex $X_p \looparrowright M$. By doing cut-and-past surgeries, we can make sure that the singular curves on X_p are far away from each other, which guarantees that the immersion $X_p \looparrowright M$ is π_1-injective.

Now we fix a closed hyperbolic 3-manifold M, and work with some very small $\epsilon > 0$ and very large $R > 0$ which will be determined later. Here we divide the construction into a few steps.

Step I. By Lemma 2.9, for any positive integer $p \geq 2$, there exists a closed geodesic γ' in M with $|\mathbf{l}(\gamma') - \frac{R+2\pi i}{p}| < \frac{\epsilon}{pDR}$ (here $D > 0$ is the constant in Lemma 2.10). Let γ be the closed geodesic which travels around γ' for p times, then γ is a nonprimitive closed geodesic with $|\mathbf{l}(\gamma) - R| < \frac{\epsilon}{DR}$, so $\gamma \in \mathbf{\Gamma}_{R,\frac{\epsilon}{DR}}$.

Step II. By Lemma 2.10, there exists an immersed $(R, \frac{\epsilon}{R})$-almost totally geodesic closed surface $f : S \looparrowright M^3$, such that for the corresponding pants decomposition $\mathcal{C}$ of S, there exists $C \in \mathcal{C}$ such that $f(C) = \gamma$. By taking a two-fold cover of S if necessary, we can suppose that C is a non-separating curve on S and the two pants adjacent to C are distinct. Let S' be the surface obtained from S by cutting along C, then S' has an induced pants decomposition $(S', \mathcal{C}')$ (here $\mathcal{C}'$ does not contain the boundary of S'). Let C_1 and C_2 be the two boundary components of S', and they are given orientations such that $[C_1] - [C_2] = 0 \in H_1(S'; \mathbb{Z})$.

Step III. Let $\rho_i : C_i \to C_i$, $i = 1, 2$ be the $\frac{2\pi}{p}$-rotation on the circle. Then we define X_p to be the 2-complex obtained from S' quotient by the ρ_i-action for $i = 1, 2$, and let c_i be the oriented embedded circle in X_p which is the image of C_i. Since C_1 and C_2 are both mapped to γ' for p times, $f : S \looparrowright M$ induces a map $f : X_p \looparrowright M$ with c_i mapped to γ' for $i = 1, 2$. Then the pants decomposition $\mathcal{C}'$ on S' and two curves C_1, C_2 induce a "pants decomposition" on X_p, which is denoted by $(X_p, \mathcal{C}', \{C_1, C_2\})$. Note that C_1 and C_2 are not embedded curves in X_p.

Step IV. Now we define a graph $G(X_p)$ from $(X_p, \mathcal{C}', \{C_1, C_2\})$. Vertices of $G(X_p)$ are pants in X_p, two vertices are connected by an edge if the corresponding two pants share some $C \in \mathcal{C}'$. $G(X_p)$ is a trivalent graph except at two vertices v_1, v_2. These two vertices correspond with the two pants in X_p containing c_1 and c_2 respectively, and both of them are degree-2 vertices.

By a path in a graph G, we mean a sequence of oriented edges in G, such that for two adjacent edges e and e' in the path, the terminal vertex of e equals the initial vertex of e'. For a path in G, its (combinatorial) length is defined to be the number of oriented edges it contains, counted with multiplicity. We say a path in G is inessential, if its initial and terminal vertices are the same vertex v, and the corresponding map between topological spaces $(I, \partial I) \to (G, \{v\})$ is homotopic to the constant map. We say a path is essential if it is not inessential.

Let $l(G(X_p))$ be the length of the shortest essential path in $G(X_p)$ with end points in $\{v_1, v_2\}$, and $n(G(X_p))$ be the number of such paths. We define the complexity of $G(X_p)$ to be

$$c(G(X_p)) = (l(G(X_p)), -n(G(X_p)))\,,$$

and we will do inductive constructions to make $G(X_p)$ more and more complicated until $l(G(X_p)) > Re^{\frac{R}{4}}$.

For any shortest essential path α with length $l \geq 1$, let $k = [\frac{l+1}{2}]$ and e be the k-th edge on α. Let $C_0 \in \mathcal{C}'$ be the curve in the pants decomposition of X_p corresponding with e, and let C_0' denote the corresponding curve in S. Take a copy of S, and pass to a two-fold cover if necessary, such that C_0' is a non-separating curve in S. Then we cut X_p and S along C_0 and C_0' respectively, and re-paste them together to get a connected 2-complex X_p' with an induced pants decomposition $\mathcal{C}''$ (such kind of surgeries have appeared in [**KM2**]). Since the pants in S and X_p have the same feet on $N^1(\sqrt{f(C_0)})$, we still have

$$\begin{cases} |\mathbf{hl}(C) - \frac{R}{2}| < \frac{\epsilon}{R}, \\ |s(C) - 1| < \frac{\epsilon}{R^2}, \end{cases}$$

for any $C \in \mathcal{C}''$.

After this surgery, the shortest essential pathes in $G(X_p)$ going through the edge e have been broken, and the corresponding length increases at least by 1 in $G(X_p')$. So the complexity $c(G(X_p')) = \big(l(G(X_p')), -n(G(X_p'))\big)$ is bigger than $c(G(X_p)) = \big(l(G(X_p)), -n(G(X_p))\big)$, i.e. either $l(G)$ increases, or $l(G)$ does not change and $-n(G)$ increases. After finitely many steps of such constructions, we can assure $l(G) > Re^{\frac{R}{4}}$. For simplicity, we still denote the 2-complex by X_p, and denote the pants decomposition by $(X_p, \mathcal{C}', \{C_1, C_2\})$.

Definition 3.1. A representation $\rho : \pi_1(X_p) \to PSL_2(\mathbb{C})$ is called a *viable representation* if the following conditions hold.

1) For each $C \in \mathcal{C} \cup \{C_1, C_2\}$, let g_C be a generator of $\pi_1(C)$, then $\rho(g_C)$ is a hyperbolic element in $PSL_2(\mathbb{C})$.
2) For each pants Π in X_p, $\rho|_{\pi_1(\Pi)}$ is an injective map, and $\rho(\pi_1(\Pi))$ is a discrete subgroup of $PSL_2(\mathbb{C})$.

3) For any two pants Π, Π' sharing some $C \in \mathcal{C} \cup \{C_1, C_2\}$, $\mathbf{hl}_{\Pi}(C) = \mathbf{hl}_{\Pi'}(C)$ holds.

Note that although $\rho(\pi_1(X_p))$ may not be a discrete subgroup of $PSL_2(\mathbb{C})$, $\mathbf{hl}_{\Pi}(C)$ can still be defined.

A map $f : X_p \to M$ is called a *viable map* if $f_* : \pi_1(X_p) \to \pi_1(M) \subset PSL_2(\mathbb{C})$ is a viable representation.

As a summary of the above construction, we have the following proposition which guarantees the existence of an immersed 2-complex $X_p \looparrowright M$.

Proposition 3.2. *For any closed hyperbolic* 3*-manifold* M*, and any positive integer* $p \geq 2$*, there exists a constant* $\hat{\epsilon} > 0$*, such that for any* $0 < \epsilon < \hat{\epsilon}$ *and any* R *sufficiently large, the following statement holds. There exists a* 2*-complex* X_p *as above with a pants decomposition* $(X_p, \mathcal{C}', \{C_1, C_2\})$*, and a viable map* $f : X_p \looparrowright M$ *such that the following conditions hold.*

1) *The induced graph* $G(X_p)$ *satisfies* $l(G(X_p)) > Re^{\frac{R}{4}}$.
2) *For any* $C \in \mathcal{C}' \cup \{C_1, C_2\}$, $|\mathbf{hl}(C) - \frac{R}{2}| < \frac{\epsilon}{R}$.
3) *For any* $C \in \mathcal{C}'$, $|s(C) - 1| < \frac{\epsilon}{R^2}$.
4) *Let* c_k *be the image of* C_k *in the* 2*-complex* X_p*, then* $|\mathbf{l}(c_k) - \frac{R+2\pi i}{p}| < \frac{\epsilon}{pR}$ *for* $k = 1, 2$.

Remark 3.3. The construction of $X_p \looparrowright M$ is very similar to the construction in [**KM2**].

In [**KM2**], Kahn and Markovic constructed immersed quasi-fuchsian surfaces $S \looparrowright M$ by pasting (generalized) good pants, the immersion satisfies inequality (1) for curves $C \in \mathcal{C}$, except bending along a sparse collection of curves in $\mathcal{C}$ that are far away from each other. Our construction is almost following the same idea with theirs, but we construct immersed 2-complexes, instead of surfaces.

Moreover, the cut-and-paste technique in Step IV of our construction also appeared in [**KM2**]. In [**KM2**], Kahn and Markovic amalgamated two immersed almost totally geodesic surfaces to one immersed quasi-fushsian surface, and they used the cut-and-paste technique to make sure that the bending curves are far away from each other.

3.2. Finite Cyclic Subgroups in Virtual Homology. The following theorem is the most technical theorem in this paper, which is an analogue of Theorem 2.2 in [**KM1**].

Theorem 3.4. *There are universal constants* $\hat{\epsilon} > 0$ *and* $\hat{R} > 0$ *depend only on* p *and* M*, such that for any* $0 < \epsilon < \hat{\epsilon}$ *and any* $R > \hat{R} > 0$*, the following statement holds. If* X_p *is a* 2*-complex with a pants decomposition* $(X_p, \mathcal{C}', \{C_1, C_2\})$ *constructed as last section, and* $\rho : \pi_1(X_p) \to PSL_2(\mathbb{C})$ *is a viable representation such that the following conditions hold.*

1) *The induced graph* $G(X_p)$ *satisfies* $l(G(X_p)) > Re^{\frac{R}{4}}$.
2) *For any* $C \in \mathcal{C}' \cup \{C_1, C_2\}$, $|\mathbf{hl}(C) - \frac{R}{2}| < \frac{\epsilon}{R}$.
3) *For any* $C \in \mathcal{C}'$, $|s(C) - 1| < \frac{\epsilon}{R^2}$.
4) $|\mathbf{l}(c_k) - \frac{R+2\pi i}{p}| < \frac{\epsilon}{pR}$ *for* $k = 1, 2$.

Then $\rho : \pi_1(X_p) \to PSL_2(\mathbb{C})$ *is an injective map and* $\rho(\pi_1(X_p))$ *is a convex-cocompact subgroup of* $Isom_+(\mathbb{H}^3)$.

The proof of Theorem 3.4 will be deferred to Section 4. In this section, we will focus on proving Theorem 1.5 by assuming Theorem 3.4.

To precisely describe a viable representation $\rho : \pi_1(X_p) \to PSL_2(\mathbb{C})$ as in Theorem 3.4, we first give parameters for such a representation. For each curve $C \in \mathcal{C}'$, it is associate with two complex numbers ξ_C and η_C such that $|\xi_C|, |\eta_C| < \epsilon$; for each $C \in \{C_1, C_2\}$, it is associate with a complex number ξ_i with $|\xi_i| < \epsilon$.

We can choose parameters such that the representation $\rho : \pi_1(X_p) \to PSL_2(\mathbb{C})$ satisfies the following conditions.

1) For any $C \in \mathcal{C}' \cup \{C_1, C_2\}$, $\mathbf{hl}(C) = \frac{R}{2} + \frac{\xi_C}{R}$.
2) For any $C \in \mathcal{C}'$, $s(C) = 1 + \frac{\eta_C}{R^2}$.
3) $\mathbf{l}(c_k) = \frac{R+2\pi i}{p} + \frac{\xi_i}{pR}$ for $k = 1, 2$.

Let $\mathbb{D}(0, 1)$ be the disc in the complex plane centered at 0 with radius 1. Then for each $\tau \in \mathbb{D}(0, 1)$, there is a small deformation of ρ, denote by $\rho_\tau : \pi_1(X_p) \to PSL_2(\mathbb{C})$, which is defined by the following conditions.

1) For any $C \in \mathcal{C}' \cup \{C_1, C_2\}$, $\mathbf{hl}(C) = \frac{R}{2} + \frac{\tau\xi_C}{R}$.
2) For any $C \in \mathcal{C}'$, $s(C) = 1 + \frac{\tau\eta_C}{R^2}$.
3) $\mathbf{l}(c_k) = \frac{R+2\pi i}{p} + \frac{\tau\xi_i}{pR}$ for $k = 1, 2$.

Then $\{\rho_\tau\}_{\tau\in\mathbb{D}(0,1)}$ is a continuous family of representations from $\pi_1(X_p)$ to $PSL_2(\mathbb{C})$, such that $\rho_1 = \rho$, and ρ_0 provides us a standard model of studying the representation $\rho : \pi_1(X_p) \to PSL_2(\mathbb{C})$.

Let $q : \widetilde{X}_p \to X_p$ be the universal cover of X_p. There is a natural map $\widetilde{f}_0 : \widetilde{X}_p \to \mathbb{H}^3$ to realize the representation ρ_0. $\widetilde{f}_0$ maps each component of $\widetilde{X}_p \setminus q^{-1}(c_1 \cup c_2)$ to a totally geodesic subsurface in $\mathbb{H}^3$, and two such totally geodesic subsurfaces sharing a geodesic has angle equal to $\frac{2k\pi}{p}$ for some integer $k \neq 0$. For each pants $\Pi \subset X_p$, the induced map $\Pi \to \mathbb{H}^3/\rho_0(\pi_1(\Pi))$ maps Π to a totally geodesic pants with $\mathbf{hl}_\Pi(C) = \frac{R}{2}$. $\widetilde{f}_0$ induces a path metric on $\widetilde{X}_p$: for any $x, y \in \widetilde{X}_p$, define $d(x, y) = inf\{l(\widetilde{f}_0(\gamma))|\ \gamma$ is a path in $\widetilde{X}_p$ with end points x and $y\}$. By using elementary hyperbolic geometry, we have the following lemma.

Lemma 3.5. *For R large enough, if the induced graph* $G(X_p)$ *satisfies* $l(G(X_p)) > Re^{\frac{R}{4}}$, *then* $\widetilde{f}_0 : (\widetilde{X}_p, d) \to (\mathbb{H}^3, d_{\mathbb{H}^3})$ *is injective and is a* *quasi-isometric embedding. In particular, $\rho : \pi_1(X_p) \to PSL_2(\mathbb{C})$ is an injective map.*

Proof. For any two points $x, y \in \widetilde{X}_p$, the shortest path α connecting x and y is a piecewise geodesic. Let $L_1, L_2, \cdots, L_m$ be consecutive geodesics in $q^{-1}(c_1 \cup c_2)$ intersecting with α, and let α_i be the segment of α between L_i and L_{i+1}. Give an arbitrary orientation for each L_i, then the angle between α_{i-1}, L_i and the angle between α_i, L_i sum to π. Since $\mathbf{l}(c_k) = \frac{R+2\pi i}{p}$, the angle between $\widetilde{f}_0(\alpha_{i-1})$ and $\widetilde{f}_0(\alpha_i)$ in $\mathbb{H}^3$ is greater or equal $\frac{2\pi}{p}$.

For any $i \in \{1, \cdots, m-1\}$, α_i lies in a component of $\widetilde{X}_p \setminus q^{-1}(c_1 \cup c_2)$ and connects two components of $q^{-1}(c_1 \cup c_2)$. So $q(\alpha_i)$ is a homotopic nontrivial path in X_p with end points in $c_1 \cup c_2$, and it induces a path β_i in the graph $G(X_p)$ with end points in $\{v_1, v_2\}$.

If β_i is an essential path in $G(X_p)$, since $l(G(X_p)) > Re^{\frac{R}{4}}$, the combinatorial length of β_i is greater than $Re^{\frac{R}{4}}$. Since the distance between two different cuffs in the pair of pants with cuff length R is roughly $2e^{-\frac{R}{4}}$, the length of α_i is greater than R. If β_i is an inessential path in $G(X_p)$, then it contains a segment $e\bar{e}$ for some oriented edge e of $G(X_p)$, or β_i is just a point. Since α_i is a geodesic in $\widetilde{X}_p$, it contains a segment which is an essential path in the pair of pants with end points lying on the same cuff. Since the pants have cuff length R, such a path has length greater than $R/2$. So in this case, α_i has length greater than $R/2$. As a summary, for any $i \in \{1, \cdots, m-1\}$, α_i has length greater than $R/2$.

Now the proof reduces to an elementary exercise in hyperbolic geometry. Let α be a piecewise geodesic in $\mathbb{H}^3$ consists of geodesic segments $\alpha_0, \cdots, \alpha_m$, and let the length of α_i be l_i. If $l_i \geq R/2$ for each $i \in \{1, \cdots, m-1\}$, and if the angle between α_{i-1} and α_i is greater or equal to $\frac{2\pi}{p}$ for $i \in \{1, \cdots, m\}$. Then for large enough R (depending on p), the distance between the end points of α in $\mathbb{H}^3$ is greater than $\frac{1}{2}\sum_{i=0}^m l_i - \frac{R}{4}$.

Since $d(x, y) = \sum_{i=0}^m l(\alpha_i)$, we have

$$d_{\mathbb{H}^3}(\widetilde{f}_0(x), \widetilde{f}_0(y)) \leq \sum_{i=0}^m d_{\mathbb{H}^3}(\widetilde{f}_0(x_i), \widetilde{f}_0(x_{i+1})) = \sum_{i=0}^m l(\alpha_i) = d(x, y)$$

and

$$\begin{aligned} &d_{\mathbb{H}^3}(\widetilde{f}_0(x), \widetilde{f}_0(y)) \geq \frac{1}{2}\sum_{i=0}^m d_{\mathbb{H}^3}(\widetilde{f}_0(x_i), \widetilde{f}_0(x_{i+1})) - \frac{R}{4} \\ =&\frac{1}{2}\sum_{i=0}^m l(\alpha_i) - \frac{R}{4} = \frac{1}{2}d(x, y) - \frac{R}{4}. \end{aligned}$$

So $\widetilde{f_0}$ is a quasi-isometry, and ρ is injective. The above inequality also implies that $\widetilde{f_0}(x) \neq \widetilde{f_0}(y)$ if α intersects with at least two components of $q^{-1}(c_1 \cup c_2)$, and the injectivity property obviously holds for the remaining cases. q.e.d.

We will first prove Theorem 1.5 for finite cyclic abelian groups. As we mentioned in the introduction, we will give two proofs here. The first proof will only prove a weaker statement: the finite cyclic group embeds into the virtual homology of any closed hyperbolic 3-manifold, but this proof is more geometric flavor. The second proof proves the original statement about virtual direct summand.

Proposition 3.6. *For any finite cyclic abelian group $\mathbb{Z}_n$, and any closed hyperbolic 3-manifold M, M admits a finite cover N, such that $\mathbb{Z}_n$ embeds into $Tor(H_1(N;\mathbb{Z}))$.*

Proof. Let $p = 2n$, then Proposition 3.2 gives an immersed 2-complex $f : X_p \looparrowright M$ with a pants decomposition $(X_p, \mathcal{C}', \{C_1, C_2\})$ such that $f_* : \pi_1(X_p) \to \pi_1(M)$ satisfies the conditions in Theorem 3.4.

By Lemma 3.5, $\rho_0(\pi_1(X_p))$ is a convex cocompact Kleinian group, so ρ_0 lies in $int(AH(\pi_1(X_p)))$. Here $AH(\pi_1(X_p))$ is the set of equivalent classes in

$$\{\rho : \pi_1(X_p) \to PSL_2(\mathbb{C})|\ \rho \text{ is a discrete, faithful representation}\}/\sim,$$

and the relation is given by conjugations.

Since the map $f : X_p \looparrowright M$ induces a viable representation $f_* : \pi_1(X_p) \to \pi_1(M) \subset PSL_2(\mathbb{C})$ satisfying the assumption of Theorem 3.4, f_* is π_1-injective.

We have pointed out that f_* lies in a continuous family of viable representations $\rho_\tau : \pi_1(X_p) \to PSL_2(\mathbb{C})$ for $\tau \in \mathbb{D}(0,1)$, with $\rho_1 = f_*$. By Theorem 3.4, $\{\rho_\tau(\pi_1(X_p))\}_{\tau\in\mathbb{D}(0,1)}$ is a continuous family of convex cocompact Kleinian groups. So f_* and ρ_0 lie in the same component of $int(AH(\pi_1(X_p)))$, and $\mathbb{H}^3/f_*(\pi_1(X_p))$ is homeomorphic to $\mathbb{H}^3/\rho_0(\pi_1(X_p))$.

Let $f_0 : X_p \to \mathbb{H}^3/\rho_0(\pi_1(X_p))$ be the map induced by $\widetilde{f_0}$. Then Lemma 3.5 implies that f_0 is an embedding, and a neighborhood $\hat{M} = N(f_0(X_p))$ is a compact core of $\mathbb{H}^3/\rho_0(\pi_1(X_p))$. It is easy to figure out the topological type of $\hat{M}$.

Recall that S' is an orientable surface with two oriented boundary components C_1, C_2 with $[C_1] - [C_2] = 0 \in H_1(S';\mathbb{Z})$, and X_p is the quotient of S'. Let V be the oriented solid torus and $\alpha \subset \partial V$ be the oriented $(p,1)$-curve, then α has a neighborhood $\alpha \times [-1,1] \subset \partial V$. Take two copies of $(V, \alpha \times [-1,1])$, and denote them by $(V_1, \alpha_1 \times [-1,1])$ and $(V_2, \alpha_2 \times [-1,1])$ respectively. Let $\psi_1 : C_1 \to \alpha_1$ and $\psi_2 : C_2 \to \alpha_2$ be two orientation preserving homeomorphisms. Let $\phi_1 : C_1 \times [-1,1] \to \alpha_1 \times [-1,1]$ equals $\psi_1 \times id$, and $\phi_2 : C_2 \times [-1,1] \to \alpha_2 \times [-1,1]$ equals

$\psi_2 \times (-id)$. Then $\hat{M} = N(f_0(X_p))$ is homeomorphic to $V_1 \cup_{\phi_1} S' \times [-1,1] \cup_{\phi_2} V_2$ and the boundary of $\hat{M}$ is incompressible. Such an $\hat{M}$ is called a "book of I-bundles" by Culler and Shalen in [**CS**].

Since $\mathbb{H}^3/f_*(\pi_1(X_p))$ is homeomorphic to $\mathbb{H}^3/\rho_0(\pi_1(X_p))$, $\mathbb{H}^3/f_*(\pi_1(X_p))$ has a submanifold $\hat{M}'$ homeomorphic to $\hat{M}$. It is known that fundamental groups of hyperbolic 3-manifolds are LERF ([**Ag**], [**Wi**]), and by Scott's criterion of LERF ([**Sc**]), M admits an intermediate cover $N \to M$ of $\mathbb{H}^3/f_*(\pi_1(X_p)) \to M$ such that $\hat{M}'$ projects to N by homeomorphism.

By the description of the topological type of $\hat{M}'$, $\hat{M}'$ has only one boundary component, and $H_1(\hat{M}';\mathbb{Z}) = \mathbb{Z}^{2g} \oplus \mathbb{Z} \oplus \mathbb{Z}_p$ for $g = g(S)$. The $\mathbb{Z}$-component is generated by $[c_1]$ and the $\mathbb{Z}_p$-component is generated by $[c_1] - [c_2]$. Since the image of $i_* : H_1(\partial \hat{M}';\mathbb{Z}) \to H_1(\hat{M}';\mathbb{Z})$ is $\mathbb{Z}^{2g} + \mathbb{Z}[pc_1] + \mathbb{Z}[[c_1]+[c_2]]$, it is easy to show that $\mathbb{Z}_p \cap i_*(H_1(\partial \hat{M}';\mathbb{Z})) = \{0, \frac{p}{2}([c_1]-[c_2])\}$.

Then an M-V sequence argument shows that $\mathbb{Z}_{p/2} = \mathbb{Z}_n$ embeds into $H_1(N;\mathbb{Z})$. q.e.d.

Now let's prove the statement for vitual direct summand.

Proposition 3.7. *For any finite cyclic abelian group $\mathbb{Z}_n$, and any closed hyperbolic 3-manifold M, M admits a finite cover N, such that $\mathbb{Z}_n$ is a direct summand of $Tor(H_1(N;\mathbb{Z}))$.*

Proof. Since M is a hyperbolic 3-manifold, $\pi_1(M)$ is virtually compact special by [**Ag**] and [**Wi**]. So we can suppose $\pi_1(M)$ is already the fundamental group of a compact special cube complex.

Let $f : X_n \looparrowright M$ be the π_1-injective immersion we have constructed in Proposition 3.6. Since $f_*(\pi_1(X_n))$ is a convex-cocompact subgroup of $PSL_2(\mathbb{C})$, it is a quasi-convex subgroup of the hyperbolic group $\pi_1(M)$.

Since $f_*(\pi_1(X_n))$ is a quasi-convex subgroup of the special group $\pi_1(M)$, $f_*(\pi_1(X_n))$ is a virtual retract of $\pi_1(M)$ ([**HW**]), i.e. M admits a finite cover N, such that the following conditions hold.

1) $f_*(\pi_1(X_n)) \subset \pi_1(N)$;

2) There exists a retraction homomorphism $r : \pi_1(N) \to \pi_1(X_n)$ such that $r \circ f_* : \pi_1(X_n) \to \pi_1(X_n)$ is identity.

So we have the induced maps on homology:

$$H_1(X_n;\mathbb{Z}) \xrightarrow{f_*} H_1(N;\mathbb{Z}) \xrightarrow{i_*} H_1(X_n;\mathbb{Z}).$$

Since the composition is the identity map and $H_1(X_n;\mathbb{Z}) = \mathbb{Z}^{2g+1} \oplus \mathbb{Z}_n$, we know that $\mathbb{Z}^{2g+1} \oplus \mathbb{Z}_n$ is a direct summand of $H_1(N;\mathbb{Z})$. In particular, $\mathbb{Z}_n$ is a direct summand of $Tor(H_1(N;\mathbb{Z}))$. q.e.d.

Remark 3.8. If n is an odd number, the proof of Proposition 3.6 also shows that $\mathbb{Z}_n$ is virtually a direct summand of the homology, by taking $p = n$.

3.3. Finite Abelian Subgroups in Virtual Homology. We will finish the proof of Theorem 1.5 in this subsection. As in the finite cyclic group case, we will also give two proofs. One for virtual embedding, and the other one for virtual direct summand.

Proposition 3.9. *For any finite abelian group A, and any closed hyperbolic 3-manifold M, M admits a finite cover N, such that A embeds into $Tor(H_1(N;\mathbb{Z}))$.*

Proof. We will prove the statement by induction on the number of generators of the finite abelian group A, and the proof of one-generator case has been done in Proposition 3.6.

For $A = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$, take $p_k = 2n_k$ for $k = 1, 2$. Then Proposition 3.2 and Theorem 3.4 provide us two immersed π_1-injective 2-complexes $f_1 : X_{p_1} \looparrowright M$ and $f_2 : X_{p_2} \looparrowright M$.

Let $G_1 = (f_1)_*(\pi_1(X_{p_1}))$ and $G_2 = (f_2)_*(\pi_1(X_{p_2}))$ be the two convex cocompact subgroups of $\pi_1(M)$ given by Theorem 3.4. Let M_k be the compact 3-manifold whose interior is homeomorphic to $\mathbb{H}^3/G_k$ for $k = 1, 2$.

Take $g \in \pi_1(M^3)$ such that both of the two limit points of g on S^2_∞ do not lie in the limit sets $\Lambda(G_1)$ and $\Lambda(G_2)$. Then for a large enough positive integer n, by the Kleinian combination Theorem, $j : G_1 * (g^n G_2 g^{-n}) \to \pi_1(M^3)$ is an embedding, and $j(G_1 * (g^n G_2 g^{-n}))$ is a convex cocompact subgroup of $\pi_1(M^3)$. So $\mathbb{H}^3/j(G_1 * (g^n G_2 g^{-n}))$ is homeomorphic to the interior of the boundary connected sum of M_1 and M_2.

Since hyperbolic 3-manifold groups are LERF, by running the argument in the proof of Proposition 3.6, we can find a finite cover N of M, such that $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$ embeds into $H_1(N;\mathbb{Z})$.

For finite abelian groups with more generators, the result can be shown by induction as the two-generator case. q.e.d.

Now we give the proof of Theorem 1.5.

Proof. We will still only show the theorem for $A = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$ case, and the proof for general finite abelian groups follows by induction.

As in the proof of Proposition 3.7, we suppose $\pi_1(M)$ is already a special group. The proof of Proposition 3.9 provides us a quasi-convex subgroup of $\pi_1(M)$ which is isomorphic to $\pi_1(X_{n_1}) * \pi_1(X_{n_2})$. Since $\pi_1(X_{n_1}) * \pi_1(X_{n_2})$ is a virtual retract of $\pi_1(M)$ and

$$Tor\,(H_1(\pi_1(X_{n_1}) * \pi_1(X_{n_2});\mathbb{Z})) \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2},$$

we know that $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$ is a direct summand of $Tor(H_1(N;\mathbb{Z}))$ for some finite cover N of M. q.e.d.

Remark 3.10. As in Remark 3.8, if the finite abelian group A has odd order, the proof of Proposition 3.9 also shows that A is a vitual direct summand.

4. The π_1-injectivily Property of Immersed 2-Complexes

This section is devoted to prove Theorem 3.4. Actually, we will show a π_1-injectivity result for more general immersed 2-complexes in closed hyperbolic 3-manifolds, with good pants as building blocks.

For the topological pair of pants Π_0, let $\partial_k\Pi_0, k = 1,2,3$ denote the three boundary components of Π_0. Suppose we have finitely many pair of pants $\mathcal{P} = \{\Pi_i\}_{i=1}^m$ and finitely many circles $\mathcal{C} = \{C_j\}_{j=1}^n$. By using these building blocks and some additional data, we will construct a 2-complex with a "pants decomposition".

Definition 4.1. For each pair $(i,k), i \in \{1,\cdots,m\}, k \in \{1,2,3\}$, suppose it is associated with a unique $j \in \{1,\cdots,n\}$ and a homeomorphism $f_{ik} : \partial_k\Pi_i \to C_j$. For each $C \in \mathcal{C}$, suppose it is associated with a positive integer $d_C > 0$.

Let $(\cup_{i=1}^m \Pi_i) \cup (\cup_{j=1}^n C_j) \to X'$ be the quotient space quotient by the relation given by $\{\phi_{ik}\}$. Let X be a further quotient space of X' quotient by the $\frac{2\pi}{d_C}$-rotation on each $C \in \mathcal{C}$, and let $q : (\cup_{i=1}^m \Pi_i) \cup (\cup_{j=1}^n C_j) \to X$ be the quotient map giving X.

For each circle $C \in \mathcal{C}$, let $D_C = d_C \cdot \#\{(k,i) |\ \partial_k\Pi_i \text{ is mapped to } C\}$. For any point $x \in p(C) \subset X$, a neighborhood of x in X is homeomorphic to the quotient space of the union of D_C half-discs, by identifying their diameters together. So D_C measures the local singularity near $p(C)$.

Let $\mathcal{C}_1 = \{C \in \mathcal{C} | D_C = 2, d_C = 1\}$ and $\mathcal{C}_2 = \{C \in \mathcal{C} | D_C > 2 \text{ or } d_C > 1\}$. If $D_C > 1$ for each $C \in \mathcal{C}$ (X does not have "boundary"), we say X is a 2-*complex with a pants decomposition* $(X, \mathcal{C}_1, \mathcal{C}_2)$.

We will call the curves in $\mathcal{C}_1$ *regular curves*, since each of these curves has a neighborhood in X which is homeomorphic to the annulus. Curves in $\mathcal{C}_2$ are called *singular curves.* We can also define a graph $G(X)$ from X as in step IV of our construction of X_p in Section 3.1. Here vertices are given by pants in X, and edges are given by regular curves. In $G(X)$, all the vertices are trivalent except those vertices corresponding with pants adjacent to singular curves. $l(G(X))$ can also be defined similarly, by considering the shortest essential path in $G(X)$ with end points corresponding with pants adjacent to singular curves. Let $S(X) = \max\{D_{C_j}\}$, which measures the maximal singularity of X.

The following definition of viable representation for more general 2-complexes is almost the same with Definition 3.1.

Definition 4.2. A representation $\rho : \pi_1(X) \to PSL_2(\mathbb{C})$ is called *viable* if the following conditions hold.

1) For each $C \in \mathcal{C}_1 \cup \mathcal{C}_1$, let g_C be a generator of $\pi_1(C)$, then $\rho(g_C)$ is a hyperbolic element in $PSL_2(\mathbb{C})$.
2) For each pants Π in X, $\rho|_{\pi_1(\Pi)}$ is an injective map, and $\rho(\pi_1(\Pi))$ is a discrete subgroup of $PSL_2(\mathbb{C})$.

3) For any two pants Π, Π' sharing some $C \in \mathcal{C}_1 \cup \mathcal{C}_2$, $\mathbf{hl}_{\Pi}(C) = \mathbf{hl}_{\Pi'}(C)$ holds.

For a singular curve $C \in \mathcal{C}_2$, let $N^1(\sqrt{C})$ be the half unit normal bundle of C (under a viable representation ρ), then $N^1(\sqrt{C}) \cong \mathbb{C}/\mathbf{hl}(C)\mathbb{Z} + 2\pi i\mathbb{Z}$. There is a flow Ψ_t on $N^1(\sqrt{C})$ along the direction of $\mathbf{hl}(C)$, and all the orbits of Ψ_t are closed orbits. Suppose $\Pi_1, \cdots, \Pi_k$ are the pants adjacent to C, then $D_C = d_C \cdot k$. Let $foot_{\Pi_j}(C) \in N^1(\sqrt{C})$ be the foot of Π_j on C.

Let $F' = \{foot_C(\Pi_j) + \frac{2l\pi i}{d_C} | \ j = 1, \cdots, k, \ l = 1, \cdots, d_C\} \subset N^1(\sqrt{C})$. If a point appears more than once in the definition of F', we will count it with multiplicity. Let $\Psi(F')$ be the union of closed orbits passing through F' under the flow Ψ_t. Let $F = \Psi(F') \cap \{ti | t \in \mathbb{R}/2\pi\mathbb{Z}\} \subset N^1(\sqrt{C})$, and we will consider F as a subset of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ (with multiplicity).

Definition 4.3. We say the pants $\Pi_1, \cdots, \Pi_n$ are *p-separated along* C if the distance between any two distinct points in F is greater or equal to $\frac{2\pi}{p}$ on S^1.

Then we can state our main technical theorem in this section.

Theorem 4.4. *Fix an positive integer $p \geq 2$, there are universal constants $\hat{\epsilon} > 0$ and $\hat{R} > 0$ depending only on p, such that for any $0 < \epsilon < \hat{\epsilon}$ and $R > \hat{R} > 0$, the following statement holds. If X is a connected 2-complex with a pants decomposition $(X, \mathcal{C}_1, \mathcal{C}_2)$, and $\rho : \pi_1(X) \to PSL_2(\mathbb{C})$ is a viable representation such that:*

1) *$S(X) \leq p$ and the induced graph $G(X)$ satisfies $l(G(X)) > Re^{\frac{R}{4}}$,*
2) *For any $C \in \mathcal{C}_1 \cup \mathcal{C}_2$, it satisfies the first modified Kahn-Markovic condition:*
$$|\mathbf{hl}(C) - \frac{R}{2}| < \frac{\epsilon}{R},$$
3) *For any $C \in \mathcal{C}_1$, it satisfies the second modified Kahn-Markovic condition:*
$$|s(C) - 1| < \frac{\epsilon}{R^2},$$
4) *For any $C \in \mathcal{C}_2$, $|\mathbf{l}(q(C)) - \frac{R+2k\pi i}{d_C}| < \frac{\epsilon}{d_C R}$ for some k coprime with d_C, and all the pants adjacent to C are p-separated along C.*

Then $\rho : \pi_1(X) \to PSL_2(\mathbb{C})$ is injective, and $\rho(\pi_1(X))$ is a convex-cocompact subgroup of $PSL_2(\mathbb{C})$.

It is obviously that Theorem 3.4 is a special case of Theorem 4.4, so we need only to prove Theorem 4.4 in the remaining part of this section.

We will prove Theorem 4.4 by showing that a partially defined map $i : \widetilde{X} \to \mathbb{H}^3$ satisfying $i \circ x = \rho(x) \circ i$ for any $x \in \pi_1(X)$ is a quasi-isometric embedding. Actually, i is defined on a 1-subcomplex of $\widetilde{X}$, and i maps each 1-cell in the 1-subcomplex to a geodesic arc in $\mathbb{H}^3$.

For each component W of $\widetilde{X} \setminus p^{-1}(\mathcal{C}_2)$, we will use the work in [**KM1**] and some classical results in hyperbolic geometry to show that $i|_W : W \to \mathbb{H}^3$ is a quasi-isometric embedding, by comparing $i|_W$ to some quasi-isometric embedding. Then we estimate the "angle" between two adjacent components of $\widetilde{X} \setminus p^{-1}(\mathcal{C}_2)$, and give it a definite lower bound. Given these estimations, we can show the globally quasi-isometric property, as the proof of Lemma 3.5.

4.1. Piecewise Linear Map is a Quasi-isometry. To prove Theorem 4.4, we need to study more detail about Kahn-Markovic surfaces under the modified condition. The following Theorem in [**KM1**] helps us to understand the boundary behavior of the universal cover of Kahn-Markovic surfaces in $\mathbb{H}^3$.

Theorem 4.5. *There are universal constants $\hat{\epsilon}, \hat{R}, K_0 > 0$, such that the following statement holds. For any ϵ, R such that $0 < \epsilon < \hat{\epsilon}$ and $R > \hat{R}$, suppose $(S, \mathcal{C})$ is a closed surface with a pants decomposition $\mathcal{C}$, and $f : S \looparrowright M$ is a viable map such that*

$$\begin{cases} |\mathbf{hl}(C) - \frac{R}{2}| < \epsilon, \\ |s(C) - 1| < \frac{\epsilon}{R} \end{cases}$$

holds for any $C \in \mathcal{C}$. Then $\rho = f_* : \pi_1(S) \to PSL_2(\mathbb{C})$ is injective and $\rho(\pi_1(S))$ is a quasi-fuchsian group.

Moreover, suppose S is endowed with the hyperbolic metric with $\mathbf{hl}(C) = \frac{R}{2}$ and $s(C) = 1$, then $\partial\widetilde{f} : \partial\widetilde{S} = \partial\mathbb{H}^2 \to \partial\widetilde{M^3} = \partial\mathbb{H}^3$ extends to a $\pi_1(S)$-equivariant $(1 + K_0\epsilon)$-quasiconformal map $\partial\widetilde{g} : \partial\mathbb{H}^3 \to \partial\mathbb{H}^3$ (by considering $\pi_1(S) \subset Isom_+(\mathbb{H}^2) \subset Isom_+(\mathbb{H}^3)$).

Since we can suppose that f satisfies the modified Kahn-Markovic condition:

$$\begin{cases} |\mathbf{hl}(C) - \frac{R}{2}| < \frac{\epsilon}{R}, \\ |s(C) - 1| < \frac{\epsilon}{R^2}, \end{cases}$$

the induced map $\partial\widetilde{f} : \partial\widetilde{S} = \partial\mathbb{H}^2 \to \partial\widetilde{M^3} = \partial\mathbb{H}^3$ extends to a $\pi_1(S)$-equivariant $(1 + \frac{K_0\epsilon}{R})$-quasiconformal map $\partial\widetilde{g} : \partial\mathbb{H}^3 \to \partial\mathbb{H}^3$.

Let $\rho_0 : \pi_1(S) \to PSL_2(\mathbb{C})$ be the representation near ρ satisfying $\mathbf{hl}(C) = \frac{R}{2}, s(C) = 1$ for any $C \in \mathcal{C}$, then the map $\partial\widetilde{g} : S^2_\infty \to S^2_\infty$ is a $(1 + \frac{K_0\epsilon}{R})$-quasiconformal conjugacy between $\rho_0(\pi_1(S))$ and $\rho(\pi_1(S))$.

Such a quasiconformal conjugacy between two Kleinian groups can be extended to a quasi-isometry from $\mathbb{H}^3/\rho_0(\pi_1(S))$ to $\mathbb{H}^3/\rho(\pi_1(S))$, and a quantitative version of this result is proved in [**Th1**] Chapter 11 and [**McM**] Corollary B.23.

Theorem 4.6. *Let $M_i = \mathbb{H}^3/\Gamma_i$, $i = 1, 2$ be two hyperbolic 3-manifolds with isomorphic fundamental groups, and let $\phi : S^2_\infty \to S^2_\infty$ be a K-quasiconformal conjugation between Γ_1 and Γ_2. Then ϕ extends to* *an equivariant* $K^{\frac{3}{2}}$*-quasi-isometry* $\Phi : \mathbb{H}^3 \to \mathbb{H}^3$*. In particular,* M_1 *and* M_2 *are* $K^{\frac{3}{2}}$*-quasi-isometric with each other.*

The proof of Theorem 4.6 is quite differential geometry style, and the L-quasi-isometry showed up in the statement means $(L, 0)$-quasi-isometry in the coarse geometry sense. In the following part of this paper, we will use the language in the coarse geometry.

Given the previous a few theorems and the modified Kahn-Markovic condition, we have the following quick corollary.

Corollary 4.7. *For any closed hyperbolic* 3*-manifold* M*, there exist constants* $K_0, \hat{\epsilon} > 0, \hat{R} > 0$ *such that the following statement hold. Suppose* S *is a hyperbolic surface with a pants decomposition* $\mathcal{C}$*, and* $f : S \looparrowright M$ *is a viable map satisfying the modified Kahn-Markovic condition*

$$\begin{cases} |\mathbf{hl}(C) - \frac{R}{2}| < \frac{\epsilon}{R}, \\ |s(C) - 1| < \frac{\epsilon}{R^2}, \end{cases}$$

for each $C \in \mathcal{C}$*, with* $0 < \epsilon < \hat{\epsilon}$ *and* $R > \hat{R}$*.*

Let $\rho = f_* : \pi_1(S) \to PSL_2(\mathbb{C})$ *be the induced map on the fundamental group, and* $\rho_0 : \pi_1(S) \to PSL_2(\mathbb{C})$ *be the representation near* ρ *and satisfying* $\mathbf{hl}(\mathbf{C}) = \frac{R}{2},\ s(C) = 1$*.*

Then there exists a $(1 + \frac{2K_0\epsilon}{R}, 0)$*-quasi-isometry* $\widetilde{g} : \mathbb{H}^3 \to \mathbb{H}^3$ *such that* $\widetilde{g} \circ \rho_0(x) = \rho(x) \circ \widetilde{g}$ *for any* $x \in \pi_1(S)$*. In particular, there exists a* $(1 + \frac{2K_0\epsilon}{R}, 0)$*-quasi-isometry* $g : M_1 \to M_2$*.*

Although we know the existence of the equivariant quasi-isometry $\widetilde{g} : \mathbb{H}^3 \to \mathbb{H}^3$, we need to know more detail about it. So we will use a more concrete partially defined map to approximate $\widetilde{g}$.

Let $(S, \mathcal{C})$ be the pants decomposition in Corollary 4.7. Endow S with the hyperbolic metric with $\mathbf{hl}(C) = \frac{R}{2}$ and $s(C) = 1$ for any $C \in \mathcal{C}$, then $\mathcal{C}$ is a union of simple closed geodesics on S.

Let $\widetilde{\mathcal{C}} \subset \mathbb{H}^2$ be the preimage of $\mathcal{C}$. For each pair of pants Π in S with three cuffs $c_i,\ i = 1, 2, 3$, there are three seams $a_i,\ i = 1, 2, 3$ in Π with a_i connects c_{i-1} and c_{i+1}. We suppose that a_i is perpendicular with both c_{i-1} and c_{i+1} in the hyperbolic surface S. Let $\mathcal{A}$ be the union of all the seams in S, and $\widetilde{\mathcal{A}} \subset \mathbb{H}^2$ be the preimage of $\mathcal{A}$. Let $Y = \widetilde{\mathcal{C}} \cup \widetilde{\mathcal{A}} \subset \mathbb{H}^2 \subset \mathbb{H}^3$, then Y is a $\rho_0(\pi_1(S))$-equivariant 1-subcomplex in $\mathbb{H}^3$, and $Y \subset \mathbb{H}^2$ gives a tessellation of $\mathbb{H}^2$ by isometric right-angled hexagons.

In Corollary 4.7, we can suppose $\widetilde{g} : \mathbb{H}^3 \to \mathbb{H}^3$ has been extended to a self-map on $\mathbb{H}^3 \cup S^2_\infty$ and we still denote it by $\widetilde{g}$.

Let Y' be the $\rho(\pi_1(S))$-equivariant 1-complex in $\mathbb{H}^3$ defined as the following. For any geodesic $\gamma \subset \widetilde{\mathcal{C}}$ with end points $x, y \in S^2_\infty$, we use $\partial\widetilde{g}(\gamma)$ to denote the geodesic with end points $\widetilde{g}(x)$ and $\widetilde{g}(y)$. For any geodesic arc $\alpha \subset \widetilde{\mathcal{A}}$ orthogonal with geodesics $\gamma_1, \gamma_2 \subset \widetilde{\mathcal{C}}$, we use $\partial\widetilde{g}(\alpha)$

to denote the common perpendicular geodesic of $\partial\widetilde{g}(\gamma_1)$ and $\partial\widetilde{g}(\gamma_2)$. Then we define $Y' = \partial\widetilde{g}(\widetilde{\mathcal{C}}) \cup \partial\widetilde{g}(\widetilde{\mathcal{A}})$.

Now we can define a piecewise linear $\pi_1(S)$-equivariant homeomorphism $h : Y \to Y'$. h maps each $\alpha \subset \widetilde{\mathcal{A}}$ to $\partial\widetilde{g}(\alpha)$ linearly, and h maps each $\gamma \subset \widetilde{\mathcal{C}}$ to $\partial\widetilde{g}(\gamma)$ piecewise linearly such that the restriction of h on each component of $\gamma \setminus \widetilde{\mathcal{A}}$ is linear.

Since the modified Kahn-Markovic condition is satisfied by ρ, it is easy to check that h is a $(1 + \frac{2\epsilon}{R}, 0)$-quasi-isometry on each geodesic segment of $Y = \widetilde{\mathcal{C}} \cup \widetilde{\mathcal{A}}$. However, we do not know whether $h : Y \to Y'$ is a globally quasi-isometry yet. By using the information given by $\widetilde{g}$, we will show that h is actually a quasi-isometry.

Theorem 4.8. *Under the induced metric from $\mathbb{H}^3$, $h : Y \to Y'$ is a $(1 + \frac{K\epsilon}{R}, K(\epsilon + \frac{1}{R})^{\frac{1}{5}})$-quasi-isometry for some universal constant $K > 0$.*

We need to show the following elementary lemma first, such kind of statements are well-known and we give a quantitative version here.

Lemma 4.9. *For small enough $\delta > 0$, suppose $\gamma : [0, n] \to \mathbb{H}^3$ is a $(1 + \delta, \delta)$-quasigeodesic in $\mathbb{H}^3$, and let $\bar{\gamma}$ be the geodesic with end points $\gamma(0)$ and $\gamma(n)$, then $\gamma \subset N_\eta(\bar{\gamma})$ for $\eta = 5\delta^{\frac{1}{5}}$.*

Proof. Suppose $\gamma(t_0)$ is the point on $\gamma([0, n])$ which is the farthest one from $\bar{\gamma}$, and let $D = d(\gamma(t_0), \bar{\gamma})$. We can suppose $D > 2\delta^{\frac{1}{5}}$, or the lemma holds trivially. In this case, $t_0, n - t_0 > \frac{2\delta^{\frac{1}{5}} - \delta}{1+\delta} > \delta^{\frac{1}{5}}$.

Let $t_1 = t_0 - \delta^{\frac{1}{5}}$, $t_2 = t_0 + \delta^{\frac{1}{5}}$, let $l_i = d(\gamma(t_0), \gamma(t_i))$ for $i = 1, 2$, and $d_i = d(\gamma(t_i), \bar{\gamma}) \leq D$. We use x_j to denote the orthogonal projection of $\gamma(t_j)$ on $\bar{\gamma}$ for $j = 0, 1, 2$.

For two points $x, y \in \mathbb{H}^3$, we will use $\overline{xy}$ to denote the geodesic segment with end points x and y.

Let ϕ_i be the angle difference between $\overline{\gamma(t_0)x_0}$ and $\overline{\gamma(t_i)x_i}$ along geodesic $\bar{\gamma}$ for $i = 1, 2$ (by using the parallel transportation from x_0 to x_i). Let θ_i be the angle between $\overline{\gamma(t_i)\gamma(t_0)}$ and $\overline{\gamma(t_0)x_0}$, and let θ be the angle between $\overline{\gamma(t_1)\gamma(t_0)}$ and $\overline{\gamma(t_2)\gamma(t_0)}$. Then $\theta \leq \theta_1 + \theta_2$.

A computation in hyperbolic geometry gives:

$$\begin{aligned} \cos\theta_i = &\frac{\sinh D \cosh l_i - \sinh d_i \cos\phi_i}{\cosh D \sinh l_i} \\ \geq &\frac{\sinh D \cosh l_i - \sinh D}{\cosh D \sinh l_i} = \tanh D \tanh \frac{l_i}{2}. \end{aligned} \tag{4}$$

Since $t_2 - t_1 = 2\delta^{\frac{1}{5}}$, and γ is a $(1+\delta, \delta)$-quasi-isometry, $d(\gamma(t_1), \gamma(t_2)) \geq \frac{2\delta^{\frac{1}{5}}}{1+\delta} - \delta$.

By the hyperbolic cosine law and (4):

$$
\begin{aligned}
&\cosh d(\gamma(t_1),\gamma(t_2))\\
&=\cosh l_1\cosh l_2-\sinh l_1\sinh l_2\cos\theta\\
&\le\cosh l_1\cosh l_2-\sinh l_1\sinh l_2\cos(\theta_1+\theta_2)\\
&\le\cosh l_1\cosh l_2+\sinh l_1\sinh l_2\Big(1-\tanh^2 D\tanh\frac{l_1}{2}\tanh\frac{l_2}{2}\Big).
\end{aligned}
\tag{5}
$$

So we have

$$
\tanh^2 D\le\frac{\cosh(l_1+l_2)-\cosh d(\gamma(t_1),\gamma(t_2))}{\sinh l_1\sinh l_2\tanh\frac{l_1}{2}\tanh\frac{l_2}{2}}.
\tag{6}
$$

Since $\frac{\delta^{\frac{1}{5}}}{1+\delta}-\delta\le l_1,l_2\le(1+\delta)\delta^{\frac{1}{5}}+\delta$ and $d(\gamma(t_1),\gamma(t_2))\ge\frac{2\delta^{\frac{1}{5}}}{1+\delta}-\delta$, we have:

(7)

$$
\tanh^2 D\le\frac{\cosh(2(1+\delta)\delta^{\frac{1}{5}}+2\delta)-\cosh(\frac{2\delta^{\frac{1}{5}}}{1+\delta}-\delta)}{\sinh^2(\frac{\delta^{\frac{1}{5}}}{1+\delta}-\delta)\tanh^2(\frac{\delta^{\frac{1}{5}}}{2(1+\delta)}-\frac{\delta}{2})}=24\delta^{\frac{2}{5}}(1+O(\delta^{\frac{1}{5}})).
$$

So $D=\sqrt{24}\delta^{\frac{1}{5}}(1+O(\delta^{\frac{1}{5}}))<5\delta^{\frac{1}{5}}$. q.e.d.

Since $\widetilde{g}:\mathbb{H}^3\to\mathbb{H}^3$ is a $(1+\frac{2K_0\epsilon}{R},0)$-quasi-isometry, by Lemma 4.9, we know that $\widetilde{g}(\widetilde{\mathcal{C}})\subset N_\eta(\partial\widetilde{g}(\widetilde{\mathcal{C}}))$ for $\eta=5(\frac{2K_0\epsilon}{R})^{\frac{1}{5}}$. Let $p:\widetilde{g}(\widetilde{\mathcal{C}})\to\partial\widetilde{g}(\widetilde{\mathcal{C}})$ be the nearest point projection from $\widetilde{g}(\gamma)$ to $\partial\widetilde{g}(\gamma)$ for each $\gamma\subset\widetilde{\mathcal{C}}$. Since p moves every point in $\widetilde{g}(\widetilde{\mathcal{C}})$ by at most η. Let $\widetilde{g}'=p\circ\widetilde{g}|_{\widetilde{\mathcal{C}}}:\widetilde{\mathcal{C}}\to\partial\widetilde{g}(\widetilde{\mathcal{C}})$, then $\widetilde{g}'$ is a $\pi_1(S)$-equivariant $(1+\frac{2K_0\epsilon}{R},2\eta)$-quasi-isometry. We will compare $\widetilde{g}'$ and h, which shows that h is a quasi-isometry.

Now we are ready to prove Theorem 4.8.

Proof. For an oriented geodesic $\gamma\in\widetilde{\mathcal{C}}$ corresponding with a curve $C\subset\mathcal{C}$, we will also use γ to denote the hyperbolic isometry corresponding with the oriented curve $C\subset S$.

For any $\alpha\in\widetilde{\mathcal{A}}$ which is orthogonal to $\gamma_1,\gamma_2\in\widetilde{\mathcal{C}}$, we give orientations for α,γ_1,γ_2 as in Figure 1. Let $x=\alpha\cap\gamma_2$, $y_1=\gamma_2(x)$ and $y_2=\gamma_1(y_1)$.

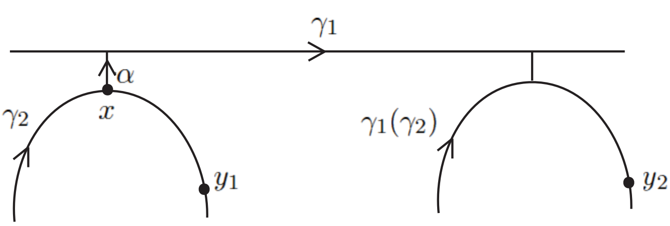

Figure 1

We will compare $\widetilde{g}'(x)$ and $h(x)$ on $\partial\widetilde{g}(\gamma_2)$, and show that they are close to each other.

Let $d_1 = d(\gamma_1, \gamma_2)$. By the hyperbolic cosine law of right-angled hexagons, and all the curves $C \subset \mathcal{C}$ satisfy $\mathbf{hl}(C) = \frac{R}{2}$, we have $\cosh d_1 = \frac{\cosh \frac{R}{2}}{\cosh \frac{R}{2} - 1}$. Let $d_2 = d(y_1, \gamma_1)$. Since $d(x, y) = R$, we have $\sinh d_2 = \sinh d_1 \cosh R$, and

$$\sinh^2 d_2 = \cosh^2 R \frac{2\cosh\frac{R}{2} - 1}{(\cosh\frac{R}{2} - 1)^2}. \tag{8}$$

By computations in hyperbolic geometry of $\mathbb{H}^2$, we have

$$\begin{aligned} &\cosh d(y_1, y_2) = \cosh^2 d_2 \cosh R - \sinh^2 d_2 \\ =&\cosh R + \cosh^2 R(\cosh R - 1)\frac{2\cosh\frac{R}{2} - 1}{(\cosh\frac{R}{2} - 1)^2} \\ =&\frac{1}{2}e^{\frac{5}{2}R}(1 + O(e^{-\frac{R}{2}})). \end{aligned} \tag{9}$$

So $d(y_1, y_2) = \frac{5}{2}R + O(e^{-\frac{R}{2}})$.

Now we think about the position of $\widetilde{g}'(x)$ on $\partial\widetilde{g}(\gamma_2)$. Let l be the oriented distance between $h(x) = h(\gamma_2) \cap h(\alpha)$ and $\widetilde{g}'(x)$ on $h(\gamma_2)$. We will prove that l is very small.

Let $d_1' = \mathbf{d}_\alpha(\gamma_2, \gamma_1)$, by the hyperbolic cosine law of right-angled hexagons,

$$d_1' = 2e^{-\frac{R}{4} + \frac{1}{2R}(\epsilon_3 - \epsilon_2 - \epsilon_1)} + O(e^{-\frac{3R}{4}}) \tag{10}$$

for complex numbers ϵ_i with $|\epsilon_i| < \epsilon$ for $i = 1, 2, 3$. Let $d_2' = d(\widetilde{g}'(y_1), \partial\widetilde{g}(\gamma_1))$, and let the real length of $\partial\widetilde{g}(\gamma_2)$ be $R + \frac{2\epsilon_4}{R}$ for some real number ϵ_4 with $|\epsilon_4| < \epsilon$, then

(11)

$$\begin{aligned} &\sinh^2 d_2' \\ =&\cosh^2\left(l + R + \frac{2\epsilon_4}{R}\right)\sinh^2(\Re(d_1')) + \sinh^2\left(l + R + \frac{2\epsilon_4}{R}\right)\sin^2(\Im(d_1')) \\ \geq&\cosh^2\left(l + R - \frac{2\epsilon}{R}\right)\sinh^2\left((2 + O(e^{-\frac{R}{2}}))e^{-\frac{R}{4} - \frac{3\epsilon}{2R}}\right). \end{aligned}$$

Here $\Re(z)$ and $\Im(z)$ are the real and imaginary part of a complex number z respectively.

Let $R_1 = \mathbf{l}(h(\gamma_1))$ be the complex translation length of $h(\gamma_1)$, then $|R_1 - R| < \frac{2\epsilon}{R}$. So by (10) and (11), we have

(12)
$$\begin{aligned}
&\cosh\left(d(g'(y_1), g'(y_2))\right)\\
=&\cosh^2 d_2' \cosh \Re(R_1) - \sinh^2 d_2' \cos \Im(R_1)\\
\geq&\cosh^2 d_2' \cosh\Big(R - \frac{2\epsilon}{R}\Big) - \sinh^2 d_2'\\
=&\cosh\Big(R - \frac{2\epsilon}{R}\Big) + \Big(\cosh\Big(R - \frac{2\epsilon}{R}\Big) - 1\Big)\sinh^2 d_2'\\
\geq&\Big(\cosh\Big(R - \frac{2\epsilon}{R}\Big) - 1\Big)\cosh^2\Big(l + R - \frac{2\epsilon}{R}\Big)\sinh^2\Big((2 + O(e^{-\frac{R}{2}}))e^{-\frac{R}{4} - \frac{3\epsilon}{2R}}\Big)\\
&+\cosh\Big(R - \frac{2\epsilon}{R}\Big)\\
=&\frac{1}{2}e^{2l + \frac{5R}{2} - \frac{9\epsilon}{R}}(1 + O(e^{-\frac{R}{2}})).
\end{aligned}$$

So $d(\widetilde{g}'(y_1), \widetilde{g}'(y_2)) \geq |2l + \frac{5R}{2} - \frac{9\epsilon}{R} + O(e^{-\frac{R}{2}})|$. Since $\widetilde{g}'$ is a $(1 + \frac{2K_0\epsilon}{R}, 2\eta)$-quasi-isometry and $d(y_1, y_2) = \frac{5}{2}R + O(e^{-\frac{R}{2}})$, the following inequality holds:

$$(13)\quad \Big(1 + \frac{2K_0\epsilon}{R}\Big)\Big(\frac{5}{2}R + O(e^{-\frac{R}{2}})\Big) + 2\eta \geq \Big|2l + \frac{5R}{2} - \frac{9\epsilon}{R} + O(e^{-\frac{R}{2}})\Big|.$$

By considering $y_1' = \gamma_2^{-1}(x)$ and $y_2' = \gamma_1^{-1}(y_1)$ and run the same argument, we have

$$(14)\quad \Big(1 + \frac{2K_0\epsilon}{R}\Big)\Big(\frac{5}{2}R + O(e^{-\frac{R}{2}})\Big) + 2\eta \geq \Big| - 2l + \frac{5R}{2} - \frac{9\epsilon}{R} + O(e^{-\frac{R}{2}})\Big|.$$

By (13) and (14) and $\eta = 5(\frac{2K\epsilon}{R})^{\frac{1}{5}}$, we get

$$(15)\qquad |l| \leq \frac{5}{2}K_0\epsilon + \frac{9\epsilon}{2R} + \eta + O(e^{-\frac{R}{2}}) \leq K_1\Big(\epsilon + \frac{1}{R} + (\frac{\epsilon}{R})^{\frac{1}{5}}\Big).$$

Since l is the oriented distance between $\widetilde{g}'(x)$ and $h(x)$, $d(\widetilde{g}'(x), h(x)) \leq K_1(\epsilon + \frac{1}{R} + (\frac{\epsilon}{R})^{\frac{1}{5}})$ holds for any $x \in \widetilde{\mathcal{C}} \cap \widetilde{\mathcal{A}}$.

Since $\widetilde{g}' : \widetilde{\mathcal{C}} \to \partial\widetilde{g}(\widetilde{\mathcal{C}})$ is a $(1 + \frac{2K_0\epsilon}{R}, 2\eta)$-quasi-isometry and the restriction of $h : \widetilde{\mathcal{C}} \to \partial\widetilde{g}(\widetilde{\mathcal{C}})$ on each single geodesic $C \subset \widetilde{\mathcal{C}}$ is a $(1 + \frac{2\epsilon}{K}, 0)$-quasi-isometry. So $d(\widetilde{g}'(y), h(y)) \leq 2\epsilon + 2K_0\epsilon + 2\eta + K_1(\epsilon + \frac{1}{R} + (\frac{\epsilon}{R})^{\frac{1}{5}})$ for each $y \in \widetilde{\mathcal{C}}$.

So $h|_{\widetilde{\mathcal{C}}} : \widetilde{\mathcal{C}} \to \partial\widetilde{g}(\widetilde{\mathcal{C}})$ is a $\pi_1(S)$-equivariant $(1 + \frac{2K_0\epsilon}{R}, K_2(\epsilon + \frac{1}{R} + (\frac{\epsilon}{R})^{\frac{1}{5}}))$-quasi-isometry. Since $\widetilde{\mathcal{C}} \subset Y$ is $2e^{-\frac{R}{4}}$-dense, $h : Y \to Y'$ is a $(1 + \frac{2K_0\epsilon}{R}, K_3(\epsilon + \frac{1}{R} + (\frac{\epsilon}{R})^{\frac{1}{5}}))$-quasi-isometry. q.e.d.

4.2. Estimation of the Angle Change. Now we consider two points $x, y \in Y$ with x lying on a geodesic $\gamma \subset \widetilde{\mathcal{C}}$ and $d(x,y) \geq \frac{R}{2}$. We give γ an arbitrary orientation. Let $\alpha \subset \widetilde{\mathcal{A}}$ be the geodesic arc which is at the same side of γ as $\overline{xy}$ on $\mathbb{H}^2$, such that it is the closest such geodesic arc to x (it is possible there are two choices and we choose either of them). Let $\vec{e}_1 \in T^1_x(\mathbb{H}^3)$ be the unit vector in the direction of γ, $\vec{e}_2 \in T^1_x(\mathbb{H}^3)$ be the unit vector in the direction of the parallel transportation of α to x, and take the third unit vector $\vec{e}_3 \in T^1_x(\mathbb{H}^3)$ such that $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is an orthonormal frame of $T_x(\mathbb{H}^3)$ and gives the orientation of $\mathbb{H}^3$. Let $\vec{e}$ be the unit vector in $T^1_x(\mathbb{H}^3)$ in the direction of $\overline{xy}$.

Now we define $\Theta(\gamma, \alpha, \overline{xy}) = (\theta, \phi) \in \mathbb{R}^2$ for $\theta = \angle(\vec{e}, \vec{e}_1) \in [0, \pi]$ and $\phi = \angle(\vec{e}, \vec{e}_3) \in [0, \pi]$. $\Theta(\gamma, \alpha, \overline{xy})$ measures the direction of $\vec{e}$ under the coordinate $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$. Since $\rho_0(\pi_1(S))$ is a Fuchsian group, $\phi = \frac{\pi}{2}$. $\Theta(h(\gamma), h(\alpha), \overline{h(x)h(y)}) = (\theta', \phi') \in \mathbb{R}^2$ is defined similarly.

The main result of this subsection is the following statement.

Theorem 4.10. *There exists constants $\hat{\epsilon} > 0$, $\hat{R} > 0$ depend only on p, such that for any $0 < \epsilon < \hat{\epsilon}$ and $R > \hat{R}$, $|\Theta(\gamma, \alpha, \overline{xy}) - \Theta(h(\gamma), h(\alpha), \overline{h(x)h(y)})| \leq \frac{1}{p}$.*

Since each hexagon in $\mathbb{H}^2 \setminus Y$ has diameter $\leq \frac{R}{2} + 1$, $\overline{xy} \cap Y = \{x_0, x_1, \cdots, x_n\}$ for $x = x_0$, $y = x_n$ and $d(x_i, x_{i+1}) \leq \frac{R}{2} + 1$. Let $h(\overline{xy})$ denote the piecewise geodesic in $\mathbb{H}^3$ which is the concatenation of $\overline{h(x_i)h(x_{i+1})}$. There is a natural piecewise linear map $h' : \overline{xy} \to h(\overline{xy})$ such that $h'(x_i) = h(x_i)$. Since $h : Y \to Y'$ is a $(1 + \mu_1, \mu_2)$-quasi-isometry for $\mu_1 = \frac{K\epsilon}{R}$, and $d(x_i, x_{i+1}) \leq \frac{R}{2} + 1$, it is easy to check that h' is a $(1 + \mu_1, 3\mu_2 + 4K\epsilon)$-quasi-isometry. Since $\mu_2 = K(\epsilon + \frac{1}{R})^{\frac{1}{5}}$, h' is a $(1 + \mu_1, 10\mu_2)$-quasi-isometry in particular.

Let x_k be the point in $\{x_0, \cdots, x_n\}$ which is the nearest one to x and such that

$$l(p) \leq d(x, x_k) \leq R \tag{16}$$

holds for $l(p) = \frac{1}{10000p^2}$. If $x_k \in \alpha \subset \widetilde{\mathcal{A}}$, let x' be one of the intersection points of $\alpha \cap \widetilde{\mathcal{C}}$. If x_k does not lie in $\widetilde{\mathcal{A}}$, simply let $x' = x_k$. By Lemma 4.9 and the choice of x', we have $d(h(x'), \overline{h(x)h(y)}) \leq 5(\mu_2)^{\frac{1}{5}} + 2e^{-\frac{R}{4}} \leq 6(\mu_2)^{\frac{1}{5}}$. Since $d(h(x), h(x')) \geq \frac{l(p)}{1+\mu_1} - \mu_2 - 2e^{-\frac{R}{4}} \geq \frac{l(p)}{2}$, by the hyperbolic sine law,

$$\begin{aligned} \sin(\angle h(x')h(x)h(y)) &= \frac{\sinh(d(h(x'), \overline{h(x)h(y)}))}{\sinh(d(h(x), h(x')))} \\ &\leq \frac{\sinh 6(\mu_2)^{\frac{1}{5}}}{\sinh \frac{l(p)}{2}} \leq \frac{15(\mu_2)^{\frac{1}{5}}}{l(p)}. \end{aligned} \tag{17}$$

So $\angle h(x')h(x)h(y) \leq \frac{20(\mu_2)^{\frac{1}{5}}}{l(p)}$, and the same argument shows $\angle x'xy \leq \frac{20(\mu_2)^{\frac{1}{5}}}{l(p)}$. Now it suffices to show that

$$|\Theta(\gamma,\alpha,\overline{xx'}) - \Theta(h(\gamma),h(\alpha),\overline{h(x)h(x')})| \leq \frac{1}{2p}.$$

Even if x' may not be same with x_k, we will abuse the notation and still use $x_0 = x, x_1, \cdots, x_k = x'$ to denote the intersection points in $\overline{xx'} \cap Y$. Moreover, we still use the notation $\Theta(\gamma,\alpha,\overline{xx'}) = (\theta,\phi)$ and $\Theta(h(\gamma),h(\alpha),\overline{h(x)h(x')}) = (\theta',\phi')$, with $\phi = \frac{\pi}{2}$.

Proposition 4.11. *Let θ and θ' be the first coordinate of $\Theta(\gamma,\alpha,\overline{xx'})$ and $\Theta(h(\gamma),h(\alpha),\overline{h(x)h(x')})$ respectively, then $|\theta-\theta'| \leq \frac{1}{4p}$.*

Proof. Let $d_1 = d(x,x')$, $d_2 = d(x',\gamma)$ and $d_1' = d(h(x),h(x'))$, $d_2' = d(h(x'),h(\gamma))$, then $d_2 \leq d_1$ and $d_2' \leq d_1'$. Since $h : Y \to Y'$ is a $(1+\mu_1,\mu_2)$-quasi-isometry, and $l(p) \leq d_1 \leq R$, we have $|d_1' - d_1| \leq K\epsilon + \mu_2$ and $|d_2' - d_2| \leq K\epsilon + \mu_2$. In particular, $d_1' \geq \frac{1}{2}l(p)$.

Since $\sin\theta = \frac{\sinh d_2}{\sinh d_1}$ and $\sin\theta' = \frac{\sinh d_2'}{\sinh d_1'}$, we have the following estimation:

$$\begin{aligned}
&|\sin\theta - \sin\theta'| \\
\leq&\frac{\sinh d_2 \cdot |\sinh d_1 - \sinh d_1'| + \sinh d_1 \cdot |\sinh d_2 - \sinh d_2'|}{\sinh d_1 \sinh d_1'} \\
\leq&\frac{|d_1 - d_1'| \cdot \cosh(\max(d_1,d_1')) + |d_2 - d_2'| \cdot \cosh(\max(d_2,d_2'))}{\sinh d_1'} \\
\leq&2(K\epsilon+\mu_2)\frac{\cosh(\max(d_1,d_1'))}{\sinh d_1'} \\
\leq&2(K\epsilon+\mu_2)e^{K\epsilon+\mu_2}\coth d_1' \\
\leq&2(K\epsilon+\mu_2)e^{K\epsilon+\mu_2}\coth\Big(\frac{1}{2}l(p)\Big).
\end{aligned} \tag{18}$$

Let $\nu = 2(K\epsilon+\mu_2)e^{K\epsilon+\mu_2}\coth(\frac{1}{2}l(p))$. If both θ and θ' are acute angles, then

$$\nu \geq |\sin\theta - \sin\theta'| = 2\sin\frac{|\theta-\theta'|}{2}\cos\frac{\theta+\theta'}{2} \geq 2\sin^2\frac{|\theta-\theta'|}{2} \geq \frac{(\theta-\theta')^2}{8}. \tag{19}$$

So in this case $|\theta-\theta'| \leq \sqrt{8\nu}$.

Without lose of generality, we can suppose that θ is an acute angle. If θ' is also acute, the above inequality gives $|\theta-\theta'| \leq \sqrt{8\nu} \leq \frac{1}{4p}$, so the lemma is proved.

If θ' is not acute, we will show that θ is very close to $\pi/2$. Let β be the subray of γ starting at x and has acute angle with $\overline{xx'}$. In this case, the angle between $\overline{h(x)h(x')}$ and $h(\beta)$ is an obtuse angle, so $d(h(x'), h(\beta)) = d(h(x'), h(x))$. By this geometric observation, we have:

$$\begin{aligned} d(x', \gamma) = d(x', \beta) \geq & \frac{d(h(x'), h(\beta))}{1 + \mu_1} - \mu_2 \\ = \frac{d(h(x'), h(x))}{1 + \mu_1} - \mu_2 \geq & \frac{d(x', x)}{(1 + \mu_1)^2} - 2\mu_2. \end{aligned} \tag{20}$$

Since $d(x', x) \leq R$ and $\mu_1 = \frac{K\epsilon}{R}$, $d(x', \gamma) \geq \frac{d(x', x)}{(1+\mu_1)^2} - 2\mu_2 \geq d(x', x) - (3K\epsilon + 2\mu_2)$. So

$$\begin{aligned} \sin\theta = & \frac{\sinh d(x', \gamma)}{\sinh d(x', x)} \geq \frac{\sinh (d(x', x) - (3K\epsilon + 2\mu_2))}{\sinh (d(x', x))} \\ \geq & \frac{\sinh (l(p) - (3K\epsilon + 2\mu_2))}{\sinh (l(p))} \geq 1 - (3K\epsilon + 2\mu_2) \coth (l(p)). \end{aligned} \tag{21}$$

So $\frac{\pi}{2} - \theta \leq 2\sqrt{(3K\epsilon + 2\mu_2) \coth l(p)} \leq \frac{1}{8p}$, and the same argument shows that $\theta' - \frac{\pi}{2} \leq \frac{1}{8p}$. So we have $|\theta' - \theta| \leq \frac{1}{4p}$. q.e.d.

Since the second coordinate of $\Theta(\gamma, \alpha, \overline{xx'}) = (\theta, \phi)$ is $\phi = \frac{\pi}{2}$, we need only to estimate the second coordinate ϕ' of $\Theta(h(\gamma), h(\alpha), \overline{h(x)h(x')}) = (\theta', \phi')$.

Proposition 4.12. *For small enough $\epsilon > 0$ and large enough R, we have $|\phi' - \frac{\pi}{2}| \leq \frac{1}{4p}$.*

We first need the following local estimation.

Lemma 4.13. *If $\alpha \subset \partial\widetilde{g}(\widetilde{\mathcal{A}})$ is the common perpendicular of geodesics $\gamma_1, \gamma_2 \subset \partial\widetilde{g}(\widetilde{\mathcal{C}})$ such that $z_i = \gamma_i \cap \alpha$, and $y \in \gamma_1$ is a point on γ_1 such that $d = d(y, z_1) \leq R$. Let P_1 be the hyperbolic plane containing γ_1 and z_2, P_2 be the hyperbolic plane containing γ_2 and y, and ψ be the angle between P_1 and P_2, then $\psi \leq \frac{10\epsilon}{R}$.*

Proof. Let $\beta = \angle z_1 z_2 y$, and $b + i\xi = \mathbf{d}_\alpha(\gamma_1, \gamma_2)$. Here we choose orientations for $\alpha, \gamma_1, \gamma_2$ such that $b > 0$ and ξ is close to 0.

By the hyperbolic cosine law of right-angled hexagons,

$$\mathbf{d}_\alpha(\gamma_1, \gamma_2) = b + i\xi = 2e^{-\frac{R}{4} + \frac{\epsilon_3 - \epsilon_2 - \epsilon_1}{2R}} + O(e^{-\frac{3R}{4}}) + O(e^{-\frac{3R}{4}})\frac{\epsilon}{R}i, \tag{22}$$

here the two $O(e^{-\frac{3R}{4}})$s are two different real functions. So $b \geq e^{-\frac{R}{4}}$, and $|\xi| \leq \frac{4\epsilon}{R} e^{-\frac{R}{4}}$.

An elementary computation gives $\cos\psi = \frac{\cos\beta\cos\xi}{\sqrt{1 - \sin^2\beta\cos^2\xi}}$, and

$$\sin^2\psi = \frac{\sin^2\xi}{1 - \sin^2\beta\cos^2\xi}. \tag{23}$$

For the angle β, we have the following estimation:

$$
\begin{aligned}
\sin^2\beta = & \frac{\sinh^2 d}{\sinh^2 d(y,z_2)} = \frac{\cosh^2 d - 1}{\cosh^2 d \cosh^2 b - 1} \\
\leq & \frac{\cosh^2 R - 1}{\cosh^2 R \cosh^2 (e^{-\frac{R}{4}}) - 1} = 1 - e^{-\frac{R}{2}} + O(e^{-R}).
\end{aligned} \tag{24}
$$

By (23) and (24)

$$
\begin{aligned}
\sin^2\psi \leq & \frac{\sin^2\xi}{1-(1-e^{-\frac{R}{2}}+O(e^{-R}))\cos^2\xi} \\
\leq & \frac{\sin^2(\frac{4\epsilon}{R}e^{-\frac{R}{4}})}{1-(1-e^{-\frac{R}{2}}+O(e^{-R}))\cos^2(\frac{4\epsilon}{R}e^{-\frac{R}{4}})} \leq \frac{32\epsilon^2}{R^2}.
\end{aligned} \tag{25}
$$

So $\psi \leq \frac{10\epsilon}{R}$.

q.e.d.

Now we are ready to prove Proposition 4.12.

Proof. Let $x = x_0, x_1, \cdots, x_k = x'$ be the consecutive intersection points of $\overline{xx'}$ with $Y = \widetilde{\mathcal{C}} \cup \widetilde{\mathcal{A}}$, and $y_i = h(x_i)$. Let C_i be the geodesic (segment) in $\partial\widetilde{g}(\widetilde{\mathcal{C}})$ or $\partial\widetilde{g}(\widetilde{\mathcal{A}})$ containing y_i.

Let P_0 be the hyperbolic plane containing $C_0 = h(\gamma)$ and $h(\alpha)$, P_0' be the hyperbolic plane containing C_0 and y_1. Let P_i be the hyperbolic plane containing C_i and y_{i-1}, and P_i' be the hyperbolic plane containing C_i and y_{i+1} for $i = 1, \cdots, k-1$. When considering about ρ_0, all the corresponding hyperbolic planes coincide with each other. So for each hyperbolic plane P_i and P_i', we can give it an orientation such that the corresponding orientations coincide with each other, when considering ρ_0.

Since $d(x, x_{k-1}) \leq l(p) \leq \frac{1}{10000p^2}$, by Lemma 2.3 of [**KM1**], $k \leq (2+R)e^5 l(p) \leq 500Rl(p)$. The possible position of y_{i-1}, y_i and geodesics C_{i-1}, C_i looks like a) or b) in Figure 2 for $i = 1, \cdots, k-1$. By the choice of x', the possible position of y_{k-1}, y_k and C_{k-1}, C_k looks like a), b) or c) in Figure 2.

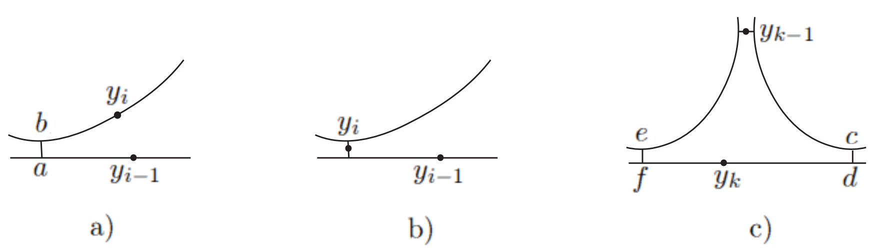

Figure 2

Now we study two possible positions of y_{i-1}, y_i and C_{i-1}, C_i.

i) For three points $a, b, c \in \mathbb{H}^3$ not lying on a geodesic, let P_{abc} be the hyperplane in $\mathbb{H}^3$ containing these three points. Then in Figure 2 a), since $d(y_{i-1}, a), d(y_i, b) < R$, Lemma 4.13 implies:

$$\tag{26} \begin{aligned} &\angle(P_{ay_{i-1}y_i}, P_{aby_{i-1}}) \leq \angle(P_{ay_{i-1}y_i}, P_{aby_i}) + \angle(P_{aby_i}, P_{aby_{i-1}}) \\ \leq &\frac{10\epsilon}{R} + \frac{10\epsilon}{R} = \frac{20\epsilon}{R}, \end{aligned}$$

and

$$\tag{27} \begin{aligned} &\angle(P'_{i-1}, P_i) = \angle(P_{ay_{i-1}y_i}, P_{by_{i-1}y_i}) \\ \leq &\angle(P_{ay_{i-1}y_i}, P_{aby_{i-1}}) + \angle(P_{aby_{i-1}}, P_{by_{i-1}y_i}) \leq \frac{30\epsilon}{R}. \end{aligned}$$

ii) In Figure 2 c), let β be the geodesic arc in $\partial\widetilde{g}(\widetilde{\mathcal{A}})$ containing y_{k-1} and we use β' to denote the bi-infinite geodesic containing β. Let $P_{\beta',c}$ be the hyperbolic plane containing β' and c, and $l = d(d, P_{\beta',c})$, then

$$\tag{28} \sinh l \leq \sin\frac{\epsilon}{R}\sinh(3e^{-\frac{R}{4}}) \leq \frac{4\epsilon}{R}e^{-\frac{R}{4}}.$$

So $l \leq \frac{4\epsilon}{R}e^{-\frac{R}{4}}$. Since $d(d, \beta') \geq \frac{R}{2} - 1$,

$$\tag{29} \sin\angle(P_{\beta',c}, P_{\beta',d}) \leq \frac{\sinh(\frac{4\epsilon}{R}e^{-\frac{R}{4}})}{\sinh(\frac{R}{2} - 1)} \leq \frac{50\epsilon}{R}e^{-\frac{3R}{4}}.$$

So $\angle(P_{\beta',c}, P_{\beta',d}) \leq \frac{100\epsilon}{R}e^{-\frac{3R}{4}}$. The same argument shows that

$$\angle(P_{\beta',e}, P_{\beta',f}) \leq \frac{100\epsilon}{R}e^{-\frac{3R}{4}}.$$

Since $\angle(P_{\beta',c}, P_{\beta',e}) \leq \frac{4\epsilon}{R}e^{-\frac{R}{4}}$ and by monotonicity, we have $\angle(P_{\beta',c}, P_{\beta',y_k}) \leq \frac{\epsilon}{R}$.

Then by a routine case-by-case argument, and using results in i) and ii), we get that:

$$\tag{30} \angle(P'_{i-1}, P_i) \leq \frac{100\epsilon}{R}$$

for $i = 1, \cdots, k-1$, and

$$\tag{31} \angle(P_i, P'_i) \leq \frac{100\epsilon}{R}$$

for $i = 0, \cdots, k-1$.

Now let $l_i = d(y_i, y_{i+1})$. Then by (16),

$$\tag{32} \sum_{i=0}^{k-2} l_i \leq 2l(p)$$

and

$$\tag{33} \frac{l(p)}{2} \leq \sum_{i=0}^{k-1} l_i \leq R + 1.$$

Let $\vec{v}_i \in T^1_{y_i}(\mathbb{H}^3)$ be the unit normal vector of P_i at y_i and $\vec{v}_i' \in T^1_{y_i}(\mathbb{H}^3)$ be the unit normal vector of P_i' at y_i. Let z_i be the orthogonal projection of y_i to P_0 and $\vec{n}_i$ be the unit normal vector of P_0 at z_i. For $x, y \in \mathbb{H}^3$ and $\vec{v} \in T_x(\mathbb{H}^3)$, we will use $\vec{v}@y$ to denote the parallel transportation of $\vec{v}$ to y along the geodesic arc $\overline{xy}$, as in [**KM1**].

Claim 1: For $i = 0, \cdots, k-1$, the following inequalities hold:

$$\angle(\vec{v}_i@y_0, n_0) \leq \frac{200i\epsilon}{R} + \sum_{j=0}^{i-1} l_j, \tag{34}$$

$$\angle(\vec{v}_i'@y_0, n_0) \leq \frac{(200i+100)\epsilon}{R} + \sum_{j=0}^{i-1} l_j. \tag{35}$$

We will prove Claim 1 by induction. The statement holds for $i = 0$ since $\angle(P_0, P_0') \leq \frac{100\epsilon}{R}$. Suppose the statement holds for $i = m$, then for $i = m+1$,

$$\begin{aligned} &\angle(\vec{v}_{m+1}@y_0, \vec{n}_0) \leq \angle(\vec{v}_{m+1}, \vec{v}_m'@y_{m+1}) + \angle(\vec{v}_m'@y_{m+1}@y_0, \vec{n}_0) \\ \leq &\angle(P_{m+1}, P_m') + \angle(\vec{v}_m'@y_0, \vec{n}_0) + \angle(\vec{v}_m'@y_{m+1}@y_0, \vec{v}_m'@y_0). \end{aligned} \tag{36}$$

The first term is less than $\frac{100\epsilon}{R}$ by (30), the second term is less than $\frac{(200m+100)\epsilon}{R} + \sum_{j=0}^{m-1} l_j$ by induction hypothesis, and the third term is less than l_m by Proposition 4.1 of [**KM1**]. So

$$\angle(\vec{v}_{m+1}@y_0, \vec{n}_0) \leq \frac{200(m+1)\epsilon}{R} + \sum_{j=0}^{m} l_j. \tag{37}$$

Since $\angle(\vec{v}_{m+1}, \vec{v}_{m+1}') \leq \frac{100\epsilon}{R}$, the second inequality holds for $i = m+1$ and the proof of Claim 1 is done.

Now we estimate $\angle(\vec{v}_m'@z_m, \vec{n}_m)$ for $m = 0, \cdots, k-1$. Since $k \leq 500Rl(p)$, we have:

$$\begin{aligned} &\angle(\vec{v}_m'@z_m, \vec{n}_m) \leq \angle(\vec{v}_m'@z_m, \vec{v}_m'@z_0@z_m) + \angle(\vec{v}_m'@z_0@z_m, \vec{n}_m) \\ \leq &d(y_0, y_m) + \angle(\vec{v}_m'@y_0, \vec{n}_0) \leq 2l(p) + \frac{(200m+100)\epsilon}{R} + \sum_{j=0}^{m-1} l_j \\ \leq &4l(p) + 2 \cdot 10^5 \epsilon l(p) \leq 10l(p). \end{aligned} \tag{38}$$

Claim 2: For $d_i = d(y_i, z_i) = d(y_i, P_0),\ i = 0, \cdots, k-1$, we have:

$$d(y_i, z_i) \leq \frac{1}{100p} \sum_{j=0}^{i-1} l_j. \tag{39}$$

We will also prove Claim 2 by induction. The inequality holds trivially for $i = 0$. Suppose the inequality holds for $i = m$. For $i = m+1$, Since $\vec{v}'_m \perp \overline{y_m y_{m+1}}$ and $\angle(\vec{v}'_m @ z_m, \vec{n}_m) \leq 10l(p)$, an elementary computation gives:

$$\begin{aligned} &\sinh d(y_{m+1}, z_{m+1}) \\ &= \sinh d(y_m, z_m) \cosh l_m + \cosh d(y_m, z_m) \sinh l_m \sin \theta_m \end{aligned} \tag{40}$$

for some θ_m with $|\theta_m| \leq \angle(\vec{v}'_m @ z_m, \vec{n}_m) \leq 10l(p)$ and $l_m \leq 2l(p)$.

Since $d(y_m, z_m)$, $l_m \leq 2l(p)$, we have:

$$\begin{aligned} &\sinh d(y_{m+1}, z_{m+1}) \\ \leq &\sinh d(y_m, z_m)(1 + l_m^2) + (1 + 4l(p^2)) \cdot 2l_m \cdot 10(p) \\ \leq &\sinh d(y_m, z_m) + 2d(y_m, z_m) l_m^2 + (1 + 4l(p^2)) \cdot 2l_m \cdot 10l(p) \\ = &\sinh d(y_m, z_m) + l_m \left(2d(y_m, z_m) l_m + 20l(p)(1 + 4l(p^2))\right) \\ \leq &\sinh d(y_m, z_m) + 100l(p) \cdot l_m. \end{aligned} \tag{41}$$

So $d(y_{m+1}, z_{m+1}) \leq d(y_m, z_m) + 100l(p) \cdot l_m \leq d(y_m, z_m) + \frac{1}{100p} l_m$, and the proof of Claim 2 is done.

The final computation is to estimate $\angle z_k y_0 y_k$.

Since h is a $(1 + \frac{K\epsilon}{R}, K(\epsilon + \frac{1}{R})^{\frac{1}{5}})$-quasi-isometry, and $\frac{l(p)}{2} \leq \sum_{i=0}^{k-1} l_i \leq R$, $d(y_0, y_k) \geq \max\left(\frac{1}{2} \sum_{i=0}^{k-1} l_i, l_{k-1} - 1\right)$. Moreover, $d(y_k, z_k)$ is given by:

$$\begin{aligned} &\sinh d(y_k, z_k) \\ = &\sinh d(y_{k-1}, z_{k-1}) \cosh l_{k-1} + \cosh d(y_{k-1}, z_{k-1}) \sinh l_{k-1} \sin \theta_{k-1} \end{aligned} \tag{42}$$

for some θ_{k-1} with $|\theta_{k-1}| \leq 10l(p)$

If $l_{k-1} \leq 2$, (41) still works with m replaced by $k-1$, so $d(y_k, z_k) \leq \frac{1}{100p} \sum_{i=0}^{k-1} l_i$. Then

$$\sin \angle z_k y_0 y_k = \frac{\sinh d(y_k, z_k)}{\sinh d(y_k, y_0)} \leq \frac{\sinh \frac{\sum_{i=0}^{k-1} l_i}{100p}}{\sinh \frac{\sum_{i=0}^{k-1} l_i}{2}} \leq \frac{1}{50p}. \tag{43}$$

If $l_{k-1} \geq 2$, we have

$$\sinh d(y_k, z_k) \leq 4l(p) \cosh l_{k-1} + 20l(p) \sinh l_{k-1} \leq 20l(p) e^{l_{k-1}}. \tag{44}$$

Then

$$\sin \angle z_k y_0 y_k = \frac{\sinh d(y_k, z_k)}{\sinh d(y_k, y_0)} \leq \frac{20l(p) e^{l_{k-1}}}{\sinh(l_{k-1} - 1)} \leq \frac{20l(p) \cdot e^2}{\sinh 1} \leq \frac{1}{50p}. \tag{45}$$

So in both of these cases, $|\psi' - \frac{\pi}{2}| = \angle z_k y_0 y_k \leq \frac{1}{4p}$. q.e.d.

4.3. Proof of Theorem 4.4. Given Theorem 4.10, we are ready to prove Theorem 4.4.

Proof. Given the estimations in Theorem 4.10, the proof here is similar to the proof of Lemma 3.5, so we will only point out the necessary modifications.

In Theorem 4.4, if conditions 2), 3) and 4) are replaced by $\mathbf{hl}(C) = \frac{R}{2}$, $s(C) = 1$ and $\mathbf{l}(q(C)) = \frac{R+2k\pi i}{d_C}$ respectively, then we denote the corresponding representation by ρ_0. Since $l(G(X)) > Re^{\frac{R}{4}}$, the same argument as in Lemma 3.5 shows that Theorem 4.4 is true for ρ_0.

Let $q : \widetilde{X} \to X$ be the universal cover. When considering ρ_0, since $l(G(X)) > Re^{\frac{R}{4}}$, there is a $\pi_1(X)$-equivariant embedding $\widetilde{X} \to \mathbb{H}^3$ with respect to ρ_0, and the image is a locally finite union of subsets of hyperbolic planes. There are two metrics on $\widetilde{X}$, one is the induced metric d_0 from $\mathbb{H}^3$ and the other one is the path metric d induced by d_0. The proof of Lemma 3.5 implies that these two metrics are quasi-isometric, and we will always endow $\widetilde{X}$ with the metric d. d is a geodesic metric on X which is locally the hyperbolic metric away from singular curves.

Let $\widetilde{\mathcal{C}} \subset \widetilde{X} \subset \mathbb{H}^3$ be the preimage of the pants decomposition $\mathcal{C}_1 \cup \mathcal{C}_2$ of X, which are union of geodesics. For each pants Π in X, there are three seams which are the common perpendiculars of the three pair of cuffs of Π. Let $\mathcal{A} \subset X$ be the union of all such geodesic arcs and let $\widetilde{\mathcal{A}} \subset \widetilde{X} \subset \mathbb{H}^3$ be the preimage of $\mathcal{A}$ in $\mathbb{H}^3$. We define $Z = \widetilde{\mathcal{C}} \cup \widetilde{\mathcal{A}}$, then the embedding of Z into $\mathbb{H}^3$ is $\pi_1(X)$-equivariant with respect to ρ_0.

Now we turn to study a general representation $\rho : \pi_1(X) \to PSL_2(\mathbb{C})$, which is a deformation of ρ_0. For each geodesic $C \subset \widetilde{\mathcal{C}}$, let C' be the corresponding geodesic in $\mathbb{H}^3$ with respect to ρ, and let $\widetilde{\mathcal{C}'}$ be the union of such C' for all $C \subset \widetilde{\mathcal{C}}$. Let $\widetilde{\mathcal{A}'}$ be the union of common perpendiculars of geodesics in $\widetilde{\mathcal{C}'}$, which correspond with geodesic arcs in $\widetilde{\mathcal{A}}$. Let $Z' = \widetilde{\mathcal{C}'} \cup \widetilde{\mathcal{A}'}$, then Z' is $\rho(\pi_1(X))$-equivariant with respect to ρ.

There is a natural piecewise linear map $h : Z \to Z'$ defined as in Section 4.1, such that the restriction of h on each geodesic arc in $\widetilde{\mathcal{A}}$ is linear, and the restriction of h on each component of $\widetilde{\mathcal{C}} \setminus \widetilde{\mathcal{A}}$ is linear. To show the convex cocompact property of ρ, we need only to show that $h : (Z, d) \to (Z', d_{\mathbb{H}^3}|_{Z'})$ is a quasi-isometry.

For any $x, y \in Z$ with $d(x, y) \geq R$, let α be the shortest path in $\widetilde{X}$ connecting x and y. If $\alpha \cup q^{-1}(\mathcal{C}_2) = \emptyset$, the Theorem 4.8 implies the quasi-isometric property. So we assume $\alpha \cap q^{-1}(\mathcal{C}_2)$ is not empty, and the intersection points are $x_1, x_2, \cdots, x_k$. Since $l(G(X)) > Re^{\frac{R}{4}}$, $d(x_i, x_{i+1}) \geq \frac{R}{2}$ for $i = 1, \cdots, k-1$. Let $x_0 = x$ if $d(x, x_1) \geq \frac{R}{2}$, or let $x_0 = x_1$. Similarly, define $x_{k+1} = y$ if $d(x_k, y) \geq \frac{R}{2}$, or let $x_{k+1} = x_k$. Now we consider about angles $\angle h(x_{i-1})h(x_i)h(x_{i+1})$.

Let γ_i be the geodesic in Z containing x_i, with a preferred orientation, and let θ be the angle between γ_i and $\overline{x_{i-1}x_i}$. Since α is a shortest path in $\widetilde{X}$, the angle between γ_i and $\overline{x_ix_{i+1}}$ equals $\pi-\theta$. Without lose of generality, we can suppose $\theta \leq \frac{\pi}{2}$.

Let θ_1 be the angle between $h(\gamma_i)$ and $\overline{h(x_{i-1})h(x_i)}$, and θ_2 be the angle between $h(\gamma_i)$ and $\overline{h(x_i)h(x_{i+1})}$. If $\theta \leq \frac{\pi}{2}-\frac{3}{2p}$, then by Theorem 4.10, we have $\theta_1 \leq \frac{\pi}{2}-\frac{1}{2p}$ and $\theta_2 \geq \frac{\pi}{2}+\frac{1}{2p}$, so $\angle h(x_{i-1})h(x_i)h(x_{i+1}) \geq \frac{1}{p}$.

If $\frac{\pi}{2}-\frac{3}{2p} \leq \theta \leq \frac{\pi}{2}$, then $\frac{\pi}{2}-\frac{5}{2p} \leq \theta_1 \leq \frac{\pi}{2}+\frac{1}{p}$ and $\frac{\pi}{2}-\frac{1}{p} \leq \theta_1 \leq \frac{\pi}{2}+\frac{5}{2p}$. Let α_i be the component of $\widetilde{\mathcal{A}}$ which is on the same component of $\widetilde{X} \setminus q^{-1}(\mathcal{C}_2)$ as x_{i-1}, intersecting with γ_i, and is the closest such arc from x_i. We also choose α_i' by the same way with x_{i-1} replaced by x_{i+1}. Let P_i be the hyperbolic plane containing $h(\gamma_i)$ and $h(\alpha_i)$, and P_i' be the hyperbolic plane containing $h(\gamma_i)$ and $h(\alpha_i')$. Since pants adjacent to the same singular curve are p-separated, $\angle(P_i, P_i') \geq \frac{2\pi}{p}$.

Let $\vec{n}_i$ and $\vec{n}_i'$ be the normal vectors of P_i and P_i' at $h(x_i)$ respectively, then $\angle(\vec{n}_i, \vec{n}_i') \geq \frac{2\pi}{p}$. Theorem 4.10 implies that, $|\angle(\overline{h(x_{i-1})h(x_i)}, \vec{n}_i) - \frac{\pi}{2}| \leq \frac{1}{p}$ and $|\angle(\overline{h(x_i)h(x_{i+1})}, \vec{n}_i') - \frac{\pi}{2}| \leq \frac{1}{p}$. An elementary computation implies that

$$\angle h(x_{i-1})h(x_i)h(x_{i+1}) \geq \cos\frac{1}{p}\sin\frac{2\pi}{p}-\frac{2}{p} \geq \Big(2\cos\frac{1}{2}-1\Big)\frac{2}{p} > 0.$$

The remaining proof is same with the proof of Lemma 3.5. q.e.d.

Remark 4.14. Given the cited theorems, our proof of Theorem 4.4 is very elementary and quite geometric flavor, but a little bit tedious. It is possible to give alternative proofs by using geometric group theory or using the method in [**Sa**].

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