

CALABI–YAU STRUCTURES ON COTANGENT BUNDLES

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Abstract

Starting with a orientable compact real-analytic Riemannian manifold (L, g) with $\chi(L) = 0$, we show that a small neighborhood $\text{Op}(L)$ of the zero section in the cotangent bundle T^*L carries a Calabi–Yau structure such that the zero section is an isometrically embedded special Lagrangian submanifold.

1. Introduction

Let (L, g) be an orientable compact real-analytic Riemannian manifold with real-analytic Riemannian metric g . According to [1] there exists a sufficiently small neighborhood $\text{Op}(L)$ of the zero section (which we identify with L) in the cotangent bundle T^*L carrying a complex structure J that we view as a integrable almost complex structure. With respect to this complex structure, $L \hookrightarrow \text{Op}(L)$ is a totally real submanifold. On $\text{Op}(L)$ there exists a strictly plurisubharmonic exhaustion function $\rho : \text{Op}(L) \rightarrow \mathbb{R}$ such that the Kähler metric \tilde{g} obtained from the Kähler form $\omega = (i/2)\partial\bar{\partial}\rho$ restricts to the Riemannian metric g when restricted to the zero section L [2]. Summarizing, we are dealing with a Kähler manifold $(\text{Op}(L), J, \omega)$ with Kähler metric \tilde{g} , such that $\omega|_L \equiv 0$ and $\tilde{g}|_L \equiv g$.

We ask if it is possible to solve the following problem.

Problem 1. Find a pair (J, ω) and a nonvanishing holomorphic $(n, 0)$ -form Ω on $\text{Op}(L)$ with the properties:

- 1) $(\text{Op}(L), J, \omega)$ is a Kähler manifold.
- 2) $\omega|_L \equiv 0, \tilde{g}|_L \equiv g$ and $\Omega|_L \equiv \text{Vol}_g$, where Vol_g is the volume form on L induced by the Riemannian metric g and \tilde{g} is the Kähler metric $\tilde{g}(\cdot, \cdot) = \omega(\cdot, J\cdot)$.
- 3) $\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \bar{\Omega}$.

Some of these structures are already uniquely determined. For example, the complex structure J is unique on a sufficiently small neighborhood of the zero section in the cotangent bundle of L up to biholomorphism

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[1]. Moreover, on a sufficiently small neighborhood $\text{Op}(L)$, there exists a unique holomorphic $(n, 0)$ -form with the property $\Omega|_L = \text{Vol}_g$. Indeed, if the Riemannian metric g on L is real-analytic, then the volume form Vol_g is given locally by the real-analytic function $\sqrt{\det(g_{ij})}$. This function can be holomorphically extended, and by the compactness of L , this extension process gives rise to a holomorphic $(n, 0)$ -form Ω such that $\Omega|_L = \text{Vol}_g$. Hence, the complex structure J together with the holomorphic $(n, 0)$ -form Ω are unique. However, the Kähler form ω is not necessarily unique, and Problem 1 reduces to finding a specific Kähler form ω satisfying the requirements (1), (2), and (3). A Kähler manifold that admits a holomorphic $(n, 0)$ -form satisfying Equation (3) is called a *Calabi–Yau* manifold. If, in addition, the conditions $\omega|_L \equiv 0$ and $\Omega|_L \equiv \text{Vol}_g$ are fulfilled, the submanifold $L \hookrightarrow \text{Op}(L)$ will be called a *special Lagrangian* submanifold. Problem 1 has up to now resisted a complete solution. However, it has been solved for some special classes of manifolds:

- 1) For a three-dimensional, compact, real-analytic Riemannian manifold, Bryant [3] solved Problem 1 by using techniques from Cartan–Kähler theory. The main idea in his proof is that every compact, real-analytic Riemannian manifold with real-analytic metric is real-analytically parallelizable, and that Problem 1 can be reduced to a problem of finding particular integral submanifolds of an exterior differential ideal.
- 2) For a compact, real-analytic Kähler manifold with real-analytic Kähler form and complex structure, Feix [4] proved that a neighborhood of the zero section of its cotangent bundle carries a HyperKähler structure, and demonstrated that by rotating the complex structures together with the Kähler forms we are led to a Calabi–Yau structure on a neighborhood of the cotangent bundle such that the zero section is a special Lagrangian submanifold.
- 3) For a compact, rank 1, globally symmetric space L , Stenzel [5] showed that Equation (3) can be reduced, by using symmetries, to a solvable ordinary differential equation, and that L is an isometrically embedded special Lagrangian submanifold.

The main result of this paper is the solution of Problem 1 in the space $\chi(L) = 0$ (generalizing Bryant’s result):

Theorem 2. *Let (L, g) be an orientable compact real-analytic Riemannian manifold with $\chi(L) = 0$. Then there exists a Calabi–Yau structure $(\text{Op}(L), \Omega, \omega)$, where $\text{Op}(L) \subset T^*L$, such that L is a isometrically embedded special Lagrangian submanifold.*

The paper is organized as follows: In Section 2, we recall some geometric properties of Calabi–Yau manifolds and special Lagrangian submanifolds. In Section 3, we prove Theorem 2.

2. Calabi–Yau manifolds and special Lagrangian submanifolds

Calabi–Yau manifolds can be defined in several ways. First, they can be regarded as Kähler manifolds equipped with a Ricci-flat Kähler metric. In the compact case, we can drop the assumption on the Ricci-flatness and define them to have vanishing first Chern class [8]. Second, they can be defined as Kähler manifolds with holonomy group contained in $SU(n)$. Third, they can be introduced via the following:

Definition 3. A Kähler manifold (X, J, ω) is called *Calabi–Yau* if there exists a nonvanishing holomorphic $(n, 0)$ -form Ω such that the following equation holds:

$$(2.1) \quad \frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \bar{\Omega}.$$

This definition will be adopted in our analysis. Some properties of Calabi–Yau manifolds are summarized below.

- 1) The constant $(-1)^{n(n-1)/2} (i/2)^n$ is a normalization constant, so that in local holomorphic coordinates (z_1, \dots, z_n) , (2.1) is of the form

$$(2.2) \quad \det(g_{i\bar{j}}) = |h|^2,$$

where $g_{i\bar{j}}$ are the coefficients of the Kähler form ω in the basis $\partial/\partial z^i, \partial/\partial \bar{z}^j$ and h is a holomorphic function such that $\Omega = h dz^1 \wedge \dots \wedge dz^n$.

- 2) A Calabi–Yau manifold is Ricci-flat. This can be seen as follows. The Ricci-form in local holomorphic coordinates is given by $\text{Ric}(\omega) = -i\partial\bar{\partial}\log(\det(g_{i\bar{j}}))$. Thus, from (2.2) we obtain

$$\begin{aligned} \text{Ric}(\omega) &= -i\partial\bar{\partial}\log(\det(g_{i\bar{j}})) = -i\partial\bar{\partial}\log(|h|^2) \\ &= -i\partial\bar{\partial}\log(h\bar{h}) = 0. \end{aligned}$$

- 3) The holonomy group of a Calabi–Yau manifold is contained in $SU(n)$. To see this, note that a Calabi–Yau manifold has trivial canonical bundle $K_X = \Lambda^{(n,0)}T^*X$. Denote by g the Kähler metric induced by the Kähler form ω . Furthermore, let ∇ be the Levi-Civita connection of the metric g . Thus, we can induce a metric on the cotangent bundle T^*X and, in particular, a Riemannian metric on the canonical bundle K_X . Denote by ∇^K the Levi-Civita connection on K_X induced by the Riemannian metric. By straightforward calculation we can show that the Ricci-form $\text{Ric}(\omega)$ is equal to $-i$ times the curvature tensor of the canonical line bundle K_X . Thus, the Ricci-form is zero if and only if there exists a parallel, hence holomorphic, form of type $(n, 0)$ in a neighborhood of any point of X . For Calabi–Yau manifolds this is obviously true. Consequently, from $\nabla g = 0, \nabla\omega = 0$, and $\nabla\Omega = 0$

we infer that the holonomy on the Levi-Civita connection induced by the Riemannian metric g is contained in $SU(n)$. For detailed calculations the reader might consult [7].

Special Lagrangian submanifolds of Calabi–Yau manifolds were first introduced by Harvey and Lawson in [9] as a particular case of calibrated submanifolds. There are some equivalent definitions of special Lagrangian submanifolds. In our analysis we will use the following:

Definition 4. A submanifold Y of a Calabi–Yau manifold (X, J, ω, Ω) is *special Lagrangian* if

$$\omega|_L = 0 \text{ and } \text{Im}(\Omega)|_L = 0.$$

3. Proof of Theorem 2

To prove Theorem 2, we use the fact that every compact manifold L with $\chi(L) = 0$ admits a nonvanishing vector field and conversely [10].

We come now to the proof. Let (L, g) be a real-analytic Riemannian manifold with real-analytic metric g . Since $\chi(L) = 0$, there exists a globally defined nonvanishing vector field X on L . According to [11], we can choose X to be real-analytic. Let $V_x = \text{span}\{X(x)\}$ and $W_x = \{Y \in T_x L | g_x(Y, X(x)) = 0\}$, and let $V = \coprod_{x \in L} V_x$ and $W = \coprod_{x \in L} W_x$. V and W are real-analytic vector bundles over L of rank 1 and $n - 1$, respectively, and it is apparent that $V \oplus W = TL$. If we regard V, W , and $V \oplus W$ as manifolds, we have that $\dim(V) = n + 1, \dim(W) = 2n - 1$, and $\dim(V \oplus W) = 2n$. Let us consider the map

$$TL = V \oplus W \xrightarrow{F} W \times \mathbb{R},$$

given by $F(p, tX(p), Y) = (p, Y, t)$. This map is a real-analytic diffeomorphism. L is identified with the zero section $L \times \{0\}$ in TL , and via F, L is identified with $L \times \{0\} \times \{0\} \subset W \times \mathbb{R}$. Consider the Kähler structure introduced by Stenzel [2] on TL , i.e., $J : T(TL) \rightarrow T(TL)$ with the Kähler form $(i/2) \partial \bar{\partial} \rho$, where ρ is a strictly plurisubharmonic exhaustion function defined on $\text{Op}(L)$. By F , we transport these structures to $W \times \mathbb{R}$, so that F becomes an isometric biholomorphism. To avoid an abundance of notations, denote again by J the complex structure on $W \times \mathbb{R}$, and denote by ρ the strictly plurisubharmonic function that defines the Kähler form. Consider now the Kähler manifold $(W \times \mathbb{R}, J, (i/2) \partial \bar{\partial} \rho)$, where the metric induced by $(i/2) \partial \bar{\partial} \rho$ restricts to g on $L \times \{0\} \times \{0\}$. We intend to find a plurisubharmonic function ϕ in a neighborhood of $L \times \{0\} \times \{0\}$ in $W \times \mathbb{R}$, such that the Kähler metric induced by $(i/2) \partial \bar{\partial} \phi$ restricts on L to g and such that

$$(3.1) \quad \left(\frac{i}{2} \partial \bar{\partial} \phi\right)^n = (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \bar{\Omega}.$$

In order to apply the Cauchy–Kovalewsky theorem, we equip (3.1) with the initial conditions $\phi(p, 0) = \rho(p, 0)$ and $\frac{\partial\phi}{\partial t}(p, 0) = \frac{\partial\rho}{\partial t}(p, 0)$. Hence we are looking for solutions to the following initial value problem:

$$(3.2) \quad \begin{cases} \left(\frac{i}{2}\partial\bar{\partial}\phi\right)^n &= (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \bar{\Omega} \\ \phi(p, 0) &= \rho(p, 0) \\ \frac{\partial\phi}{\partial t}(p, 0) &= \frac{\partial\rho}{\partial t}(p, 0). \end{cases}$$

For $p_0 \in L$, let (x^1, \dots, x^n) be the normal coordinates in a neighborhood U of p_0 in L . Moreover, let (Y_1, \dots, Y_{n-1}, X) be an orthonormal frame in U , with respect to g , such that

$$(Y_1(p_0), \dots, Y_{n-1}(p_0), X(p_0)) = \left(\frac{\partial}{\partial x^1}\Big|_{p_0}, \dots, \frac{\partial}{\partial x^{n-1}}\Big|_{p_0}, \frac{\partial}{\partial x^n}\Big|_{p_0}\right).$$

Hence,

$$(3.3) \quad (x^1, \dots, x^n, y^1, \dots, y^{n-1}, t) \mapsto (x^1, \dots, x^n, \sum_{i=1}^{n-1} y^i Y_i + tX)$$

is a coordinate chart around p_0 in TL with the property that

$$(x^1, \dots, x^n, y^1, \dots, y^{n-1}) \mapsto (x^1, \dots, x^n, \sum_{i=1}^{n-1} y^i Y_i)$$

is a trivialization of the bundle $W \rightarrow L$. The coordinates (3.3) are real-analytic on $W \times \mathbb{R}$, where t is the global coordinate on \mathbb{R} . In these coordinates, we have

$$(3.4) \quad J_{p_0} \frac{\partial}{\partial x^i}\Big|_{p_0} = \frac{\partial}{\partial y^i}\Big|_{p_0} \text{ for } i = 1, \dots, n-1$$

$$(3.5) \quad J_{p_0} \frac{\partial}{\partial x^n}\Big|_{p_0} = \frac{\partial}{\partial t}\Big|_{p_0}$$

and the complex structure takes the form

$$J = \begin{pmatrix} J_{x^1}^{x^1} & J_{x^2}^{x^1} & \cdots & J_{x^n}^{x^1} & J_{y^1}^{x^1} & \cdots & J_{y^{n-1}}^{x^1} & J_t^{x^1} \\ J_{x^1}^{x^2} & J_{x^2}^{x^2} & \cdots & J_{x^n}^{x^2} & J_{y^1}^{x^2} & \cdots & J_{y^{n-1}}^{x^2} & J_t^{x^2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ J_{x^1}^{x^n} & J_{x^2}^{x^n} & \cdots & J_{x^n}^{x^n} & J_{y^1}^{x^n} & \cdots & J_{y^{n-1}}^{x^n} & J_t^{x^n} \\ J_{x^1}^{y^1} & J_{x^2}^{y^1} & \cdots & J_{x^n}^{y^1} & J_{y^1}^{y^1} & \cdots & J_{y^{n-1}}^{y^1} & J_t^{y^1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ J_{x^1}^{y^{n-1}} & J_{x^2}^{y^{n-1}} & \cdots & J_{x^n}^{y^{n-1}} & J_{y^1}^{y^{n-1}} & \cdots & J_{y^{n-1}}^{y^{n-1}} & J_t^{y^{n-1}} \\ J_{x^1}^t & J_{x^2}^t & \cdots & J_{x^n}^t & J_{y^1}^t & \cdots & J_{y^{n-1}}^t & J_t^t \end{pmatrix}.$$

To compute $(i/2)\partial\bar{\partial}\phi$, we note that

$$\frac{\partial}{\partial z^i} = \frac{1}{2} \left(\frac{\partial}{\partial x^i} - iJ \frac{\partial}{\partial x^i} \right) = \frac{1}{2} \left(\frac{\partial}{\partial x^i} - i \left(J_{x^i}^{x^k} \frac{\partial}{\partial x^k} + J_{x^i}^{y^k} \frac{\partial}{\partial y^k} + J_{x^i}^t \frac{\partial}{\partial t} \right) \right)$$

and that

$$\frac{\partial}{\partial \bar{z}^i} = \frac{1}{2} \left(\frac{\partial}{\partial x^i} + iJ \frac{\partial}{\partial x^i} \right) = \frac{1}{2} \left(\frac{\partial}{\partial x^i} + i \left(J_{x^i}^{x^k} \frac{\partial}{\partial x^k} + J_{x^i}^{y^k} \frac{\partial}{\partial y^k} + J_{x^i}^t \frac{\partial}{\partial t} \right) \right),$$

for $i = 1, \dots, n$. Then, we obtain

$$\begin{aligned} \frac{\partial^2 \phi}{\partial \bar{z}^j \partial z^i} &= \frac{1}{2} \frac{\partial}{\partial \bar{z}^j} \left(\frac{\partial \phi}{\partial x^i} - iJ_{x^i}^{x^k} \frac{\partial \phi}{\partial x^k} - iJ_{x^i}^{y^k} \frac{\partial \phi}{\partial y^k} - iJ_{x^i}^t \frac{\partial \phi}{\partial t} \right) \\ &= \frac{1}{4} \left(\frac{\partial}{\partial x^j} + iJ_{x^j}^{x^l} \frac{\partial}{\partial x^l} + iJ_{x^j}^{y^l} \frac{\partial}{\partial y^l} + iJ_{x^j}^t \frac{\partial}{\partial t} \right) \\ &\quad \times \left(\frac{\partial \phi}{\partial x^i} - iJ_{x^i}^{x^k} \frac{\partial \phi}{\partial x^k} - iJ_{x^i}^{y^k} \frac{\partial \phi}{\partial y^k} - iJ_{x^i}^t \frac{\partial \phi}{\partial t} \right) \\ &= \frac{1}{4} \left(F_{i,j} + J_{x^i}^t J_{x^j}^t \frac{\partial^2 \phi}{\partial t^2} \right), \end{aligned}$$

where

$$\begin{aligned} F_{i,j} &= \frac{\partial^2 \phi}{\partial x^j \partial x^i} - i\partial_{x^j} J_{x^i}^{x^k} \frac{\partial \phi}{\partial x^k} - iJ_{x^i}^{x^k} \frac{\partial^2 \phi}{\partial x^j \partial x^k} - i\partial_{x^j} J_{x^i}^{y^k} \frac{\partial \phi}{\partial y^k} - iJ_{x^i}^{y^k} \frac{\partial^2 \phi}{\partial x^j \partial y^k} \\ &\quad - i\partial_{x^j} J_{x^i}^t \frac{\partial \phi}{\partial t} - iJ_{x^i}^t \frac{\partial^2 \phi}{\partial x^j \partial t} + iJ_{x^j}^{x^l} \frac{\partial^2 \phi}{\partial x^l \partial x^i} + J_{x^j}^{x^l} \partial_{x^l} J_{x^i}^{x^k} \frac{\partial \phi}{\partial x^k} + J_{x^j}^{x^l} J_{x^i}^{x^k} \frac{\partial^2 \phi}{\partial x^l \partial x^k} \\ &\quad + J_{x^j}^{x^l} \partial_{x^l} J_{x^i}^{y^k} \frac{\partial \phi}{\partial y^k} + J_{x^j}^{x^l} J_{x^i}^{y^k} \frac{\partial^2 \phi}{\partial x^l \partial y^k} + J_{x^j}^{x^l} \partial_{x^l} J_{x^i}^t \frac{\partial \phi}{\partial t} + J_{x^j}^{x^l} J_{x^i}^t \frac{\partial^2 \phi}{\partial x^l \partial t} \\ &\quad + iJ_{x^j}^{y^l} \frac{\partial^2 \phi}{\partial y^l \partial x^i} + J_{x^j}^{y^l} \partial_{y^l} J_{x^i}^{x^k} \frac{\partial \phi}{\partial x^k} + J_{x^j}^{y^l} J_{x^i}^{x^k} \frac{\partial^2 \phi}{\partial y^l \partial x^k} + J_{x^j}^{y^l} \partial_{y^l} J_{x^i}^{y^k} \frac{\partial \phi}{\partial y^k} \\ &\quad + J_{x^j}^{y^l} J_{x^i}^{y^k} \frac{\partial^2 \phi}{\partial y^l \partial y^k} + J_{x^j}^{y^l} \partial_{y^l} J_{x^i}^t \frac{\partial \phi}{\partial t} + J_{x^j}^{y^l} J_{x^i}^t \frac{\partial^2 \phi}{\partial y^l \partial t} \\ &\quad + iJ_{x^j}^t \frac{\partial^2 \phi}{\partial t \partial x^i} + J_{x^j}^t \partial_t J_{x^i}^{x^k} \frac{\partial \phi}{\partial x^k} + J_{x^j}^t J_{x^i}^{x^k} \frac{\partial^2 \phi}{\partial t \partial x^k} \\ &\quad + J_{x^j}^t \partial_t J_{x^i}^{y^k} \frac{\partial \phi}{\partial y^k} + J_{x^j}^t J_{x^i}^{y^k} \frac{\partial^2 \phi}{\partial t \partial y^k} + J_{x^j}^t \partial_t J_{x^i}^t \frac{\partial \phi}{\partial t}. \end{aligned}$$

In local coordinates we have $\Omega = h dz^1 \wedge \dots \wedge dz^n$ for a holomorphic function h and Equation (3.1) is equivalent to

$$\det \left(\frac{\partial^2 \phi}{\partial \bar{z}^j \partial z^i} \right) = |h|^2.$$

Consider now the function

$$\begin{aligned} G &:= \det \left(\frac{\partial^2 \phi}{\partial \bar{z}^j \partial z^i} \right) - |h|^2 \\ &= \det \left(\frac{1}{4} \left(F_{i,j} + J_t^{x^i} J_t^{x^j} \frac{\partial^2 \phi}{\partial t^2} \right) \right) - |h|^2. \end{aligned}$$

Obviously, G is a polynomial of order n in $\partial^2 \phi / \partial t^2$ with functions defined on $W \times \mathbb{R}$ as coefficients. In general, G can be regarded as a

function

$$G = G(p, t, \phi_x, \phi_y, \phi_t, \phi_{xx}, \phi_{yy}, \phi_{xy}, \phi_{tx}, \phi_{ty}, \phi_{tt}),$$

where $(p, t) \in W \times \mathbb{R}$. By (3.4) and (3.5), there holds

$$(3.6) \quad J|_{T_{p_0}W \times \mathbb{R}} = J_{\text{st}} = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix},$$

and we deduce that $J_t^{x^i} = 0$ for $i = 1, \dots, n - 1$, and $J_t^{x^n} = 1$. Therefore, the equation $G = 0$, restricted on L , can be solved for $\partial^2\phi/\partial t^2$ because of

$$(D_{\phi_{tt}}G)|_L = \det \begin{pmatrix} g_{11} & \cdots & g_{1(n-1)} \\ \vdots & \ddots & \vdots \\ g_{(n-1)1} & \cdots & g_{(n-1)(n-1)} \end{pmatrix} \neq 0,$$

where g_{ij} are the metric coefficients of the metric g on L . Here we have used the initial condition $\phi|_L = \rho|_L$ and the fact that the Kähler metric induced by $(i/2)\partial\bar{\partial}\rho$ restricts to g on L . Thus, by the real-analytic implicit function theorem [9], the equation $G = 0$ can be solved locally for $\partial^2\phi/\partial t^2$, i.e., there exists an analytic function $H = H(p, t, \phi_x, \phi_y, \phi_{xx}, \phi_{yy}, \phi_{xy}, \phi_t, \phi_{xt}, \phi_{yt})$, such that $\partial^2\phi/\partial t^2 = H$ and $G(p, t, \phi_x, \phi_y, \phi_{xx}, \phi_{yy}, \phi_{xy}, \phi_t, \phi_{xt}, \phi_{yt}, H) = 0$ in a neighborhood of L in $W \times \mathbb{R}$. As a result, the initial-value problem

$$(3.7) \quad \begin{cases} \det \left(\frac{\partial^2 \phi}{\partial \bar{z}^j \partial z^i} \right) & = |h|^2 \\ \phi(p, 0) & = \rho(p, 0) \\ \frac{\partial \phi}{\partial t}(p, 0) & = \frac{\partial \rho}{\partial t}(p, 0) \end{cases}$$

is locally equivalent to

$$(3.8) \quad \begin{cases} \frac{\partial^2 \phi}{\partial t^2} & = H(p, t, \phi_x, \phi_y, \phi_{xx}, \phi_{yy}, \phi_{xy}, \phi_t, \phi_{xt}, \phi_{yt}) \\ \phi(p, 0) & = \rho(p, 0) \\ \frac{\partial \phi}{\partial t}(p, 0) & = \frac{\partial \rho}{\partial t}(p, 0). \end{cases}$$

Since all coefficient functions are real-analytic, by the Cauchy-Kovalewsky theorem [12] has a unique solution in a neighborhood in $W \times \mathbb{R}$ of each point $x \in L$. As ρ is defined globally and the local solutions of the Cauchy-Kovalewsky type are unique, we obtain a solution ϕ of (3.2) in a neighborhood of L in $W \times \mathbb{R}$. Now, we check that ϕ is strictly plurisubharmonic. Locally, we expand ϕ in a power series in t , i.e.,

$$\phi(x, y, t) = \phi_0(x, y) + \phi_1(x, y)t + \phi_2(x, y)t^2 + \dots$$

From $\phi(x, y, 0) = \rho(x, y, 0)$ and $\frac{\partial \phi}{\partial t}(x, y, 0) = \frac{\partial \rho}{\partial t}(x, y, 0)$, we find that

$$\phi(x, y, t) = \rho(x, y, 0) + \frac{\partial \rho}{\partial t}(x, y, 0)t + t^2g(x, y, t),$$

where $g(x, y, t) = \sum_{i=0}^{\infty} \phi_{i+2}(x, y)t^i$. Consequently, we obtain

$$\begin{aligned} \frac{\partial^2 \phi}{\partial x^j \partial x^k}(x, 0, 0) &= \frac{\partial^2 \rho}{\partial x^j \partial x^k}(x, 0, 0) \\ \frac{\partial^2 \phi}{\partial x^j \partial y^k}(x, 0, 0) &= \frac{\partial^2 \rho}{\partial x^j \partial y^k}(x, 0, 0) \\ \frac{\partial^2 \phi}{\partial x^j \partial t}(x, 0, 0) &= \frac{\partial^2 \rho}{\partial x^j \partial t}(x, 0, 0) \\ \frac{\partial^2 \phi}{\partial y^j \partial t}(x, 0, 0) &= \frac{\partial^2 \rho}{\partial y^j \partial t}(x, 0, 0) \\ \frac{\partial \phi}{\partial x^j}(x, 0, 0) &= \frac{\partial \rho}{\partial x^j}(x, 0, 0) \\ \frac{\partial \phi}{\partial y^j}(x, 0, 0) &= \frac{\partial \rho}{\partial y^j}(x, 0, 0) \\ \frac{\partial \phi}{\partial t}(x, 0, 0) &= \frac{\partial \rho}{\partial t}(x, 0, 0) \end{aligned}$$

and

$$\frac{\partial^2 \phi}{\partial t^2}(x, 0, 0) = 2g(x, 0, 0).$$

Accounting of (3.6), we end up with

$$\left(\frac{\partial^2 \phi}{\partial \bar{z}^j \partial z^i} \right) |_L = \begin{pmatrix} \frac{\partial^2 \rho}{\partial z^1 \partial \bar{z}^1} & \cdots & \frac{\partial^2 \rho}{\partial z^1 \partial \bar{z}^{n-1}} & \frac{\partial^2 \rho}{\partial z^1 \partial \bar{z}^n} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial^2 \rho}{\partial z^{n-1} \partial \bar{z}^1} & \cdots & \frac{\partial^2 \rho}{\partial z^{n-1} \partial \bar{z}^{n-1}} & \frac{\partial^2 \rho}{\partial z^{n-1} \partial \bar{z}^n} \\ \frac{\partial^2 \rho}{\partial z^n \partial \bar{z}^1} & \cdots & \frac{\partial^2 \rho}{\partial z^n \partial \bar{z}^{n-1}} & 2g(x, 0, 0) \end{pmatrix} |_L.$$

The function ϕ is strictly plurisubharmonic if the matrix on the left-hand side of the above equation is positive definite. In fact, it is enough to show that this matrix is positive definite on L , since in this case it follows that it is positive definite in a neighborhood of L . As ρ is strictly plurisubharmonic, the first $n - 1$ principal minors of the matrix on the right-hand side of the above equation are positive, and it remains to show that

$$\det \left(\frac{\partial^2 \phi}{\partial \bar{z}^j \partial z^i} \right) |_L = \det \begin{pmatrix} \frac{\partial^2 \rho}{\partial z^1 \partial \bar{z}^1} & \cdots & \frac{\partial^2 \rho}{\partial z^1 \partial \bar{z}^{n-1}} & \frac{\partial^2 \rho}{\partial z^1 \partial \bar{z}^n} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial^2 \rho}{\partial z^{n-1} \partial \bar{z}^1} & \cdots & \frac{\partial^2 \rho}{\partial z^{n-1} \partial \bar{z}^{n-1}} & \frac{\partial^2 \rho}{\partial z^{n-1} \partial \bar{z}^n} \\ \frac{\partial^2 \rho}{\partial z^n \partial \bar{z}^1} & \cdots & \frac{\partial^2 \rho}{\partial z^n \partial \bar{z}^{n-1}} & 2g(x, 0, 0) \end{pmatrix} |_L > 0.$$

However, as ϕ solves (3.7) and $|h|^2$ is positive on L , the conclusion readily follows, so ϕ is strictly plurisubharmonic. Finally, we prove that the metric obtained from the Kähler form $(i/2) \partial \bar{\partial} \phi$ restricts on L to

the Riemannian metric on L . Because of

$$\det \left(\frac{\partial^2 \phi}{\partial \bar{z}^j \partial z^i} \right) |_{L} = \det \begin{pmatrix} g_{11} & \cdots & g_{1(n-1)} & g_{nn} \\ \vdots & \ddots & \vdots & \vdots \\ g_{(n-1)1} & \cdots & g_{(n-1)(n-1)} & g_{(n-1)n} \\ g_{n1} & \cdots & g_{n(n-1)} & 2g(x, 0, 0) \end{pmatrix} |_{L}$$

and

$$\det \left(\frac{\partial^2 \phi}{\partial \bar{z}^j \partial z^i} \right) |_{L} = |h|^2 |_{L} = \det(g_{ij}),$$

we obtain $2g(x, 0, 0) = g_{nn}(x)$.

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