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AN ALGEBRAIC PROOF OF THE HYPERPLANE PROPERTY OF THE GENUS ONE GW-INVARIANTS OF QUINTICS

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Abstract

Li-Zinger's *hyperplane property for reduced genus one GW-invariants of quintics* states that the genus one GW-invariants of the quintic threefold is the sum of its reduced genus one GW-invariants and 1/12 times its genus zero GW-invariants. We apply the theory of GW-invariants of stable maps with fields to give an algebro-geometric proof of this *hyperplane property*.

1. Introduction

GW-invariants of a smooth projective variety X are "virtual enumerations" of stable maps to X subject to geometric constraints. More precisely, for $d \in H_2(X,\mathbb{Z})$, the moduli space $\overline{M}_g(X,d)$ of genus g stable morphisms to X of fundamental class (degree) d is a proper, separated DM-stack, and carries a canonical virtual fundamental cycle $[\overline{M}_g(X,d)]^{\mathrm{vir}}$. When X is a Calabi-Yau threefold, this class is a dimension zero cycle, and its degree is the degree d genus g GW-invariant of X:

$$N_g(d)_X = \deg\,[\overline{M}_g(X,d)]^{\mathrm{vir}}.$$

Investigating GW-invariants of Calabi-Yau threefolds is one of the main focuses in the subject of Mirror Symmetry. In case the Calabi-Yau threefold X is a complete intersection in a projective space $\mathbb{P}$, by [**Ko**], its genus zero degree d GW-invariant is the integral of the top Chern class

$$\deg\,[\overline{M}_0(X,d)]^{\mathrm{vir}} = \int_{[\overline{M}_0(\mathbb{P},d)]} c_{\mathrm{top}}(\mathbb{E}_{0,d}) \tag{1.1}$$

of a vector bundle $\mathbb{E}_{0,d}$ on $\overline{M}_0(\mathbb{P},d)$. The identity (1.1) is called the "*hyperplane property of the GW-invariants*". The techniques developed in [**Gi**, **LLY**], etc., allow one to completely solve the genus zero invariants of such X.

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Generalizing this to high genus requires a new approach, in part because the *hyperplane property* stated fails for positive genus invariants. In [**Zi2**, **LZ**] Zinger and the second named author introduced the reduced $g = 1$ GW-invariants $N_1(d)_Q^{\mathrm{red}}$ of the quintic $Q \subset \mathbb{P}^4$ using an analogous "hyperplane property" and showed that they relate to the ordinary GW-invariants $N_1(d)_Q$ by a simple linear relation.

In this paper, we give an algebraic proof of this relation.

Theorem 1.1 ([**LZ**]). *The reduced and the ordinary genus one GW-invariants of a quintic Calabi-Yau $Q \subset \mathbb{P}^4$ are related by*

$$N_1(d)_Q = N_1(d)_Q^{red} + \frac{1}{12} N_0(d)_Q.$$

In algebraic geometry, the reduced genus one GW-invariants of quintics take the following form. For simplicity, in this paper we abbreviate $\mathcal{X} = \overline{M}_1(\mathbb{P}^4, d)$. Let

$$(f_{\mathcal{X}}, \pi_{\mathcal{X}}) : \mathcal{C} \longrightarrow \mathbb{P}^4 \times \mathcal{X} = \mathbb{P}^4 \times \overline{M}_1(\mathbb{P}^4, d)$$

be the universal family of $\mathcal{X}$; let $\mathcal{X}_{\mathrm{pri}} \subset \mathcal{X}$ be the primary component of $\mathcal{X}$ that is the closure of the open substack of all stable morphisms with smooth domains. We pick a DM-stack $\tilde{\mathcal{X}}_{\mathrm{pri}}$ and a proper birational morphism $\varphi : \tilde{\mathcal{X}}_{\mathrm{pri}} \to \mathcal{X}_{\mathrm{pri}}$ so that with $(f_{\tilde{\mathcal{X}}_{\mathrm{pri}}}, \pi_{\tilde{\mathcal{X}}_{\mathrm{pri}}}) : \tilde{\mathcal{C}}_{\mathrm{pri}} \to \mathbb{P}^4 \times \tilde{\mathcal{X}}_{\mathrm{pri}}$ the pull back of $(f_{\mathcal{X}}, \pi_{\mathcal{X}})$, the direct image sheaf

$$\pi_{\tilde{\mathcal{X}}_{\mathrm{pri}}*} f^*_{\tilde{\mathcal{X}}_{\mathrm{pri}}} \mathcal{O}_{\mathbb{P}^4}(5)$$

is locally free on $\tilde{\mathcal{X}}_{\mathrm{pri}}$. In [**VZ**], (see also [**HL**],) such φ is constructed by a modular blowing-up. We state the working definition of the reduced invariants of Q.

Definition 1.2 ([**LZ**]). We define the reduced $g = 1$ GW-invariants of Q to be

$$N_1(d)_Q^{\mathrm{red}} = \int_{[\tilde{\mathcal{X}}_{\mathrm{pri}}]} c_{\mathrm{top}}(\pi_{\tilde{\mathcal{X}}_{\mathrm{pri}}*} f^*_{\tilde{\mathcal{X}}_{\mathrm{pri}}} \mathcal{O}_{\mathbb{P}^4}(5)). \tag{1.2}$$

To prove Theorem 1.1, we will separate $[\overline{M}_1(Q, d)]^{\mathrm{vir}}$ into its "primary" and "ghost" parts, and show that the "primary" part can be evaluated using (1.2) while the "ghost" part contributes to $\frac{1}{12} N_0(d)_Q$.

The original proof of this theorem was via analysis, which achieves the desired separation by perturbing the complex structure of Q to a generic almost complex structure, worked out in details in [**Zi3**].

The proof presented in this paper uses "the GW-invariants of stable maps with p-field" worked out by the authors in [**CL1**], generalizing the Guffin-Sharpe-Witten's (GSW) $g = 0$ invariants of complete intersection Calabi-Yau threefolds, which we briefly outline now.

Given a positive integer d, we form the moduli $\overline{M}_1(\mathbb{P}^4,d)^p$ of genus 1 degree d stable morphisms to $\mathbb{P}^4$ with p-fields:

$$\overline{M}_1(\mathbb{P}^4,d)^p = \{[f,C,p] \mid [f,C]\in\overline{M}_1(\mathbb{P}^4,d),\ p\in\Gamma(f^*\mathcal{O}_{\mathbb{P}^4}(-5)\otimes\omega_C)\}/\sim.$$

It is a Deligne-Mumford stack with a perfect obstruction theory. The polynomial $\mathbf{w} = x_1^5+\ldots+x_5^5$ (or any general quintic polynomial) induces a cosection (homomorphism) of its obstruction sheaf

$$\sigma : \mathcal{O}b_{\overline{M}_1(\mathbb{P}^4,d)^p} \longrightarrow \mathcal{O}_{\overline{M}_1(\mathbb{P}^4,d)^p}, \tag{1.3}$$

whose non-surjective locus (called the degeneracy locus) is

$$\overline{M}_1(Q,d)\subset\overline{M}_1(\mathbb{P}^4,d)^p,\quad Q=(x_1^5+\ldots+x_5^5=0)\subset\mathbb{P}^4,$$

which is proper. The cosection-localized virtual fundamental class construction of Kiem-Li defines a localized virtual cycle

$$[\overline{M}_1(\mathbb{P}^4,d)^p]^{\mathrm{vir}}_\sigma\in A_0\overline{M}_1(Q,d).$$

(For convention of cosection-localized virtual fundamental classes, see discussion after (2.6).) The GW-invariant of $\overline{M}_1(\mathbb{P}^4,d)^p$ is defined to be

$$N_1(d)^p_{\mathbb{P}^4} = \deg\,[\overline{M}_1(\mathbb{P}^4,d)^p]^{\mathrm{vir}}_\sigma\in\mathbb{Q}.$$

Theorem 1.3 ([**CL1**]). *For $d>0$, the GW-invariant of $\overline{M}_1(\mathbb{P}^4,d)^p$ coincides with the GW-invariant $N_1(d)_Q$ of the quintic Q up to a sign:*

$$N_1(d)^p_{\mathbb{P}^4} = (-1)^{5d}\cdot N_1(d)_Q.$$

By this theorem, to prove Theorem 1.1 it suffices to study the cycle $[\overline{M}_1(\mathbb{P}^4,d)^p]^{\mathrm{vir}}_\sigma$. Following the recipe in [**VZ**, **HL**], we form the modular blow-up $\tilde{\mathcal{Y}}$ of $\overline{M}_1(\mathbb{P}^4,d)^p$. It is a union of $\tilde{\mathcal{Y}}_{\mathrm{pri}}$, which is birational to the primary component of $\overline{M}_1(\mathbb{P}^4,d)$, with other smooth components $\tilde{\mathcal{Y}}_\mu$, indexed by partitions μ of d:

$$\tilde{\mathcal{Y}} = \tilde{\mathcal{Y}}_{\mathrm{pri}}\ \cup\ \Big(\bigcup_{\mu\vdash d}\tilde{\mathcal{Y}}_\mu\Big).$$

Geometrically, a general element of $\tilde{\mathcal{Y}}_{\mathrm{pri}}$ is a stable morphism in $\overline{M}_1(\mathbb{P}^4,d)$ having smooth domain; a general element of $\tilde{\mathcal{Y}}_\mu$ (with $\mu = (d_1,\cdots,d_\ell)$) lies over a stable morphism $[f,C]\in\overline{M}_1(\mathbb{P}^4,d)$ whose domain C is a smooth elliptic curve together with ℓ $\mathbb{P}^1$ attached to it, and the morphism f is constant along the elliptic curve and has degree d_i along the i-th $\mathbb{P}^1$ attached.

It is convenient to work with the obstruction theory relative to the Artin stack $\mathcal{D}$ of pairs (C,L) of degree d line bundles L on connected genus one nodal curves C. Using the universal family of $\overline{M}_1(\mathbb{P}^4,d)^p$, we obtain a forgetful morphism $\overline{M}_1(\mathbb{P}^4,d)^p\to\mathcal{D}$. We then perform a parallel modular blow-up of $\mathcal{D}$ to obtain $\tilde{\mathcal{D}}\to\mathcal{D}$, which reconstructs $\tilde{\mathcal{Y}}$ via the Cartesian product $\tilde{\mathcal{Y}} = \overline{M}_1(\mathbb{P}^4,d)^p\times_{\mathcal{D}}\tilde{\mathcal{D}}$. By working out the relative perfect obstruction theory of $\overline{M}_1(\mathbb{P}^4,d)^p\to\mathcal{D}$, we obtain the relative perfect obstruction theory of $\tilde{\mathcal{Y}} \to \tilde{\mathcal{D}}$ and its obstruction complex $E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$. We form its intrinsic normal cone $\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$, which is a subcone of $h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})$. (cf. [**BF**, **LT**]).

By picking a homogeneous quintic polynomial, say $x_1^5 + \cdots + x_5^5$, we construct a cosection (homomorphism) $\tilde{\sigma} : h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}) \to \mathcal{O}_{\tilde{\mathcal{Y}}}$. By a cosection-localized version of [**Cos**, Thm 5.0.1], we prove

$$\deg\, [\overline{M}_1(\mathbb{P}^4, d)^p]^{\mathrm{vir}}_{\sigma} = \deg 0^!_{\tilde{\sigma},\mathrm{loc}}[\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}]. \tag{1.4}$$

(cf. Proposition 2.5.) Using the explicit local defining equation of $\tilde{\mathcal{Y}}$ obtained in [**Zi1**, **HL**], we conclude that as cycles

$$[\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}] = [\mathbf{C}_{\mathrm{pri}}] + \sum_{\mu \vdash d} [\mathbf{C}_{\mu}],$$

where $\mathbf{C}_{\mathrm{pri}}$ is an irreducible cycle lies over $\tilde{\mathcal{Y}}_{\mathrm{pri}}$; and $\mathbf{C}_{\mu}$ are cycles lying over $\tilde{\mathcal{Y}}_{\mu}$. Thus

$$\deg 0^!_{\tilde{\sigma},\mathrm{loc}}[\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}] = 0^!_{\tilde{\sigma},\mathrm{loc}}[\mathbf{C}_{\mathrm{pri}}] + \sum_{\mu \vdash d} 0^!_{\tilde{\sigma},\mathrm{loc}}[\mathbf{C}_{\mu}]. \tag{1.5}$$

After working out a structure result of three cones in Section four, in Section five, we show that $0^!_{\tilde{\sigma},\mathrm{loc}}[\mathbf{C}_{\mathrm{pri}}]$ equals the reduced invariant $N_1(d)^{\mathrm{red}}_Q$. In Section six and seven, we develop a method to attack the remainder terms in (1.5); in Section eight, we show that $0^!_{\tilde{\sigma},\mathrm{loc}}[\mathbf{C}_{\mu}] = 0$ for all partitions $\mu \neq (d)$, where (d) is the partition of d into a single block. Finally in Section nine, we prove that $0^!_{\tilde{\sigma},\mathrm{loc}}[\mathbf{C}_{(d)}] = \frac{(-1)^{5d}}{12} N_0(d)_Q$. Together with (1.4), (1.5) and Theorem 1.3, these prove Theorem 1.1.

Earlier, the authors developed an algebro-geometric approach to prove Theorem 1.1 using an auxiliary closed substack $\mathcal{Z} \subset \overline{M}_1(\mathbb{P}^4, d)$ to capture the contribution $\frac{1}{12} N_1(d)_Q$. This approach was far from satisfactory, and its writing up was never finalized. The current approach was developed after the authors introduced the GW-invariants of stable maps with p-field. We expect that this approach can be generalized to prove a conjecture of Zinger and the second named author on high genus reduced GW-invariants of quintics and other complete intersection Calabi-Yau threefolds in the product of projective spaces.

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2. Moduli of stable morphisms with fields

We begin with a brief introduction GW-invariant of stable maps with p-fields. Let $Q = (\mathbf{w} = 0) \subset \mathbb{P}^4$ be the smooth Calabi-Yau manifold, where $\mathbf{w} = x_1^5 + \cdots + x_5^5$; let $K_{\mathbb{P}^4}$ be the total space of the canonical line bundle of $\mathbb{P}^4$. The polynomial $\mathbf{w}$ defines a map $\mathbf{w}_{\mathbb{P}^4} : K_{\mathbb{P}^4} \to \mathbb{C}$ whose critical locus is the Calabi-Yau manifold $Q \subset \mathbb{P}^4$ (in the zero section of $K_{\mathbb{P}^4}$). In physics literature, the pair $(K_{\mathbb{P}^4}, \mathbf{w}_{\mathbb{P}^4})$ is called a (non-linear) Landau-Ginzburg Model. In [**GS**], Guffin-Sharpe constructed a path integral for genus zero A-twisted theory of the Landau-Ginzburg space $(K_{\mathbb{P}^4}, \mathbf{w}_{\mathbb{P}^4})$ and showed that it gives the genus zero GW invariants of Q. In short, Guffin-Sharpe's theory is built on the space of smooth maps $f : C \to \mathbb{P}^4$ together with sections of the pullback bundles $f^*K_{\mathbb{P}^4}$ twisted by ω_C. In [**CL1**], this theory has been worked in more general setting mathematically, and proved to give all genus GW-invariants of Q. This is the GW invariants of stable maps with p-fields ψ in Definition 2.1.

We start with the moduli of stable maps. Let $\overline{M}_1(\mathbb{P}^4, d)$ be the moduli of genus one degree d stable maps to $\mathbb{P}^4$. In this paper, we abbreviate it to $\mathcal{X} := \overline{M}_1(\mathbb{P}^4, d)$, with $g = 1$ and d implicitly understood. We denote its universal family by

$$(f_{\mathcal{X}}, \pi_{\mathcal{X}}) : \mathcal{C}_{\mathcal{X}} \to \mathbb{P}^4 \times \mathcal{X}.$$

We recall the definition of the moduli of stable morphisms with fields.

Definition 2.1. Let $\overline{M}_1(\mathbb{P}^4, d)^p$ be the groupoid that associates to any scheme S the set $\overline{M}_1(\mathbb{P}^4, d)^p(S)$ of all S-families $(f_S, \mathcal{C}_S, \psi_S)$ where $[f_S, \mathcal{C}_S] \in \overline{M}_1(\mathbb{P}^4, d)(S)$, and $\psi_S \in \Gamma(\mathcal{C}_S, f_S^*\mathcal{O}(-5) \otimes \omega_{\mathcal{C}_S/S})$. An arrow between two S-families $(f_S, \mathcal{C}_S, \psi_S)$ and $(f'_S, \mathcal{C}'_S, \psi'_S)$ consists of an arrow η from $(f_S, \mathcal{C}_S)$ to $(f'_S, \mathcal{C}'_S)$ such that $\eta^*(\psi'_S) = \psi_S$.

Proposition 2.2. *The groupoid $\overline{M}_1(\mathbb{P}^4, d)^p$ is represented by a separated Deligne-Mumford stack of finite type.*

The proof is given in [**CL1**, (3,1)] and [**CL1**, Prop 2.2]. Note that the forgetful map $\overline{M}_1(\mathbb{P}^4, d)^p \to \mathcal{X}$ is not proper: its fiber over $[f, C] \in \mathcal{X}$ is $H^0(f^*\mathcal{O}(-5) \otimes \omega_C)$, which is non-trivial for some $[f, C]$.

To study its obstruction theory, we form the moduli stack of curves with line bundles. We let $\mathcal{D}$ be the groupoid that associates any scheme the set $\mathcal{D}(S)$ of all pairs $\tau = (\mathcal{C}_\tau, \mathcal{L}_\tau)$ of which $\pi_\tau : \mathcal{C}_\tau \to S$ are flat families of genus one connected nodal curves and $\mathcal{L}_\tau$ are fiberwise degree d line bundles on $\mathcal{C}_\tau$; an arrow from τ to τ' in $\mathcal{D}(S)$ consists of an S-isomorphism $\phi_1 : \mathcal{C}_\tau \to \mathcal{C}_{\tau'}$ and an isomorphism $\phi_2 : \mathcal{L}_\tau \to \phi_1^*\mathcal{L}_{\tau'}$ (cf. [**CL1**, Def 2.6]). In [**CL1**, (3,1)], this moduli space is constructed as a direct image cone over $\mathcal{D}$, which gives us the deformation theories of $\mathcal{X} (= \overline{M}_1(\mathbb{P}^4, d))$ and $\overline{M}_1(\mathbb{P}^4, d)^p$ relative to $\mathcal{D}$.

We let $\pi : \mathcal{C} \to \mathcal{D}$ with $\mathcal{L}$ on $\mathcal{C}$ be the universal curve and line bundle of $\mathcal{D}$. Then the invertible sheaf $\mathcal{L}_{\mathcal{X}} := f_{\mathcal{X}}^*\mathcal{O}(1)$ on $\mathcal{C}_{\mathcal{X}}$ induces a tautological morphism $\mathcal{X} \to \mathcal{D}$ so that the pull back of $(\mathcal{C}, \mathcal{L})$ is canonically isomorphic to $(\mathcal{C}_{\mathcal{X}}, \mathcal{L}_{\mathcal{X}})$. We introduce an auxiliary invertible sheaf $\mathcal{P} = \mathcal{L}^{\otimes(-5)} \otimes \omega_{\mathcal{C}/\mathcal{D}}$.

We recall the definition of the direct image cone stack

$$C(\pi_*(\mathcal{L}^{\oplus 5} \oplus \mathcal{P})) \tag{2.1}$$

defined in [**CL1**, Def 2.1]. For any scheme S, an object in $C(\pi_*(\mathcal{L}^{\oplus 5} \oplus \mathcal{P}))(S)$ consists of data $(\mathcal{C}_\tau, \mathcal{L}_\tau, u, \psi)$, where $\tau \in \mathcal{D}(S)$ with $\pi_\tau : \mathcal{C}_\tau \to S$ and $\mathcal{L}_\tau$ on $\mathcal{C}_\tau$ its associated families, $u = (u_1, \cdots, u_5) \in \Gamma(\pi_{\tau *}\mathcal{L}_\tau^{\oplus 5})$ and $\psi \in \Gamma(\pi_{\tau *}\mathcal{P}_\tau)$. An arrow from $(\mathcal{C}_\tau, \mathcal{L}_\tau, u, \psi)$ to $(\mathcal{C}_{\tau'}, \mathcal{L}_{\tau'}, u', \psi')$ (over the same S) consists of an arrow $\tau \to \tau'$ in $\mathcal{D}(S)$ so that (u, ψ) is equal to the pullback of (u', ψ') under the given arrow $\tau \to \tau'$. By construction, $C(\pi_*(\mathcal{L}^{\oplus 5} \oplus \mathcal{P}))$ is a stack over $\mathcal{D}$. (Indeed, it is linear over $\mathcal{D}$ in the following sense: it admits a scalar multiplication by $c \in \mathbb{C}$ that sends $(\mathcal{C}_\tau, \mathcal{L}_\tau, u, \psi)$ to $(\mathcal{C}_\tau, \mathcal{L}_\tau, c \cdot u, c \cdot \psi)$; It admits an addition that sends any pair $\big((\mathcal{C}_\tau, \mathcal{L}_\tau, u, \psi), (\mathcal{C}_\tau, \mathcal{L}_\tau, u', \psi')\big)$ (over the same $(\mathcal{C}_\tau, \mathcal{L}_\tau)$ in $\mathcal{D}$) to $(\mathcal{C}_\tau, \mathcal{L}_\tau, u + u', \psi + \psi')$.)

For simplicity, in this paper we abbreviate $\mathcal{Y} = \overline{M}_1(\mathbb{P}^4, d)^p$, with $g = 1$ and d implicitly understood. Like before, we let

$$(f_\mathcal{Y}, \pi_\mathcal{Y}) : \mathcal{C}_\mathcal{Y} \longrightarrow \mathbb{P}^4 \times \mathcal{Y},\ \psi_\mathcal{Y} \in \Gamma(\mathcal{C}_\mathcal{Y}, \mathcal{P}_\mathcal{Y}) \text{ where } \mathcal{P}_\mathcal{Y} = f_\mathcal{Y}^* \mathcal{O}(-5) \otimes \omega_{\mathcal{C}_\mathcal{Y}/\mathcal{Y}}$$

be the universal family of $\mathcal{Y}$. We denote $\mathcal{L}_\mathcal{Y} = f_\mathcal{Y}^*\mathcal{O}(1)$. After fixing a homogeneous coordinates $[z_1, \cdots, z_5]$ of $\mathbb{P}^4$, the morphism $f_\mathcal{Y}$ is given by $u_\mathcal{Y} := (u_{\mathcal{Y},i})_{1 \le i \le 5}$, $u_{\mathcal{Y},i} = f_\mathcal{Y}^* z_i$. The data $(\mathcal{C}_\mathcal{Y}, \mathcal{L}_\mathcal{Y}, u_\mathcal{Y}, \psi_\mathcal{Y})$ induces a morphism $\overline{M}_1(\mathbb{P}^4, d)^p \to C(\pi_*(\mathcal{L}^{\oplus 5} \oplus \mathcal{P}))$.

Lemma 2.3. *The morphism $\overline{M}_1(\mathbb{P}^4, d)^p \to C(\pi_*(\mathcal{L}^{\oplus 5} \oplus \mathcal{P}))$ is an open embedding.*

Proof. This follows from [**CL2**, Prop 2.7 and (3.1)]. q.e.d.

Let p to be the composite

$$p : \mathcal{Y} = \overline{M}_1(\mathbb{P}^4, d)^p \xrightarrow{\subset} C(\pi_*(\mathcal{L}^{\oplus 5} \oplus \mathcal{P})) \xrightarrow{\mathrm{pr}} \mathcal{D}.$$

Using the obstruction theory of $C(\pi_*(\mathcal{L}^{\oplus 5} \oplus \mathcal{P}))$ relative to $\mathcal{D}$ ([**CL1**, Prop 3.1]), we obtain a perfect relative obstruction theory of $\mathcal{Y} \to \mathcal{D}$:

$$\phi_{\mathcal{Y}/\mathcal{D}} : (E_{\mathcal{Y}/\mathcal{D}})^\vee \longrightarrow L^\bullet_{\mathcal{Y}/\mathcal{D}}, \quad E_{\mathcal{Y}/\mathcal{D}} := R^\bullet \pi_{\mathcal{Y}*}(\mathcal{L}_\mathcal{Y}^{\oplus 5} \oplus \mathcal{P}_\mathcal{Y}). \tag{2.2}$$

In the same spirit, we obtain perfect relative obstruction theory of $\mathcal{X} \to \mathcal{D}$ ([**CL1**, Prop 2.5 and 2.7]):

$$\phi_{\mathcal{X}/\mathcal{D}} : (E_{\mathcal{X}/\mathcal{D}})^\vee \longrightarrow L^\bullet_{\mathcal{X}/\mathcal{D}}, \quad E_{\mathcal{X}/\mathcal{D}} := R^\bullet \pi_{\mathcal{X}*}\mathcal{L}_\mathcal{X}^{\oplus 5}, \tag{2.3}$$

where $\mathcal{L}_\mathcal{X} = f_\mathcal{X}^*\mathcal{O}(1)$. Following standard convention, we call the cohomology sheaf

$$\mathcal{O}b_{\mathcal{Y}/\mathcal{D}} := H^1(E_{\mathcal{Y}/\mathcal{D}}) = R^1\pi_{\mathcal{Y}*}(\mathcal{L}_\mathcal{Y}^{\oplus 5} \oplus \mathcal{P}_\mathcal{Y})$$

the relative obstruction sheaf of $\phi_{\mathcal{Y}/\mathcal{D}}$.

Since $\mathcal{Y}$ is non-proper, we use Kiem-Li's cosection-localized virtual class to construct its GW-invariants. In [**CL1**, (3,7)], the authors have constructed a cosection of $\mathcal{O}b_{\mathcal{Y}/\mathcal{D}}$ (i.e. a homomorphism)

$$\sigma : \mathcal{O}b_{\mathcal{Y}/\mathcal{D}} \longrightarrow \mathcal{O}_{\mathcal{Y}}, \tag{2.4}$$

based on a choice of a quintic polynomial, say $\mathbf{w} = x_1^5 + \cdots + x_5^5$. It was verified in the same paper that this cosection lifted to a cosection $\bar{\sigma} : \mathcal{O}b_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{Y}}$ of the (absolute) obstruction sheaf $\mathcal{O}b_{\mathcal{Y}}$, where $\mathcal{O}b_{\mathcal{Y}}$ is defined by the exact sequence

$$p^*\Omega_{\mathcal{D}}^{\vee} \longrightarrow \mathcal{O}b_{\mathcal{Y}/\mathcal{D}} \longrightarrow \mathcal{O}b_{\mathcal{Y}} \longrightarrow 0.$$

It was also verified that the degeneracy locus $D(\sigma)$ of σ, which is the locus where σ is not surjective, is the closed substack

$$D(\sigma) = \overline{M}_1(Q,d) \subset \overline{M}_1(\mathbb{P}^4,d)^p = \mathcal{Y}. \tag{2.5}$$

Here $Q = (\mathbf{w} = 0) \subset \mathbb{P}^4$ is the quintic hypersurface $(x_1^5 + \cdots + x_5^5 = 0)$ of $\mathbb{P}^4$ and $\mathbf{w} = x_1^5 + \cdots + x_5^5$ was used ([**CL1**, (3,7)]) to construct the cosection σ; $\overline{M}_1(Q,d)$ is the moduli of stable morphisms to Q, and the embedding is via the tautological embedding $\overline{M}_1(Q,d) \subset \overline{M}_1(\mathbb{P}^4,d)$ composed with the embedding $\overline{M}_1(\mathbb{P}^4,d) \subset \mathcal{Y} = \overline{M}_1(\mathbb{P}^4,d)^p$ defined by assigning $\psi = 0$.

The cosection σ induces a morphism of vector bundle stacks (see [**BF**, Sec 2] for notation of h^0/h^1)

$$\sigma : h^1/h^0(E_{\mathcal{Y}/\mathcal{D}}) \longrightarrow \mathcal{O}_{\mathcal{Y}} \tag{2.6}$$

that is surjective over $\mathcal{U} = \mathcal{Y} - D(\sigma)$. (By abuse of notation, we use the same σ; also, we use $\mathcal{O}_{\mathcal{Y}}$ to denote the rank one trivial line bundle on $\mathcal{Y}$.) We let

$$\begin{aligned} h^1/h^0(E_{\mathcal{Y}/\mathcal{D}})_\sigma = \\ \big(h^1/h^0(E_{\mathcal{Y}/\mathcal{D}}) \times_{\mathcal{Y}} D(\sigma)\big) \cup \ker\{\sigma|_{\mathcal{U}} : h^1/h^0(E_{\mathcal{Y}/\mathcal{D}})|_{\mathcal{U}} \to \mathcal{O}_{\mathcal{U}}\}, \end{aligned} \tag{2.7}$$

endowed with the reduced stack structure.

Applying [**KL**, Prop 4.3] on cosection-localized virtual fundamental class, we know that the cycle of the intrinsic normal cone $[\mathbf{C}_{\mathcal{Y}/\mathcal{D}}] \in Z_*h^1/h^0(E_{\mathcal{Y}/\mathcal{D}})$ lies in

$$[\mathbf{C}_{\mathcal{Y}/\mathcal{D}}] \in Z_*h^1/h^0(E_{\mathcal{Y}/\mathcal{D}})_\sigma; \tag{2.8}$$

applying the localized Gysin map

$$0^!_{\sigma,\mathrm{loc}} : A_*h^1/h^0(E_{\mathcal{Y}/\mathcal{D}})_\sigma \longrightarrow A_{*-n}D(\sigma),$$

where $-n = \operatorname{rank} E_{\mathcal{Y}/\mathcal{D}}$, we obtain a localized virtual fundamental class

$$[\mathcal{Y}]^{\mathrm{vir}}_{\mathrm{loc}} := 0^!_{\sigma,\mathrm{loc}}[\mathbf{C}_{\mathcal{Y}/\mathcal{D}}] \in A_0 D(\sigma) = A_0\overline{M}_1(Q,d). \tag{2.9}$$

We define its degree to be the GW-invariants of stable morphisms to $\mathbb{P}^4$ with fields:

$$N_1(d)^p_{\mathbb{P}^4} = \deg\,[\mathcal{Y}]^{\mathrm{vir}}_{\mathrm{loc}}.$$

This is well-defined since $\overline{M}_1(Q,d)$ is proper.

Theorem 2.4. [**CL1**, Thm 5.7] *Let $N_1(d)_Q$ be the GW-invariants of genus one degree d stable morphisms to Q, then we have*

$$N_1(d)^p_{\mathbb{P}^4} = (-1)^{5d} N_1(d)_Q.$$

We remark that this theorem holds for all genus g. For our purpose, we state it in the case $g=1$. We will also use a modular blow-up of $\mathcal{X} = \overline{M}(\mathbb{P}^4,d)$ to study $N_1(d)^p_{\mathbb{P}^4}$. The version we use is that worked out by Hu and the second named author [**HL**], following the original construction of Vakil-Zinger's modular blow-up of the primary component of $\mathcal{X}$ [**VZ**].

In this paper, a weighted nodal curve is a connected nodal curve with its irreducible components assigned non-negative integer weights; the total weight of a weighted nodal curve is the sum of the weights of all of its irreducible components. Let $\mathcal{M}^{\mathrm{w}}$ be the Artin stack of weighted genus one nodal curves. By replacing $\mathcal{L}$ (of the universal family $(\mathcal{C},\mathcal{L})$ of $\mathcal{D}$) by its degrees along irreducible components of fibers of $\mathcal{C}\to\mathcal{D}$, we obtain a family of genus one weighted nodal curves over $\mathcal{D}$, which induces a morphism $\mathcal{D}\to\mathcal{M}^{\mathrm{w}}$. Let

$$\mathcal{X} \longrightarrow \mathcal{M}^{\mathrm{w}}, \quad \mathcal{Y} \longrightarrow \mathcal{M}^{\mathrm{w}} \tag{2.10}$$

be the composites of $\mathcal{X}$, $\mathcal{Y}\to\mathcal{D}$ with $\mathcal{D}\to\mathcal{M}^{\mathrm{w}}$ (cf. [**HL**, Sec 2.1]).

Let $\tilde{\mathcal{M}}^{\mathrm{w}}\to\mathcal{M}^{\mathrm{w}}$ be the modular blow-up of $\mathcal{M}^{\mathrm{w}}$ described in [**HL**, Sec 2.6]. We define the modular blow-ups of $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{D}$ to be

$$\tilde{\mathcal{X}} = \mathcal{X}\times_{\mathcal{M}^{\mathrm{w}}}\tilde{\mathcal{M}}^{\mathrm{w}}, \quad \tilde{\mathcal{Y}} = \mathcal{Y}\times_{\mathcal{M}^{\mathrm{w}}}\tilde{\mathcal{M}}^{\mathrm{w}}, \quad \text{and} \quad \tilde{\mathcal{D}} = \mathcal{D}\times_{\mathcal{M}^{\mathrm{w}}}\tilde{\mathcal{M}}^{\mathrm{w}}. \tag{2.11}$$

We denote $\mathcal{C}_{\tilde{\mathcal{Y}}} = \mathcal{C}_{\mathcal{Y}}\times_{\mathcal{Y}}\tilde{\mathcal{Y}}$ and let $f_{\tilde{\mathcal{Y}}}:\mathcal{C}_{\tilde{\mathcal{Y}}}\to\mathbb{P}^4$ be the composition of $\mathcal{C}_{\tilde{\mathcal{Y}}}\to\mathcal{C}_{\mathcal{Y}}$ with $f_{\mathcal{Y}}:\mathcal{C}_{\mathcal{Y}}\to\mathbb{P}^4$. We call $(f_{\tilde{\mathcal{Y}}},\pi_{\tilde{\mathcal{Y}}}):\mathcal{C}_{\tilde{\mathcal{Y}}}\to\mathbb{P}^4\times\tilde{\mathcal{Y}}$ the tautological family of $\tilde{\mathcal{Y}}$. Note that if we let $(f_{\tilde{\mathcal{X}}},\pi_{\tilde{\mathcal{X}}}):\mathcal{C}_{\tilde{\mathcal{X}}}\to\mathbb{P}^4\times\tilde{\mathcal{X}}$ be the similarly defined tautological family of $\tilde{\mathcal{X}}$, and let $\mathcal{L}_{\tilde{\mathcal{X}}} := f^*_{\tilde{\mathcal{X}}}\mathcal{O}(1)$ and $\mathcal{P}_{\tilde{\mathcal{X}}} = \mathcal{L}_{\tilde{\mathcal{X}}}^{\otimes(-5)}\otimes\omega_{\mathcal{C}_{\tilde{\mathcal{X}}}/\tilde{\mathcal{X}}}$, then we have a canonical isomorphism and its induced projection

$$\mathfrak{p}:\tilde{\mathcal{Y}}\cong C(\pi_{\tilde{\mathcal{X}}*}\mathcal{P}_{\tilde{\mathcal{X}}})\longrightarrow\tilde{\mathcal{X}}. \tag{2.12}$$

By adding $\psi=0$ to elements in $\tilde{\mathcal{X}}$, we can realize $\tilde{\mathcal{X}}$ as the zero section of this tautological projection. In this paper, whenever we say $\tilde{\mathcal{X}}\subset\tilde{\mathcal{Y}}$ we mean this embedding by the zero section.

Let $\zeta:\tilde{\mathcal{Y}}\to\mathcal{Y}$ be the projection. Since $\tilde{\mathcal{Y}}$ is derived from $\mathcal{Y}\to\mathcal{D}$ by the base change $\tilde{\mathcal{D}}\to\mathcal{D}$, the relative perfect obstruction theory of

$\mathcal{Y} \to \mathcal{D}$ pulls back to a relative perfect obstruction theory of $\tilde{\mathcal{Y}} \to \tilde{\mathcal{D}}$:

$$\phi_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}} : (E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})^{\vee} \longrightarrow L^{\bullet}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}, \quad E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}} = \zeta^* E_{\mathcal{Y}/\mathcal{D}}. \tag{2.13}$$

The relative obstruction sheaf is $\mathcal{O}b_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}} = \zeta^* \mathcal{O}b_{\mathcal{Y}/\mathcal{D}}$, and the cosection σ pullbacks to a cosection

$$\tilde{\sigma} = \zeta^* \sigma : \mathcal{O}b_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}} \to \mathcal{O}_{\tilde{\mathcal{Y}}}, \tag{2.14}$$

whose degeneracy locus $D(\tilde{\sigma})$ is also proper, and fits into the Cartesian product:

$$D(\tilde{\sigma}) = D(\sigma) \times_{\mathcal{Y}} \tilde{\mathcal{Y}} = \overline{M}_1(Q,d) \times_{\mathcal{M}^{\mathrm{w}}} \tilde{\mathcal{M}}^{\mathrm{w}}. \tag{2.15}$$

We define $h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})_{\tilde{\sigma}}$ parallel to that defined after (2.6). Then by (2.8), we have

$$[\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}] \in Z_* h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})_{\tilde{\sigma}}.$$

We define $[\tilde{\mathcal{Y}}]^{\mathrm{vir}}_{\mathrm{loc}} = 0^!_{\tilde{\sigma},\mathrm{loc}}[\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}] \in A_0 D(\tilde{\sigma})$ to be the cosection-localized virtual fundamental class.

Proposition 2.5. *We have the identity*

$$\deg\, [\tilde{\mathcal{Y}}]^{\mathrm{vir}}_{\mathrm{loc}} = \deg\, [\mathcal{Y}]^{\mathrm{vir}}_{\mathrm{loc}} = (-1)^{5d} N_1(d)_Q.$$

Proof. By [**HL**, Sec 2.6], the blow-up $\tilde{\mathcal{M}}^{\mathrm{w}} \to \mathcal{M}^{\mathrm{w}}$ is the result after successively blowing up along smooth loci; therefore $\tilde{\mathcal{M}}^{\mathrm{w}} \to \mathcal{M}^{\mathrm{w}}$, and consequently $\iota_{\mathcal{D}} : \tilde{\mathcal{D}} = \mathcal{D} \times_{\mathcal{M}^{\mathrm{w}}} \tilde{\mathcal{M}}^{\mathrm{w}} \to \mathcal{D}$, are proper and l.c.i. morphisms between smooth stacks of identical pure dimensions.

From (2.15), the degeneracy locus $D(\tilde{\sigma})$ is $\overline{M}_1(Q,d)^{\sim} := \overline{M}_1(Q,d) \times_{\mathcal{M}^{\mathrm{w}}} \tilde{\mathcal{M}}^{\mathrm{w}}$, which is contained in $\tilde{\mathcal{Y}}$ and fits into the Cartesian diagram

$$\begin{array}{ccc} \overline{M}_1(Q,d)^{\sim} \subset \tilde{\mathcal{Y}} & \xrightarrow{\iota'_{\mathcal{D}}} & \overline{M}_1(Q,d) \subset \mathcal{Y} \\ \downarrow & & \downarrow \\ \tilde{\mathcal{D}} & \xrightarrow{\iota_{\mathcal{D}}} & \mathcal{D}. \end{array} \tag{2.16}$$

We apply [**CLL**, Sec 4.1] to the fiber square to obtain

$$\iota^!_{\mathcal{D}}[\mathcal{Y}]^{\mathrm{vir}}_{\mathrm{loc}} = [\tilde{\mathcal{Y}}]^{\mathrm{vir}}_{\mathrm{loc}}, \tag{2.17}$$

where $\iota^!_{\mathcal{D}} : A_* \overline{M}_1(Q,d) \to A_* \overline{M}_1(Q,d)^{\sim}$ is the Gysin map associated to the square (2.16). We comment that although that the bases in [**CLL**, Sec 4.1] are assumed to be DM stacks, the same proof works for the case where the bases are Artin stacks of finite presentations. Thus (2.17) holds.

By [**CL1**, Thm 5.7], we have $[\mathcal{Y}]^{\mathrm{vir}}_{\mathrm{loc}} = (-1)^{5d}[\overline{M}_1(Q,d)]^{\mathrm{vir}}$, thus

$$[\tilde{\mathcal{Y}}]^{\mathrm{vir}}_{\mathrm{loc}} = (-1)^{5d} \iota^!_{\mathcal{D}} [\overline{M}_1(Q,d)]^{\mathrm{vir}}. \tag{2.18}$$

Using the Cartesian product (2.16), we obtain a perfect relative obstruction theory of $\overline{M}_1(Q,d)^{\sim} \to \tilde{\mathcal{D}}$. Because the same proof in [**BF**, Prop

7.2 (2)] applies to the case when the immersions of bases, like $\iota_{\mathcal{D}}$, are l.c.i. morphisms, we can apply [**BF**, Prop 7.2 (2)] to conclude

$$[\overline{M}_1(Q,d)\tilde{\ }]^{\mathrm{vir}} = \iota_{\mathcal{D}}^{!}[\overline{M}_1(Q,d)]^{\mathrm{vir}}.$$

Thus $[\tilde{\mathcal{Y}}]^{\mathrm{vir}}_{\mathrm{loc}} = (-1)^{5d}[\overline{M}_1(Q,d)\tilde{\ }]^{\mathrm{vir}}$.

On the other hand, applying [**Cos**, Thm 5.0.1] to (2.16) we obtain

$$\iota'_{\mathcal{D}*}([\overline{M}_1(Q,d)\tilde{\ }]^{\mathrm{vir}}) = [\overline{M}_1(Q,d)]^{\mathrm{vir}} \in A_0(D(\sigma)),$$

where $\iota'_{\mathcal{D}}$ is as in (2.16). Thus $\deg[\overline{M}_1(Q,d)\tilde{\ }]^{\mathrm{vir}} = \deg[\overline{M}_1(Q,d)]^{\mathrm{vir}}$, and

$$\begin{aligned}\deg[\tilde{\mathcal{Y}}]^{\mathrm{vir}}_{\mathrm{loc}} &= \deg(-1)^{5d}[\overline{M}_1(Q,d)\tilde{\ }]^{\mathrm{vir}}\\ &= (-1)^{5d}\deg[\overline{M}_1(Q,d)]^{\mathrm{vir}} = (-1)^{5d}N_1(d)_Q.\end{aligned}$$

This proves the Proposition. q.e.d.

3. A structure result of the modular blow-ups

Following [**HL**, Sec 2.8], we know that the modular blow-up $\tilde{\mathcal{X}}$ is a union of smooth irreducible proper DM stacks: one of them is $\tilde{\mathcal{X}}_{\mathrm{pri}}$, birational to $\mathcal{X}_{\mathrm{pri}} \subset \mathcal{X}$ under $\tilde{\mathcal{X}} \to \mathcal{X}$ (cf. (2.11)); the others are $\tilde{\mathcal{X}}_\mu$, indexed by partitions μ of d, characterized by that its image in $\mathcal{X}$ is the closure of the set of all stable morphisms $[f, C]$ whose domains C are ℓ $\mathbb{P}^1$'s attached to smooth elliptic curves and the morphisms f are constant along the elliptic curves and have degrees d_i along the i-th attached $\mathbb{P}^1$'s.

We will show that the corresponding stack $\tilde{\mathcal{Y}}$ has similar structure. Let $\mathfrak{p} : \tilde{\mathcal{Y}} \longrightarrow \tilde{\mathcal{X}}$ be the canonical projection, as in (2.12). We let $\tilde{\mathcal{Y}}_\alpha = \tilde{\mathcal{Y}} \times_{\tilde{\mathcal{X}}} \tilde{\mathcal{X}}_\alpha$, where $\alpha = \mathrm{pri}$ or $\mu \vdash d$; we denote $\tilde{\mathcal{X}}_{\mathrm{gst}} = \cup_{\mu\vdash d}\tilde{\mathcal{X}}_\mu \subset \tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}_{\mathrm{gst}} = \cup_{\mu\vdash d}\tilde{\mathcal{Y}}_\mu$ $(= \tilde{\mathcal{Y}} \times_{\tilde{\mathcal{X}}} \tilde{\mathcal{X}}_{\mathrm{gst}})$. Thus

$$\tilde{\mathcal{X}} = \tilde{\mathcal{X}}_{\mathrm{pri}} \cup \tilde{\mathcal{X}}_{\mathrm{gst}} \quad \text{and} \quad \tilde{\mathcal{Y}} = \tilde{\mathcal{Y}}_{\mathrm{pri}} \cup \tilde{\mathcal{Y}}_{\mathrm{gst}}. \tag{3.1}$$

We collect the property of this decomposition as follows. We call $\mathcal{S} \to \mathcal{X}$ a smooth chart if $\mathcal{S}$ is a scheme and $\mathcal{S} \to \mathcal{X}$ is smooth.

Proposition 3.1. *For any closed $\tilde{x} \in \tilde{\mathcal{X}}$, we can find a smooth chart $\tilde{S} \to \tilde{\mathcal{X}}$ containing $\tilde{x}$, an embedding (All embeddings in this section are locally closed embeddings.) $\tilde{S} \to Z$ into a smooth scheme, and an embedding $\tilde{T} := \tilde{S} \times_{\tilde{\mathcal{X}}} \tilde{\mathcal{Y}} \to Z' := Z \times \mathbb{A}^1$, such that*

1) *the projection $\tilde{T} \to \tilde{S}$ commutes with the projection $p_Z : Z' = Z \times \mathbb{A}^1 \to Z$, and there is a smooth morphism $Z \to \tilde{\mathcal{M}}^w$ so that the composite $\tilde{T} \to Z \to \tilde{\mathcal{M}}^w$ is identical to the composite $\tilde{T} \to \tilde{\mathcal{Y}} \to \tilde{\mathcal{M}}^w$;*
2) *there are regular functions $w_1, \ldots, w_4$ and $z_\mu \in \Gamma(\mathcal{O}_Z)$, $\mu \vdash d$, such that $(w_1 = \cdots = w_4 = 0) \subset Z$ is smooth, and all $(z_\mu = 0) \subset Z$ have at worst normal crossing singularities;*

3) *denote* $\tilde T_\alpha = \tilde T \times_{\tilde{\mathcal Y}} \tilde{\mathcal Y}_\alpha$*, where* $\alpha = \mathrm{pri}, \mathrm{gst}$ *or* $\mu \vdash d$*; let* $t \in \Gamma(\mathcal O_{Z'})$ *be the pull back of the standard coordinate function of* $\mathbb A^1$ *via* $p_{\mathbb A^1} : Z' \to \mathbb A^1$*, and let* $z = \prod_{\mu \vdash d} z_\mu$*, then as subschemes of* Z'*, we have* $\tilde T \subset (w_1 \cdot z = \cdots = w_4 \cdot z = t \cdot z = 0)$*;* $\tilde T_{\mathrm{pri}} = \tilde T \cap (w_1 = \cdots = w_4 = z \cdot t = 0)$*;* $\tilde T_\mu = \tilde T \cap (z_\mu = 0)$*, and* $\tilde T_{\mathrm{gst}} = \tilde T \cap (z = 0)$ *are open subschemes of* $(w_1 = \cdots = w_4 = z \cdot t = 0)$*,* $(z_\mu = 0)$*, and* $(z = 0)$ *respectively;*
4) *identify* $Z \cong Z' \cap (t = 0)$*, then as subschemes of* Z*,* $\tilde S = \tilde T \cap (t = 0)$*, and* $\tilde S_\alpha = \tilde T_\alpha \cap (t = 0)$*, where* $\alpha = \mathrm{pri}$ *or* $\mu \vdash d$*; further for any* $\mu \vdash d$*,* $(z_\mu = 0) \cap \tilde S_{\mathrm{pri}}$ *is a smooth divisor in* $\tilde S_{\mathrm{pri}}$*.*

We begin our proof of Proposition 3.1 by outlining the derivation of the embedding and the defining equations of $\tilde S \to Z$, following [**HL**].

We let $x \in \mathcal X$ be the image of $\tilde x$ under $\tilde{\mathcal X} \to \mathcal X$. We pick a homogeneous coordinates $[z_0, \cdots, z_4]$ of $\mathbb P^4$ so that $x \in \mathcal X_{z_0}$, where $\mathcal X_{z_0}$ is the open substack of $[f, C] \in \mathcal X$ so that $(f^* z_0 = 0) \subset C$ are divisors disjoint from the singularities of C. Recall the composite $\mathcal X_{z_0} \subset \mathcal X \to \mathcal D$ is via sending $[f, C]$ to $(C, f^*\mathcal O(1))$. We show that it factors through the Artin stack $\mathcal K$ of pairs (C, D) of connected genus one nodal curves C with effective divisors $D \subset C$ disjoint from the nodes of C

Indeed, let $(f_{\mathcal X_{z_0}}, \mathcal C_{\mathcal X_{z_0}})$ be the restriction of the tautological family of $\mathcal X$ to $\mathcal X_{z_0}$. Then $(f^*_{\mathcal X_{z_0}} z_0)^{-1}(0) \subset \mathcal C_{\mathcal X_{z_0}}$ is a family of pairs of nodal curves with effective divisors away from the singularities of the fibers, which induces a morphism $\mathcal X_{z_0} \to \mathcal K$. Next, let $(\mathcal C_{\mathcal K}, D_{\mathcal K})$ be the universal family over $\mathcal K$; the line bundle $\mathcal O_{\mathcal C_{\mathcal K}}(D_{\mathcal K})$ paired with $\mathcal C_{\mathcal K}$ defines a morphism $\mathcal K \to \mathcal D$. Clearly, the composite $\mathcal X_{z_0} \to \mathcal K \to \mathcal D$ coincides with $\mathcal X_{z_0} \subset \mathcal X \to \mathcal D$. Lastly, we let $\mathcal K \to \mathcal M^{\mathrm w}$ be the composite $\mathcal K \to \mathcal D \to \mathcal M^{\mathrm w}$; it is a smooth morphism.

We pick a smooth affine chart $M \to \mathcal M^{\mathrm w}$. Let $(\mathcal C_M, \mathbf w)$ be the tautological family over M. We let $\mathbf{Pic}_{\mathcal C_M/M}$ be the relative Picard scheme, and let $\mathbf{Pic}_{\mathcal C_M/M, \mathbf w}$ be its open substack of line bundles over fibers of $\mathcal C_M \to M$ of degrees $\mathbf w$ along irreducible components of the fibers of $\mathcal C_M \to M$. Because $\mathbf{Pic}_{\mathcal C_M/M, \mathbf w}$ is representable, it is represented by a scheme, say U, which is smooth over M because $\mathcal C_M \to M$ is a flat family of proper nodal curves. Using the tautological line bundle on $U \times_M \mathcal C_M$, we obtain a morphism $U \to \mathcal D$ making the rightmost square below commutative:

$$
\begin{array}{ccccccc}
S = \mathcal X_{z_0} \times_{\mathcal D} U & \longrightarrow & K = \mathcal K \times_{\mathcal D} U & \longrightarrow & U & \longrightarrow & M \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal X_{z_0} & \longrightarrow & \mathcal K & \longrightarrow & \mathcal D & \longrightarrow & \mathcal M^{\mathrm w}.
\end{array}
\tag{3.2}
$$

Because $\mathcal K \to \mathcal D, \mathcal X \to \mathcal D$ are representable and $U \to \mathcal D$ is smooth, $\mathcal K \times_{\mathcal D} U$ and $\mathcal X_{z_0} \times_{\mathcal D} U$ are smooth charts of $\mathcal K$ and $\mathcal X_{z_0}$, respectively.

We denote $K := \mathcal{K} \times_{\mathcal{D}} U$ and $S := \mathcal{X}_{z_0} \times_{\mathcal{K}} K$. Using the smoothness of $U \to \mathcal{D} \times_{\mathcal{M}^{\mathrm{w}}} M$ and $\mathcal{K} \to \mathcal{M}^{\mathrm{w}}$, we know $K \to M$ is also smooth.

We next pick charts of $\tilde{\mathcal{X}}_{z_0} := \mathcal{X}_{z_0} \times_{\mathcal{X}} \tilde{\mathcal{X}}$, etc.. We pick an affine open subscheme $\tilde{M} \subset M \times_{\mathcal{M}^{\mathrm{w}}} \tilde{\mathcal{M}}^{\mathrm{w}}$; let $\tilde{U} = \tilde{M} \times_{(M \times_{\mathcal{M}^{\mathrm{w}}} \tilde{\mathcal{M}}^{\mathrm{w}})} (U \times_{\mathcal{D}} \tilde{\mathcal{D}})$. We next denote $\tilde{\mathcal{K}} = \mathcal{K} \times_{\mathcal{M}^{\mathrm{w}}} \tilde{\mathcal{M}}^{\mathrm{w}}$; pick a connected affine open subscheme $\tilde{K} \subset \tilde{U} \times_{\tilde{\mathcal{D}}} \tilde{\mathcal{K}}$, and and form $\tilde{S} = \tilde{\mathcal{X}}_{z_0} \times_{\tilde{\mathcal{K}}} \tilde{K}$. They are smooth charts of $\tilde{\mathcal{M}}^{\mathrm{w}}$, $\tilde{\mathcal{D}}$, $\tilde{\mathcal{K}}$ and $\tilde{\mathcal{X}}$, respectively, such that $\tilde{M}, \tilde{K}$ are affine, and (the induced) $\tilde{K} \to \tilde{M}$ is smooth. For our goal, we require that $\tilde{S} \to \tilde{\mathcal{X}}$ is a chart of $\tilde{x}$.

Let $(f_S, \mathcal{C}_S)$ be the pullback of $(f_{\mathcal{X}}, \mathcal{C}_{\mathcal{X}})$ to S. We give a canonical presentation of f_S using $[z_0, \cdots, z_4]$. We let $D_S = (f_S^* z_0)^{-1}(0)$, a family of divisors in $\mathcal{C}_S$, and fix $f_S^* \mathcal{O}(1) \cong \mathcal{O}_{\mathcal{C}_S}(D_S)$ so that $f_S^* z_0$ is the constant section 1 via the tautological inclusion $\mathcal{O}_{\mathcal{C}_S} \subset \mathcal{O}_{\mathcal{C}_S}(D_S)$. Then $u_{S,i} = f_S^* z_i$ are sections of $f_S^* \mathcal{O}(1)$,

$$f_S = [u_{S,0}, \cdots, u_{S,4}] : \mathcal{C}_S \longrightarrow \mathbb{P}^4. \qquad u_{S,i} = f_S^* z_i; \tag{3.3}$$

and a closed $s \in S$ associates to the data $(D_s \subset \mathcal{C}_s, (u_{s,1}, \cdots, u_{s,4}))$, where $u_{s,i} = u_{S,i}|_{\mathcal{C}_s}$.

We let $D_K \subset \mathcal{C}_K$ be the tautological family over K. By the universality of the tautological families over $\mathcal{X}$ and $\mathcal{K}$, we have the induced isomorphism and identity

$$\mathcal{C}_S \cong \mathcal{C}_K \times_K S \quad \text{and} \quad D_S = D_K \times_{\mathcal{C}_K} \mathcal{C}_S,$$

where the second identity holds under the first isomorphism. Let $p_{\tilde{K}} : C_{\tilde{K}} \to \tilde{K}$ with $D_{\tilde{K}} \subset C_{\tilde{K}}$ be the pull back of $D_K \subset C_K \to K$.

For a partition μ, we let $\mathcal{M}_\mu \subset \mathcal{M}^{\mathrm{w}}$ be the closed substack defined to be the closure of the locally closed substack of all $(C, \mathbf{w}) \in \mathcal{M}^{\mathrm{w}}$ so that C are ℓ $\mathbb{P}^1$'s attached to smooth elliptic curves, and $\mathbf{w}$ take value 0 on the elliptic curves and take value $\mu_1, \cdots, \mu_\ell$ on the ℓ $\mathbb{P}^1$'s. We let $\tilde{\mathcal{M}}_\mu$ be the proper transforms of $\mathcal{M}_\mu$ under the blowing up morphism $\tilde{\mathcal{M}}^{\mathrm{w}} \to \mathcal{M}^{\mathrm{w}}$. We let $\tilde{K}_\mu = \tilde{K} \times_{\tilde{\mathcal{M}}^{\mathrm{w}}} \tilde{\mathcal{M}}_\mu$.

Proposition 3.2. *By shrinking $\tilde{K}$ if necessary, we can find a $\xi \in \Gamma(\mathcal{O}_{\tilde{K}})$ so that*

$$R^\bullet p_{\tilde{K}*} \mathcal{O}_{\tilde{K}}(D_{\tilde{K}}) \cong_{q.i.} [\mathcal{O}_{\tilde{K}} \xrightarrow{\times \xi} \mathcal{O}_{\tilde{K}}] \oplus [\mathcal{O}_{\tilde{K}}^{\oplus d} \longrightarrow 0], \tag{3.4}$$

and $(\xi = 0)$ is a normal crossing divisor in $\tilde{K}$ of the form $(\xi = 0) = \cup_{\mu \vdash d} \tilde{K}_\mu$. Further, we can make ξ the pullback of a $\xi' \in \Gamma(\mathcal{O}_{\tilde{M}})$.

Proof. The proposition is essentially proved in [**HL**], where an explicit perfect two-term complex quasi-isomorphic to $R^\bullet p_{\tilde{K}*} \mathcal{O}_{\tilde{K}}(D_{\tilde{K}})$ is derived. We follow the notation of the proof of [**HL**, Theorem 2.11]. On [**HL**, page 671], it is shown that $R^\bullet p_{\tilde{K}*} \mathcal{O}_{\tilde{K}}(D_{\tilde{K}})$ is quasi-isomorphic to $\mathcal{O}_{\tilde{K}} \oplus [\mathcal{O}_{\tilde{K}}^{\oplus d} \to \mathcal{O}_{\tilde{K}}]$ with the arrow $\mathcal{O}_{\tilde{K}}^{\oplus d} \to \mathcal{O}_{\tilde{K}}$ given by $\oplus^d \beta^* \varphi_i$ (cf. [**HL**, (5.23)]), and the structure of $\beta^* \varphi_i$ is given by [**HL**, page 962 line

18]. Using the divisibility property of $\beta^*\varphi_i$ shown on [**HL**, page 962], we obtain the quasi-isomorphism (3.4) and the property (1) and (2) stated. We prove that ξ can be the pullback of a $\xi' \in \Gamma(\mathcal{O}_{\tilde{M}})$. Indeed, by the construction of $\tilde{\mathcal{M}}^{\mathrm{w}} \to \mathcal{M}^{\mathrm{w}}$, the union of $\tilde{\mathcal{M}}_\mu$ of all $\mu = (\mu_1, \cdots, \mu_\ell)$ with $\ell \geq 2$ form the exceptional divisor of the blowing up $\tilde{\mathcal{M}}^{\mathrm{w}} \to \mathcal{M}^{\mathrm{w}}$; also for $\mu = (d)$, $\tilde{\mathcal{M}}_\mu$ is a smooth divisor of $\tilde{\mathcal{M}}^{\mathrm{w}}$. Further the union $\cup_{\mu \vdash d} \tilde{\mathcal{M}}_\mu$ is a reduced divisor in $\tilde{\mathcal{M}}^{\mathrm{w}}$ with at worst normal crossing singularities (cf. [**HL**, Section 2.6]). Therefore, letting $\tilde{M}_\mu = \tilde{M} \times_{\tilde{\mathcal{M}}^{\mathrm{w}}} \tilde{\mathcal{M}}_\mu$, then $\cup_{\mu \vdash d} \tilde{M}_\mu$ is a reduced divisor in $\tilde{M}$, and its preimage in $\tilde{K}$ is $\xi = 0$. Since $\tilde{M}$ is affine, we can find a section $\xi' \in \Gamma(\mathcal{O}_{\tilde{M}})$ so that $\xi' = 0$ is $\cup_{\mu \vdash d} \tilde{M}_\mu$. Therefore, by a change of trivializations of $\mathcal{O}_{\tilde{K}}$ in (3.4), we can make ξ the pullback of ξ'. This proves the Proposition. q.e.d.

We construct the desired embedding $\tilde{S} \to Z$. Let $(u_{\tilde{S},1}, \cdots, u_{\tilde{S},4})$ be the pull back of $(u_{S,1}, \cdots, u_{S,4})$. We quote the result showing that the defining equation of $\tilde{S}$ is dictated by the derived object

$$R^\bullet p_{\tilde{K}*} \mathcal{O}_{\tilde{K}}(D_{\tilde{K}})^{\oplus 4} \cong_{q.i.} [\mathcal{O}_{\tilde{K}} \xrightarrow{\times \xi} \mathcal{O}_{\tilde{K}}]^{\oplus 4} \oplus [\mathcal{O}_{\tilde{K}}^{\oplus 4d} \longrightarrow 0]. \tag{3.5}$$

We let $Z = \mathbb{A}^4 \times \mathbb{A}^{4d} \times \tilde{K}$ be the total space of $\mathcal{O}_{\tilde{K}}^{\oplus 4} \oplus \mathcal{O}_{\tilde{K}}^{\oplus 4d}$, where the copy $\mathcal{O}_{\tilde{K}}^{\oplus 4}$ stands for the four $\mathcal{O}_{\tilde{K}}$'s in the 0-th place in the factor $[\mathcal{O}_{\tilde{K}} \to \mathcal{O}_{\tilde{K}}]^{\oplus 4}$, etc.. We let $w_1, \cdots, w_4$ be the standard coordinates of $\mathbb{A}^4$, viewed also as regular functions on Z via pull back. Let $\bar{p} : Z \to \tilde{K}$ be the projection.

Proposition 3.3 ([**HL**, Theorem 2.19])**.** *The data $(u_{\tilde{S},1}, \cdots, u_{\tilde{S},4})$ defines a lifting $\tilde{S} \to Z$ (of $\tilde{S} \to \tilde{K}$) that factors through an open embedding*

$$F : \tilde{S} \longrightarrow (w_1 \cdot \bar{p}^*(\xi) = \cdots = w_4 \cdot \bar{p}^*(\xi) = 0) \subset Z.$$

We remark that the $\bar{p}^*(\xi)$ is the product $\xi_1 \cdots \xi_d$ in [**HL**, Theorem 2.19]. By Proposition 3.2, $(\bar{p}^*(\xi) = 0) \subset Z$ is a normal crossing singularity. Our construction implies each w_i has smooth vanishing locus and $F(\tilde{S})$ has (at worst) normal crossing singularities (also cf. [**HL**, Theorem 2.19]).

We let $T = \mathcal{Y} \times_{\mathcal{X}} S$, and let $\tilde{T} = \tilde{\mathcal{Y}} \times_{\tilde{\mathcal{X}}} \tilde{S}$. Thus $T \to \mathcal{Y}$ and $\tilde{T} \to \tilde{\mathcal{Y}}$ are smooth charts. We now derive the embedding $\tilde{\mathcal{Y}} \to Z'$. Like Proposition 3.3, the structure of $\tilde{T}$ can be given by the structure of a perfect two term complex quasi-isomorphic to

$$R^\bullet p_{\tilde{K}*} \big(\mathcal{O}_{\tilde{K}}(D_{\tilde{K}})^{\oplus 4} \oplus \mathcal{O}_{\tilde{K}}(-5D_{\tilde{K}}) \otimes \omega_{C_{\tilde{K}}/\tilde{K}} \big).$$

Lemma 3.4. *We have a quasi-isomorphism*

$$R^\bullet p_{\tilde{K}*} \mathcal{O}_{\tilde{K}}(-5D_{\tilde{K}}) \otimes \omega_{C_{\tilde{K}}/\tilde{K}} \cong_{q.i.} [\mathcal{O}_{\tilde{K}} \xrightarrow{\times \xi} \mathcal{O}_{\tilde{K}}] \oplus [0 \longrightarrow \mathcal{O}_{\tilde{K}}^{\oplus 5d}], \tag{3.6}$$

where the ξ is that in Proposition 3.2.

Proof. Let $(\mathcal{C}_{\mathcal{K}}, D_{\mathcal{K}})$ be the universal family over $\mathcal{K}$. Since $(\mathcal{C}_{\mathcal{K}}, 5D_{\mathcal{K}})$ is also a family of pairs of divisors in genus one nodal curves, it induces a morphism $\rho : \mathcal{K} \to \mathcal{K}$ so that $\rho^*(\mathcal{C}_{\mathcal{K}}, D_{\mathcal{K}}) \cong (\mathcal{C}_{\mathcal{K}}, 5D_{\mathcal{K}})$. Clearly, ρ is a closed embedding of stacks.

Let $\tilde{K}$ be the smooth chart of $\tilde{\mathcal{K}}$ as in the statement of the lemma. Without loss of generality, we can find a smooth chart $K' \to \mathcal{K}$ so that $\rho : \mathcal{K} \to \mathcal{K}$ lifts to $\phi : K \to K'$. We then pick an affine smooth chart $\tilde{K}'$ of $K' \times_{\mathcal{M}^{\mathrm{w}}} \tilde{\mathcal{M}}^{\mathrm{w}}$, possibly after shrinking the chart $\tilde{K}$ if necessary, so that the $\phi : K \to K'$ lifts to a morphism $\tilde{\phi} : \tilde{K} \to \tilde{K}'$ by the following fiber diagram:

$$\begin{array}{ccc} \tilde{K} & \xrightarrow{\tilde{\phi}} & \tilde{K}' \\ \Big\downarrow r & & \Big\downarrow \\ K & \xrightarrow{\phi} & K'. \end{array} \tag{3.7}$$

Without loss of generality, we can assume that ϕ is a locally closed immersion, thus $\tilde{\phi}$ is a locally closed immersion. Let $(\mathcal{C}_{\tilde{K}'}, D_{\tilde{K}'})$ be the pull back of the tautological family over $\tilde{K}$. We have the following isomorphism and identity

$$C_{\tilde{K}} \cong C_{\tilde{K}'} \times_{\tilde{K}'} \tilde{K} \quad \text{and} \quad 5D_{\tilde{K}} = D_{\tilde{K}'} \times_{C_{\tilde{K}'}} C_{\tilde{K}}. \tag{3.8}$$

(The last identity holds under the given isomorphism.) Let $p_{\tilde{K}'} : C_{\tilde{K}'} \to \tilde{K}'$ be the projection. Then by Proposition 3.2, we can find a section $\xi' \in H^0(\mathcal{O}_{\tilde{K}'})$ such that

$$R^{\bullet} p_{\tilde{K}'*} \mathcal{O}_{\tilde{K}'}(D_{\tilde{K}'}) \cong_{q.i.} [\mathcal{O}_{\tilde{K}'} \xrightarrow{\times \xi'} \mathcal{O}_{\tilde{K}'}] \oplus [\mathcal{O}_{\tilde{K}'}^{\oplus 5d} \longrightarrow 0],$$

and $(\xi' = 0)$ satisfies the properties stated in Proposition 3.2.

Therefore, because of (3.8), we have

$$\begin{aligned} R^{\bullet} p_{\tilde{K}*} \mathcal{O}_{\tilde{K}}(5D_{\tilde{K}}) &\cong_{q.i.} \phi^* R^{\bullet} p_{\tilde{K}'*} \mathcal{O}_{\tilde{K}'}(D_{\tilde{K}'}) \\ &\cong_{q.i.} [\mathcal{O}_{\tilde{K}} \xrightarrow{\times \tilde{\phi}^*(\xi')} \mathcal{O}_{\tilde{K}}] \oplus [\mathcal{O}_{\tilde{K}}^{\oplus 5d} \longrightarrow 0]. \end{aligned}$$

Finally, using the description of the vanishing locus $(\xi = 0)$ in Theorem 3.2, we see that $(\xi = 0)$ and $(\tilde{\phi}^*(\xi') = 0)$ are identical as sets. Since both are reduced divisors, we conclude that $(\xi = 0) = (\tilde{\phi}^*(\xi') = 0)$. Thus $\tilde{\phi}^*(\xi')/\xi$ is nowhere vanishing. In particular, by an appropriate isomorphism $\mathcal{O}_{\tilde{K}} \cong \mathcal{O}_{\tilde{K}}$, we can make $\tilde{\phi}^*(\xi') = \xi$.

Finally, by Serre duality $R^{\bullet} p_{\tilde{K}*} \mathcal{O}_{\tilde{K}}(-5D_{\tilde{K}}) \otimes \omega_{C_{\tilde{K}}/\tilde{K}}$ is dual to $R^{\bullet} p_{\tilde{K}*} \mathcal{O}_{\tilde{K}}(5D_{\tilde{K}})[-1]$. This proves the Lemma. q.e.d.

We let $Z' = \mathbb{A}^4 \times \mathbb{A}^1 \times \mathbb{A}^{4d} \times \tilde{K}$, where the factors $\mathbb{A}^4$, $\mathbb{A}^{4d}$ and $\tilde{K}$ are as in Z, and the factor $\mathbb{A}^1$ corresponds to the (0-th place) factor $\mathcal{O}_{\tilde{K}}$ in $[\mathcal{O}_{\tilde{K}} \to \mathcal{O}_{\tilde{K}}]$ in the quasi-isomorphism (3.6). As before, we let $(w_1, \cdots, w_4)$ be the standard coordinate variables of $\mathbb{A}^4$; we let t be the standard coordinate variable of $\mathbb{A}^1$ in $Z' = \mathbb{A}^4 \times \mathbb{A}^1 \times \mathbb{A}^{4d} \times \tilde{K}$. (As before, we view w_i and t as functions on Z'.) Let $\bar q : Z' \to \tilde K$ be the projection and $\iota : Z \to Z'$ be the embedding via setting $t = 0$.

Lemma 3.5. *The morphism $\tilde T \to \tilde K$ lifts to an open embedding F' making the following square commutative:*

$$\begin{array}{lccc} & \tilde T & \xrightarrow{F'} & (w_1 \cdot \bar q^*(\xi) = \cdots = w_4 \cdot \bar q^*(\xi) = t \cdot \bar q^*(\xi) = 0) \subset Z' \\ (3.9) & \Big\uparrow{\scriptstyle 0-\text{section}} & & \Big\uparrow{\scriptstyle t=0} \\ & \tilde S & \xrightarrow{F} & (w_1 \cdot \bar p^*(\xi) = \cdots = w_4 \cdot \bar p^*(\xi) = 0) \subset Z. \end{array}$$

Proof. The proof is a repetition of [**HL**, Theorem 2.17], and will be omitted. q.e.d.

Proof of Proposition 3.1. Let $z = \bar q^*(\xi)$; then (3.9) gives us the defining equation of $\tilde T \subset Z'$ and $\tilde S = \tilde T \cap (t = 0)$. Recall that $\tilde S_{\text{pri}}$, which is a chart of the primary part of $\tilde M_1(\mathbb{P}^n, d)$ (cf. [**HL**, Theorem 2.9]), is smooth and irreducible (cf. [**HL**, Section 2.8]). Since general elements in $\tilde S_{\text{pri}}$ correspond to stable maps with smooth domains, z restricted to $\tilde S_{\text{pri}}$ is non-trivial. Thus since $\tilde S$ under the embedding F is open in $(w_1 z = \cdots w_4 z = 0) \subset Z$, we have $F(\tilde S_{\text{pri}})$ is contained and dense in $(w_1 = \cdots = w_4 = 0) \subset Z$. Because $(w_1 = \cdots = w_4 = 0)$ is smooth, we conclude that $F(\tilde S_{\text{pri}}) \subset (w_1 = \cdots = w_4 = 0)$ is an open subscheme. Consequently, using $\tilde T_{\text{pri}} = \tilde T \times_{\tilde S} \tilde S_{\text{pri}}$, and that $F'(\tilde T) \subset (w_1 z = \cdots = w_4 z = tz = 0) \subset Z'$ is an open subscheme, we conclude that $F'(\tilde T_{\text{pri}}) \subset (w_1 = \cdots = w_4 = tz = 0) \subset Z'$ is an open subscheme.

From [**HL**], we know that $\tilde{\mathcal{X}}$ is reduced; its primary component $\tilde{\mathcal{X}}_{\text{pri}}$ is smooth of dimension $5d$, and the remainder components, indexed by partitions $\mu \vdash d$, are of pure dimensions $5d+3$ (cf. [**HL**, Theorem 2.9]). Since $F(\tilde S_{\text{pri}})$ is open in $(w_1 = \cdots = w_4 = 0) \subset Z$, and since $\tilde K$ is smooth and connected, we know that $F(\tilde S_{\text{gst}})$ is open in $(z = 0) \subset Z$ and $F'(\tilde T_{\text{gst}})$ is open in $(z = 0) \subset Z'$. Further, after writing $\xi = \prod_{\mu \vdash d} \xi_\mu$ so that $(\xi_\mu = 0) = \tilde K_\mu$ in the notation of Proposition 3.2, and letting $z_\mu = \bar q^*(\xi_\mu)$ we have that $F(\tilde S_\mu)$ is open in $(z_\mu = 0) \subset Z$ and $F'(\tilde T_\mu)$ is open in $(z_\mu = 0) \subset Z'$.

Finally, (2) of the Proposition follows from that $(\xi = 0)$ is a normal crossing divisor (Proposition 3.2), and the identity describing $\tilde T_{\text{pri}}, \tilde T_\mu$, $\tilde T_{\text{gst}}$ of the Proposition follows from the description of $F'(\tilde T_{\text{pri}}), F'(\tilde T_\mu)$, $F'(\tilde T_{\text{gst}})$ above and [**HL**, Thm. 2.9]. This proves the Proposition. q.e.d.

4. The decomposition of cones

We keep the notation introduced in the previous section, like the U, K and S defined in (3.2). We have a commutative diagram of tautological morphisms:

$$
\begin{array}{ccccccc}
\tilde{T} & \xrightarrow{\tilde{\varphi}_1} & \tilde{K} & \xrightarrow{\tilde{\varphi}_2} & \tilde{U} & \xrightarrow{\tilde{\varphi}_3} & \tilde{M} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
T & \xrightarrow{\varphi_1} & K & \xrightarrow{\varphi_2} & U & \xrightarrow{\varphi_3} & M,
\end{array} \tag{4.1}
$$

constructed in paragraphs before and following (3.2), and such that the induced $\tilde{T} \to T \times_K \tilde{K}$ and $\tilde{K} \to K \times_U \tilde{U}$ are open embeddings, and $\tilde{U} \cong U \times_M \tilde{M}$.

Lemma 4.1. *The sheaves $R^i\pi_{\tilde{\mathcal{X}}_{\rm gst}*}\mathcal{L}_{\tilde{\mathcal{X}}_{\rm gst}}$ and $R^i\pi_{\tilde{\mathcal{X}}_{\rm gst}*}\mathcal{P}_{\tilde{\mathcal{X}}_{\rm gst}}$ are locally free over $\tilde{\mathcal{X}}_{\rm gst}$. The sheaf $\pi_{\tilde{\mathcal{X}}_{\rm gst}*}\mathcal{P}_{\tilde{\mathcal{X}}_{\rm gst}}$ is an invertible sheaf over $\tilde{\mathcal{X}}_{\rm gst}$.*

Proof. We pick a smooth chart $\tilde{S}$ of $\tilde{\mathcal{X}}$ as in Proposition 3.1. By (3) of the same Proposition, $\tilde{S}_{\rm gst}$ is an open subscheme in $(z=0) \subset Z$. By definition, $z = \bar{p}^*(\xi)$; thus applying Lemma 3.4 we conclude that

$$\big(R^\bullet\pi_{\tilde{\mathcal{X}}*}\mathcal{L}_{\tilde{\mathcal{X}}}\big)|_{\tilde{S}_{\rm gst}} \cong_{q.i.} [\mathcal{O}_{\tilde{S}_{\rm gst}}^{\oplus(5d+1)} \xrightarrow{\times 0} \mathcal{O}_{\tilde{S}_{\rm gst}}]$$

and

$$\big(R^\bullet\pi_{\tilde{\mathcal{X}}*}\mathcal{P}_{\tilde{\mathcal{X}}}\big)|_{\tilde{S}_{\rm gst}} \cong_{q.i.} [\mathcal{O}_{\tilde{S}_{\rm gst}} \xrightarrow{\times 0} \mathcal{O}_{\tilde{S}_{\rm gst}}^{\oplus(5d+1)}].$$

The Lemma follows. q.e.d.

Let $\mathfrak{p} : \tilde{\mathcal{Y}} \to \tilde{\mathcal{X}}$ be the base change of the forgetful morphism $\mathcal{Y} \to \mathcal{X}$ (cf. (2.12)).

Proposition 4.2. *We have the following structure results of $\tilde{\mathcal{Y}}_{\rm pri}$ and $\tilde{\mathcal{Y}}_{\rm gst}$:*
(1). $\tilde{\mathcal{Y}}_{\rm gst}$ is canonically isomorphic to the (total space of the) line bundle $\pi_{\tilde{\mathcal{X}}_{\rm gst}*}\mathcal{P}_{\tilde{\mathcal{X}}_{\rm gst}}$; $\mathfrak{p}|_{\tilde{\mathcal{Y}}_{\rm gst}} : \tilde{\mathcal{Y}}_{\rm gst} \to \tilde{\mathcal{X}}_{\rm gst}$ is identical to the line bundle projection $\pi_{\tilde{\mathcal{X}}_{\rm gst}*}\mathcal{P}_{\tilde{\mathcal{X}}_{\rm gst}} \to \tilde{\mathcal{X}}_{\rm gst}$.
(2). Under the zero section embedding $\tilde{\mathcal{X}} \to \tilde{\mathcal{Y}}$, $\tilde{\mathcal{Y}}_{\rm pri} = \tilde{\mathcal{X}}_{\rm pri} \cup (\tilde{\mathcal{Y}}_{\rm gst} \times_{\tilde{\mathcal{X}}} \tilde{\mathcal{X}}_{\rm pri})$.

Proof. For the first statement, by the definition of direct image cones ([**CL1**, Def 2.1]) we conclude

$$\tilde{\mathcal{Y}}_{\rm gst} = \tilde{\mathcal{Y}} \times_{\tilde{\mathcal{X}}} \tilde{\mathcal{X}}_{\rm gst} \cong C((\pi_{\tilde{\mathcal{X}}_{\rm gst}})_*\mathcal{P}_{\tilde{\mathcal{X}}_{\rm gst}}). \tag{4.2}$$

Since $R^i\pi_{\tilde{\mathcal{X}}_{\rm gst}*}\mathcal{P}_{\tilde{\mathcal{X}}_{\rm gst}}$ are locally free for $i=0,1$, we see that (4.2) is the total space of the line bundle of $\pi_{\tilde{\mathcal{X}}_{\rm gst}*}\mathcal{P}_{\tilde{\mathcal{X}}_{\rm gst}}$. Finally, that $\mathfrak{p}|_{\tilde{\mathcal{Y}}_{\rm gst}} : \tilde{\mathcal{Y}}_{\rm gst} \to \tilde{\mathcal{X}}_{\rm gst}$ coincides with the projection of $\pi_{\tilde{\mathcal{X}}_{\rm gst}*}\mathcal{P}_{\tilde{\mathcal{X}}_{\rm gst}}$ to $\tilde{\mathcal{X}}_{\rm gst}$ follows from the definition of $\mathfrak{p}$.

The second statement follows from Proposition 3.1. q.e.d.

We remark that indeed all $\mathcal{X}_\alpha$ are smooth, which was proved in [**HL**] without being explicitly stated. For us, the property of having at worst normal crossing singularities is sufficient.

Lemma 4.3. *The intrinsic normal cone* $\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}} \subset h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})$ *embedded via the obstruction theory* $\phi_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$ *of* $\tilde{\mathcal{Y}} \to \tilde{\mathcal{D}}$ *has the following properties:*

1) *over* $\tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_{\mathrm{gst}}$*, it is the zero section of* $h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})|_{\tilde{\mathcal{Y}}-\tilde{\mathcal{Y}}_{\mathrm{gst}}}$*;*
2) *over* $\tilde{\mathcal{Y}}-\tilde{\mathcal{Y}}_{\mathrm{pri}}$*, it is a rank two subbundle stack of* $h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})|_{\tilde{\mathcal{Y}}-\tilde{\mathcal{Y}}_{\mathrm{pri}}}$.

Proof. We first prove that the tautological morphism $p : \tilde{\mathcal{Y}} \to \tilde{\mathcal{D}}$ is smooth along $\tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_{\mathrm{gst}}$. We pick a smooth chart $\tilde{T} \to \tilde{\mathcal{Y}}$ together with its locally closed embedding $\tilde{T} \to Z' = \mathbb{A}^4 \times \mathbb{A}^1 \times \mathbb{A}^{4d} \times \tilde{K}$ as in (3.9) and Proposition 3.1. Using the defining equation of $\tilde{T} \subset Z'$ in (3.9) we know z is nowhere vanishing away from $\tilde{\mathcal{Y}}_{\mathrm{gst}}$. Because w_i are coordinate variables of the factors of $\mathbb{A}^4$ in Z', we conclude that $\tilde{\varphi}_1|_{\tilde{T}-\tilde{T}_{\mathrm{gst}}} : \tilde{T} - \tilde{T}_{\mathrm{gst}} \to \tilde{K}$ is smooth (cf. (4.1)).

We next claim that φ_2 is smooth along $\varphi_1(T - T_{\mathrm{gst}})$. Let $\xi \in \varphi_1(T - T_{\mathrm{gst}})$ whose image in $\mathcal{K}$ is $D \subset C$, by the definition of $\mathcal{Y}_{\mathrm{gst}}$ we know that D intersects the minimal (arithmetic) genus one subcurve of C. Because $g(C) = 1$, we conclude that $H^1(\mathcal{O}_C(D)) = 0$. Using the exact sequence $0 \to \mathcal{O}_C \to \mathcal{O}_C(D) \to \mathcal{O}_D(D) \to 0$ and its induced long exact sequence of cohomology groups, we conclude that $H^0(\mathcal{O}_D(D)) \to H^1(\mathcal{O}_C)$ is surjective. This implies that any first order deformation of the pair $(C, \mathcal{O}_C(D))$ can be lifted to a first order deformation of (C, D), proving that φ_2 is smooth near ξ.

Since $\tilde{\varphi}_2$ is the base change of φ_2, $\tilde{\varphi}_2$ is smooth near $\tilde{\varphi}_1(\tilde{T} - \tilde{T}_{\mathrm{gst}})$. Therefore, $\tilde{\varphi}_2 \circ \tilde{\varphi}_1$ is smooth along $\tilde{T} - \tilde{T}_{\mathrm{gst}}$, and consequently $p : \tilde{\mathcal{Y}} \to \tilde{\mathcal{D}}$ is smooth along $\tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_{\mathrm{gst}}$. This proves (1).

To prove (2), we first prove a sublemma.

Sublemma 4.4. *The image* $p(\tilde{\mathcal{Y}}_{\mathrm{gst}}) \subset \tilde{\mathcal{D}}$ *is a locally closed, codimension two l.c.i. substack, and* $p|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}} : \tilde{\mathcal{Y}}_{\mathrm{gst}} \to p(\tilde{\mathcal{Y}}_{\mathrm{gst}})$ *is smooth.*

Proof. Let $\tilde{T} \to \tilde{\mathcal{Y}}$ be the smooth chart as before, and let $\tilde{\varphi} := \tilde{\varphi}_3 \circ \tilde{\varphi}_2 \circ \tilde{\varphi}_1 : \tilde{T} \to \tilde{M}$ (cf. (4.1)). By our constructions and Proposition 3.1, 3.2, the morphism $\tilde{\phi}|_{\tilde{T}_{\mathrm{gst}}} : \tilde{T}_{\mathrm{gst}} \to \tilde{M}$ factors through the following fiber diagrams

$$\begin{array}{ccccccc}
\tilde{T}_{\mathrm{gst}} & \longrightarrow & (z=0) & \longrightarrow & (\xi=0) & \longrightarrow & (\xi'=0) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & Z' & \xrightarrow{\bar{q}} & \tilde{K} & \xrightarrow{\tilde{\phi}_3\circ\tilde{\phi}_2} & \tilde{M}
\end{array}$$

where $z = \bar{q}^*\xi, \xi = (\tilde{\phi}_3 \circ \tilde{\phi}_2)^*\xi', \xi' \in \Gamma(\mathcal{O}_{\tilde{M}})$, and the two arrows in the lower row are smooth. Since $\tilde{T}_{\rm gst} \to (z = 0)$ is an open embedding, by Proposition 3.1 (2) we conclude that $\tilde{\varphi}|_{\tilde{T}_{\rm gst}} : \tilde{T}_{\rm gst} \to \tilde{\varphi}(\tilde{T}_{\rm gst}) \subset \tilde{M}$ is smooth and $\tilde{\varphi}(\tilde{T}_{\rm gst}) \subset \tilde{M}$ is a locally closed divisor with at worst normal crossing singularities. For simplicity, we denote $\Lambda = \tilde{\varphi}(\tilde{T}_{\rm gst}) \subset \tilde{M}$. We claim that we can find a section

$$\tilde{\zeta} : \Lambda \longrightarrow \tilde{\varphi}_3^{-1}(\Lambda)$$

of $\tilde{\varphi}'_3 = \tilde{\varphi}_3|_{\tilde{\varphi}_3^{-1}(\Lambda)} : \tilde{\varphi}_3^{-1}(\Lambda) \to \Lambda$ so that $\tilde{\varphi}|_{\tilde{T}_{\rm gst}}$ factors through $\tilde{\varphi}_3|_{\tilde{\zeta}(\Lambda)} : \tilde{\zeta}(\Lambda) \to \Lambda$. Because $\tilde{\varphi}'_3$ is smooth of relative dimension one, and $\Lambda \subset \tilde{M}$ is a locally closed divisor with at worst normal crossing singularities, the claim implies that $\tilde{\varphi}_2 \circ \tilde{\varphi}_1(\tilde{T}_{\rm gst}) = \tilde{\zeta}(\Lambda) \subset \tilde{U}$ is a locally closed codimension two l.c.i. substack(scheme), and that $\tilde{\varphi}_2 \circ \tilde{\varphi}_1|_{\tilde{T}_{\rm gst}} : \tilde{T}_{\rm gst} \to \tilde{\zeta}(\Lambda)$ is smooth, thus proving the Sublemma.

We now prove the claim. Let $M_\Lambda \subset M$ be the image stack of $\tilde{T}_{\rm gst} \to M$. Let $U_\Lambda = U \times_M M_\Lambda \subset U$. We claim that we can find a section

$$\zeta_M : M_\Lambda \longrightarrow U_\Lambda$$

of the tautological projection $U_\Lambda \cong \mathbf{Pic}_{\mathcal{C}_{M_\Lambda}/M_\Lambda} \to M_\Lambda$. Indeed, given any point $\xi \in M_\Lambda$ represented by a weighted curve $(C, \mathbf{w})$, by the decomposition result Proposition 8.2, C is a union of a nodal elliptic curve (arithmetic genus one) E with a collection of $\mathbb{P}^1$ so that $\mathbf{w}$ takes value zero on every component of E and takes non-negative integer value on every $\mathbb{P}^1 \subset C$ not in E. We let L be a line bundle on C so that $L|_E \cong \mathcal{O}_E$, and for every $\mathbb{P}^1 \cong R \subset C$ we have $\deg L|_R$ equals the value of $\mathbf{w}$ on $[R]$. Such L is unique. We define $\zeta_M(\xi) \in U_\Lambda$ to be the unique element lies over $(C, L) \in \mathcal{D}$ and $(C, \mathbf{w}) \in M_\Lambda$. Using that $U = \mathbf{Pic}_{\mathcal{C}_M/M} \to M$ is representable, that M_Λ is reduced, and the decomposition Proposition 8.2, we conclude that this point-wise definition of ζ_M defines a morphism ζ_M as desired. Then because $\tilde{\varphi}'_3 : \tilde{\varphi}_3^{-1}(\Lambda) \to \Lambda$ is a base change of $U_\Lambda \to M_\Lambda$, ζ_M lifts to the desired section $\tilde{\zeta}$.

Finally, using the defining equation of $\tilde{\mathcal{T}}_{\rm gst}$, we see that $\tilde{\varphi}_1(\tilde{\mathcal{T}}_{\rm gst}) \subset (\tilde{\xi} = 0) \subset \tilde{K}$. By Proposition 3.2 the image of general points of $(\tilde{\xi} = 0)$ in $\mathcal{K}$ are represented by $[D \subset C] \in \mathcal{K}$ with D does not intersect elliptic curve $E \subset C$. Since this property is preserved after taking closure, we have $\tilde{\varphi}_2 \circ \tilde{\varphi}_1(\tilde{\mathcal{T}}_{\rm gst}) \subset \tilde{\zeta}(\Lambda)$. Thus $\tilde{\varphi}|_{\tilde{\mathcal{T}}_{\rm gst}}$ factors through $\tilde{\varphi}_3|_{\tilde{\zeta}(\Lambda)}$. This proves the Sublemma. q.e.d.

Now the proof of (2) of Lemma 4.3 follows from that $\tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_{\rm pri}$ is open in $\tilde{\mathcal{Y}}$, $p(\tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_{\rm pri}) \subset \tilde{\mathcal{D}}$ is a locally closed, codimension two l.c.i. substack of $\tilde{\mathcal{D}}$, and $p|_{\tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_{\rm pri}} : \tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_{\rm pri} \to p(\tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_{\rm pri})$ is smooth. q.e.d.

We will see that to each $\alpha = \text{pri}$ or $\mu \vdash d$, the cone $\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$ contains a unique integral component that dominates $\tilde{\mathcal{Y}}_\alpha$. For $\tilde{\mathcal{Y}}_{\text{pri}}$, it is the closure of the zero section of $h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})|_{\tilde{\mathcal{Y}}-\tilde{\mathcal{Y}}_{\text{gst}}}$ (in $h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})|_{\tilde{\mathcal{Y}}_{\text{pri}}}$); we denote it by $\mathbf{C}_{\text{pri}}$. For $\mu \vdash d$, it is the closure of the rank two subbundle stack in $h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})|_{\tilde{\mathcal{Y}}_\mu}$ given in Lemma 4.3; we denote this subcone by $\mathbf{C}'_\mu$.

There are possibly other irreducible components of $\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$ lying over $\tilde{\mathcal{Y}}_{\text{pri}} \cap \tilde{\mathcal{Y}}_{\text{gst}}$; we group them (not necessarily unique) into $\bigcup_{\mu\vdash d} \mathbf{C}''_\mu$ such that $\mathbf{C}''_\mu$ lies over $\tilde{\mathcal{Y}}_{\text{pri}} \times_{\tilde{\mathcal{Y}}} \tilde{\mathcal{Y}}_\mu$. We write $\mathbf{C}_\mu = \mathbf{C}'_\mu \cup \mathbf{C}''_\mu$. Therefore

$$[\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}] = [\mathbf{C}_{\text{pri}}] + \sum_{\mu\vdash d} [\mathbf{C}_\mu] \in Z_* h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}). \tag{4.3}$$

Consequently, (denoting $[\mathbf{C}_{\text{gst}}] = \sum_{\mu\vdash d} [\mathbf{C}_\mu]$,)

$$[\tilde{\mathcal{Y}}]^{\text{vir}}_{\text{loc}} = 0^!_{\tilde{\sigma},\text{loc}}[\mathbf{C}_{\text{pri}}] + 0^!_{\tilde{\sigma},\text{loc}}[\mathbf{C}_{\text{gst}}] = 0^!_{\tilde{\sigma},\text{loc}}[\mathbf{C}_{\text{pri}}] + \sum_{\mu\vdash d} 0^!_{\tilde{\sigma},\text{loc}}[\mathbf{C}_\mu]. \tag{4.4}$$

Let $N_1(d)^{\text{red}}_Q$ be the reduced genus one GW-invariants of the quintic Q introduced in [**LZ**] and stated in (1.2). We state the key Propositions whose proofs will occupy the remainder of this paper.

Proposition 4.5. *We have* $\deg 0^!_{\tilde{\sigma},\text{loc}}[\mathbf{C}_{\text{pri}}] = (-1)^{5d} N_1(d)^{red}_Q$.

Let (d) be the partition of d into a single part; i.e., the non-partition of d.

Proposition 4.6. *For $\mu \vdash d$ and $\mu \neq (d)$, we have* $\deg 0^!_{\tilde{\sigma},\text{loc}}[\mathbf{C}_\mu] = 0$.

Proposition 4.7. *We have* $\deg 0^!_{\tilde{\sigma},\text{loc}}[\mathbf{C}_{(d)}] = \frac{(-1)^{5d}}{12} N_0(d)_Q$.

These three Propositions, the identity (4.4) and Proposition 2.5 combined give an algebraic proof of the hyperplane property of genus one GW-invariants of quintics proved originally via analytic method in [**LZ**, **VZ**, **Zi1**].

5. Contribution from the primary component

We begin with more notations. From now on, we view $\tilde{\mathcal{X}}_{\text{pri}}$ as a closed substack of $\tilde{\mathcal{Y}}_{\text{pri}}$ via the zero section embedding $\tilde{\mathcal{X}} \to \tilde{\mathcal{Y}}$ and (2) of Proposition 4.2. We denote $\tilde{\sigma}_{\text{pri}} := \tilde{\sigma}|_{\tilde{\mathcal{X}}_{\text{pri}}} : \mathcal{O}b_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}|_{\tilde{\mathcal{X}}_{\text{pri}}} \to \mathcal{O}_{\tilde{\mathcal{X}}_{\text{pri}}}$. Using $\mathcal{O}b_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}|_{\tilde{\mathcal{X}}_{\text{pri}}} = H^1(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}|_{\tilde{\mathcal{X}}_{\text{pri}}})$, and the definition (2.7), we define the substack

$$h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}|_{\tilde{\mathcal{X}}_{\text{pri}}})_{\tilde{\sigma}_{\text{pri}}} \subset h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}|_{\tilde{\mathcal{X}}_{\text{pri}}}).$$

Note that $D(\tilde\sigma_{\mathrm{pri}}) = \tilde{\mathcal X}_{\mathrm{pri}} \cap D(\tilde\sigma)$. Since $[\mathbf{C}_{\mathrm{pri}}]$ is supported on $\tilde{\mathcal X}_{\mathrm{pri}} \subset \tilde{\mathcal Y}_{\mathrm{pri}}$,

$$[\mathbf{C}_{\mathrm{pri}}] \in Z_*(h^1/h^0(E_{\tilde{\mathcal Y}/\tilde{\mathcal D}}) \times_{\tilde{\mathcal Y}_{\mathrm{pri}}} \tilde{\mathcal X}_{\mathrm{pri}}) = Z_*(h^1/h^0(E_{\tilde{\mathcal Y}/\tilde{\mathcal D}}|_{\tilde{\mathcal X}_{\mathrm{pri}}})_{\tilde\sigma_{\mathrm{pri}}}).$$

We then apply the localized Gysin map

$$0^!_{\tilde\sigma_{\mathrm{pri}},\mathrm{loc}} : A_*(h^1/h^0(E_{\tilde{\mathcal Y}/\tilde{\mathcal D}}|_{\tilde{\mathcal X}_{\mathrm{pri}}})_{\tilde\sigma_{\mathrm{pri}}}) \longrightarrow A_*D(\tilde\sigma_{\mathrm{pri}}) \tag{5.1}$$

to it to obtain $0^!_{\tilde\sigma_{\mathrm{pri}},\mathrm{loc}}[\mathbf{C}_{\mathrm{pri}}] \in A_0D(\tilde\sigma_{\mathrm{pri}})$. By [**KL**, Lemma 2.5], its image under $A_0D(\tilde\sigma_{\mathrm{pri}}) \longrightarrow A_0(D(\tilde\sigma))$ induced by the inclusion is $0^!_{\tilde\sigma,\mathrm{loc}}([\mathbf{C}_{\mathrm{pri}}])$. This implies

$$\deg 0^!_{\tilde\sigma_{\mathrm{pri}},\mathrm{loc}}[\mathbf{C}_{\mathrm{pri}}] = \deg 0^!_{\tilde\sigma,\mathrm{loc}}[\mathbf{C}_{\mathrm{pri}}].$$

Let $(f_{\tilde{\mathcal Y}}, \pi_{\tilde{\mathcal Y}}) : \mathcal C_{\tilde{\mathcal Y}} \to \mathbb P^4 \times \tilde{\mathcal Y}$ with $\psi_{\tilde{\mathcal Y}} \in \Gamma(\mathcal C_{\tilde{\mathcal Y}}, \mathcal P_{\tilde{\mathcal Y}})$ be the tautological family of $\tilde{\mathcal Y}$, where $\mathcal L_{\tilde{\mathcal Y}} = f^*_{\tilde{\mathcal Y}}\mathcal O(1)$ and $\mathcal P_{\tilde{\mathcal Y}} = f^*_{\tilde{\mathcal Y}}\mathcal O(-5) \otimes \omega_{\mathcal C_{\tilde{\mathcal Y}}/\tilde{\mathcal Y}}$. Recall from (2.13) that the deformation complex of the relative obstruction theory of $\tilde{\mathcal Y} \to \tilde{\mathcal D}$ is $E_{\tilde{\mathcal Y}/\tilde{\mathcal D}} = R^\bullet\pi_{\tilde{\mathcal Y}*}(\mathcal L_{\tilde{\mathcal Y}}^{\oplus 5} \oplus \mathcal P_{\tilde{\mathcal Y}})$. Let

$$\mathbf H_1 = h^1/h^0\big((R^\bullet\pi_{\tilde{\mathcal Y}*}\mathcal L^{\oplus 5}_{\tilde{\mathcal Y}})|_{\tilde{\mathcal X}_{\mathrm{pri}}}\big) \quad \text{and} \quad \mathbf H_2 = h^1/h^0\big((R^\bullet\pi_{\tilde{\mathcal Y}*}\mathcal P_{\tilde{\mathcal Y}})|_{\tilde{\mathcal X}_{\mathrm{pri}}}\big).$$

By the base change property of the h^1/h^0-construction (The base change property is as follows. Let $\varphi : V \to U$ be a morphism of DM stacks; let $G^\bullet \in D^b(U)$ be quasi-isomorphic to $[G^0 \to G^1]$, where G^i are vector bundles over U. The h^1/h^0 construction ([**BF**, p 57]) defines $h^1/h^0(G^\bullet) = [G^1/G^0]$, a quotient stack is smooth over U. Since $\varphi^*G^\bullet \cong_{q.i} [\varphi^*G^0 \to \varphi^*G^1]$, we have canonical isomorphism $h^1/h^0(\varphi^*G^\bullet) \cong [\varphi^*G^1/\varphi^*G^0] \cong [G^1/G^0] \times_U V$.), we have

$$h^1/h^0(E_{\tilde{\mathcal Y}/\tilde{\mathcal D}})|_{\tilde{\mathcal X}_{\mathrm{pri}}} \cong \mathbf H_1 \times_{\tilde{\mathcal X}_{\mathrm{pri}}} \mathbf H_2. \tag{5.2}$$

We need the notion of the closure of the zero section of a vector bundle stack. For M an integral DM stack and $G^\bullet = [\alpha : G^0 \to G^1]$ a two-term complex of locally free sheaves placed at $[0,1]$, the zero section $0_{\mathbf H}$ of $\mathbf H = [G^0/G^1]$ is the substack $0_{\mathbf H} : [G^0/G^0] \to [G^1/G^0]$, where $G^0 \to G^0$ is the identity map; we define its closure to be

$$\bar 0_{\mathbf H} := [\mathrm{cl}\big(\alpha(G^0)\big)/G^0] \subset \mathbf H, \tag{5.3}$$

where $\alpha(G^0)$ is the image of $\alpha : G^0 \to G^1$ and $\mathrm{cl}\big(\alpha(G^0)\big)$ is the closure of $\alpha(G^0)$ in G^1. For $G^\bullet \in D^b(M)$ locally represented by two-term complexes of locally free sheaves placed at $[0,1]$, we can cover M by open substacks U_i such that $G^\bullet|_{U_i} \cong_{q.i.} [G^0_i \to G^1_i]$ as stated and define $0_{\mathbf H}|_{U_i}$ and $\bar 0_{\mathbf H}|_{U_i}$ in $\mathbf H|_{U_i} = h^0/h^1(G^\bullet|_{U_i})$ as before. A direct check shows that $\{0_{\mathbf H}|_{U_i}\}$ and $\{\bar 0_{\mathbf H}|_{U_i}\}$ patch together to substacks $0_{\mathbf H}$ and $\bar 0_{\mathbf H} \subset \mathbf H$. Since M is integral, $\bar 0_{\mathbf H}$ is integral.

We apply this construction to $(R^{\bullet}\pi_{\tilde{\mathcal{Y}}*}\mathcal{L}_{\tilde{\mathcal{Y}}}^{\oplus 5})|_{\tilde{\mathcal{X}}_{\rm pri}}$ and $(R^{\bullet}\pi_{\tilde{\mathcal{Y}}*}\mathcal{P}_{\tilde{\mathcal{Y}}})|_{\tilde{\mathcal{X}}_{\rm pri}} \in D^b(\tilde{\mathcal{X}}_{\rm pri})$ to obtain $\bar{0}_{\mathbf{H}_1} \subset \mathbf{H}_1$ and $\bar{0}_{\mathbf{H}_2} \subset \mathbf{H}_2$. As $\tilde{\mathcal{X}}_{\rm pri}$ is integral, both $\bar{0}_{\mathbf{H}_1}$ and $\bar{0}_{\mathbf{H}_2}$ are integral.

Lemma 5.1. *Under the tautological inclusion* $\mathbf{H}_1 \times_{\tilde{\mathcal{X}}_{\rm pri}} \mathbf{H}_2 \to h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})$*, we have*

$$[\mathbf{C}_{\rm pri}] = [\mathbf{H}_1 \times_{\tilde{\mathcal{X}}_{\rm pri}} \bar{0}_{\mathbf{H}_2}] \in Z_* h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}).$$

Proof. For convenience, we abbreviate $\tilde{\mathcal{X}}^{\circ}_{\rm pri} := \tilde{\mathcal{X}} - \tilde{\mathcal{X}}_{\rm gst}$ and $\tilde{\mathcal{Y}}^{\circ}_{\rm pri} = \tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_{\rm gst}$; by (2) of Proposition 4.2, we know $\tilde{\mathcal{X}}^{\circ}_{\rm pri} = \tilde{\mathcal{Y}}^{\circ}_{\rm pri}$, and combined with (1) of Proposition 4.2, we know that the closure of $\tilde{\mathcal{Y}}^{\circ}_{\rm pri}$ in $\tilde{\mathcal{Y}}$ is $\tilde{\mathcal{X}}_{\rm pri}$.

Because $R^1\pi_{\tilde{\mathcal{Y}}*}\mathcal{L}_{\tilde{\mathcal{Y}}}^{\oplus 5}|_{\tilde{\mathcal{Y}}^{\circ}_{\rm pri}} = R^1\pi_{\tilde{\mathcal{Y}}*}\mathcal{P}_{\tilde{\mathcal{Y}}}|_{\tilde{\mathcal{Y}}^{\circ}_{\rm pri}} = 0$, we have

$$\mathbf{H}_1|_{\tilde{\mathcal{Y}}^{\circ}_{\rm pri}} = 0_{\mathbf{H}_1}|_{\tilde{\mathcal{Y}}^{\circ}_{\rm pri}} \quad \text{and} \quad \mathbf{H}_2|_{\tilde{\mathcal{Y}}^{\circ}_{\rm pri}} = 0_{\mathbf{H}_2}|_{\tilde{\mathcal{Y}}^{\circ}_{\rm pri}} = \bar{0}_{\mathbf{H}_2}|_{\tilde{\mathcal{Y}}^{\circ}_{\rm pri}}.$$

Thus by (1) of Lemma 4.3,

$$\mathbf{C}_{\rm pri}|_{\tilde{\mathcal{Y}}^{\circ}_{\rm pri}} = 0_{\mathbf{H}_1 \times_{\tilde{\mathcal{X}}_{\rm pri}} \mathbf{H}_2}|_{\tilde{\mathcal{Y}}^{\circ}_{\rm pri}} = (\mathbf{H}_1 \times_{\tilde{\mathcal{X}}_{\rm pri}} \bar{0}_{\mathbf{H}_2})|_{\tilde{\mathcal{Y}}^{\circ}_{\rm pri}} \subset h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})|_{\tilde{\mathcal{Y}}^{\circ}_{\rm pri}}.$$

Since $\bar{0}_{\mathbf{H}_2}$ is integral and $\mathbf{H}_1$ is a bundle-stack over $\tilde{\mathcal{X}}_{\rm pri}$ which is smooth over $\tilde{\mathcal{X}}_{\rm pri}$, the stack $\mathbf{H}_1 \times_{\tilde{\mathcal{X}}_{\rm pri}} \bar{0}_{\mathbf{H}_2}$ is integral and thus is identical to the closure of $(\mathbf{H}_1 \times_{\tilde{\mathcal{X}}_{\rm pri}} \bar{0}_{\mathbf{H}_2})|_{\tilde{\mathcal{Y}}^{\circ}_{\rm pri}}$ in $h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})$. Because $\mathbf{C}_{\rm pri}$ is the closure of $\mathbf{C}_{\rm pri}|_{\tilde{\mathcal{Y}}^{\circ}_{\rm pri}}$ in $h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})$, the Lemma follows. q.e.d.

Let $0^!_{\tilde{\sigma}_{\rm pri},\rm loc}$ be the localized Gysin map in (5.1); let $0^!_{\mathbf{H}_2} : A_*\mathbf{H}_2 \to A_*\tilde{\mathcal{X}}_{\rm pri}$ be the Gysin map by intersecting with the zero section of $\mathbf{H}_2$.

Corollary 5.2. *Let* $\iota : D(\tilde{\sigma}_{\rm pri}) \to \tilde{\mathcal{X}}_{\rm pri}$ *be the inclusion. Then*

$$\deg 0^!_{\tilde{\sigma}_{\rm pri},\rm loc}([\mathbf{C}_{\rm pri}]) = \deg 0^!_{\mathbf{H}_2}([\bar{0}_{\mathbf{H}_2}]). \tag{5.4}$$

Proof. Let

$$\jmath : h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})_{\tilde{\sigma}} \times_{\tilde{\mathcal{Y}}} \tilde{\mathcal{X}}_{\rm pri} \longrightarrow h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}) \times_{\tilde{\mathcal{Y}}} \tilde{\mathcal{X}}_{\rm pri} = \mathbf{H}_1 \times_{\tilde{\mathcal{X}}_{\rm pri}} \mathbf{H}_2$$

be induced by the inclusion $h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})_{\tilde{\sigma}} \subset h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})$ and the identity (5.2). By [**KL**, Prop 1.3], we have $\iota_* \circ 0^!_{\tilde{\sigma}_{\rm pri},\rm loc} = 0^!_{\mathbf{H}_1 \times_{\tilde{\mathcal{X}}_{\rm pri}} \mathbf{H}_2} \circ \jmath_*$. By Lemma 5.1, we have

$$\iota_* 0^!_{\tilde{\sigma}_{\rm pri},\rm loc}([\mathbf{C}_{\rm pri}]) = 0^!_{\mathbf{H}_1 \times_{\tilde{\mathcal{X}}_{\rm pri}} \mathbf{H}_2}([\mathbf{H}_1 \times_{\tilde{\mathcal{X}}_{\rm pri}} \bar{0}_{\mathbf{H}_2}]).$$

Because $0^!_{\mathbf{H}_1 \times_{\tilde{\mathcal{X}}_{\rm pri}} \mathbf{H}_2}[\mathbf{H}_1 \times_{\tilde{\mathcal{X}}_{\rm pri}} \bar{0}_{\mathbf{H}_2}] = 0^!_{\mathbf{H}_2}[\bar{0}_{\mathbf{H}_2}]$, we obtain

$$\iota_* 0^!_{\tilde{\sigma}_{\rm pri},\rm loc}([\mathbf{C}_{\rm pri}]) = 0^!_{\mathbf{H}_2}([\bar{0}_{\mathbf{H}_2}]).$$

Finally, because $\tilde{\mathcal{X}}_{\rm pri}$ is proper, the homomorphism ι_* preserves the degrees, which implies (5.4). This proves the Corollary. q.e.d.

We prove a useful result. In the following, for a locally free sheaf $\mathcal{E}$ we will use the same symbol to denote its associated vector bundle.

Lemma 5.3. *Let $R = [\mathcal{R}_0 \to \mathcal{R}_1]$ be a complex of locally free sheaves of amplitude $[0,1]$ on an integral Deligne-Mumford stack M, and let $\mathbf{B} = h^1/h^0(R^\vee[-1]) = [\mathcal{R}_0^\vee/\mathcal{R}_1^\vee]$. Suppose $H^1(R)$ is a torsion sheaf on M and the image sheaf of $\mathcal{R}_0 \to \mathcal{R}_1$ is locally free, then $H^0(R)$ is locally free and*

$$0^!_{\mathbf{B}}[\bar{0}_{\mathbf{B}}] = e(H^0(R)^\vee) \in A_*M.$$

Proof. Let $\mathcal{K}_i = H^i(R)$, which fit into the exact sequence

$$0 \longrightarrow \mathcal{K}_0 \xrightarrow{\alpha} \mathcal{R}_0 \xrightarrow{\beta} \mathcal{R}_1 \xrightarrow{\gamma} \mathcal{K}_1 \longrightarrow 0, \tag{5.5}$$

which breaks up a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{K}_0 \longrightarrow \mathcal{R}_0 \longrightarrow \operatorname{Im}(\beta) \longrightarrow 0.$$

Because $\operatorname{Im}(\beta)$ is locally free by assumption, we see that $\mathcal{K}_0$ is locally free and $\alpha^\vee : \mathcal{R}_0^\vee \to \mathcal{K}_0^\vee$ is surjective.

We next consider the dual $\beta^\vee : \mathcal{R}_1^\vee \to \mathcal{R}_0^\vee$, and let $Z = \mathrm{cl}(\beta^\vee(\mathcal{R}_1^\vee))$ (cf. (5.3)). By (5.3), we have $\bar{0}_{\mathbf{B}} = [Z/\mathcal{R}_1^\vee] \subset [\mathcal{R}_0^\vee/\mathcal{R}_1^\vee] = \mathbf{B}$. Let $U \subset M$ be the largest open substack so that $\mathcal{K}_1|_U = 0$. Then $\beta^\vee|_U : \mathcal{R}_1^\vee|_U \to \mathcal{R}_0^\vee|_U$ is a sub-bundle homomorphism, and then $Z \times_M U = \beta^\vee(\mathcal{R}_1^\vee|_U)$. Since M is integral and $H^1(R)$ is torsion, Z is identical to the closure of $Z \times_M U$ in $\mathcal{R}_0^\vee$.

We denote by $b : \mathcal{R}_0^\vee \longrightarrow [\mathcal{R}_0^\vee/\mathcal{R}_1^\vee]$ the quotient morphism, then $b^{-1}(\bar{0}_{\mathbf{B}}) = Z$, and

$$0^!_{\mathbf{B}}([\bar{0}_{\mathbf{B}}]) = 0^!_{\mathcal{R}_0^\vee}([Z]) \in A_*M.$$

Thus to prove the lemma we only need to show that $0^!_{\mathcal{R}_0^\vee}([Z]) = e(\mathcal{K}_0^\vee)$.

Because $\mathcal{K}_1|_U = 0$, we have the exact sequence of locally free sheaves

$$0 \longrightarrow \mathcal{R}_1^\vee|_U \xrightarrow{\beta^\vee} \mathcal{R}_0^\vee|_U \xrightarrow{\alpha^\vee} \mathcal{K}_0^\vee|_U \longrightarrow 0.$$

Hence $Z = \mathrm{cl}\big(\beta^\vee(\mathcal{R}_1^\vee|_U)\big) = \mathrm{cl}\big(\ker(\alpha^\vee)|_U\big)$ in $\mathcal{R}_0^\vee$. Since $\alpha^\vee : \mathcal{R}_0^\vee \to \mathcal{K}_0^\vee$ is a surjection of vector bundles, $\ker\alpha^\vee$ is integral in $\mathcal{R}_0^\vee$, thus $Z = \ker\alpha^\vee$ and $0^!_{\mathcal{R}_0^\vee}([Z]) = e(\mathcal{K}_0^\vee)$. This proves the Lemma. q.e.d.

Let $(f_{\tilde{\mathcal{X}}_{\rm pri}}, \pi_{\tilde{\mathcal{X}}_{\rm pri}}) = (f_{\tilde{\mathcal{Y}}}, \pi_{\tilde{\mathcal{Y}}})|_{\tilde{\mathcal{X}}_{\rm pri}}$, and let $R = R^\bullet\pi_{\tilde{\mathcal{X}}_{\rm pri}*} f^*_{\tilde{\mathcal{X}}_{\rm pri}}\mathcal{O}_{\mathbb{P}^4}(5)$. By Serre duality, $R^\vee \cong (R^\bullet\pi_{\tilde{\mathcal{Y}}*}\mathcal{P}_{\tilde{\mathcal{Y}}})[1]|_{\tilde{\mathcal{X}}_{\rm pri}}$. Since $f^*_{\tilde{\mathcal{X}}_{\rm pri}}\mathcal{O}_{\mathbb{P}^4}(5)$ is locally free on $\mathcal{C}_{\tilde{\mathcal{X}}_{\rm pri}}$, it is known that $R^\bullet\pi_{\tilde{\mathcal{X}}_{\rm pri}*} f^*_{\tilde{\mathcal{X}}_{\rm pri}}\mathcal{O}_{\mathbb{P}^4}(5) \cong_{q.i.} [\mathcal{R}_0 \to \mathcal{R}_1]$ for $\mathcal{R}_i$ locally free sheaves on $\tilde{\mathcal{X}}_{\rm pri}$ (cf. [**Beh**, Prop 5]). On the other hand, by [**HL**, Thm 2.11] (or Prop. 3.1), $\pi_{\tilde{\mathcal{X}}_{\rm pri}*} f^*_{\tilde{\mathcal{X}}_{\rm pri}}\mathcal{O}_{\mathbb{P}^4}(5)$ is locally free of rank $5d$, and $R^1\pi_{\tilde{\mathcal{X}}_{\rm pri}*} f^*_{\tilde{\mathcal{X}}_{\rm pri}}\mathcal{O}(5)$ is torsion. The image of $\mathcal{R}_0 \to \mathcal{R}_1$ is locally free by taking dual of Lemma 3.4. Applying Corollary 5.2 and Lemma 5.3, we obtain

$$\deg 0^!_{\tilde\sigma,\mathrm{loc}}[\mathbf{C}_{\mathrm{pri}}] = c_{5d}\big((\pi_{\tilde{\mathcal{X}}_{\mathrm{pri}}*} f^*_{\tilde{\mathcal{X}}_{\mathrm{pri}}} \mathcal{O}_{\mathbb{P}^4}(5))^\vee\big) =$$

$$(-1)^{5d} c_{5d}\big(\pi_{\tilde{\mathcal{X}}_{\mathrm{pri}}*} f^*_{\tilde{\mathcal{X}}_{\mathrm{pri}}} \mathcal{O}_{\mathbb{P}^4}(5)\big).$$

Proof of Proposition 4.5. Note that the right hand side of the previous identities is the reduced genus one GW-invariants introduced in [**LZ**] and stated in (1.2). This identity proves Proposition 4.5. q.e.d.

6. Reducing cosection-localized Gysin map

In the previous section, using that $\tilde{\mathcal{X}}_{\mathrm{pri}}$ is proper we have reduced the contribution from the primary component $0^!_{\tilde\sigma_{\mathrm{pri}},\mathrm{loc}}[\mathbf{C}_{\mathrm{pri}}]$ to an expression using ordinary Gysin maps (cf. Corollary 5.2 and (5.4)). In this section, we achieve the same goal for $0^!_{\tilde\sigma,\mathrm{loc}}[\mathbf{C}_{\mathrm{gst}}]$ by working with a compactification of $\tilde{\mathcal{Y}}_{\mathrm{gst}}$ and the extension of $\mathbf{C}_{\mathrm{gst}}$ to the compactification. The structure of this extension is made simple by studying the homogeneity of the cone $\mathbf{C}_{\mathrm{gst}}$ along the fibers of $\tilde{\mathcal{Y}}_{\mathrm{gst}} \to \tilde{\mathcal{X}}_{\mathrm{gst}}$.

We abbreviate

$$(6.1)\quad V_1 = R^1\pi_{\tilde{\mathcal{X}}_{\mathrm{gst}}*}\mathcal{L}^{\oplus 5}_{\tilde{\mathcal{X}}_{\mathrm{gst}}}, \quad V_2 = R^1\pi_{\tilde{\mathcal{X}}_{\mathrm{gst}}*}\mathcal{P}_{\tilde{\mathcal{X}}_{\mathrm{gst}}}, \quad \text{and} \quad V = V_1 \oplus V_2,$$

and abbreviate $L = \pi_{\tilde{\mathcal{X}}_{\mathrm{gst}}*}\mathcal{P}_{\tilde{\mathcal{X}}_{\mathrm{gst}}}$. By Lemma 4.1, they are vector bundles (locally free sheaves) on $\tilde{\mathcal{X}}_{\mathrm{gst}}$. By Proposition 4.2, $\tilde{\mathcal{Y}}_{\mathrm{gst}}$ is the total space of the line bundle L. Let $\gamma : \tilde{\mathcal{Y}}_{\mathrm{gst}} = \mathrm{Tot}(L) \to \tilde{\mathcal{X}}_{\mathrm{gst}}$ be the tautological projection, which is identical to the restriction $\mathfrak{p}|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}} : \tilde{\mathcal{Y}}_{\mathrm{gst}} \to \tilde{\mathcal{X}}_{\mathrm{gst}}$, mentioned in (1) of Proposition 4.2.

We let $f_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}, \pi_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}, \mathcal{C}_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}, \mathcal{L}_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}$ and $\mathcal{P}_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}$ be objects over $\tilde{\mathcal{Y}}_{\mathrm{gst}}$ defined similarly as that over $\tilde{\mathcal{X}}_{\mathrm{gst}}$, and define

$$(6.2)\quad \tilde V_1 = R^1\pi_{\tilde{\mathcal{Y}}_{\mathrm{gst}}*}\mathcal{L}^{\oplus 5}_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}. \quad \tilde V_2 = R^1\pi_{\tilde{\mathcal{Y}}_{\mathrm{gst}}*}\mathcal{P}_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}, \quad \text{and} \quad \tilde V = \tilde V_1 \oplus \tilde V_2.$$

Lemma 6.1. *We have canonical isomorphisms* $\gamma^* V_i = \tilde V_i$ *for* $i = 1, 2$.

Proof. Since we have a canonical isomorphism $\mathcal{C}_{\tilde{\mathcal{Y}}_{\mathrm{gst}}} \cong \mathcal{C}_{\tilde{\mathcal{X}}_{\mathrm{gst}}} \times_{\tilde{\mathcal{X}}_{\mathrm{gst}}} \tilde{\mathcal{Y}}_{\mathrm{gst}}$, letting $\tilde\gamma : \mathcal{C}_{\tilde{\mathcal{Y}}_{\mathrm{gst}}} \to \mathcal{C}_{\tilde{\mathcal{X}}_{\mathrm{gst}}}$ be the induced projection, we have $\tilde\gamma^*\mathcal{L}_{\tilde{\mathcal{X}}_{\mathrm{gst}}} = \mathcal{L}_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}$, and the same for $\mathcal{P}_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}$. Applying the base change formula (using $R^2\pi_{\tilde{\mathcal{X}}_{\mathrm{gst}}*}(\cdot) = 0$) we obtain the canonical isomorphisms

$$\tilde V_1 = R^1\pi_{\tilde{\mathcal{Y}}_{\mathrm{gst}}*}\mathcal{L}^{\oplus 5}_{\tilde{\mathcal{Y}}_{\mathrm{gst}}} = R^1\pi_{\tilde{\mathcal{Y}}_{\mathrm{gst}}*}\tilde\gamma^*\mathcal{L}^{\oplus 5}_{\tilde{\mathcal{X}}_{\mathrm{gst}}} \cong \gamma^* R^1\pi_{\tilde{\mathcal{X}}_{\mathrm{gst}}*}\mathcal{L}^{\oplus 5}_{\tilde{\mathcal{X}}_{\mathrm{gst}}} = \gamma^* V_1.$$

The same reason gives a canonical $\tilde V_2 \cong \gamma^* V_2$. q.e.d.

For the $\tilde\sigma$ given in (2.14), we form

$$\tilde\sigma_{\mathrm{gst}} := \tilde\sigma|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}} : \tilde V \longrightarrow \mathcal{O}_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}. \tag{6.3}$$

It has the following reconstruction result. We recall the construction of σ in (1.3) given by [**CL2**, (3.8)]: let $x \in \tilde{\mathcal{X}}_{\mathrm{gst}}$ and fix a homogeneous coordinates $[z_i]$ of $\mathbb{P}^4$; for any $y \in \gamma^{-1}(x) \subset \tilde{\mathcal{Y}}_{\mathrm{gst}}$ associated with $([f,C],\psi)$, we let $u_i = f^*x_i \in H^0(f^*\mathcal{O}(1))$ and $\psi \in H^0(f^*\mathcal{O}(-5)\otimes\omega_C)$. We have

$$\tilde V_1|_y = H^1(f^*\mathcal{O}(1)^{\oplus 5}) \quad \text{and} \quad \tilde V_2|_y = H^1(f^*\mathcal{O}(-5)\otimes\omega_C),$$

and $\tilde\sigma(y) : \tilde V_1|_y \oplus \tilde V_2|_y \to \mathbb{C}$ is

$$\tilde\sigma(\tilde v_i, \tilde\phi) = 5\psi \sum_{i=1}^5 u_i^4 \tilde v_i + \tilde\phi \sum_{i=1}^5 u_i^5, \qquad (\tilde v_i) \in \tilde V_1|_y,\ \tilde\phi \in \tilde V_2|_y. \tag{6.4}$$

We define

$$\xi_1(x)((v_i)\otimes\phi) = 5\phi \sum u_i^4 v_i, \quad \text{and}$$

$$\xi_2(x)(\phi) = \phi\sum u_i^5, \qquad (v_i) \in V_1|_x,\ \phi \in V_2|_x.$$

Since $\tilde{\mathcal{X}}_{\mathrm{gst}}$ is reduced, this pointwise definition of $\xi_i(x)$ for $x \in \tilde{\mathcal{X}}_{\mathrm{gst}}$ defines homomorphisms

$$\xi_1 : V_1 \otimes L \to \mathcal{O}_{\tilde{\mathcal{X}}_{\mathrm{gst}}}, \quad \text{and} \quad \xi_2 : V_2 \to \mathcal{O}_{\tilde{\mathcal{X}}_{\mathrm{gst}}}. \tag{6.5}$$

Also ξ_1 is surjective because $\xi_1(x)$ is the Serre pairing $V_1|_x \otimes L|_x \to \mathbb{C}$, which is nondegenerate (cf. [**CL1**, Prop 3.4]).

Lemma 6.2. *Let ξ_1 and ξ_2 be as defined; let $\tilde\xi_1 := \gamma^*(\xi_1)(\cdot\otimes\epsilon) : \tilde V_1 := \gamma^* V_1 \to \mathcal{O}_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}$ be defined by the tautological (identity) section $\epsilon \in \Gamma(\tilde{\mathcal{Y}}_{\mathrm{gst}}, \gamma^* L)$ paired with $\gamma^*(\xi_1)$, and let $\tilde\xi_2 = \gamma^*(\xi_2) : \tilde V_2 := \gamma^* V_2 \to \mathcal{O}_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}$. Then*

$$\tilde\sigma_{\mathrm{gst}} = (\tilde\xi_1, \tilde\xi_2) : \tilde V := \tilde V_1 \oplus \tilde V_2 \longrightarrow \mathcal{O}_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}. \tag{6.6}$$

Proof. This follows from the construction of ξ_1 and ξ_2 and the definition of $\tilde\sigma$ in (6.4). q.e.d.

The cosection $\tilde\sigma_{\mathrm{gst}}$ defines a cosection-localized Gysin map. Since ξ_1 is surjective, $\tilde\xi_1$ is surjective away from the zero section $0_{\tilde{\mathcal{Y}}_{\mathrm{gst}}} = \tilde{\mathcal{X}}_{\mathrm{gst}} \subset \tilde{\mathcal{Y}}_{\mathrm{gst}}$; thus the non-surjective locus $D(\tilde\sigma_{\mathrm{gst}})$ of $\tilde\sigma_{\mathrm{gst}}$ is contained in the zero section $0_{\tilde{\mathcal{Y}}_{\mathrm{gst}}} \subset \tilde{\mathcal{Y}}_{\mathrm{gst}}$. We let $U = \tilde{\mathcal{Y}}_{\mathrm{gst}} - D(\tilde\sigma_{\mathrm{gst}})$, and form

$$\tilde V_{\tilde\sigma_{\mathrm{gst}}} = \tilde V|_{D(\tilde\sigma_{\mathrm{gst}})} \cup \ker\{\tilde\sigma_{\mathrm{gst}}|_U : \tilde V|_U \to \mathcal{O}_U\} \subset \tilde V. \tag{6.7}$$

(This is consistent with the convention (2.7).) The cosection $\tilde\sigma_{\mathrm{gst}}$ defines a localized Gysin map (cf. [**KL**, Coro 2.9])

$$0^!_{\tilde\sigma_{\mathrm{gst}},\mathrm{loc}} : A_*\tilde V_{\tilde\sigma_{\mathrm{gst}}} \longrightarrow A_* D(\tilde\sigma_{\mathrm{gst}}), \tag{6.8}$$

which is useful because of the following Proposition, to be proved in the next section.

Proposition 6.3. *There is a cycle* $[C_{\mathrm{gst}}] \in Z_*\tilde V(\tilde\sigma_{\mathrm{gst}})$ *such that*

$$0^!_{\tilde\sigma,\mathrm{loc}}[\mathbf{C}_{\mathrm{gst}}] = 0^!_{\tilde\sigma_{\mathrm{gst}},\mathrm{loc}}[C_{\mathrm{gst}}] \in A_*D(\tilde\xi).$$

We remark that $[\mathbf{C}_{\mathrm{gst}}]$ is a cycle in the bundle-stack $h^1/h^0(E_{\tilde{\mathcal Y}/\tilde{\mathcal D}})|_{\tilde{\mathcal Y}_{\mathrm{gst}}}$ over $\tilde{\mathcal Y}_{\mathrm{gst}}$ while $[C_{\mathrm{gst}}]$ is a cycle in the vector bundle $\tilde V$ over $\tilde{\mathcal Y}_{\mathrm{gst}}$. Working with cycles in $\tilde V$ will free us from those technicalities due to working with $h^1/h^0(\cdot)$.

As indicated in the beginning of this section, we will form a compactification of $\tilde{\mathcal Y}_{\mathrm{gst}}$ and reduce $0^!_{\tilde\sigma_{\mathrm{gst}},\mathrm{loc}}$ to a classical Gysin map. We let

$$\bar\gamma : \tilde{\mathcal Y}^{\mathrm{cpt}}_{\mathrm{gst}} = \mathbf{P}(L \oplus \mathcal{O}_{\tilde{\mathcal X}_{\mathrm{gst}}}) \longrightarrow \tilde{\mathcal X}_{\mathrm{gst}} \tag{6.9}$$

be the obvious compactification of $\tilde{\mathcal Y}_{\mathrm{gst}}$. (Here the superscript "cpt" stands for "compactification".) We let $\mathcal D_\infty = \mathbf{P}(L \oplus 0)$, called the infinite-divisor of $\tilde{\mathcal Y}^{\mathrm{cpt}}_{\mathrm{gst}}$; thus $\tilde{\mathcal Y}_{\mathrm{gst}} = \tilde{\mathcal Y}^{\mathrm{cpt}}_{\mathrm{gst}} - \mathcal D_\infty$. We still view $\tilde{\mathcal X}_{\mathrm{gst}}$ as a substack of $\tilde{\mathcal Y}^{\mathrm{cpt}}_{\mathrm{gst}}$ via $\tilde{\mathcal X}_{\mathrm{gst}} = \mathbf{P}(0 \oplus \mathcal O_{\tilde{\mathcal X}_{\mathrm{gst}}}) \subset \tilde{\mathcal Y}^{\mathrm{cpt}}_{\mathrm{gst}}$, consistent with the 0-section embedding $\tilde{\mathcal X}_{\mathrm{gst}} \subset \tilde{\mathcal Y}_{\mathrm{gst}}$.

We extend $\tilde V_1$ and $\tilde V_2$ to $\tilde{\mathcal Y}^{\mathrm{cpt}}_{\mathrm{gst}}$ via

$$\tilde V_1^{\mathrm{cpt}} = \bar\gamma^* V_1(-\mathcal D_\infty) \quad \text{and} \quad \tilde V_2^{\mathrm{cpt}} = \bar\gamma^* V_2. \tag{6.10}$$

We let $\bar\xi_2 = \bar\gamma^*\xi_2$, which is the extension of $\tilde\xi_2$. Because of the expression (6.5), $\tilde\xi_1$ extends to a homomorphism $\bar\xi_1 : \tilde V_1^{\mathrm{cpt}} = \bar\gamma^* V_1(-\mathcal D_\infty) \to \mathcal O_{\tilde{\mathcal Y}^{\mathrm{cpt}}_{\mathrm{gst}}}$. Let

$$\bar\xi = (\bar\xi_1, \bar\xi_2) : \tilde V^{\mathrm{cpt}} := \tilde V_1^{\mathrm{cpt}} \oplus \tilde V_2^{\mathrm{cpt}} \longrightarrow \mathcal O_{\tilde{\mathcal Y}^{\mathrm{cpt}}_{\mathrm{gst}}}. \tag{6.11}$$

Because $\tilde\xi_1$ has the form stated in Lemma 6.2, and ξ_1 is surjective, $\bar\xi_1$ is surjective along $\mathcal D_\infty$. Consequently, the non-surjective locus of $\bar\xi$ and $\tilde\sigma_{\mathrm{gst}}$ are identical; namely $D(\bar\xi) = D(\tilde\sigma_{\mathrm{gst}}) \subset \tilde{\mathcal X}_{\mathrm{gst}}$.

Lemma 6.4. *Let* $\iota_! : Z_*\tilde V_{\tilde\sigma_{\mathrm{gst}}} \to Z_*\tilde V^{\mathrm{cpt}}$ *be the linear map that sends any closed integral (cycle)* $[C] \in Z_*\tilde V_{\tilde\sigma_{\mathrm{gst}}}$ *to the cycle of its closure in* $\tilde V^{\mathrm{cpt}}$*:* $\iota_![C] = [\overline{C}] \in Z_*\tilde V^{\mathrm{cpt}}$. *Let* $\tau : D(\tilde\sigma_{\mathrm{gst}}) \to \tilde{\mathcal X}_{\mathrm{gst}}$ *be the inclusion. Then we have*

$$\bar\gamma_* \circ 0^!_{\tilde V^{\mathrm{cpt}}} \circ \iota_! = \tau_* \circ 0^!_{\tilde\sigma_{\mathrm{gst}},\mathrm{loc}} : Z_*\tilde V_{\tilde\sigma_{\mathrm{gst}}} \longrightarrow A_*\tilde{\mathcal X}_{\mathrm{gst}}.$$

Proof. We let $\tilde V^{\mathrm{cpt}}_{\bar\xi}$ be defined similar to (2.7). Let

$$0^!_{\bar\xi,\mathrm{loc}} : Z_*\tilde V^{\mathrm{cpt}}_{\bar\xi} \longrightarrow A_*D(\bar\xi) = A_*D(\tilde\sigma_{\mathrm{gst}})$$

be the localized Gysin map associated to $\bar\xi$. Let $\tau' : \tilde V^{\rm cpt}_{\bar\xi} \to \bar V$ and $\tau'' : D(\bar\xi) \to \tilde{\mathcal Y}^{\rm cpt}_{\rm gst}$ be the inclusions. By [**KL**, Prop 1.3], we have the commutative square

$$\begin{array}{ccc} Z_*\tilde V^{\rm cpt}_{\bar\xi} & \xrightarrow{0^!_{\bar\xi,\rm loc}} & A_*D(\bar\xi) \\ \Big\downarrow{\scriptstyle \tau'_*} & & \Big\downarrow{\scriptstyle \tau''_*} \\ Z_*\tilde V^{\rm cpt} & \xrightarrow{0^!_{\tilde V^{\rm cpt}}} & A_*\tilde{\mathcal Y}^{\rm cpt}_{\rm gst}. \end{array}$$

On the other hand, since $\bar\xi$ is an extension of $\tilde\sigma_{\rm gst}$, the homomorphism $\iota_! : Z_*\tilde V_{\tilde\sigma_{\rm gst}} \to Z_*\tilde V^{\rm cpt}$ factors through $\iota'_! : Z_*\tilde V_{\tilde\sigma_{\rm gst}} \to Z_*\tilde V^{\rm cpt}(\bar\xi)$. Composing, we obtain

$$\tau''_* \circ 0^!_{\bar\xi,\rm loc} \circ \iota'_! = 0^!_{\tilde V^{\rm cpt}} \circ \tau'_* \circ \iota'_! = 0^!_{\tilde V^{\rm cpt}} \circ \iota_!.$$

Since $\bar\xi$ is an extension of $\tilde\sigma_{\rm gst}$ and $D(\bar\xi) = D(\tilde\sigma_{\tilde{\mathcal Y}_{\rm gst}})$, tracing through the construction of the localized Gysin maps in [**KL**, Sec 2], we conclude $0^!_{\bar\xi,\rm loc} \circ \iota'_! = 0^!_{\tilde\sigma_{\rm gst},\rm loc}$. Composed with $\bar\gamma_* : A_*\tilde{\mathcal Y}^{\rm cpt}_{\rm gst} \to A_*\tilde{\mathcal X}_{\rm gst}$, we obtain

$$\bar\gamma_* \circ 0^!_{\tilde V^{\rm cpt}} \circ \iota_! = \bar\gamma_* \circ \tau''_* \circ 0^!_{\bar\xi,\rm loc} \circ \iota'_! = \tau_* \circ 0^!_{\tilde\sigma_{\rm gst},\rm loc} : Z_*\tilde V_{\tilde\sigma_{\rm gst}} \longrightarrow A_*\tilde{\mathcal X}_{\rm gst}.$$

This proves the Lemma. q.e.d.

Corollary 6.5. *Let $C \subset \tilde V_{\tilde\xi}$ be any closed integral substack; let $\overline C \subset \tilde V^{\rm cpt}$ be its closure, and let $\overline C_b = \overline C \cap (0 \oplus \tilde V_2^{\rm cpt})$. We let $N_{\overline C_b}\overline C$ be the normal cone to $\overline C_b$ in $\overline C$, which is a closed substack in $\tilde V^{\rm cpt}$. Then we have*

$$0^!_{\tilde\sigma_{\rm gst},\rm loc}[C] = 0^!_{\tilde V^{\rm cpt}}[\overline C] = 0^!_{\tilde V^{\rm cpt}}[N_{\overline C_b}\overline C]. \tag{6.12}$$

To make use of this Corollary, we need to know more about the intersection $\overline C_b$. It turns out that $\overline C_b$ has a simple answer in case C is homogeneous which we define now. Since $\tilde{\mathcal Y}_{\rm gst}$ is the total space of the line bundle L and $\tilde V_i = \gamma^* V_i$, we define the dilation morphism $\mathbf m_t$ and the homomorphism $\Phi_{i,0}(t)$ to be

$$\begin{array}{ccc} \tilde V_i & \xrightarrow{\Phi_{i,0}(t)} & \tilde V_i \\ \Big\downarrow{\scriptstyle \rm pr} & & \Big\downarrow{\scriptstyle \rm pr} \\ \tilde{\mathcal Y}_{\rm gst} & \xrightarrow{\mathbf m_t} & \tilde{\mathcal Y}_{\rm gst}, \end{array} \tag{6.13}$$

where $\mathbf m_t$ sends $x \in L|_{x'}$ (over $x' \in \tilde{\mathcal X}_{\rm gst}$) to $tx \in L|_{x'} \subset W$; $\Phi_{i,0}(t)$ keeps the above square commutative and leaves $\gamma^* e$ invariant for any $e \in V_i$.

The collection $\{\mathbf m_t\}_{t\in\mathbb C^*}$ defines a $\mathbb C^*$ action on $\tilde{\mathcal Y}_{\rm gst}$, making the projection $\gamma : \tilde{\mathcal Y}_{\rm gst} \to \tilde{\mathcal X}_{\rm gst}$ a $\mathbb C^*$-equivariant morphism with $\mathbb C^*$ acting trivially on $\tilde{\mathcal X}_{\rm gst}$. The fixed locus $(\tilde{\mathcal Y}_{\rm gst})^{\mathbb C^*}$ is the 0-section $\tilde{\mathcal X}_{\rm gst} \subset \tilde{\mathcal Y}_{\rm gst}$. For

$k \in \mathbb{Z}$, we define

$$\Phi_{i,k}(t) = t^k \cdot \Phi_{i,0}(t) : \tilde{V}_i \longrightarrow \tilde{V}_i. \tag{6.14}$$

Definition 6.6. A closed integral substack $C \subset \tilde{V}$ is homogeneous of weight (k_1, k_2) if it is invariant under $(\Phi_{1,k_1}(t), \Phi_{2,k_2}(t))$ for all $t \in \mathbb{C}^*$. We say a cycle $\alpha \in Z_*\tilde{V}$ is homogeneous of weight (k_1, k_2) if each of its integral components is homogeneous of weight (k_1, k_2).

First we state the following Proposition, which we will prove in the next section.

Proposition 6.7. *The cycle* $[C_{\mathrm{gst}}] \in Z_*\tilde{V}$ *mentioned in Proposition* 6.3 *can be made to be homogeneous of weight* $(0, 1)$.

We let $C \subset \tilde{V}$ be an integral component of C_{gst}, and let $\overline{C} \subset \tilde{V}^{\mathrm{cpt}}$ be its closure. We intend to find the structure of $\overline{C} \cap (0 \oplus \tilde{V}_2^{\mathrm{cpt}})$. We let $\phi_2 : \tilde{V}_2^{\mathrm{cpt}} \to V_2$ be the projection induced by the isomorphism $\tilde{V}_2^{\mathrm{cpt}} = \bar{\gamma}^* V_2$ (cf. (6.10)).

Lemma 6.8. *Let* $C \subset \tilde{V}$ *be an integral component of* C_{gst} *and let* $\overline{C} \subset \tilde{V}^{\mathrm{cpt}}$ *be its closure. Then there is a closed substack* $B \subset V_2$ *such that*

$$\overline{C} \cap (0 \oplus \tilde{V}_2^{\mathrm{cpt}})|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}^{\mathrm{cpt}} - \tilde{\mathcal{X}}_{\mathrm{gst}}} = \phi_2^{-1}(B)|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}^{\mathrm{cpt}} - \tilde{\mathcal{X}}_{\mathrm{gst}}}.$$

Proof. To prove the Lemma, we only need to show that $\tilde{\mathcal{X}}_{\mathrm{gst}}$ can be covered by étale charts $M \to \tilde{\mathcal{X}}_{\mathrm{gst}}$ so that for each such chart we can find a closed $B_M \subset V_2|_M$ such that

$$\phi_2^{-1}(B_M)|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}^{\mathrm{cpt}} - \tilde{\mathcal{X}}_{\mathrm{gst}}} = \big((\overline{C} \cap (0 \oplus \tilde{V}_2^{\mathrm{cpt}})) \times_{\tilde{\mathcal{X}}_{\mathrm{gst}}} M\big)|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}^{\mathrm{cpt}} - \tilde{\mathcal{X}}_{\mathrm{gst}}}.$$

Let $M = \operatorname{Spec} A \to \tilde{\mathcal{X}}_{\mathrm{gst}}$ be an affine étale chart; we abbreviate $U_i = \tilde{V}_i^{\mathrm{cpt}} \times_{\tilde{\mathcal{X}}_{\mathrm{gst}}} M$ and $U = \tilde{V}^{\mathrm{cpt}} \times_{\tilde{\mathcal{X}}_{\mathrm{gst}}} M$; they are vector bundles over $\overline{W} := \tilde{\mathcal{Y}}_{\mathrm{gst}}^{\mathrm{cpt}} \times_{\tilde{\mathcal{X}}_{\mathrm{gst}}} M$. We denote $W = \tilde{\mathcal{Y}}_{\mathrm{gst}} \times_{\tilde{\mathcal{X}}_{\mathrm{gst}}} M$ and $L|_M = L \times_{\tilde{\mathcal{X}}_{\mathrm{gst}}} M$, etc..

Possibly after shrinking M if necessary, we can find trivializations $L|_M \cong \mathcal{O}_M$ and $V_i|_M \cong \mathcal{O}_M^{\oplus n_i}$. Using such trivializations, we have induced isomorphisms

$$\overline{W} = M \times \mathbb{P}^1, \quad U_i \cong (M \times \mathbb{P}^1) \times \mathbb{A}^{n_i}, \quad \text{and} \quad U \cong (M \times \mathbb{P}^1) \times \mathbb{A}^{n_1} \times \mathbb{A}^{n_2}. \tag{6.15}$$

We denote $0_W = M \times 0 \subset W$, and continue to denote $D_\infty = M \times \{\infty\} = \overline{W} - W$.

We let t (resp. $x = (x_i)$; resp. $y = (y_j)$) be the standard coordinate variable(s) of $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$ (resp. of $\mathbb{A}^{n_1}$; resp. of $\mathbb{A}^{n_2}$). Let $C_M = C \times_{\tilde{\mathcal{X}}_{\mathrm{gst}}} M$; because it is homogeneous of weight $(0, 1)$, the ideal of $C_M|_{W - 0_W} \subset U|_{W - 0_W}$ is generated by elements

$$\{f(x_1, \cdots, x_{n_1}, t^{-1}y_1, \cdots, t^{-1}y_{n_2}) \mid f \in J \subset A[x, y]\},$$

where J is an ideal in the polynomial ring $A[x,y] = A[x_1,\cdots,x_{n_1}, y_1,\cdots,y_{n_2}]$.

We now pick a new trivialization of U_1 over $\overline{W}-0_W$. We let $\epsilon_1,\cdots,\epsilon_{n_1}$ be the basis of U_1 that is the pullback of the standard basis of $V_1|_M \cong \mathcal{O}_M^{\oplus n_1}$. As t^{-1} extends to a regular function near D_∞ and with order one vanishing along D_∞, the collection $\{e_i = t^{-1}\cdot\epsilon_i\}_{1\le i\le n_1}$ forms a basis of $U_1|_{\overline{W}-0_W}$.

We let

$$U|_{\overline{W}-0_W} \cong M\times(\mathbb{P}^1-0)\times\mathbb{A}^{n_1}\times\mathbb{A}^{n_2} \tag{6.16}$$

be the isomorphism induced by the given trivializations of $L|_M$ and $V_2|_M$, and the new trivialization of U_1 using the basis e_i. We let $x'=(x'_i)$ be the standard coordinate variables of $\mathbb{A}^{n_1}$ in (6.16) (under the basis e_i). Then x_i and x'_i are related by $x_i = t^{-1}x'_i$. Thus in the coordinates (x'_i, y_j), $C_M|_{W-0_W}$ is defined by the ideal generated by

$$\{f(t^{-1}x'_1,\cdots,t^{-1}x'_{n_1},t^{-1}y_1,\cdots,t^{-1}y_{n_2}) \mid f\in J\subset A[x,y]\}. \tag{6.17}$$

Since $C\subset\tilde{V}$ is a cone, J is a homogeneous ideal; thus the same ideal is also generated by

$$\{f(x'_1,\cdots,x'_{n_1},y_1,\cdots,y_{n_2}) \mid f\in J\subset A[x,y]\}. \tag{6.18}$$

Therefore, denoting $\overline{C}_M = \overline{C}\times_{\tilde{\mathcal{X}}_{\mathrm{gst}}} M$, $\overline{C}_M\cap U|_{W-0_W}\subset U|_{W-0_W}$ is defined by the ideal generated by (6.18). Thus $\overline{C}\cap U|_{\overline{W}-0_W} = \phi^{-1}(B')|_{\overline{W}-0_W}$ for a $B'\subset M\times\mathbb{A}^{n_1}\times\mathbb{A}^{n_2}$ and $\phi: U|_{\overline{W}-0_W}\to M\times\mathbb{A}^{n_1}\times\mathbb{A}^{n_2}$ the tautological projection. Intersecting with $U_2|_{\overline{W}-0_W}$ proves the Lemma. q.e.d.

7. The cycle C_{gst}

We first construct the desired cycle C_{gst}, which proves Propositions 6.3. As $E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}$ has locally free H^0 and H^1, we have a canonical smooth quotient morphism

$$h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}} \longrightarrow H^1(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}} = \tilde{V} \tag{7.1}$$

which fiberwise is the morphism $[H^1(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}|_y)/H^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}|_y)] \to H^1(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}|_y)$ obtained by taking coarse moduli. The desired cycle $[C_{\mathrm{gst}}]\in Z_*(\tilde{V})$ will be constructed as "the coarse moduli" of $[\mathbf{C}_{\mathrm{gst}}]\in Z_*(h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}})$.

As mentioned before, we can find a complex of locally free sheaves $[F^0\to F^1]$ of $\mathcal{O}_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}$-modules so that $[F^0\to F^1]\cong_{q.i.} R^\bullet\pi_{\tilde{\mathcal{Y}}*}(\mathcal{L}_{\tilde{\mathcal{Y}}}^{\oplus 5}\oplus\mathcal{P}_{\tilde{\mathcal{Y}}})|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}$. We introduce

$$\tilde{\mathbf{V}}_1 = h^1/h^0((R^\bullet\pi_{\tilde{\mathcal{Y}}*}\mathcal{L}_{\tilde{\mathcal{Y}}}^{\oplus 5})|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}),\quad \tilde{\mathbf{V}}_2 = h^1/h^0((R^\bullet\pi_{\tilde{\mathcal{Y}}*}\mathcal{P}_{\tilde{\mathcal{Y}}})|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}),$$

$$\tilde{\mathbf{V}} = \tilde{\mathbf{V}}_1 \times_{\tilde{\mathcal{Y}}_{\mathrm{gst}}} \tilde{\mathbf{V}}_2.$$

By the base change property of the h^1/h^0-construction, canonically $[F^1/F^0] \cong \tilde{\mathbf{V}} = h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}$. And by the construction of $\mathbf{C}_{\mathrm{gst}}$, we have $[\mathbf{C}_{\mathrm{gst}}] \in Z_*\tilde{\mathbf{V}}$.

We now construct $[C_{\mathrm{gst}}] \in Z_*\tilde{V}$. Let

$$\beta : F^1 \longrightarrow \tilde{\mathbf{V}} \quad \text{and} \quad \beta' : F^1 \longrightarrow \tilde{V} = R^1\pi_{\tilde{\mathcal{Y}}*}(\mathcal{L}_{\tilde{\mathcal{Y}}}^{\oplus 5} \oplus \mathcal{P}_{\tilde{\mathcal{Y}}})|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}} \tag{7.2}$$

be the tautological projections. By their definitions, both are flat. Because $R^1\pi_{\tilde{\mathcal{Y}}*}(\mathcal{L}_{\tilde{\mathcal{Y}}}^{\oplus 5} \oplus \mathcal{P}_{\tilde{\mathcal{Y}}})|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}$ is locally free, for $\alpha : F^0 \to F^1$ the arrow in $[F^0 \to F^1]$, $\alpha(F^0) \subset F^1$ is a vector subbundle. Let $[C_F] = \beta^*[\mathbf{C}_{\mathrm{gst}}]$ be the flat pullback. Then $[C_F]$ is invariant under the $\alpha(F^0)$-action on F^1 (via the subbundle structure $\alpha(F^0) \subset F^1$). Since $\alpha(F^0)$ acts freely on F^1 and such that $\tilde{V} = F^1/\alpha(F^0)$, we define

$$[C_{\mathrm{gst}}] := [C_F]/\alpha(F^0) \in Z_*\big(F^1/\alpha(F^0)\big) = Z_*\tilde{V}. \tag{7.3}$$

Proof of Proposition 6.3. We will show that $[C_{\mathrm{gst}}] \in Z_*\tilde{V}_{\tilde{\xi}}$ and $0^!_{\tilde{\sigma},\mathrm{loc}}[\mathbf{C}_{\mathrm{gst}}] = 0^!_{\tilde{\xi},\mathrm{loc}}[C_{\mathrm{gst}}]$. First, because the cosection $\tilde{\sigma} : h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}) \to \mathcal{O}_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}$ is induced by the cosection $\tilde{\xi} : \tilde{V} = h^1(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}) \to \mathcal{O}_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}$, for β and β' in (7.2), we have $\tilde{\sigma} \circ \beta = \tilde{\xi} \circ \beta'$. Therefore,

$$D(\tilde{\sigma} \circ \beta) = D(\tilde{\xi} \circ \beta') = D(\tilde{\xi}).$$

Here the second equality holds because β' is surjective.

Since $[\mathbf{C}_{\mathrm{gst}}] \in Z_*h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})_{\tilde{\sigma}}$ and $[C_F] = \beta^*[\mathbf{C}_{\mathrm{gst}}]$, we have $[C_F] \in Z_*F^1_{\tilde{\sigma}\circ\beta}$ and

$$0^!_{\tilde{\sigma},\mathrm{loc}}[\mathbf{C}_{\mathrm{gst}}] = 0^!_{\tilde{\sigma}\circ\beta,\mathrm{loc}}[C_F] \in A_*D(\tilde{\sigma} \circ \beta),$$

which is equal to $0^!_{\tilde{\xi}\circ\beta',\mathrm{loc}}[C_F]$ because $\tilde{\sigma} \circ \beta = \tilde{\xi} \circ \beta'$. Finally, since $[C_F] = \beta'^*[C_{\mathrm{gst}}]$ by (7.3), we have $[C_{\mathrm{gst}}] \in Z_*\tilde{V}_{\tilde{\xi}}$, and

$$0^!_{\tilde{\xi}\circ\beta',\mathrm{loc}}[C_F] = 0^!_{\tilde{\xi},\mathrm{loc}}[C_{\mathrm{gst}}] \in A_*D(\tilde{\xi}).$$

This proves the Proposition. q.e.d.

We next show that the cycle C_{gst} constructed is homogeneous of weight $(0, 1)$.

Proof of Proposition 6.7. As the obstruction theory of $\tilde{\mathcal{Y}} \to \tilde{\mathcal{D}}$ is the pullback of that of $\mathcal{Y} \to \mathcal{D}$, and the later is via the open $\mathcal{D}$-embedding (cf. (2.1))

$$\jmath : \mathcal{Y} = \overline{M}_1(\mathbb{P}^4, d)^p \xrightarrow{\subset} \mathfrak{S} := C(\pi_*(\mathcal{L}^{\oplus 5} \oplus \mathcal{P})), \tag{7.4}$$

we will prove a corresponding result for $\mathfrak{S}$.

Let $\pi_{\mathfrak{S}}:\mathcal{C}_{\mathfrak{S}}\to\mathfrak{S}$, $\mathcal{L}_{\mathfrak{S}}$ and $\mathcal{P}_{\mathfrak{S}}$ be the pullback of $(\pi:\mathcal{C}\to\mathcal{D},\mathcal{L},\mathcal{P})$. The construction of $\mathfrak{S}$ provides us a universal section

$$(u_{\mathfrak{S},1},\cdots,u_{\mathfrak{S},5},\psi_{\mathfrak{S}})\in\Gamma(\mathcal{C}_{\mathfrak{S}},\mathcal{L}_{\mathfrak{S}}^{\oplus 5}\oplus\mathcal{P}_{\mathfrak{S}})=\Gamma(\mathfrak{S},\pi_{\mathfrak{S}*}(\mathcal{L}_{\mathfrak{S}}^{\oplus 5}\oplus\mathcal{P}_{\mathfrak{S}})).\tag{7.5}$$

Namely, over each closed $z=(\mathcal{C}_z,u_{z,i},\psi_z)\in\mathfrak{S}$, we have $u_{\mathfrak{S},i}|_z\equiv u_{z,i}$ and $\psi_{\mathfrak{S}}|_z\equiv\psi_z$.

We define

$$\mathfrak{a}_t:\mathcal{L}^{\oplus 5}\oplus\mathcal{P}\longrightarrow\mathcal{L}^{\oplus 5}\oplus\mathcal{P},\quad t\in\mathbb{C}^*,$$

so that it is $\mathrm{id}_{\mathcal{L}^{\oplus 5}}$ when restricted to the summand $\mathcal{L}^{\oplus 5}$, and is $t\cdot\mathrm{id}_{\mathcal{P}}$ when restricted to the summand $\mathcal{P}$; they define a $\mathbb{C}^*$-action on $\mathcal{L}^{\oplus 5}\oplus\mathcal{P}$, where $\mathbb{C}^*$ acts trivially on $\mathcal{D}$.

Following the construction of $\mathfrak{S}$, the automorphism $\mathfrak{a}_t$ induces a $\mathcal{D}$-automorphism $\tilde{\mathfrak{a}}_t:\mathfrak{S}\to\mathfrak{S}$; it sends $(\mathcal{C}_z,\mathcal{L}_z,u_{z,i},\psi_z)\in\mathfrak{S}(\mathbb{C})$ to $(\mathcal{C}_z,\mathcal{L}_z,u_{z,i},\psi_z)^t=(\mathcal{C}_z,\mathcal{L}_z,u_{z,i},t\cdot\psi_z)$. The collection $\{\tilde{\mathfrak{a}}_t\mid t\in\mathbb{C}^*\}$ defines a $\mathbb{C}^*$-action on $\mathfrak{S}$; making $\mathfrak{S}\to\mathcal{D}$ a $\mathbb{C}^*$-equivariant projection.

Because $\mathcal{C}_{\mathfrak{S}}=\mathcal{C}\times_{\mathcal{D}}\mathfrak{S}$, and because $\mathfrak{S}\to\mathcal{D}$ is $\mathbb{C}^*$-equivariant, the trivial $\mathbb{C}^*$-action on $\mathcal{C}$ and the $\mathbb{C}^*$-action $\tilde{\mathfrak{a}}_t$ on $\mathfrak{S}$ lifts to a $\mathbb{C}^*$-action on $\mathcal{C}_{\mathfrak{S}}\to\mathfrak{S}$. We denote this action by $\varphi_0(t):\mathcal{C}_{\mathfrak{S}}\to\mathcal{C}_{\mathfrak{S}}$. Then since $\mathcal{L}_{\mathfrak{S}}$ and $\mathcal{P}_{\mathfrak{S}}$ are pullbacks of $\mathcal{L}$ and $\mathcal{P}$ on $\mathcal{C}$, $\mathcal{L}_{\mathfrak{S}}$ and $\mathcal{P}_{\mathfrak{S}}$ admit the obvious $\mathbb{C}^*$-linearizations

$$\varphi_1(t):\mathcal{L}_{\mathfrak{S}}\longrightarrow\mathcal{L}_{\mathfrak{S}},\quad\varphi_2(t):\mathcal{P}_{\mathfrak{S}}\longrightarrow\mathcal{P}_{\mathfrak{S}}\tag{7.6}$$

lifting the $\mathbb{C}^*$-action φ_0 on $\mathcal{C}_{\mathfrak{S}}$ so that their $\mathbb{C}^*$-invariant sections are pullback sections of $\mathcal{L}$ and $\mathcal{P}$, respectively.

We define another pair of $\mathbb{C}^*$-linearizations:

$$\bar\varphi_1(t)=\varphi_1(t):\mathcal{L}_{\mathfrak{S}}\to\mathcal{L}_{\mathfrak{S}},\quad\bar\varphi_2(t)=t\cdot\varphi_2(t):\mathcal{P}_{\mathfrak{S}}\to\mathcal{P}_{\mathfrak{S}}\tag{7.7}$$

lifting the $\mathbb{C}^*$-action φ_0 on $\mathcal{C}_{\mathfrak{S}}$. By this construction, the universal section (7.5) is $\mathbb{C}^*$-invariant under the linearization $(\bar\varphi_1,\bar\varphi_2)$:

$$(\bar\varphi_1(t)(u_{\mathfrak{S},i}),\bar\varphi_2(t)(\psi_{\mathfrak{S}}))=(u_{\mathfrak{S},i}\circ\varphi_0(t),\psi_{\mathfrak{S}}\circ\varphi_0(t)).\tag{7.8}$$

Let $\phi_{\mathfrak{S}/\mathcal{D}}:L^{\bullet}_{\mathfrak{S}/\mathcal{D}}\to(E_{\mathfrak{S}/\mathcal{D}})^{\vee}$ be the perfect relative obstruction theory constructed in [**CL1**, Prop 2.5] using $(u_{\mathfrak{S},i},\psi_{\mathfrak{S}})$. Tracing through the construction of the obstruction theory $\phi_{\mathfrak{S}/\mathcal{D}}$ in [**CL1**, Prop 2.5], using that the universal section $(u_{\mathfrak{S},i},\psi_{\mathfrak{S}})$ is invariant under the $\mathbb{C}^*$-linearization $(\bar\varphi_1,\bar\varphi_2)$, we conclude that the obstruction theory $\phi_{\mathfrak{S}/\mathcal{D}}$ is $\mathbb{C}^*$-equivariant under $(\bar\varphi_1,\bar\varphi_2)$.

We now show that this invariance of $\phi_{\mathfrak{S}/\mathcal{D}}$ implies the $\mathbb{C}^*$-invariance of $\phi_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$. We consider the composite

$$\tilde\jmath:\tilde{\mathcal{Y}}\longrightarrow\mathcal{Y}\xrightarrow{\jmath}\mathfrak{S}$$

of the tautological $\tilde{\mathcal{Y}}\to\mathcal{Y}$ with the open embedding $\jmath$ in (7.4). Since the obstruction theory of $\tilde{\mathcal{Y}}\to\tilde{\mathcal{D}}$ is the pullback of that of $\mathfrak{S}\to\mathcal{D}$, we have canonical isomorphism

$$\tilde{\jmath}^* E_{\mathfrak{S}/\mathcal{D}} \xrightarrow{\cong} E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}. \tag{7.9}$$

Since $\tilde{\jmath}$ is a $\mathcal{D}$-morphism and $\tilde{\mathcal{Y}}$ is constructed from $\mathcal{Y} \to \mathcal{D}$ via a base change $\tilde{\mathcal{D}} \to \mathcal{D}$. the $\mathbb{C}^*$-action on $\mathfrak{S}$ lifts to a $\mathbb{C}^*$-action on $\tilde{\mathcal{Y}}$ making $\tilde{\jmath}$ $\mathbb{C}^*$-equivariant. Using (7.9), we endow $E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$ the $\mathbb{C}^*$-linearlization induced by $(\bar{\varphi}_1, \bar{\varphi}_2)$. This way, since $\phi_{\mathfrak{S}/\mathcal{D}}$ is $\mathbb{C}^*$-equivariant (via $(\bar{\varphi}_1, \bar{\varphi}_2)$), the obstruction theory $\phi_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}} : (E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})^\vee \to L^\bullet_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$ is $\mathbb{C}^*$-equivariant.

Finally, a direct check shows that the introduced $\mathbb{C}^*$-action on $\tilde{\mathcal{Y}}$ restricting to $\tilde{\mathcal{Y}}_{\mathrm{gst}}$ is the $\mathbb{C}^*$-action $\mathfrak{m}_t$ constructed in (6.13); also, by [**CL1**, Prop 2.5], $E_{\mathfrak{S}/\mathcal{D}} = R^\bullet \pi_{\mathfrak{S}*}(\mathcal{L}_{\mathfrak{S}}^{\oplus 5} \oplus \mathcal{P}_{\mathfrak{S}})$, and the introduced $\mathbb{C}^*$-linearization on $E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}} \cong \tilde{\jmath}^* E_{\mathfrak{S}/\mathcal{D}}$ restricting to $\tilde{\mathcal{Y}}_{\mathrm{gst}}$ is the linearization $(\Phi_{1,0}, \Phi_{2,1})$ of $\tilde{V}_1 \oplus \tilde{V}_2$, in the notation of (6.14).

Since $\phi_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$ is $\mathbb{C}^*$-equivariant, the intrinsic normal cone $\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}} \subset h^1/h^1(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})$ is $\mathbb{C}^*$-equivariant; therefore C_{gst} is $\mathbb{C}^*$-equivariant under the $\mathbb{C}^*$-action $(\Phi_{1,0}, \Phi_{2,1})$, which is equivalent to say that C_{gst} is homogeneous of weight $(0,1)$. q.e.d.

To prove that only one component of $\mathcal{Y}_{\mathrm{gst}}$ has non-trivial contribution to the GW-invariant $N_1(d)$, we need a finer result on the intersection $C_{\mathrm{gst}} \cap (0 \oplus \tilde{V}_2)$. Its statement and proof will occupy the remainder of this section.

We introduce more notation. Let $\Delta_{\tilde{\mathcal{X}}} = \tilde{\mathcal{X}}_{\mathrm{pri}} \cap \tilde{\mathcal{X}}_{\mathrm{gst}}$, and recall

$$\Delta_{\tilde{\mathcal{Y}}} := \tilde{\mathcal{Y}} \times_{\tilde{\mathcal{X}}} \Delta_{\tilde{\mathcal{X}}} = \tilde{\mathcal{Y}}_{\mathrm{gst}} \times_{\tilde{\mathcal{X}}_{\mathrm{gst}}} \Delta_{\tilde{\mathcal{X}}}$$

is a line bundle over $\Delta_{\tilde{\mathcal{X}}}$. We continue to denote by $\gamma : \tilde{\mathcal{Y}}_{\mathrm{gst}} \to \tilde{\mathcal{X}}_{\mathrm{gst}}$ the projection, and denote $0_{\tilde{V}_2} = \tilde{\mathcal{Y}}_{\mathrm{gst}}$ the zero section of $\tilde{V}_2$.

Proposition 7.1. *There is a sub-line bundle $F \subset V_2|_{\Delta_{\tilde{\mathcal{X}}}}$ so that for $\tilde{F} = \gamma^* F \subset \tilde{V}_2|_{\Delta_{\tilde{\mathcal{Y}}}}$,*

$$C_{\mathrm{gst}} \cap (0 \oplus \tilde{V}_2) \subset 0_{\tilde{V}_2} \cup \tilde{F} = \tilde{\mathcal{Y}}_{\mathrm{gst}} \cup \tilde{F}.$$

Here by abuse of notation we use $\gamma^* F$ to denote the pullback of F via $\gamma|_{\Delta_{\tilde{\mathcal{Y}}}} : \Delta_{\tilde{\mathcal{Y}}} \to \Delta_{\tilde{\mathcal{X}}}$. Thus $\gamma^* F$ is a line bundle over $\Delta_{\tilde{\mathcal{Y}}}$.

The proposition will be proved via a sequence of Lemmas. First, following the argument in [**CL1**, Sec 5.2], the relative obstruction theories of the triple $(\tilde{\mathcal{Y}}, \tilde{\mathcal{X}}, \tilde{\mathcal{D}})$ fit into a compatible diagram of distinguished triangles:

$$\begin{tikzcd}
\mathfrak{p}^*(E_{\tilde{\mathcal{X}}/\tilde{\mathcal{D}}})^\vee \arrow[r] \arrow[d, "\mathfrak{p}^*\phi_{\tilde{\mathcal{X}}/\tilde{\mathcal{D}}}"] & (E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})^\vee \arrow[r] \arrow[d, "\phi_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}"] & (E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}})^\vee \arrow[r, "+1"] \arrow[d, "\phi_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}}"] & {} \\
\mathfrak{p}^* L^\bullet_{\tilde{\mathcal{X}}/\tilde{\mathcal{D}}} \arrow[r] & L^\bullet_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}} \arrow[r] & L^\bullet_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}} \arrow[r, "+1"] & {}
\end{tikzcd} \tag{7.10}$$

Here $E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$ is given by (2.13); $E_{\tilde{\mathcal{X}}/\tilde{\mathcal{D}}} = R^\bullet\pi_{\tilde{\mathcal{X}}*}\mathcal{L}_{\tilde{\mathcal{X}}}^{\oplus 5}$ and $E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}} = R^\bullet\pi_{\tilde{\mathcal{Y}}*}\mathcal{P}_{\tilde{\mathcal{Y}}}$. (As usual, $\mathcal{L}_{\tilde{\mathcal{Y}}} = f_{\tilde{\mathcal{Y}}}^*\mathcal{O}(1)$, $\mathcal{P}_{\tilde{\mathcal{Y}}} = \mathcal{L}_{\tilde{\mathcal{Y}}}^{\otimes(-5)} \otimes \omega_{\mathcal{C}_{\tilde{\mathcal{Y}}}/\tilde{\mathcal{Y}}}$, and $\mathfrak{p} : \tilde{\mathcal{Y}} \to \tilde{\mathcal{X}}$ is the projection.) Taking the cohomologies of the duals of the top row restricting to $\tilde{\mathcal{Y}}_{\mathrm{gst}}$, we obtain the exact sequences of sheaves on $\tilde{\mathcal{Y}}_{\mathrm{gst}}$:

$$\begin{array}{ccccc} H^1(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}}|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}) & \longrightarrow & H^1(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}) & \longrightarrow & H^1(\mathfrak{p}^*E_{\tilde{\mathcal{X}}/\tilde{\mathcal{D}}}|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}) \\ \| & & \| & & \| \\ \tilde{V}_2 & \xrightarrow{\tilde{\beta}_1} & \tilde{V} = \tilde{V}_1 \oplus \tilde{V}_2 & \xrightarrow{\tilde{\beta}_2} & \tilde{V}_1. \end{array} \tag{7.11}$$

Here the vertical identities are given by the explicit form of the complexes $E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$, etc.; $\tilde{\beta}_1$ (resp. $\tilde{\beta}_2$) is the tautological inclusion (resp. projection), and that the diagram commutes follows from the proof of [**CL1**, Prop 2.5 and 3.1].

Let $\mathfrak{N} = h^1/h^0((L^\bullet_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}})^\vee)$; because $\phi_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}}$ is a relative perfect obstruction theory of $\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}$, $\mathfrak{N}$ is a closed subcone-stack of $h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}})$. Similar to the perfect resolution of deformation complexes given in [**LT**, **Beh**], we can find a complex of locally free sheaves of $\mathcal{O}_{\tilde{\mathcal{Y}}}$-modules $[\alpha : F^0 \to F^1]$ so that $[F^0 \to F^1] \cong_{q.i.} E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}}$. Thus using the flat $\beta : F^1 \to h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}})$, we obtain the pullback $\beta^*(\mathfrak{N}) \subset F^1$. Restricting to $\tilde{\mathcal{Y}}_{\mathrm{gst}}$, and using that $h^1([F^0 \to F^1]|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}})$ is locally free, we obtain a closed subcone

$$\mathcal{N} := \big(\beta^*(\mathfrak{N}) \times_{\tilde{\mathcal{Y}}} \tilde{\mathcal{Y}}_{\mathrm{gst}}\big)/\alpha(F^0|_{\tilde{\mathcal{Y}}_{\mathrm{gst}}}) \subset \tilde{V}_2. \tag{7.12}$$

Here we used that $\beta^*(\mathfrak{N})$ is invariant under the (free) group action $\alpha(F^0)$ on F^1.

We quote a useful result. For any closed $y \in \tilde{\mathcal{Y}}_{\mathrm{gst}}$, letting $T^i_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}},y} = H^i((L^\bullet_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}})^\vee)|_y)$, then the paragraph before [**BF**, Lemma 4.6] gives

$$\mathfrak{N}_y := \mathfrak{N}|_y = h^1/h^0((L^\bullet_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}})^\vee)|_y = [T^1_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}},y}/T^0_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}},y}] = h^1/h^0((L^\bullet_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}})^\vee|_y), \tag{7.13}$$

where $T^0_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}},y}$ acts on $T^1_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}},y}$ trivially. Consequently, $\mathcal{N}|_y = T^1_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}},y} \subset \tilde{V}_2|_y$, where the inclusion is induced by $H^1(\phi^\vee_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}})$.

Lemma 7.2. *Viewing $\mathcal{N}$ as a substack of $\tilde{V} = \tilde{V}_1 \oplus \tilde{V}_2$ via $\tilde{\beta}_1$ in (7.11), then*

$$\mathrm{Supp}(C_{\mathrm{gst}}) \cap (0 \oplus \tilde{V}_2) \subset \mathcal{N}. \tag{7.14}$$

Proof. It suffices to show that for any closed $y \in \tilde{\mathcal{Y}}_{\mathrm{gst}}$, we have

$$\big(\mathrm{Supp}(C_{\mathrm{gst}}) \cap (0 \oplus \tilde{V}_2)\big)|_y \subset \mathcal{N}_y.$$

Dualizing (7.10), restricting it to y and taking its cohomology groups, we obtain the following commutative diagram

(7.15)
$$\begin{array}{ccccc} H^1((L^\bullet_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}})^\vee|_y) & \longrightarrow & H^1((L^\bullet_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})^\vee|_y) & \longrightarrow & H^1(\mathfrak{p}^*(L^\bullet_{\tilde{\mathcal{X}}/\tilde{\mathcal{D}}})^\vee|_y) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \varrho_y} & & \Big\downarrow \\ \tilde{V}_2|_y & \xrightarrow{\tilde{\beta}_1|_y} & (\tilde{V}_1\oplus\tilde{V}_2)|_y & \xrightarrow{\tilde{\beta}_2|_y} & \tilde{V}_1|_y\,, \end{array}$$

where the vertical arrows are H^1 of $\phi^\vee_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}}|_y$, $\phi^\vee_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}|_y$ and $\mathfrak{p}^*\phi^\vee_{\tilde{\mathcal{X}}/\tilde{\mathcal{D}}}|_y$, respectively, and the bottom line follows from (7.11). Since $\phi_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$, etc., are perfect obstruction theories, the vertical arrows in (7.15) are injective.

We now prove (7.14). First, the inclusion $\mathbf{C}_{\mathrm{gst}} \subset h^1/h^0((L^\bullet_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})^\vee)$ induces an inclusion of their respective fibers over y

$$\mathbf{C}_{\mathrm{gst}}|_y \subset h^1/h^0((L^\bullet_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})^\vee)|_y = h^1/h^0((L^\bullet_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})^\vee|_y),$$

where the equality follows from (7.13). Tracing through the construction of C_{gst} (from $\mathbf{C}_{\mathrm{gst}}$), we have $C_{\mathrm{gst}}|_y \subset H^1((L^\bullet_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})^\vee|_y)$. Hence

$$\big(\mathrm{Supp}(C_{\mathrm{gst}})\cap(0\oplus\tilde{V}_2)\big)|_y = \mathrm{Supp}(C_{\mathrm{gst}})|_y\cap(0\oplus\tilde{V}_2|_y) \subset \mathrm{Im}\,(\varrho_y)\cap\ker(\tilde{\beta}_2|_y).$$

Because in (7.15) all vertical arrows are injective and the squares are commutative, $\mathrm{Im}\,(\varrho_y)\cap\ker(\tilde{\beta}_2|_y) \subset 0\oplus\mathcal{N}_y \subset 0\oplus\tilde{V}_2|_y$. This proves (7.14). q.e.d.

Proof of Proposition 7.1. We only need to show that there is a sub-line bundle $F \subset V_2|_{\Delta_{\tilde{\mathcal{X}}}}$ so that $\mathcal{N} \subset \tilde{V}_2$ is the union of the zero section $0_{\tilde{V}_2} \subset \tilde{V}_2$ with the total space $\mathrm{Tot}(\gamma^*F) \subset \gamma^*(V_2|_{\Delta_{\tilde{\mathcal{X}}}}) = \tilde{V}_2|_{\Delta_{\tilde{\mathcal{Y}}}}$.

Let $\iota : \tilde{\mathcal{T}} \to \tilde{\mathcal{Y}}$ be a smooth chart of $\tilde{\mathcal{Y}}$ over a smooth chart $\tilde{\mathcal{S}} \to \tilde{\mathcal{X}}$ with $\tilde{\mathcal{T}} = \tilde{\mathcal{Y}} \times_{\tilde{\mathcal{X}}} \tilde{\mathcal{S}}$ and the embedding data $\tilde{\mathcal{Y}} \to Z'$, etc., given in Proposition 3.1. Then by base change,

$$\iota^* h^1/h^0((L^\bullet_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}})^\vee)|_{\tilde{\mathcal{T}}_{\mathrm{gst}}} \cong h^1/h^0((L^\bullet_{\tilde{\mathcal{T}}/\tilde{\mathcal{S}}})^\vee)|_{\tilde{\mathcal{T}}_{\mathrm{gst}}}. \tag{7.16}$$

Following the notation of Proposition 3.1, $Z'_{\tilde{\mathcal{S}}} := \tilde{\mathcal{S}} \times_Z Z' = \tilde{\mathcal{S}} \times \mathbb{A}^1$, and $\tilde{\mathcal{T}} \to Z'$ factor through a closed $\tilde{\mathcal{S}}$-embedding $\tilde{\mathcal{T}} \to Z'_{\tilde{\mathcal{S}}}$, defined by the vanishing of tz. Let $I = (tz)$ be the ideal of $\tilde{\mathcal{T}}$ in $Z'_{\tilde{\mathcal{S}}}$. By definition,

$$L^{\bullet\geq -1}_{\tilde{\mathcal{T}}/\tilde{\mathcal{S}}} \cong [I/I^2 \xrightarrow{\delta} \Omega_{Z'_{\tilde{\mathcal{S}}}/\tilde{\mathcal{S}}}|_{\tilde{\mathcal{T}}}], \tag{7.17}$$

where $\delta(\cdot) = d_{\tilde{\mathcal{S}}}(\cdot)$ is the relative differential that annihilates elements in $\mathcal{O}_{\tilde{\mathcal{S}}}$. We denote $\mathrm{S}^\bullet(\cdot)$ the symmetric product, and denote

$$N := \mathrm{Spec}\,\mathrm{S}^\bullet_{\mathcal{O}_{\tilde{\mathcal{T}}_{\mathrm{gst}}}}((I/I^2)\otimes_{\mathcal{O}_{\tilde{\mathcal{T}}}} \mathcal{O}_{\tilde{\mathcal{T}}_{\mathrm{gst}}}).$$

Following [**BF**, p 67], since $h^1/h^0((L^\bullet_{\tilde{\mathcal{T}}/\tilde{\mathcal{S}}})^\vee) = h^1/h^0((L^{\bullet\geq -1}_{\tilde{\mathcal{T}}/\tilde{\mathcal{S}}})^\vee)$, we have

$$(7.18)\quad \begin{aligned} h^1/h^0((L^\bullet_{\tilde{\mathcal{T}}/\tilde{\mathcal{S}}})^\vee)|_{\tilde{\mathcal{T}}_{\rm gst}} &= [(\operatorname{Spec} \mathrm{S}^\bullet_{\mathcal{O}_{\tilde{\mathcal{T}}}}(I/I^2))/(T_{Z'_{\tilde{\mathcal{S}}}/\tilde{\mathcal{S}}}|_{\tilde{\mathcal{T}}})]|_{\tilde{\mathcal{T}}_{\rm gst}} \\ &= [N/(T_{Z'_{\tilde{\mathcal{S}}}/\tilde{\mathcal{S}}}|_{\tilde{\mathcal{T}}_{\rm gst}})]. \end{aligned}$$

Here the $T_{Z'_{\tilde{\mathcal{S}}}/\tilde{\mathcal{S}}}|_{\tilde{\mathcal{T}}_{\rm gst}}$-action on N is induced by the arrow δ in (7.17).

We claim that the $T_{Z'_{\tilde{\mathcal{S}}}/\tilde{\mathcal{S}}}|_{\tilde{\mathcal{T}}_{\rm gst}}$-action on N is trivial. Indeed, since $I = (zt) \subset \mathcal{O}_{Z'_{\tilde{\mathcal{S}}}} = \mathcal{O}_{\tilde{\mathcal{S}}}[t]$, using $\tilde{\mathcal{T}}_{\rm gst} = (z=0)\cap Z'_{\tilde{\mathcal{S}}}$ and $z \in \Gamma(\mathcal{O}_{\tilde{\mathcal{S}}})$, we obtain $\delta|_{\tilde{\mathcal{T}}_{\rm gst}}(zt) = d_{\tilde{\mathcal{S}}}(zt)|_{\tilde{\mathcal{T}}_{\rm gst}} = z\cdot dt|_{\tilde{\mathcal{T}}_{\rm gst}} = 0$. This proves the claim.

We show that

$$(7.19)\qquad N \cong \mathcal{N} \times_{\tilde{\mathcal{Y}}_{\rm gst}} \tilde{\mathcal{T}}_{\rm gst}.$$

First, because $\tilde{\mathcal{T}}$ is affine and (in the notation of (7.12)) $H^1([F^0 \to F^1]|_{\tilde{\mathcal{Y}}_{\rm gst}})$ is locally free, we have

$$\begin{aligned} [F^0 \xrightarrow{\alpha} F^1]|_{\tilde{\mathcal{T}}_{\rm gst}} &= [F^0|_{\tilde{\mathcal{T}}_{\rm gst}} \xrightarrow{\alpha|_{\tilde{\mathcal{T}}_{\rm gst}}} F^1|_{\tilde{\mathcal{T}}_{\rm gst}}] \\ &\cong [\ker(\alpha|_{\tilde{\mathcal{T}}_{\rm gst}}) \xrightarrow{0} F^1|_{\tilde{\mathcal{T}}_{\rm gst}}/(\alpha(F^0|_{\tilde{\mathcal{T}}_{\rm gst}}))]. \end{aligned}$$

Thus the inclusion $\mathfrak{N}|_{\tilde{\mathcal{T}}_{\rm gst}} \subset h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}}|_{\tilde{\mathcal{T}}_{\rm gst}})$ is given by

$$[\mathcal{N}|_{\tilde{\mathcal{T}}_{\rm gst}}/H^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}}|_{\tilde{\mathcal{T}}_{\rm gst}})] \subset [H^1(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}}|_{\tilde{\mathcal{T}}_{\rm gst}})/H^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}}|_{\tilde{\mathcal{T}}_{\rm gst}})],$$

where the quotients are via trivial actions. Combined with (7.16) and (7.18), we conclude

$$[N/(T_{Z'_{\tilde{\mathcal{S}}}/\tilde{\mathcal{S}}}|_{\tilde{\mathcal{T}}_{\rm gst}})] \cong [\mathcal{N}|_{\tilde{\mathcal{Y}}_{\rm gst}}/H^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}}|_{\tilde{\mathcal{T}}_{\rm gst}})].$$

As both sides are quotients by trivial action of bundles over $\tilde{\mathcal{T}}_{\rm gst}$, we conclude (7.19).

We now determine N. Define $f: \mathcal{O}_{Z'} \to I/I^2 \otimes_{\mathcal{O}_{\tilde{\mathcal{T}}}} \mathcal{O}_{\tilde{\mathcal{T}}_{\rm gst}}$ to be the homomorphism via $f(1) = zt\otimes 1$; because $I = (zt)$, f is surjective; because $z = 0 \in \mathcal{O}_{\tilde{\mathcal{T}}_{\rm gst}}$, $f(z) = z^2t\otimes 1 = zt\otimes z = 0$, and because $w_iz = 0$ in $\mathcal{O}_{\tilde{\mathcal{S}}}$, $f(w_i) = w_izt\otimes 1 = 0$. Further, using that $\tilde{\mathcal{S}}_{\rm pri} = (w_1 = w_2 = w_3 = w_4 = 0) \subset Z$ is smooth and $(z=0)\cap\tilde{\mathcal{S}}_{\rm pri}$ is a divisor in $\tilde{\mathcal{S}}_{\rm pri}$ ((2) of Proposition 3.1), A direct check shows that $\ker f = (z, w_1, \cdots, w_4)$. Hence $I/I^2 \otimes_{\mathcal{O}_{\tilde{\mathcal{T}}}} \mathcal{O}_{\tilde{\mathcal{T}}_{\rm gst}} \cong \mathcal{O}_{\tilde{\mathcal{T}}_\Delta}$, where $\tilde{\mathcal{T}}_\Delta = \tilde{\mathcal{T}} \times_{\tilde{\mathcal{Y}}} \Delta_{\tilde{\mathcal{Y}}}$. Therefore,

$$N = \operatorname{Spec} \mathrm{S}^\bullet_{\mathcal{O}_{\tilde{\mathcal{T}}_{\rm gst}}}\big(I/I^2 \otimes_{\mathcal{O}_{\tilde{\mathcal{T}}}} \mathcal{O}_{\tilde{\mathcal{T}}_{\rm gst}}\big) \cong \operatorname{Spec} \mathrm{S}^\bullet_{\mathcal{O}_{\tilde{\mathcal{T}}_{\rm gst}}}(\mathcal{O}_{\tilde{\mathcal{T}}_\Delta})$$

(as $\tilde{\mathcal{T}}_{\rm gst}$ scheme) is the union of $\tilde{\mathcal{T}}_{\rm gst}$ with a line bundle $\tilde{F}_{\tilde{\mathcal{T}}}$ over $\tilde{\mathcal{T}}_\Delta$ so that $\tilde{\mathcal{T}}_{\rm gst}\cap \tilde{F}_{\tilde{\mathcal{T}}}$ is the zero section $0_{\tilde{F}_{\tilde{\mathcal{T}}}} \subset \tilde{F}_{\tilde{\mathcal{T}}}$.

Since $\tilde{\mathcal{T}} \to \tilde{\mathcal{Y}}$ is an arbitrary smooth chart, the above argument proves that there is a line bundle $\tilde{F}$ on $\Delta_{\tilde{\mathcal{Y}}}$ so that as stacks over $\tilde{\mathcal{Y}}_{\rm gst}$, $\mathcal{N}$ is the union of $\tilde{\mathcal{Y}}_{\rm gst}$ and (the total space of) $\tilde{F}$ so that $\tilde{\mathcal{Y}}_{\rm gst} \cap \tilde{F} = 0_{\tilde{F}} \subset \tilde{F}$.

Finally, we show that we can make $\tilde{F}$ a subline bundle of $\tilde{V}_2|_{\Delta_{\tilde{\mathcal{Y}}}}$ so that $\mathcal{N} \subset \tilde{V}_2$ is $0_{\tilde{V}_2} \cup \mathrm{Tot}(\tilde{F})$, and further that there is a subline bundle $F \subset V_2|_{\Delta_{\tilde{\mathcal{X}}}}$ so that $\tilde{F} = \gamma^* F \subset \tilde{V}_2|_{\Delta_{\tilde{\mathcal{Y}}}}$. First, since $\mathcal{N} \subset \tilde{V}_2$ is a subcone, $\tilde{\mathcal{F}} = \mathcal{N} \times_{\tilde{\mathcal{Y}}_{\rm gst}} \Delta_{\tilde{\mathcal{Y}}} \subset \tilde{V}_2|_{\Delta_{\tilde{\mathcal{Y}}}}$ is also a subcone. Therefore, because $\mathcal{N} \times_{\tilde{\mathcal{Y}}_{\rm gst}} \Delta_{\tilde{\mathcal{Y}}}$ is the total space of a line bundle over $\Delta_{\tilde{\mathcal{Y}}}$, $\mathcal{N} \times_{\tilde{\mathcal{Y}}_{\rm gst}} \Delta_{\tilde{\mathcal{Y}}} \subset \tilde{V}_2|_{\Delta_{\tilde{\mathcal{Y}}}}$ is the total space of a subline bundle of $\tilde{V}_2|_{\Delta_{\tilde{\mathcal{Y}}}}$. Without loss of generality, we can assume that $\tilde{F}$ is a subline bundle of $\tilde{V}_2|_{\Delta_{\tilde{\mathcal{Y}}}}$ so that $\mathcal{N} = 0_{\tilde{V}_0} \cup \mathrm{Tot}(\tilde{F}) \subset \tilde{V}_2$.

We now prove the further part. To this end, we use the $\mathbb{C}^*$-action on $\tilde{\mathcal{Y}}_{\rm gst}$ introduced in the proof of Lemma 6.7. By Lemma 6.7, $\tilde{\mathcal{Y}}_{\rm gst} - \tilde{\mathcal{X}}_{\rm gst} \to \tilde{\mathcal{X}}_{\rm gst}$ is a $\mathbb{C}^*$-quotient morphism. By its construction, the relative obstruction theory of $\tilde{\mathcal{Y}} \to \tilde{\mathcal{X}}$ is $\mathbb{C}^*$-equivariant. Thus $\tilde{F}|_{\tilde{\mathcal{Y}}_{\rm gst} - \tilde{\mathcal{X}}_{\rm gst}} \subset \tilde{V}_2|_{\tilde{\mathcal{Y}}_{\rm gst} - \tilde{\mathcal{X}}_{\rm gst}}$ is $\mathbb{C}^*$-equivariant, where the $\mathbb{C}^*$-action on $\tilde{V}_2|_{\tilde{\mathcal{Y}}_{\rm gst} - \tilde{\mathcal{X}}_{\rm gst}}$ is via the linearization $\Phi_{2,1}$.

As $\Phi_{2,1}$ differs from $\Phi_{2,0}$ by a scalar multiplication, and because $\tilde{F} \subset \tilde{V}_2|_{\Delta_{\tilde{\mathcal{Y}}}}$ is a subline bundle, $\tilde{F}|_{\Delta_{\tilde{\mathcal{Y}}}} \subset \tilde{V}_2|_{\Delta_{\tilde{\mathcal{Y}}}}$ is also invariant under the linearization $\Phi_{2,0}$. Since $\Phi_{2,0}$ is induced from the pullback $\tilde{V}_2 = \gamma^* V_2$, by descent theory, $\tilde{F}|_{\Delta_{\tilde{\mathcal{Y}}} - \Delta_{\tilde{\mathcal{X}}}} \subset \tilde{V}_2|_{\Delta_{\tilde{\mathcal{Y}}} - \Delta_{\tilde{\mathcal{X}}}}$ descends to a subline bundle $F \subset V_2|_{\Delta_{\tilde{\mathcal{X}}}}$. (As usual, we view $\Delta_{\tilde{\mathcal{X}}} \subset \Delta_{\tilde{\mathcal{Y}}}$ via the zero section of $W = \mathrm{Tot}(L)$.) Thus $\tilde{F}|_{\Delta_{\tilde{\mathcal{Y}}} - \Delta_{\tilde{\mathcal{X}}}} = \gamma^* F|_{\Delta_{\tilde{\mathcal{Y}}} - \Delta_{\tilde{\mathcal{X}}}} \subset \tilde{V}_2|_{\Delta_{\tilde{\mathcal{Y}}} - \Delta_{\tilde{\mathcal{X}}}}$. Since $\Delta_{\tilde{\mathcal{Y}}} - \Delta_{\tilde{\mathcal{X}}}$ is dense in $\Delta_{\tilde{\mathcal{Y}}}$, we conclude $\tilde{F} = \gamma^* F \subset \tilde{V}_2|_{\Delta_{\tilde{\mathcal{Y}}}}$. This proves the Proposition. q.e.d.

8. Contributions from the ghost components I

We now apply Corollary 6.5 to individual components $\tilde{\mathcal{Y}}_\mu \subset \tilde{\mathcal{Y}}_{\rm gst}$ and $\tilde{\mathcal{X}}_\mu \subset \tilde{\mathcal{X}}_{\rm gst}$. Accordingly, we will use the subscript μ to denote the corresponding objects restricting to $\tilde{\mathcal{X}}_\mu$, $\tilde{\mathcal{Y}}_\mu$, or $\tilde{\mathcal{Y}}_\mu^{\rm cpt} = \tilde{\mathcal{Y}}_{\rm gst}^{\rm cpt} \times_{\tilde{\mathcal{X}}} \tilde{\mathcal{X}}_\mu$. For instance, $V_{i,\mu} = V_i|_{\tilde{\mathcal{X}}_\mu}$, $\tilde{V}_{i,\mu}^{\rm cpt} = \tilde{V}_i^{\rm cpt}|_{\tilde{\mathcal{Y}}_\mu^{\rm cpt}}$, $\Delta_{\tilde{\mathcal{X}}_\mu} = \Delta_{\tilde{\mathcal{X}}} \cap \tilde{\mathcal{X}}_\mu$ and $\Delta_{\tilde{\mathcal{Y}}_\mu} = \Delta_{\tilde{\mathcal{Y}}_{\rm gst}} \times_{\tilde{\mathcal{Y}}_{\rm gst}} \tilde{\mathcal{Y}}_\mu$, etc. Thus $\tilde{\mathcal{Y}}_\mu^{\rm cpt}$ is a $\mathbb{P}^1$ bundles over $\tilde{\mathcal{X}}_\mu$. Restricting $\bar{\gamma} : \tilde{\mathcal{Y}}_{\rm gst}^{\rm cpt} \to \tilde{\mathcal{X}}_{\rm gst}$ defined in (6.9) to $\tilde{\mathcal{Y}}_\mu^{\rm cpt}$, we obtain the projection

$$\bar{\gamma}_\mu = \bar{\gamma}|_{\tilde{\mathcal{Y}}_\mu^{\rm cpt}} : \tilde{\mathcal{Y}}_\mu^{\rm cpt} \longrightarrow \tilde{\mathcal{X}}_\mu. \tag{8.1}$$

Following (4.4), the cycle $[C_{\rm gst}] \in Z_* \tilde{V}$ has a decomposition $[C_{\rm gst}] = \sum_{\mu \vdash d} \iota_{\mu *}[C_\mu]$, where $\iota_\mu : \tilde{V}_\mu \to \tilde{V}$ is the inclusion and $[C_\mu] = \sum_k n_{\mu,k}[C_{\mu,k}] \in Z_* \tilde{V}_\mu$ such that $C_{\mu,k}$ are integral (cf. before (4.3)).

Let $\overline{C}_{\mu,k}$ be the closure of $C_{\mu,k}$ in $\tilde V_\mu^{\mathrm{cpt}}$, and let $\overline{C}_\mu=\sum_k n_{\mu,k}[\overline{C}_{\mu,k}]$. We let $\overline{C}_{\mu,k,b}=\overline{C}_{\mu,k}\cap(0\oplus\tilde V_{2,\mu}^{\mathrm{cpt}})$, and let $N_{\overline{C}_{\mu,k,b}}\overline{C}_{\mu,k}$ be the normal cone to $\overline{C}_{\mu,k,b}$ in $\overline{C}_{\mu,k}$. We form

$$R_\mu=\sum_k n_{\mu,k}[N_{\overline{C}_{\mu,k,b}}\overline{C}_{\mu,k}].$$

By Corollary 6.5, we have

$$\deg 0^!_{\tilde\sigma_{\mathrm{gst}},\mathrm{loc}}[C_{\mathrm{gst}}]=\deg 0^!_{\tilde V^{\mathrm{cpt}}}[\overline{C}_{\mathrm{gst}}]=\sum_{\mu\vdash d}\deg 0^!_{\tilde V_\mu^{\mathrm{cpt}}}(R_\mu). \tag{8.2}$$

We now investigate the individual terms $0^!_{\tilde V_\mu^{\mathrm{cpt}}}(R_\mu)$. Following the notations in Proposition 7.1, we let $F_\mu=F|_{\Delta_{\tilde{\mathcal X}_\mu}}\subset V_{2,\mu}|_{\Delta_{\tilde{\mathcal X}_\mu}}=V_2|_{\Delta_{\tilde{\mathcal X}_\mu}}$; let

$$\delta_\mu=\bar\gamma|_{\Delta_{\tilde{\mathcal Y}_\mu^{\mathrm{cpt}}}}:\Delta_{\tilde{\mathcal Y}_\mu^{\mathrm{cpt}}}:=\tilde{\mathcal Y}_\mu^{\mathrm{cpt}}\times_{\tilde{\mathcal X}_\mu}\Delta_{\tilde{\mathcal X}_\mu}\to\Delta_{\tilde{\mathcal X}_\mu}$$

be the tautological projection (a $\mathbb P^1$-bundle), then $\delta_\mu^*F_\mu\subset\tilde V_{2,\mu}^{\mathrm{cpt}}|_{\Delta_{\tilde{\mathcal Y}_\mu^{\mathrm{cpt}}}}$ is induced by $F_\mu\subset V_{2,\mu}|_{\Delta_{\tilde{\mathcal X}_\mu}}$. We use $\bar F_\mu$ to denote the total space of $\delta_\mu^*F_\mu$; thus $\bar F_\mu$ is a subline bundle of $V_{2,\mu}|_{\Delta_{\tilde{\mathcal X}_\mu}}$, and let

$$Z_\mu=0_{\tilde V_{2,\mu}^{\mathrm{cpt}}}\cup\bar F_\mu, \tag{8.3}$$

be a closed substack of $\tilde V_{2,\mu}^{\mathrm{cpt}}$. By viewing $\tilde V_{1,\mu}^{\mathrm{cpt}}\times_{\tilde{\mathcal Y}_\mu^{\mathrm{cpt}}}\tilde V_{2,\mu}^{\mathrm{cpt}}$ as a bundle over $\tilde V_{2,\mu}^{\mathrm{cpt}}$ and applying Proposition 7.1, $\overline{C}_{\mu,k,b}$ all lie over Z_μ. Thus $R_\mu\in Z_*\big(\tilde V_{1,\mu}^{\mathrm{cpt}}\times_{\tilde{\mathcal Y}_\mu^{\mathrm{cpt}}}Z_\mu\big)$.

We claim that $\dim\tilde{\mathcal Y}_\mu^{\mathrm{cpt}}=5d+4$, $\dim C_{\mu,k}=5d+6$, $\operatorname{rank}\tilde V_{1,\mu}^{\mathrm{cpt}}=5$ and $\operatorname{rank}\tilde V_{2,\mu}^{\mathrm{cpt}}=5d+1$. Indeed, by Proposition 3.1, we know that all $\tilde{\mathcal X}_\mu$ have equal pure dimensions. Thus

$$\dim\tilde{\mathcal X}_\mu=\dim\tilde{\mathcal X}_\mu=\dim\tilde{\mathcal X}_{(d)}=\dim\overline{M}_{0,1}(\mathbb P^4,d)+\dim\overline{M}_{1,1}=5d+3.$$

As $\tilde{\mathcal Y}_\mu$ is a line bundle over $\tilde{\mathcal X}_\mu$, we obtain $\dim\tilde{\mathcal Y}_\mu^{\mathrm{cpt}}=\dim\tilde{\mathcal Y}_\mu=5d+4$. For $\dim C_{\mu,k}$, we only need to verify $\dim C_{\mu,k}=\dim\tilde{\mathcal Y}_\mu+2$, but this follows from (2) of Lemma 4.3. The remainder two identities follow from Riamenn-Roch theorem.

We denote by $|\overline{C}_{\mu,b}|$ the support of $\overline{C}_{\mu,b}$, which is the union $\cup_k\overline{C}_{\mu,k,b}$. Since R_μ is the union of normal cones, by their constructions, the support of R_μ is contained inside $\tilde V_{1,\mu}^{\mathrm{cpt}}\times_{\tilde{\mathcal Y}_\mu^{\mathrm{cpt}}}|\overline{C}_{\mu,b}|$. Therefore, we have

$$0^!_{\tilde V_{1,\mu}^{\mathrm{cpt}}}(R_\mu)\in A_{5d+1}|\overline{C}_{\mu,b}|. \tag{8.4}$$

We write $0^!_{\tilde V_{1,\mu}^{\mathrm{cpt}}}(R_\mu)=P_{\mu,1}+P_{\mu,2}$, where $P_{\mu,1}\in A_*(0_{\tilde V_\mu^{\mathrm{cpt}}})$ and $P_{\mu,2}\in A_*(\bar F_\mu)$.

Lemma 8.1. *We have the following vanishings. (1).* $\deg 0^!_{\tilde V^{\rm cpt}_{2,\mu}}(P_{\mu,2}) = 0$ *for any* μ*; (2).* $\deg 0^!_{\tilde V^{\rm cpt}_{2,\mu}}(P_{\mu,1}) = 0$ *when* $\mu \neq (d)$.

Proof. We prove the first case. Let $\eta_\mu : \bar F_\mu \to F_\mu$ be the projection; it is proper since $\Delta_{\tilde{\mathcal{Y}}^{\rm cpt}_\mu} \to \Delta_{\tilde{\mathcal{X}}_\mu}$ is proper. Then by the projection formula,

$$\deg 0^!_{\tilde V^{\rm cpt}_{2,\mu}}(P_{\mu,2}) = \deg 0^!_{V_{2,\mu}}(\eta_{\mu *}(P_{\mu,2})).$$

Since $\eta_{\mu*}(P_{\mu,2}) \in A_{5d+1}F_\mu$ and $\dim F_\mu = \dim \Delta_{\tilde{\mathcal{X}}_\mu} + 1 = (\dim \tilde{\mathcal{X}}_{\rm pri} - 1) + 1 = 5d$, we have $\eta_{\mu*}(P_{\mu,2}) = 0$. This proves the first vanishing.

We now prove the second vanishing. To do this, we will construct a proper DM stack $\tilde{\mathcal{B}}_\mu$, a vector bundle $\tilde{\mathcal{K}}_\mu$ on $\tilde{\mathcal{B}}_\mu$, a proper morphism $\rho_\mu : \tilde{\mathcal{X}}_\mu = \tilde{\mathcal{X}}_\mu \to \tilde{\mathcal{B}}_\mu$ and an isomorphism $\rho_\mu^* \tilde{\mathcal{K}}_\mu^\vee \cong V_{2,\mu}$ such that $\dim \tilde{\mathcal{B}}_{(d)} = 5d+1$, and $\dim \tilde{\mathcal{B}}_\mu \le 5d$ for $\mu \neq (d)$. Once $(\tilde{\mathcal{B}}_\mu, \tilde{\mathcal{K}}_\mu)$ is constructed, we let $\lambda_\mu : \tilde V^{\rm cpt}_{2,\mu} = \bar\gamma_\mu^* V_{2,\mu} \to \tilde{\mathcal{K}}_\mu^\vee$ be the projection induced by the isomorphism in (8.7). Then, for $\mu \neq (d)$, because $P_{\mu,1} \in Z_*(0_{\tilde V^{\rm cpt}_{2,\mu}})$, $\lambda_\mu(0_{\tilde V^{\rm cpt}_{2,\mu}}) \subset \tilde{\mathcal{B}}_\mu$, and $A_{5d+1}\tilde{\mathcal{B}}_\mu = 0$ for dimension reason, applying the projection formula, we obtain

$$\deg 0^!_{\tilde V^{\rm cpt}_{2,\mu}}(P_{\mu,1}) = \deg 0^!_{\mathcal{K}^\vee_\mu}(\lambda_{\mu*}(P_{\mu,1})) = 0.$$

Constructing $\tilde{\mathcal{B}}_\mu$ and $\tilde{\mathcal{K}}_\mu$ with the required properties will occupy the remainder of this section. q.e.d.

We first state a decomposition result, which follows from the construction in [**HL**]. Let $\mu = (d_1, \cdots, d_\ell)$ be the partition of d as before and let $(f_{\tilde{\mathcal{X}}_\mu}, \pi_{\tilde{\mathcal{X}}_\mu}, \mathcal{C}_{\tilde{\mathcal{X}}_\mu})$ be the tautological family of $\tilde{\mathcal{X}}_\mu$. By the construction of $\tilde{\mathcal{X}}_{\rm gst}$, the map associated to a closed point in $\tilde{\mathcal{X}}_\mu$ is by attaching to an ℓ-pointed stable elliptic curve ℓ one-pointed $[u_i, C_i, p_i] \in \overline{M}_{0,1}(\mathbb{P}^4, d_i)$ such that $u_i(p_i) = u_j(p_j)$ for all i, j. We state this in the family version.

Proposition 8.2. *The tautological family* $\mathcal{C}_{\tilde{\mathcal{X}}_\mu} \to \tilde{\mathcal{X}}_\mu$ *admits an* ℓ*-section* $\Sigma_\mu \subset \mathcal{C}_{\tilde{\mathcal{X}}_\mu}$ *(i.e.* Σ_μ *is a codimension one closed substack, and a proper* ℓ*-étale cover of* $\tilde{\mathcal{X}}_\mu$*) that lies in the locus of nodal points of the fibers of* $\mathcal{C}_{\tilde{\mathcal{X}}_\mu}/\tilde{\mathcal{X}}_\mu$*, and splits* $\mathcal{C}_{\tilde{\mathcal{X}}_\mu}$ *into two families of curves:* $\mathcal{C}_{\tilde{\mathcal{X}}_\mu,\rm pr}$ *and* $\mathcal{C}_{\tilde{\mathcal{X}}_\mu,\rm tl}$ *(*$\subset \mathcal{C}_{\tilde{\mathcal{X}}_\mu}$*), such that*

1) *the pair* $(\mathcal{C}_{\tilde{\mathcal{X}}_\mu,\rm pr}, \Sigma_\mu)$ *is a family of (unordered)* ℓ*-pointed stable genus one curves; the morphism* $f_{\tilde{\mathcal{X}}_\mu}$ *is constant along fibers of* $\mathcal{C}_{\tilde{\mathcal{X}}_\mu,\rm pr} \to \tilde{\mathcal{X}}_\mu$*;*
2) *the pair* $(\mathcal{C}_{\tilde{\mathcal{X}}_\mu,\rm tl}, \Sigma_\mu)$ *is a family of (unordered)* ℓ*-pointed nodal rational curves over* $\tilde{\mathcal{X}}_\mu$ *such that each closed fiber of* $\mathcal{C}_{\tilde{\mathcal{X}}_\mu,\rm tl} \to \tilde{\mathcal{X}}_\mu$

has ℓ connected components and each such connected component contains one marked point.

Here the subscript "pr" stands for the "principal part" and the subscript "tl" stands for the "tail". We comment that the total space $\overline{\mathcal{C}}_{\tilde{\mathcal{X}}_\mu,\mathrm{tl}}$ may have less than ℓ connected components.

Proof. The proof follows from the modular construction of $\tilde{\mathcal{X}}_\mu$ in [**HL**]. q.e.d.

We now assume $\ell \geq 2$. Using this decomposition, we can relate $\tilde{\mathcal{X}}_\mu$ to a stack that parameterizes the tails of $[u, C] \in \tilde{\mathcal{X}}_\mu$. We take the moduli of genus zero stable morphisms $\overline{M}_{0,1}(\mathbb{P}^4, d_i)$, considered as a stack over $\mathbb{P}^4$ via the evaluation morphism $\mathrm{ev}_i : \overline{M}_{0,1}(\mathbb{P}^4, d_i) \to \mathbb{P}^4$ (of the marked points); we form

$$\mathcal{B}_\mu = \overline{M}_{0,1}(\mathbb{P}^4, d_1) \times_{\mathbb{P}^4} \cdots \times_{\mathbb{P}^4} \overline{M}_{0,1}(\mathbb{P}^4, d_\ell).$$

We let S_μ be the subgroup of permutations $\alpha \in S_\ell$ that leave the ℓ-tuple $(d_1, \cdots, d_\ell)$ invariant (i.e. $d_i = d_{\alpha(i)}$ for all $1 \leq i \leq \ell$). Each $\alpha \in S_\mu$ acts as an automorphism of $\mathcal{B}_\mu$ by permuting the i-th and the $\alpha(i)$-th factors of $\mathcal{B}_\mu$. This gives an S_μ-action on $\mathcal{B}_\mu$. We define the stacky quotient

$$\tilde{\mathcal{B}}_\mu = [\mathcal{B}_\mu / S_\mu].$$

Since $\overline{M}_{0,1}(\mathbb{P}^4, d_i)$ are proper DM-stacks and have dimensions $5d_i + 2$, $\tilde{\mathcal{B}}$ is a proper DM-stack and

$$\dim \tilde{\mathcal{B}}_\mu = (5d + 2l) - (4l - 4) = 5d - 2l + 4. \tag{8.5}$$

We denote the universal family of $\overline{M}_{0,1}(\mathbb{P}^4, d_i)$ by $(f_i, \pi_i) : \mathcal{C}_i \to \mathbb{P}^4 \times \overline{M}_{0,1}(\mathbb{P}^4, d_i)$ with $s_i : \overline{M}_{0,1}(\mathbb{P}^4, d_i) \to \mathcal{C}_i$ the section of marked points. We introduce

$$\mathcal{K}_i = \pi_{i*} f_i^* \mathcal{O}(5). \tag{8.6}$$

Because $R^1\pi_{i*} f_i^* \mathcal{O}(5) = 0$, by Riemann-Roch $\mathcal{K}_i$ is a rank $5d_i + 1$ locally free sheaf on $\overline{M}_{0,1}(\mathbb{P}^4, d_i)$.

By its construction, $\mathcal{K}_i$ comes with an evaluation homomorphism $e_i : \mathcal{K}_i \to \mathrm{ev}_i^* \mathcal{O}(5)$. We let $v_i : \mathcal{B}_\mu \to \overline{M}_{0,1}(\mathbb{P}^4, d_i)$ be the i-th projection. Since $\mathcal{B}_\mu$ is constructed as the fiber-product using the evaluations ev_i, the collection $\{\mathrm{ev}_i\}_{i=1}^\ell$ descends to a single evaluation morphism $\mathrm{ev} : \mathcal{B}_\mu \to \mathbb{P}^4$. We form a sheaf $\mathcal{K}_\mu$ on $\mathcal{B}_\mu$ fitting into the exact sequence

$$0 \longrightarrow \mathcal{K}_\mu \longrightarrow \oplus_{i=1}^\ell v_i^* \mathcal{K}_i \xrightarrow{\beta} \mathrm{ev}^*\big(\mathcal{O}(5)^{\oplus \ell} / \mathcal{O}(5)\big) \longrightarrow 0,$$

where $\mathcal{O}(5) \subset \mathcal{O}(5)^{\oplus \ell}$ is the image subsheaf of the diagonal homomorphism $\mathcal{O}(5) \to \mathcal{O}(5)^{\oplus \ell}$; β is the composite of

$$(v_1^* e_1, v_2^* e_2, \cdots, v_\ell^* e_\ell) : \oplus_{i=1}^\ell v_i^* \mathcal{K}_i \longrightarrow \oplus_{j=1}^\ell v_i^* \mathrm{ev}_i^* \mathcal{O}(5) = \mathrm{ev}^* \mathcal{O}(5)^{\oplus \ell}$$

with the quotient $\mathrm{ev}^* \mathcal{O}(5)^{\oplus \ell} \to \mathrm{ev}^*\big(\mathcal{O}(5)^{\oplus \ell} / \mathcal{O}(5)\big)$.

Since each $e_i : \mathcal{K}_i \to \mathrm{ev}_i^* \mathcal{O}(5)$ is surjective, β is surjective and thus $\mathcal{K}_\mu$ is locally free. Further, because β is equivariant under S_μ, its kernel $\mathcal{K}_\mu$ is also S_μ equivariant. Therefore, $\mathcal{K}_i$, descends to a locally free sheaf $\tilde{\mathcal{K}}_\mu$ on $\tilde{\mathcal{B}}_\mu$. As $\operatorname{rank} \mathcal{K}_i = 5d_i + 1$, $\operatorname{rank} \tilde{\mathcal{K}}_\mu = 5d + 1$.

We define the desired morphism

$$\rho_\mu : \tilde{\mathcal{X}}_\mu \longrightarrow \tilde{\mathcal{B}}_\mu \tag{8.7}$$

Given any closed $x \in \tilde{\mathcal{X}}_\mu$, we let $[f_x, \mathcal{C}_x]$ with $\Sigma_x \subset \mathcal{C}_x$ be the restriction of $f_{\tilde{\mathcal{X}}_\mu}$ and Σ_μ to $\mathcal{C}_x = \mathcal{C}_{\tilde{\mathcal{X}}_\mu} \times_{\tilde{\mathcal{X}}_\mu} x$. By Proposition 8.2, Σ_x divides $\mathcal{C}_x$ into one genus one curve and ℓ one-pointed genus zero curves, and the restriction of f_x to these ℓ genus zero curves form ℓ one-pointed genus zero stable maps to $\mathbb{P}^4$, of degrees $d_1, \cdots, d_\ell$, respectively. We label these ℓ-stable maps as $[u_i, C_i, p_i]$ so that $[u_i] \in \overline{M}_{0,1}(\mathbb{P}^4, d_i)$. We define $\rho_\mu(x)$ to be the equivalence class in $\tilde{\mathcal{B}}_\mu$ of $([u_1], \cdots, [u_\ell]) \in \mathcal{B}_\mu$. It is routine to check that this pointwise definition defines a morphism ρ_μ as stated.

We verify that $\rho_\mu^* \tilde{\mathcal{K}}_\mu^\vee \cong V_{2,\mu}$. First, by Serre duality,

$$V_{2,\mu} = R^1 \pi_{\tilde{\mathcal{X}}_\mu *} \mathcal{P}_{\tilde{\mathcal{X}}_\mu} \cong (\pi_{\tilde{\mathcal{X}}_\mu *} f^*_{\tilde{\mathcal{X}}_\mu} \mathcal{O}(5))^\vee.$$

Let $f_{\tilde{\mathcal{X}}_\mu, \mathrm{tl}}$ and $\pi_{\tilde{\mathcal{X}}_\mu, \mathrm{tl}}$ be the restrictions of $f_{\tilde{\mathcal{X}}_\mu}$ and $\pi_{\tilde{\mathcal{X}}_\mu}$ to $\mathcal{C}_{\tilde{\mathcal{X}}_\mu, \mathrm{tl}}$. We obtain the restriction homomorphism

$$\pi_{\tilde{\mathcal{X}}_\mu *} f^*_{\tilde{\mathcal{X}}_\mu} \mathcal{O}(5) \longrightarrow (\pi_{\tilde{\mathcal{X}}_\mu, \mathrm{tl}})_* f^*_{\tilde{\mathcal{X}}_\mu, \mathrm{tl}} \mathcal{O}(5). \tag{8.8}$$

Since fibers of $\mathcal{C}_{\tilde{\mathcal{X}}_\mu, \mathrm{pr}} \to \tilde{\mathcal{X}}_\mu$ are connected, and $f_{\tilde{\mathcal{X}}_\mu}$ restricting to them are constants, (8.8) is injective and its cokernel is the difference of the evaluations along $\Sigma_\mu \subset \mathcal{C}_{\tilde{\mathcal{X}}_\mu, \mathrm{tl}}$. On the other hand, denoting $(\oplus_{i=1}^\ell v_i^* \mathcal{K}_i)/S_\mu$ the quotient of $\oplus_{i=1}^\ell v_i^* \mathcal{K}_i$ over $\mathcal{B}_\mu$ by S_μ, (which is its descent to $\tilde{\mathcal{B}}_\mu$,) then via ρ_μ, we have a canonical isomorphism

$$\rho_\mu^*((\oplus_{i=1}^\ell v_i^* \mathcal{K}_i)/S_\mu) \cong (\pi_{\tilde{\mathcal{X}}_\mu, \mathrm{tl}})_* f^*_{\tilde{\mathcal{X}}_\mu, \mathrm{tl}} \mathcal{O}(5).$$

A direct inspection shows that under this isomorphism, $\rho_\mu^* \tilde{\mathcal{K}}_\mu \cong \pi_{\tilde{\mathcal{X}}_\mu *} f^*_{\tilde{\mathcal{X}}_\mu} \mathcal{O}(5)$. This proves $\rho_\mu^* \tilde{\mathcal{K}}_\mu^\vee \cong V_{2,\mu}$.

In case $\mu = (d)$, we let $\tilde{\mathcal{B}}_{(d)} = \overline{M}_0(\mathbb{P}^4, d)$, and all others are the same.

Completing the proof of Lemma 8.1. The pair $(\tilde{\mathcal{B}}_\mu, \mathcal{K}_\mu)$ satisfies the requirements stated in the proof of Lemma 8.1, except the dimensions part. By (8.5), for $\ell \geq 2$, we have $\dim \tilde{\mathcal{B}}_\mu \leq 5d$; for $\mu = (d)$, we have $\dim \tilde{\mathcal{B}}_{(d)} = \dim \overline{M}_0(\mathbb{P}^4, d) = 5d + 1$, which are as required. This completes the proof of Lemma 8.1. q.e.d.

9. Contribution from the ghost components II

In this section, we evaluate the remainder term $\deg 0^!_{\tilde{V}^{\mathrm{cpt}}_{2,(d)}}(P_{(d),1})$. First by (8.4), $P_{(d),1} \in A_{5d+1}(0_{\tilde{V}^{\mathrm{cpt}}_{(d)}})$; using the morphism and the projection (of $\mathbb{P}^1$ bundle)

$$\rho_{(d)} : \tilde{\mathcal{X}}_{(d)} \longrightarrow \tilde{\mathcal{B}}_{(d)} = \overline{M}_0(\mathbb{P}^4, d) \quad \text{and} \quad \bar{\gamma}_{(d)} = \bar{\gamma}|_{\tilde{\mathcal{Y}}^{\mathrm{cpt}}_{(d)}} : \tilde{\mathcal{Y}}^{\mathrm{cpt}}_{(d)} \longrightarrow \tilde{\mathcal{X}}_{(d)}$$

constructed in (8.7) and (8.1) for $\mu = (d)$, we obtain

$$(\rho_{(d)} \circ \bar{\gamma}_{(d)})_*(P_{(d),1}) = c[\tilde{\mathcal{B}}_{(d)}] \tag{9.1}$$

for a $c \in \mathbb{Q}$. Here we used that both $\rho_{(d)}$ and $\bar{\gamma}_{(d)}$ are proper and $\tilde{\mathcal{B}}_{(d)} = \overline{M}_0(\mathbb{P}^4, d)$ is irreducible of dimension $5d+1$.

By (8.7) and (6.10), we have $(\rho_{(d)} \circ \bar{\gamma}_{(d)})^* \tilde{\mathcal{K}}_{(d)} = \bar{\gamma}^*_{(d)} V_{2,(d)} = V^{\mathrm{cpt}}_{2,(d)}$. Applying the projection formula, we obtain

$$\begin{aligned} \deg 0^!_{\tilde{V}^{\mathrm{cpt}}_{2,(d)}}(P_{(d),1}) &= \deg 0^!_{\tilde{\mathcal{K}}_{(d)}}[(\rho_{(d)} \circ \bar{\gamma}_{(d)})_*(P_{(d),1})] \\ &= c \cdot \deg c_{5d+1}(\tilde{\mathcal{K}}_{(d)})[\tilde{\mathcal{B}}_{(d)}], \end{aligned} \tag{9.2}$$

where the second equality follows from (9.1). Using $\tilde{\mathcal{B}}_{(d)} = \overline{M}_0(\mathbb{P}^4, d)$, (8.6) and the genus zero hyperplane property in [**Ko**], we obtain

$$\deg 0^!_{\tilde{V}^{\mathrm{cpt}}_{2,(d)}}(P_{(d),1}) = c \cdot (-1)^{5d+1} N_0(d)_Q.$$

Therefore, all we need is to determine c.

Our approach is to work with an open substack of $\tilde{\mathcal{Y}}_{(d)}$ over which the cone $C_{(d)} \subset \tilde{V}|_{\tilde{\mathcal{Y}}_{(d)}}$, etc., can be described explicitly. We let $M_\circ \subset \overline{M}_1(\mathbb{P}^4, d)$ be the open substack consisting of stable maps $[f, C]$ such that the domains $C = E \cup \mathbb{P}^1$ are unions of smooth elliptic curves E with $\mathbb{P}^1$, and the restrictions $f|_E = \mathrm{const.}$ and $f|_{\mathbb{P}^1} : \mathbb{P}^1 \to \mathbb{P}^4$ are regular embeddings. By [**HL**, Sec 2.8], $M_\circ$ does not meet the blowing-up loci of $\tilde{\mathcal{X}} \to \mathcal{X} = \overline{M}_1(\mathbb{P}^4, d)$. Thus, both projections

$$\tilde{\mathcal{X}} \times_{\mathcal{X}} M_\circ \longrightarrow M_\circ \quad \text{and} \quad W_\circ := \tilde{\mathcal{Y}} \times_{\mathcal{X}} M_\circ \longrightarrow \mathcal{Y} \times_{\mathcal{X}} M_\circ \subset \tilde{\mathcal{Y}}_{(d)}$$

are isomorphisms. Further, because $M_\circ \subset \mathcal{X}$ is open and dense, $W_\circ \subset \tilde{\mathcal{Y}}_{(d)}$ is open and dense.

Since the relative obstruction theories of $\tilde{\mathcal{Y}}$ and $\tilde{\mathcal{X}}$ are pull backs from that of $\mathcal{Y}$ and $\mathcal{X}$, to work with $W_\circ$ and $M_\circ$, we might as well view them as substacks of $\mathcal{Y}$ and $\mathcal{X}$, which we will do from now on. Therefore, as $W_\circ \subset \mathcal{Y}$ is open, restricting the obstruction theory $\phi_{\mathcal{Y}/\mathcal{D}}$ to $W_\circ$ gives a perfect relative obstruction theory $\phi_{\mathcal{Y}/\mathcal{D}}|_{W_\circ} : (E_{\mathcal{Y}/\mathcal{D}})^\vee|_{W_\circ} \to L^\bullet_{W_\circ/\mathcal{D}}$.

Lemma 9.1. *Restricting to $W_\circ$, we have*

$$C_{(d)}|_{W_\circ} = H^1((L^\bullet_{W_\circ/\mathcal{D}})^\vee) \longrightarrow H^1(E_{\mathcal{Y}/\mathcal{D}}|_{W_\circ}) = \tilde{V}|_{W_\circ}.$$

Further, $H^1((L^\bullet_{W_\circ/\mathcal{D}})^\vee)$ is a rank two locally free sheaf of $\mathcal{O}_{W_\circ}$-modules, and the arrow above is injective with locally free cokernel.

Proof. Let $q_\circ : W_\circ \to \mathcal{D}$ be the projection induced by $\mathcal{Y} \to \mathcal{D}$; let $\mathcal{D}_\circ = q_\circ(W_\circ)$ be its image stack. By the description of $W_\circ$ and Sublemma 4.4, $\mathcal{D}_\circ \subset \mathcal{D}$ is a smooth codimension two locally closed substack. By [**Il**, Chap. III Prop. 3.2.4], $H^1(L^\vee_{\mathcal{D}_\circ/\mathcal{D}})$ is isomorphic to the normal sheaf to $\mathcal{D}_\circ$ in $\mathcal{D}$, which is a rank two locally free sheaf on $\mathcal{D}_\circ$ since $\mathcal{D}$ is smooth.

On the other hand, since $W_\circ \to \mathcal{D}_\circ$ is smooth, [**Il**, Chap. III Prop. 3.1.2] implies that $H^i((L^\bullet_{W_\circ/\mathcal{D}_\circ})^\vee) = 0$ for $i \geq 1$. Taking H^1 of the distinguished triangle

$$(L^\bullet_{W_\circ/\mathcal{D}_\circ})^\vee \longrightarrow (L^\bullet_{W_\circ/\mathcal{D}})^\vee \longrightarrow q_\circ^*(L^\bullet_{\mathcal{D}_\circ/\mathcal{D}})^\vee \xrightarrow{+1},$$

we obtain canonical an isomorphism of two rank two locally free sheaves on $W_\circ$:

$$q_\circ^* H^1((L^\bullet_{\mathcal{D}_\circ/\mathcal{D}})^\vee) \cong H^1((L^\bullet_{W_\circ/\mathcal{D}})^\vee).$$

Next, for any closed point $y \in \mathcal{Y}$, since $\phi_{\mathcal{Y}/\mathcal{D}}$ is a perfect obstruction theory,

$$H^1(\phi^\vee_{\mathcal{Y}/\mathcal{D}}|_y) : H^1((L^\bullet_{W_\circ/\mathcal{D}})^\vee|_y) \longrightarrow H^1(E_{\mathcal{Y}/\mathcal{D}}|_y)$$

is injective. Combined with that $W_\circ$ is smooth, it shows that the arrow in the statement of Lemma 9.1 is injective with locally free cokernel.

Finally, we show that $C_{(d)}|_{W_\circ} = H^1((L^\bullet_{W_\circ/\mathcal{D}})^\vee) \subset \tilde{V}|_{W_\circ}$. Because $W_\circ \to \mathcal{D}_\circ$ is smooth and $\mathcal{D}_\circ \subset \mathcal{D}$ is smooth and of codimension two, we conclude that $L^\bullet_{W_\circ/\mathcal{D}}$ is perfect of amplitude $[0,1]$, and $H^i((L^\bullet_{W_\circ/\mathcal{D}})^\vee)$ are locally free. We pick an affine smooth chart $U \to W_\circ$; using the argument similar to those after (7.19), we conclude that

$$\begin{array}{ccc}
(L^\bullet_{W_\circ/\mathcal{D}})^\vee|_U & \xrightarrow{\cong} & [H^0((L^\bullet_{W_\circ/\mathcal{D}})^\vee|_U) \xrightarrow{0} H^1(L^\bullet_{W_\circ/\mathcal{D}})^\vee|_U] \\
\Big\downarrow{\scriptstyle \phi^\vee_{\mathcal{Y}/\mathcal{D}}|_U} & & \Big\downarrow{\scriptstyle H^\bullet(\phi^\vee_{\mathcal{Y}/\mathcal{D}}|_U)} \\
E_{\mathcal{Y}/\mathcal{D}}|_U & \xrightarrow{\cong} & [H^0(E_{\mathcal{Y}/\mathcal{D}}|_U) \xrightarrow{0} H^1(E_{\mathcal{Y}/\mathcal{D}}|_U)]
\end{array}$$

is commutative and such that

$$\begin{aligned}
(\mathbf{C}_{\mathcal{Y}/\mathcal{D}}|_{W_\circ})|_U &= h^1/h^0((L^\bullet_{W_\circ/\mathcal{D}})^\vee|_U) \\
&= [H^1((L^\bullet_{W_\circ/\mathcal{D}})^\vee|_U)/H^0((L^\bullet_{W_\circ/\mathcal{D}})^\vee|_U)]
\end{aligned}$$

is a substack of $[H^1(E_{\mathcal{Y}/\mathcal{D}}|_U)/H^0(E_{\mathcal{Y}/\mathcal{D}}|_U)]$. By an argument analogous to the constructions (7.12) and (7.19), we conclude that $C_{(d)}|_{W_\circ} = H^1((L^\bullet_{W_\circ/\mathcal{D}})^\vee)$. This proves the Lemma. q.e.d.

Let $\gamma_\circ : W_\circ \to M_\circ$ be the projection induced by $\mathcal{Y} \to \mathcal{X}$. We denote by $\mathcal{M}$ the Artin stack of genus one nodal curves; (which is consistent with that $\mathcal{M}^{\mathrm{w}}$ is the stack of weighted genus one nodal curves). Since

$\mathcal{M}^{\mathrm{w}} \to \mathcal{M}$ is étale, the obstruction theory of $\mathcal{X} \to \mathcal{M}$ is the same as that of $\mathcal{X} \to \mathcal{M}^{\mathrm{w}}$.

We determine the subsheaf $H^1((L^\bullet_{W_\circ/\mathcal{D}})^\vee) \subset \tilde{V}|_{W_\circ}$ by studying the following diagrams:

(9.3)

$$\begin{array}{ccccc}
H^1((L^\bullet_{W_\circ/\mathcal{D}})^\vee) & \xrightarrow{\alpha_1} & \gamma_\circ^* H^1((L^\bullet_{M_\circ/\mathcal{D}})^\vee) & \xrightarrow{\alpha_2} & \gamma_\circ^* H^1((L^\bullet_{M_\circ/\mathcal{M}})^\vee) \\
\Big\downarrow {\scriptstyle H^1(\phi^\vee_{\mathcal{Y}/\mathcal{D}})} & & \Big\downarrow {\scriptstyle H^1(\phi^\vee_{\mathcal{X}/\mathcal{D}})} & & \Big\downarrow {\scriptstyle H^1(\phi^\vee_{\mathcal{X}/\mathcal{M}})} \\
H^1(E_{\mathcal{Y}/\mathcal{D}}|_{W_\circ}) & \xrightarrow{\tilde{\beta}_2|_{W_\circ}} & \gamma_\circ^* H^1(E_{\mathcal{X}/\mathcal{D}}|_{M_\circ}) & \xrightarrow{\gamma_\circ^*\beta_\circ} & \gamma_\circ^* H^1(E_{\mathcal{X}/\mathcal{M}}|_{M_\circ}).
\end{array}$$

Here, $\tilde{\beta}_2$ is defined in (7.11) and $\beta_\circ$ is the tautological projection induced by the comparison of the obstruction theories of $\mathcal{X} \to \mathcal{D}$ and $\mathcal{X} \to \mathcal{M}$:

$$\beta_\circ : H^1(E_{\mathcal{X}/\mathcal{D}}|_{M_\circ}) \longrightarrow H^1(E_{\mathcal{X}/\mathcal{M}}|_{M_\circ}). \tag{9.4}$$

We comment that the left square is commutative because it is induced by the obstruction theories (relative to $\mathcal{D}$) of $\mathcal{Y} \subset C(\pi_*(\mathcal{L}^{\oplus 5} \oplus \mathcal{P}))$ and of $\mathcal{X} \subset C(\pi_*(\mathcal{L}^{\oplus 5}))$, which are compatible under $C(\pi_*(\mathcal{L}^{\oplus 5} \oplus \mathcal{P})) \to C(\pi_*(\mathcal{L}^{\oplus 5}))$ induced by the projection $\mathrm{pr} : \mathcal{L}^{\oplus 5} \oplus \mathcal{P} \to \mathcal{L}^{\oplus 5}$. The commutativity of the right square follows from [**CL1**, Lemm 2.8].

We let $(f_{M_\circ}, \pi_{M_\circ}) : \mathcal{C}_{M_\circ} \to \mathbb{P}^4 \times M_\circ$ be the universal family of $M_\circ \subset \mathcal{X}$.

Lemma 9.2. *All sheaves in the diagram* (9.3) *are locally free sheaves of $\mathcal{O}_{W_\circ}$-modules; all vertical arrows are injective with locally free cokernels; the arrow α_1 is an isomorphism; the arrow α_2 is surjective and has rank one kernel; the arrow $\tilde{\beta}_2|_{W_\circ}$ is the obvious projection $\tilde{V}|_{W_\circ} = (\tilde{V}_1 \oplus \tilde{V}_2)|_{W_\circ} \to \tilde{V}_1|_{W_\circ}$; the arrow $\beta_\circ$ is the projection*

$$\begin{aligned}
\beta_\circ : H^1(E_{\mathcal{X}/\mathcal{D}}|_{M_\circ}) &= R^1\pi_{M_\circ *} f^*_{M_\circ} \mathcal{O}(1)^{\oplus 5} \\
&\longrightarrow H^1(E_{\mathcal{X}/\mathcal{M}}|_{M_\circ}) = R^1\pi_{M_\circ *} f^*_{M_\circ} T_{\mathbb{P}^4}
\end{aligned}$$

induced by the tautological projection $\mathcal{O}(1)^{\oplus 5} \to T_{\mathbb{P}^4}$ in the Euler sequence of $\mathbb{P}^4$. Finally, the cokernels of the middle and the third vertical arrows are isomorphic.

Proof. We let $\mathcal{M}_\circ \subset \mathcal{M}$ be the image stack of $M_\circ \to \mathcal{M}$. By the description of $M_\circ$, $\mathcal{M}_\circ$ is a locally closed smooth divisor in $\mathcal{M}$. Thus the normal sheaf $\mathcal{N}_{\mathcal{M}_\circ/\mathcal{M}}$ is invertible and the canonical

$$\mathcal{N}_{\mathcal{D}_\circ/\mathcal{D}} \longrightarrow \mathcal{N}_{\mathcal{M}_\circ/\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}_\circ}} \mathcal{O}_{\mathcal{D}_\circ} \tag{9.5}$$

is surjective.

Following the proof of Lemma 9.1, $H^1((L^\bullet_{M_\circ/\mathcal{D}})^\vee)$ and $H^1((L^\bullet_{M_\circ/\mathcal{M}})^\vee)$ are canonically isomorphic to the pullbacks of the normal sheaves $\mathcal{N}_{\mathcal{D}_\circ/\mathcal{D}}$ and $\mathcal{N}_{\mathcal{M}_\circ/\mathcal{M}}$, respectively, and the arrow α_1 and α_2 are induced by the identity map of $\mathcal{N}_{\mathcal{D}_\circ/\mathcal{D}}$ and (9.5), respectively. This proves the statements about the sheaves and arrows in the top horizontal line.

Parallel to the proof Lemma 9.1, we obtain that the vertical arrows are injective with locally free cokernels.

By definition of $\tilde{V}_i$, the first two sheaves in the lower horizontal line are $\tilde{V}|_{W_\circ}$ and $\gamma_\circ^* V_1|_{M_\circ} = \tilde{V}_1|_{W_\circ}$, and the arrow $\tilde{\beta}_2|_{W_\circ}$ is the obvious projection as stated. The statement about $\gamma_\circ^*\beta_\circ$ and $\beta_\circ$ follows from [**CL1**, Lemm 2.8].

A direct calculation shows that $R^1\pi_{M_\circ *} f_{M_\circ}^* \mathcal{O}(1)^{\oplus 5}$ and $R^1\pi_{M_\circ *} f_{M_\circ}^* T_{\mathbb{P}^4}$ have rank five and four, respectively, and $\gamma_\circ^*\beta_\circ$ is surjective, therefore $\ker(\gamma_\circ^*\beta_\circ)$ is an invertible sheaf, and is isomorphic to $\ker(\alpha_2)$, using that the middle and the third vertical arrows are injective with locally free cokernel. Consequently, the cokernels of the middle and the third vertical arrows are isomorphic. These complete the proof of the Lemma.

q.e.d.

We now determine the image sheaf of the third vertical arrow in (9.3). Let $\xi = [f, C] \in M_\circ$ be a closed point. By the description of $M_\circ$, $C = E \cup R$ such that E (resp. R) is a smooth genus one curve (resp. $R \cong \mathbb{P}^1$); $p = E \cap R$ is the node of C, and $f|_E = $ const. and $f|_R : R \to \mathbb{P}^4$ is a regular embedding. Let $\underline{\xi} \in \mathcal{M}$ be the image of ξ under the tautological $M_\circ \to \mathcal{M}$. From the definition of $M_\circ$, we know that the image stack $\mathcal{M}_\circ := \mathrm{Im}\,(M_\circ \to \mathcal{M})$ is a locally closed smooth Cartier divisor of $\mathcal{M}$. Therefore, $H^1((L^\bullet_{M_\circ/\mathcal{M}})^\vee|_\xi) = N_{\mathcal{M}_\circ/\mathcal{M},\underline{\xi}}$ is one dimensional and is spanned by the image of any $v \in T_{\mathcal{M},\underline{\xi}} - T_{\mathcal{M}_\circ,\underline{\xi}}$ under the quotient map

$$T_{\mathcal{M},\underline{\xi}} = H^0((L^\bullet_{\mathcal{M}})^\vee|_{\underline{\xi}}) \longrightarrow H^1((L^\bullet_{M_\circ/\mathcal{M}})^\vee|_\xi).$$

Applying the deformation theory of $\mathcal{X}/\mathcal{M}$, the image of

$$H^1(\phi^\vee_{\mathcal{X}/\mathcal{M}})|_\xi : H^0((L^\bullet_{\mathcal{X}/\mathcal{M}})^\vee)|_\xi \longrightarrow H^1(E_{\mathcal{X}/\mathcal{M}})|_\xi$$

is the linear span of the image of any $v \in T_{\mathcal{M},\underline{\xi}} - T_{\mathcal{M}_\circ,\underline{\xi}}$ under the composite

$$\text{(9.6)} \qquad \mathrm{ob}_{\mathbb{P}^4} : H^0((L^\bullet_{\mathcal{M}})^\vee|_{\underline{\xi}}) \longrightarrow H^1((L^\bullet_{M_\circ/\mathcal{M}})^\vee|_\xi) \longrightarrow H^1(E_{\mathcal{X}/\mathcal{M}}|_\xi)$$

induced by the obstruction theory $\phi_{\mathcal{X}/\mathcal{M}}$. (It is the obstruction assignment map.) Note that because $f|_E$ is constant, we have

$$H^1(E_{\mathcal{X}/\mathcal{M}}|_\xi) = H^1(f^*T_{\mathbb{P}^4}) = H^1(\mathcal{O}_E)\otimes_{\mathbb{C}}(f^*T_{\mathbb{P}^4})|_p = H^1(\mathcal{O}_E)\otimes_{\mathbb{C}} T_{\mathbb{P}^4,f(p)}.$$

Lemma 9.3. *Let $\xi = [f, C]$, where $C = E \cup R$ and $p = E \cap R$ be as before; let $v \in T_{\mathcal{M},\underline{\xi}} - T_{\mathcal{M}_\circ,\underline{\xi}}$. The linear span of the image of v in $H^1(E_{\mathcal{X}/\mathcal{M}}|_\xi)$ is*

$$H^1(\mathcal{O}_E) \otimes_{\mathbb{C}} u_*(T_{R,p}) \subset H^1(\mathcal{O}_E) \otimes_{\mathbb{C}} T_{\mathbb{P}^4,f(p)}.$$

Proof. Let $H = f(R) \subset \mathbb{P}^4$. Since $f|_R$ is a regular embedding, $H \subset \mathbb{P}^4$ is a smooth rational curve. We let $f' : C \to H$ be the factorization of $f : C \to \mathbb{P}^4$. Thus $\xi' = [f', C] \in \overline{M}_1(H, d')$, where $d' = f'_*[R] \in H_2(H, \mathbb{Z})$

is a generator. We let $\overline{M}_1(H,d') \to \mathcal{M}$ be the tautological projection; thus $\underline{\xi} \in \mathcal{M}$ is also the image of ξ'. By the description of $\overline{M}_1(H,d')$ in [**Zi1**] or [**HL**], there is no first order deformation of $[f',C]$ in $\overline{M}_1(H,d')$ whose image in $T_{\underline{\xi}}\mathcal{M}$ is v. Let $H^1(E_{\overline{M}_1(H,d')/\mathcal{M}}|_{\xi'}) = H^1(C, u'^*T_{\mathbb{P}^4})$ be the obstruction space of the standard relative obstruction theory of $\overline{M}_1(H,d')/\mathcal{M}$. Then the image of v under the obstruction assignment

$$\mathrm{ob}_H : H^0((L^\bullet_{\mathcal{M}})^\vee)|_{\underline{\xi}}) \longrightarrow H^1(E_{\overline{M}_1(H,d')/\mathcal{M}}|_{\xi'}) = H^1(\mathcal{O}_E) \otimes_{\mathbb{C}} T_{H,f'(p)}$$

is non-zero. Since $\dim H^1(\mathcal{O}_E) = \dim T_{H,f'(p)} = 1$, $\mathrm{ob}_H(v)$ spans $H^1(\mathcal{O}_E) \otimes_{\mathbb{C}} T_{H,f'(p)}$.

Then, because the obstruction theories of moduli spaces of stable morphisms to schemes are compatible via morphisms between schemes, we conclude that the linear span of the image $\mathrm{ob}_{\mathbb{P}^4}(v) \subset H^1(E_{\mathcal{X}/\mathcal{M}}|_\xi)$ is identical to the image of the linear span of $\mathrm{ob}_H(f')$ under the canonical

$$H^1(E_{\overline{M}_1(H,d')/\mathcal{M}}|_{\xi'}) \longrightarrow H^1(E_{\overline{M}_1(\mathbb{P}^4,d)/\mathcal{M}}|_\xi) = H^1(E_{\mathcal{X}/\mathcal{M}}|_\xi).$$

Adding $T_{H,f'(p)} = f_*(T_{R,p})$ as subspaces in $T_{\mathbb{P}^4,f(p)}$, we prove the Lemma. q.e.d.

We consider the middle vertical arrow $H^1((L^\bullet_{M_\circ/\mathcal{D}})^\vee) \to H^1(E_{\mathcal{X}/\mathcal{D}}|_{M_\circ}) = V_1|_{M_\circ}$ in (9.3). By Lemma 9.2, $H^1((L^\bullet_{M_\circ/\mathcal{D}})^\vee)$ is a rank two locally free sheaf on $M_\circ$, and the arrow is injective with locally free cokernel. Let $S_\circ \subset V_1|_{M_\circ}$ be this image sub-vector bundle.

We continue to denote by $\bar\gamma : \tilde{\mathcal{Y}}^{\mathrm{cpt}}_{\mathrm{gst}} \to \tilde{\mathcal{X}}_{\mathrm{gst}}$ the projection (cf. (6.9)). We let $\overline{W}_\circ = \tilde{\mathcal{Y}}^{\mathrm{cpt}}_{\mathrm{gst}} \times_{\tilde{\mathcal{X}}_{\mathrm{gst}}} M_\circ$ and $\bar\gamma_\circ : \overline{W}_\circ \to M_\circ$ the projection. Recall that $\tilde V_1^{\mathrm{cpt}} = \bar\gamma^* V_1(-D_\infty)$. Thus $S_\circ \subset V_1|_{M_\circ}$ provides a subbundle

$$\overline{S}_\circ = \bar\gamma_\circ^* S_\circ(-D_\infty) \subset \tilde V_1^{\mathrm{cpt}}|_{\overline{W}_\circ}. \tag{9.7}$$

We let

$$\eta_\circ : \tilde V_1^{\mathrm{cpt}}|_{\overline{W}_\circ} \longrightarrow \tilde V^{\mathrm{cpt}}|_{\overline{W}_\circ} = (\tilde V_1^{\mathrm{cpt}} \oplus \tilde V_2^{\mathrm{cpt}})|_{\overline{W}_\circ}$$

be the inclusion; let $j_\circ : \overline{W}_\circ \to \tilde{\mathcal{Y}}^{\mathrm{cpt}}_{\mathrm{gst}}$ be the open embedding, thus flat. Recall $R_{(d)} = N_{\overline{C}_{(d),b}} \overline{C}_{(d)}$ (cf. before (8.2), see also (8.4)).

Lemma 9.4. *As cycles, we have*

$$j_\circ^*[R_{(d)}] = \eta_{\circ *}[\overline{S}_\circ] \in Z_*(\tilde V^{\mathrm{cpt}}|_{\overline{W}_\circ}). \tag{9.8}$$

Proof. Lemma 9.1 shows that $C_{(d)} \times_W W_\circ$ is a rank two subbundle of $\tilde V|_{W_\circ}$. By Proposition 7.1 and Lemma 6.8, we have $\overline{C}_{(d),b} \subset \tilde V_2^{\mathrm{cpt}}$, $\overline{C}_{(d),b} \cap \tilde V_2^{\mathrm{cpt}}|_{\overline{W}_\circ} = 0_{\tilde V_2^{\mathrm{cpt}}} \times_{\tilde{\mathcal{Y}}^{\mathrm{cpt}}_{\mathrm{gst}}} \overline{W}_\circ$, and $\overline{C}_{(d)} \times_{\tilde{\mathcal{Y}}^{\mathrm{cpt}}_{\mathrm{gst}}} \overline{W}_\circ$ is a rank two subbundle of $\tilde V^{\mathrm{cpt}}|_{\overline{W}_\circ}$. Further, they fit into the commutative diagram

(the left one is a Cartesian product)

$$\begin{array}{ccccc}
0_{\tilde{V}^{\mathrm{cpt}}} \times_{\tilde{\mathcal{Y}}^{\mathrm{cpt}}_{\mathrm{gst}}} \overline{W}_\circ & \longrightarrow & \overline{C}_{(d)} \times_{\tilde{\mathcal{Y}}^{\mathrm{cpt}}_{\mathrm{gst}}} \overline{W}_\circ & \xrightarrow{\cong} & \overline{S}_\circ \\
\Big\downarrow \subseteq & & \Big\downarrow \subseteq & & \Big\downarrow \subseteq \\
\tilde{V}_2^{\mathrm{cpt}}|_{\overline{W}_\circ} & \xrightarrow{\bar{\beta}_1|_{\overline{W}_\circ}} & \tilde{V}^{\mathrm{cpt}}|_{\overline{W}_\circ} = (\tilde{V}_1^{\mathrm{cpt}} \oplus \tilde{V}_2^{\mathrm{cpt}})|_{\overline{W}_\circ} & \xrightarrow{\bar{\beta}_2|_{\overline{W}_\circ}} & \tilde{V}_1^{\mathrm{cpt}}|_{\overline{W}_\circ},
\end{array}$$

where $\bar{\beta}_1|_{\overline{W}_\circ}$ and $\bar{\beta}_2|_{\overline{W}_\circ}$ are the obvious inclusion and projection. This implies that

$$R_{(d)} \times_{\tilde{\mathcal{Y}}^{\mathrm{cpt}}_{\mathrm{gst}}} \overline{W}_\circ = (N_{\overline{C}_{(d),b}} \overline{C}_{(d)}) \times_{\tilde{\mathcal{Y}}^{\mathrm{cpt}}_{\mathrm{gst}}} \overline{W}_\circ = \eta_\circ(\overline{S}_\circ) \subset \tilde{V}^{\mathrm{cpt}}|_{\overline{W}_\circ}.$$

Since $j_\circ : \overline{W}_\circ \to \tilde{\mathcal{Y}}^{\mathrm{cpt}}_{\mathrm{gst}}$ is an open embedding,

$$j_\circ^*[R_{(d)}] = [R_{(d)} \times_{\tilde{\mathcal{Y}}^{\mathrm{cpt}}_{\mathrm{gst}}} \overline{W}_\circ] = [\eta_\circ(\overline{S}_\circ)] = \eta_{\circ *}[\overline{S}_\circ] \in Z_*(\tilde{V}^{\mathrm{cpt}}|_{\overline{W}_\circ}).$$

This proves the Lemma. q.e.d.

We pick a degree d regular embedding $h : \mathbb{P}^1 \to \mathbb{P}^4$, viewed as a closed point in $\tilde{\mathcal{B}}_{(d)} = \overline{M}_0(\mathbb{P}^4, d)$. We form

$$M_h = \{[f, C] \in M_\circ \mid f|_R \cong h\} \subset M_\circ.$$

Using the convention introduced in the proof of Lemma 8.1, we have that $\tilde{\mathcal{Y}}^{\mathrm{cpt}}_{(d)} \cup \bar{F}_{(d)} \subset \tilde{V}^{\mathrm{cpt}}_{2,(d)}$, where $\tilde{\mathcal{Y}}^{\mathrm{cpt}}_{(d)}$ is the zero section of $\tilde{V}^{\mathrm{cpt}}_{2,(d)}$. We form $\overline{W}_h = \overline{W}_\circ \times_{M_\circ} M_h$ and the inclusions

$$j_h : \overline{W}_h \longrightarrow \tilde{\mathcal{Y}}^{\mathrm{cpt}}_{(d)} \cup \bar{F}_{(d)} \quad \text{and} \quad J_h : \tilde{V}_1^{\mathrm{cpt}}|_{\overline{W}_h} \longrightarrow \tilde{V}_1^{\mathrm{cpt}} \times_{\tilde{\mathcal{Y}}^{\mathrm{cpt}}_{(d)}} (\tilde{\mathcal{Y}}^{\mathrm{cpt}}_{(d)} \cup \bar{F}_{(d)}),$$

where the last term is viewed as a vector bundle over $\tilde{\mathcal{Y}}^{\mathrm{cpt}}_{(d)} \cup \bar{F}_{(d)}$. Since $j_h(\overline{W}_h) \cap \bar{F}_{(d)} = \emptyset$, both j_h and J_h are proper, regular embeddings; thus the Gysin map $j_h^!$ and $J_h^!$ are well-defined.

Lemma 9.5. *The constant c in* (9.1) *is* $c = \deg e(j_h^* \tilde{V}_1^{\mathrm{cpt}} / j_h^* \overline{S}_\circ)$.

Proof. Let $\phi_h : \overline{W}_h \to [h]$ be the projection to the point, and let $\phi_{(d)}$ be the projection that fits into the Cartesian product

$$\begin{array}{ccc}
\overline{W}_h & \xrightarrow{j_h} & Z_{(d)} = \tilde{\mathcal{Y}}^{\mathrm{cpt}}_{(d)} \cup \overline{F}_{(d)} \\
\Big\downarrow \phi_h & & \Big\downarrow \phi_{(d)} \\
[h] & \xrightarrow{\iota_h} & \tilde{\mathcal{B}}_{(d)}.
\end{array}$$

(Here $\phi_{(d)}$ is the composite of $Z_{(d)} \to \tilde{\mathcal{Y}}^{\mathrm{cpt}}_{(d)}$ mentioned after (8.3), the morphism $\bar{\gamma}_{(d)} : \tilde{\mathcal{Y}}^{\mathrm{cpt}}_{(d)} \to \tilde{\mathcal{X}}_{(d)}$, and the $\rho_{(d)} : \tilde{\mathcal{X}}_{(d)} \to \tilde{\mathcal{B}}_{(d)}$ constructed in (8.7).)

Since $j_h(\overline{W}_h) \cap \overline{F}_{(d)} = \emptyset$, $j_h^!(P_{(d),2}) = 0$. Thus

$$j_h^!(P_{(d),1}) = j_h^!((P_{(d),1}) + (P_{(d),2})) = j_h^! 0^!_{\tilde{V}^{\mathrm{cpt}}_{1,(d)}}[R_{(d)}].$$

Since Gysin maps commute, we obtain

$$j_h^!(P_{(d),1}) = j_h^! 0^!_{\tilde V^{\mathrm{cpt}}_{1,(d)}}[R_{(d)}] = 0^!_{j_h^*\tilde V_1^{\mathrm{cpt}}}[J_h^! R_{(d)}]$$

$$= 0^!_{j_h^*\tilde V_1^{\mathrm{cpt}}}[\overline{S}_\circ|_{\overline{W}_h}] = e(j_h^*\tilde V_1^{\mathrm{cpt}}/j_h^*\overline{S}_\circ).$$

On the other hand, since the Gysin maps commute with proper push-forwards,

$$\phi_{h*}\big(e(j_h^*\tilde V_1^{\mathrm{cpt}}/j_h^*\overline{S}_\circ)\big) = \phi_{h*}\big(j_h^!(P_{(d),1})\big)$$

$$= \iota_h^!\big(\phi_{(d)*}(P_{(d),1})\big) = \iota_h^!\big(c[\tilde{\mathcal{B}}_{(d)}]\big) = c.$$

This proves the Lemma. q.e.d.

Finally, we evaluate c. Let $\mathcal{E} \to \overline{M}_{1,1}$ with $s_1 : \overline{M}_{1,1} \to \mathcal{E}$ be the universal family of (the moduli of pointed elliptic curves) $\overline{M}_{1,1}$. We form $B = \overline{M}_{1,1} \times \mathbb{P}^1$ and construct a family of stable morphisms in M_h, as follows. First, for any $(a,b) \in B = \overline{M}_{1,1} \times \mathbb{P}^1$, with (E_a, s_a) the pointed elliptic curve associated to $a \in \overline{M}_{1,1}$, we identify $s_a \in E_a$ with $b \in \mathbb{P}^1$ to obtain a genus one (connected) curve $E_{(a,b)}$ that has two irreducible components E_a and $\mathbb{P}^1$. We then define $f_{(a,b)} : E_{(a,b)} \to \mathbb{P}^4$ so that $f_{(a,b)}|_{\mathbb{P}^1} = h : \mathbb{P}^1 \to \mathbb{P}^4$, and $f_{(a,b)}$ restricting to E_a is constant. Clearly, this construction can be carried out in family version, thus resulting a family of stable morphisms

$$(\pi_B, f_B) : \mathcal{C}_B \longrightarrow B \times \mathbb{P}^4.$$

Thus it induces a morphism $B \to \mathcal{X}$ that factors through $M_h \subset \mathcal{X}$ and induces an isomorphism $B \cong M_h$. In the following, we will not distinguish B from M_h; in particular, $(\mathcal{C}_B, f_B)$ is the tautological family on $B \cong M_h \subset \mathcal{X}$.

Let q_1 and q_2 be the first and the second projections from B to $\overline{M}_{1,1}$ and $\mathbb{P}^1$ respectively; let $\mathcal{H}_B = \pi_{B*}\omega_{\mathcal{C}_B/B} \cong (R^1\pi_{B*}\mathcal{O}_{\mathcal{C}_B})^\vee$ to be the Hodge bundle over B. Using the family $(\mathcal{C}_B, f_B)$, and because of (9.3), Lemma 9.2 and Lemma 9.3, we have

$$(V_1/S_\circ)|_{M_h} = \mathcal{H}_B^\vee \otimes_{\mathcal{O}_B} q_2^*(h^*T_{\mathbb{P}^4}/T_{\mathbb{P}^1}) = \mathcal{H}_B^\vee \otimes_{\mathcal{O}_B} q_2^* N_{R/\mathbb{P}^4},$$

where $R = h(\mathbb{P}^1) \subset \mathbb{P}^4$, and $N_{R/\mathbb{P}^4}$ is the normal bundle to R in $\mathbb{P}^4$. Also, for the line bundle L on M defined before (6.1) that gives $W = \mathrm{Tot}(L)$, we have

$$L_B := L|_{M_h} = \pi_{B*}(f_B^*\mathcal{O}(-5) \otimes \omega_{\mathcal{C}_B/B}) \cong \mathcal{H}_B \otimes_{\mathcal{O}_B} q_2^* h^*\mathcal{O}(-5). \tag{9.9}$$

Thus, $\overline{W}_h = \mathbb{P}_B(L_B \oplus \mathcal{O}_B)$, and for $j_h : \overline{W}_h \to Z_{(d)} \supset \tilde{\mathcal{Y}}^{\mathrm{cpt}}_{(d)}$, we have $D_B := \mathbb{P}(L_B \oplus 0) = j_h^{-1}(D_\infty)$. Denote $\bar\gamma_h : \overline{W}_h \to M_h$ to be the $\mathbb{P}^1$-bundle projection. Following the construction of $\tilde V_1^{\mathrm{cpt}}$ and $\overline{S}_\circ$, we have

$$j_h^*\tilde V_1^{\mathrm{cpt}}/j_h^*\overline{S}_\circ \cong \bar\gamma_h^*(q_2^* N_{R/\mathbb{P}^4} \otimes \mathcal{H}_B^\vee)(-D_B). \tag{9.10}$$

Let $\zeta = 24 \cdot c_1(\mathcal{H}) \in A^1\overline{M}_{1,1}$ and $\xi = \frac{1}{2}(c(T_{\mathbb{P}^1}) - 1) \in A^1\mathbb{P}^1$, where $\mathcal{H}$ is the Hodge bundle on $\overline{M}_{1,1}$. We calculate $c(h^*T_{\mathbb{P}^4}) = 1 + 5d\xi$, and $c(N_{R/\mathbb{P}^4}) = 1 + (5d-2)\xi$. Let $\bar{\xi} = \bar{\gamma}_h^* q_2^* \xi$ and $\bar{\zeta} = \bar{\gamma}_h^* q_1^* \zeta \in A^1\overline{W}_h$ to be the pullbacks of ξ and ζ via $\overline{W}_h$ to $\mathbb{P}^1$ and to $\overline{M}_{1,1}$, respectively. We calculate

$$c(\bar{\gamma}_h^* q_2^* N_{R/\mathbb{P}^4}) = 1 + (5d-2)\bar{\xi} \quad \text{and} \quad c(\bar{\gamma}_h^* q_1^* \mathcal{H}^\vee) = 1 - \frac{1}{24}\bar{\zeta}. \tag{9.11}$$

Let $F \in A^2\overline{W}_h$ be the Poincare dual of the fiber class of $\bar{\gamma}_h : \overline{W}_h \to M_h$. Using $\bar{\xi}\bar{\zeta} = F$ and $\bar{\xi}^2 = \bar{\zeta}^2 = 0$, (9.11) gives

$$c\big(\bar{\gamma}_h^*(q_2^* N_{R/\mathbb{P}^4} \otimes q_1^* \mathcal{H}^\vee)\big) = 1 + \left((5d-2)\bar{\xi} - \frac{1}{8}\bar{\zeta}\right) - \frac{5d-2}{12} \cdot F.$$

Hence the euler class

$$\begin{aligned} &e(j_h^* \tilde{V}_1^{\mathrm{cpt}} / j_h^* \overline{S}_\circ) \\ &\quad = [-D_B]^3 + \left((5d-2)\bar{\xi} - \frac{1}{8}\bar{\zeta}\right) \cdot [D_B]^2 - (5d-2)/12 \cdot F \cdot [-D_B]. \end{aligned} \tag{9.12}$$

(Here we view $[D_B] \in A^1\overline{W}_h$ as the Poincare dual of the cycle D_B in $\overline{W}_h$.)

Let $\tau : D_B \to B = M_h$ be the projection (isomorphism). We compute each term in the above formula. First, direct calculations using (9.9) give

$$c_1(N_{D_B/\overline{W}_h}) = -\frac{1}{24}\tau^* q_1^* \zeta + 5d\tau^* q_2^* \xi,$$

$$\bar{\xi} \cdot [D_B]^2 = \frac{-1}{24} \quad \text{and} \quad \bar{\zeta} \cdot [D_B]^2 = 5d.$$

Thus the middle term in (9.12) is $\frac{1}{12} - \frac{5d}{6}$. The $c_1(N_{D_B/\overline{W}_h})$ just calculated implies $[D_B] \cdot \tau^* q_2^* \xi = \frac{-1}{24}$ and $[D_B] \cdot \tau^* q_1^* \zeta = 5d$. Using $[D_B]^2 = c_1(N_{D_B/\overline{W}_h})$, we obtain $-[D_B]^3 = -[D_B]^2 \cdot [D_B] = \frac{5d}{24} + \frac{5d}{24} = \frac{5d}{12}$. Using $[F] \cdot [D_B] = 1$ we conclude that (9.12) is equal to $-\frac{1}{12}$. By Lemma 9.5, $c = -\frac{1}{12}$.

Proof of Proposition 4.7. By (8.2), Lemma 8.1 and (9.2) we conclude

$$\deg 0^!_{\tilde{\sigma},\mathrm{loc}}[\mathbf{C}_{(d)}] = \deg 0^!_{\tilde{V}^{\mathrm{cpt}}_{2,(d)}}(P_{(d),1})$$

$$= c \cdot \deg c_{5d+1}(\tilde{\mathcal{K}}_{(d)})[\tilde{\mathcal{B}}_{(d)}] = \frac{(-1)^{5d}}{12} N_0(d)_Q.$$

This completes the algebro-geometric proof of the hyperplane property of the reduced genus one GW-invariants of the quintics. q.e.d.

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