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NON-HYPERBOLIC CLOSED GEODESICS ON FINSLER SPHERES

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Abstract

In this paper, we prove that for every bumpy Finsler $2k$*-sphere* (S^{2k}, F) *with reversibility* λ *and flag curvature* K *satisfying the pinching condition* $\left(\frac{\lambda}{\lambda+1}\right)^2 < K \leq 1$*, either there exist infinitely many closed geodesics or there exist at least* $2k$ *non-hyperbolic closed geodesics. Due to the example* [**Kat**] *of A. B. Katok, this estimate is sharp.*

1. Introduction and main results

This paper is devoted to a study on closed geodesics on Finsler spheres. Let us recall firstly the definition of Finsler metrics.

Definition 1.1. (cf. [**BCS**], [**She**]) Let M be a finite dimensional smooth manifold. A function $F : TM \to [0, +\infty)$ is a Finsler metric if it satisfies

(F1) F is C^∞ on $TM \setminus \{0\}$.
(F2) $F(x, \lambda y) = \lambda F(x, y)$ for all $y \in T_xM$, $x \in M$, and $\lambda > 0$.
(F3) For every $y \in T_xM \setminus \{0\}$, the quadratic form

$$g_{x,y}(u, v) \equiv \frac{1}{2}\frac{\partial^2}{\partial s \partial t}F^2(x, y + su + tv)|_{t=s=0}, \qquad \forall u, v \in T_xM,$$

is positive definite.

In this case, (M, F) is called a Finsler manifold. F is reversible if $F(x, -y) = F(x, y)$ holds for all $y \in T_xM$ and $x \in M$. F is Riemannian if $F(x, y)^2 = \frac{1}{2}G(x)y \cdot y$ for some symmetric positive definite matrix function $G(x) \in GL(T_xM)$ depending on $x \in M$ smoothly.

A closed curve on a Finsler manifold is a closed geodesic if it is locally the shortest path connecting any two nearby points on this curve (cf. [**She**]). As usual, on any Finsler n-sphere $S^n = (S^n, F)$, a closed geodesic $c : S^1 = \mathbf{R}/\mathbf{Z} \to S^n$ is *prime* if it is not a multiple covering (i.e., iteration) of any other closed geodesics. Here the m-th iteration c^m of c is defined by $c^m(t) = c(mt)$. The inverse curve c^{-1} of c is defined by $c^{-1}(t) = c(1 - t)$ for $t \in \mathbf{R}$. Note that on a non-reversible Finsler

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manifold, the inverse curve of a closed geodesic is not a closed geodesic in general. We call two prime closed geodesics c and d *distinct* if there is no $\theta \in (0,1)$ such that $c(t)=d(t+\theta)$ for all $t\in \mathbf{R}$. We shall omit the word *distinct* when we talk about more than one prime closed geodesic. On a symmetric Finsler (or Riemannian) n-sphere, two closed geodesics c and d are called *geometrically distinct* if $c(S^1)\neq d(S^1)$, i.e., their image sets in S^n are distinct.

For a closed geodesic c on $(S^n,\,F)$, denote by P_c the linearized Poincaré map of c. Then $P_c\in \mathrm{Sp}(2n-2)$ is symplectic. For any $M\in \mathrm{Sp}(2k)$, we define the *elliptic height* $e(M)$ of M to be the total algebraic multiplicity of all eigenvalues of M on the unit circle $\mathbf{U}=\{z\in \mathbf{C}|\ |z|=1\}$ in the complex plane $\mathbf{C}$. Since M is symplectic, $e(M)$ is even and $0\le e(M)\le 2k$. A closed geodesic c is called *elliptic* if $e(P_c)=2(n-1)$, i.e., all the eigenvalues of P_c locate on $\mathbf{U}$; *hyperbolic* if $e(P_c)=0$, i.e., all the eigenvalues of P_c locate away from $\mathbf{U}$; and *non-degenerate* if 1 is not an eigenvalue of P_c. A Finsler sphere $(S^n,\,F)$ is called *bumpy* if all the closed geodesics on it are non-degenerate.

Following H.-B. Rademacher in [**Rad3**], the reversibility $\lambda=\lambda(M,\,F)$ of a compact Finsler manifold $(M,\,F)$ is defined to be

$$\lambda := \max\{F(-X)\,|\,X\in TM,\,F(X)=1\}\ge 1.$$

There is a famous conjecture in Riemannian geometry which claims there exist infinitely many geometrically distinct closed geodesics on any compact Riemannian manifold. By the contribution of many mathematicians, e.g., R. Bott [**Bot**], D. Gromol & W. Meyer [**GrM**], W. Klingenberg [**Kli1**], M. Vigué-Poirrier & D. Sullivan [**VSu**], and W. Ziller [**Zil1**], this conjecture has been proved except for CROSSs (compact rank one symmetric spaces). In [**Hin**] of 1984, N. Hingston proved that a Riemannian metric on a CROSS all of whose closed geodesics are hyperbolic carries infinitely many geometrically distinct closed geodesics. The results of J. Franks [**Fra**] in 1992 and V. Bangert [**Ban**] in 1993 imply this conjecture is true for any Riemannian 2-sphere.

When one considers the Finsler case, the above conjecture becomes false. It was quite surprising when Katok [**Kat**] in 1973 found some non-reversible Finsler metrics on CROSS with only finitely many prime closed geodesics and with all closed geodesics non-degenerate and elliptic. The smallest number of closed geodesics on S^n that one obtains in these examples is $2[\frac{n+1}{2}]$ (cf. [**Zil2**]). Then D. V. Anosov in I.C.M. of 1974 conjectured that the lower bound of the number of closed geodesics on any Finsler sphere $(S^n,\,F)$ should be $2[\frac{n+1}{2}]$, i.e., the number of closed geodesics in Katok's example.

For the stability of closed geodesics, there is a problem: Suppose M is a compact Riemannian manifold with finite fundamental group; then does there exist a non-hyperbolic closed geodesic on M? In [**BTZ1**] and [**BTZ2**], W. Ballmann, G. Thorbergsson, and W. Ziller studied the existence and stability of closed geodesics on positively curved Riemannian manifolds. In particular, they proved that for a Riemannian metric on S^n with sectional curvature $1/4 \leq K \leq 1$ there exist $g(n)$ geometrically distinct closed geodesics, and $\frac{n(n+1)}{2}$ geometrically distinct closed geodesics if the metric is bumpy; for a Riemannian metric on S^n with sectional curvature $\left(\frac{2n-2}{2n-1}\right)^2 \leq K \leq 1$ there exist $g(n)-1$ non-hyperbolic closed geodesics, and $]n^2/2[$ non-hyperbolic closed geodesics if the metric is bumpy.

We have the following main result in this paper.

Theorem 1.2. *For every bumpy Finsler $2k$-sphere $(S^{2k},\, F)$ with reversibility λ and flag curvature K satisfying $\left(\frac{\lambda}{\lambda+1}\right)^2 < K \leq 1$, there exist at least $2k$ non-hyperbolic prime closed geodesics, provided the number of prime closed geodesics on $(S^{2k},\, F)$ is finite.*

Remark 1.3. Note that Katok metrics on S^n have constant flag curvature 1 (cf. p. 764 in [**Rad5**]); moreover, all the closed geodesics on it are non-hyperbolic. Hence the lower bound for the number of non-hyperbolic closed geodesics in Theorem 1.2 is sharp. In [**BaL**], V. Bangert and Y. Long proved that on any Finsler 2-sphere (S^2, F), there exist at least two prime closed geodesics, which solves Anosov's conjecture for the S^2 case. In [**LoW2**] of Y. Long and the author, they further proved the existence of at least two irrationally elliptic prime closed geodesics on every Finsler 2-sphere $(S^2,\, F)$, provided the number of prime closed geodesics is finite. In [**Rad4**], H.-B. Rademacher studied the existence and stability of closed geodesics on positively curved Finsler manifolds. In particular, he proved the existence of at least $n/2-1$ prime closed geodesics with *length* $< 2n\pi$ on every Finsler n-sphere $(S^n,\, F)$ satisfying $\left(\frac{\lambda}{\lambda+1}\right)^2 < K \leq 1$. In a series of papers [**LoD**], [**DuL1**]–[**DuL3**], Y. Long and H. Duan proved there exist two prime closed geodesics on any compact simply connected Finsler or Riemannian manifold. In [**Rad6**], H.-B. Rademacher proved there exist two prime closed geodesics on any bumpy n-sphere. In [**W1**], the author proved there exist three prime closed geodesics on any (S^3, F) satisfying $(\frac{\lambda}{\lambda+1})^2 < K \leq 1$. In [**W2**], the author proved there exist $2[\frac{n+1}{2}]$ prime closed geodesics on any bumpy (S^n, F) satisfying $(\frac{\lambda}{\lambda+1})^2 < K \leq 1$.

Our proof of Theorem 1.2 contains mainly three ingredients: the common index jump theorem of Y. Long, Morse theory, and the mean index equality of H.-B. Rademacher. In fact, we get $2k-2$ non-hyperbolic prime closed geodesics as a straightforward consequence of the common index jump theorem and equivariant Morse theory (cf. Step 1 in §5). Then we use the mean index equality and Morse inequality to get the next-to-the-last elliptic prime closed geodesic (cf. Step 2 in §5). Finally we get the last non-hyperbolic prime closed geodesic by the index iteration theory and Morse inequality (cf. Step 3 in §5).

Fix a Finsler metric F on S^n. Let $\Lambda = \Lambda S^n$ be the free loop space of S^n, which is a Hilbert manifold. For definition and basic properties of Λ, we refer readers to [**Kli2**] and [**Kli3**]. Let $E(c) = \frac{1}{2}\int_0^1 F(\dot{c}(t))^2 dt$ be the energy functional on Λ. In this paper for $\kappa \in \mathbf{R}$ we denote

$$\Lambda^\kappa = \{d \in \Lambda \mid E(d) \le \kappa\}, \tag{1.1}$$

and consider the quotient space Λ/S^1. Since the energy functional E is S^1-invariant, the negative gradient flow of E induces a flow on Λ/S^1, so we can apply Morse theory on Λ/S^1. By a result of H.-B. Rademacher in [**Rad1**] of 1989, we get the Morse series of the space pair $(\Lambda/S^1, \Lambda^0/S^1)$ with rational coefficients. The reason we use $(\Lambda/S^1, \Lambda^0/S^1)$ instead of (Λ, Λ^0) is that the Morse series of the first is lacunary. One other important reason is that in the non-degenerate case on a Finsler manifold if we take the homology of Λ/S^1 (or equivariant homology for the group S^1) with rational coefficients, the local critical $\mathbf{Q}$-module of a closed geodesic has rank 1, with a generator in dimension of the index.

In this paper, let $\mathbf{N}$, $\mathbf{N}_0$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, and complex numbers respectively. We use only singular homology modules with $\mathbf{Q}$-coefficients. For terminologies in algebraic topology we refer to [**GrH**]. For $k \in \mathbf{N}$, we denote by $\mathbf{Q}^k$ the direct sum $\mathbf{Q} \oplus \cdots \oplus \mathbf{Q}$ of k copies of $\mathbf{Q}$ and $\mathbf{Q}^0 = 0$. For an S^1-space X, we denote by $\overline{X}$ the quotient space X/S^1. We define the functions

$$\left\{\begin{array}{ll} [a] = \max\{k \in \mathbf{Z} \mid k \le a\}, & \mathcal{E}(a) = \min\{k \in \mathbf{Z} \mid k \ge a\}, \\ \varphi(a) = \mathcal{E}(a) - [a], & \{a\} = a - [a]. \end{array}\right. \tag{1.2}$$

Especially, $\varphi(a) = 0$ if $a \in \mathbf{Z}$, and $\varphi(a) = 1$ if $a \notin \mathbf{Z}$.

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2. Morse theory for closed geodesics

In this section, we review the variational structures of closed geodesics; all the details can be found in [**Rad2**] or [**BaL**].

On a compact Finsler manifold (M, F), we choose an auxiliary Riemannian metric. This endows the space $\Lambda = \Lambda M$ of H^1-maps $\gamma : S^1 \to M$ with a natural structure of Riemannian Hilbert manifold on which the group $S^1 = \mathbf{R}/\mathbf{Z}$ acts continuously by isometries; cf. [**Kli2**], Chapters 1 and 2. This action is defined by translating the parameter, i.e.,

$$(s \cdot \gamma)(t) = \gamma(t + s)$$

for all $\gamma \in \Lambda$ and $s, t \in S^1$. The Finsler metric F defines an energy functional E and a length functional L on Λ by

$$E(\gamma) = \frac{1}{2} \int_{S^1} F(\dot{\gamma}(t))^2 dt, \quad L(\gamma) = \int_{S^1} F(\dot{\gamma}(t)) dt. \tag{2.1}$$

Both functionals are invariant under the S^1-action. The critical points of E of positive energies are precisely the closed geodesics $c : S^1 \to M$ of the Finsler structure. If $c \in \Lambda$ is a closed geodesic, then c is a regular curve, i.e., $\dot{c}(t) \neq 0$ for all $t \in S^1$, and this implies that the second differential $E''(c)$ of E at c exists. As usual we define the index $i(c)$ of c as the maximal dimension of subspaces of $T_c\Lambda$ on which $E''(c)$ is negative definite, and the nullity $\nu(c)$ of c so that $\nu(c) + 1$ is the dimension of the null space of $E''(c)$.

For $m \in \mathbf{N}$ we denote the m-fold iteration map $\phi^m : \Lambda \to \Lambda$ by

$$\phi^m(\gamma)(t) = \gamma(mt) \qquad \forall\, \gamma \in \Lambda, t \in S^1. \tag{2.2}$$

We also use the notation $\phi^m(\gamma) = \gamma^m$. For a closed geodesic c, the mean index is defined to be:

$$\hat{i}(c) = \lim_{m \to \infty} \frac{i(c^m)}{m}. \tag{2.3}$$

If $\gamma \in \Lambda$ is not constant then the multiplicity $m(\gamma)$ of γ is the order of the isotropy group $\{s \in S^1 \mid s \cdot \gamma = \gamma\}$. If $m(\gamma) = 1$ then γ is called *prime*. Hence $m(\gamma) = m$ if and only if there exists a prime curve $\tilde{\gamma} \in \Lambda$ such that $\gamma = \tilde{\gamma}^m$.

For a closed geodesic c we set

$$\Lambda(c) = \{\gamma \in \Lambda \mid E(\gamma) < E(c)\}.$$

If $A \subseteq \Lambda$ is invariant under some subgroup Γ of S^1, we denote by A/Γ the quotient space of A with respect to the action of Γ.

Using singular homology with rational coefficients, we will consider the following critical $\mathbf{Q}$-module of a closed geodesic $c \in \Lambda$:

$$\overline{C}_*(E, c) = H_*\left((\Lambda(c) \cup S^1 \cdot c)/S^1, \Lambda(c)/S^1\right). \tag{2.4}$$

We say a closed geodesic satisfies the isolation condition, if the following holds:

(Iso) For all $m \in \mathbf{N}$, the orbit $S^1 \cdot c^m$ is an isolated critical orbit of E.

Note that if the number of prime closed geodesics on a Finsler manifold is finite, then all the closed geodesics satisfy (Iso).

The following propositions were proved in [**Rad2**] and [**BaL**].

Proposition 2.1. *(cf. Satz 6.11 of* [**Rad2**] *or Proposition 3.12 of* [**BaL**]*) Let c be a prime closed geodesic on a bumpy Finsler manifold (M,F) satisfying (Iso). Then we have*

$$\overline{C}_q(E,c^m) = \begin{cases} \mathbf{Q}, & \text{if } \ i(c^m)-i(c)\in 2\mathbf{Z}, \text{ and } q=i(c^m), \\ 0, & \text{otherwise}. \end{cases} \tag{2.5}$$

Now we briefly describe the relative homological structure of the quotient space $\overline{\Lambda}\equiv\overline{\Lambda}S^n$. Here we have $\overline{\Lambda}^0S^n=\{\text{constant curves in } S^n\}\cong S^n$.

Theorem 2.2 (cf. p. 104 of [**Hin**], Theorem 2.4 of [**Rad1**])**.** *We have the Poincaré series: Let $n=2k$ be even; then we have*

$$\begin{aligned} P(\overline{\Lambda}S^n,\overline{\Lambda}^0S^n)(t) &= t^{n-1}\left(\frac{1}{1-t^2}+\frac{t^{n(m+1)-2}}{1-t^{n(m+1)-2}}\right)\frac{1-t^{nm}}{1-t^n} \\ &= t^{2k-1}\left(\frac{1}{1-t^2}+\frac{t^{4k-2}}{1-t^{4k-2}}\right), \end{aligned} \tag{2.6}$$

where $m=1$ by Theorem 2.4 of [**Rad1**]*. Thus for $q\in\mathbf{Z}$ and $l\in\mathbf{N}_0$, we have*

$$\begin{aligned} b_q &= b_q(\overline{\Lambda}S^n,\overline{\Lambda}^0S^n) \\ &= \operatorname{rank} H_q(\overline{\Lambda}S^n,\overline{\Lambda}^0S^n) \\ &= \begin{cases} 2, & \text{if } q\in\{6k-3+2l,\ l=0 \bmod 2k-1\}, \\ 1, & \text{if } q\in\{2k-1\}\cup\{2k-1+2l,\ l\neq 0 \bmod 2k-1\}, \\ 0 & \text{otherwise}. \end{cases} \end{aligned} \tag{2.7}$$

We have the following version of the Morse inequalities.

Theorem 2.3 (Theorem 6.1 of [**Rad2**])**.** *Suppose that there exist only finitely many prime closed geodesics $\{c_j\}_{1\le j\le p}$ on (M,F), and $0\le a<b\le\infty$ are regular values of the energy functional E. Define for each $q\in\mathbf{Z}$,*

$$\begin{aligned} M_q(\overline{\Lambda}^b,\overline{\Lambda}^a) &= \sum_{1\le j\le p,\ a<E(c_j^m)<b} \operatorname{rank}\overline{C}_q(E,c_j^m) \\ b_q(\overline{\Lambda}^b,\overline{\Lambda}^a) &= \operatorname{rank} H_q(\overline{\Lambda}^b,\overline{\Lambda}^a). \end{aligned}$$

Then there holds

$$\begin{aligned} &M_q(\overline{\Lambda}^b,\overline{\Lambda}^a)-M_{q-1}(\overline{\Lambda}^b,\overline{\Lambda}^a)+\cdots+(-1)^qM_0(\overline{\Lambda}^b,\overline{\Lambda}^a) \\ &\qquad \ge b_q(\overline{\Lambda}^b,\overline{\Lambda}^a)-b_{q-1}(\overline{\Lambda}^b,\overline{\Lambda}^a)+\cdots+(-1)^qb_0(\overline{\Lambda}^b,\overline{\Lambda}^a), \end{aligned} \tag{2.8}$$

$$M_q(\overline{\Lambda}^b,\overline{\Lambda}^a)\ge b_q(\overline{\Lambda}^b,\overline{\Lambda}^a). \tag{2.9}$$

3. Index iteration theory for closed geodesics

Let c be a closed geodesic on a Finsler n-sphere $S^n = (S^n,\, F)$. Denote the linearized Poincaré map of c by $P_c \in \mathrm{Sp}(2n-2)$. Then P_c is a symplectic matrix. Note that the index iteration formulae in [**Lon3**] of 2000 (cf. Chap. 8 of [**Lon4**]) work for Morse indices of iterated closed geodesics (cf. [**LiL**], Chap. 12 of [**Lon4**]). Since every closed geodesic on a sphere must be orientable, then by Theorem 1.1 of [**Liu**] of C. Liu (cf. also [**Wil**]), the initial Morse index of a closed geodesic c on a n-dimensional Finsler sphere coincides with the index of a corresponding symplectic path introduced by C. Conley, E. Zehnder, and Y. Long in 1984–1990 (cf. [**Lon4**]).

Note that the precise index iteration formulae of Y. Long (cf. Theorem 8.3.1 of [**Lon4**]) are established upon the decomposition of the end matrix $\gamma(\tau)$ of the symplectic path $\gamma : [0, \tau] \to \mathrm{Sp}(2n)$ within $\Omega^0(\gamma(\tau))$ in Theorem 1.8.10 and the first part of Theorem 8.3.1 of [**Lon4**], which leads to the 2×2 or 4×4 basic normal form decomposition of $\gamma(\tau)$. Especially it is proved in Lemma 9.1.5 of [**Lon4**] that the splitting numbers of M are constants on $\Omega^0(M)$, where

$$\begin{aligned}\Omega(M) = \{N \in \mathrm{Sp}(2n) \mid \sigma(N) \cap \mathbf{U} = \sigma(M) \cap \mathbf{U},\\ \dim_{\mathbf{C}} \ker_{\mathbf{C}}(N - \lambda I) = \dim_{\mathbf{C}} \ker_{\mathbf{C}}(M - \lambda I),\ \forall \lambda \in \sigma(M) \cap \mathbf{U}\},\end{aligned}$$

where $\mathbf{U} = \{z \in \mathbf{C} \,|\, |z| = 1\}$. $\Omega^0(M)$ is defined to be the path connected component of $\Omega(M)$ which contains M. The Bott iteration formulae in [**Bot**] and [**BTZ1**] are based on decomposition of the end matrix $\gamma(\tau)$ of the symplectic path $\gamma : [0, \tau] \to \mathrm{Sp}(2n)$ within $[\gamma(\tau)]$, the conjugate set of $\gamma(\tau)$. Especially it is proved that the splitting numbers of M in [**Bot**] and [**BTZ1**] are constants on $[M] \equiv \{P^{-1}MP \,|\, P \in \mathrm{Sp}(2n)\}$. Note that $[M]$ is a proper subset of $\Omega^0(M)$ in general for $M \in \mathrm{Sp}(2n)$. Note also that there are only 11 basic normal forms (cf. [**Lon4**]), and they are only 2×2 or 4×4 matrices. Thus they are simpler than usual normal forms, and then it is possible to use different patterns of the iteration formula Theorem 8.3.1 of [**Lon4**] to classify symplectic paths as well as closed geodesics to carry out proofs. This is a major difference between formulae established in [**Lon3**] and Bott-type formulae established in [**Bot**], [**BTZ1**], and [**Lon2**]. Hence in this section we recall briefly the index theory for symplectic paths. All the details can be found in [**Lon4**] or [**LoZ**].

As usual, the symplectic group $\mathrm{Sp}(2n)$ is defined by

$$\mathrm{Sp}(2n) = \{M \in \mathrm{GL}(2n, \mathbf{R}) \,|\, M^T J M = J\},$$

whose topology is induced from that of $\mathbf{R}^{4n^2}$, where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ and I_n is the identity matrix in $\mathbf{R}^n$. For $\tau > 0$ we are interested in paths in $\mathrm{Sp}(2n)$:

$$\mathcal{P}_\tau(2n) = \{\gamma \in C([0,\tau], \mathrm{Sp}(2n)) \,|\, \gamma(0) = I_{2n}\},$$

which is equipped with the topology induced from that of $\mathrm{Sp}(2n)$. The following real function was introduced in [**Lon2**]:

$$D_\omega(M) = (-1)^{n-1}\overline{\omega}^n \det(M - \omega I_{2n}), \qquad \forall \omega \in \mathbf{U},\ M \in \mathrm{Sp}(2n).$$

Thus for any $\omega \in \mathbf{U}$ the following codimension 1 hypersurface in $\mathrm{Sp}(2n)$ is defined in [**Lon2**]:

$$\mathrm{Sp}(2n)^0_\omega = \{M \in \mathrm{Sp}(2n) \,|\, D_\omega(M) = 0\}.$$

For any $M \in \mathrm{Sp}(2n)^0_\omega$, we define a co-orientation of $\mathrm{Sp}(2n)^0_\omega$ at M by the positive direction $\frac{d}{dt}Me^{t\epsilon J}|_{t=0}$ of the path $Me^{t\epsilon J}$ with $0 \le t \le 1$ and $\epsilon > 0$ being sufficiently small. Let

$$\begin{aligned} \mathrm{Sp}(2n)^*_\omega &= \mathrm{Sp}(2n) \setminus \mathrm{Sp}(2n)^0_\omega, \\ \mathcal{P}^*_{\tau,\omega}(2n) &= \{\gamma \in \mathcal{P}_\tau(2n) \,|\, \gamma(\tau) \in \mathrm{Sp}(2n)^*_\omega\}, \\ \mathcal{P}^0_{\tau,\omega}(2n) &= \mathcal{P}_\tau(2n) \setminus \mathcal{P}^*_{\tau,\omega}(2n). \end{aligned}$$

For any two continuous arcs ξ and $\eta : [0,\tau] \to \mathrm{Sp}(2n)$ with $\xi(\tau) = \eta(0)$, it is defined as usual:

$$\eta * \xi(t) = \begin{cases} \xi(2t), & \text{if } 0 \le t \le \tau/2, \\ \eta(2t-\tau), & \text{if } \tau/2 \le t \le \tau. \end{cases}$$

Given any two $2m_k \times 2m_k$ matrices of square block form $M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$ with $k = 1, 2$, as in [**Lon4**], the $\diamond$-product of M_1 and M_2 is defined by the following $2(m_1+m_2) \times 2(m_1+m_2)$ matrix $M_1 \diamond M_2$:

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

Denote by $M^{\diamond k}$ the k-fold $\diamond$-product $M \diamond \cdots \diamond M$. Note that the $\diamond$-product of any two symplectic matrices is symplectic. For any two paths $\gamma_j \in \mathcal{P}_\tau(2n_j)$ with $j = 0$ and 1, let $\gamma_0 \diamond \gamma_1(t) = \gamma_0(t) \diamond \gamma_1(t)$ for all $t \in [0,\tau]$.

A special path ξ_n is defined by

$$\xi_n(t) = \begin{pmatrix} 2 - \frac{t}{\tau} & 0 \\ 0 & (2 - \frac{t}{\tau})^{-1} \end{pmatrix}^{\diamond n} \qquad \text{for } 0 \le t \le \tau. \tag{3.1}$$

Definition 3.1. (cf. [**Lon3**], [**Lon4**]) For any $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, define

$$\nu_\omega(M) = \dim_{\mathbf{C}} \ker_{\mathbf{C}}(M - \omega I_{2n}). \tag{3.2}$$

For any $\tau > 0$ and $\gamma \in \mathcal{P}_\tau(2n)$, define

$$\nu_\omega(\gamma) = \nu_\omega(\gamma(\tau)). \tag{3.3}$$

If $\gamma \in \mathcal{P}^*_{\tau,\omega}(2n)$, define

$$i_\omega(\gamma) = [\mathrm{Sp}(2n)^0_\omega : \gamma * \xi_n], \tag{3.4}$$

where the right hand side of (3.4) is the usual homotopy intersection number, and the orientation of $\gamma * \xi_n$ is its positive time direction under homotopy with fixed end points.

If $\gamma \in \mathcal{P}^0_{\tau,\omega}(2n)$, we let $\mathcal{F}(\gamma)$ be the set of all open neighborhoods of γ in $\mathcal{P}_\tau(2n)$, and define

$$i_\omega(\gamma) = \sup_{U \in \mathcal{F}(\gamma)} \inf\{i_\omega(\beta) \mid \beta \in U \cap \mathcal{P}^*_{\tau,\omega}(2n)\}. \tag{3.5}$$

Then

$$(i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbf{Z} \times \{0, 1, \ldots, 2n\}$$

is called the index function of γ at ω.

For any symplectic path $\gamma \in \mathcal{P}_\tau(2n)$ and $m \in \mathbf{N}$, we define its m-th iteration $\gamma^m : [0, m\tau] \to \mathrm{Sp}(2n)$ by

(3.6)

$$\gamma^m(t) = \gamma(t - j\tau)\gamma(\tau)^j, \qquad \text{for} \quad j\tau \le t \le (j+1)\tau,\ j = 0, 1, \ldots, m-1.$$

We still denote the extended path on $[0, +\infty)$ by γ.

Definition 3.2. (cf. [**Lon3**], [**Lon4**]) For any $\gamma \in \mathcal{P}_\tau(2n)$, we define

$$(i(\gamma, m), \nu(\gamma, m)) = (i_1(\gamma^m), \nu_1(\gamma^m)), \qquad \forall m \in \mathbf{N}. \tag{3.7}$$

The mean index $\hat{i}(\gamma, m)$ per $m\tau$ for $m \in \mathbf{N}$ is defined by

$$\hat{i}(\gamma, m) = \lim_{k \to +\infty} \frac{i(\gamma, mk)}{k}. \tag{3.8}$$

For any $M \in \mathrm{Sp}(2n)$ and $\omega \in \mathbf{U}$, the *splitting numbers* $S^\pm_M(\omega)$ of M at ω are defined by

$$S^\pm_M(\omega) = \lim_{\epsilon \to 0^+} i_{\omega \exp(\pm\sqrt{-1}\epsilon)}(\gamma) - i_\omega(\gamma), \tag{3.9}$$

for any path $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau) = M$.

For a given path $\gamma \in \mathcal{P}_\tau(2n)$, we consider deforming it to a new path η in $\mathcal{P}_\tau(2n)$ so that

$$i_1(\gamma^m) = i_1(\eta^m), \quad \nu_1(\gamma^m) = \nu_1(\eta^m), \quad \forall m \in \mathbf{N}, \tag{3.10}$$

and that $(i_1(\eta^m), \nu_1(\eta^m))$ is easy enough to compute. This leads to finding homotopies $\delta : [0,1] \times [0,\tau] \to \mathrm{Sp}(2n)$ starting from γ in $\mathcal{P}_\tau(2n)$ and keeping the end points of the homotopy always in a certain suitably chosen maximal subset of $\mathrm{Sp}(2n)$ so that (3.10) always holds. In fact, this set was first discovered in [**Lon2**] as the path connected component $\Omega^0(M)$ containing $M = \gamma(\tau)$ of the set

$$\begin{aligned} \Omega(M) = \{N \in \mathrm{Sp}(2n) \quad | \quad & \sigma(N) \cap \mathbf{U} = \sigma(M) \cap \mathbf{U} \text{ and} \\ & \nu_\lambda(N) = \nu_\lambda(M),\ \forall\, \lambda \in \sigma(M) \cap \mathbf{U}\}. \end{aligned} \tag{3.11}$$

Here $\Omega^0(M)$ is called the *homotopy component* of M in $\mathrm{Sp}(2n)$.

In [**Lon2**]–[**Lon4**], the following symplectic matrices were introduced as *basic normal forms*:

$$D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad \lambda = \pm 2, \tag{3.12}$$

$$N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \qquad \lambda = \pm 1, b = \pm 1, 0, \tag{3.13}$$

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in (0,\pi)\cup(\pi,2\pi), \tag{3.14}$$

$$N_2(\omega, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}, \qquad \theta \in (0,\pi)\cup(\pi,2\pi), \tag{3.15}$$

where $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ with $b_i \in \mathbf{R}$ and $b_2 \neq b_3$.

Splitting numbers possess the following properties:

Lemma 3.3 (cf. [**Lon3**] and Lemma 9.1.5 of [**Lon4**])**.** *Splitting numbers $S^{\pm}_M(\omega)$ are well defined, i.e., they are independent of the choice of the path $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau) = M$, which appeared in (*3.9*). For $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, splitting numbers $S^{\pm}_N(\omega)$ are constant for all $N \in \Omega^0(M)$.*

Lemma 3.4 (cf. Lemma 9.1.5 of [**Lon3**] and List 9.1.12 of [**Lon4**])**.** *For $M \in \mathrm{Sp}(2n)$ and $\omega \in \mathbf{U}$, there hold*

$$S^{\pm}_M(\omega) = 0, \qquad \text{if } \omega \notin \sigma(M). \tag{3.16}$$

$$S^{+}_{N_1(1,a)}(1) = \begin{cases} 1, & \text{if } a \geq 0, \\ 0, & \text{if } a < 0. \end{cases} \tag{3.17}$$

For any $M_i \in \mathrm{Sp}(2n_i)$ with $i = 0$ and 1, there holds

$$S^{\pm}_{M_0 \diamond M_1}(\omega) = S^{\pm}_{M_0}(\omega) + S^{\pm}_{M_1}(\omega), \qquad \forall\, \omega \in \mathbf{U}. \tag{3.18}$$

We have the following:

Theorem 3.5 (cf. [**Lon3**] and Theorem 1.8.10 of [**Lon4**])**.** *For any $M \in \mathrm{Sp}(2n)$, there is a path $f : [0,1] \to \Omega^0(M)$ such that $f(0) = M$ and*

$$f(1) = M_1 \diamond \cdots \diamond M_k, \tag{3.19}$$

*where each M_i is a basic normal form listed in (*3.12*)–(*3.15*) for $1 \leq i \leq k$.*

The next theorem is the precise index iteration formulae of Y. Long (cf. Theorem 8.3.1 and Corollary 8.3.2 of [**Lon4**], §6 of [**LoZ**]).

Theorem 3.6. *Let $\gamma \in \mathcal{P}_\tau(2n)$. Then there exists a path $f \in C([0,1], \Omega^0(\gamma(\tau))$ such that $f(0) = \gamma(\tau)$ and*

$$\begin{aligned} f(1) = \quad & N_1(1,1)^{\diamond p_-} \diamond I_{2p_0} \diamond N_1(1,-1)^{\diamond p_+} \diamond \\ & N_1(-1,1)^{\diamond q_-} \diamond (-I_{2q_0}) \diamond N_1(-1,-1)^{\diamond q_+} \\ & \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_r) \diamond N_2(\omega_1, u_1) \diamond \cdots \diamond N_2(\omega_{r_*}, u_{r_*}) \\ & \diamond N_2(\lambda_1, v_1) \diamond \cdots \diamond N_2(\lambda_{r_0}, v_{r_0}) \diamond M_0 \end{aligned} \tag{3.20}$$

*where $N_2(\omega_j, u_j)$s are non-trivial and $N_2(\lambda_j, v_j)$s are trivial basic normal forms; $\sigma(M_0) \cap U = \emptyset$; p_-, p_0, p_+, q_-, q_0, q_+, r, r_*, and r_0 are non-negative integers; $\omega_j = e^{\sqrt{-1}\alpha_j}$, $\lambda_j = e^{\sqrt{-1}\beta_j}$; θ_j, α_j, $\beta_j \in (0,\pi) \cup (\pi, 2\pi)$; these integers and real numbers are uniquely determined by $\gamma(\tau)$. Then using the functions defined in (*1.2*),*

$$\begin{aligned} i(\gamma, m) = \quad & m(i(\gamma,1) + p_- + p_0 - r) + 2\sum_{j=1}^{r} \mathcal{E}\left(\frac{m\theta_j}{2\pi}\right) - r - p_- - p_0 \\ & - \frac{1+(-1)^m}{2}(q_0 + q_+) + 2\left(\sum_{j=1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right) - r_*\right), \end{aligned} \tag{3.21}$$

$$\begin{aligned} \nu(\gamma, m) = \quad & \nu(\gamma,1) + \frac{1+(-1)^m}{2}(q_- + 2q_0 + q_+) + 2(r + r_* + r_0) \\ & -2\left(\sum_{j=1}^{r} \varphi\left(\frac{m\theta_j}{2\pi}\right) + \sum_{j=1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right) + \sum_{j=1}^{r_0} \varphi\left(\frac{m\beta_j}{2\pi}\right)\right), \end{aligned} \tag{3.22}$$

$$\hat{i}(\gamma, 1) = i(\gamma, 1) + p_- + p_0 - r + \sum_{j=1}^{r} \frac{\theta_j}{\pi}. \tag{3.23}$$

Where $N_1(1, \pm 1) = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$, $N_1(-1, \pm 1) = \begin{pmatrix} -1 & \pm 1 \\ 0 & -1 \end{pmatrix}$, $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $N_2(\omega, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}$ with some $\theta \in (0,\pi) \cup (\pi, 2\pi)$, and $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \mathbf{R}^{2\times 2}$, such that $(b_2 - b_3)\sin\theta > 0$, if $N_2(\omega, b)$ is trivial; $(b_2 - b_3)\sin\theta < 0$, if $N_2(\omega, b)$ is non-trivial. We have $i(\gamma, 1)$ is odd if $f(1) = N_1(1,1)$, I_2, $N_1(-1,1)$, $-I_2$, $N_1(-1,-1)$, and $R(\theta)$; $i(\gamma,1)$ is even if $f(1) = N_1(1,-1)$ and $N_2(\omega, b)$; $i(\gamma, 1)$ can be any integer if $\sigma(f(1)) \cap \mathbf{U} = \emptyset$.

We have the following properties in the index iteration theory.

Theorem 3.7 (cf. Theorem 2.2 of [**LoZ**]). *Let $\gamma \in \mathcal{P}_\tau(2n)$; then for any $m \in \mathbf{N}$, there holds*

$$\nu(\gamma, m) - \frac{e(M)}{2} \le i(\gamma, m+1) - i(\gamma, m) - i(\gamma, 1) \le \nu(\gamma, 1) - \nu(\gamma, m+1) + \frac{e(M)}{2}$$

where $e(M)$ *is the elliptic height defined in* §*1.*

The following is the common index jump theorem of Y. Long and C. Zhu.

Theorem 3.8 (cf. Theorems 4.1–4.3 of [**LoZ**])**.** *Let* $\gamma_k \in \mathcal{P}_{\tau_k}(2n)$ *for* $k = 1, \dots, q$ *be a finite collection of symplectic paths. Let* $M_k = \gamma(\tau_k)$*. Suppose* $\hat{i}(\gamma_k, 1) > 0$*, for all* $k = 1, \dots, q$*. Then there exist infinitely many* $(N, m_1, \dots, m_q) \in \mathbf{N}^{q+1}$ *such that*

$$\nu(\gamma_k, 2m_k - 1) = \nu(\gamma_k, 1), \tag{3.24}$$

$$\nu(\gamma_k, 2m_k + 1) = \nu(\gamma_k, 1), \tag{3.25}$$

$$\begin{aligned} &i(\gamma_k, 2m_k - 1) + \nu(\gamma_k, 2m_k - 1) \\ &\qquad = 2N - \Big(i(\gamma_k, 1) + 2S^{+}_{M_k}(1) - \nu(\gamma_k, 1)\Big), \end{aligned} \tag{3.26}$$

$$i(\gamma_k, 2m_k + 1) = 2N + i(\gamma_k, 1), \tag{3.27}$$

$$i(\gamma_k, 2m_k) \geq 2N - \frac{e(M_k)}{2} \geq 2N - n, \tag{3.28}$$

$$\begin{aligned} &i(\gamma_k, 2m_k) + \nu(\gamma_k, 2m_k) \\ &\qquad \leq 2N + \frac{e(M_k)}{2} \leq 2N - n, \end{aligned} \tag{3.29}$$

for every $k = 1, \dots, q$*. Moreover, we have*

$$\min\left\{\left\{\frac{m_k\theta}{\pi}\right\}, 1 - \left\{\frac{m_k\theta}{\pi}\right\}\right\} < \delta, \tag{3.30}$$

whenever $e^{\sqrt{-1}\theta} \in \sigma(M_k)$ *and* δ *can be chosen as small as we want (cf. (4.43) of* [**LoZ**]*). More precisely, by (4.10) and (4.40) in* [**LoZ**]*, we have*

$$m_k = \left(\left[\frac{N}{M\hat{i}(\gamma_k, 1)}\right] + \xi_k\right) M, \quad 1 \leq k \leq q, \tag{3.31}$$

where $\xi_k = 0$ *or* 1 *for* $1 \leq k \leq q$ *and* $\frac{M\theta}{\pi} \in \mathbf{Z}$ *whenever* $e^{\sqrt{-1}\theta} \in \sigma(M_k)$ *and* $\frac{\theta}{\pi} \in \mathbf{Q}$ *for some* $1 \leq k \leq q$*. By (4.20) in Theorem 4.1 of* [**LoZ**]*, for any* $\epsilon > 0$*, we can choose* N *and* $\{\xi_k\}_{1 \leq k \leq q}$ *such that*

$$\left|\frac{N}{M\hat{i}(\gamma_k, 1)} - \left[\frac{N}{M\hat{i}(\gamma_k, 1)}\right] - \xi_k\right| < \epsilon, \quad 1 \leq k \leq q. \tag{3.32}$$

Furthermore, given $M_0 \in \mathbf{N}$*, by the proof of Theorem 4.1 of* [**LoZ**] *(Theorem 11.1.1 of* [**Lon4**]*), we may further require* $M_0 | N$ *(since the closure of the set* $\{\{Nv\} : N \in \mathbf{N},\ M_0 | N\}$ *is still a closed additive subgroup of* $\mathbf{T}^h$ *for some* $h \in \mathbf{N}$*, where we use notations as (4.21) in* [**LoZ**]*, then we can use the proof of Step 2 in Theorem 4.1 of* [**LoZ**] *to get* N*).*

By Theorems 6.2.7 and 9.2.1 of [**Lon4**], we have the following symplectic additivity property for the index iteration formula:

Theorem 3.9. *For any* $\gamma_i \in \mathcal{P}_\tau(2n_i)$ *with* $i = 1, 2$, *we have*

$$i(\gamma_1 \diamond \gamma_2, m) = i(\gamma_1, m) + i(\gamma_2, m), \qquad \forall m \in \mathbf{N}. \tag{3.33}$$

4. A mean index equality on (S^n, F)

In this section, we recall the mean index equality obtained in [**Rad1**]. Suppose that there are only finitely many prime closed geodesics $\{c_j\}_{1\le j\le p}$ on a bumpy (S^n, F) with $\hat{i}(c_j) > 0$ for $1 \le j \le p$.

Lemma 4.1. *Let* c *be a prime closed geodesic on a bumpy* (S^n, F). *Then we have*

$$i(c^{q+2}) - i(c^q) \in 2\mathbf{Z}, \qquad \forall q \in \mathbf{N}. \tag{4.1}$$

Proof. This follows directly from Theorem 3.6. q.e.d.

Definition 4.2. Suppose c is a closed geodesic on (S^n, F). The Euler characteristic $\chi(c^m)$ of c^m is defined by

$$\begin{aligned} \chi(c^m) &\equiv \chi\left((\Lambda(c^m) \cup S^1 \cdot c^m)/S^1, \Lambda(c^m)/S^1\right), \\ &\equiv \sum_{q=0}^{\infty} (-1)^q \dim \overline{C}_q(E, c^m). \end{aligned} \tag{4.2}$$

Here $\chi(A, B)$ denotes the usual Euler characteristic of the space pair (A, B).

The average Euler characteristic $\hat{\chi}(c)$ of c is defined by

$$\hat{\chi}(c) = \lim_{N\to\infty} \frac{1}{N} \sum_{1\le m\le N} \chi(c^m). \tag{4.3}$$

The following remark shows that $\hat{\chi}(c)$ is well defined and is a rational number.

Remark 4.3. By Proposition 2.1 and Lemma 4.1 we have

$$\begin{aligned} \hat{\chi}(c) &= \lim_{N\to\infty} \frac{1}{N} \sum_{1\le m\le N} (-1)^{i(c^m)} \dim \overline{C}_{i(c^m)}(E, c^m) \\ &= \lim_{s\to\infty} \frac{1}{2s} \sum_{\substack{1\le m\le 2 \\ 0\le p<s}} (-1)^{i(c^{2p+m})} \dim \overline{C}_{i(c^{2p+m})}(E, c^m) \\ &= \frac{1}{2} \sum_{1\le m\le 2} (-1)^{i(c^m)} \dim \overline{C}_{i(c^m)}(E, c^m) = \frac{1}{2} \sum_{1\le m\le 2} \chi(c^m). \end{aligned} \tag{4.4}$$

Therefore $\hat{\chi}(c)$ is well defined and is a rational number.

The following is the mean index equality of H.-B. Rademacher (Theorem 7.9 in [**Rad2**]).

Theorem 4.4. *Suppose that there exist only finitely many prime closed geodesics* $\{c_j\}_{1\le j\le p}$ *with* $\hat{i}(c_j)>0$ *for* $1\le j\le p$ *on* (S^n,F)*. Then the following equality holds:*

$$\sum_{1\le j\le p}\frac{\hat{\chi}(c_j)}{\hat{i}(c_j)}=B(n,1)=\begin{cases}\frac{n+1}{2(n-1)}, & n\quad odd,\\ \frac{-n}{2(n-1)}, & n\quad even.\end{cases}\tag{4.5}$$

5. Proof of the main theorem

In this section, we give the proof of Theorem 1.2 by using the mean index equality in Theorem 4.4, Morse inequality, and the index iteration theory developed by Y. Long and his coworkers.

In the following for the notation introduced in §2, we use especially $M_j=M_j(\overline{\Lambda}S^n,\overline{\Lambda}^0S^n)$ and $b_j=b_j(\overline{\Lambda}S^n,\overline{\Lambda}^0S^n)$ for $j=0,1,2,\ldots$.

First note that if the flag curvature K of (S^n,F) satisfies $\left(\frac{\lambda}{\lambda+1}\right)^2<K\le 1$, then every nonconstant closed geodesic must satisfy

$$i(c)\ \ge\ n-1,\tag{5.1}$$

$$\hat{i}(c)\ >\ n-1,\tag{5.2}$$

where (5.1) follows from Theorem 3 and Lemma 3 of [**Rad3**], and (5.2) follows from Lemma 2 of [**Rad4**]. Now it follows from Theorem 3.7 that

$$i(c^{m+1})-i(c^m)\ge i(c)-\frac{e(P_c)}{2}\ge 0,\quad \forall m\in\mathbf{N}.\tag{5.3}$$

Here the last inequality holds by (5.1) and the fact that $e(P_c)\le 2(n-1)$.

In the rest of this paper, we will assume the following:

(F) There are only finitely many prime closed geodesics $\{c_j\}_{1\le j\le p}$ **on a bumpy Finsler** n**-sphere** (S^n,F)**.**

By (5.2), we can use Theorem 3.8 and (5.3) to obtain some $(N,m_1,\ldots,m_p)\in\mathbf{N}^{p+1}$ such that

$$i(c_j^{2m_j})\ \ge\ 2N-\frac{e(P_{c_j})}{2}\ge 2N-(n-1),\tag{5.4}$$

$$i(c_j^{2m_j})\ \le\ 2N+\frac{e(P_{c_j})}{2}\le 2N+(n-1),\tag{5.5}$$

$$i(c_j^{2m_j-m})\ \le\ 2N-(i(c_j)+2S^+_{P_{c_j}}(1)-\nu(c_j)),\quad \forall m\in\mathbf{N},\tag{5.6}$$

$$i(c_j^{2m_j+m})\ \ge\ 2N+i(c_j),\quad \forall m\in\mathbf{N},\tag{5.7}$$

where $S^+_{P_{c_j}}(1)$ denotes the splitting number of c_j at 1. Since c_j is nondegenerate, 1 is not an eigenvalue of P_{c_j} for $1\le j\le p$. Thus we have $S^+_{P_{c_j}}(1)=0$ and $\nu(c_j)=0$ by Lemma 3.4. Hence (5.6) becomes

$$i(c_j^{2m_j-m})\le 2N-i(c_j),\quad \forall m\in\mathbf{N}.\tag{5.8}$$

Moreover, we have

$$\min\left\{\left\{\frac{m_j\theta}{\pi}\right\}, 1-\left\{\frac{m_j\theta}{\pi}\right\}\right\} < \delta, \tag{5.9}$$

whenever $e^{\sqrt{-1}\theta} \in \sigma(P_{c_j})$ and δ can be chosen as small as we want. More precisely, we have

$$m_j = \left(\left[\frac{N}{M\hat{i}(c_j)}\right] + \xi_j\right) M, \quad 1 \le j \le p, \tag{5.10}$$

where $\xi_j = 0$ or 1 for $1 \le j \le p$, and for any $\epsilon > 0$, we can choose N and $\{\xi_j\}_{1\le j\le p}$ such that

$$\left|\frac{N}{M\hat{i}(c_j)} - \left[\frac{N}{M\hat{i}(c_j)}\right] - \xi_j\right| < \epsilon < \frac{1}{1+\sum_{1\le j\le p} 4M|\hat{\chi}(c_j)|}, \tag{5.11}$$

for $1 \le j \le p$.

In order to prove Theorem 1.2, we need the following two lemmas. Denote $n = 2k$.

Lemma 5.1. *There exists a prime closed geodesic* c_{j_0} *such that* $i(c_{j_0}^{2m_{j_0}}) = 2N + (n-1)$. *In particular, we have* $\overline{C}_{2N+n-1}(E, c_{j_0}^{2m_{j_0}}) \ne 0$.

Proof. Suppose the contrary. Then by (5.5), we have

$$i(c_j^{2m_j}) < 2N + (n-1), \qquad 1 \le j \le p. \tag{5.12}$$

Now by (5.1), (5.3), (5.7), and (5.12), we have

$$i(c_j^m) \le i(c_j^{2m_j}), \quad \forall m < 2m_j, \tag{5.13}$$

$$i(c_j^{2m_j}) \le 2N + n - 2, \tag{5.14}$$

$$i(c_j^m) \ge 2N + n - 1, \quad \forall m > 2m_j. \tag{5.15}$$

By Theorem 4.4, we have

$$\sum_{1\le j\le p} \frac{\hat{\chi}(c_j)}{\hat{i}(c_j)} = B(n,1) \in \mathbf{Q}. \tag{5.16}$$

Note that by Theorem 3.8, we can require that $N \in \mathbf{N}$ further satisfies

$$2NB(n,1) \in \mathbf{Z}. \tag{5.17}$$

Multiplying both sides of (5.16) by $2N$ yields

$$\sum_{1\le j\le p} \frac{2N\hat{\chi}(c_j)}{\hat{i}(c_j)} = 2NB(n,1). \tag{5.18}$$

Claim 1. We have

$$\sum_{1\le j\le p} 2m_j\hat{\chi}(c_j) = 2NB(n,1). \tag{5.19}$$

In fact, by (5.10) and (5.18), we have

$$\begin{aligned}
&2NB(n,1)\\
=\ &\sum_{1\le j\le p}\frac{2N\hat\chi(c_j)}{\hat i(c_j)}\\
=\ &\sum_{1\le j\le p}2\hat\chi(c_j)\left(\left[\frac{N}{M\hat i(c_j)}\right]+\xi_j\right)M\\
&+\sum_{1\le j\le p}2\hat\chi(c_j)\left(\frac{N}{M\hat i(c_j)}-\left[\frac{N}{M\hat i(c_j)}\right]-\xi_j\right)M\\
=\ &\sum_{1\le j\le p}2m_j\hat\chi(c_j)+\sum_{1\le j\le p}2M\hat\chi(c_j)\epsilon_j. \qquad (5.20)
\end{aligned}$$

By (4.4) we have

$$2m_j\hat\chi(c_j)\in\mathbf{Z},\quad 1\le j\le p. \qquad (5.21)$$

Now Claim 1 follows by (5.11), (5.17), (5.20), and (5.21).

Claim 2. We have

$$\sum_{1\le j\le p}2m_j\hat\chi(c_j)=M_0-M_1+M_2-\cdots+(-1)^{2N+n-2}M_{2N+n-2}, \qquad (5.22)$$

where $M_q\equiv M_q(\overline{\Lambda},\overline{\Lambda}^0)$ for $q\in\mathbf{Z}$.

In fact, by definition, the right hand side of (5.22) is

$$RHS=\sum_{\substack{q\le 2N+n-2\\ m\ge1,\ 1\le j\le p}}(-1)^q\dim\overline{C}_q(E,c_j^m). \qquad (5.23)$$

By (5.13)–(5.15) and Proposition 2.1, we have

$$\begin{aligned}
RHS &= \sum_{\substack{1\le j\le p,\ 1\le m\le 2m_j\\ q\le 2N+n-2}}(-1)^q\dim\overline{C}_q(E,c_j^m), && (5.24)\\
&= \sum_{1\le j\le p,\ 1\le m\le 2m_j}\chi(c_j^m), && (5.25)
\end{aligned}$$

where the second equality follows from (4.2).

By (4.4), we have

$$\begin{aligned}
\sum_{1\le m\le 2m_j}\chi(c_j^m) &= \sum_{0\le s<m_j,\,1\le m\le 2}\chi(c_j^{2s+m})\\
&= m_j\sum_{1\le m\le 2}\chi(c_j^m)\\
&= 2m_j\hat\chi(c_j). \qquad (5.26)
\end{aligned}$$

This proves Claim 2.

Now we let $n = 2k$. We have by (4.5)

$$B(n,1) = \frac{-n}{2(n-1)} = \frac{-k}{2k-1}. \tag{5.27}$$

By Theorem 3.8 we may further assume $N = (2k-1)s$ for some $s \in \mathbf{N}$.

Thus by (5.19), (5.22), and (5.27), we have

$$M_0 - M_1 + M_2 - \cdots + (-1)^{2N+n-2} M_{2N+n-2} = -2sk. \tag{5.28}$$

On the other hand, we have by (2.7)

$$\begin{aligned} & b_0 - b_1 + b_2 - \cdots + (-1)^{2N+n-2} b_{2N+n-2} \\ = \quad & -b_{2k-1} - (b_{2k+1} + \cdots + b_{6k-3} + \cdots + b_{(s-1)(4k-2)+2k+1} \\ & + \cdots + b_{s(4k-2)+2k-1}) + b_{s(4k-2)+2k-1} \\ = \quad & -1 - s(2k-2+2) + 2 \\ = \quad & -2sk + 1. \end{aligned} \tag{5.29}$$

In fact, we cut off the sequence $\{b_{2k+1}, \ldots, b_{s(4k-2)+2k-1}\}$ into s pieces; each of them contains $2k-1$ terms. Moreover, each piece contains 1 for $2k-2$ times and 2 for one time. Thus (5.29) holds.

Now by (5.28), (5.29), and Theorem 2.3, we have

$$\begin{aligned} -2sk &= M_{2N+n-2} - M_{2N+n-3} + \cdots + M_1 - M_0 \\ &\geq b_{2N+n-2} - b_{2N+n-3} + \cdots + b_1 - b_0 \\ &= -2sk + 1. \end{aligned} \tag{5.30}$$

This contradiction yields the lemma. q.e.d.

Lemma 5.2. *We have*

$$i(c_j^{2m_j-2}) < 2N - (n-1), \tag{5.31}$$

for $1 \leq j \leq p$.

Proof. By (5.3) and (5.8), if $i(c_j) > n-1$, then (5.31) holds. Thus it remains to consider the case $i(c_j) = n-1$. By (3.23) and (5.2), we have

$$\begin{aligned} \hat{i}(c_j) &= i(c_j) + p_- + p_0 - r + \sum_{i=1}^{r} \frac{\theta_i}{\pi} \\ &= i(c_j) - r + \sum_{i=1}^{r} \frac{\theta_i}{\pi} > n-1, \end{aligned} \tag{5.32}$$

where the second equality follows from $p_- = 0 = p_0$, which holds since c_j is non-degenerate. Plugging $i(c_j) = n-1$ into (5.32) yields

$$\sum_{i=1}^{r} \left(\frac{\theta_i}{\pi} - 1 \right) > 0. \tag{5.33}$$

Hence we can write

$$P_{c_j} = R(\theta) \diamond M, \tag{5.34}$$

for some $\theta \in (\pi, 2\pi)$ and $M \in Sp(2n-4)$. Thus by Theorem 3.6 and the assumption that c_j^ms are non-degenerate for $m \in \mathbf{N}$, we have

$$\begin{aligned} i(c_j^m) &= m(i(c_j)-r) + 2\sum_{i=1}^{r} \mathcal{E}\left(\frac{m\theta_i}{2\pi}\right) - r \\ &= 2\mathcal{E}\left(\frac{m\theta}{2\pi}\right) - 1 + i(\gamma, m), \end{aligned} \tag{5.35}$$

where $\gamma \in \{\xi \in C([0,\tau], Sp(2n-4)) \mid \xi(0) = I\}$ satisfies $\gamma(\tau) = M$ and $i(\gamma, 1) = n-2$. The second equality follows from Theorem 3.9. Note that it follows from Theorem 3.7 that

$$i(\gamma, m+1) - i(\gamma, m) \geq i(\gamma, 1) - \frac{e(M)}{2} \geq 0, \ \forall m \in \mathbf{N}. \tag{5.36}$$

Here the last inequality holds from $i(\gamma, 1) = n-2$ and the fact that $e(M) \leq 2(n-2)$. By (5.3) and (5.8), in order to prove (5.31), it is sufficient to prove

$$i(c_j^{2m_j-2}) < i(c_j^{2m_j-1}). \tag{5.37}$$

By (5.35) and (5.36), in order to prove (5.37), it is sufficient to prove

$$\mathcal{E}\left(\frac{(2m_j-2)\theta}{2\pi}\right) < \mathcal{E}\left(\frac{(2m_j-1)\theta}{2\pi}\right). \tag{5.38}$$

In order to satisfy (5.38), it is sufficient to choose

$$\delta < \min\left\{\frac{\theta}{\pi} - 1,\ 1 - \frac{\theta}{2\pi}\right\}, \tag{5.39}$$

where δ is given by (5.9). This proves the lemma. q.e.d.

Proof of Theorem 1.2. By the main theorem of [**LoW2**], Theorem 1.2 is true for $n = 2$. Thus it remains to consider the case $n \geq 4$.

Note that by Theorem 2.3, we have

$$M_q \equiv M_q(\overline{\Lambda}, \overline{\Lambda}^0) = \sum_{m\geq 1,\, 1\leq j\leq p} \mathrm{rank}\overline{C}_q(E, c_j^m), \qquad \forall q \in \mathbf{Z}. \tag{5.40}$$

The proof contains three steps:

Step 1. There are $n-2$ distinct prime closed geodesics c_j with the property that $2N-(n-1) < i(c_j^{2m_j}) < 2N+(n-1)$. Each closed geodesic c_j has a nontrivial critical $\mathbf{Q}$-module in dimension $i(c_j^{2m_j})$. Moreover, all of these closed geodesics are non-hyperbolic.

As in Lemma 5.1, we may assume $N = (2k-1)s$ for some $s \in \mathbf{N}$. Then by Theorem 2.2, we have

$$b_{2N-(n-1)+2m} = 1, \quad 1 \leq m \leq n-2. \tag{5.41}$$

Thus by Theorem 2.3, we have

$$M_{2N-(n-1)+2m} \geq b_{2N-(n-1)+2m} = 1, \quad 1 \leq m \leq n-2. \tag{5.42}$$

By Proposition 2.1 and (5.1), (5.7), and (5.8), we have

$$M_{2N-(n-1)+2m} = \sum_{1\le j\le p} \mathrm{rank}\overline{C}_{2N-(n-1)+2m}(E, c_j^{2m_j})$$

$$=^{\#} \{j | i(c_j^{2m_j}) - i(c_j) \in 2\mathbf{Z},\ i(c_j^{2m_j}) = 2N-(n-1)+2m\}, \tag{5.43}$$

for $1 \le m \le n-2$. Hence we have $p \ge n-2$ by (5.41)–(5.43) and Proposition 2.1. In fact, only the $2m_j$-th iteration $c_j^{2m_j}$ of c_j contributes at most 1 to

$$\sum_{1\le m\le n-2} M_{2N-(n-1)+2m} \ge \sum_{1\le m\le n-2} b_{2N-(n-1)+2m} = n-2$$

for each $1 \le j \le p$.

By (5.4) and (5.5), a hyperbolic closed geodesic c_j must satisfy $i(c_j^{2m_j}) = 2N$. Hence closed geodesics c_js satisfying $\overline{C}_{2N-(n-1)+2m}(E, c_j^{2m_j}) \ne 0$ are non-hyperbolic. Clearly, the number of these closed geodesics is

$$\sum_{1\le m\le n-2} M_{2N-(n-1)+2m} \ge \sum_{1\le m\le n-2} b_{2N-(n-1)+2m} = n-2.$$

This yields Step 1.

Step 2. There exists a prime closed geodesic c_{j_0} with

$$i(c_{j_0}^{2m_{j_0}}) = 2N + (n-1).$$

The closed geodesic c_{j_0} has a nontrivial critical $\mathbf{Q}$-module in dimension $i(c_{j_0}^{2m_{j_0}})$. In particular, the closed geodesic c_{j_0} is elliptic.

Now we denote the $n-2$ prime closed geodesics obtained in Step 1 by $\{c_{j_1}, \ldots, c_{j_{n-2}}\}$. Then by (5.40), (5.43), and Proposition 2.1, we have

$$i(c_{j_l}^{2m_{j_l}}) = 2N-(n-1)+2\tau_{j_l},$$
$$\overline{C}_{2N-(n-1)+2\tau_{j_l}}(E, c_{j_l}^{2m_{j_l}}) \ne 0, \tag{5.44}$$

for some $1 \le \tau_{j_l} \le n-2$ and $1 \le l \le n-2$. On the other hand, by Lemma 5.1, there exists a closed geodesic c_{j_0} such that

$$i(c_{j_0}^{2m_{j_0}}) = 2N+(n-1),\ \overline{C}_{2N+n-1}(E, c_{j_0}^{2m_{j_0}}) \ne 0. \tag{5.45}$$

Hence we have $c_{j_0} \notin \{c_{j_1}, \ldots, c_{j_{n-2}}\}$ by (5.44) and (5.45). By (5.45) and (5.5), the closed geodesic c_{j_0} is elliptic. This yields Step 2.

Step 3. There exists a non-hyperbolic prime closed geodesic $c_\star \notin \{c_{j_0}, \ldots, c_{j_{n-2}}\}$.

Denote the $n-1$ prime closed geodesics obtained in Steps 1 and 2 by $\{c_{j_0}, \ldots, c_{j_{n-2}}\}$.

By Theorems 2.2 and 2.3, we have

$$M_{n-1} = \sum_{1\le j\le p,\, m\ge 1} \mathrm{rank}\overline{C}_{n-1}(E, c_j^m) \ge b_{n-1} = 1. \tag{5.46}$$

Thus it follows from (5.1) and (5.3) that there exists at least one closed geodesic c_j such that $i(c_j) = n-1$. We have the following two cases:

Case 1. We have ${}^{\#}\{j | i(c_j) = n-1\} = 1$, i.e., there is only one prime closed geodesic which has index $n-1$.

Denote the prime closed geodesic which has index $n-1$ by c_*. Then we have

$$i(c_l) > n-1, \qquad l \in \{1, \ldots, p\} \setminus \{*\}. \tag{5.47}$$

Thus it follows from (5.8) that

$$i(c_l^{2m_l - m}) < 2N - (n-1), \ \forall m \in \mathbf{N}, \ l \in \{1, \ldots, p\} \setminus \{*\}. \tag{5.48}$$

By Lemma 5.2 and (5.3), we have

$$i(c_*^{2m_* - m}) < 2N - (n-1), \quad \forall m \geq 2. \tag{5.49}$$

Then by (5.1), (5.7), (5.44), (5.45), (5.48), (5.49), and Proposition 2.1, we have

$$\sum_{0 \leq l \leq n-2,\, m \geq 1} \mathrm{rank}\overline{C}_{2N-(n-1)}(E, c_{j_l}^m) \leq 1. \tag{5.50}$$

In fact, the only possible non-zero term is $\mathrm{rank}\overline{C}_{2N-(n-1)}(E, c_*^{2m_*-1})$.

By Theorems 2.2 and 2.3, we have

$$\begin{aligned} M_{2N-(n-1)} &= \sum_{1 \leq j \leq p,\, m \geq 1} \mathrm{rank}\overline{C}_{2N-(n-1)}(E, c_j^m) \\ &\geq b_{2N-(n-1)} = 2. \end{aligned} \tag{5.51}$$

Hence there must be another closed geodesic $c_\star \notin \{c_{j_0}, \ldots, c_{j_{n-2}}\}$ by (5.50) and (5.51). Especially, we have

$$i(c_\star^{2m_\star}) = 2N - (n-1) \tag{5.52}$$

by (5.1), (5.7), (5.48), and Proposition 2.1. By (5.52) and (5.4), the closed geodesic $c_\star$ is non-hyperbolic. This yields Case 1.

Case 2. We have ${}^{\#}\{j | i(c_j) = n-1\} > 1$.

By Proposition 2.1 we have

$$M_{n-1} = \sum_{1 \leq j \leq p,\, m \geq 1} \mathrm{rank}\overline{C}_{n-1}(E, c_j^m) \geq 2. \tag{5.53}$$

By (5.1), Proposition 2.1, and Theorems 2.2 and 2.3, we have

$$\begin{aligned} M_n - M_{n-1} &= M_n - M_{n-1} + \cdots + (-1)^n M_0 \\ &\geq b_n - b_{n-1} + \cdots + (-1)^n b_0 = b_n - b_{n-1} = -1. \end{aligned} \tag{5.54}$$

Thus we have

$$M_n = \sum_{1 \leq j \leq p,\, m \geq 1} \mathrm{rank}\overline{C}_n(E, c_j^m) \geq 1. \tag{5.55}$$

Then we must have a prime closed geodesic c_* with

$$i(c_*) = n. \tag{5.56}$$

In fact, by (5.55), we have

$$\overline{C}_n(E, c_*^m) \neq 0 \tag{5.57}$$

for some $1 \leq * \leq p$ and some $m \in \mathbf{N}$. By (5.1) and (5.3), if $i(c_*) \neq n$, we must have $i(c_*) = n - 1$. Thus it follows from Proposition 2.1 that $\overline{C}_n(E, c_*^m) = 0$ for any $m \in \mathbf{N}$. This contradicts (5.57) and yields (5.56). Now it follows from Proposition 2.1 and (5.56) that

$$\overline{C}_{2N-(n-1)+2m}(E, c_*^m) = 0, \qquad \forall m \in \mathbf{Z}. \tag{5.58}$$

Thus we have $c_* \notin \{c_{j_0}, \ldots, c_{j_{n-2}}\}$ by (5.44) and (5.45).

Now we prove Step 3 as the following: First by the above argument, we have $\Delta \equiv \{c_j\}_{1\leq j\leq p} \setminus \{c_{j_0}, \ldots, c_{j_{n-2}}\} \neq \emptyset$. Then we show that there must be a non-hyperbolic closed geodesic $c_\star \in \Delta$. We prove this by contradiction, i.e., assume all the closed geodesics $c_j \in \Delta$ are hyperbolic. By Step 1, (5.4), (5.5), (5.56), and Proposition 2.1, we have

$$M_{2N-(n-1)+2m} = b_{2N-(n-1)+2m} = 1, \quad 1 \leq m \leq n-2, \tag{5.59}$$

$$M_{2N} = \sum_{m\geq 1,\, 1\leq j\leq p} \mathrm{rank}\overline{C}_{2N}(E, c_j^m) \geq \mathrm{rank}\overline{C}_{2N}(E, c_*^{2m_*}) \geq 1. \tag{5.60}$$

In fact, since c_* is hyperbolic by assumption, we have $i(c_*^{2m_*}) = 2N$ by (5.4) and (5.5). On the other hand, we have $i(c_*) = n$ is even by (5.56). Thus we have $\overline{C}_{2N}(E, c_*^{2m_*}) \neq 0$ by Proposition 2.1. By Theorem 2.3, we have

$$\begin{aligned} &M_{2N+1} - M_{2N} + \cdots + M_1 - M_0 \\ &\qquad \geq b_{2N+1} - b_{2N} + \cdots + b_1 - b_0, \end{aligned} \tag{5.61}$$

$$\begin{aligned} &M_{2N} - M_{2N-1} + \cdots - M_1 + M_0 \\ &\qquad \geq b_{2N} - b_{2N-1} + \cdots - b_1 + b_0, \end{aligned} \tag{5.62}$$

$$\begin{aligned} &M_{2N-1} - M_{2N-2} + \cdots + M_1 - M_0 \\ &\qquad \geq b_{2N-1} - b_{2N-2} + \cdots + b_1 - b_0, \end{aligned} \tag{5.63}$$

$$\begin{aligned} &M_{2N-2} - M_{2N-3} + \cdots - M_1 + M_0 \\ &\qquad \geq b_{2N-2} - b_{2N-3} + \cdots - b_1 + b_0. \end{aligned} \tag{5.64}$$

Thus, together with (5.59) and the fact $n \geq 4$, from (5.61)–(5.62) and (5.63)–(5.64), respectively, we obtain

$$\begin{aligned} &M_{2N} - M_{2N-1} + \cdots - M_1 + M_0 \\ &\qquad = b_{2N} - b_{2N-1} + \cdots - b_1 + b_0, \end{aligned} \tag{5.65}$$

$$\begin{aligned} &M_{2N-1} - M_{2N-2} + \cdots + M_1 - M_0 \\ &\qquad = b_{2N-1} - b_{2N-2} + \cdots + b_1 - b_0. \end{aligned} \tag{5.66}$$

Hence $M_{2N} = b_{2N} = 0$ by Theorem 2.2. This contradicts (5.60) and proves Case 2.

The proof of Theorem 1.2 is complete. q.e.d.

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