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ON THE SPECTRUM OF BOUNDED IMMERSIONS

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Abstract

In this article, we investigate some of the relations between the spectrum of a non-compact, extrinsically bounded submanifold $\varphi\colon M^m \to N^n$ and the Hausdorff dimension of its limit set $\lim \varphi$. In particular, we prove that if $\varphi\colon M^2 \to \mathbb{R}^3$ is a (complete) minimal surface immersed into an open, bounded, strictly convex subset Ω with C^3-boundary, then M has discrete spectrum, provided that $\mathcal{H}_\Psi(\lim \varphi \cap \Omega) = 0$, where $\mathcal{H}_\Psi$ is the Hausdorff measure of order $\Psi(t) = t^2|\log t|$. Our main theorem, Thm. 2.4, applies to a number of examples recently constructed, by many authors, in the light of Nadirashvili's discovery of complete bounded minimal disks in $\mathbb{R}^3$, as well as to solutions of Plateau's problems for non-rectifiable Jordan curves, giving a fairly complete answer to a question posed by S.T. Yau in his Millenium Lectures [**64**], [**65**].

On the other hand, we present a simple criterion, called *the ball property*, whose fulfilment guarantees the existence of elements in the essential spectrum. As an application, we show that some of the examples of Jorge-Xavier [**36**] and Rosenberg-Toubiana [**60**] of complete minimal surfaces between two planes have essential spectrum $\sigma_{\text{ess}}(-\Delta) = [0, \infty)$.

1. Introduction

An interesting problem in geometry is the study of the spectrum of the Laplacian Δ of a Riemannian manifold in terms of its Riemannian invariants. There is a huge literature studying the spectrum of complete Riemannian manifolds under various geometric conditions. To have a glimpse, see these few references with geometric restrictions implying that the spectrum is purely continuous: [**22**], [**23**], [**27**], [**37**], [**56**], [**62**]; or these implying that the spectrum is discrete: [**5**], [**24**], [**33**], [**38**], [**39**]. However, in the study of the spectrum of submanifolds, the relevant geometric restrictions are related to extrinsic bounds, ambient manifold curvature bounds, and the mean curvature of the submanifold; see [**8**], [**9**], [**10**], [**15**]. A particularly interesting problem is the spectrum-related part of the *Calabi-Yau conjectures on minimal hypersurfaces*. In his Millennium Lectures ([**64**], [**65**]), S. T. Yau, revisiting the E. Calabi

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conjectures on the existence of bounded minimal surfaces ([**13**], [**16**]), in the light of Jorge-Xavier and Nadirashvili's counter-examples ([**36**], [**48**]), proposed a new set of questions about bounded minimal surfaces of $\mathbb{R}^3$.

He wrote: "*It is known* [**48**] *that there are complete minimal surfaces properly immersed into the* [*open*] *ball. What is the geometry of these surfaces? Can they be embedded? Since the curvature must tend to minus infinity, it is important to find the precise asymptotic behavior of these surfaces near their ends. Are their* [*Laplacian*] *spectra discrete?*"

This set of questions is known in the literature as the Calabi-Yau conjectures on minimal surfaces. They motivated the construction of a large number of exotic examples of minimal surfaces in $\mathbb{R}^3$; see [**1**], [**2**], [**3**], [**28**], [**40**], [**41**], [**43**], [**44**], [**45**], [**46**], [**63**].

One of the purposes of this article is to shed light on the study of the essential spectrum of bounded submanifolds, in particular, the spectrum of those *exotic* examples constructed after the Calabi-Yau conjectures. The new fact we found is that the size of the limit sets of bounded immersions plays an important role in the existence of elements in their essential spectrum. Before we state our main results with precision, we will present some of the examples, after the Calabi-Yau conjectures, that motivate our work. The problem about the existence of bounded, complete, *embedded* minimal surfaces in $\mathbb{R}^3$ was negatively answered by T. Colding and W. Minicozzi in the finite topology case; see [**17**]. Although Yau's question suggests that Nadirashvili's example [**48**] is *properly* immersed into an open ball $B_r(0) \subset \mathbb{R}^3$, this is not clear from his construction, and the question "*Does there exist a complete minimal surface properly immersed into a ball of* $\mathbb{R}^3$*?*" may be considered as the first problem of the Calabi-Yau conjectures. This question was answered by F. Martin and S. Morales in [**43**]; see below. Recall that the limit set of an isometric immersion $\varphi\colon M \to \Omega \subset N$ is the set

$$\lim \varphi\colon = \{y \in \overline{\Omega};\ \exists\, \{x_n\} \subseteq M \text{ divergent in } M, \text{ s.t. } \varphi(x_n) \to y \text{ in } N\},$$

and that φ is proper in Ω if and only if $\lim \varphi \subset \partial\Omega$. The question "*What is the geometry of these surfaces?*" motivated the construction of bounded complete minimal surfaces of arbitrary topology in $\mathbb{R}^3$ and the understanding of their shape and the size of their limit sets stimulated intense research in the last fifteen years; see [**2**], [**10**], [**17**], [**18**], [**28**], [**36**], [**40**], [**41**], [**43**], [**44**], [**45**], [**46**], [**48**], [**63**]. We briefly recall the main achievements:

(i) For each convex domain $\Omega \subseteq \mathbb{R}^3$, not necessarily bounded or smooth, Martin and Morales constructed a complete minimal disk $\varphi\colon \mathbb{D} \hookrightarrow \Omega$ properly immersed into Ω; see [**43**].

(ii) In [**63**] M. Tokuomaru, constructed a complete minimal annulus $\varphi\colon A \hookrightarrow \mathbb{R}^3$ properly immersed into the unit ball of $B_1^{\mathbb{R}^3}(0) \subset \mathbb{R}^3$.

(iii) Martin and Morales [**44**] improved the results of [**43**], showing that, if Ω is a bounded, strictly convex domain of $\mathbb{R}^3$, with $\partial\Omega$ of class $C^{2,\alpha}$, then there exists a complete, minimal disk properly immersed into Ω whose limit set is close to a prescribed Jordan curve on $\partial\Omega$.

(iv) A. Alarcon, L. Ferrer, and F. Martin [**2**, Thm. B.] extended the results of [**43**] and [**63**]. They showed that for any convex domain $\Omega \subset \mathbb{R}^3$, not necessarily bounded or smooth, there exists a proper minimal immersion $\varphi\colon M \to \Omega$ of a complete non-compact surface M with arbitrary finite topology into Ω.

(v) Ferrer, Martin, and Meeks [**28**] improved the main results of [**44**], proving that given a bounded smooth domain $\Omega \subset \mathbb{R}^3$ and given any open surface M, there exists a complete, proper, minimal immersion $\varphi\colon M \to \Omega$ with the property that the limit sets of different ends are disjoint, compact, connected subsets of $\partial\Omega$. It should be remarked that the Ferrer-Martin-Meeks surfaces [**28**] immersed into a bounded smooth domain Ω can have either finite or infinite topology. They can have uncountably many ends and be either orientable or non-orientable. Moreover, the convexity of Ω is not a necessary hypothesis, although they need its smoothness. In fact, one can not drop the convexity and the smoothness of Ω altogether; see [**42**] for a counter-example. They also prove that for every convex open set Ω and every non-compact, orientable surface M, there exists a complete, proper minimal immersion $\varphi\colon M \to \Omega$ such that $\lim \varphi \equiv \partial\Omega$; see [**28**, Prop. 1].

(vi) In [**46**], Martin and Nadirashvili constructed complete minimal immersions $\varphi\colon \mathbb{D} \to \mathbb{R}^3$ of the disk $\mathbb{D} \subset \mathbb{C}$, admitting continuous extensions $\overline{\varphi}\colon \overline{\mathbb{D}} \to \mathbb{R}^3$ such that $\overline{\varphi}_{|\partial\mathbb{D}}\colon \partial\mathbb{D} = \mathbb{S}^1 \to \overline{\varphi}(\mathbb{S}^1)$ is an homeomorphism and $\overline{\varphi}(\mathbb{S}^1)$ is a non-rectifiable Jordan curve of Hausdorff dimension $\dim_{\mathcal{H}}(\overline{\varphi}(\mathbb{S}^1)) = 1$. They also showed that the set of Jordan curves $\overline{\varphi}(\mathbb{S}^1)$ constructed via this procedure is dense in the space of Jordan curves of $\mathbb{R}^3$ with respect to the Hausdorff metric.

(vii) Alarcon proved in [**1**] that for any arbitrary finite topological type there exists a compact Riemann surface $\mathcal{M}$, an open domain $M \subset \mathcal{M}$ with this topological type, and a conformal complete minimal immersion $\varphi\colon M \to \mathbb{R}^3$ that admits an extension to a continuous map $\overline{\varphi}\colon \overline{M} \to \mathbb{R}^3$ such that $\overline{\varphi}_{|\partial M}$ is an embedding and the Hausdorff dimension of $\overline{\varphi}(\partial M)$ is 1.

In this paper, we will address Yau's question whether the spectrum of bounded minimal surfaces of $\mathbb{R}^3$ is discrete or not. We provide a sharp, general criterion that applies to each of the examples in (i), ..., (vii). A preliminary answer was given by Bessa-Jorge-Montenegro in [**10**], where they proved that the spectrum of a complete minimal surface properly immersed into a ball of $\mathbb{R}^3$ is discrete. Despite the great generality of this result, the "bounding" convex domains $\Omega \subset \mathbb{R}^3$ were restricted to geodesic balls. Moreover, their proof uses, in a fundamental way, the properness condition that cannot be modified to deal with non-proper immersions. On the other hand, if the limit set of a bounded minimal immersion has Hausdorff dimension 1, thus resembling a curve as the Martin-Nadirashvili examples in (vi) or the examples of Alarcon [**1**], we can think the fact of that the minimal surface is not too far from a compact set with boundary and thus has discrete spectrum. We will show that that is the case. In Theorem 2.4, we show that the spectrum of a bounded minimal surface is discrete provided its limit set has zero Hausdorff measure of order $\Psi(t) = t^2|\log t|$. Moreover, we will consider bounded immersions where the "bounding"set satisfies a weaker convexity notion. On the other hand, we will set a simple geometric criterion that implies that the essential spectrum is not empty. In particular, we will show that the essential spectrum of non-proper isometric immersions with locally bounded geometry is non-empty. We will also study the spectrum of the examples Jorge-Xavier and Rosenberg-Toubiana give of complete minimal surfaces between two planes.

The structure of this paper is as follows. In section 2 we state our main result and its corollaries. In section 3 we introduce the notation and the necessary material to prove all results. Finally, in section 5 we present all the proofs.

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2. Main results

We start with the definition of j-convex open subsets.

Definition 2.1. An open subset $\Omega \subset N^n$ with smooth C^2-boundary is strictly j-convex, $j \in \{1, \dots, n-1\}$, if for every $q \in \partial\Omega$, the ordered eigenvalues $\xi_1(q) \leq \cdots \leq \xi_{n-1}(q)$ of the second fundamental form α of the boundary $\partial\Omega$ at q with respect to the unit normal vector field ν pointing toward Ω satisfies $\xi_1(q) + \cdots + \xi_j(q) > 0$. If for all $q \in \partial\Omega$ and some constant $c > 0$, the eigenvalues satisfy $\xi_1(q) + \ldots + \xi_j(q) \geq c$, then we say that Ω is strictly j-convex with constant c.

A result of J. Hadamard [**32**] states that if a compact immersed hypersurface $M \subset \mathbb{R}^n$ has positive definite second fundamental form at all $p \in M$, then M is embedded as the boundary $M = \partial\Omega$ of a strictly convex body Ω. In other words, a compact 1-convex subset $\Omega \subset \mathbb{R}^n$ is a convex body in the sense that any two points in Ω can be joined by a segment contained in Ω. The classical notions of convexity and mean convexity are respectively 1-convexity and $(n-1)$-convexity. The following example due to Jorge-Tomi [**35**] shows that a set can be 2-convex without being 1-convex. Let

$$\mathbb{T}^n(r_1,r_2)=\{(z,w)\in\mathbb{R}^2\times\mathbb{R}^{n-2}\colon (|z|-r_2)^2+|w|^2\leq r_1^2\},\quad 0<r_1<r_2$$

be the solid torus homeomorphic to $\mathbb{S}^1\times\mathbb{B}^{n-1}$ where $\mathbb{B}^{n-1}$ is the unit ball of $\mathbb{R}^{n-1}$. It was shown in [**35**] that $\mathbb{T}^n$ is 2-convex whenever this relation $r_1\leq r_2/2$ is satisfied. Regarding these notions of j-convexity, we shall show that strictly j-convexity of an open set Ω with constant $c>0$ and C^3-smooth boundary $\partial\Omega$ is equivalent to the existence of suitable j-subharmonic C^2-function $f\colon\Omega\to\mathbb{R}$ see Lemma 4.3 for details.

2.1. Discrete spectrum. In this section we start setting up the basic notation in order to state our main result.

Let $\Omega\subset N$ be a bounded open set in a Riemannian manifold. For a given $r>0$ let $T_r(\Omega)=\{y\in N\colon \mathrm{dist}_N(y,\Omega)\leq r\}$ be the closed tube around Ω and let

$$b=\sup\{K_N(z),z\in T_{\mathrm{diam}(\Omega)}(\Omega)\}. \tag{1}$$

For each $y\in\Omega$ define $r(y)=\min\{\mathrm{inj}_N(y),\pi/2\sqrt{b}\}$, where $\pi/2\sqrt{b}$ is replaced by $+\infty$ if $b\leq 0$. Set $r_{\Omega}=\inf_{y\in\Omega}r(y)$.

Definition 2.2. A bounded domain $\Omega\subset N$ is said to be totally regular if $\mathrm{diam}_N(\Omega)<r_{\Omega}$.

Example 2.3. Any bounded domain $\Omega\subset N$ of a Hadamard manifold is totally regular. On the other hand, $\Omega\subset\mathbb{S}^n(1)$ is totally regular if and only if $\mathrm{diam}_{\mathbb{S}^n(1)}(\Omega)<\pi/2$.

For $b\in\mathbb{R}$, define the function $\mu_b\colon[0,\infty)\to\mathbb{R}$ by

$$\mu_b(t)=\begin{cases}\dfrac{1}{\sqrt{b}}\tan(\sqrt{b}t) & \text{if}\quad b>0\\ t & \text{if}\quad b=0\\ \dfrac{1}{\sqrt{-b}}\tanh(\sqrt{-b}t) & \text{if}\quad b<0\end{cases} \tag{2}$$

We will need the notion of *generalized Hausdorff measures* or simply the Ψ-Hausdorff measures $\mathcal{H}_{\Psi}$, where $\Psi\colon[0,\infty)\to[0,\infty)$ is a continuous function, given by the Caratheodory construction. The Caratheodory construction can be found (with every detail) in the beautiful book of P. Mattila [**47**, Chapter 4]; see also Definition 3.1 later on in this article. Our first result is the following theorem that gives sufficient conditions on the size of the limit set of a bounded submanifold for its spectrum be discrete.

Theorem 2.4. *Let $\varphi\colon M\to N$ be an isometric immersion of a Riemannian m-manifold M into a Riemannian n-manifold N with mean* *curvature vector H. Suppose that $\varphi(M) \subset \Omega$, a bounded, totally regular, open subset of N, and let b be as in (1) and μ_b as defined in (2). Assume that*

$$\|H\|_{L^\infty(M)} < \frac{m-1}{m \cdot \mu_b(\mathrm{diam}(\Omega))}. \tag{3}$$

Define $\theta = \big[m - 1 - m \cdot \mu_b(\mathrm{diam}(\Omega)) \cdot \|H\|_{L^\infty(M)}\big] > 0$ and $\Psi \in C^0([0,\infty])$ given by

$$\Psi(t) = \begin{cases} t^2 & \text{if} \quad \theta > 1 \\ t^2|\log t| & \text{if} \quad \theta = 1 \\ t^{\theta+1} & \text{if} \quad \theta \in (0,1). \end{cases} \tag{4}$$

If one of the following conditions holds:

1) $\lim \varphi \cap \partial\Omega = \emptyset$ *and* $\mathcal{H}_\Psi(\lim \varphi) = 0$,

2) $\lim \varphi \cap \partial\Omega \neq \emptyset$, $\mathcal{H}_\Psi(\lim \varphi \cap \Omega) = 0$, *$\Omega$ is strictly m-convex with constant $c > 0$, $\partial\Omega$ is of class C^3, and the mean curvature vector H satisfies the further restriction,*

$$\|H\|_{L^\infty(M)} < \frac{c}{m}, \tag{5}$$

then the spectrum of $-\Delta$ is discrete.

We shall make a few comments about Theorem 2.4.

- We remark that in item 2, the Hausdorff measure of $\lim \varphi \cap \partial\Omega$ does not need to be zero. In particular, the examples of Ferrer, Martin, Meeks [**28**] of complete, proper minimal immersions $\varphi\colon M \to \Omega$ such that $\lim \varphi \equiv \partial\Omega \subset \mathbb{R}^3$ have discrete spectrum, provided Ω is strictly 2-convex. One illustrative example is the 2-convex solid torus $\mathbb{T}^2(r_1, r_2)$, $r_1 \leq r_2/2$ described in [**35**]. If M is any open surface, then there exists a complete, proper minimal immersion $\varphi\colon M \to \mathbb{T}^2(r_1, r_2)$, [**28**, Prop. 1], such that $\lim \varphi \equiv \partial\mathbb{T}^2(r_1, r_2)$; hence by Theorem 2.4, item 2, its spectrum is discrete.
- A more technical observation is: our definition of Ω being totally regular implies that $\mu_b(\mathrm{diam}(\Omega)) > 0$ and thus (3) is meaningful, where $b = \sup\{K_N(z), z \in T_{\mathrm{diam}(\Omega)}(\Omega)\}$. However, if one knows only an upper bound for the sectional curvatures $b_0 > b$ instead, then Theorem 2.4 is still valid, provided $\mu_{b_0}(\mathrm{diam}(\Omega)) > 0$.
- The case that $\lim \varphi \cap \Omega = \emptyset$ is equivalent to the properness of φ in Ω; therefore the statement of Theorem 2.4 extends in many aspects the main result of [**10**].
- Theorem 2.4 also applies to non-orientable manifolds M. In fact, its proof can be applied to the two-sheeted oriented covering of M yielding the same conclusions (We thank an anonymous referee for pointing this out.)

- The Riemannian manifold M may be geodesically incomplete and the statement regards the spectrum of the Friedrichs extension of $\Delta\colon C_c^\infty(M) \to C_c^\infty(M)$.

The minimal surfaces in the examples (i), (ii), (iii), and (iv) are properly immersed in 1-convex domains Ω of $\mathbb{R}^3$, whereas the minimal surfaces in (v) are properly immersed in smooth domains Ω. In those examples $\lim \varphi \cap \Omega = \emptyset$ thus $\mathcal{H}_\Psi(\lim \varphi \cap \Omega) = 0$. The examples in (vi) and (vii) are bounded and $\lim \varphi$ is a non-rectifiable Jordan curve of Hausdorff dimension 1. Thus $\mathcal{H}_\Psi(\lim \varphi \cap \Omega) = 0$ for $\Psi(t) = t^2|\log(t)|$. By Theorem 2.4, all of those examples of (i), (ii), (iii), (iv), (v), (vi), and (vii) have discrete spectrum, provided Ω is bounded strictly 2-convex with C^3-boundary. That can be summarized in the following corollary as follows.

Corollary 2.5. *Let $\varphi\colon M^m \to N^n$ be a minimal m-submanifold, possibly incomplete, immersed into a bounded open m-convex subset Ω of a Hadamard manifold with constant $c > 0$. Suppose that $\partial\Omega$ is C^3-smooth and $\Psi(t) = t^2$ if $m \geq 3$ and $\Psi(t) = t^2|\log(t)|$ if $m = 2$. If $\mathcal{H}_\Psi(\lim \varphi \cap \Omega) = 0$ then the spectrum of $-\Delta$ is discrete. In particular, those minimal surfaces constructed in* (i), (ii), (iii), (iv), (v), (vi), *and* (vii) *have discrete spectrum provided Ω is bounded, strictly 2-convex with C^3-boundary.*

Let γ be a Jordan curve and let a_γ be the infimum of all the areas of the disks spanning γ. It is well known that J. Douglas [**25**] and T. Radó [**54**] proved the existence of minimal disks $\varphi\colon \mathbb{D} \to \mathbb{R}^3$ spanning γ (Douglas proved the existence of minimal disks spanning Jordan curves in $\mathbb{R}^n$) if area $a_\gamma < +\infty$, and therefore the spectrum for each is discrete (provided $\mathcal{H}_\Psi(\gamma) = 0$). When $a_\gamma = +\infty$, there is no sense to speak about the least area surface spanning γ, however, Douglas [**26**] proved that there exists a globally stable minimal disk $\varphi\colon \mathbb{D} \to \mathbb{R}^3$ with infinite area spanning γ. On the other hand, the set $\mathbb{J} = \{\gamma\colon \mathbb{S}^1 \to \mathbb{R}^3\}$ of all non-rectifiable Jordan curves of $\mathbb{R}^3$ coming from the Martin-Nadirashvili's procedure is dense in the set of Jordan curves of $\mathbb{R}^3$ with respect to the Hausdorff metric, see [**46**]. Hence, the globally stable minimal disks D_γ of Douglas spanning non-rectifiable Jordan curves $\gamma \in \mathbb{J}$ cannot be complete since complete stable minimal surfaces (either orientable or non-orientable) of $\mathbb{R}^3$ are planar by Do Carmo-Peng, Fischer-Colbrie-Schoen, Pogorelov, and Ros's Theorem (A. Ros proved this characterization of the plane in the non-orientable case) [**20**], [**29**], [**53**], [**58**], [**61**]. For the same $\gamma \in \mathbb{J}$ there exists a complete minimal disk M_γ spanning γ by Martin-Nadirashvili's result [**46**]. Hence, every non-rectifiable curve $\gamma \in \mathbb{J}$ considered by Martin-Nadirashvili is spanned by at least two minimal disks. This together with our main result yields the following corollary that has interest on its own.

Corollary 2.6. *Let $\gamma \in \mathbb{J}$ be a non-rectifiable Jordan curve spanning a Martin-Nadirashvili minimal surface M_γ as in* [**46**]*. Then*

1) *γ is spanned by at least two minimal disks: a non-flat geodesically complete minimal surface M_γ given by Martin-Nadirashvili's result and a geodesically incomplete but globally stable minimal surface D_γ given by Douglas's result* [**26**].
2) *Any Douglas's solution D_γ for the classical Plateau problem for $\gamma \in \mathbb{J}$, as well as M_γ, has discrete spectrum.*
3) *Any minimal surface spanning a Jordan curve γ with $\mathcal{H}_\Psi(\gamma) = 0$, $\Psi(t) = t^2 |\log t|$, has discrete spectrum.*

There are examples of embedded continuous curves $\gamma\colon [0,1] \to \mathbb{R}^2$ with $\dim_{\mathcal{H}}(\gamma([0,1])) = 2$; see [**26**], [**49**]. It would be interesting to know the spectrum of the minimal solutions of the Plateau problem spanning such curve γ with Hausdorff dimension $\dim_{\mathcal{H}}(\gamma) \geq 2$.

Remark 2.7. The hypothesis concerning the measure of the limit set $\lim \varphi$ in Theorem 2.4 is sharp. Consider a bounded, complete proper minimal annulus $\varphi\colon M \to B_1^{\mathbb{R}^3}(0)$ as in [**63**] with $\lim \varphi \cap \Omega = \emptyset$, and thus with discrete spectrum by Theorem 2.4 or [**10**, Thm. 1]. Considering the universal cover $\pi\colon \widetilde{M} \to M$ and setting $\phi = \varphi \circ \pi\colon \widetilde{M} \to \mathbb{R}^3$ one has a bounded, complete minimal surface with non-empty essential spectrum. In fact, if $\pi\colon (\widetilde{M}, \pi^* \mathrm{d}s^2) \to (M, \mathrm{d}s^2)$ is an infinite sheeted covering then the induced metric $\pi^* \mathrm{d}s^2$ satisfies the "ball property" (see Definition 2.8); therefore the essential spectrum of $(\widetilde{M}, \pi^* ds^2)$ is non-empty, regardless of the spectrum of (M, ds^2). Observe that the immersed submanifold $\varphi(M) = \phi(\widetilde{M})$ but the limit sets are different, $\lim \varphi \neq \lim \phi = \overline{\phi(M)}$ and Theorem 2.4 could not be applied since the Hausdorff dimension $\dim_{\mathcal{H}}(\lim \phi \cap B_1^{\mathbb{R}^3}(0)) \geq 2$.

2.2. Ball property. A counterpart to Theorem 2.4 would be a set of (natural) geometric conditions implying that the essential spectrum of a Riemannian manifold is not empty. In the second part of this paper we will establish a criterion that does not involve curvatures and thus can be used to study the spectrum of the complete minimal surfaces, for instance, those immersed into a slab of $\mathbb{R}^3$ constructed by Jorge-Xavier [**36**] and Rosenberg-Toubiana [**60**]. We begin with the following simple definition.

Definition 2.8. A Riemannian manifold M has the ***ball property*** if there exists $R > 0$ and a collection of disjoint balls $\{B_R^M(x_j)\}_{j=1}^{+\infty}$ of radius R centered at x_j such that for some constants $C > 0$, $\delta \in (0,1)$, possibly depending on R,

$$\mathrm{vol}\big(B_{\delta R}^M(x_j)\big) \geq C^{-1} \mathrm{vol}\big(B_R^M(x_j)\big) \qquad \forall\, j \in \mathbb{N}. \tag{6}$$

Observe that (6) is *not* a doubling condition since it needs to hold only along the sequence $\{x_j\}$ and the constant C may depend on R. The importance of the ball property is that its validity implies that the essential spectrum is non-empty.

Theorem 2.9. *If a Riemannian manifold M has the ball property* (*with parameters R, δ, C*), *then*

$$\inf \sigma_{\text{ess}}(-\Delta) \leq \frac{C}{R^2(1-\delta)^2}. \tag{7}$$

The well-known Bishop-Gromov volume comparison theorem (see [**12**], [**31**], [**51**]) shows that any complete non-compact Riemannian m-manifold M with Ricci curvature bounded from below has the ball property, and therefore it has non-empty essential spectrum. This was known to H. Donnelly, who proved sharp results in the class of manifolds with Ricci curvature bounded below. He showed that the essential spectrum of a complete non-compact Riemannian m-manifold M with Ricci curvature $\mathrm{Ric}_M \geq -(m-1)c^2 > -\infty$ intersects the interval $[0, (m-1)^2c^2/4]$; see [**21**, Thm. 3.1]. However, there are examples of complete non-compact Riemannian manifolds with the ball property and $\inf \mathrm{Ric} = -\infty$: for instance, the examples of Jorge-Xavier of minimal surfaces between two planes that have Ricci curvature satisfying $\inf \mathrm{Ric} = -\infty$ (see [**7**], [**59**]) some of them have the ball property and therefore have non-empty essential spectrum. H. Rosenberg and E. Toubiana, in [**60**], constructed a complete minimal annulus between two parallel planes of $\mathbb{R}^3$ such that the immersion is proper in the slab. The Jorge-Xavier and Rosenberg-Toubiana examples are constructed with a flexible method depending on a chosen set of parameters and we will show that, depending on this choice of parameters, the spectrum of the complete minimal surfaces immersed in the slab can be the half-line $[0, \infty)$.

There are other examples of manifolds with the ball property, for instance, the non-proper submanifolds with locally bounded geometry. An isometric immersion $\varphi \colon M \to N$ is said to have *locally bounded geometry* if for each compact set $W \subset N$ there is a constant $\Lambda = \Lambda(W)$ such that

$$\|\alpha_\varphi\|_{L^\infty(\varphi^{-1}(W))} \leq \Lambda.$$

Here α_φ is the second fundamental form of the immersion φ.

To complete this section about the ball property, we will prove the following result about the spectrum of non-proper submanifolds with locally bounded geometry.

Theorem 2.10. *Let $\varphi \colon M \to N$ be an isometric immersion with locally bounded geometry of an open Riemannian m-manifold M into a complete Riemannian n-manifold N. If the immersion is non-proper then M has the ball property. Thus it has non-empty essential spectrum.*

2.2.1. Spectrum of complete minimal surfaces in the slab. We will need to give a brief description of the complete minimal surfaces between two parallel planes, constructed by Jorge-Xavier.

Jorge and Xavier in [**36**] constructed a complete minimal immersion of the disk $\varphi\colon \mathbb{D} \to \mathbb{R}^3$ with $\mathbb{R}^3$, $\varphi(M) \subset \{(x, y, z) \in \mathbb{R}^3 \colon |z| < 1\}$. Let $\{D_n \subset \mathbb{D}\}$ be a sequence of closed disks centered at the origin such that $D_n \subset \mathrm{int}(D_{n+1})$, $\cup D_n = \mathbb{D}$. Let $K_n \subset D_n$ be a compact set so that $K_n \cap D_{n-1} = \emptyset$ and $D_n \setminus K_n$ is connected as in Figure 1 below.

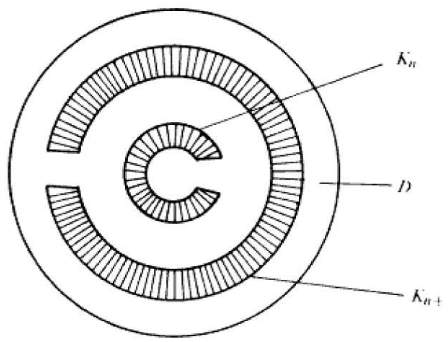

Fig. 1. The compact sets K_n.

By Runge's Theorem [**34**, p. 96], there exists a holomorphic function $h\colon \mathbb{D} \to \mathbb{C}$ such that $|h - c_n| < 1$ on K_n, for each n. Letting $g = e^h$ and $f = e^{-h}$ and setting $\phi = (f(1-g^2)/2, i \cdot f(1+g^2)/2, fg)$, by the Weierstrass representation, one has that $\varphi = \mathrm{Re} \int \phi\colon \mathbb{D} \to \mathbb{R}^3$ is a minimal surface with bounded third coordinate. Let r_n denote the Euclidean distance between the inner and the outer circle of K_n and for each n choose a constant c_n such that

$$\sum_{n \text{ even}}^{+\infty} r_n e^{c_n - 1} = +\infty, \qquad \sum_{n \text{ odd}}^{+\infty} r_n e^{c_n - 1} = +\infty. \tag{8}$$

Condition (8) implies that this minimal surface is complete.

The induced metric $\mathrm{d}s^2$ by this minimal immersion is conformal to the Euclidean metric $|\mathrm{d}z|^2$ given by $\mathrm{d}s^2 = \lambda^2 |\mathrm{d}z|^2$, where

$$\lambda(z) = \frac{1}{2}\left(|e^{h(z)}| + |e^{-h(z)}|\right). \tag{9}$$

The choice of the compact subsets $K_n \subset D_n$ with width r_n and the set of constants c_n satisfying (8) and yielding a complete minimal surface of $\mathbb{R}^3$ between two parallel planes is what we are calling a choice of parameters, $(\{(r_n, c_n)\})$, in Jorge-Xavier's construction. We should give a brief description of Rosenberg-Toubiana construction of a complete minimal annulus properly immersed into a slab of $\mathbb{R}^3$; see details in [**60**]. They start considering a labyrinth in the annulus

$$\mathbb{A}(1/c, c) = \{z \in \mathbb{C} \colon 1/c < |z| < c\}, \; c > 1$$

composed by compact sets K_n contained in the annulus $\mathbb{A}(1, c)$ and compact sets $L_n = \{1/\overline{z}\colon -z \in K_n\}$ contained in the annulus $\mathbb{A}(1/c, 1)$ as in Figure 2 below. The compact sets L_n are converging to the boundary $|z| = 1/c$ and the compact sets K_n are converging to the boundary $|z| = c$.

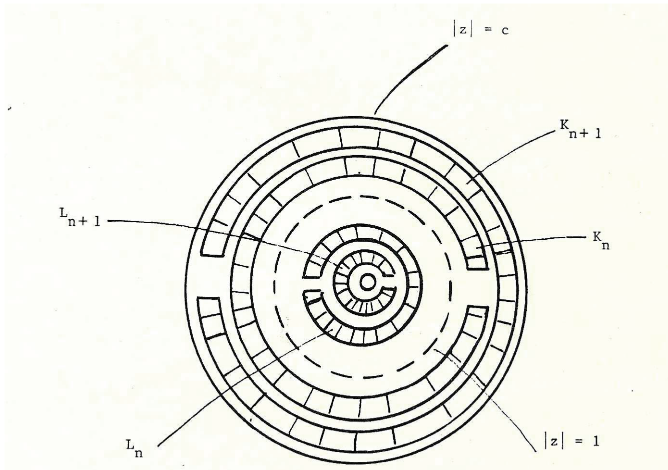

Fig. 2.

They (Rosenberg and Toubiana) needed two non-vanishing holomorphic functions $f, g\colon \mathbb{A}(1/c, c) \to \mathbb{C}$, in order to construct a minimal surface via Weierstrass representation formula, so that the resulting minimal surface is geodesically complete and properly immersed into a slab. They construct f and g satisfying $f(z) \cdot g(z) = 1/z$ where $|g(z) - e^{2c_n}| < 1$ on K_n and $|g(z) - e^{-2c_n}| < 1$ on L_n, where $\{c_n\}$ is a sequence of positive numbers such that

$$\sum_n^{\infty} r_n e^{2c_n} = \infty, \quad \sum_n^{\infty} s_n e^{2c_n} = \infty$$

and r_n and s_n are the width of K_n and L_n respectively. The induced metric by the immersion on the annulus $\mathbb{A}(1/c, c)$ is $ds^2 = \lambda^2 |dz|^2$ where

$$\lambda = \frac{1}{2|z|} \left(\frac{1}{|g(z)|} + |g(z)| \right).$$

On K_n we have

$$e^{2c_n} \geq \left(1 + \frac{e^{2c_n}}{2} \right) \geq \lambda \geq \frac{1}{2|c|} \left(e^{2c_n} - 1 \right). \tag{10}$$

The choice of parameters $\{(r_n, c_n)\}$ in Jorge-Xavier's construction or $\{(r_n, s_n, c_n)\}$ in Rosenberg-Toubiana's construction gives information about the essential spectrum of the resulting surfaces. In the next result, set $\lambda_n := \sup_{z \in K_n} \lambda(z)$.

Theorem 2.11. *Let $\varphi\colon \mathbb{D}, \mathbb{A}(1/c, c) \to \mathbb{R}^3$ be either Jorge-Xavier's or Rosenberg-Toubiana's complete minimal surface immersed into the slab with parameters $\{(r_n, c_n)\}$ or $\{(r_n, s_n, c_n)\}$. If $\limsup \lambda_n r_n = \infty$ then $\sigma_{\text{ess}}(-\Delta) = [0, \infty)$. And if $\limsup \lambda_n r_n > 0$ then $\varphi(\mathbb{D})$ or $\varphi(\mathbb{A}(1/c, c))$ has the ball property and $\sigma_{\text{ess}}(-\Delta) \neq \emptyset$.*

At points $z \in K_n$ we have $e^{1+c_n} \geq \lambda(z) \geq \frac{1}{2}e^{c_n-1}$; therefore

$$e^{c_n+1} \geq \lambda_n \geq e^{c_n}/2e.$$

If $c_n = -\log(r_n^2)$ we have that the parameters $\{(r_n, c_n)\}$ satisfy (8) and $\lambda_n r_n = 1/(2er_n)$. Thus $\limsup \lambda_n r_n = \infty$, yielding a complete minimal surface between two parallel planes with spectrum $\sigma(-\Delta) = [0, \infty)$. In the original construction in [**36**], Jorge-Xavier choose $c_n = -\log r_n$ that yields $e \geq r_n\lambda_n \geq 1/2e$ and the resulting minimal surfaces have non-empty essential spectrum.

3. Preliminaries

We will denote by $\varphi\colon M \to N$ an isometric immersion of a complete Riemannian m-manifold M into a Riemannian n-manifold N. The Levi-Civita connections of N and M are denoted by $\overline{\nabla}$ and ∇ respectively. The second fundamental form $\alpha = \overline{\nabla}\mathrm{d}\varphi^{\perp}$ and mean curvature vector $H = \mathrm{tr}\alpha/m$. The gradient of a function $g\colon N \to \mathbb{R}$ is denoted by $\overline{\nabla} g$ whereas $\nabla(g \circ \varphi) = (\overline{\nabla} g)^{\top}$ is the gradient of $g \circ \varphi$, the restriction of g to M. The hessian of g is denoted by $\overline{\nabla}\mathrm{d}g$ and the hessian $\nabla\mathrm{d}(g \circ \varphi)$ of $g \circ \varphi$ are related by

$$\nabla\mathrm{d}(g \circ \varphi) = \overline{\nabla}\mathrm{d}g + \langle \overline{\nabla}\mathrm{d}\varphi^{\perp}, \overline{\nabla} g\rangle. \tag{11}$$

The symbol $B_r^N(x)$ denotes the geodesic ball of N centered at $x \in N$ with radius r. However, the unit ball $B_1^{\mathbb{R}^2}(0)$ of $\mathbb{R}^2$ will be denoted by $\mathbb{D}$. Similarly, for $X \subset N$ the symbol $T_r^N(X)$, called the tube of radius r around X, denotes the open set of points (in N) whose distance from X is less than r. Finally, denote $\mathbb{R}^+ = (0, +\infty)$ and $\mathbb{R}_0^+ = [0, +\infty)$.

3.1. Carathéodory's Construction.

In this section we shall review the notion of generalized Ψ-Hausdorff measures. Here, we do follow the elegant exposition of P. Mattila in [**47**, Chap. 4].

Definition 3.1 (Carathéodory's Construction)**.** Let X be a metric space, $\mathcal{J}$ a family of subsets of X, and $\zeta \geq 0$ a non-negative function on $\mathcal{J}$. Make the following assumptions.

1. For every $\delta > 0$ there are $E_1, E_2, \ldots \in \mathcal{J}$ such that $X = \bigcup_{i=1}^{\infty} E_i$ and $\mathrm{diam}(E_i) \leq \delta$.
2. For all $\delta > 0$ there is $E \in \mathcal{J}$ such that $\zeta(E) \leq \delta$ and $\mathrm{diam}(E) \leq \delta$.

For $0 < \delta \leq \infty$ and $A \subset X$ we define

$$\zeta_\delta(A) = \inf \left\{ \sum_{i=1}^{\infty} \zeta(E_i) : A \subset \bigcup_{i=1}^{\infty} E_i,\ \mathrm{diam}(E_i) \leq \delta,\ E_i \in \mathcal{J} \right\}.$$

It is easy to see that $\zeta_\delta(A) \leq \zeta_\epsilon(A)$ whenever $0 < \epsilon < \delta \leq \infty$. Therefore,

$$\mathcal{H}_\zeta(A) = \lim_{\delta \to 0} \zeta_\delta(A) = \sup_{\delta > 0} \zeta_\delta(A)$$

defines the generalized ζ-Hausdorff measure $\mathcal{H}_\zeta$.

In this construction, let X be a complete Riemannian manifold M and let $\mathcal{J}$ be the family of Borel subsets of M. Let $\Psi\colon [0,\infty) \to [0,\infty)$ be a continuous function such that $\Psi(0) = 0$. The Ψ-Hausdorff measure is defined by $\mathcal{H}_\Psi(A) = \mathcal{H}_\zeta(A)$ where $\zeta(A) = \Psi(\mathrm{diam}(A))$ and it is Borel regular; see [**47**, Thm. 4.2]. Taking $\mathcal{J} = \{\text{open subsets of } M\}$ instead of the Borel sets and the same Ψ, the generalized Hausdorf measures obtained by the Carathéodory construction coincide, i.e. they are the same Ψ-Hausdorff measure $\mathcal{H}_\Psi$; see [**47**, Thm. 4.4]. The choice $\Psi(t) = t^\beta$, for some fixed $\beta > 0$, gives the standard β-dimensional Hausdorff measure $\mathcal{H}_{t^\beta} = \mathcal{H}^\beta$.

Remark 3.2. If $\mathcal{J}$ is the family of geodesic balls of M, the resulting measure $\overline{\mathcal{H}}_\Psi$ does not coincide, in general, with generalized Hausdorff measure $\mathcal{H}_\Psi$; see [**47**, Chap. 5]. However, if for some constant $c > 0$ the inequality $\Psi(2t) \leq c \cdot \Psi(t)$ holds, then $\mathcal{H}_\Psi \leq \overline{\mathcal{H}}_\Psi \leq c\mathcal{H}_\Psi$.

The first inequality $\mathcal{H}_\Psi \leq \overline{\mathcal{H}}_\Psi$ is obvious from the definition. To prove $\overline{\mathcal{H}}_\Psi \leq c\mathcal{H}_\Psi$ we proceed as follows. Since every open set E_j is contained in a ball $B^M_{r_j}(x_j)$ of radius $r_j = \mathrm{diam}(E_j)$, we have that for every covering $\{E_j\}$ of $A \subseteq M$ with $\mathrm{diam}(E_j) < \delta$,

$$\sum_{i=1}^{+\infty} \Psi\big(\mathrm{diam}(E_j)\big) \geq \frac{1}{c} \cdot \sum_{i=1}^{+\infty} \Psi\big(2\mathrm{diam}(E_j)\big) = \frac{1}{c} \cdot \sum_{i=1}^{+\infty} \Psi\big(\mathrm{diam}(B^M_{r_j}(x_j))\big).$$

Taking the infimum in the right-hand side with respect to all covering $\{B^M_{r_j}(x_j)\}$ by balls of diameter less than 2δ and taking the infimum in the left-hand side with respect to E_i we have $\overline{\zeta}_\delta \leq c \cdot \zeta_\delta$, ($\zeta = \Psi(\mathrm{diam})$. Letting $\delta \downarrow 0$, we obtain the desired $\overline{\mathcal{H}}_\Psi \leq c\mathcal{H}_\Psi$.

3.2. Strategy of proof of Theorem 2.4. Let M be a Riemannian manifold. The Laplace operator $\Delta = \mathrm{div} \circ \mathrm{grad}$ acting on C^∞_0, the space of smooth functions with compact support, is symmetric with respect to the L^2-scalar product. If M is complete, it is known that Δ is essentially self-adjoint; thus it has a unique (unbounded) self-adjoint extension to an operator on $L^2(M)$, also denoted by Δ whose domain $D(\Delta) = \{f \in L^2(M) : \Delta f \in L^2(M)\}$. If M is not geodesically complete then Δ may fail to be essentially self-adjoint in $C_c^\infty(M)$ and in this case we will consider the Friedrichs extension of Δ (that is, the unique self-adjoint extension of $(\Delta, C_c^\infty(M))$ whose domain lies in that of the closure of the associated quadratic form). Moreover, $-\Delta$ is positive semi-definite so that the spectrum of $-\Delta$ is contained in $[0,\infty)$. The spectrum of a self-adjoint operator $-\Delta$, denoted by $\sigma(-\Delta)$, is formed by all $\lambda \in \mathbb{R}$ for which $-(\Delta+\lambda I)$ is not injective or the inverse operator $-(\Delta+\lambda I)^{-1}$ is unbounded; see [**19**]. The set of all eigenvalues of $\sigma(M)$ is the *point spectrum* $\sigma_p(M)$, while the *discrete spectrum* $\sigma_d(M)$ is the set of all isolated eigenvalues of finite multiplicity. The complement of the discrete spectrum is the *essential spectrum*, $\sigma_{\mathrm{ess}}(M) = \sigma(M) \setminus \sigma_d(M)$.

To show that $-\Delta$ has discrete spectrum we rely on the well-known characterization (12) of the essential spectrum (see [**24**], [**50**, Thm. 2.1]) and Barta's eigenvalue lower bound (see [**6**], [**9**]). This characterization relates the infimum $\inf \sigma_{\mathrm{ess}}(-\Delta)$ of the essential spectrum of $-\Delta$ to the fundamental tone of the complements of compact sets. This is

$$\inf \sigma_{\mathrm{ess}}(-\Delta) = \sup_{K \subset M} \lambda^*(M \backslash K) \tag{12}$$

where K is compact and $\lambda^*(M\backslash K)$ is the bottom of the spectrum of the Friedrichs extension of $(-\Delta, C_c^\infty(M\backslash K))$, given by

$$\lambda^*(M\backslash K) = \inf\left\{ \frac{\int_{M\backslash K} |\nabla u|^2}{\int_{M\backslash K} u^2}, \, 0 \neq u \in C_0^\infty(M\backslash K) \right\}.$$

On the other hand, Barta inequality gives a lower bound for $\lambda^*(M\backslash K)$ via positive functions; this is

$$\lambda^*(M\backslash K) \geq \inf_{M\backslash K} \frac{-\Delta w}{w} \qquad \text{for every } \ 0 < w \in C^2(M\backslash K). \tag{13}$$

To prove that $-\Delta$ has discrete spectrum or, equivalently, to prove that $\inf \sigma_{\mathrm{ess}}(-\Delta) = +\infty$, it is enough to find, for each small $\epsilon > 0$, a compact set $K_\epsilon \subset M$ and a function $0 < w_\epsilon \in C^2(M \setminus K_\epsilon)$ such that

$$\frac{-\Delta w_\epsilon}{w_\epsilon} \geq c(\epsilon) \ \text{ on } M\backslash K_\epsilon, \tag{14}$$

where $c(\epsilon) \to +\infty$ as $\epsilon \to 0$. Each w_ϵ will be constructed as a sum of suitable strictly positive superharmonic functions, depending on a good covering of $\lim \varphi$ by balls.

4. Technical lemmas

4.1. Main Lemma. Let $\varphi\colon M \to N$ be an isometric immersion of a complete Riemannian m-manifold M into a Riemannian n-manifold N, with mean curvature vector H. Suppose that $\varphi(M) \subset \Omega$, a bounded, totally regular subset, and let $b = \sup\{K_N(z), z \in T_{\mathrm{diam}(\Omega)}(\Omega)\}$. Fix

$\bar{a} > 0$ such that $(\log(\bar{a}))^2 > \log(\operatorname{diam}(\Omega))$ and if $b > 0$, suppose in addition that $\bar{a} \leq \min\{\pi/3\sqrt{b}, \pi/2(1+\theta)\sqrt{b}\}$. Recalling that

$$\theta = \left[m - 1 - m \cdot \mu_b(\operatorname{diam}(\Omega)) \cdot \|H\|_{L^\infty(M)}\right]$$

we have the following lemma.

Lemma 4.1 (Main Lemma)**.** *For each $a \in (0, \bar{a}/3]$ and $x \in \Omega$ such that $\varphi(M) \subset B^N_{\operatorname{diam}(\Omega)}(x)$, if $\theta > 0$ there exists $u \in C^\infty(M)$ satisfying these four conditions.*

i. *$u \geq 0$ and $u(p) = 0$ if and only if $\varphi(p) = x$.*
ii. *$\Delta u \geq \theta/3$ on $\varphi^{-1}(B^N_a(x))$ if $\varphi^{-1}(B^N_a(x)) \neq \emptyset$.*
iii. *$\Delta u \geq 0$ on M.*
iv.

$$\|u\|_{L^\infty}(M) \leq \begin{cases} Ca^2 & \text{if} \quad \theta > 1 \\ Ca^2|\log a| & \text{if} \quad \theta = 1 \\ Ca^{\theta+1} & \text{if} \quad 0 < \theta < 1 \end{cases}$$

Where C is positive constant depending on m, $\operatorname{diam}(\Omega)$, $\|H\|_{L^\infty(M)}$.

Proof. Fix $x \in \Omega$ such that $\varphi(M) \subset B^N_{\operatorname{diam}(\Omega)}(x) \subset B^N_{r_\Omega}(x)$. Thus, the distance function $\rho(y) = \operatorname{dist}_N(x, y)$ is smooth (except at $y = x$) and the geodesic ball $B^N_{\operatorname{diam}(\Omega)}(x)$ is 1-convex. In fact, by the hessian comparison theorem, [**11**, Theorem 1.15],

$$\overline{\nabla}\mathrm{d}\rho \geq \frac{h'(\rho)}{h(\rho)}\Big(\langle\, ,\, \rangle - \mathrm{d}\rho \otimes \mathrm{d}\rho\Big), \tag{15}$$

where $h\colon [0, \infty) \to [0, \infty)$ given by

$$h(t) = \begin{cases} \dfrac{1}{\sqrt{b}}\sin(\sqrt{b}t) & if \quad b > 0 \\[2ex] t & if \quad b = 0 \\[2ex] \dfrac{1}{\sqrt{-b}}\sinh(\sqrt{-b}t) & if \quad b < 0. \end{cases}$$

Let $f \in C^2(N)$ be defined by $f(y) = g(\rho(y))$ for some $g \in C^2(\mathbb{R}^+_0)$ that will be chosen later. The chain rule applied to the composition $f \circ \varphi \in C^2(M)$ implies that

$$\nabla\mathrm{d}(f \circ \varphi) = \overline{\nabla}\mathrm{d}f(\mathrm{d}\varphi, \mathrm{d}\varphi) + \mathrm{d}f(\overline{\nabla}\mathrm{d}\varphi^\perp)$$

where $\nabla, \overline{\nabla}$ are the connections of M and N respectively and $\overline{\nabla}\mathrm{d}\varphi^\perp$ is the second fundamental form of the immersion. Let $\{e_i, e_\alpha\}$ be a local

Darboux frame along φ, with $\{e_i\}$ tangent to M. Tracing the above equality, it yields

$$\Delta(f\circ\varphi)=\sum_{j=1}^{m}\overline{\nabla}\mathrm{d}f(e_j,e_j)+m\mathrm{d}f(H). \tag{16}$$

On the other hand,

$$\overline{\nabla}\mathrm{d}f=g''(\rho)\mathrm{d}\rho\otimes\mathrm{d}\rho+g'(\rho)\overline{\nabla}\mathrm{d}\rho.$$

If $g'\geq 0$ and by (15),

$$\overline{\nabla}\mathrm{d}f\geq\frac{g'(\rho)h'(\rho)}{h(\rho)}\Big(\langle\,,\,\rangle-\mathrm{d}\rho\otimes\mathrm{d}\rho\Big)+g''(\rho)\mathrm{d}\rho\otimes\mathrm{d}\rho. \tag{17}$$

Using $|\mathrm{d}\rho|=1$ and by (17),

$$\begin{aligned}\sum_{j=1}^{m}\overline{\nabla}\mathrm{d}f(e_j,e_j)+m\mathrm{d}f(H) &= \frac{g'h'}{h}\Big(m-\sum_{j=1}^{m}\mathrm{d}\rho(e_j)^2\Big)+g''\sum_{j=1}^{m}\mathrm{d}\rho(e_j)^2\\ &\quad +mg'\mathrm{d}\rho(H)\\ &\geq \frac{g'h'}{h}\Big(m-\sum_{j=1}^{m}\mathrm{d}\rho(e_j)^2-m\frac{h}{h'}\|H\|\Big)\\ &\quad +g''\sum_{j=1}^{m}\mathrm{d}\rho(e_j)^2\\ &\geq \frac{g'h'}{h}\Big(\overbrace{m-1-m\mu_b(\mathrm{diam}(\Omega))\|H\|_{L^\infty(M)}}^{=\theta}\Big)\\ &\quad +g''\sum_{j=1}^{m}\mathrm{d}\rho(e_j)^2\\ &= \frac{g'h'}{h}\theta+g''\sum_{j=1}^{m}\mathrm{d}\rho(e_j)^2.\end{aligned} \tag{18}$$

In other words,

$$\Delta(f\circ\varphi)\geq\frac{g'h'}{h}\theta+g''\sum_{j=1}^{m}\mathrm{d}\rho(e_j)^2. \tag{19}$$

Define $\omega\colon[0,\infty)\to\mathbb{R}$ by

$$\omega(t)=\begin{cases}(1-\dfrac{t}{3a(1+\theta)})(\theta+1)h'(t) & \text{if } t\leq 3a(1+\theta)\\ 0 & \text{if } t\geq 3a(1+\theta)\end{cases}$$

where $3a\leq\bar{a}$. Setting

$$g(t)=\int_0^t\frac{1}{h(s)^\theta}\left[\int_0^s h(\sigma)^\theta\omega(\sigma)\mathrm{d}\sigma\right]\mathrm{d}s, \tag{20}$$

we have that g is a solution of

$$g'(t)\frac{h'(t)}{h(t)}\theta + g''(t) = \omega(t). \tag{21}$$

It is easy to show that $g \in C^2([0,\infty))$. From (21) we have that if $t \leq 3a(1+\theta)$ then

$$\begin{aligned} g''(t) &= \omega(t) - \frac{\theta h'(t)}{h(t)^{1+\theta}(t)}\int_0^t (1 - \frac{s}{3a(1+\theta)})\frac{\mathrm{d}}{\mathrm{d}s}(h^{1+\theta}(s))\mathrm{d}s \\ &= \omega(t) - \theta h'(t) + \frac{\theta h'(t)}{h^{(\theta+1)}(t)}\int_0^t \frac{s}{3a}h^{\theta}(s)h'(s)\mathrm{d}s \\ &= (1-\frac{t}{3a})h'(t) + \frac{\theta h'(t)}{h^{(\theta+1)}(t)}\int_0^t \frac{s}{3a}h^{\theta}(s)h'(s)\mathrm{d}s. \end{aligned} \tag{22}$$

From (22) we have that $g''(t) \geq 0$ if $t \leq 3a$. Moreover, $h'(t) \geq 1/2$ if $t \leq 3a$. Then at any $x' \in \varphi^{-1}(B_a^N(x))$ we have from (19)

$$\begin{aligned} \Delta f \circ \varphi(x') &\geq \frac{g'h'}{h}\theta + g''\sum_{j=1}^{m}\mathrm{d}\rho(e_j)^2 \\ &\geq g'(\rho(\varphi(x)))\frac{h'}{h}\rho(\varphi(x))\theta \\ &\geq (1-\frac{\rho(\varphi(x))}{3a(1+\theta)})\theta h'(\rho(\varphi(x))) \\ &\geq \frac{1}{2}(1-\frac{\rho(\varphi(x))}{3a(1+\theta)})\theta \\ &\geq \frac{\theta}{3}. \end{aligned} \tag{23}$$

Decompose

$$M = \{y \in M\colon g''(\rho(\varphi(y))) \geq 0\} \cup \{y \in M\colon g''(\rho(\varphi(y))) < 0\} = A \cup B.$$

We have that the inequality (23) shows that if $x' \in A$ then $\Delta f \circ \varphi(x) \geq 0$.

On the other hand, at any point $x' \in B$ we have by (19) and by the fact that

$$|\overline{\nabla}\rho|^2 = 1 = \sum_{j=1}^{m}\mathrm{d}\rho(e_j)^2 + \sum_{\alpha=m+1}^{n}\mathrm{d}\rho(e_\alpha)^2 \geq \sum_{j=1}^{m}\mathrm{d}\rho(e_j)^2,$$

that

$$
\begin{aligned}
\Delta f\circ\varphi(x) &\geq \left[g'\frac{h'}{h}\theta+g''\sum_{j=1}^{m}\mathrm{d}\rho(e_j)^2\right],\\
&\geq g'\frac{h'}{h}\theta+g''\\
&\geq \omega \qquad\qquad (24)\\
&\geq 0.
\end{aligned}
$$

Observe that

$$
\int_0^t h(s)^\theta\omega(s)\mathrm{d}s \leq \begin{cases} h(t)^{1+\theta} & if \quad 0\leq t\leq 3a(1+\theta)\\ \\ h(t_1)^{1+\theta} & if \quad t>t_1=3a(1+\theta). \end{cases} \tag{25}
$$

Taking into account that $c_1\cdot t\leq h(t)\leq c_2\cdot t$, $t\in[0,\mathrm{diam}(\Omega)]$ for some positive constants c_1, c_2, we have the following upper bounds for g.

If $0\leq t\leq t_1=3a(1+\theta)$,

$$
\begin{aligned}
g(t) &= \int_0^t\frac{1}{h(s)^\theta}\left[\int_0^s h(\sigma)^\theta\omega(\sigma)\mathrm{d}\sigma\right]\mathrm{d}s\\
&\leq \int_0^t h(s)\mathrm{d}s \qquad\qquad (26)\\
&\leq c_2\frac{(t_1)^2}{2}=9\cdot c_2\cdot\frac{(1+\theta)^2}{2}\cdot a^2.
\end{aligned}
$$

If $t\geq t_1=3a(1+\theta)$,

$$
\begin{aligned}
g(t) &= \int_0^a\frac{1}{h(s)^\theta}\left[\int_0^s h(\sigma)^\theta\omega(\sigma)\mathrm{d}\sigma\right]\mathrm{d}s+\int_a^t\frac{1}{h(s)^\theta}\left[\int_0^{t_1} h(\sigma)^\theta\omega(\sigma)\mathrm{d}\sigma\right]\mathrm{d}s\\
&\leq \int_0^a h(s)\mathrm{d}s+h^{1+\theta}(t_1)\int_a^t\frac{1}{h(s)^\theta}\mathrm{d}s\\
&\leq \frac{c_2}{2}\cdot a^2+\frac{c_2^{(1+\theta)}(3a(1+\theta))^{(1+\theta)}}{c_1}\int_a^t\frac{1}{s^\theta}\mathrm{d}s\\
&= c_3\cdot a^2+c_4\cdot a^{(\theta+1)}\int_a^t\frac{1}{s^\theta}\mathrm{d}s\\
&\leq c_3\cdot a^2+c_4\cdot a^{(\theta+1)}\begin{cases} \dfrac{a^{1-\theta}}{\theta-1} & \text{if} \quad \theta>1\\ \\ c_5\cdot|\ln a| & \text{if} \quad \theta=1\\ \\ \dfrac{t^{1-\theta}}{1-\theta}\leq\dfrac{\mathrm{diam}(\Omega)^{1-\theta}}{1-\theta} & \text{if} \quad 0<\theta<1 \end{cases}
\end{aligned}
$$

We can deduce from (26) and (**??**) that there exists a positive constant C depending on $m, \mathrm{diam}(\Omega), b$ and $\|H\|_{L^\infty(M)}$ such that

$$\|g\|_{L^\infty([0,\mathrm{diam}(\Omega)])} \leq \begin{cases} Ca^2 & \text{if } \theta > 1 \\ Ca^2|\log a| & \text{if } \theta = 1 \\ Ca^{\theta+1} & \text{if } \theta \in (0,1). \end{cases} \tag{27}$$

Taking $u = f \circ \varphi\colon M^m \to \mathbb{R}$ we have that:

- By construction $u(p) = 0$ if and only if $\varphi(p) = x$.
- By (23) and (25) we have $\Delta u \geq \theta/3$ on $\varphi^{-1}(B_a^N(x))$ and $\Delta u \geq 0$ on M, respectively.
- By (27) we have
$$\|u\|_{L^\infty(M)} \leq \|f\|_{L^\infty(\varphi^{-1}(B^N_{\mathrm{diam}(\Omega)}(x)))} = \|g\|_{L^\infty([0,\mathrm{diam}(\Omega)])}.$$

This proves Lemma 4.1. q.e.d.

4.1.1. Strictly m-convex domains. A strictly m-convex domain Ω with constant $c > 0$ is related to the existence of strictly m-subharmonic functions on Ω.

Definition 4.2. A C^2-function $\phi\colon \Omega \to \mathbb{R}$ is said to be **strictly m-subharmonic** with constant $c > 0$; if $\lambda_1(p) \leq \lambda_2(p) \leq \cdots \leq \lambda_n(p)$ are the ordered eigenvalues of the hessian $\overline{\nabla}\mathrm{d}\phi(p)$ then there exists an $\epsilon > 0$ such that

$$\begin{cases} \lambda_1(p) + \cdots + \lambda_m(p) & \geq & c,\ \forall p \in T^N_\epsilon(\partial\Omega) = \{y \in N\colon \mathrm{dist}_N(y,\partial\Omega) \leq \epsilon\} \\ \lambda_1(p) + \cdots + \lambda_m(p) & \geq & 0,\ \forall p \in \Omega. \end{cases}$$

Let $\Omega \subset N$ be a strictly m-convex domain of N with constant $c > 0$ and $\Gamma = \partial\Omega$ of class C^3. Let $t\colon N \to \mathbb{R}$ be the oriented distance function to Γ with orientation outward Ω. That is,

$$t(y) = \begin{cases} -\mathrm{dist}_N(y,\partial\Omega) & if \quad y \in \Omega \\ \mathrm{dist}_N(y,\partial\Omega) & if \quad y \in N \setminus \Omega. \end{cases} \tag{28}$$

The oriented distance $t(y)$ is Lipschitz in N and of class C^2 in a tubular neighborhood $T^N_{\epsilon_0}(\partial\Omega)$ for some ϵ_0. Let α_s be the shape operator of the parallel hypersurface $\Gamma_s = t^{-1}(s)$, $|s| \leq \epsilon_0$ with respect to the normal vector field $-\overline{\nabla}t$. At each point of Γ_s there is an orthonormal basis of $T\Gamma_s$ such that α_s is diagonalized:

$$\alpha_s = \mathrm{diag}\left(\xi_1^s, \xi_2^s, \ldots, \xi_{n-1}^s\right),$$

where $\xi_1^s \leq \xi_2^s \leq \ldots \leq \xi_{n-1}^s$. By the uniform continuity of each ξ_j^s and the compactness of $\overline{T^N_{\epsilon_0}(\partial\Omega)}$, for each $\delta \in (0,1)$ one can choose ϵ_0 small enough to have

$$\xi_1^s(y) + \cdots + \xi_m^s(y) \geq \delta c$$

,$\forall y \in T^N_{\epsilon_0}(\partial\Omega)$. Let ϵ_1 be a positive number so that

$$\epsilon_1 < \min\left\{1, \epsilon_0, \|\alpha_s\|^{-1}_{L^\infty(T^N_{\epsilon_0}(\partial\Omega))}\right\}. \tag{29}$$

Define $\Phi_\epsilon \colon N \to \mathbb{R}$, $0 < \epsilon < \epsilon_1/2$, by

$$\Phi_\epsilon(y) = \begin{cases} -2\epsilon & \text{if} \quad t(y) \le -2\epsilon \\ \\ 2\epsilon\left[\left(\dfrac{t(y)}{2\epsilon} + 1\right)^3 - 1\right] & \text{if} \quad t(y) \ge -2\epsilon. \end{cases} \tag{30}$$

The function Φ_ϵ is Lipschitz in N and C^2 in the tubular neighborhood $\overline{T^N_{\epsilon_0}(\Omega)} = t^{-1}\left((-\infty, \epsilon_0]\right)$.

For $t(y) \le \epsilon_0$, we can compute the gradient and the hessian of Φ_ϵ as follows.

$$\overline{\nabla}\Phi_\epsilon(y) = \begin{cases} 0 & \text{if} \quad t(y) \le -2\epsilon \\ \\ 3\left(\dfrac{t(y)}{2\epsilon} + 1\right)^2 \overline{\nabla} t(y) & \text{if} \quad -2\epsilon \le t(y) \le \epsilon_0 \end{cases}$$

$$\overline{\nabla}\mathrm{d}\Phi_\epsilon(y)(X,Y) = \begin{cases} 0 & \text{if} \qquad t(y) \le -2\epsilon \\ 3\left(\dfrac{t(y)}{2\epsilon} + 1\right)^2 \overline{\nabla}\mathrm{d}t(y)(X,Y) & \\ & \text{if} \quad -2\epsilon \le t(y) \le \epsilon_0 \\ \quad +\dfrac{3}{\epsilon}\left(\dfrac{t(y)}{2\epsilon} + 1\right) X(t)Y(t) & \end{cases}$$

Writing $\overline{\nabla}\mathrm{d}\Phi_\epsilon(y)(X,Y) = \langle S(X), Y\rangle$, for an appropriate symmetric endomorphism $S \colon TN \to TN$, we have that for $-2\epsilon \le t(y) \le 2\epsilon$, $S(y)$ can be represented by a diagonal matrix,

$$S(y) = \mathrm{diag}\left(3\left(\frac{t(y)}{2\epsilon} + 1\right)^2 \xi^t_1(y), \ldots, 3\left(\frac{t(y)}{2\epsilon} + 1\right)^2 \xi^t_{n-1}(y), \frac{3}{\epsilon}\left(\frac{t(y)}{2\epsilon} + 1\right)\right),$$

since

$$\begin{aligned} 3\left(\frac{t(y)}{2\epsilon} + 1\right)^2 \xi^t_j(y) - \frac{3}{\epsilon}\left(\frac{t(y)}{2\epsilon} + 1\right) &= 3\left(\frac{t(y)}{2\epsilon} + 1\right)\left[\left(\frac{t(y)}{2\epsilon} + 1\right)\xi^t_j(y) - \frac{1}{\epsilon}\right] \\ &\le 6\left[2\xi^t_j - \frac{1}{\epsilon_1}\right] \\ &\le 12\left(\xi^t_j(y) - 2\|\alpha_t\|_{L^\infty(T^N_{\epsilon_0}(\partial\Omega))}\right) \\ &\le 0. \end{aligned}$$

We obtain $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, $\lambda_j = 3\left(\frac{t}{2\epsilon}+1\right)\xi_j^t$, $j = 1,\dots n-1$, $\lambda_n(y) = \dfrac{3}{\epsilon}\left(\dfrac{t(y)}{2\epsilon}+1\right)$ with $S = \mathrm{diag}\,(\lambda_1, \lambda_2, \dots \lambda_n)$. By Lemma 2.3 of [**35**], we have that for any subspace $V \subset T_yN$, $y \in T^N_{2\epsilon}(\partial\Omega)$ and $1 \leq \dim V = m \leq n-1$ that

$$\begin{aligned} \mathrm{Trace}\left(\overline{\nabla}\mathrm{d}\Phi_\epsilon(y)|V\right) &\geq \lambda_1(y) + \cdots + \lambda_m(y) \\ &\geq 3\left(\frac{t(y)}{2\epsilon}+1\right)^2\left[\xi_1^{t(y)} + \cdots + \xi_m^{t(y)}\right] \\ &\geq 3\left(\frac{t(y)}{2\epsilon}+1\right)^2\delta c. \end{aligned} \tag{31}$$

Then

- if $t(y) \leq 2\epsilon$, we obtain that $\mathrm{Trace}(\overline{\nabla}\mathrm{d}\Phi_\epsilon(y)|V) \geq 0$, and
- for $|t(y)| \leq \epsilon(1-\sqrt{\delta})$, we obtain

$$\mathrm{Trace}\left(\overline{\nabla}\mathrm{d}\Phi_\epsilon(y)|V\right) \geq 3(1+\sqrt{\delta})^2\delta c/4.$$

This proves the following lemma.

Lemma 4.3. *Let Ω be a strictly m-convex domain, $1 \leq m \leq n-1$, with constant $c > 0$. There exists a Lipschitz function $\Phi_\epsilon \colon N \to \mathbb{R}$, that is, C^2 in $T_{2\epsilon}(\Omega)$, where $2\epsilon < \epsilon_1$, ϵ_1 is a positive number depending on the geometry of $\partial\Omega$ see* (29) *and such that*

1. $\Phi_\epsilon^{-1}((-\infty, 0)) = \Omega$, $\Phi_\epsilon^{-1}(0) = \partial\Omega$.
2. $|\Phi_\epsilon| \leq 2\epsilon$ *in* $\overline{\Omega}$.
3. $\mathrm{Trace}\left(\overline{\nabla}\mathrm{d}\Phi_\epsilon(y)|V\right) \geq 3(1+\sqrt{\delta})^2\delta c/4$, *for* $|t(y)| \leq \epsilon(1-\sqrt{\delta})$ *and any m-subspace $V \subset T_yN$.*
4. $\mathrm{Trace}\left(\overline{\nabla}\mathrm{d}\Phi_\epsilon(y)|V\right) \geq 0$ *in Ω for any m-subspace $V \subset T_yN$.*

In other words, Φ_ϵ is a strictly m-subharmonic function with constant $3(1+\sqrt{\delta})^2\delta c/4$.

We will need the following lemma for the proof of Theorem 2.4.

Lemma 4.4. *Let $\varphi \colon M^m \to N^n$ be an isometric immersion such that there exists a bounded, totally regular, strictly m-convex domain $\Omega \subset N$ with constant $c > 0$ and C^3-boundary $\partial\Omega$ such that $\varphi(M) \subset \Omega$, $\mathcal{H}_\Psi(\lim\varphi \cap \Omega) = 0$, and*

$$\|H\|_{L^\infty(M)} < \min\{\frac{m-1}{m \cdot \mu_b(\mathrm{diam}(\Omega))}, \frac{c}{m}\}. \tag{32}$$

Take $\delta \in (0,1)$ such that

$$\|H\|_{L^\infty}(M) < \frac{\delta^2 c}{m}$$

and let $\epsilon < \epsilon_1/2$ as above in Lemma 4.3. *Then the function $u \colon M^m \to \mathbb{R}$ given by $u = \Phi_\epsilon \circ \varphi$, where Φ_ϵ is also given in Lemma* 4.3, *satisfies*

1) $|u(x)| \leq 2\epsilon$ *for all* $x \in M$.
2) $\Delta u(x) \geq 0$ *for all* $x \in M$.
3) $\Delta u(x) \geq C_\delta$, *if* $|t(\varphi(x))| \leq \epsilon(1-\sqrt{\delta})$, *where* $C_\delta = 3c \cdot \delta \cdot (1-\delta) \cdot (1+\sqrt{\delta})^2/4$.
4) $\varphi(M) \cap \partial\Omega = \emptyset$.

Proof. Taking $u = \Phi_\epsilon \circ \varphi$ the item 1 holds by the item 2 of Lemma 4.3 and the fact that $\varphi(M) \subset \Omega$. On the other hand, we have by (31)

$$\begin{aligned} \Delta u(x) &= \mathrm{Trace}\left(\overline{\nabla} \mathrm{d}\Phi_\epsilon | T_{\varphi(x)} M\right) + < \nabla \Phi_\epsilon, mH > \\ &\geq 3\left(\frac{t(\varphi(x))}{2\epsilon} + 1\right)^2 \delta c - 3\left(\frac{t(\varphi(x))}{2\epsilon} + 1\right)^2 \delta^2 c \\ &= 3\left(\frac{t(\varphi(x))}{2\epsilon} + 1\right)^2 \delta c(1-\delta) \\ &\geq 0. \end{aligned} \tag{33}$$

This proves item 2. If $|t(\varphi(x))| \leq \epsilon(1-\sqrt{\delta})$ we get

$$\Delta u(x) \;\; \geq \;\; \frac{3}{4}(1+\sqrt{\delta})^2(1-\delta)\delta c \tag{34}$$

and that proves item 3. If there exists an $x \in \varphi^{-1}(\varphi(M) \cap \partial\Omega)$ then $\Delta u(x) > 0$ by (33). On the other hand, u has a maximum at x; therefore $\Delta u(x) \leq 0$, a contradiction. This proves item 4 and finishes the proof of Lemma 4.4. q.e.d.

5. Proof of the results

5.1. Proof of Theorem 2.4. Let $\varphi\colon M \to N$ be an isometric immersion of a Riemannian m-manifold M into a Riemannian n-manifold N with mean curvature vector H. Suppose that $\varphi(M) \subset \Omega$ for a bounded totally regular subset Ω. Let $b = \sup\{K_N(z), z \in T_{\mathrm{diam}(\Omega)}(\Omega)\}$ and $\|H\|_{L^\infty(M)} < (m-1)/m \cdot \mu_b(\mathrm{diam}(\Omega))$. First we will prove Theorem 2.4 under the assumptions of item 1. Suppose that $\mathcal{H}_\Psi(\lim \varphi) = 0$. Choose a positive number $\bar{a} > 0$ such that $(\log(\bar{a}))^2 > \log(\mathrm{diam}(\Omega))$ and if $b > 0$ take $\bar{a} \leq \min\{\pi/3\sqrt{b}, \pi/2(1+\theta)\sqrt{b}\}$, where

$$\theta = m - 1 - m\mu(\mathrm{diam}(\Omega))\|H\|_{L^\infty(M)}.$$

Observe that $\Omega \subset B^N_{\mathrm{diam}(\Omega)}(x_0)$ for $x_0 \in \Omega$. Then choose $r_1 \ll \mathrm{diam}(\Omega)$ such that the $2r_1$-tubular neighborhood $T_{2r_1}(\lim \varphi) \subset B^N_{\mathrm{diam}(\Omega)}(x_0)$. Fix $\epsilon \in (0, r_1)$. Since $\mathcal{H}_\Psi(\lim \varphi) = 0$ and in light of Remark 3.2, there is $a > 0$ and a countable covering of $\lim \varphi$ by balls $B_j = B^N_{a_j}(y_j) \subset N$ of radius $2a_j \leq a \leq \min\{r_1, \bar{a}/3\}$ such that

$$\lim \varphi \subset \bigcup_j B_j \qquad \text{and} \qquad \left|\sum_j \Psi(2a_j)\right| < \epsilon. \tag{35}$$

Since $\lim\varphi$ is compact we can extract a finite sub-covering $\{B_j\}_{j=1}^k$ of $\lim\varphi$ such that (35) holds, and each $B_j \subset T_{2r_1}(\lim\varphi)$ for all $j = 1,\ldots,k$. Applying Lemma 4.1, we construct, for every $j = 1,\ldots,k$, a function $u_j\colon M \to \mathbb{R}$ such that

$$\left\{\begin{array}{l} u_j \geq 0, \quad u_j(p) = 0 \text{ if and only if } \varphi(p) = y_j, \\ \|u_j\|_{L^\infty(M)} \leq C\Psi(2a_j) \\ \Delta u_j \geq 0 \text{ on } M, \quad \Delta u_j \geq \theta/3 \text{ on } \varphi^{-1}(B_j), \end{array}\right. \tag{36}$$

where C is positive constant depending on $m, \operatorname{diam}(\Omega), \|H\|_{L^\infty(M)}$.

Let $w_1 = \sum_{j=1}^{k_1}(2\|u_j\|_{L^\infty} - u_j) > 0$. By the boundedness of $\varphi(M)$ the set

$$K_\epsilon = M \setminus \varphi^{-1}\Big(\bigcup_{j=1}^{k_1} B_j\Big)$$

is compact in M. Now, by (13) the fundamental tone

$$\lambda^*(M \setminus K_\epsilon) \geq \inf_{M\setminus K_\epsilon}(-\frac{\Delta_M w_1}{w_1}).$$

Let $q \in M \setminus K_\epsilon$, then $\varphi(q) \in \bigcup_{j=1}^{k_1} B_j$. Let j' be such that $\varphi(q) \in B_{j'}$. Then $\Delta_M u_{j'}(q) \geq \theta/3$ and $\Delta_M u_j(q) \geq 0$ for all other $j's$. Therefore,

$$\begin{aligned} -\frac{\Delta w_1}{w_1}(q) &\geq \frac{\sum_j \Delta_M u_j(q)}{2\sum_j \|u_j\|_{L^\infty}} \\ &\geq \frac{\Delta_M u_{j'}(q)}{2C\sum_j \Psi(2a_j)} \\ &\geq \frac{\theta}{6C\epsilon}. \end{aligned} \tag{37}$$

Here $C = C(m, R_1, \|H\|_{L^\infty(M)})$. This shows that $\lambda^*(M \setminus K_\epsilon) \geq \dfrac{\theta}{6C\epsilon}$ for each $\epsilon \in (0, r_1)$. Therefore $\lambda^*(M \setminus K_\epsilon) \to +\infty$ if $\epsilon \to 0$ and proves item 1.

To prove item 2 we recall that we have an isometric immersion $\varphi\colon M^m \to N^n$ of a Riemannian manifold M into a Riemannian manifold N with mean curvature vector H such that $\varphi(M) \subset \Omega$, a totally regular, strictly m-convex domain with constant $c > 0$ and C^3-boundary $\partial\Omega$ and Ψ-Hausdorff measure $\mathcal{H}_\Psi(\lim\varphi \cap \Omega) = 0$. The mean curvature vector is assumed to satisfy $\|H\|_{L^\infty(M)} < \min\{(m-1)/m \cdot \mu_b(\operatorname{diam}(\Omega)),\, c/m\}$. We may assume that $\lim\varphi \cap \partial\Omega \neq \emptyset$; otherwise we can apply item 1. By Lemma 4.4, there exist positive numbers $\delta = \delta(\varphi)$, $C_\delta > 0$, and $\epsilon_1 = \epsilon_1(\Omega)$ such that for any $\epsilon < \epsilon_1/2$, there exists a C^2 function $u\colon M \to \mathbb{R}$, such that

1. $u^{-1}(-\infty, 0)) = M$.

2. $|u(x)| \leq 2\epsilon$ in M.
3. $\Delta u(x) \geq 0$ for all $x \in M$.
4. $\Delta u(x) \geq C_\delta$, if $\varphi(x) \in T_{\epsilon(1-\sqrt{\delta})}(\partial\Omega)$.

Fix one ϵ, $0 < \epsilon < \epsilon_1/2$ and set $K = \lim \varphi \setminus T_{\epsilon(1-\sqrt{\delta})}(\partial\Omega)$. We have $K \subset \lim \varphi \cap \Omega$ compact $\mathcal{H}_\Psi(K) = 0$. By the first part of this proof we have finite functions $u_j \colon M \to \mathbb{R}$ and balls $B_j \subset \Omega$ (covering K) such that (35) and (36) hold. Take $w_1 = \sum_{j=1}^{k_1}(2\|u_j\|_{L^\infty} - u_j) > 0$ (related to K) and $u \colon M \to \mathbb{R}$ given by Lemma 4.4. Define $\omega : M \to \mathbb{R}$ by

$$\omega(x) = \omega_1(x) + \epsilon - u(x), \qquad x \in M$$

and

$$K_\epsilon = M \setminus \varphi^{-1}\left((\cup_{j=1}^{k} B_j) \cup T_{\epsilon(1-\sqrt{\delta})}(\partial\Omega) \right).$$

The set K_ϵ is compact and for $x \in M \setminus K_\epsilon$ we get

$$-\Delta\omega \geq c_0 = \min\{\frac{\theta}{3}, C_\delta\} > 0.$$

Since $0 < \omega(x) < (2C+3)\epsilon$, $x \in M$, we get

$$-\frac{\Delta\omega}{\omega} \geq \frac{c_0}{(2C+3)\epsilon}.$$

Then $\lambda^*(M \setminus K_\epsilon) \to \infty$ if $\epsilon \to 0$, which proves item 2.

5.2. Proof of Theorem 2.9. In this section we show that the ball property, introduced in Definition 2.8, implies the existence of elements in the essential spectrum of $-\Delta$. Let M be a Riemannian manifold with the ball property, that is, there exists $R > 0$ and a collection of disjoint balls $\{B_R^M(x_j)\}_{j=1}^\infty$ such that for some constants $C > 0$ and $\delta \in (0,1)$ the inequalities

$$\mathrm{vol}(B_{\delta R}^M(x_j)) \geq C^{-1}\mathrm{vol}(B_R^M(x_j)), \ \ j = 1, 2, \ldots$$

hold. For each j, define the compactly supported, Lipschitz function $\phi_j(x) = \zeta(\rho_j(x))$, where $\rho_j(x) = \mathrm{dist}(x, x_j)$ and

$$\zeta(t) = \begin{cases} 1 & \text{if} \quad t \leq \delta R \\ \dfrac{R-t}{R(1-\delta)} & \text{if} \quad t \in [\delta R, R] \\ 0 & \text{if} \quad t \geq R. \end{cases} \tag{38}$$

Observe that $|\zeta'| \le \dfrac{1}{R(1-\delta)}\cdot$ By the ball property (6),

$$
\begin{aligned}
I_\lambda(\phi_j,\phi_j) &= \int_{B_R^M(x_j)} |\nabla\phi_j|^2 - \lambda \int_{B_R^M(x_j)} \phi_j^2 \\
&\le \frac{\mathrm{vol}(B_R^M(x_j))}{R^2(1-\delta)^2} - \lambda \mathrm{vol}\big(B_{\delta R}^M(x_j)\big) \\
&\le \mathrm{vol}(B_R^M(x_j)) \left(\frac{1}{R^2(1-\delta)^2} - \lambda C^{-1}\right) \\
&< 0,
\end{aligned} \tag{39}
$$

provided that $\lambda > C/(R^2(1-\delta)^2)$.

Since $\{\phi_j\}$ span an infinite-dimensional subspace of the domain of $-\Delta$, the Friedrichs extension of the operator $-(\Delta+\lambda)$ has infinite index, or equivalently, $-\Delta$ has infinite eigenvalues below λ, for each $\lambda > C/(R(1-\delta))^2$. By the Min-Max Theorem (see [**21**, Prop. 2.1 & 2.2], [**52**, Section 3] or [**55**]), the inequality $\inf \sigma_{\mathrm{ess}}(-\Delta) \le C/(R(1-\delta))^2$ follows.

Remark 5.1. In virtue of the well-known Bishop-Gromov volume comparison theorem, [**12**], [**31**], [**51**], all Riemannian n-manifolds M with Ricci curvature bounded below $Ric_M \ge -(n-1)k^2$ have the ball property. In fact, if we denote by $\mathrm{vol}_k(r)$ the volume of a geodesic ball of radius r in the hyperbolic space $\mathbb{H}^n(-k^2)$ of constant sectional curvature $-k^2$, by the Bishop-Gromov volume comparison theorem, the ratio $\mathrm{vol}(B_r(x_j))/\mathrm{vol}_k(r)$ is non-increasing on $[0,R]$. Hence, for each $\delta > 0$,

$$
\mathrm{vol}\big(B_{\delta R}^M(x_j)\big) \ge \frac{\mathrm{vol}_k(\delta R)}{\mathrm{vol}_k(R)} \mathrm{vol}\big(B_R^M(x_j)\big) = C(\delta,R)^{-1}\mathrm{vol}\big(B_R^M(x_j)\big).
$$

5.2.1. Application of the ball property. Now we will show that, for a suitable choice of their parameters, the Jorge-Xavier and Rosenberg-Toubiana complete minimal surfaces immersed into slabs of $\mathbb{R}^3$ have the ball property. Denoting by $\varphi\colon \mathbb{D} \to \{(x_1,x_2,x_3)\colon |x_3| < 1\}$ and $\varphi\colon \mathbb{A}(1/c/c) \to \{(x_1,x_2,x_3\colon 1/c < x_3 < c\}$ with parameters $\{(r_n,c_n)\}$, $\{(r_n,s_n,c_n)\}$ respectively, the examples of Jorge-Xavier and Rosenberg-Toubiana, we shall show that with the choice $c_n = -\log(r_n^2)$, we have that $0 = \inf \sigma_{\mathrm{ess}}(-\Delta)$ in both surfaces. The induced metric $\mathrm{d}s^2$ in Jorge-Xavier minimal immersion is conformal to the Euclidean metric $|\mathrm{d}z|^2$. More precisely, $\mathrm{d}s^2 = \lambda^2|\mathrm{d}z|^2$, where

$$
\lambda = \frac{1}{2}\left(|e^h| + |e^{-h}|\right).
$$

At points of K_n,

$$
e^{1+c_n} \ge \lambda \ge \frac{1}{2}e^{c_n-1}
$$

and thus,

$$e^{2+2c_n}|\mathrm{d}z|^2 \geq \mathrm{d}s^2 = \lambda^2|\mathrm{d}z|^2 \geq \frac{1}{4}e^{2c_n-2}|\mathrm{d}z|^2.$$

Choosing $c_n = -\log(r_n^2)$ and letting I_n be the segment of the real axis that crosses K_n, one has that the length $\ell(I_n)$ of this segment in the metric $\mathrm{d}s^2$ has the following lower and upper bound:

$$\frac{e^2}{r_n^4} \geq \ell(I_n) \geq r_n e^{c_n-1} \geq \frac{e^{-1}}{r_n}.$$

Let p_n be the center of the I_n and denote by $B_R^{\mathrm{d}s^2}(p_n)$ and $B_R^{|\mathrm{d}z|^2}(p_n)$ the geodesic balls of radius R and center p_n with respect to the metric $\mathrm{d}s^2$ and the metric $|\mathrm{d}z|^2$ respectively. Giving $R>0$, there exists $n_{_R}$ such that for all $n \geq n_{_R}$ the geodesic ball $B_R^{\mathrm{d}s^2}(p_n) \subset K_n$ for all $n \geq n_R$. Indeed, since $r_n \to 0$ as $n \to \infty$, just choose $n_{_R}$ such that $r_{n_{_R}} \leq \frac{e^{-1}}{3R}$. Moreover, these inclusions hold:

$$B^{|\mathrm{d}z|^2}_{R/(e^{1+c_n})}(p_n) \subset B_R^{\mathrm{d}s^2}(p_n) \subset B^{|\mathrm{d}z|^2}_{2R/(e^{c_n-1})}(p_n).$$

Therefore, for $\delta \in (0,1)$, we have

$$\begin{aligned} \mathrm{vol}_{\mathrm{d}s^2}(B_{\delta R}^{\mathrm{d}s^2}(p_n)) &\geq \mathrm{vol}_{\mathrm{d}s^2}(B^{|\mathrm{d}z|^2}_{\delta R/(e^{1+c_n})}(p_n)) \\ &\geq \frac{1}{4}e^{2c_n-2}\mathrm{vol}_{|\mathrm{d}z|^2}(B^{|\mathrm{d}z|^2}_{\delta R/(e^{1+c_n})}(p_n)) \\ &= \frac{1}{4e^4}\mathrm{vol}_{|\mathrm{d}z|^2}(B_{\delta R}^{|\mathrm{d}z|^2}(p_n)) \end{aligned} \tag{40}$$

$$\begin{aligned} \mathrm{vol}_{\mathrm{d}s^2}(B_R^{\mathrm{d}s^2}(p_n)) &\leq \mathrm{vol}_{\mathrm{d}s^2}(B^{|\mathrm{d}z|^2}_{2R/(e^{c_n-1})}(p_n)) \\ &\leq e^{2c_n+2}\mathrm{vol}_{|\mathrm{d}z|^2}(B^{|\mathrm{d}z|^2}_{2R/(e^{c_n-1})}(p_n)) \\ &= 4e^4\mathrm{vol}_{|\mathrm{d}z|^2}(B_R^{|\mathrm{d}z|^2}(p_n)). \end{aligned} \tag{41}$$

From (40) and (41) we have

$$\mathrm{vol}_{\mathrm{d}s^2}(B_{\delta R}^{\mathrm{d}s^2}(p_n)) \geq \frac{\delta^2}{e^{10}} \cdot \mathrm{vol}_{\mathrm{d}s^2}(B_R^{\mathrm{d}s^2}(p_n)). \tag{42}$$

This shows that Jorge-Xavier minimal surfaces with those choices of c_n above have the ball property (along the sequence p_n, for $n \geq n_R$), with parameters R, δ and $C = e^{10}/\delta^2$. By Theorem 2.9,

$$\inf \sigma_{ess}(-\Delta) \leq \frac{C}{R^2(1-\delta)^2}.$$

Letting $R \to \infty$, we conclude that $0 \in \sigma_{ess}(-\Delta)$.

Likewise, the induced metric in the Rosenberg-Toubiana's complete minimal annulus properly immersed into a slab of $\mathbb{R}^3$ is $\lambda^2|\mathrm{d}z|^2$, where $\lambda = \frac{1}{2|z|}\left(\frac{1}{|g(z)|} + |g(z)|\right)$. On K_n we have

$$e^{2c_n} \geq \left(1 + \frac{e^{2c_n}}{2}\right) \geq \lambda \geq \frac{1}{2|c|}\left(e^{2c_n} - 1\right).$$

Letting I_n be the segment of the real axis crossing K_n, and p_n the middle point of I_n, we have that the geodesic ball (in the metric $\mathrm{d}s^2$) with radius $R > 0$ and center p_n is contained in K_n, for sufficiently large n,

$$B_R^{\mathrm{d}s^2}(p_n) \subset K_n$$

Moreover,

$$B^{|\mathrm{d}z|^2}_{\frac{1}{e^{2c_n}}R}(p_n) \subset B_R^{\mathrm{d}s^2}(p_n) \subset B^{|\mathrm{d}z|^2}_{\frac{2|c|}{e^{2c_n}-1}R}(p_n).$$

Thus

$$\mathrm{vol}_{\mathrm{d}s^2}(B_{\delta R}^{\mathrm{d}s^2}(p_n)) \geq \mathrm{vol}_{\mathrm{d}s^2}(B^{|\mathrm{d}z|^2}_{\frac{1}{e^{2c_n}}\delta R}(p_n)) \geq \frac{(e^{2c_n}-1)^2}{4|c|^2 e^{4c_n}}\mathrm{vol}_{|\mathrm{d}z|^2}(B^{|\mathrm{d}z|^2}_{\delta R}(p_n))$$

and

$$\mathrm{vol}_{\mathrm{d}s^2}(B_{R}^{\mathrm{d}s^2}(p_n)) \leq \mathrm{vol}_{\mathrm{d}s^2}(B^{|\mathrm{d}z|^2}_{\frac{2|c|}{e^{2c_n}-1}R}(p_n)) \leq \frac{4|c|^2 e^{4c_n}}{(e^{2c_n}-1)^2}\mathrm{vol}_{|\mathrm{d}z|^2}(B^{|\mathrm{d}z|^2}_{R}(p_n)).$$

Therefore, for n so that $1 - r_n \geq 2/3$, we have

$$\mathrm{vol}_{\mathrm{d}s^2}(B_{\delta R}^{\mathrm{d}s^2}(p_n)) \geq \frac{\delta^2}{81|c|^4}\mathrm{vol}_{\mathrm{d}s^2}(B_{R}^{\mathrm{d}s^2}(p_n)).$$

This shows that Rosenberg-Toubiana minimal surfaces with those choices of c_n have the ball property (along the sequence p_n), with parameters R, δ and $C = 81|c|^4/\delta^2$. By Theorem 2.9,

$$\inf \sigma_{ess}(-\Delta) \leq \frac{C}{R^2(1-\delta)^2}.$$

Again, letting $R \to \infty$, we conclude that $0 \in \sigma_{ess}(-\Delta)$. This finishes the proof.

We conclude this section by calling attention to an example of a bounded minimal surface $\varphi\colon M \to \mathbb{R}^3$ with $\dim_{\mathcal{H}}(\varphi(M)) = 3$ which is not a covering, and $\sigma_{\mathrm{ess}}(-\Delta) \neq \emptyset$. In [**4**] P. Andrade constructed a complete minimal immersion $\varphi\colon \mathbb{C} \to \mathbb{R}^3$ with bounded curvature and with the property that $\overline{\varphi(\mathbb{C})}$ was an unbounded subset of the Euclidean space $\mathbb{R}^3$ with $\mathrm{vol}_3(\overline{\varphi(\mathbb{C})}) = \infty$; see also [**57**]. In other words, he constructed a dense complete minimal surface with bounded curvature and thus with the ball property. However, the restriction of the parametrization of Andrade's surface to a strip $U = \{u + iv \in \mathbb{C} : |u| < 1\}$ yields a bounded, simply connected minimal immersion with the ball property and dense in a bounded subset of $\mathbb{R}^3$. To give more details, we will keep Andrade's notation; thus, here and only here, H will be a holomorphic function.

Example 5.2. Choose $r_1, r_2 > 0$ such that r_1/r_2 is irrational and $r_1/r_2 < 1$, and set $\mathrm{d} = r_2 - r_1$. Define the map $\chi\colon \mathbb{C} \to \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, $\chi(z) = (L(z) - \overline{H(z)}, h(z))$, for the following choice of holomorphic functions L, H and harmonic function h:

$$L(z) = (r_1 - r_2)e^z, \quad H(z) = -\mathrm{d}e^{\left(\frac{r_1}{r_2}-1\right)z}, \text{ and}$$

$$h(z) = 4\left(\tfrac{\mathrm{d}}{r_2}\right)^{1/2}\left|\tfrac{r_2}{r_1}\right| |r_2 - r_1| \Re\left(ie^{\frac{r_1}{2r_2}z}\right),$$

where $\Re$ means the real part. Then a straightforward computation gives that

$$|L'(z)| + |H'(z)| > 0, \qquad L'H' = \left(\frac{\partial h}{\partial z}\right)^2 \qquad \text{on } \mathbb{C},$$

the necessary and sufficient conditions on χ to be a conformal minimal immersion of $\mathbb{C}$ in $\mathbb{R}^3$. Restricting χ to the region $U = \{u + iv\colon |u| < 1\}$, we get a bounded, simply connected minimal immersion $\varphi = \chi_{|U}$. For each fixed $u \in (-1, 1)$, $\varphi(u + iv)$ is a dense immersed trochoid in the cylinder $\Gamma_u = \left[B_{s_1(u)} \backslash B_{s_2(u)}\right] \times (-l(u), l(u))$, where s_1, s_2, l are explicit functions of u depending on r_1 and r_2. Therefore, $\lim \varphi$ is dense in the open subset $\bigcup_{u \in (-1,1)} \Gamma_u$ of $\mathbb{R}^3$, which gives $\dim_{\mathcal{H}}(\lim \varphi) = 3$. Moreover, the induced metric $\mathrm{d}s^2$ satisfies

$$\begin{aligned} \mathrm{d}s^2 &= (|L'| + |H'|)^2\, |\mathrm{d}z|^2 \\ &= \left(|r_2 - r_1|e^u + de^{\left(\frac{r_1}{r_2}-1\right)u}\right)^2 |\mathrm{d}z|^2 \\ &\geq 4(r_2 - r_1)^2 |\mathrm{d}z|^2. \end{aligned} \tag{43}$$

Considering $z_k = 2ik \in U$, each of the unit balls $B_1^{|\mathrm{d}z|^2}(z_k) \subseteq U$ in the metric $|\mathrm{d}z|^2$ contains a ball $B_R(z_k)$ in the metric $\mathrm{d}s^2$ of radius at least $R = 2|r_2 - r_1|$. Since the sectional curvature of χ satisfies

$$K = -c_1\left(e^{\left(1 - \frac{r_1}{4r_2}\right)u} + c_2 e^{\left(\frac{3r_1}{4r_2} - 1\right)u}\right)^{-4},$$

for some positive constants c_1, c_2, and $1 - \frac{r_1}{4r_2}$ and $\frac{3r_1}{4r_2} - 1$ have opposite signs, then χ has globally bounded curvature. In particular, $\{B_R(z_k)\}$ is a collection of disjoint balls in $(U, \mathrm{d}s^2)$ with uniformly bounded sectional curvature; therefore, $\sigma_{\mathrm{ess}}(-\Delta) \neq \emptyset$ on $(U, \mathrm{d}s^2)$, by Theorem 2.9 and Remark 5.1.

5.3. Proof of Theorem 2.10. Let us consider a non-proper isometric immersion $\varphi\colon M \to N$ with locally bounded geometry of a complete Riemannian manifold into a complete Riemannian manifold N. We are going to show that there exists a sequence $\{x_j\} \subset M$, a radius R, a constant $C > 0$ and $\delta \in (0,1)$ such that

$$\mathrm{vol}_M(B^M_{\delta R}(x_j)) \geq C^{-1}\mathrm{vol}_M(B^M_R(x_j)).$$

In other words, M has the ball property. Let $y_0 \in \lim \varphi$ and let $W \subset N$ be a compact subset with $y_0 \in \mathrm{int}(W)$. Let $\Lambda_0 = \Lambda_0(W)$ be such that $\|\alpha_\varphi\|_{L^\infty(\varphi^{-1}(W))} \leq \Lambda_0$. The Gauss equation and the upper bound $\sup_W |K_N| < \infty$ of the sectional curvatures of N on W give a positive number $b_0 > 0$ such that

$$\sup_{x\in\varphi^{-1}(W)} |K_M(x)| \leq 2\Lambda_0^2 + \sup_W |K_N| \leq b_0$$

where K_M are the sectional curvatures of M. Therefore, each connected component $U \subset \varphi^{-1}(W)$ has uniformly bounded sectional curvatures $|K_U| \leq b_0$. Set

$$2r_0 = \min\{i_W, (2\Lambda_0)^{-1}, b_0^{-1/2}\cdot\cot^{-1}(1/(2\sqrt{b_0})), \mathrm{dist}_N(y_0, N\backslash W)\} \tag{44}$$

where $i_W = \inf\{\mathrm{inj}_N(x), x \in W\}$. Let $B_0 = \overline{B^N_{r_0}}(y_0)$ be the closure of the geodesic ball of N with radius r_0 and center y_0. There exists a sequence of points $q_j \in M$, $q_j \to \infty$ in M such that $\varphi(q_j) \to y_0$ in N. Passing to a subsequence if necessary, we may assume that $q_j \in B_0$ and $q_j \neq q_{j'}$ if $j \neq j'$. Define $\rho_{y_0}\colon N \to \mathbb{R}$ by $\rho_{y_0}(z) = \mathrm{dist}_N(y_0, z)^2/2,\ z \in N$. Since $r_0 < \mathrm{inj}_N(y_0)$, the function $z \to \rho_{y_0}(z) \in C^2$ if $\rho_{y_0}(z) \leq r_0$. If we let $\mathrm{d}_{b_0}(x) = \mathrm{dist}_{\mathbb{N}(b_0)}(0, x)$ be the distance to an origin 0 in a simply connected n-space form $\mathbb{N}^n(b_0)$ of constant sectional curvature b_0, then by the hessian comparison theorem we obtain

$$\begin{aligned}\mathrm{Hess}\rho_{y_0}(z)(Y,Y) &\geq \mathrm{Hess}\frac{1}{2}\mathrm{d}_{b_0}(p_0,p)^2(Y',Y') \\ &\geq \sqrt{b_0}\cot(\sqrt{b_0}\,r_0)|Y'|^2 \\ &\geq \frac{1}{2}|Y|^2,\end{aligned} \tag{45}$$

where $\mathrm{d}_N(y_0, z) = \mathrm{d}_{b_0}(p_0, p) \leq r_0$, $|Y| = |Y'|$, $Y \perp \nabla\rho_y$ and $Y' \perp \nabla d_{b_0}$. We need part of the following result that might have interest on its own.

Lemma 5.3. *Let $r \leq r_0/8$. Then*

i. *For each $x \in \varphi^{-1}(B_0)$ we have $\mathrm{inj}_M(x) > r_0$.*

ii. *Let U_j be a connected component of $\varphi^{-1}(\overline{B^N_{4r}(y_0)})$ containing q_j, then*

$$\mathrm{dist}_N(\varphi(z_1), \varphi(z_2)) \leq \mathrm{dist}_M(z_1, z_2) \leq 2\mathrm{dist}_N(\varphi(z_1), \varphi(z_2)),\ \forall\, z_1, z_2 \in U.$$

Thus the map $\varphi_{|U_j}\colon U_j \to N$ is an embedding.

iii. *Take $x_j \in U_j$ such that $\mathrm{dist}_N(y_0, \varphi(x_j)) = \mathrm{dist}_N(y_0, \varphi(U_j))$. If j is large enough, then $B^M_{3r}(x_j) \subset U_j \subset B^M_{10r}(x_j)$.*

Proof. Let $x \in \varphi^{-1}(B_0)$. Suppose that $\mathrm{dist}_M(x, \mathrm{cut}_M(x)) < r_0$. Let $z \in \mathrm{cut}_M(x)$ such that $\mathrm{dist}_M(x,z) = \mathrm{dist}_M(x, \mathrm{cut}_M(x))$. By (44), z is not conjugated to x; thus, there are two distinct minimal geodesics γ_1 and γ_2 joining x to z, making a geodesic loop $\gamma = \gamma_1 \cup \gamma_2$ based at x [**14**, Lemma 5.6]. Since $r_0 > \mathrm{dist}_M(x,z) \geq \mathrm{dist}_N(\varphi(x), \varphi(z))$, the closed curve $\varphi(\gamma)$ is the region in N where ρ_{y_0} is C^2. The function $h(s) = \rho_{y_0}(\varphi(\gamma(t)))$ has a maximum at $s = \mathrm{inj}_M(x)$; however,

$$\begin{aligned} h''(s) &= \nabla \mathrm{d}\rho_{y_0}(d\varphi\gamma', \mathrm{d}\varphi\gamma') + \langle \nabla\rho_{y_0}, \alpha(\gamma',\gamma')\rangle \\ &\geq 1/2 - r_0\Lambda_0 \\ &\geq 1/4, \qquad 0 \leq s \leq 2\mathrm{inj}_M(x). \end{aligned} \tag{46}$$

This contradiction proves item (i). To prove (ii), let $U_j \subset \varphi^{-1}(\overline{B^N_{4r}(y_0)})$ be a connected component containing q_j. Let $z_1, z_2 \in U_j$ and $y_1 = \varphi(z_1)$ and $y_2 = \varphi(z_2)$. Let $\gamma(s)$, $s \in [0, \mathrm{dist}_M(z_1,z_2)]$ be a minimal geodesic in M joining z_1 to z_2. We may assume without loss of generality that $\mathrm{dist}_N(y_0,y_1) \leq \mathrm{dist}_N(y_0,y_2)$. Observe that $\rho_{y_0}(\varphi(\gamma(s))) \leq \rho_{y_0}(y_2)$ for all s. Otherwise, $s \mapsto \rho_{y_0}(\varphi(\gamma(s)))$ has a maximum at some interior point $s_0 \in (0, \mathrm{dist}_M(z_1,z_2))$ and $\mathrm{dist}_N(y_0, \varphi_\iota(\gamma(s_0))) < r_0$. Taking the second derivative at this point of maximum, we get a contradiction, as above, and that proves our assertion. Moreover, $s \mapsto \rho_{y_1}(\varphi(\gamma(s)))$ is of class at least C^2. It is clear that $(\rho_{y_1}(\varphi_\iota(\gamma(s))))'' \geq 1/4$ for all $s \in [0, t = \mathrm{dist}_M(z_1,z_2)]$. Then

$$\begin{aligned} \frac{\mathrm{dist}^2_N(y_1,y_2)}{2} &= \rho_{y_1}(\varphi(\gamma(t))) \\ &= \rho_{y_1}(\varphi(\gamma(0))) + t\rho_{y_1}(\varphi(\gamma(s)))'|_{s=0} \\ &\quad + \int_0^1 (1-s)\left(\rho_{y_1}(\varphi(\gamma(st)))\right)'' \mathrm{d}s \\ &\geq \frac{t^2}{4}\int_0^1 (1-s)\mathrm{d}s \\ &= \frac{t^2}{8}. \end{aligned}$$

It follows that $\mathrm{dist}_M(z_1,z_2) \leq 2\mathrm{dist}_N(\varphi(z_1), \varphi(z_2))$. To prove item (iii), pick $x_j \in U_j$ such that $\mathrm{dist}_N(y_0, \varphi(x_j)) = \mathrm{dist}_N(y_0, \varphi(U_j))$. We may choose j large enough so that $\mathrm{dist}_N(y_0, \varphi(x_j)) < r$. Let $x \in B^M_{3r}(x_j)$. Then

$$\begin{aligned} \mathrm{dist}_N(\varphi(x), y_0) &\leq \mathrm{dist}_N(\varphi(x), \varphi(x_j)) + \mathrm{dist}_N(\varphi(x_j), y) \\ &< \mathrm{dist}_M(x, x_j) + r \\ &\leq 4r. \end{aligned}$$

Now let $x \in U$; then

$$\begin{aligned}\mathrm{dist}_M(x_j, x) &\leq 2\mathrm{dist}_N(\varphi(x_j), \varphi(x)) \\ &\leq 2\left[\mathrm{dist}_N(\varphi(x_j), y_0) + \mathrm{dist}_N(y_0, \varphi(x))\right] \\ &< 10r.\end{aligned}$$

By Lemma 5.3, there exists a sequence $x_j \in M$ such that

$$B^M_{3r}(x_j) \subset U_j \subset B^M_{10r}(x_j), \ \ \forall j.$$

Observe that $\mathrm{dist}_N(q_j, y_0) \geq \mathrm{dist}_N(\varphi(x_j), y_0) \to 0$ as $j \to \infty$ and then $y_0 \in \lim \varphi$. Therefore, passing to a subsequence, we have that $x_j \neq x_{j+k}$ for all $k \geq 1$. Recall that the sectional curvatures of U_j are bounded below $K_{U_j} \geq -b_0$. Let $\mathbb{N}^m(-b_0)$ be the simply connected space form of constant sectional curvature $-b_0$. Choose any $\delta \in (0,1)$. By the Bishop-Gromov volume comparison theorem, we have

$$\begin{aligned}\mathrm{vol}\big(B^M_{\delta 3r}(x_j)\big) &\geq \frac{\mathrm{vol}\big(B^{\mathbb{N}^m(-b_0)}_{\delta 3r}\big)}{\mathrm{vol}\big(B^{\mathbb{N}^m(-b_0)}_{3r}\big)}\mathrm{vol}\big(B^M_{3r}(x_j)\big) \\ &= C(b_0, m, \delta, 3r)^{-1}\mathrm{vol}\big(B^M_{3r}(x_j)\big).\end{aligned}$$

This shows that M has the ball property with respect to the parameters $\{x_j\}$, $R = 3r$, $C^{-1} = \mathrm{vol}\big(B^{\mathbb{N}^m(-b_0)}_{\delta 3r}\big)/\mathrm{vol}\big(B^{\mathbb{N}^m(-b_0)}_{3r}\big)$ and any $\delta \in (0,1)$. Since $3r \in (0, 3r_0/8)$ and $\delta \in (0,1)$ we have by Theorem 2.9 (taking $\delta = 1/2$) that

$$\inf \sigma_{ess}(-\Delta) \leq \frac{256}{9r_0^2} \cdot \frac{\mathrm{vol}\big(B^{\mathbb{N}^m(-b_0)}_{3r}\big)}{\mathrm{vol}\big(B^{\mathbb{N}^m(-b_0)}_{\delta 3r/2}\big)}.$$

q.e.d.

5.4. Proof of Theorem 2.11. To prove Theorem 2.11, we will apply the following proposition derived from the Spectral Theorem; see details in [**21**, Prop. 2], [**30**, pp. 13–15].

Proposition 5.4. *Let M be a Riemannian manifold. A necessary and sufficient condition for $(\eta - \epsilon, \eta + \epsilon) \cap \sigma_{\mathrm{ess}}(-\Delta) \neq \emptyset$ is that there exists an infinite dimensional subspace G_ϵ of the domain $D(-\Delta)$ of $-\Delta$, for which $\|(\Delta + \eta I)\psi\|_{L^2(M)} < \epsilon\|\psi\|_{L^2(M)}$, $\psi \in G_\epsilon$.*

To show that $\eta \geq 0$ belongs to $\sigma_{\mathrm{ess}}(-\Delta)$, we need to take a sequence $\upsilon_n \to 0$ as $n \to \infty$ and a sequence of functions $\psi_n \in C^\infty_0(M)$ satisfying $\|(\Delta+\eta I)\psi_n\|_{L^2(M)} < \upsilon_n\|\psi_n\|_{L^2(M)}$ with $\mathrm{supp}\,\psi_n \cap \mathrm{supp}\,\psi_{n'} = \emptyset$ if $n \neq n'$.

Consider a sequence of compact subsets $K_n \subset D_n$ with Euclidean width $r_n \to 0$ as $n \to \infty$ and the set of constants c_n satisfying (8) in Jorge-Xavier's or Rosenberg-Toubiana's construction. The induced metric on the minimal surface is conformal to the Euclidean metric $|\mathrm{d}z|^2$ on the disk $\mathbb{D}$ that is, $\mathrm{d}s^2 = \lambda^2|\mathrm{d}z|^2$. Set $\lambda_n = \sup_{K_n} \lambda(z)$ and

$\zeta_n = \lambda_n/(\inf_{K_n}\lambda(z))$ so that $\lambda_n/\zeta_n \le \lambda \le \lambda_n$ in K_n. Let I_n be the segment of the real axis that crosses K_n. The length $\ell_{\mathrm{d}s^2}(I_n)$ of I_n in the metric $\mathrm{d}s^2$ has the following lower and upper bound:

$$\frac{\lambda_n r_n}{\zeta_n} \le \ell_{\mathrm{d}s^2}(I_n) \le \lambda_n r_n.$$

Let p_n be the center of the I_n and denote by $B_t^{\mathrm{d}s^2}(p_n)$ and $B_t^{|\mathrm{d}z|^2}(p_n)$ the geodesic balls of radius t and center p_n with respect to the metrics $\mathrm{d}s^2$ and $|\mathrm{d}z|^2$ respectively. Denote by $\Delta^{|\mathrm{d}z|^2}$ and by $\mathrm{d}x$, respectively, the Laplace operator and the Lebesgue measure of $\mathbb{R}^2$ with respect to the metric $|\mathrm{d}z|^2$, and denote by $\Delta^{\mathrm{d}s^2}$ and by $\lambda^2\mathrm{d}x$ the Laplace operator and the Riemannian measure on M with respect to the metric $\mathrm{d}s^2$. The Laplace operators $\Delta^{|\mathrm{d}z|^2}$ and $\Delta^{\mathrm{d}s^2}$ are related, on $\mathbb{D}$, by $\Delta^{\mathrm{d}s^2} = \frac{1}{\lambda^2}\Delta^{|\mathrm{d}z|^2}$. Given $\eta > 0$, let $f \in C_0^\infty(B_{r_n}^{|\mathrm{d}z|^2}(p_n))$ be a smooth function with compact support in $B_{r_n}^{|\mathrm{d}z|^2}(p_n) \subset K_n$ to be chosen later. We have that

$$\begin{aligned}
\|\Delta^{\mathrm{d}s^2} f + \eta f\|^2_{L^2(M)} &= \int_{B_{r_n}^{|\mathrm{d}z|^2}(p_n)} \left(\frac{1}{\lambda^2}\Delta^{|\mathrm{d}z|^2} f + \eta f\right)^2 \lambda^2 \mathrm{d}x \\
&= \int_{B_{r_n}^{|\mathrm{d}z|^2}(p_n)} \frac{1}{\lambda^2}(\Delta^{|dz|^2} f)^2 \mathrm{d}x + \eta^2 \int_{B_{r_n}^{|\mathrm{d}z|^2}(p_n)} f^2\lambda^2 \mathrm{d}x \\
&\quad + 2\eta \int_{B_{r_n}^{|\mathrm{d}z|^2}(p_n)} f\Delta^{|\mathrm{d}z|^2} f \mathrm{d}x \\
&\le \int_{B_{r_n}^{|\mathrm{d}z|^2}(p_n)} \frac{\epsilon_n^2}{\lambda_n^2}\left(\Delta^{|\mathrm{d}z|^2} f\right)^2 \mathrm{d}x \\
&\quad + \eta^2\zeta_n^2 \int_{B_{r_n}^{|\mathrm{d}z|^2}(p_n)} f^2\lambda_n^2 \mathrm{d}x \qquad (47)\\
&\quad + 2\eta\zeta_n^2 \int_{B_{r_n}^{|\mathrm{d}z|^2}(p_n)} f\Delta^{|\mathrm{d}z|^2} f \mathrm{d}x \\
&\quad + 2\eta(\zeta_n^2 - 1)\int_{B_{r_n}^{|\mathrm{d}z|^2}(p_n)} |\nabla^{|\mathrm{d}z|^2} f|^2 \mathrm{d}x \\
&= \zeta_n^2 \int_{B_{r_n}^{|\mathrm{d}z|^2}(p_n)} \left(\frac{1}{\lambda_n^2}\Delta^{|\mathrm{d}z|^2} f + \eta f\right)^2 \lambda_n^2 \mathrm{d}x \\
&\quad + 2\eta(\zeta_n^2 - 1)\int_{B_{r_n}^{|\mathrm{d}z|^2}(p_n)} |\nabla^{|\mathrm{d}z|^2} f|^2 \mathrm{d}x.
\end{aligned}$$

Let us consider the ball $B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n) = p_n + B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(0) \subset \mathbb{R}^2$ of radius $\lambda_n r_n$ and center p_n and the map $\xi\colon B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n) \to B_{r_n}^{|\mathrm{d}z|^2}(p_n)$ given by $\xi(p_n + x) = p_n + x/\lambda_n$, and define $h\colon B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n) \to \mathbb{R}$ by $h = f\circ\xi$. We have that $\Delta^{|\mathrm{d}z|^2} h = \Delta^{|\mathrm{d}z|^2} f(\xi)/\lambda_n^2$ and the Jacobian $J(\xi)(x) = 1/\lambda_n^2$. Making the change of variables $x = \xi(y)$, we have that

- $\int_{B_{r_n}^{|\mathrm{d}z|^2}(p_n)}\Big(\frac{1}{\lambda_n^2}\Delta^{|\mathrm{d}z|^2}f+\eta f\Big)^2\lambda_n^2\mathrm{d}x=\int_{B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n)}\Big(\Delta^{|\mathrm{d}z|^2}h+\eta h\Big)^2\mathrm{d}x$
- $\int_{B_{r_n}^{|\mathrm{d}z|^2}(p_n)}|\nabla^{|\mathrm{d}z|^2}f|^2\mathrm{d}x=\int_{B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n)}|\nabla^{|\mathrm{d}z|^2}h|^2\mathrm{d}x$

Thus from (47) and the change of variable above, we have the following inequality:

$$
\begin{aligned}
\|\Delta^{\mathrm{d}s^2}f+\eta f\|_{L^2(M)} &\leq \zeta_n\|\Delta^{|\mathrm{d}z|^2}h+\eta h\|_{L^2(B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n))}\\
&\qquad\qquad + \sqrt{2\eta(\zeta_n^2-1)}\|\nabla^{|\mathrm{d}z|^2}h\|_{L^2(B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n))}.
\end{aligned} \tag{48}
$$

Where $f\colon B_{r_n}^{|\mathrm{d}z|^2}(p_n)\subset K_n\to\mathbb{R}$, $h=f\circ\xi\colon B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n)\to\mathbb{R}$ is defined by $h(p_n+x)=f(p_n+x/\lambda_n)$. Observe that $f=h\circ\xi^{-1}\colon B_{r_n}^{|\mathrm{d}z|^2}(p_n)\to\mathbb{R}$ so that $f(p_n+x)=h(p_n+\lambda_n x)$, $x\in B_{r_n}^{|\mathrm{d}z|^2}(0)$.

Therefore, given $h\in C_0^\infty(B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n))$, we obtain $f\in C_0^\infty(B_{r_n}^{|\mathrm{d}z|^2}(p_n))$ and vice-versa, satisfying the inequality (48).

Since $\sigma(-\Delta^{|\mathrm{d}z|^2})=\sigma_{\mathrm{ess}}(-\Delta^{|\mathrm{d}z|^2})=[0,\infty)$, given a positive number $\eta>0$, we have that $\eta\in\sigma_{\mathrm{ess}}(-\Delta^{|\mathrm{d}z|^2})$. Therefore for each $\delta>0$ there exists (by Proposition 5.4) $h\in C_0^\infty(\mathbb{R}^2)$ such that

$$
\|\Delta^{|\mathrm{d}z|^2}h+\eta h\|_{L^2(\mathbb{R}^2)}<\delta\|h\|_{L^2(\mathbb{R}^2)}. \tag{49}
$$

Suppose that $\limsup_{n\to\infty} r_n\lambda_n=\infty$. Then there exists n_0 such that for all $n\geq n_0$ the ball $B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n)$ contains the support of h since for large n we have $1\leq e_n<2$ and the length $\ell_{ds^2}(I_n)\geq\lambda_n r_n/\zeta_n\to\infty$. For this function $h\in C_0^\infty(B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n))$ we have

- $\displaystyle\int_{B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n)}|\nabla^{|\mathrm{d}z|^2}h|^2\mathrm{d}x\leq\mu_1(n)\int_{B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n)}h^2\mathrm{d}y$, where $\mu_1(n)$ is the first Dirichlet eigenvalue of the ball $B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n)$.
- Letting $f(p_n+x)=h(p_n+\lambda_n x)\in C_0^\infty(B_{r_n}^{|\mathrm{d}z|^2}(p_n))$, we have

$$
\begin{aligned}
\int_{B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n)}h^2\mathrm{d}y &= \int_{B_{r_n}^{|\mathrm{d}z|^2}(p_n)}\lambda_n^2 f^2\mathrm{d}x\\
&\leq 4\int_{B_{r_n}^{|\mathrm{d}z|^2}(p_n)}f^2\lambda^2\mathrm{d}x\\
&= 4\|f\|^2_{L^2(M)},
\end{aligned} \tag{50}
$$

  since $\lambda_n\leq 2\lambda$.
- Putting together this information, we have

$$
\int_{B_{\lambda_n r_n}^{|\mathrm{d}z|^2}(p_n)}|\nabla^{|\mathrm{d}z|^2}h|^2\mathrm{d}x\leq 4\|f\|^2_{L^2(M)}
$$

From the inequality (48) we then have

$$\|\Delta^{ds^2} f + \eta f\|_{L^2(M)} \leq \left(2\zeta_n\delta + 2\sqrt{2\eta(\zeta_n^2 - 1)\mu_1(n)}\right)\|f\|_{L^2(M)}.$$

We are ready to conclude that each $\eta > 0$ belongs to $\sigma_{\mathrm{ess}}(-\Delta^{\mathrm{d}s^2})$. Let us consider a sequence of positive numbers $\upsilon_i \to 0$. For each i, choose n such that $2\sqrt{2\eta(\epsilon_{n_i}^2 - 1)\mu_1(n_i)} < \upsilon_i/2$. This n exists since $\mu_1(n) = \lambda_1(B^{|\mathrm{d}z|^2}_{\lambda_n r_n}(p_n)) = c/(\lambda_n r_n)^2 \to 0$ and $\epsilon_n \to 1$ as $n \to \infty$. Take $\delta < \upsilon_i/4$ and choose $h_i \in C_0^\infty(\mathbb{R}^2)$ such that (48) holds, choosing n_i large enough so that $\operatorname{supp} h_i \subset B^{|\mathrm{d}z|^2}_{\lambda_{n_i} r_{n_i}}(p_n)$. Then the function f_i associated to h_i satisfies

$$\|\Delta^{ds^2} f_i + \eta f_i\|_{L^2(M)} < \upsilon_i\|f_i\|_{L^2(M)}.$$

It is clear that we can choose the family h_i with support in different balls. All this shows that $\eta \in \sigma_{\mathrm{ess}}(-\Delta^{\mathrm{d}s^2})$. To finish the proof of Theorem 2.11, we need to address the case that $\limsup r_n\lambda_n > 0$. Observe that in K_n we have that

$$\frac{\lambda_n}{\zeta_n} \leq \lambda \leq \lambda_n.$$

This implies that, in K_n,

$$\left(\frac{\lambda_n}{\zeta_n}\right)^2 |\mathrm{d}z|^2 \leq \mathrm{d}s^2 \leq \lambda_n^2|\mathrm{d}z|^2.$$

From this point on, is easy to see that $(\mathbb{D}, \mathrm{d}s^2)$ or $(\mathbb{A}(1/c, c), \mathrm{d}s^2)$ has the ball property; see details in the application of subsection 5.2.1. Thus $\sigma_{\mathrm{ess}}(\mathrm{d}s^2) \neq \emptyset$. This finishes the proof of Theorem 2.11.

5.5. Open problems.

1) We presented an example of a complete bounded surface with non-empty essential spectrum and limit set with positive 2-dimensional Hausdorff measure; see Remark 2.7. This shows that Theorem 2.4 is sharp. However, for submanifolds of dimension $m \geq 3$, it seems that requiring that the 2-dimensional Hausdorff measure of the limit set be zero is a technical requirement of our proof.

   If we consider a bounded, minimally immersed submanifold $\varphi\colon M^m \to N^n$ of dimension $m \geq 3$ of a Hadamard manifold N, that is, $\varphi(M) \subset \Omega$ for a bounded open subset $\Omega \subset N$ and if $\mathcal{H}^m(\lim \varphi \cap \Omega) = 0$, does $-\Delta$ has discrete spectrum?
2) Infinite sheeted coverings of complete bounded minimal surfaces always have non-empty essential spectrum. However, Example 5.2 establishes the existence of incomplete minimal surfaces with $\sigma_{\mathrm{ess}}(-\Delta) \neq \emptyset$ and whose immersion map φ is not a Riemannian covering. One could naturally ask: is it possible to find a complete, bounded minimal surface $\varphi\colon M \to \mathbb{R}^3$ with non-empty essential spectrum and such that φ is not a Riemannian covering map?

3) Although Theorem 2.4 can be applied for each of the examples (i), ... ,(vii), it is still unapplicable to the original example of Nadirashvili [**48**]. Is it possible to find a choice of parameters in Nadirashvili's construction such that the essential spectrum of the resulting minimal surface is not empty?
4) Jorge-Xavier's or Rosenberg-Toubiana's construction, what can be said about the essential spectrum if the choice of the parameters $\{(r_n, c_n)\}$ is such that $\limsup r_n\lambda_n = 0$?

References

[1] A. Alarcón, *Compact complete minimal immersions in* $\mathbb{R}^3$, Trans. Amer. Math. Soc. **362** (2010), 4063–4076, MR 2608395, Zbl 1196.53006.

[2] A. Alarcón, L. Ferrer & F. Martin, *Density theorems for complete minimal surfaces in* $\mathbb{R}^3$, Geom. Funct. Anal. **18** (2008), no. 1, 1–49, MR 2399094, Zbl 1144.53013.

[3] A. Alarcón & F.J. López, *Null curves in* $\mathbb{C}^3$ *and Calabi-Yau conjectures*, Math. Ann. **355**, no. 2, 429–455, MR 3010135, Zbl 1269.53061.

[4] P. Andrade, *A wild minimal plane in* $\mathbb{R}^3$, Proc. Amer. Math. Soc. **128** (2000), no. 5, 1451–1457, MR 1664289, Zbl 0958.53005.

[5] A. Baider, *Noncompact Riemannian manifolds with discrete spectra*, J. Diff. Geom. **14** (1979), 41–57, MR 0577877, Zbl 0411.58022.

[6] J. Barta, *Sur la vibration fundamentale d'une membrane*, C. R. Acad. Sci. **204** (1937), 472–473, JFM 63.0762.02.

[7] G.P. Bessa, L.P. Jorge & G. Oliveira-Filho, *Half-Space Theorems for Minimal Surfaces with Bounded Curvature*, J. Differential Geometry **57** (2001), 493–508, MR 1882666, Zbl 1041.53003.

[8] G.P. Bessa & J.F. Montenegro, *Eigenvalue estimates for submanifolds with locally bounded mean curvature*, Ann. Global Anal. and Geom. **24**, (2003), 279–290, MR 1996771, Zbl 1060.53063.

[9] G.P. Bessa & J.F. Montenegro, *An extension of Barta's Theorem and geometric applications*, Ann. Global Anal. and Geom. **31** (2007), 345–362, MR 2325220, Zbl 1116.58006.

[10] G.P. Bessa, L. Jorge & J.F. Montenegro, *The Spectrum of the Martin-Morales-Nadirashvili Minimal Surfaces is Discrete*. J. Geom. Anal. **20** (2010), 63–71, MR 2574720, Zbl 1187.58009.

[11] B. Bianchini, L. Mari & M. Rigoli, *On some aspects of Oscillation Theory and Geometry*, Mem. Amer. Math. Soc. **225**, (2013) no. 1056, vi+195 pp. ISBN: 978-0-8218-8799-8 MR 3112813.

[12] R.L. Bishop & R.J. Crittenden, *Geometry of manifolds*, Pure and Applied Mathematics, Vol. XV Academic Press, New York-London (1964), ix+273 pp. MR 0169148, Zbl 0132.16003.

[13] E. Calabi, *Problems in Differential Geometry*, (S. Kobayashi and J. Eells, Jr., eds.) Proc. of the United States-Japan Seminar in Differential Geometry, Kyoto, Japan, 1965, Nippon Hyoronsha Co. Ltd., Tokyo (1966) 170.

[14] J. Cheeger & D. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland Mathematical Library, Vol. 9. North-Holland Publishing Co.,

Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, 1975. viii+174 pp. MR 0458335, Zbl 0309.53035.

[15] S.Y. Cheng, P. Li & S.T. Yau, *Heat equations on minimal submanifolds and their applications*, Amer. J. Math. **106**, (1984), 1033–1065, MR 0761578, Zbl 0575.53038.

[16] S.S. Chern, *The geometry of G-structures*, Bull. Amer. Math. Soc. **72** (1966), 167–219, MR 0192436, Zbl 0136.17804.

[17] T.H. Colding & W.P. Minicozzi, *The Calabi-Yau conjectures for embedded surfaces*, Ann. of Math. (2) **167**, no. 1 (2008), 211–243, MR 2373154, Zbl 1142.53012.

[18] P. Collin & H. Rosenberg, *Notes sur la démonstration de N. Nadirashvili des conjectures de Hadamard et Calabi-Yau*, Bull. Sci. Math. **123**, no. 7, (1999), 563–575, MR 1713304, Zbl 0949.53009.

[19] E.B. Davies, *Spectral theory and differential operators*, Cambridge Studies in Advanced Mathematics, 42. Cambridge University Press, Cambridge, 1995, x+182 pp. ISBN: 0-521-47250-4 MR 1349825, Zbl 0893.47004.

[20] M. DoCarmo & C.K. Peng, *Stable complete minimal surfaces in $\mathbb{R}^3$ are planes*, Bull. Amer. Math. Soc. (N.S.) **1**, no. 6 (1979), 903–906, MR 0546314, Zbl 0442.53013.

[21] H. Donnelly, *On the essential spectrum of a complete Riemannian manifold*, Topology **20** (1981), no. 1, 1–14, MR 0592568, Zbl 0463.53027.

[22] H. Donnelly, *Negative curvature and embedded eigenvalues*, Math. Z. **203**, (1990), 301–308, MR 1033439, Zbl 0699.53052.

[23] H. Donnelly, N. Garofalo, *Riemannian manifolds whose Laplacian have purely continuous spectrum*, Math. Ann. **293**, (1992), 143–161, MR 1162679, Zbl 0735.58033.

[24] H. Donnelly & P. Li, *Pure point spectrum and negative curvature for noncompact manifolds*, Duke Math. J. **46** (1979), 497–503, MR 0544241, Zbl 0416.58025.

[25] J. Douglas, *Solution of the problem of Plateau*, Trans. Amer. Math. Soc. **33** (1931), 263–321, MR 1501590, Zbl 0001.14102.

[26] J. Douglas, *The least area property of the minimal surface determined by an arbitrary Jordan contour*, Proc. Natl. Acad. Sci. U.S.A. **17**, (1931), no. 4, 211–216, Zbl 0001.34001.

[27] J.F. Escobar, *On the spectrum of the Laplacian on complete Riemannian manifolds*, Comm. Partial Differ. Equations **11**, (1985), 63–85, MR 0814547, Zbl 0585.58046.

[28] L. Ferrer, F. Martin & W. Meeks III, *Existence of proper minimal surfaces of arbitrary topological type*, Adv. Math. **231**, no. 1, (2012), 378–413, MR 2935393, Zbl 1246.53006.

[29] D. Fischer-Colbrie & R. Schoen, *The structure of complete stable minimal surfaces in* 3*-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. **33** (1980), 199–211, MR 0562550, Zbl 0439.53060.

[30] I.M. Glazman, *Direct methods of qualitative spectral analysis of singular differential operators*, Translated from the Russian by the IPST staff Israel Program for Scientific Translations, Jerusalem, 1965; Daniel Davey & Co., Inc., New York 1966 ix+234 pp. MR 0190800, Zbl 0143.36505.

[31] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Based on the 1981 French original. With appendices by M. Katz, P. Pansu and S.

Semmes. Translated from the French by Sean Michael Bates. Reprint of the 2001 English edition. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2007. xx+585 pp. ISBN: 978-0-8176-4582-3; 0-8176-4582-9 MR 2307192, Zbl 1113.53001.

[32] J. Hadamard, *Les surfaces à courbures opposées et leurs linges géodésiquese*, J. Math. Pures Appl. **4** (1898), 27–73, JFM 29.0522.01. **23** (1970), 97–114.

[33] M. Harmer, *Discreteness of the spectrum of the Laplacian and Stochastic incompleteness*, J. Geom. Anal. **19** (2009), 358–372, MR 2481965, Zbl 1175.58012.

[34] K. Hoffman, *Banach spaces of analytic functions*, Reprint of the 1962 original. Dover Publications, Inc., New York, 1988. viii+216 pp. ISBN: 0-486-65785-X MR 1102893, Zbl 0734.46033.

[35] L.P. Jorge & F. Tomi, *The barrier principle for minimal submanifolds of arbitrary codimension*, Ann. Global Anal. Geom. **24** (2003), no. 3, 261–267, MR 1996769, Zbl 1027.53007.

[36] L.P. Jorge & F. Xavier, *A complete minimal surface in $\mathbb{R}^3$ between two parallel planes*, Annals of Math. (2) **112** (1980), 203–206, MR 0584079, Zbl 0455.53004.

[37] L. Karp, *Noncompact manifolds with purely continuous spectrum*, Mich. Math. J. **31**, (1984), 339–347, MR 0767613, Zbl 0578.53034.

[38] R. Kleine, *Discreteness conditions for the Laplacian on complete non-compact Riemannian manifolds*. Math. Z. **198** (1988), 127–141, MR 0938034, Zbl 0622.53024.

[39] R. Kleine, *Warped products with discrete spectra*, Results Math. **15** (1989), 81–103, MR 0979446, Zbl 0669.58030.

[40] F. Lopez, F. Martin & S. Morales, *Adding handles to Nadirashvili's surfaces*, J. Diff. Geom. **60**, no. 1, (2002), 155–175, MR 1924594, Zbl 1069.53008.

[41] F. Lopez, F. Martin & S. Morales, *Complete nonorientable minimal surfaces in a ball of $\mathbb{R}^3$*, Trans. Amer. Math. Soc. **358**, no. 9, (2006), 3807–3820, MR 2219000, Zbl 1095.53011.

[42] F. Martín, W. Meeks III, N. Nadirashvili, *Bounded domains which are universal for minimal surfaces*, Amer. J. Math. **129** (2007), no. 2, 455–461, MR 2306042, Zbl 1119.53007.

[43] F. Martín & S. Morales, *Complete proper minimal surfaces in convex bodies of $\mathbb{R}^3$*, Duke Math. J. **128** (2005), 559–593, MR 2145744, Zbl 1082.53009.

[44] F. Martín & S. Morales, *Complete proper minimal surfaces in convex bodies of $\mathbb{R}^3$. II. The behavior of the limit set*, Comment. Math. Helv. **81** (2006), 699–725, MR 2250860, Zbl 1110.53007.

[45] F. Martín & S. Morales, *Construction of a complete bounded minimal annulus in $\mathbb{R}^3$*. (Spanish) Proceedings of the Meeting of Andalusian Mathematicians, Vol. II (Spanish) (Sevilla, 2000), 649–653, Colecc. Abierta, 52, Univ. Sevilla Secr. Publ., Seville, 2001. MR 1899773.

[46] F. Martín & N. Nadirashvili, *A Jordan curve spanned by a complete minimal surface*, Arch. Ration. Mech. Anal. **184** (2007), no. 2, 285–301, MR 2299764, Zbl 1114.49039.

[47] P. Mattila, *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability*, Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, Cambridge, 1995. xii+343 pp. ISBN: 0-521-46576-1; 0-521-65595-1 MR 1333890, Zbl 0819.28004.

[48] N. Nadirashvili, *Hadamard's and Calabi-Yau's conjectures on negatively curved and minimal surfaces*, Invent. Math. **126** (1996), 457–465, MR 1419004, Zbl 0881.53053.

[49] W.F. Osgood, *A Jordan curve of positive area*, Trans. Amer. Math. Soc. **4** (1903), no. 1, 107–112, MR 1500628, JFM 34.0533.02.

[50] A. Persson, *Bounds for the discrete part of the spectrum of a semi-bounded Schrödinger operator*, Math. Scand. **8** (1960), 143–153, MR 0133586, Zbl 0145.14901.

[51] P. Petersen, *Riemannian geometry*, Second edition. Graduate Texts in Mathematics, 171. Springer, New York, 2006. xvi+401 pp. ISBN: 978-0387-29246-5; 0-387-29246-2 MR 2243772, Zbl 1220.53002.

[52] S. Pigola, M. Rigoli & A.G. Setti, *Vanishing and finiteness results in geometric analysis. A generalization of the Bochner technique*, Progress in Mathematics, 266. Birkhäuser Verlag, Basel, 2008. xiv+282 pp. ISBN: 978-3-7643-8641-2 MR 2401291, Zbl 1150.53001.

[53] A.V. Pogorelov, *On the stability of minimal surfaces*, (Russian) Dokl. Akad. Nauk SSSR **260** (1981), no. 2, 293–295. MR 0630142, Zbl 0495.53005.

[54] T. Radó, *On the Problem of Plateau* Ann. of Math. (2) **31** (1930), no. 3, 457–469.

[55] M. Reed & B. Simon, *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978. xv+396 pp. ISBN: 0-12-585004-2 MR 0493421, Zbl 0401.47001.

[56] F. Rellich, *Über das asymptotische Verhalten der Lösungen von $\Delta u + \lambda u = 0$ in unendlichen Gebieten*, (German) Jber. Deutsch. Math. Verein. 53, (1943). 57–65, MR 0017816, Zbl 0028.16401.

[57] R.M. Mesquita, *O Conjunto Limite da Superficie de P. Andrade*, Master's thesis, Universidade Federal do Ceará-UFC, (1999).

[58] A. Ros, *One-sided complete stable minimal surfaces*, J. Diff. Geom. **74** (2006), 69–92, MR 2260928, Zbl 1110.53009.

[59] H. Rosenberg, *Intersection of minimal surfaces of bounded curvature*, Bull. Sci. Math. **125** (2) (2001), 161–168, MR 1812162, Zbl 0982.53053.

[60] H. Rosenberg & E. Toubiana, *A cylindrical type complete minimal surface in the slab of* $\mathbb{R}^3$, Bull. Sci. Math. (2) **111**, no. 3, (1987), 241–245, MR 0912952, Zbl 0631.53012.

[61] R. Schoen, *Estimates for stable minimal surfaces in three dimensional manifolds*, Seminar on minimal submanifolds, 111–126, Ann. of Math. Stud. **103**, Princeton Univ. Press, Princeton, NJ, (1983), MR 0795231, Zbl 0532.53042.

[62] T. Tayoshi, *On the spectrum of the Laplace-Beltrami operator on a noncompact surface*, Proc. Japan Acad. **47**, (1971), 579–585, MR 0295151, Zbl 0225.35080.

[63] M. Tokuomaru, *Complete minimal cylinders properly immersed in the unit ball*, Kyushu J. Math. **61** (2007), no. 2, 373–394, MR 2362891, Zbl 1143.53010.

[64] S.T. Yau, *Review of geometry and analysis*, Kodaira's issue. Asian J. Math. **4** (2000), 235–278, MR 1803723, Zbl 1031.53004.

[65] S.T. Yau, *Review of geometry and analysis. Mathematics: Frontier and perspectives*, Amer. Math. Soc. Providence. RI (2000), 353–401, MR 1754787, Zbl 0969.53001.

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