LARGE TIME BEHAVIOR OF THE HEAT KERNEL

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Abstract

In this paper, we study the large time behavior of the heat kernel on complete Riemannian manifolds with nonnegative Ricci curvature, which was studied by P. Li with additional maximum volume growth assumption. Following Y. Ding's original strategy, by blowing down the metric, using Cheeger and Colding's theory about limit spaces of Gromov-Hausdorff convergence, combining with the Gaussian upper bound of heat kernel on limit spaces, we succeed in reducing the limit behavior of the heat kernel on manifold to the values of heat kernels on tangent cones at infinity of manifold with renormalized measure. As one application, we get the consistent large time limit of heat kernel in more general context, which generalizes the former result of P. Li. Furthermore, by choosing different sequences to blow down the suitable metric, we show the first example manifold whose heat kernel has inconsistent limit behavior, which answers an open question posed by P. Li negatively.

Contents

1. Introduction

On (M^n, g), we consider the fundamental solution $H(x, y, t)$, which solves the heat equation with initial data:

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta\right) F(x, t) & = 0 \qquad on \ M^n \times (0, \infty) \\ \qquad\quad F(x, 0) & = f(x) \qquad on \ M^n \end{cases}$$

by setting

$$F(x, t) = \int_{M^n} H(x, y, t) f(y) dy.$$

It is well-known that there exists a minimal positive fundamental solution of (M^n, g) (cf. Theorem 12.4 in [Li12]). In [Dod83], J. Dodziuk showed that if the Ricci curvature (M^n, g) is bounded from below, then the minimal positive fundamental solution of (M^n, g) is the unique positive fundamental solution of (M^n, g). In this case, we say that the unique positive fundamental solution $H(x, y, t)$ is the **heat kernel** of (M^n, g).

Especially, when (M^n, g) has non-negative Ricci curvature, in [LY86], P. Li and S-T. Yau proved that for all $\epsilon > 0$, there exists constants $C(\epsilon) > 0$, such that

(1.1)
$$\frac{C(\epsilon)^{-1}}{V(\sqrt{t})} \exp\left(-\frac{d^2(x, y)}{(4 - \epsilon)t}\right) \le H(x, y, t) \le \frac{C(\epsilon)}{V(\sqrt{t})} \exp\left(-\frac{d^2(x, y)}{(4 + \epsilon)t}\right)$$

where the terms $V(\sqrt{t})$ and $d(x, y)$ denote the volume of the geodesic ball centered at y of radius $\sqrt{t}$ and the geodesic distance from x to y, respectively.

In particular, there are constants $C_1(n)$ and $C_2(n)$ depending only on dimension n of M^n, such that

(1.2) $\quad C_1(n) \le \varliminf_{t \to \infty} V(\sqrt{t}) H(x, y, t) \le \varlimsup_{t \to \infty} V(\sqrt{t}) H(x, y, t) \le C_2(n).$

For smooth manifold M^n with non-negative Ricci curvature, Bishop-Gromov volume comparison theorem asserts that the relative volume $\frac{V(r)}{r^n}$ is decreasing in the radius r. As $r \to \infty$, it converges a non-negative number Θ, which is called asymptotic volume ratio. If $\Theta > 0$, then we say that M^n has maximal volume growth.

In [Li86], P. Li initiated the study of large time behavior of heat kernel on open manifolds with $Rc \ge 0$ and maximal volume growth. Among other things, he proved the following theorem:

Theorem 1.1 (P. Li). *If (M^n, g) has $Rc \geq 0$ and maximal volume growth, then*

$$(1.3) \qquad \lim_{t \to \infty} V\left(\sqrt{t}\right) H(x, y, t) = \omega(n)(4\pi)^{-\frac{n}{2}}$$

where $\omega(n)$ is the volume of the unit n-ball in $\mathbb{R}^n$.

The key of the proof is Li-Yau's Harnack inequality established in [LY86] and the Bishop-Gromov Volume Comparison Theorem.

Inspired by the above work, in [CM97b] T. Colding and W. Minicozzi studied the large scale behavior of the Green's function $G(x, y)$. Among other things, they proved

Theorem 1.2 (T. Colding and W. Minicozzi). *If M^n, $n \geq 3$ has nonnegative Ricci curvature and maximal volume growth, then for a fixed $x \in M^n$,*

$$\lim_{d(x,y) \to \infty} \frac{G(x, y)}{G_{\mathbb{R}^n}(x, y)} = \frac{\omega(n)}{\Theta}$$

where $G_{\mathbb{R}^n}(x, y)$ is the Green's function on $\mathbb{R}^n$.

And they also pointed out that the geometric motivation behind of Theorem 1.2 is the fact: every tangent cone at infinity of a manifold satisfying the assumptions of Theorem 1.2 is a metric cone, which was shown in [CC96].

Let us recall that for a complete noncompact manifold M^n with $Rc \geq 0$, a metric space M_∞ is a tangent cone at infinity of M^n if it is a Gromov-Hausdorff limit of a sequence of rescaled manifolds $(M^n, p, t_j^{-2} g)$, where $t_j \to \infty$. By Gromov's compactness theorem, [Gro99], any sequence $t_j \to \infty$, has a subsequence, also denoted as $t_j \to \infty$, such that the rescaled manifolds $(M^n, p, t_j^{-2} g)$ converge to some M_∞ in the Gromov-Hausdorff sense. Example of Perelman ([Per97]) shows that tangent cone at infinity is not unique in general even if the manifold with $Rc \geq 0$ has maximal volume growth and quadratic curvature decay. We refer the reader to [CC97] for more examples including collapsing case. Note tangent cones at infinity of M^n reflect the geometry at infinity of manifoold M^n.

Later on, in [LTW97], in addition to providing another proof of Theorem 1.2, P. Li, L. Tam and J. Wang proved the sharp bound of the heat kernel under the assumption in Theorem 1.1. Their sharp bound of heat kernel shows that the coefficients $\frac{C(\epsilon)^{-1}}{V\left(\sqrt{t}\right)}$ and $\frac{C(\epsilon)}{V\left(\sqrt{t}\right)}$ in (1.1) have some relationship with the asymptotic volume ratio Θ.

As the asymptotic volume ratio is one quantity reflecting the geometry at infinity of manifolds, combined with the above observation about the Green's function and tangent cones at infinity of manifold, it is reasonable to speculate that Theorem 1.1 has one proof from the view

point of tangent cones at infinity of manifold. In other words, the large time behavior of the heat kernel should have close relationship with the geometry at infinity of manifolds.

In [Din02], under the maximum volume growth assumption, Y. Ding reduced the study of large scale behavior of the Green's function and large time behavior of the heat kernel, to the analysis on tangent cones at infinity of manifolds, where all tangent cones are metric cones and the Gromov-Hausdorff convergence is non-collapsing. Note the analysis on metric cones had been done by J. Cheeger [Che83] in different context. By the above strategy, Y. Ding provided one alternative proof for Theorem 1.1 and Theorem 1.2 in unified way.

However, as pointed out in [Li86], the answer to the following question was still unknown:

Question 1.3. *Does* $\lim\limits_{t \to \infty} V(\sqrt{t})H(x, y, t)$ *exist generally without the assumption of maximal volume growth?*

To study the above question, we firstly set up the setting as the following:

Blow Down Setup: Note that (M^n, g, μ) is a complete Riemannian manifold with $Rc \geq 0$, where μ is the volume element determined by the metric g. We can define (M_i, y, ρ_i, ν_i), where M_i is the same differential manifold as M^n, ρ_i is the metric defined as $\rho_i = t_i^{-1}g$, $\{t_i\}_{i=1}^\infty$ is an increasing positive sequence whose limit is ∞, and y is a fixed point on $M_i = M^n$. ν_i is a Borel regular measure defined by

$$(1.4) \qquad \nu_i(A) \doteq \left(\int_{B_i(1)} 1 d\mu_i \right)^{-1} \left(\int_A 1 d\mu_i \right) = t_i^{\frac{n}{2}} V(\sqrt{t_i})^{-1} \mu_i(A)$$

where $A \subset M_i$, $B_i(1) \doteq \{z \in M_i | \ d_{\rho_i}(z, y) \leq 1\}$, and μ_i is the volume element determined by ρ_i. Then by Gromov's compactness theorem (see [Gro99]) and Theorem 1.6 in [CC97], after passing to a suitable subsequence, we have $(M_i, y, \rho_i, \nu_i) \xrightarrow{d_{GH}} (M_\infty, y_\infty, \rho_\infty, \nu_\infty)$ in the measured Gromov-Hausdorff sense, where ν_∞ is the renormalized limit measure defined as in Section 1 of [CC97].

Unless otherwise mentioned, in this paper (M^n, y, g, μ), $(M_i^n, y, \rho_i, \nu_i)$ and $(M_\infty, y_\infty, \rho_\infty, \nu_\infty)$ are as in the above **Blow Down Setup** and $n \geq 3$.

A main result of this paper is the following:

Theorem 1.4. *Assume* $(M_i, y, \rho_i, \nu_i) \xrightarrow{d_{GH}} (M_\infty, y_\infty, \rho_\infty, \nu_\infty)$ *as in the above* **Blow Down Setup** *and* $n \geq 3$, *then*

$$(1.5) \qquad\qquad \lim\limits_{i \to \infty} V(\sqrt{t_i})H(x, y, t_i) = p_\infty(y_\infty, y_\infty, 1)$$

where p_∞ *is the heat kernel on the metric measure space* $(M_\infty, y_\infty, \rho_\infty, \nu_\infty)$, *and the convergence is point-wise convergence.*

Remark 1.5. In fact, after some suitable modification, it is not hard to show that the results of this paper also hold on complete Riemann surface, i.e. the $n = 2$ case. For space reason, we will not discuss the $n = 2$ case separately here.

To prove Theorem 1.4, we follow Y. Ding's strategy loosely. However, by combining K.-T. Sturm's study about heat kernel on metric spaces (see [Stu94], [Stu95], [Stu96], [Stu98]), with Cheeger-Colding's theory about limit spaces with Ricci curvature bounded from below (see [CC96], [CC97], [CC00a], [CC00b], [Che99]), we manage to overcome the difficulties caused by collapsing during Gromov-Hausdorff convergence.

More concretely, in [Din02], the assumption of maximum volume growth was needed to get the Li-Yau's estimate for the Green's function on tangent cones at infinity of manifolds, then the reduction for the Green's function from manifolds to limit space under Gromov-Hausdorff convergence can be obtained, finally the reduction for the heat kernel as in Theorem 1.4 follows from the integral formula connecting the heat kernel with the Green's function.

Our approach is kind of direct by avoiding the discussion of the Green's function. Note in Ding's proof, the Li-Yau's estimate for the Green's function on the limit spaces (metric cones) plays the essential role in getting the reduction for the Green's function. To get the reduction for the heat kernel, we need such an estimate for the heat kernel on the general limit spaces (metric measure spaces). Following K.-T. Sturm's method, we proved the general existence result and Gaussian-type upper bounds of heat kernel on M_∞, which is enough for our use.

Note on compact domains, the heat kernel has the expansion determined by eigenvalues and eigenfunctions. On the other hand, J. Cheeger and T. Colding [CC00b] (also see [Che99] for some technical details) had proved that the eigenvalues and eigenfunctions on compact metric measure spaces behave continuously under measured Gromov-Hausdorff convergence, which was originally conjectured by K. Fukaya in [Fuk87]. Combining the suitable modifications of these two facts about heat kernel, eigenvalues and eigenfunctions on bounded domains, we can get the reduction of the heat kernel on bounded domains over complete manifolds, see Theorem 7.3.

Then applying the crucial Gaussian-type upper bounds of heat kernel on tangent cones at infinity of manifolds and the family of blowing down manifolds, using the suitable compact exhaustion of these complete blowing down manifolds, we succeed in getting the above reduction generally for the heat kernel on complete manifolds, from the reduction of the heat kernel on bounded domains over complete manifolds. Note the role of Gaussian-type upper bounds of heat kernel on tangent cones at infinity of manifolds and on blowing down manifolds, in getting our reduction, is analogous to the role that the uniform integrable function

bound of measurable functions plays to guarantee two limit processes commute in Lebesgue's Dominated Convergence Theorem.

A byproduct of the above general reduction result is, a generalization of the former results of P. Li and Y. Ding about the consistent large time behavior of heat kernel. More concretely, we have the following theorem.

Theorem 1.6. *Assume that (M^n, g) is a complete manifold with cone structures at infinity, y is some fixed point on M^n and $n \geq 3$. Furthermore assume that for any $r > 0$, any two positive sequence $\{s_i\}$, $\{l_i\}$ with the following property:*

(1.6)

$$\lim_{i \to \infty} s_i = \lim_{i \to \infty} l_i = \infty \ , \quad \lim_{i \to \infty} \frac{V_y(\sqrt{s_i}r)}{V_y(\sqrt{s_i})} = h(r) \ , \quad \lim_{i \to \infty} \frac{V_y(\sqrt{l_i}r)}{V_y(\sqrt{l_i})} = \tilde{h}(r)$$

where $h(r)$, $\tilde{h}(r)$ are positive functions, the following equation holds:

(1.7)
$$\frac{h''(r)}{h'(r)} = \frac{\tilde{h}''(r)}{\tilde{h}'(r)}.$$

Then

(1.8)
$$\lim_{t \to \infty} V_y(\sqrt{t}) \cdot H(x, y, t) = p_\infty(y_\infty, y_\infty, 1)$$

where p_∞ is the heat kernel on any tangent cone at infinity of manifold M^n with renormalized measure, and the value of the right hand side is consistent.

The concept of manifolds with cone structures at infinity will be defined in Section 8. Especially, the manifolds with nonnegative Ricci curvature and maximal volume growth satisfy the assumptions in Theorem 1.6, in fact $h(r) = \tilde{h}(r) = r^n$ in this case.

Furthermore, we construct the first example of manifold with $Rc \geq 0$, where the limit in Question 1.3 does not exist. More precisely, we have the following theorem.

Theorem 1.7. *There exists a complete Riemannian manifold (M^8, g) with $Rc \geq 0$, such that on (M^8, g),*

$$\varliminf_{t \to \infty} V(\sqrt{t}) H(x, y, t) < \varlimsup_{t \to \infty} V(\sqrt{t}) H(x, y, t).$$

Following Cheeger and Colding's strategy in Section 8 of [CC97], we modify the examples there to construct our example. Note that not every two different tangent cones at infinity of manifold will give different values of $p_\infty(y, y, 1)$. The different renormalized measures on tangent cones at infinity of manifold are the key point to result in the inconsistent limit behavior of heat kernel.

The organization of this paper is as the following. In Section 2, we state some background facts about Gromov-Hausdorff convergence,

which are needed for later sections. For this part, we mainly refer to
[CC96], [CC97], [Gro99]. And we also review the results about the first
order differentiation, Sobolev spaces and Laplacian operator on metric
measure spaces, which were proved in [Che99] and [CC00b].

In Section 3, we proved a Harnack's convergence theorem in Gromov-
Hausdorff topology (Theorem 3.1), which roughly says that the limit (if
it exists) of harmonic functions on manifolds, is a harmonic function
on limit spaces under some gradient bounds assumption. Theorem 3.1
was originally due to Y. Ding (see Section 3 of [Din02]). For reader's
convenience, we provide a detailed proof here.

In Section 4, as in [Din02], combining with the well-known estimates
of eigenvalues and eigenfunctions, the convergence of eigenvalues and
eigenfunctions in Gromov-Hausdorff sense follows from the Harnack's
convergence theorem proved in Section 3.

In Section 5, the heat equation on metric measure space M_∞ is dis-
cussed. Using the theory of abstract Cauchy problem developed in
[LM72], we get the existence of the solutions of heat equation on M_∞ as
in [Stu95]. In addition, some mean value inequality of the heat equation
solutions are obtained, whose proof imitates L. Saloff-Coste's argument
on smooth manifolds (cf. see [SC02]).

In Section 6, we follow closely the argument of K.-T. Sturm in [Stu95]
(also see [Stu94], [Stu96] and [Stu98]) and L. Saloff-Coste in [SC02] (also
see [SC92a], [SC92b]) to prove the existence and Gaussian upper bound
of heat kernel on metric measure space $(M_\infty, \rho_\infty, \nu_\infty)$. We believe that
some results in this section are well-known to experts in this field in more
general context, but we provide the details here to make our argument
self-contained.

In Section 7, using the results established in the former sections, we
manage to reduce the $\lim_{i \to \infty} V(\sqrt{t_i})H(x, y, t_i)$ to the heat kernel value
$p_\infty(y, y, 1)$ on (M_∞, ν_∞), where M_∞ is any tangent cone at infinity of
complete manifold M^n with $Rc \geq 0$ and ν_∞ is the renormalized measure
on M_∞.

In Section 8, by the general reduction results obtained in Section
7, the general criterion in Theorem 1.6 is given to determine whether
the limit behavior of heat kernel is consistent. This general criterion
includes the former related results of P. Li and Y. Ding as a special
case.

In Section 9, using the generalized Hopf fibration of $\mathbb{S}^7$, we construct
the example (M^8, g) by modifying the metric on $\mathbb{R}^8$ step by step. When
M_∞ have cone structure $dr^2 + f(r)^2 dX$, one key point to get different
heat kernel values $p_\infty(y, y, 1)$ on (M_∞, ν_∞) is, to assure that (1.7) does
not hold for two specially chosen positive sequences whose limits are
infinity. The computation involved in the construction of this example
is long but straightforward, we give the details for completeness.

Finally in Appendix A, some L^p-convergence results in Gromov-Hausdorff sense are stated, and the proof of the Rellich-type compactness theorem is also provided for reader's convenience.

2. Preliminaries on Cheeger-Colding's theory

In this section we review some background material about Gromov-Hausdorff convergence and analysis on limit spaces, which were established in [Gro99] and [CC97], [CC00a], [CC00b], [Che99]. Especially, the doubling condition and local Poincaré inequality on limit spaces are showed. Also the existence of self-adjoint Laplacian operator on limit spaces is established. Those two results are used repeatedly through the whole paper.

Let $\{(M_i^n, y_i, \rho_i)\}$ be a sequence of pointed Riemannian manifolds, where $y_i \in M_i^n$ and ρ_i is the metric on M_i^n. If $\{(M_i^n, y_i, \rho_i)\}$ converges to $(M_\infty, y_\infty, \rho_\infty)$ in the Gromov-Hausdorff sense, we write $(M_i^n, y_i, \rho_i) \xrightarrow{d_{GH}} (M_\infty, y_\infty, \rho_\infty)$. See [Gro99] for the definition and basic facts concerning Gromov-Hausdorff convergence.

Obviously if a sequence of pointed metric spaces converges to a pointed space (X, p) in the Gromov-Hausdorff sense, it also converges to its completion. We will only consider complete metric spaces as Gromov-Hausdorff limits. Then, similarly to the case of ordinary convergence, a Gromov-Hausdorff limit of pointed spaces is essentially unique. For general background on metric space and length space, we refer the reader to [BBI01].

Let $(X_i, p_i) \xrightarrow{d_{GH}} (X, p)$ where X_i are length spaces and X is a complete metric space, from Theorem 8.1.9 in [BBI01], X is a complete length space.

From the above argument, we get that $(M_\infty, y_\infty, \rho_\infty, \nu_\infty)$ is a complete length space.

A metric space is said to be boundedly compact if all closed bounded sets in it are compact. By Exercise 8.1.8 in [BBI01], (M_∞, ρ_∞) is also boundedly compact.

We define the convergence concept for functions on manifolds $\{M_i^n\}$ as the following, it is so called "uniform convergence in Gromov-Hausdorff topology", for simplification, sometimes it is written as "uniform convergence in G-H topology".

Definition 2.1 (Uniform Convergence in G-H topology)**.** Suppose

$$K_i \subset M_i^n \xrightarrow{d_{GH}} K_\infty \subset M_\infty$$

Assume that $\{f_i\}_{i=1}^\infty$ are functions on M_i^n, f_∞ is a function on M_∞. and $\Phi_i : K_\infty \to K_i$ are ϵ_i-Gromov-Hausdorff approximations, $\lim_{i\to\infty} \epsilon_i = 0$. If $f_i \circ \Phi_i$ converge to f_∞ uniformly, we say that $f_i \to f_\infty$ uniformly over $K_i \xrightarrow{d_{GH}} K_\infty$.

As in Section 9 of [Che99], we have the following definition.

Definition 2.2. If ν_i, ν_∞ are Borel regular measures on M_i^n, M_∞, we say that $(M_i^n, y_i, \rho_i, \nu_i)$ converges to $(M_\infty, y_\infty, \rho_\infty, \nu_\infty)$ **in the measured Gromov-Hausdorff sense**, if $(M_i^n, y_i, \rho_i) \xrightarrow{d_{GH}} (M_\infty, y_\infty, \rho_\infty)$, in addition, for any $x_i \to x_\infty$, $(x_i \in M_i^n$, $x_\infty \in M_\infty)$, $r > 0$, we have

$$\nu_i\Big(B_i(x_i, r)\Big) \to \nu_\infty\Big(B_\infty(x_\infty, r)\Big)$$

where (M_∞, ρ_∞) is a length space with length metric ρ_∞, and

$$B_i(x_i, r) = \{z \in M_i^n | \ d_{\rho_i}(z, x_i) \leq r\} \ ,$$
$$B_\infty(x_\infty, r) = \{z \in M_\infty | \ d_{\rho_\infty}(z, x_\infty) \leq r\}.$$

In the rest of this section, we assume that $\{M_i^n\}$ is a sequence of complete noncompact manifolds with non-negative Ricci curvature, ν_i is the renormalized measure on M_i^n defined as $\nu_i(A) = \frac{\mu_i(A)}{\mu_i(B_i(1))}$, where μ_i is the volume element determined by ρ_i. And $(M_i^n, y_i, \rho_i, \nu_i)$ converges to $(M_\infty, y_\infty, \rho_\infty, \nu_\infty)$ in the measured Gromov-Hausdorff sense. Note from Theorem 1.6 in [CC97], any sequence (M_i, y_i, ρ_i) with $Rc \geq 0$, there is a subsequence, $(M_i^n, y_i, \rho_i, \nu_i)$, convergent to some $(M_\infty, y_\infty, \rho_\infty, \nu_\infty)$ in the measured Gromov-Hausdorff sense.

Before discussing the analysis on M_∞, we firstly consider the general metric measure space (X, m), where X is a metric space and m is a Borel regular measure on X. Hence $(M_\infty, \rho_\infty, \nu_\infty)$ is a special case of (X, m). Fixed a set $A \subset X$, let f be a function on A with values in the extended real numbers.

Definition 2.3. An **upper gradient**, g, for f is an extended real valued Borel function, $g : A \to [0, \infty]$, such that for all points, $z_1, z_2 \in A$, and all continuous rectifiable curves, $\gamma : [0, l] \to A$, parameterized by arc-length, s, with $\gamma(0) = z_1$, $\gamma(l) = z_2$, we have

$$(2.1) \qquad\qquad |f(z_1) - f(z_2)| \leq \int_0^l g(\gamma(s))ds.$$

Fix an open set $U \subset X$, and until further notice, write L^p for $L^p(U)$. For $f \in L^p$, we set

$$(2.2) \qquad |f|_{1,p} \doteq |f|_{L^p} + \inf_{\{g_i\}} \liminf_{i \to \infty} |g_i|_{L^p}$$

where the inf is taken over all sequences $\{g_i\}$, for which there exists a sequence, $f_i \xrightarrow{L^p} f$, such that g_i is an upper gradient for f_i, for all i.

Definition 2.4. For $p \geq 1$, the Sobolev space $W^{1,p}(U)$ is the subspace of $L^p(U)$ consisting of functions, f, for which $|f|_{1,p} < \infty$, equipped with the norm $| \cdot |_{1,p}$.

Let $0 \to W^{1,p} \xrightarrow{\mathbf{i}} L^p$ denote the natural map, $U_\eta \subset U$ denote the set of points at distance $\geq \eta$ from ∂U. Let $\mathcal{K}(U)$ denote the subset of $W^{1,p}(U)$ consisting of those functions, f, for which there exists $\eta > 0$, such that $\mathbf{i}(f)$, the image of f, in $L^p(U)$, has a representative with support in U_η.

Definition 2.5. The Sobolev space $W_0^{1,p}(U) \subset W^{1,p}(U)$, is the closure of the space $\mathcal{K}(U)$ in $W^{1,p}(U)$.

From Definition 2.8, 2.9 and Theorem 2.10 in [Che99], we have the following theorem.

Theorem 2.6 (Cheeger). *For all $1 < p < \infty$ and $f \in W^{1,p}(U)$, there exists a unique $g_f \in L^p(U)$ (up to modification on subsets of measure zero) such that*

$$(2.3) \qquad |f|_{1,p} = |f|_{L^p} + |g_f|_{L^p}$$

and there exist sequences, $f_i \xrightarrow{L^p} f$, $g_i \xrightarrow{L^p} g_f$, where g_i is an upper gradient for f_i, for all i.

g_f is called a **minimal generalized upper gradient** for f, which may depend on the choice of p and U.

When $p = 2$, the above Sobolev spaces become Hilbert spaces, we use the following notations:

$$H^1 \doteq W^{1,2} , \quad H_0^1 \doteq W_0^{1,2}.$$

We define the following properties:

Property $(\mathcal{B})$(**the doubling condition**): For all balls $B_{2r}(x) \subset X$, we have

$$(2.4) \qquad m\Big(B_{2r}(x)\Big) \leq 2^n \cdot m\Big(B_r(x)\Big)$$

Property $(\mathcal{C})$: There exists a constant $C = C(n)$ such that for all balls $B_{2r}(x) \subset X$, we have

$$(2.5) \qquad \int_{B_r(x)} |f - f_{x,r}|^2 dm \leq Cr^2 \int_{B_{2r}(x)} |g_f|^2 dm$$

for all $f \in H^1(X, m)$, and

$$(2.6) \qquad f_{x,r} = \frac{1}{m(B_r(x))} \int_{B_r(x)} f \, dm.$$

We have the following proposition about $(M_\infty, y_\infty, \rho_\infty, \nu_\infty)$.

Proposition 2.7. *Property* $(\mathscr{B})$, $(\mathscr{C})$ *hold on* $(M_\infty, \rho_\infty, y, \nu_\infty)$.

Proof: It follows from Volume Comparison Theorem that Property $(\mathscr{B})$ holds on $(M_i^n, y_i, \rho_i, \nu_i)$.

By $Rc \geq 0$ on M_i^n, from Theorem 5.6.5 in [SC02], we have

$$(2.7)$$
$$\int_{B_i(z,r)} |f - f_{z,r}|^{\frac{3}{2}} d\nu_i \leq C(n) r^{\frac{3}{2}} \int_{B_i(z,r)} |\nabla f|^{\frac{3}{2}} d\nu_i \ , \quad f \in H^1(M_i, \nu_i).$$

Using Hölder inequality, we obtain that

$$(2.8) \qquad \left(|f - f_{z,r}| \right)_{z,r} \leq C(n) r \left[\left(|\nabla f|^{\frac{3}{2}} \right)_{z,r} \right]^{\frac{2}{3}} , \quad f \in H^1(M_i, \nu_i).$$

By Theorem 9.6 in [Che99], we get Property $(\mathscr{B})$ and the following inequality holds on $(M_\infty, y_\infty, \rho_\infty, \nu_\infty)$:

$$(2.9) \qquad \left(|f - f_{z,r}| \right)_{z,r} \leq C(n) r \left[\left(|g|^2 \right)_{z,r} \right]^{\frac{1}{2}}$$

where $f \in H^1(M_\infty, \nu_\infty)$ and g is any upper gradient for f.

Using Theorem 2.6, there exist sequences, $f_i \xrightarrow{L^2} f$, $g_i \xrightarrow{L^2} g_f$, and g_i is an upper gradient for f_i. From (2.9), we get

$$\left(|f_i - (f_i)_{z,r}| \right)_{z,r} \leq C(n) r \left[\left(|g_i|^2 \right)_{z,r} \right]^{\frac{1}{2}}$$

taking $i \to \infty$ in the above inequality, we have

$$(2.10) \qquad \left(|f - f_{z,r}| \right)_{z,r} \leq C(n) r \left[\left(|g_f|^2 \right)_{z,r} \right]^{\frac{1}{2}} , \quad f \in H^1(M_\infty, \nu_\infty).$$

From the argument in the beginning of Section 2, we know that (M_∞, ρ_∞) is a complete boundedly compact length space. By Corollary 1 in [HK95], $B_\infty(z, r)$ satisfies the $C(\lambda, M)$ condition (defined in [HK95]) for $\lambda = 1$ and some independent constant M. Then we can use (2.10) and Theorem 1 in [HK95] to get

$$(2.11) \qquad \left[\left(|f - f_{z,r}|^{2\chi} \right)_{z,r} \right]^{\frac{1}{2\chi}} \leq \tau r \left[\left(|g_f|^2 \right)_{z,r} \right]^{\frac{1}{2}} , \quad f \in H^1(M_\infty, \nu_\infty)$$

where $\chi = \chi(n) > 1$, $\tau = \tau(n, \chi) > 0$ are some constants.

By (2.11) and Hölder inequality, we conclude that

$$(2.12)$$
$$\int_{B_\infty(z,r)} |f - f_{z,r}|^2 d\nu_\infty \leq C(n) r^2 \int_{B_\infty(z,r)} |g_f|^2 d\nu_\infty \ , \quad f \in H^1(M_\infty, \nu_\infty)$$

which implies Property ($\mathscr{C}$) on $(M_\infty, y_\infty, \rho_\infty, \nu_\infty)$.   q.e.d.

We have the following theorem about "*df*":

Theorem 2.8 ([Che99], [CC00b]). *$f \in H^1(M_\infty)$ $\left(H^1_0(M_\infty)\right)$, if and only if there exists a sequence of Lipschitz functions (compactly supported Lipschitz functions) $f_i \xrightarrow{L^2} f$ and $df_i \xrightarrow{L^2} \omega$ for some L^2-section ω of $T^* M_\infty$, and ω is unique.*

Proof: By Theorem 4.47 in [Che99] (also see Theorem 6.7 in [CC00b]) and Proposition 2.7 above, we get our conclusion.   q.e.d.

Remark 2.9. ω in Theorem 2.8 is called a strong L^2 exterior derivative of f in [CC00b], we can define $df \doteq \omega$ for $f \in H^1_0(M_\infty)$, then df is the L^∞ section of $T^* M_\infty$ (the cotangent tensor bundle) determined by f, which is called the differential of f. From the Theorem above, it is well defined.

We define

$$\mathscr{L}(U) = \{f| \ f \ is \ Lipschitz \ function \ on \ U\}$$
$$\mathscr{L}_c(U) = \{f| \ f \ is \ compactly \ supported \ Lipschitz \ function \ on \ U\}.$$

From Theorem 2.8 above, we know that $\mathscr{L}_c(U)$ is dense in $H^1_0(U)$. We define $H^1_0(M_\infty)$ as the closure of $\mathscr{L}_c(M_\infty)$ in $H^1(M_\infty)$.

It is easy to see $C_c(U)$ is dense in $L^2(U)$, from the fact that any compactly supported continuous function can be uniformly approximated by compactly supported Lipschitz functions, we get that $\mathscr{L}_c(U)$ is dense in $L^2(U)$. Then $H^1_0(U)$ is also dense in $L^2(U)$.

Because the operator d is well defined on $\mathscr{L}(M_\infty)$, we can view the operator d on $L^2(M_\infty)$ as a densely defined unbounded operator. By Theorem 2.8, this operator is closable as an operator on $L^2(M_\infty)$. We have the existence of self-adjoint operator Δ_∞ on M_∞ as the following.

Theorem 2.10 ([CC00b]). *The bilinear form $\int_{M_\infty} < df_1, df_2 > d\nu_\infty$ is a densely defined, closed symmetric form on $L^2(M_\infty)$. Hence, there is a unique self-adjoint operator, Δ_∞, (associated to the minimal closure), such that*

(2.13)
$$\int_{M_\infty} |df|^2 d\nu_\infty = \int_{M_\infty} < (-\Delta_\infty)^{\frac{1}{2}} f, (-\Delta_\infty)^{\frac{1}{2}} f > d\nu_\infty \ , \quad f \in H^1_0(M_\infty).$$

Proof: It follows from Theorem 2.23 of [Kat95].   q.e.d.

3. Harnack's convergence theorem in the Gromov-Hausdorff sense

In this section, we will show that under uniform gradient bound assumption, the uniform limit of solutions, of Poisson equations on a sequence of convergent manifolds (in Gromov-Hausdorff sense), if it exists, will be the solution of Poisson equation on the limit space. The result of this section will only be needed in Section 4.

Compared with the classical Harnack's convergence theorem (cf. Theorem 2.9 in [GT01]), which says that the limit of monotonic increasing bounded harmonic functions is still harmonic, where monotonicity is used to apply Harnack estimate on harmonic functions. With the uniform gradient bound assumption replacing Harnack estimate, one may think of our theorem (Theorem 3.1) as Harnack's convergence theorem in the Gromov-Hausdorff sense, which is crucial in the proof of Proposition 4.5.

On Riemannian manifold (M_i^n, ρ_i, ν_i), one solves the Poisson equation

$$\begin{cases} \Delta_{\rho_i} u = f \\ u\big|_{\partial B_i(x_i, r)} = h \end{cases}$$

for Lipschitz functions f, h on $B_i(x_i, r) \subset M_i^n$. By the Dirichlet's principle, u is the unique minimizer of the functional

$$I(u, \nu_i, x_i, r) = \int_{B_i(x_i, r)} \left(\frac{1}{2} |\nabla u|^2 + fu \right) d\nu_i$$

within the space $\mathscr{H}_i \doteq h + H_0^1\Big(B_i(x_i, r)\Big)$.

Similarly, for $(M_\infty, \rho_\infty, \nu_\infty)$, by Theorem 2.10, the solution of the Poisson equation

$$\begin{cases} \Delta_\infty u = f \\ u\big|_{\partial B_\infty(x_\infty, r)} = h \end{cases}$$

is the unique minimizer of the functional

$$I(u, \nu_\infty, x_\infty, r) = \int_{B_\infty(x_\infty, r)} \left(\frac{1}{2} |du|^2 + fu \right) d\nu_\infty$$

within the space $\mathscr{H}_\infty \doteq h + H_0^1\Big(B_\infty(x_\infty, r)\Big)$.

The following theorem was originally proved by Y. Ding. We present a detailed proof here for completeness, which is loosely based on that in [Din02].

Theorem 3.1. *Suppose u_i, f_i are C^2 functions over $B_i(x_i, 2r) \subset (M_i^n, y_i, \rho_i, \nu_i)$, where $B_i(x_i, 2r) = \{z \in M_i^n | \, d_{\rho_i}(z, x_i) \leq 2r\}$; $\Delta_{\rho_i} u_i = f_i$ on $B_i(x_i, 2r)$ and r is some fixed positive constant. Also assume $u_i \to u_\infty$, $f_i \to f_\infty$ uniformly over the sequence of converging balls*

$B_i(x_i, 2r) \to B_\infty(x_\infty, 2r) \subset (M_\infty, y_\infty, \rho_\infty, \nu_\infty)$, and there exists $L > 0$ such that for any i:

(3.1) $|\nabla u_i(x)| \leq L$, $|\nabla f_i(x)| \leq L$ for $x \in B_i(x_i, 2r)$

Then

(3.2) $\Delta_\infty u_\infty = f_\infty$ on $B_\infty(x_\infty, r)$.

Proof: To prove the theorem, we need the following lemma:

Lemma 3.2. *Let u_∞, f_∞ be as in Theorem 3.1, then we have*

(3.3) $I(u_\infty, \nu_\infty, x_\infty, r) \leq \liminf_{i \to \infty} I(u_i, \nu_i, x_i, r)$

where

$$I(u_\infty, \nu_\infty, x_\infty, r) = \int_{B_\infty(x_\infty, r)} \left(\frac{1}{2}|du_\infty|^2 + f_\infty u_\infty\right) d\nu_\infty$$

$$I(u_i, \nu_i, x_i, r) = \int_{B_i(x_i, r)} \left(\frac{1}{2}|\nabla u_i|^2 + f_i u_i\right) d\nu_i.$$

The proof of the Lemma is deferred to the end of this section. We assume that Lemma 3.2 holds, and prove the theorem by contradiction. Assume $\Delta_\infty u_\infty = f_\infty$ is not true over $B_\infty(x, s) \subset\subset B_\infty(x_\infty, r)$.

By solving the Dirichlet problem on $B_\infty(x, s)$ (see Theorem 7.8 and Remark 7.11 in [Che99]), we can find $\tilde{u}_\infty$ with the same boundary value as u_∞ over $\partial B_\infty(x, s)$ and

(3.4) $I(\tilde{u}_\infty, \nu_\infty, x, s) < I(u_\infty, \nu_\infty, x, s) - 2\delta$

where $\delta > 0$ is some constant.

By Lemma 3.2, assume that $x^{(i)} \to x$, then there exists $i_1 > 0$, for $i > i_1$,

(3.5) $I(u_\infty, \nu_\infty, x, s) \leq I(u_i, \nu_i, x^{(i)}, s) + \delta.$

By Lemma 10.7 in [Che99], we can find a sequence of Lipschitz functions $\tilde{u}_i : B_i(x^{(i)}, s) \to \mathbb{R}$, such that $\tilde{u}_i$ converges uniformly to $\tilde{u}_\infty$ and

$$\overline{\lim_{i \to \infty}} \int_{B_i(x^{(i)}, s)} |\nabla \tilde{u}_i|^2 d\nu_i \leq \int_{B_\infty(x, s)} |d\tilde{u}_\infty|^2 d\nu_\infty.$$

Hence there exists $i_2 > 0$, for $i > i_2$,

(3.6) $I(\tilde{u}_i, \nu_i, x^{(i)}, s) < I(\tilde{u}_\infty, \nu_\infty, x, s) + \frac{1}{2}\delta.$

By (3.4), (3.5) and (3.6), we get that for $i > i_0$, where $i_0 = \max\{i_1, i_2\}$,

(3.7) $I(\tilde{u}_i, \nu_i, x^{(i)}, s) < I(u_i, \nu_i, x^{(i)}, s) - \frac{1}{2}\delta.$

When $i > i_0$, solve the following Dirichlet problem:

$$\begin{cases} \Delta \hat{u}_i = f_i & on\ B_i(x^{(i)}, s) \\ \hat{u}_i = \tilde{u}_i & on\ \partial B_i(x^{(i)}, s) \end{cases}$$

then by Dirichlet principle and (3.7), we get that

$$(3.8) \qquad I(\hat{u}_i, \nu_i, x^{(i)}, s) \leq I(\tilde{u}_i, \nu_i, x^{(i)}, s) < I(u_i, \nu_i, x^{(i)}, s) - \frac{1}{2}\delta.$$

Note in fact we have

$$\begin{cases} \Delta(\hat{u}_i - u_i) = 0 & on \ B_i(x^{(i)}, s) \\ (\hat{u}_i - u_i) = (\tilde{u}_i - u_i) & on \ \partial B_i(x^{(i)}, s) \end{cases}$$

and

$$\lim_{i \to \infty} \sup_{\partial B_i(x^{(i)}, s)} |\tilde{u}_i - u_i| = \sup_{\partial B_\infty(x,s)} |\tilde{u}_\infty - u_\infty| = 0.$$

By maximum principle, we get

(3.9)
$$\lim_{i \to \infty} \sup_{z \in B_i(x^{(i)}, s)} |(\hat{u}_i - u_i)(z)| \leq \lim_{i \to \infty} \sup_{z \in \partial B_i(x^{(i)}, s)} |(\tilde{u}_i - u_i)(z)| = 0.$$

From (3.8) and (3.9), there exists $i_3 > 0$, such that for $i > i_3$,

$$\frac{1}{2} \int_{B_i(x^{(i)}, s)} |\nabla \hat{u}_i|^2 d\nu_i < \frac{1}{2} \int_{B_i(x^{(i)}, s)} |\nabla u_i|^2 d\nu_i - \frac{1}{4}\delta.$$

By $|\nabla u_i| \leq L$ in (3.1) and volume convergence of $B_i(x^{(i)}, s)$, there exists $i_4 > 0$ and $s_1 \in (0, s)$, such that for $i > i_4$,

$$\int_{B_i(x^{(i)}, s) \setminus B_i(x^{(i)}, s_1)} |\nabla u_i|^2 d\nu_i < \frac{1}{100}\delta$$

hence for $i > i_4$, we have

$$(3.10) \qquad \int_{B_i(x^{(i)}, s)} |\nabla \hat{u}_i|^2 d\nu_i < \int_{B_i(x^{(i)}, s_1)} |\nabla u_i|^2 d\nu_i - \frac{1}{4}\delta.$$

On $B_i(x^{(i)}, s_1) \subset\subset B_i(x^{(i)}, s)$, from Cheng-Yau's gradient estimate (also see Lemma 4.4 later), we get

$$(3.11) \qquad \sup_{B_i(x^{(i)}, s_1)} |\nabla \hat{u}_i - \nabla u_i| \leq \frac{C(n)}{s - s_1} \sup_{B_i(x^{(i)}, s)} |\hat{u}_i - u_i|.$$

From (3.9), (3.11) and $|\nabla u_i| \leq L$, there exists $i_5 > 0$, for $i > i_5$,

$$(3.12) \qquad \int_{B_i(x^{(i)}, s_1)} |\nabla u_i|^2 d\nu_i - \int_{B_i(x^{(i)}, s_1)} |\nabla \hat{u}_i|^2 \leq \frac{1}{100}\delta.$$

From (3.10) and (3.12), we get

$$\int_{B_i(x^{(i)}, s) \setminus B_i(x^{(i)}, s_1)} |\nabla \hat{u}_i|^2 < -\frac{1}{8}\delta.$$

That is contradiction, the theorem is proved.   q.e.d.

Proof of Lemma 3.2*:* Recall the Bochner formula:

$$(3.13) \qquad \frac{1}{2}\Delta\Big(|\nabla u_i|^2\Big) = \Big|\nabla^2 u_i\Big|^2 + <\nabla\Delta u_i, \nabla u_i> + Rc(\nabla u_i, \nabla u_i).$$

Multiply by a cut-off function ϕ with $supp(\phi) \subset B_i(x_i, 2r)$, $\phi|_{B_i(x_i, \frac{3}{2}r)} = 1$, $|\Delta\phi| \leq C(n,r)$, $\frac{|\nabla\phi|^2}{\phi} \leq C(n,r)$ (see Theorem 6.33 of [CC96]):

(3.14)
$$\frac{1}{2}\phi\Delta\Big(|\nabla u_i|^2\Big) = \phi\Big|\nabla^2 u_i\Big|^2 + \phi Rc(\nabla u_i, \nabla u_i) + \phi<\nabla\Delta u_i, \nabla u_i>.$$

Integration by parts, using $Rc \geq 0$, we get

$$\frac{1}{2}\int_{B_i(x_i,2r)} |\nabla u_i|^2 \Delta\phi d\nu_i$$

$$\geq \int_{B_i(x_i,2r)} \Big[\phi\Big|\nabla^2 u_i\Big|^2 - \phi|\Delta u_i|^2 - \Delta u_i\Big(\nabla\phi\cdot\nabla u_i\Big)\Big]d\nu_i$$

$$\geq \int_{B_i(x_i,2r)} \Big[\phi\Big|\nabla^2 u_i\Big|^2 - \frac{3}{2}\phi|\Delta u_i|^2 - \frac{|\nabla\phi|^2}{2\phi}|\nabla u_i|^2\Big]d\nu_i$$

$$\geq \int_{B_i(x_i,2r)} \Big[\phi\Big|\nabla^2 u_i\Big|^2 - C(n,r)|\nabla u_i|^2 - \frac{3}{2}\phi|f_i|^2\Big]d\nu_i.$$

Hence when i is big enough,

$$\int_{B_i(x_i,2r)} \phi\Big|\nabla^2 u_i\Big|^2 d\nu_i \leq C(n,r)\int_{B_i(x_i,2r)} |\nabla u_i|^2\Big(|\Delta\phi|+1\Big)d\nu_i$$

$$+ \frac{3}{2}\int_{B_\infty(x_\infty,2r)} |f_\infty|^2 d\nu_\infty + 1$$

$$\leq C(n,r)L\cdot\Big[\nu_\infty\big(B_\infty(x_\infty,2r)\big)+1\Big] + \frac{3}{2}\int_{B_\infty(x_\infty,2r)} |f_\infty|^2 d\nu_\infty + 1.$$

We get a uniform upper bound of $\int_{B_i(x_i,\frac{3}{2}r)} \Big|\nabla^2 u_i\Big|^2 d\nu_i$.

By Theorem A.5 in the Appendix, we can get that some subsequence of $|\nabla u_i|$ converges to a function Γ on $B_\infty(x_\infty,r)$ in $L^2\Big(B_\infty(x_\infty,r),\nu_\infty\Big)$, from (3.1) we also know that $\Gamma \in L^\infty\Big(B_\infty(x_\infty,r),\nu_\infty\Big)$. By Lusin's theorem for general topological spaces with measure and $\Gamma \in L^2\Big(B_\infty(x_\infty,r),\mu_\infty\Big)$, for any $\epsilon > 0$, there exists $K_\epsilon \subset\subset B_\infty\big(x_\infty,r\big)$ and $\nu_\infty\Big(B_\infty(x_\infty,r)\backslash K_\epsilon\Big) < \epsilon$, Γ is continuous on K_ϵ, note K_ϵ is ν_∞-measurable.

Note ν_∞ satisfies the doubling condition, which implies the Vitali Covering Theorem $\Big($see Chapter 2 of [Mat95]$\Big)$, hence the Lebesgue

Differentiation Theorem holds for measure ν_∞. Then

$$(3.15) \qquad \lim_{s \to 0} \frac{\nu_\infty\Big(B_\infty(x,s) \cap K_\epsilon\Big)}{\nu_\infty\Big(B_\infty(x,s)\Big)} = 1 \qquad \nu_\infty \ a.e. \ x \in K_\epsilon.$$

For $x \in K_\epsilon$ satisfying (3.15), we will show

$$(3.16) \qquad \Big|du_\infty(x)\Big| \leq \Gamma(x).$$

Finally for $x \in \bigcup\limits_{i=1}^{\infty} K_{2^{-i}}$, (3.16) is valid. Hence for ν_∞ a.e. $x \in B_\infty(x_\infty, r)$, (3.16) is valid, which implies (3.3) holds.

To prove (3.16), it is enough to prove that for any $\delta > 0$, there exists $1 > \epsilon(\delta) > 0$, when $d_{\rho_\infty}(y,x) < \epsilon(\delta)$, the following holds:

$$(3.17) \qquad \Big|u_\infty(x) - u_\infty(y)\Big| \leq d_{\rho_\infty}(y,x)\Big(\Gamma(x) + 7\delta\Big).$$

By contradiction. Then there is $1 > \delta_0 > 0$, $\{y_i\}_{i=1}^{\infty}$, $y_i \in B_\infty(x_\infty, r)$, such that $d_{\rho_\infty}(y_i, x) = \ell_i \to 0$, and

$$(3.18) \qquad \Big|u_\infty(x) - u_\infty(y_i)\Big| > d_{\rho_\infty}(y_i, x)\Big[\Gamma(x) + 7\delta_0\Big].$$

Then for $z \in B_\infty\Big(x, \frac{\ell_i \delta_0}{L}\Big)$, $y \in B_\infty\Big(y_i, \frac{\ell_i \delta_0}{L}\Big)$, we have

$$\Big|u_\infty(z) - u_\infty(y)\Big|$$
$$\geq \Big|u_\infty(x) - u_\infty(y_i)\Big| - \Big|u_\infty(z) - u_\infty(x)\Big| - \Big|u_\infty(y_i) - u_\infty(y)\Big|$$
$$> \ell_i\Big[\Gamma(x) + 7\delta_0\Big] - L \cdot d_{\rho_\infty}(y_i, y) - L \cdot d_{\rho_\infty}(z, x)$$
$$(3.19) \qquad \geq \ell_i\Big[\Gamma(x) + 5\delta_0\Big].$$

Pick $\tilde{x}_j$, $y_{j,i} \in M_j^n$, $\tilde{x}_j \to x$, $y_{j,i} \to y_i$, and $d(\tilde{x}_j, y_{j,i}) = d(x, y_i)$. When j is big enough, for all $z_j \in B_j\Big(\tilde{x}_j, \frac{\ell_i \delta_0}{L}\Big)$, $\tilde{y}_j \in B_j\Big(y_{j,i}, \frac{\ell_i \delta_0}{L}\Big)$ and all minimal geodesic γ_j connecting z_j, $\tilde{y}_j$, by (3.19), we have

$$(3.20) \qquad \int_{\gamma_j} |\nabla u_j| d\rho_j \geq \ell_i\Big[\Gamma(x) + 4\delta_0\Big].$$

Since $|\nabla u_j| \leq L$, a simple computation shows along every γ_j,

$$(3.21) \qquad |\nabla u_j| > \Gamma(x) + 2\delta_0$$

on a subset of γ_j, which has 1-dim Hausdorff measure at least $\frac{2\delta_0 \ell_i}{L}$.

By $Rc \geq 0$ and Theorem 2.11 in [CC96], we get that the global segment inequality holds on $\left(M_j^n, \rho_j, y, \nu_j \right)$:

$$\int_{A_1 \times A_2} \left(\int_0^{d_{\rho_j}(p,q)} e(\gamma_{p,q}(s)) ds \right) dp dq$$

(3.22)
$$\leq C(n) D \left[\nu_j(A_1) + \nu_j(A_2) \right] \cdot \left(\int_W e d\nu_j \right)$$

where e is any nonnegative integrable function on $W \subset M_j^n$, and $\gamma_{p,q}$ is a minimal geodesic from p to q,

$$D \doteq \max_{p \in A_1, q \in A_2} d_{\rho_j}(p,q) , \quad A_1, A_2 \subset M_j^n , \quad \underset{p,q}{\cup} \gamma_{p,q} \subset W.$$

Choose $A_1 = B_j \left(\tilde{x}_j, \frac{\ell_i \delta_0}{L} \right)$, $A_2 = B_j \left(y_{j,i}, \frac{\ell_i \delta_0}{L} \right)$ and $e = \chi_{E_j^i}$ in (3.22), where

$$E_j^i \doteq \left\{ z | \ z \in B_j \left(\tilde{x}_j, \ell_i \left(1 + \frac{\delta_0}{L} \right) \right), \ |\nabla u_j(z)| > \Gamma(x) + 2\delta_0 \right\}$$

then we get

$$\nu_j(E_j^i) \cdot C(n) \left[1 + \frac{\delta_0}{L} \right] \ell_i \left[\nu_j \left(B_j(\tilde{x}_j, \frac{\ell_i \delta_0}{L}) \right) + \nu_j \left(B_j(y_{j,i}, \frac{\ell_i \delta_0}{L}) \right) \right]$$

$$\geq \frac{2\delta_0 \ell_i}{L} \cdot \nu_j \left(B_j(\tilde{x}_j, \frac{\ell_i \delta_0}{L}) \right) \cdot \nu_j \left(B_j(y_{j,i}, \frac{\ell_i \delta_0}{L}) \right).$$

Using the Bishop-Gromov volume comparison theorem, we get that for any i, if j big enough,

(3.23)
$$\frac{\nu_j(E_j^i)}{\nu_j \left(B_j(\tilde{x}_j, \ell_i(1 + \frac{\delta_0}{L})) \right)} \geq C(\delta_0, L, \Gamma(x), n).$$

From (3.23), we obtain that there exists

$$\mathscr{C}_i \subset \mathscr{B}_i \doteq B_\infty \left(x, \ell_i \left(1 + \frac{\delta_0}{L} \right) \right)$$

such that $\nu_\infty(\mathscr{C}_i) \geq \delta_1 \nu_\infty(\mathscr{B}_i)$, where $\delta_1 = \frac{1}{2} C(\delta_0, L, \Gamma(x), n)$, and

$$F_j^i \subset E_j^i , \quad F_j^i \xrightarrow{d_{GH}} \mathscr{C}_i \text{ as } j \to \infty.$$

For fixed i, we further assume $\varphi_j : F_j^i \to \mathscr{C}_i$ is a measure approximation and an ϵ_j-Gromov-Hausdorff approximation for some $\epsilon_j \to 0$.

Let $\tau_1 = \frac{\delta_1}{10} \nu_\infty(\mathscr{B}_i)$, $\tau_2 = \frac{\delta_1 \delta_0^2}{40} \nu_\infty(\mathscr{B}_i)$.

Let $h_j = |\nabla u_j|$, note that h_j converges to Γ in L^2 on $B_\infty(x_\infty, r)$. By Definition A.4, on $\mathscr{C}_i \subset B_\infty(x_\infty, r)$ there exists $h_\infty^{(k)} : \mathscr{C}_i \to \mathbb{R}$, such that

(3.24)
$$\lim_{k \to \infty} \int_{\mathscr{C}_i} |h_\infty^{(k)} - \Gamma|^2 d\nu_\infty = 0$$

and

$$(3.25) \qquad \lim_{k\to\infty} \overline{\lim_{j\to\infty}} \int_{F_j^i} |h_j - h_\infty^{(k)} \circ \varphi_j|^2 d\nu_j = 0.$$

For τ_1, from (3.24) and Egoroff's Theorem, there exists $A \subset \mathscr{C}_i$, such that $\nu_\infty(A) < \tau_1$, and on $\mathscr{C}_i - A$, $h_\infty^{(k)} \to \Gamma$ uniformly.

Note there exists $C_0 > 0$, such that $\nu_j(B_j(\tilde{x}_j, \ell_i(1 + \frac{\delta_0}{L}))) \leq C_0$ for any i, j. And there exists $k_1 > 0$, if $k > k_1$,

$$(3.26) \qquad |h_\infty^{(k)} - \Gamma| \leq \sqrt{\frac{\tau_2}{C_0}} \qquad on \ \mathscr{C}_i - A.$$

For $\tau_2 > 0$, from (3.25), there exists $k_2 > k_1 > 0$, if $k \geq k_2$,

$$\overline{\lim_{j\to\infty}} \int_{F_j^i} |h_j - h_\infty^{(k)} \circ \varphi_j|^2 d\nu_j \leq \frac{\tau_2}{2}$$

hence, there exists $j_1 > 0$, if $j > j_1$, then

$$(3.27) \qquad \int_{F_j^i} |h_j - h_\infty^{(k_2)} \circ \varphi_j|^2 d\nu_j < \tau_2.$$

Let $Q_j^i = F_j^i - \varphi_j^{-1}(A)$, then when $j > j_1$,

$$\int_{Q_j^i} |h_j - \Gamma \circ \varphi_j|^2 d\nu_j$$

$$\leq 2\Big[\int_{Q_j^i} |h_\infty^{(k_2)} \circ \varphi_j - \Gamma \circ \varphi_j|^2 + \int_{Q_j^i} |h_j - h_\infty^{(k_2)} \circ \varphi_j|^2 \Big]$$

$$(3.28) \qquad \leq 4\tau_2$$

the last inequality above follows from (3.26) and (3.27).

Define

$$\mathscr{W}_\infty = \{z| \ \Gamma(z) \leq \Gamma(x) + \delta_0 \ , \ z \in \mathscr{C}_i - A\}$$

and

$$\mathscr{W}_j = \varphi_j^{-1}(\mathscr{W}_\infty) \subset F_j^i - \varphi_j^{-1}(A) = Q_j^i$$

hence on $\mathscr{W}_j$, $h_j(z) > \Gamma(x) + 2\delta_0$, and $(\Gamma \circ \varphi_j)(z) \leq \Gamma(x) + \delta_0$, we get

$$(3.29) \qquad \int_{\mathscr{W}_j} |h_j - \Gamma \circ \varphi_j|^2 \geq \int_{\mathscr{W}_j} \delta_0^2 = \delta_0^2 \nu_j(\mathscr{W}_j).$$

From (3.28) and (3.29),

$$\nu_j(\mathscr{W}_j) \leq \frac{4\tau_2}{\delta_0^2} = \frac{\delta_1}{10} \nu_\infty(\mathscr{B}_i).$$

Hence

$$\nu_\infty(\mathscr{W}_\infty) = \lim_{j\to\infty} \nu_j(\mathscr{W}_j) \leq \frac{\delta_1}{10} \nu_\infty(\mathscr{B}_i).$$

Define $\mathscr{A}_i = \{z \in \mathscr{B}_i | \; \Gamma(z) > \Gamma(x) + \delta_0\}$, note that on $\mathscr{C}_i - A - \mathscr{W}_\infty \subset \mathscr{B}_i$, $\Gamma(z) > \Gamma(x) + \delta_0$, hence

$$(3.30) \qquad \nu_\infty(\mathscr{A}_i) \geq \nu_\infty(\mathscr{C}_i) - \nu_\infty(A) - \nu_\infty(\mathscr{W}_\infty) \geq \frac{4}{5}\delta_1\nu_\infty(\mathscr{B}_i).$$

Note $\delta_1 = \frac{1}{2}C(\delta_0, L, \Gamma(x), n)$, we get

$$(3.31) \qquad \frac{\nu_\infty(\mathscr{A}_i)}{\nu_\infty(\mathscr{B}_i)} \geq C(\delta_0, L, \Gamma(x), n) > 0$$

where $C(\delta_0, L, \Gamma(x), n)$ in different lines may be different.

Now we have

$$0 < C(\delta_0, L, \Gamma(x), n) \leq \frac{\nu_\infty(\mathscr{A}_i)}{\nu_\infty(\mathscr{B}_i)} = \frac{\nu_\infty(\mathscr{A}_i \cap K_\epsilon) + \nu_\infty(\mathscr{A}_i \backslash K_\epsilon)}{\nu_\infty(\mathscr{B}_i)}$$

$$(3.32) \qquad \leq \frac{\nu_\infty(\mathscr{B}_i \backslash K_\epsilon)}{\nu_\infty(\mathscr{B}_i)} + \frac{\nu_\infty(\mathscr{A}_i \cap K_\epsilon)}{\nu_\infty(\mathscr{B}_i)} = (I)_i + (II)_i.$$

From (3.15) and the choice of x, we get $\lim_{i\to\infty}(I)_i = 0$. Because Γ is continuous on K_ϵ, it is easy to see that $(II)_i = 0$ when i is big enough. We take $i \to \infty$ in (3.32), it is contradiction. Hence (3.17) holds for any $\delta > 0$, (3.16) holds ν_∞ a.e. $B_\infty(x_\infty, r)$. We are done.   q.e.d.

4. The convergence of eigenfunctions in the Gromov-Hausdorff sense

In this section, we will show that the eigenvalues, eigenfunctions on the convergent sequence of manifolds converge (subsequentially) to eigenvalues, eigenfunctions on limit space under Gromov-Hausdorff convergence. The main tools are eigenvalue and eigenfunction estimates obtained by P. Li, S-Y. Cheng, S-T. Yau and Harnack's convergence theorem in the Gromov-Hausdorff sense (Theorem 3.1).

Write $\lambda_{j,i}^{(R)}$ for the j-th Dirichlet eigenvalue over $B_i(R) \subset (M_i, y, \rho_i, \nu_i)$, and $\phi_{j,i}^{(R)}$ is the corresponding eigenfunction satisfying the following:

$$(4.1) \qquad \begin{cases} \Delta_{\rho_i}\phi_{j,i}^{(R)} = -\lambda_{j,i}^{(R)}\phi_{j,i}^{(R)} & on \; B_i(R) \\ \phi_{j,i}^{(R)}(x) = 0 & on \; \partial B_i(R) \end{cases}$$

and $\int_{B_i(R)} \phi_{j,i}^{(R)} \cdot \phi_{k,i}^{(R)} d\nu_i = \delta_{jk}$, where Δ_{ρ_i} is the Laplace operator with respect to the metric ρ_i.

From Theorem 3.1 in [SC92a], for any $f \in H_0^1(B_i(R))$, we get

$$\left[\int_{B_i(R)} |f|^{\frac{2n}{n-2}} d\mu_i \right]^{\frac{n-2}{n}}$$

$$(4.2) \qquad \leq C(n) \frac{R^2}{\mu_i\left(B_i(R)\right)^{\frac{2}{n}}} \cdot \left[\int_{B_i(R)} \left(|\nabla f|^2 + R^{-2}f^2 \right) d\mu_i \right].$$

Using Corollary 1.1 in [LS84],

$$(4.3) \qquad R^{-2} \int_{B_i(R)} f^2 d\mu_i \le C(n) \int_{B_i(R)} |\nabla f|^2 d\mu_i.$$

By (4.2) and (4.3), we have

$$\int_{B_i(R)} |\nabla f|^2 d\mu_i \ge C(n)\mu_i\Big(B_i(R)\Big)^{\frac{2}{n}} R^{-2} \cdot \Big[\int_{B_i(R)} |f|^{\frac{2n}{n-2}} d\mu_i\Big]^{\frac{n-2}{n}}$$

$$(4.4) \qquad = C_{\mathscr{SD}}\Big[\int_{B_i(R)} |f|^{\frac{2n}{n-2}} d\mu_i\Big]^{\frac{n-2}{n}}$$

where

$$(4.5) \qquad C_{\mathscr{SD}} \doteqdot C(n)\mu_i\Big(B_i(R)\Big)^{\frac{2}{n}} R^{-2}.$$

Lemma 4.1. *There exists a constant $C(n)$ such that*

$$(4.6) \qquad C(n)^{-1} \cdot R^{-2} \cdot j^{\frac{1}{n}} \le \lambda_{j,i}^{(R)} \le C(n) \cdot R^{-2} \cdot j^2.$$

Proof: Define $C_1(n) \doteqdot \sum_{\ell=0}^{\infty} \frac{1}{2\beta^\ell - 1}$, where $\beta = \frac{n}{n-2}$. Then we have $\frac{n}{4} \le C_1(n) \le \frac{n}{2}$. By the argument of (10.9) in [Li12] (also see [Li80]), we get the lower bound of $\lambda_{j,i}^{(R)}$ as the following:

$$(4.7) \qquad \lambda_{j,i}^{(R)} \ge C(n)j^{\frac{1}{2C_1(n)}} C_{\mathscr{SD}} \cdot \mu_i\Big(B_i(R)\Big)^{-\frac{2}{n}}$$

combining with the definition of $C_{\mathscr{SD}}$ in (4.5), we have

$$(4.8) \qquad \lambda_{j,i}^{(R)} \ge C(n)j^{\frac{1}{2C_1(n)}} R^{-2} \ge C(n) \cdot R^{-2} \cdot j^{\frac{1}{n}}.$$

By the similar argument of Theorem 2 on page 105 of [SY94] (also see [Che75]), we get the upper bound of $\lambda_{j,i}^{(R)}$. q.e.d.

The following lemma is standard, for completeness, we provide the proof following the argument of Theorem 10.1 in [Li12].

Lemma 4.2. *If $R > 2$, we have*

$$(4.9) \qquad \|\phi_{j,i}^{(R)}\|_{L^\infty(\nu_i)} \le C(n,R)j^{\frac{n}{2}}$$

where $\|\cdot\|_{L^k(\nu_i)}$ denotes the L^k norm with respect to the measure ν_i.

Proof: We observe that for a C^∞ function u, from Lemma 7.6 and Lemma 7.7 in [GT01], $|\nabla u|^2 = |\nabla|u||^2$ for μ_i a.e. x. The identities

$$\Delta(u^2) = 2u\Delta u + 2|\nabla u|^2$$

and

$$\Delta(|u|^2) = 2|u|\Delta|u| + 2|\nabla|u||^2$$

imply $u\Delta u = |u|\Delta|u|$ a.e. Hence we have

$$(4.10) \qquad |\phi_{j,i}^{(R)}|\Delta_{\rho_i}|\phi_{j,i}^{(R)}| = \phi_{j,i}^{(R)}\Delta_{\rho_i}\phi_{j,i}^{(R)} = -\lambda_{j,i}^{(R)}|\phi_{j,i}^{(R)}|^2.$$

For any constant $k \geq 2$, by (4.10), (4.4) and integration by parts,

$$\int_{B_i(R)} \left|\phi_{j,i}^{(R)}\right|^k d\mu_i = -\frac{1}{\lambda_{j,i}^{(R)}} \int_{B_i(R)} \left|\phi_{j,i}^{(R)}\right|^{k-1} \cdot \Delta_{\rho_i} \left|\phi_{j,i}^{(R)}\right| d\mu_i$$

$$= \frac{4(k-1)}{\lambda_{j,i}^{(R)} \cdot k^2} \int_{B_i(R)} \left|\nabla\left(|\phi_{j,i}^{(R)}|^{\frac{k}{2}}\right)\right|^2 d\mu_i$$

$$\geq \frac{2C_{\mathscr{SD}}}{k\lambda_{j,i}^{(R)}} \left(\int_{B_i(R)} \left|\phi_{j,i}^{(R)}\right|^{\frac{kn}{n-2}} d\mu_i\right)^{\frac{n-2}{n}}.$$

Denote $\beta = \frac{n}{n-2}$, then for all $k \geq 2$,

$$\|\phi_{j,i}^{(R)}\|_{L^k(\mu_i)} \geq \left(\frac{2C_{\mathscr{SD}}}{k\lambda_{j,i}^{(R)}}\right)^{\frac{1}{k}} \|\phi_{j,i}^{(R)}\|_{L^{k\beta}(\mu_i)}.$$

Setting $k = 2\beta^s$ for $s = 0, 1, 2, \ldots$, we have

$$\|\phi_{j,i}^{(R)}\|_{L^{2\beta^{s+1}}(\mu_i)} \leq \left(\frac{\beta^j \lambda_{j,i}^{(R)}}{C_{\mathscr{SD}}}\right)^{\frac{1}{2\beta^s}} \cdot \|\phi_{j,i}^{(R)}\|_{L^{2\beta^s}(\mu_i)}.$$

Iterating this estimate and using

$$\|\phi_{j,i}^{(R)}\|_{L^2(\mu_i)} = t_i^{-\frac{n}{4}} V(\sqrt{t_i})^{\frac{1}{2}} \|\phi_{j,i}^{(R)}\|_{L^2(\nu_i)} = t_i^{-\frac{n}{4}} V(\sqrt{t_i})^{\frac{1}{2}}$$

we conclude that

$$\|\phi_{j,i}^{(R)}\|_{L^{2\beta^{s+1}}(\mu_i)} \leq \left[\prod_{l=0}^{s} \left(\frac{\beta^l \lambda_{j,i}^{(R)}}{C_{\mathscr{SD}}}\right)^{\frac{1}{2\beta^l}}\right] \cdot t_i^{-\frac{n}{4}} V(\sqrt{t_i})^{\frac{1}{2}}.$$

Let $s \to \infty$ and applying the fact that

$$\|\phi_{j,i}^{(R)}\|_{L^\infty(\nu_i)} = \|\phi_{j,i}^{(R)}\|_{L^\infty(\mu_i)} = \lim_{p\to\infty} \|\phi_{j,i}^{(R)}\|_{L^p(\mu_i)}.$$

We obtain

$$\|\phi_{j,i}^{(R)}\|_{L^\infty(\nu_i)} \leq \left(C_{\mathscr{SD}}\right)^{-\frac{n}{4}} \cdot \left(\lambda_{j,i}^{(R)}\right)^{\frac{n}{4}} \cdot C(n) t_i^{-\frac{n}{4}} V(\sqrt{t_i})^{\frac{1}{2}}$$

$$= C(n) \left[\frac{V(\sqrt{t_i})R^n}{V(\sqrt{t_i}R)}\right]^{\frac{1}{2}} \cdot \left(\lambda_{j,i}^{(R)}\right)^{\frac{n}{4}}$$

$$(4.11) \qquad\qquad \leq C(n) R^{\frac{n}{2}} \left(\lambda_{j,i}^{(R)}\right)^{\frac{n}{4}}.$$

Combining with Lemma 4.1, we get

$$\|\phi_{j,i}^{(R)}\|_{L^\infty(\nu_i)} \leq C(n,R) j^{\frac{n}{2}}.$$

$$\text{q.e.d.}$$

Note that the volume element ν_i of (M_i, y, ρ_i, ν_i) is not determined by the metric ρ_i, the heat kernel of (M_i, y, ρ_i, ν_i) is

$$(4.12) \quad H_i(x, y, s) = t_i^{\frac{n}{2}} \mu_i\left(B_i(1)\right) \cdot H(x, y, t_i s) = V(\sqrt{t_i}) \cdot H(x, y, t_i s)$$

where $H(x, y, s)$ is the heat kernel of (M^n, y, g, μ), μ is the volume element determined by g, $V(\sqrt{t_i}) \doteq \int_{B(t_i)} 1 d\mu$, and $B(t_i) \doteq \{z \in M^n|\ d_g(z, y) \le t_i\}$. Note $H_i(x, y, s)$ is different from the heat kernel $\tilde{H}_i(x, y, s)$ of (M_i, y, ρ_i, μ_i), which is $t_i^{\frac{n}{2}} H(x, y, t_i s)$.

Hence we have

(4.13)
$$\lim_{i \to \infty} V(\sqrt{t_i}) H(x, y, t_i) = \lim_{i \to \infty} H_i(x, y, 1)$$

and by (1.1),

(4.14)

$$H_i(x, y, t) = V(\sqrt{t_i}) H(x, y, t_i t) \le C(n) V(\sqrt{t_i}) V(\sqrt{t_i t})^{-1} e^{-\frac{d_g^2(x, y)}{5 t_i t}}.$$

Let us denote by $H_R(x, y, t)$ the Dirichlet heat kernel on the metric ball

$$B(R) = \{z \in M^n|\ d_g(z, y) \le R\} \subset (M^n, g, \mu)$$

where $R > 0$ is a constant, and put $H_R = 0$ outside of $B(R)$. Similarly, we denote by $H_{R,i}(x, y, t)$ the Dirichlet heat kernel on $B_i(R) \subset (M_i, y, \rho_i, \nu_i)$.

From Lemma 4.1 and Lemma 4.2, using similar argument in the proof of Theorem 10.1 in [Li12], it is easy to get the following eigenfunction expansion of $H_{R,i}(x, y, t)$:

(4.15)
$$H_{R,i}(x, y, t) = \sum_{j=1}^{\infty} e^{-\lambda_{j,i}^{(R)} t} \phi_{j,i}^{(R)}(x) \phi_{j,i}^{(R)}(y).$$

Lemma 4.3. *For any $N > 0$, there exists a function $\epsilon(N, R, \delta)$ such that for any fixed $R > 2$, $\lim_{\delta \to 0} \epsilon(N, R, \delta) = 0$. And for j satisfying $\lambda_{j,i}^{(R)} < N$, we have*

(4.16)
$$\int_{A_i(R-\delta, R)} \left| \phi_{j,i}^{(R)} \right|^2 d\nu_i \le \epsilon(n, N, R, \delta) \quad \text{for } 0 < \delta \le 1$$

where $A_i(R - \delta, R) \doteq \{z \in M_i|\ R - \delta \le d_{\rho_i}(z, y) \le R\}$.

Proof: Using (4.15) and (4.14), we get

$$\int_{A_i(R-\delta, R)} \left| \phi_{j,i}^{(R)}(x) \right|^2 d\nu_i(x) \le \int_{A_i} e^{\lambda_{j,i}^{(R)}} H_{R,i}(x, x, 1) d\nu_i(x)$$

$$\le e^{\lambda_{j,i}^{(R)}} \int_{A_i} H_i(x, x, 1) d\nu_i(x)$$

$$\le C(n) e^N \int_{A_i} e^{-\frac{1}{5 t_i}} d\nu_i(x)$$

$$\le C(n, N) \frac{\mu(A_i)}{V(\sqrt{t_i})} \le C(n, N) \left[R^n - (R - \delta)^n \right]$$

in the last inequality above, we used the Bishop-Gromov inequality. Our conclusion is proved.   q.e.d.

The following lemma follows from a standard argument of Cheng-Yau in [CY75], which is needed in the proof of Proposition 4.5.

Lemma 4.4. *Assume that (M^n, g) is a complete manifold with $Rc \geq 0$, if $\Delta u = -\lambda u$ on $B_p(2r) \subset M^n$ and $\lambda \geq 0$, then we have*

$$|\nabla u|(x) \leq C(n)[r^{-1} + \lambda] \cdot \sup_{x \in B_p(2r)} |u(x)| , \quad x \in B_p(r)$$

where $B_p(r) = \{z \in M^n | \ d_g(z, p) \leq r\}$.

Proof: Let $\mathscr{M} = \sup_{x \in B_p(2r)} |u(x)|$, $f(x) = u(x) + \mathscr{M}$, without loss of generality, assume $\mathscr{M} > 0$. It is easy to get $\Delta f = -\lambda f + \lambda \mathscr{M}$ on $B_p(2r)$, and $f \geq 0$.

Apply Theorem 6 in [CY75] to $f(x)$, we get

(4.17) $|\nabla f(x)| \leq C(n)[r^{-1} + \lambda] \cdot [f(x) + \mathscr{M}] , \quad x \in B_p(r).$

By the definition of $f(x)$ and $\mathscr{M}$, our conclusion follows from (4.17).
                                    q.e.d.

Proposition 4.5. *For fixed j, $k > 0$, assume (for a subsequence of the eigenvalues) $\lambda_{j,i}^{(R)} \to \lambda_{j,\infty}^{(R)}$, $\lambda_{k,i}^{(R)} \to \lambda_{k,\infty}^{(R)}$ as $i \to \infty$. Then there is a subsequence (denoted also by $\phi_{j,i}^{(R)}$, $\phi_{k,i}^{(R)}$) that converges uniformly on compact subsets of $\mathring{B}_\infty(R)$, and also in $L^2\big(B_\infty(R)\big)$, to two compactly supported Lipschitz functions $\phi_{j,\infty}^{(R)}$, $\phi_{k,\infty}^{(R)}$ on $B_\infty(R)$, where $B_\infty(R) = \{z \in M_\infty | \ d_{\rho_\infty}(z, y) \leq R\}$, $\mathring{B}_\infty(R)$ denotes the interior of $B_\infty(R)$. Moreover,*

(4.18) $\Delta_\infty \phi_{j,\infty}^{(R)} = -\lambda_{j,\infty}^{(R)} \phi_{j,\infty}^{(R)} , \quad \Delta_\infty \phi_{k,\infty}^{(R)} = -\lambda_{k,\infty}^{(R)} \phi_{k,\infty}^{(R)} ,$

(4.19) $\displaystyle \int_{B_\infty(R)} \phi_{j,\infty}^{(R)} \phi_{k,\infty}^{(R)} d\nu = \delta_{j,k}.$

Proof: Locally uniform convergence follows from Lemma 4.2 and 4.4. The L^2 convergence and (4.19) are implied by locally uniform convergence and Lemma 4.3. Finally, (4.18) follows from Theorem 3.1 and Lemma 4.4.   q.e.d.

5. Solutions of the heat equations on metric measure spaces

In this section, on metric measure spaces, we will show the existence of the solution of the heat equations and the parabolic mean value inequality. For smooth manifolds, all these results are well-known. On metric measure spaces, our setup is closely related with the discussion in [Stu95].

Assume $U \subset M_\infty$, and U is open. We will be concerned with the following Banach spaces.

- $L^2\big((0,T); H_0^1(U)\big)$ is the Hilbert space consisting of functions $u(x, t)$, measurable on $(0,T)$ with range in $H_0^1(U)$ (for the Lebesgue measure dt on $(0,T)$), for any $t \in (0,T)$, $u(\cdot, t) \in H_0^1(U)$ and the norm of the space is $\Big(\int_0^T |u(\cdot, t)|_{H_0^1(U)}^2 dt \Big)^{\frac{1}{2}}$.

- $H^1\big((0,T); H_0^1(U)^*\big)$ is the Sobolev space of functions u, where $H_0^1(U)^*$ is the dual space of $H_0^1(U)$, and $u \in L^2\big((0,T); H_0^1(U)^*\big)$, and it has distributional time derivative $\frac{\partial}{\partial t} u \in L^2\big((0,T); H_0^1(U)^*\big)$ equipped with the norm

$$\Big(\int_0^T |u(\cdot, t)|_{H_0^1(U)^*}^2 + |\frac{\partial}{\partial t} u(\cdot, t)|_{H_0^1(U)^*}^2 dt \Big)^{\frac{1}{2}}.$$

- $\mathscr{F}\big((0,T) \times U\big) \doteq L^2\big((0,T); H_0^1(U)\big) \cap H^1\big((0,T); H_0^1(U)^*\big)$. We mention the following important result from [RR93]:

$$\mathscr{F}\big((0,T) \times U\big) \subset C\big([0,T], L^2(U)\big).$$

- Similarly, $\mathscr{G}\big((0,T) \times U\big) \doteq L^2\big((0,T); H^1(U)\big) \cap H^1\big((0,T); H^1(U)^*\big)$.

Definition 5.1. A function u is called a **Dirichlet solution** of the heat equation on $(0,T) \times U$:

(5.1) $$\frac{\partial}{\partial t} u = \Delta_\infty u \quad \text{on } (0,T) \times U$$

iff $u \in \mathscr{F}\big((0,T) \times U\big)$, and for all $\phi \in \mathscr{F}\big((0,T) \times U\big)$:

(5.2) $$\int_0^T \int_U < du, d\phi > d\nu_\infty dt + \int_0^T \int_U \frac{\partial u}{\partial t} \cdot \phi d\nu_\infty dt = 0.$$

Remark 5.2. For $u \in \mathscr{G}\big((0,T) \times U\big)$, we say that

$$\Big(\frac{\partial}{\partial t} - \Delta_\infty \Big) u = (\leq) 0 \qquad \text{on } (0,T) \times U$$

if for almost all $t \in (0,T)$ except a subset of $(0,T)$ with Lebesgue measure 0,

$$\int_U < du, d\phi > d\nu_\infty + \int_U \frac{\partial u}{\partial t} \cdot \phi d\nu_\infty = (\leq) 0$$

holds for all non-negative $\phi \in H_0^1(U)$. Such u is also called a **solution (subsolution) of the heat equation** on $(0,T) \times U$.

Definition 5.3. Given a function $f \in L^2(U)$, the function u is called a **Dirichlet solution of the initial value problem** on $[0,T) \times U$:

(5.3) $$\begin{cases} \frac{\partial}{\partial t} u = \Delta_\infty u & \text{on } (0,T) \times U \\ u(\cdot, 0) = f(\cdot) & \text{on } U \end{cases}$$

iff u is a Dirichlet solution of (5.1) and $\lim_{t\to 0}\int_U |u(x,t)-f(x)|^2 d\nu_\infty = 0$.

Proposition 5.4. *For every $f \in L^2(U)$, there exists a unique Dirichlet solution $u \in \mathscr{F}\big((0,T)\times U\big)$ of the initial value problem (5.3).*

Proof: It follows from Theorem 4.1 and Remark 4.3 in Chapter 3 of [LM72].   q.e.d.

For the solutions of heat equations on M_∞, we have the following mean value inequalities.

Theorem 5.5. *If $\frac{\partial u}{\partial t} - \Delta_\infty u = 0$ in Q_1, then for any $0 < \delta < 1$, we have*

$$(5.4)\qquad \sup_{z\in Q_\delta} u^2(z) \le \frac{C(n)}{(1-\delta)^{n+2}r^2\nu_\infty(B)}\int_{Q_1} u^2 d\nu_\infty dt$$

$$(5.5)\qquad \sup_{z\in Q_\delta} u(z) \le \frac{C(n)}{(1-\delta)^{n+2}r^2\nu_\infty(B)}\int_{Q_1} u d\nu_\infty dt$$

where $B = B_\infty(x,r)$, $s > r^2 > 0$, $\tau > 0$ is a fixed positive constant, and

$$Q_1 \doteq (s-r^2,s)\times B_\infty(x,r)\ ,\qquad Q_\delta \doteq (s-\delta r^2,s)\times B_\infty(x,\delta r).$$

Remark 5.6. The parabolic mean value inequality on smooth manifold were firstly proved in [LT91], however the proof there used the upper bound of heat kernel, which is the target we want to prove. The conclusion on metric measure spaces was essentially obtained in [SC02], although the context there are smooth manifolds. The following argument is just slight modification of the original argument there, hence it is sketchy. For the complete details, we refer the reader to that book.

Proof: Firstly, from the argument of Lemma 5.3.2, Lemma 5.2.5 in [SC02] and Proposition 2.7, we can get the following Dirichlet Poincaré Inequality:

There exists positive constant $C(n) > 0$, such that for any $B = B_\infty(x,r) \subset M_\infty$,

$$(5.6)\qquad\qquad |f|_{L^2} \le C(n)r|g_f|_{L^2}\ ,\quad f \in H^1_0(B).$$

Secondly, from the argument of Theorem 5.3.3 in [SC02], Proposition 2.7 and (5.6), we can obtain Local Sobolev Inequality as the following:

There exists $C(n) > 0$, such that for any $B = B_\infty(x,r) \subset M_\infty$, we have

$$(5.7)$$
$$\left(\int_B |f|^{\frac{2n}{n-2}} d\nu_\infty\right)^{\frac{n-2}{n}} \le C(n)\frac{r^2}{\nu_\infty(B)^{\frac{2}{n}}}\left(\int_B |g_f|^2 d\nu_\infty\right)\ ,\quad f \in H^1_0(B).$$

Next, employing (5.7), we can use almost exactly the same argument of Theorem 5.2.9 in [SC02] to get the following two inequalities:

If $\frac{\partial u}{\partial t} - \Delta_\infty u \leq 0$ in Q_1 and $u \geq 0$, then for any $0 < \delta < 1$, (5.4) and (5.5) hold.

Finally, for any $\epsilon > 0$, it is easy to show that $v \doteq \sqrt{u^2 + \epsilon}$ is the solution of the heat equation, which was defined in Remark 5.2, and $v \geq 0$. By the above argument,

$$(5.8) \qquad \sup_{z \in Q_\delta} (u^2 + \epsilon)(z) \leq \frac{C(n)}{(1 - \delta)^{n+2} r^2 \nu_\infty(B)} \int_{Q_1} (u^2 + \epsilon) d\nu_\infty dt.$$

Let $\epsilon \to 0$ in (5.8), we get (5.4).

Similar argument yields (5.5). q.e.d.

6. The existence and Gaussian upper bound of heat kernel on limit spaces

In this section we will prove the existence of heat kernel on limit spaces under Gromov-Hausdorff convergence, and establish Gaussian upper bound of heat kernel.

To prove the existence of heat kernel on limit spaces, we are inspired by the method of K.-T. Sturm in [Stu95]. Firstly, from Proposition 5.4, there exists a uniquely determined operator:

$$(6.1) \qquad T : L^2(M_\infty) \to \mathscr{F}((0, T) \times M_\infty)$$

with the property that for every $f \in L^2(M_\infty)$, the unique Dirichlet solution of (5.3) ($U = M_\infty$ there) is given by $u(x, t) = [Tf](x, t)$.

We also define $[T_t f](x) = [Tf](x, t)$ for every $t \in (0, T)$, then

$$(6.2) \qquad T_t : L^2(M_\infty) \to L^2(M_\infty).$$

Lemma 6.1. *There exists $C(n) > 0$ such that for any $t \in (0, 8R^2)$,*

$$\sup_{x \in B_\infty(R)} |(T_t f)(x)|$$

$$\leq C(n) \Big(\frac{R}{\sqrt{t}}\Big)^{n+2} \nu_\infty(B_\infty(R))^{-\frac{1}{2}} |f|_{L^2(M_\infty)}, \quad \forall f \in L^2(M_\infty)$$

where $R > 0$ is any positive constant.

Proof: We will apply Theorem 5.5 on $T_t f$ for given $t \in (0, 4R^2)$. Let $r = 2R$, $\delta = 1 - \frac{t}{10R^2}$, $s = (2R)^2 + \frac{1}{2}t$, $\tau = 1$ in (5.4), note that $t \in (s - \delta r^2, s)$, then we get

$$\sup_{x \in B_\infty(R)} |(T_t f)(x)| \leq \sup_{Q_\delta} |(T_t f)(x)|$$

$$\leq C(n) \Big(\frac{1}{1 - \delta}\Big)^{\frac{n+2}{2}} \Big(\frac{1}{(2R)^2 \nu_\infty(B_\infty(2R))} \int_{Q_1} |T_t f|^2\Big)^{\frac{1}{2}}$$

$$\leq C(n) \Big(\frac{R}{\sqrt{t}}\Big)^{n+2} \nu_\infty(B_\infty(R))^{-\frac{1}{2}} |f|_{L^2(M_\infty)}$$

in the last inequality, we used that

$$\int_{M_\infty} |T_t f|^2 d\nu_\infty \leq \int_{M_\infty} |f|^2 d\nu_\infty \ , \quad \forall t > 0$$

which follows from (5.2).   q.e.d.

We also have the following parabolic maximum principle on M_∞ (for the proof, see Proposition 4.11 in [GH08]).

Lemma 6.2 ([GH08]). *Assume h is a solution of the heat equation on $(0, T+1) \times B_\infty(z, R)$, and*

$$(6.3) \qquad \lim_{t \to 0} \int_{B_\infty(z,R)} h^2(x, t) d\nu_\infty(x) = 0 \ , \quad h|_{\partial B_\infty(z,R) \times (0,T]} \leq 0$$

for any $f(x) \in L^2\big(B_\infty(z, R)\big)$. Then $h \leq 0$ on $(0, T] \times B_\infty(z, R)$.

The following result is one modification of classical result in functional analysis, which was due to J-X Hu and Grigor'yan (see Lemma 3.3 in [GH]).

Lemma 6.3 ([GH]). *Let $K : L^2(Y) \to L^\infty(X)$ be a bounded linear operator, with the norm bounded by C, that is, for any $f \in L^2(Y)$,*

$$(6.4) \qquad\qquad \sup_X |Kf| \leq C|f|_2.$$

There exists a mapping $k : X \to L^2(Y)$ such that, for all $f \in L^2(Y)$, and almost all $x \in X$,

$$(6.5) \qquad\qquad Kf(x) = (k(x), f).$$

Moreover, for all $x \in X$, $\|k(x)\|_{L^2(Y)} \leq C$. Furthermore, there is a function $k(x, y)$ that is jointly measurable in $(x, y) \in M \times M$, such that, for almost all $x \in X$, $k(x, \cdot) = k(x)$ almost everywhere on Y.

Now we can prove the existence of the heat kernel with respect to the Dirichlet boundary condition on M_∞.

Theorem 6.4. *There exists a nonnegative measurable function*

$$p_\infty : M_\infty \times M_\infty \times \mathbb{R}^+ \to [0, \infty]$$

with the following properties:

1) *On $[0, \infty) \times M_\infty$, the function*

$$u(x, t) = \int_{M_\infty} p_\infty(x, z, t) f(z) d\nu_\infty(z)$$

   is a solution of (5.3), where $f \in L^2\big(M_\infty\big)$.
2) *For any fixed $w \in M_\infty$, any $T > 0$,*

   $$p_\infty(x, w, t) \in L^2\big((0, T); H_0^1(M_\infty)\big) \cap H^1\big((0, T); H_0^1(M_\infty)^*\big)$$

   is a Dirichlet solution of the heat equation (defined as in Definition 5.1).

Remark 6.5. Such p_∞ is called the **heat kernel of** $(M_\infty, \rho_\infty, \nu_\infty)$.

Proof: By Lemma 6.1 and Lemma 6.3, there exists $p_\infty(x, z, t)$, which is jointly measurable in $(x, z) \in M_\infty \times M_\infty$, such that

$$T_t(f)(x) = \int_{M_\infty} p_\infty(x, z, t) f(z) d\nu_\infty(z).$$

From Lemma 6.2, we get that if $f \geq 0$, $T_t(f) \geq 0$. It follows from Lemma 3.2 in [GH], $p_\infty(x, z, t) \geq 0$. Then $p_\infty \geq 0$ and the conclusion in (1) above are proved.

For any $f \in L^2\big(B_\infty(R)\big)$, from the uniqueness of solution in Proposition 5.4 and the definition of T, T_t, we get

$$
\begin{aligned}
\big[T_{t+s}f\big](z) = \big[Tf\big](z, t+s) &= T\big[(Tf)(\cdot, s)\big](z, t) \\
&= T_t\big[(Tf)(\cdot, s)\big](z) = \int_{M_\infty} p_\infty(z, x, t) \cdot \big[Tf\big](x, s) d\nu_\infty(x) \\
&= \int_{M_\infty} p_\infty(z, x, t)\Big(\int_{M_\infty} p_\infty(x, w, s) f(w) d\nu_\infty(w)\Big) d\nu_\infty(x) \\
&= \int_{M_\infty} \Big(\int_{M_\infty} p_\infty(z, x, t) p_\infty(x, w, s) d\nu_\infty(x)\Big) \cdot f(w) d\nu_\infty(w).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
p_\infty(z, w, t+s) &= \int_{M_\infty} p_\infty(z, x, t) p_\infty(x, w, s) d\nu_\infty(w) \\
&= \Big[T_t\big(p_\infty(\cdot, w, s)\big)\Big](z) = \Big[T\big(p_\infty(\cdot, w, s)\big)\Big](z, t).
\end{aligned}
$$

By the definition of T, and $t+s$ can be chosen as any positive number, we get that $p_\infty(z, w, t)$ is a Dirichlet solution of the heat equation on $(0, \infty) \times M_\infty$. q.e.d.

And we have the following theorem about the upper bound of $p_\infty(x, y, t)$. We will follow the method developed by E. B. Davies on smooth manifolds (see [Dav89], also [SC02]), our proof is just slight modification of the proof given in [SC02], and it is presented here for completeness and reader's convenience.

Theorem 6.6. *Assume that $p_\infty(x, y, t)$ is the heat kernel of $\big(M_\infty, y, \rho_\infty, \nu_\infty\big)$, then*

$$(6.6) \qquad p_\infty(x, y, t) \leq C(n)\nu_\infty\Big(B_\infty(y, \sqrt{t})\Big)^{-1} e^{-\frac{1}{5t}\rho_\infty^2(x,y)}$$

where $C(n)$ is the positive constant depending only on n.

We firstly need to prove a lemma.

Lemma 6.7. *For any function $\phi \in H_0^1(M_\infty)$ with $|g_\phi| \leq 1$ and any $\alpha \in \mathbb{R}$, we define the operator $H_t^{\alpha,\phi}$ as the following:*

(6.7)
$$H_t^{\alpha,\phi} f(x) \doteqdot e^{-\alpha\phi(x)} \int_{M_\infty} p_\infty(x,y,t) e^{\alpha\phi(y)} f(y) d\nu_\infty(y) , \quad f \in L^2(M_\infty).$$

Then as an operator from $L^2(M_\infty)$ to $L^2(M_\infty)$, $H_t^{\alpha,\phi}$ satisfies $\|H_t^{\alpha,\phi}\| \leq e^{\alpha^2 t}$.

Proof: For any $f \in L^2(M_\infty)$, set $u(t) = |H_t^{\alpha,\phi} f|_{L^2}^2$, then

$$\begin{aligned}
u'(t) &= 2 \int_{M_\infty} \frac{\partial}{\partial t}\left(H_t^{\alpha,\phi} f\right) \cdot H_t^{\alpha,\phi} f \\
&= \int_{M_\infty} e^{-\alpha\phi(x)} \Delta\left(e^{\alpha\phi(x)} H_t^{\alpha,\phi} f(x)\right) H_t^{\alpha,\phi} f(x) d\nu_\infty(x) \\
&= -2 \int_{M_\infty} < d\left(e^{\alpha\phi(x)} H_t^{\alpha,\phi} f(x)\right), d\left(e^{-\alpha\phi(x)} H_t^{\alpha,\phi} f(x)\right) > \\
&= 2\left[\alpha^2 \int_{M_\infty} |d\phi|^2 |H_t^{\alpha,\phi} f|^2 - \int_{M_\infty} |H_t^{\alpha,\phi} f|^2\right] \leq 2\alpha^2 u(t).
\end{aligned}$$

Hence $u(t) \leq e^{2\alpha^2 t} u(0)$, note $u(0) = \left|f\right|_{L^2}^2$, we get

$$\left|H_t^{\alpha,\phi} f\right|_{L^2}^2 \leq e^{2\alpha^2 t} \left|f\right|_{L^2}^2.$$

The conclusion follows from the above inequality.   q.e.d.

Proof of Theorem 6.6: Fix x, $y \in M_\infty$, and r_1, $r_2 > 0$. Let χ_1 (respectively χ_2) be the function equal to 1 on $B_1 = B_\infty(x, r_1)$ (respectively $B_2 = B_\infty(y, r_2)$) and equal to 0 otherwise. Then

$$\int_{B_1} \int_{B_2} p_\infty(\xi, \zeta, t) e^{-\alpha(\phi(\xi)-\phi(\zeta))} d\zeta d\xi = \int_{M_\infty} \chi_1(\xi)\left(H_t^{\alpha,\phi} \chi_2\right)(\xi) d\xi$$

$$\leq \|H_t^{\alpha,\phi}\| \cdot \|\chi_1\|_{L^2} \cdot \|\chi_2\|_{L^2} \leq e^{\alpha^2 t} \nu_\infty(B_1)^{\frac{1}{2}} \nu_\infty(B_2)^{\frac{1}{2}}.$$

Using $|d\phi| \leq 1$, we get

$$\int_{B_1} \int_{B_2} p_\infty(\xi, \zeta, t) d\zeta d\xi$$

$$\leq \int_{B_1} \int_{B_2} p_\infty e^{-\alpha\left[(\phi(\xi)-\phi(x))-(\phi(\zeta)-\phi(y))\right]} \cdot e^{|\alpha|(r_1+r_2)}$$

$$\leq \left[\nu_\infty(B_1)\nu_\infty(B_2)\right]^{\frac{1}{2}} \exp\{\alpha^2 t + \alpha[\phi(x) - \phi(y)] + |\alpha|(r_1 + r_2)\}.$$

As $p_\infty(x, \cdot, t)$ is a Dirichlet solution of heat equation in $(0, \infty) \times M_\infty$, assume $t \geq \frac{1}{4} r_2^2$ and applying Theorem 5.5, we obtain

$$p_\infty(\xi, y, t) \leq \frac{C(n)}{r_2^2 \nu_\infty(B_2)} \int_{t - \frac{1}{4} r_2^2}^t \int_{B_2} p_\infty(\xi, \zeta, s) d\zeta ds.$$

Thus

$$\int_{B_1} p_\infty(\xi, y, t) d\xi$$

$$\leq \frac{C(n) \nu_\infty(B_1)^{\frac{1}{2}}}{\nu_\infty(B_2)^{\frac{1}{2}}} \cdot \exp\left\{\alpha^2 t + \alpha[\phi(x) - \phi(y)] + |\alpha|(r_1 + r_2)\right\}.$$

Assume $t \geq \frac{1}{4}(r_1^2 + r_2^2)$, by Theorem 5.5 again, combining with the above inequality, we get

$$p_\infty(x, y, t) \leq \frac{C(n)}{r_1^2 \nu_\infty(B_1)} \int_{t - \frac{1}{4} r_1^2}^t \int_{B_1} p_\infty(\xi, y, s) d\xi ds$$

$$\leq \frac{C(n)}{\left[\nu_\infty(B_1) \nu_\infty(B_2)\right]^{\frac{1}{2}}} \exp\left\{\alpha^2 t + \alpha[\phi(x) - \phi(y)] + |\alpha|(r_1 + r_2)\right\}.$$

Taking $\alpha = \frac{\phi(y) - \phi(x)}{2t}$, $r_1 = r_2 = \frac{t}{\sqrt{t} + \rho_\infty(x, y)}$, we obtain

(6.8)

$$p_\infty \leq \frac{C(n)}{\left[\nu_\infty(B_1) \nu_\infty(B_2)\right]^{\frac{1}{2}}} \cdot \exp\left\{-\frac{\left(\phi(x) - \phi(y)\right)^2}{4t} + \frac{|\phi(x) - \phi(y)|}{\sqrt{t} + \rho_\infty(x, y)}\right\}.$$

Choosing $\phi(\cdot) = \rho_\infty(x, \cdot)$ in (6.8) gives

$$p_\infty(x, y, t) \leq \frac{C(n)}{\left[\nu_\infty(B_1) \nu_\infty(B_2)\right]^{\frac{1}{2}}} \exp\left\{-\frac{\rho_\infty^2(x, y)}{4t}\right\}$$

$$\leq \frac{C(n)\left(1 + \frac{\rho_\infty(x, y)}{\sqrt{t}}\right)^{\frac{n}{2}}}{\left[\nu_\infty\left(B_\infty(x, \sqrt{t})\right) \nu_\infty\left(B_\infty(y, \sqrt{t})\right)\right]^{\frac{1}{2}}} \exp\left\{-\frac{\rho_\infty^2(x, y)}{4t}\right\}$$

$$\leq \frac{C(n)}{\nu_\infty\left(B_\infty(y, \sqrt{t})\right)} \exp\left\{-\frac{\rho_\infty^2(x, y)}{5t}\right\}.$$

The conclusion is proved.   q.e.d.

Corollary 6.8. *For positive constant $T > 0$, there exists a positive constant $\epsilon(n, T, R)$ with $\lim_{R \to \infty} \epsilon(n, T, R) = 0$ such that for $t \in (0, T]$:*

(6.9)
$$\int_{M_\infty \setminus B_\infty(R)} p_\infty(x, y, t) d\nu_\infty(x) \leq \epsilon(n, T, R).$$

Proof: From (6.6) and Property $(\mathscr{B})$ on $(M_\infty, \rho_\infty, \nu_\infty)$, we get

$$
\int_{M_\infty \backslash B_\infty(R)} p_\infty(x, y, t) d\nu_\infty(x)
$$

$$
\leq C(n) \int_{M_\infty \backslash B_\infty(R)} \nu_\infty\left(B_\infty(\sqrt{t})\right)^{-1} e^{-\frac{\rho_\infty^2(x,y)}{5t}} d\nu_\infty
$$

$$
\leq \frac{C(n)}{\nu_\infty\left(B_\infty(\sqrt{t})\right)} \sum_{k=0}^{\infty} \int_{B_\infty(2^{k+1}R) \backslash B_\infty(2^k R)} e^{-\frac{\rho_\infty^2(x,y)}{5t}} d\nu_\infty
$$

$$
\leq \frac{C(n)}{\nu_\infty\left(B_\infty(\sqrt{t})\right)} \sum_{k=0}^{\infty} e^{-\frac{(2^k R)^2}{5t}} \cdot (2^k R)^n
$$

$$
\leq C(n, T) \sum_{k=0}^{\infty} e^{-\frac{1}{5}\left(\frac{2^k R}{\sqrt{t}}\right)^2} \cdot \left(\frac{2^k R}{\sqrt{t}}\right)^n.
$$

Without loss of generality, we can assume $R \geq \sqrt{T}$. Then from the above,

$$
\int_{M_\infty \backslash B_\infty(R)} p_\infty(x, y, t) d\nu_\infty(x) \leq C(n, T) \int_{\frac{R}{\sqrt{T}}}^{\infty} e^{-\frac{1}{5}s^2} s^n ds \leq \epsilon(n, T, R).
$$

q.e.d.

7. The convergence of heat kernels in the Gromov-Hausdorff sense

In this section, we will prove one main theorem of this paper, Theorem 1.4. The eigenfunction expansion of heat kernel and Proposition 4.5 provides the bridge between local Dirichlet heat kernels on bounded regions of M_i and M_∞. Combined with Gaussian upper bounds of heat kernels on M_i, M_∞, maximum principle leads to the convergence of local Dirichlet heat kernel to global Dirichlet heat kernel on M_i, M_∞. From all these, the hear kernels' convergence in the Gromov-Hausdorff sense is proved.

Lemma 7.1. *For positive constant $T > 0$, there exists $\epsilon(n, T, R) > 0$ with $\lim_{R \to \infty} \epsilon(n, T, R) = 0$, such that for $t \in (0, T]$:*

$$
(7.1) \qquad \int_{M_i \backslash B_i(R)} H_i(x, y, t) d\nu_i(x) \leq \epsilon(n, T, R).
$$

Proof: Without loss of generality, assume $R \geq \sqrt{T}$, then from $\nu_i = \frac{1}{V(\sqrt{t_i})}\mu$ and (4.14), we get

$$
\int_{M_i \setminus B_i(R)} H_i(x, y, t) d\nu_i(x)
$$

$$
\leq C(n) \int_{M^n \setminus B(\sqrt{t_i}R)} V(\sqrt{t_i t})^{-1} e^{-\frac{d^2}{5t_i t}} d\mu
$$

$$
= \frac{C(n)}{V(\sqrt{t_i t})} \int_{\sqrt{t_i}R}^{\infty} e^{-\frac{r^2}{5t_i t}} A(r) r^{n-1} dr
$$

$$
\leq C(n) \frac{A(\sqrt{t_i T}) \cdot (t_i t)^{\frac{n}{2}}}{V(\sqrt{t_i t})} \cdot \left(\int_{\frac{R}{\sqrt{t}}}^{\infty} e^{-\frac{1}{5}s^2} s^{n-1} ds \right)
$$

$$
\leq C(n) \frac{A(\sqrt{t_i T})}{A(\sqrt{t_i t})} \cdot \left(\int_{\frac{R}{\sqrt{T}}}^{\infty} e^{-\frac{1}{5}s^2} s^{n-1} ds \right)
$$

$$
\leq C(n) \cdot \int_{\frac{R}{\sqrt{T}}}^{\infty} e^{-\frac{1}{5}s^2} s^{n-1} ds \leq \epsilon\left(n, T, R\right)
$$

where $A(r)r^{n-1}$ in the first equality is the surface area element of $\partial B(r)$, in the second inequality above we used the fact that $A(r)$ is non-increasing (from Bishop-Gromov inequality) and $R \geq \sqrt{T}$; the third inequality from the end follows from the fact $V(\sqrt{t_i t}) \geq \frac{1}{n} A(\sqrt{t_i t})(t_i t)^{\frac{n}{2}}$.

<div align="right">q.e.d.</div>

Proposition 7.2.

$$
(7.2) \qquad \lim_{R \to \infty} H_{R,i}(\cdot, y, t) = H_i(\cdot, y, t).
$$

The convergence is uniform on $x \in M_i^n$, $i = 1, 2, \ldots$, and uniform in $L^1(\nu_i)$ on any finite time interval $(0, T]$.

Proof: Assume $R \geq 1$, put

$$
(7.3) \qquad M_{R,i} \doteqdot \sup\{H_i(x, y, t) \mid x \in \partial B_i(R), \ 0 < t \leq T\}.
$$

By (4.14) and Volume Comparison Theorem, we have

$$
M_{R,i} \leq \sup_{0 < t \leq T} C(n) V(\sqrt{t_i}) V(\sqrt{t_i t})^{-1} e^{-\frac{R^2}{5t}}
$$

$$
(7.4) \qquad \leq C(n) \cdot \max\left\{ e^{-\frac{R^2}{5T}}, \ \sup_{0 < t \leq 1} t^{-\frac{n}{2}} e^{-\frac{R^2}{5t}} \right\}
$$

$$
(7.5) \qquad \leq C(n) \max\left\{ e^{-\frac{R^2}{5T}}, \ R^{-n} \right\}.
$$

By the maximum principle, when $x \in B_i(R)$,

$$
(7.6) \qquad H_i(x, y, t) - M_{R,i} \leq H_{R,i}(x, y, t) \leq H_i(x, y, t).
$$

From (7.5) and (7.6), we get $\lim\limits_{R\to\infty} H_{R,i}(\cdot,y,t) = H_i(\cdot,y,t)$ uniformly on $(0,T]\times B_i(R)$, $i=1,2,3,\cdots$. Combining with (4.14), we get that the convergence is uniform on $(0,T]\times M_i^n$ and $i=1,2,3,\cdots$.

From (7.4) and Volume Comparison Theorem, we get

$$\lim_{R\to\infty} M_{R,i}\nu_i\Big(B_i(R)\Big) \le \lim_{R\to\infty} C(n)R^n \cdot \max\left\{e^{\frac{R^2}{5T}},\ \sup_{0<t\le 1} t^{-\frac{n}{2}}e^{-\frac{R^2}{5t}}\right\}$$

$$(7.7)\qquad\qquad \le \lim_{R\to\infty} C(n)\max\left\{R^n e^{-\frac{R^2}{5T}},\ \sup_{s\ge R^2} s^{\frac{n}{2}}e^{-\frac{s}{5}}\right\} = 0.$$

Combining (7.6), (7.7) with Lemma 7.1, we have

$$(7.8)\qquad\qquad \|H_{R,i}(\cdot,y,t) - H_i(\cdot,y,t)\|_{L^1(\nu_i)} \le \epsilon(n,T,R)$$

and $\lim\limits_{R\to\infty} \epsilon(n,T,R) = 0$. $\hfill$ q.e.d.

By Lemma 4.1, Lemma 4.2 and Proposition 4.5, we can assume, after passing to a subsequence of $\{i\}_{i=1}^\infty$, that for every j, eigenvalue and eigenfunction converge:

$$(7.9)\qquad\qquad \lim_{i\to\infty} \lambda_{j,i}^{(R)} = \lambda_{j,\infty}^{(R)},\quad \lim_{i\to\infty} \phi_{j,i}^{(R)} = \phi_{j,\infty}^{(R)}.$$

Theorem 7.3.

$$(7.10)\qquad\qquad H_{R,\infty}(x,y,t) \doteq \sum_{j=1}^\infty e^{-\lambda_{j,\infty}^{(R)}t}\phi_{j,\infty}^{(R)}(x)\phi_{j,\infty}^{(R)}(y)$$

is well defined on $B_\infty(R)\times B_\infty(R)\times (0,\infty)$, where $R>2$. And

$$(7.11)\qquad\qquad \lim_{i\to\infty} H_{R,i}(\cdot,y,t) = H_{R,\infty}(\cdot,y,t)$$

where the convergence is in $L^2\big(B_\infty(R)\big)$, and is also locally uniform on $\mathring{B}_\infty(R)$. Furthermore, $H_{R,\infty}(\cdot,y,t)$ is locally Lipschitz on $\mathring{B}_\infty(R)$.

Remark 7.4. $H_{R,\infty}$ may depend on the choice of subsequence of $\{M_i^n\}$.

Proof: By (4.11) and (7.9), we get

$$(7.12)\qquad\qquad \|\phi_{j,\infty}^{(R)}\|_{L^\infty(\nu_\infty)} \le C(n)R^{\frac{n}{2}}\Big(\lambda_{j,\infty}^{(R)}\Big)^{\frac{n}{4}}.$$

Using (7.12), when $t\in[t_0,\infty)$, $t_0>0$ is any positive constant, we can obtain

$$\|e^{-\lambda_{j,\infty}^{(R)}t}\phi_{j,\infty}^{(R)}(x)\phi_{j,\infty}^{(R)}(y)\|_{L^\infty(\nu_\infty)}$$

$$\le e^{-\lambda_{j,\infty}^{(R)}t}\|\phi_{j,\infty}^{(R)}\|_{L^\infty(\nu_\infty)}^2$$

$$(7.13)\qquad \le C(n,R)e^{-\lambda_{j,\infty}^{(R)}t}\Big(\lambda_{j,\infty}^{(R)}\Big)^{\frac{n}{2}} \le C(n,R,t_0)e^{-\frac{\lambda_{j,\infty}^{(R)}t}{2}}.$$

Applying (7.9) and Lemma 4.1, we conclude that

$$\sum_{j=1}^{\infty} \left| e^{-\lambda_{j,\infty}^{(R)} t} \phi_{j,\infty}^{(R)}(x) \phi_{j,\infty}^{(R)}(y) \right| \leq C(n, R, t_0) \sum_{j=1}^{\infty} e^{-\left[C(n,R) j^{\frac{1}{n}} t \right]}$$

which clearly converges uniformly on $B_\infty(R) \times B_\infty(R) \times [t_0, \infty)$ for any $t_0 > 0$. Hence the kernel $H_{R,\infty}(x, y, t)$ is well defined and locally Lipschitz on $\mathring{B}_\infty(R)$.

Similar as (7.13), it is easy to see

$$(7.14) \qquad \| e^{-\lambda_{j,i}^{(R)} t} \phi_{j,i}^{(R)}(x) \phi_{j,i}^{(R)}(y) \|_{L^\infty(\nu_i)} \leq C(n, R, t_0) e^{-\left[C(n,R) j^{\frac{1}{n}} t \right]}$$

when $t \in [t_0, \infty)$. Then (7.11) follows from (4.15), (7.10), (7.13), (7.14) and Proposition 4.5.   q.e.d.

Fix one increasing sequence $R_k \to \infty$, by a diagonal argument, we can choose one subsequence of $\{M_i^n\}$, also denoted as $\{M_i^n\}$, such that for each k, $H_{R_k,i} \to H_{R_k,\infty}$ in $L^2\big(B_\infty(R_k)\big)$ and also locally uniform on $\mathring{B}_\infty(R_k)$.

On M_i^n, for $R_j < R_k$, we have

$$(7.15) \quad H_{R_j,i}(x, y, t) \leq H_{R_k,i}(x, y, t) \leq H_i(x, y, t) \leq \frac{C(n)}{\nu_i(\sqrt{t})} e^{-\frac{d_{\rho_i}^2(x,y)}{5t}}$$

where $\nu_i(\sqrt{t}) \doteq \nu_i\big(B_i(\sqrt{t})\big)$. Taking $i \to \infty$ in (7.15), we get

$$(7.16) \qquad 0 \leq H_{R_j,\infty}(x, y, t) \leq H_{R_k,\infty}(x, y, t) \leq \frac{C(n)}{\nu_\infty(\sqrt{t})} e^{-\frac{d_{\rho_\infty}^2(x,y)}{5t}}$$

where $\nu_\infty(\sqrt{t}) = \nu_\infty\big(B_\infty(\sqrt{t})\big)$. Thus we can get that the non-decreasing sequence $H_{R_j,\infty}$ converges pointwise to some function H_∞:

$$(7.17) \qquad H_\infty(x, y, t) = \lim_{k \to \infty} H_{R_k,\infty}(x, y, t) = \lim_{k \to \infty} \lim_{i \to \infty} H_{R_k,i}(x_i, y, t)$$

for some subsequence of $\{M_i^n\}_{i=1}^\infty$, $\{R_k\}_{k=1}^\infty$ and any $x_i \to x$.

Proposition 7.5. $H_{R,\infty}$ is a Dirichlet solution of the heat equation

$$(7.18) \qquad \begin{cases} \left(\frac{\partial}{\partial t} - \Delta_\infty \right) H_{R,\infty} & = 0 \\[2mm] \lim_{t \to 0} H_{R,\infty}(x, y, t) & = \delta_y(x) \end{cases}$$

on $B_\infty(R) \subset \Big(M_\infty, y, \rho_\infty, \nu_\infty \Big)$.

Proof: By Lemma 4.1, 4.2, 4.4 and Proposition 4.5, we have

$$(7.19) \qquad \lim_{k \to \infty} \sum_{j=k}^{\infty} \left| e^{-\lambda_j t} \big[d\phi_{j,\infty}(x) \big] \phi_{j,\infty}(y) \right| = 0 , \qquad x \in \mathring{B}_\infty(R).$$

Hence $H_{R,\infty}$ is a Dirichlet solution of the heat equation by directly checking that (5.2) holds for it.

From (7.11), (7.16) and the definition of $H_{R,i}$, using the similar argument as in Lemma 7.1, we get

$$(7.20) \qquad \lim_{t \to 0} \int_{M_\infty} H_{R,\infty}(x,y,t) f(x) d\nu_\infty(x) = f(y)$$

where f is any Lipschitz function with compact support on M_∞. q.e.d.

Proposition 7.6.

$$(7.21) \qquad \lim_{R \to \infty} H_{R,\infty}(\cdot,y,t) = p_\infty(\cdot,y,t).$$

The convergence is uniform on $x \in M_\infty$, and uniform in $L^1(\nu_\infty)$ on any finite time interval $(0,T]$.

Proof: Assume $R \geq 1$, put

$$(7.22) \qquad M_{R,\infty} \doteqdot \sup\{p_\infty(x,y,t)|\ x \in \partial B_\infty(R),\ 0 < t \leq T\}.$$

By (6.6) and Property $(\mathscr{B})$ on M_∞ (from Proposition 2.7), we have

$$M_{R,\infty} \leq \sup_{0<t\leq T} C(n)\nu_\infty\Big(B_\infty(\sqrt{t})\Big)^{-1} e^{-\frac{R^2}{5t}}$$

$$\leq C \cdot \max\Big\{e^{-\frac{R^2}{5T}},\ \sup_{0<t\leq 1} \nu_\infty\big(B_\infty(\sqrt{t})\big) e^{-\frac{R^2}{5t}}\Big\}$$

$$(7.23) \qquad \leq C \cdot \max\Big\{e^{-\frac{R^2}{5T}},\ \sup_{0<t\leq 1} t^{-\frac{n}{2}} e^{-\frac{R^2}{5t}}\Big\}$$

$$(7.24) \qquad \leq C(n) \max\Big\{e^{-\frac{R^2}{5T}},\ R^{-n}\Big\}.$$

From Proposition 7.5 and comparison inequalities for heat kernels on metric measure spaces (see Proposition 4.1 in [GHL10]), we get

$$(7.25) \qquad p_\infty(x,y,t) - M_{R,\infty} \leq H_{R,\infty}(x,y,t) \leq p_\infty(x,y,t).$$

From (7.24) and (7.25), $\lim_{R\to\infty} H_{R,\infty}(\cdot,y,t) = p_\infty(\cdot,y,t)$ uniformly on $B_\infty(R)$. Combining with (6.6), the convergence is uniform on $(0,T] \times M_\infty$.

From (7.23) and Property $(\mathscr{B})$, note $\nu_\infty\big(B_\infty(1)\big) = 1$, we get

$$\lim_{R\to\infty} M_{R,\infty}\nu_\infty\Big(B_\infty(R)\Big) \leq \lim_{R\to\infty} C(n)R^n \cdot \max\Big\{e^{\frac{R^2}{5T}},\ \sup_{0<t\leq 1} t^{-\frac{n}{2}} e^{-\frac{R^2}{5t}}\Big\}$$

$$(7.26) \qquad \leq \lim_{R\to\infty} C(n) \max\Big\{R^n e^{-\frac{R^2}{5T}},\ \sup_{s\geq R^2} s^{\frac{n}{2}} e^{-\frac{s}{5}}\Big\} = 0.$$

Combining (7.26) with Corollary 6.8, we have

$$(7.27) \qquad \|H_{R,\infty}(\cdot,y,t) - p_\infty(\cdot,y,t)\|_{L^1(\nu_\infty)} \leq \epsilon(n,T,R)$$

and $\lim_{R\to\infty} \epsilon(n,T,R) = 0$.   q.e.d.

Proposition 7.7. *Assume $x_i \to x$ as $(M_i, y, \rho_i, \nu_i) \xrightarrow{d_{GH}} (M_\infty, y_\infty, \rho_\infty, \nu_\infty)$, then*

$$(7.28) \qquad \lim_{i \to \infty} H_i(x_i, y, t) = p_\infty(x, y, t) \qquad t \in (0, \infty).$$

The convergence is locally uniform on M_∞.

Remark 7.8. H_∞ in (7.17) is equal to p_∞ in (7.28).

Proof: For any sequence $(M_i, y, \rho_i, \nu_i) \xrightarrow{d_{GH}} (M_\infty, y_\infty, \rho_\infty, \nu_\infty)$, we can get a subsequence of $\{M_i^n\}$ as before, also denoted as $\{M_i^n\}$, such that, there exists increasing sequence $R_k \to \infty$, and

$$\lim_{i \to \infty} H_{R_k, i}(\cdot, y, t) = H_{R_k, \infty}(\cdot, y, t) \qquad k = 1, 2, 3, \cdots$$

where the convergence is as in Theorem 7.3.

Then

$$|H_i(x_i, y, t) - p_\infty(x, y, t)| \leq \Big(|H_i(x_i, y, t) - H_{R_k, i}(x_i, y, t)|$$

$$(7.29)$$

$$+ |H_{R_k, \infty}(x, y, t) - p_\infty(x, y, t)| \Big) + |H_{R_k, i}(x_i, y, t) - H_{R_k, \infty}(x, y, t)|.$$

For any $\epsilon > 0$, from Proposition 7.2 and 7.6, we get the first two terms on the right side of (7.29) will be less than $\frac{1}{3}\epsilon$ when k is big enough. Now fixed k such that $x \in \mathring{B}_\infty(R_k)$ and

$$\Big(|H_i(x_i, y, t) - H_{R_k, i}(x_i, y, t)| + |H_{R_k, \infty}(x, y, t) - p_\infty(x, y, t)| \Big) < \frac{2}{3}\epsilon.$$

Using Theorem 7.3, if i is big enough (which may depend on k we chose above), then we get

$$|H_{R_k, i}(x_i, y, t) - H_{R_k, \infty}(x, y, t)| < \frac{1}{3}\epsilon.$$

By the above argument, we get that for such subsequence of $\{M_i^n\}$,

$$\lim_{i \to \infty} H_i(x_i, y, t) = p_\infty(x, y, t).$$

However, any subsequence of $\{M_i^n\}$ must contain a subsequence whose limit is also p_∞ by the above argument. Hence, in fact we prove that for the original sequence $\{M_i^n\}$, (7.28) holds. q.e.d.

Proof of Theorem 1.4: From (4.12), (7.28) and $x \to y_\infty$ as $i \to \infty$ for any $x \in M_i$. q.e.d.

8. Analysis on manifolds with cone structures at infinity

In this section we will discuss large time behavior of the heat kernel on manifolds with cone structures at infinity (see Definition below), and prove Theorem 1.6.

Definition 8.1. Assume that (M^n, g) is a complete manifold with $Rc \geq 0$, y is some fixed point on M^n, and for any $t_i \to \infty$, define (M_i, y, ρ_i, ν_i) as in **Blow Down Setup**, such that

$$(8.1) \qquad (M_i, y, \rho_i, \nu_i) \xrightarrow{d_{GH}} (M_\infty, y_\infty, \rho_\infty, \nu_\infty).$$

If $(M_\infty, y_\infty, \rho_\infty)$ (may be different for different choice of $\{t_i\}$) always has the cone structure, i.e.

$$(8.2) \qquad \rho_\infty = dr^2 + l(r)^2 dX$$

where X is some compact metric space, $l(r) > 0$ is some function of r, then we say that M^n is a **manifold with cone structures at infinity**.

Proof of Theorem 1.6: Assume that $s_i \to \infty$, blowing down the metric g by s_i^{-1} instead of t_i^{-1}, define (M_i, y, ρ_i, ν_i) as in **Blow Down Setup**, and the following holds:

$$(M_i, y, \rho_i, \nu_i) \xrightarrow{d_{GH}} (M_\infty, y_\infty, \rho_\infty, \nu_\infty).$$

From (1.4) and (1.6), it is easy to get $\nu_\infty(B_\infty(y_\infty, r)) = h(r)$. By the assumption that M^n is a complete manifold with cone structures at infinity, we get that the heat kernel p_∞ on $(M_\infty, y_\infty, \rho_\infty, \nu_\infty)$, only depends on $r = \rho(x, y_\infty)$ and t, denoted as $p_\infty(r, t)$.

It is easy to get

$$\Delta p_\infty(r, t) = \frac{\partial^2 p_\infty}{\partial r^2} + \left(\frac{h''(r)}{h'(r)} \right) \cdot \frac{\partial p_\infty}{\partial r}.$$

Hence $p_\infty(r, t)$ is the unique positive solution of

$$\begin{cases} \frac{\partial p_\infty}{\partial t} = (p_\infty)_{rr} + \left(\frac{h''}{h'} \right)(p_\infty)_r \\ \\ \lim_{t \to 0} p_\infty(r, t) = \delta_{y_\infty}(x). \end{cases}$$

From the above, it is easy to see that $p_\infty(r, t)$ is uniquely determined by $\left(\frac{h''}{h'} \right)(r)$. The conclusion follows from Theorem 1.4, the above argument and (1.7). q.e.d.

Remark 8.2. Note (1.7) is equivalent to the assumption that $\frac{h'(r)}{\overline{h}'(r)}$ is a constant independent of r. Although the tangent cones at infinity of manifold M^n may be different metric measure spaces for different choices of s_i, p_∞ only depends on the function $h'(r)$ when the tangent cone at infinity $(M_\infty, y_\infty, \rho_\infty)$ has the cone structure as in (8.2).

Theorem 8.3. *Assume that (M^n, g) is a complete manifold with nonnegative sectional curvature, $n \geq 3$, y is some fixed point on M^n, and for any $r > 0$,*

$$(8.3) \qquad \lim_{s \to \infty} \frac{V_y(sr)}{V_y(s)} = h(r)$$

where $h(r) > 0$ is some positive function.

*Then there exists a unique $(M_\infty, y_\infty, \rho_\infty, \nu_\infty)$, where $\nu_\infty(B_\infty(y_\infty, r)) = h(r)$, such that for any $t_i \to \infty$, define (M_i, y, ρ_i, ν_i) as in **Blow Down Setup**, we have*

$$(8.4) \qquad (M_i, y, \rho_i, \nu_i) \overset{d_{GH}}{\longrightarrow} (M_\infty, y_\infty, \rho_\infty, \nu_\infty)$$

$$(8.5) \qquad \lim_{t \to \infty} V_y(\sqrt{t}) \cdot H(x, y, t) = p_\infty(y_\infty, y_\infty, 1)$$

where p_∞ is the heat kernel on $(M_\infty, y_\infty, \rho_\infty, \nu_\infty)$.

Proof: Because M^n has non-negative sectional curvature, from Theorem $I.26$ in [CCG$^+$10], we know that the tangent cone at infinity $(M_\infty, y_\infty, \rho_\infty)$ is the unique metric cone. Hence M^n is a manifold with cone structures at infinity and (8.4) is obtained.

From the assumption (8.3) and the above argument, we can apply Theorem 1.6, (8.5) is obtained. q.e.d.

As an application of the above theorem, we have the following interesting result about nonnegatively curved manifolds with asymptotic polynomial volume growth.

Corollary 8.4. *Assume that (M^n, g) is a complete manifold with nonnegative sectional curvature, $n \geq 3$ and it has asymptotic polynomial volume growth, i.e.*

$$\lim_{r \to \infty} \frac{V(r)}{r^k} = C_0$$

where $k \geq 1$ and $C_0 > 0$ are constants. Then (8.5) holds.

Proof: The proof follows directly from Theorem 8.3. q.e.d.

9. Example with $\underline{\lim}_{t \to \infty} V(\sqrt{t})H(x, y, t) < \overline{\lim}_{t \to \infty} V(\sqrt{t})H(x, y, t)$

In this section we will construct the first example, which is a complete manifold with nonnegative Ricci curvature and

$$\underline{\lim_{t \to \infty}} V(\sqrt{t})H(x, y, t) < \overline{\lim_{t \to \infty}} V(\sqrt{t})H(x, y, t).$$

From Theorem 1.4, the example should have different tangent cones at infinity of the manifold with renormalized measure. Furthermore, from Theorem 1.6 and its proof, if two tangent cones at infinity of (M^n, g) have the cone structure as defined in Definition 8.1, **only different renormalized measure** will result in the inconsistent limit behavior

of heat kernel. Note in this context, if there exists $r > 0$, such that $\nu_\infty\big(B_\infty(r)\big) \neq \tilde{\nu}_\infty\big(\tilde{B}_\infty(r)\big)$, where $B_\infty(r) \subset M_\infty = C(X)$ and $\tilde{B}_\infty(r) \subset \tilde{M}_\infty = C(\tilde{X})$ are two balls with the same radius r in different metric tangent cones $C(X)$, $C(\tilde{X})$; we say that the renormalized measures ν_∞, $\tilde{\nu}_\infty$ are different.

Hence, the different structure of tangent cones at infinity alone can not guarantee the inequality the inconsistent limit behavior of heat kernel. As mentioned in the introduction of this paper, Perelman ([Per97]) had constructed the manifold with $Rc \geq 0$, maximal volume growth and quadratic curvature decay, where the tangent cone at infinity is not unique. However it is not hard to see that the renormalized measure on those different tangent cones (in fact, metric cones) are the same, so will not lead to inconsistent limit behavior of heat kernel on such manifolds.

In fact, from Theorem 1.1, the example manifold must be collapsing case. The construction of the following example is inspired by the related discussion in Section 8 of [CC97]. However, we need to do some suitable modifications to assure the different renormalized measure on different tangent cones at infinity of manifold.

Let us start from the generalized Hopf fibration of $\mathbb{S}^7$ as the following:

$$\mathbb{S}^3 \longrightarrow \mathbb{S}^7 \xrightarrow{\pi} \mathbb{S}^4 \ , \quad g^{\mathbb{S}^7} = k_1 + k_2$$

where $\mathbb{S}^3$, $\mathbb{S}^7$, $\mathbb{S}^4$ carry the metrics $g^{\mathbb{S}^3}$, $g^{\mathbb{S}^7}$, $\frac{1}{4}g^{\mathbb{S}^4}$; π is a Riemannian submersion with totally geodesic fibers and $k_1 = g^{\mathbb{S}^3}$, $k_2 = \pi^*\big(\frac{1}{4}g^{\mathbb{S}^4}\big)$; $g^{\mathbb{S}^n}$ denotes the canonical metric of curvature $\equiv 1$ on $\mathbb{S}^n$.

Define $\tilde{g} = f^2 k_1 + h^2 k_2$, then the following formulas are well-known (for example, see Section 2 in [BKN12]):

$$(9.1) \qquad Rc(\tilde{g})|_{k_1} = \left(\frac{2}{f^2} + \frac{4f^2}{h^4}\right)I \ , \quad Rc(\tilde{g})|_{k_2} = \frac{6(2h^2 - f^2)}{h^4}I.$$

Other mixed $Rc(\tilde{g}) = 0$.

Then for metric $g = dr^2 + f^2(r)k_1 + h^2(r)k_2$ on M^8, which is diffeomorphic to $\mathbb{R}^8$, from (8.13) in [CC97] and (9.1), we have

$$(9.2) \qquad Rc(g)|_{k_1} = \frac{2\big(1 - (f')^2\big)}{f^2} - \frac{f''}{f} + \frac{4f^2}{h^4} - 4\frac{f'h'}{fh}$$

$$(9.3) \qquad Rc(g)|_{k_2} = \frac{6(2h^2 - f^2)}{h^4} - \frac{h''}{h} - 3\frac{(h')^2}{h^2} - 3\frac{f'h'}{fh}$$

$$(9.4) \qquad Rc(g)(\vec{n}, \vec{n}) = -\left[3\frac{f''}{f} + 4\frac{h''}{h}\right].$$

Our construction will be broken into four steps in subsections 9.1-9.4 separately, we will verify that our example (M^8, g) has the property $\varliminf_{t \to \infty} H(x, y, t) < \varlimsup_{t \to \infty} H(x, y, t)$ in subsection 9.4.

9.1. Step (I).
Initial approximation $\bar{f}$, $\bar{h}$ to the functions f, h will be constructed inductively at this stage. These approximations have jump discontinuities at the points b_i; see (9.29), (9.30), (9.31), (9.32). However, the left- and right-hand limits of the first derivatives do agree at all b_i, $i \geq 1$, see (9.11) and (9.19).

We can define $\bar{f}(r)$ as the following:

(9.5)
$$\bar{f}(r) = \begin{cases} \beta_{2i} b_{2i+1}^{-\omega_{2i}} r^{1-\eta_1} & r \in (b_{2i}, b_{2i+1}] \\ \\ \beta_{2i+1} b_{2i+2}^{\omega_{2i+1}} r^{1-\eta_2} & r \in (b_{2i+1}, b_{2i+2}] \end{cases}$$

where for $i = 0, 1, 2, \cdots$ we have:

Assumption 1.

(9.6) $1 - \epsilon_0 > \dfrac{1-\eta_2}{1-\eta_1} \geq \dfrac{99}{100}$, $\quad 1 > \eta_2 > \eta_1 > \dfrac{1}{2}(1+\epsilon_0)$

(9.7) $\beta_{2i+2} > \beta_{2i} > 0$, $\quad \beta_{2i+1} > \beta_{2i+3} > 0$, $\quad \beta_0 \geq \dfrac{99}{100}$, $\quad \beta_1 \leq \dfrac{1}{100}$

(9.8) $\lim\limits_{i\to\infty} \beta_{2i} = 1$, $\quad \lim\limits_{i\to\infty} \beta_{2i+1} = 0$

β_i, ω_i, ϵ_0 are positive constants to be determined later and satisfy

Assumption 2. $\lim\limits_{i\to\infty} \omega_i = 0$, $\quad \dfrac{\eta_2 - \eta_1}{100} > \omega_0 > \omega_1 > \omega_2 > \cdots$

We have the following equations for $i = 0, 1, 2, \cdots$

Assumption 3.

(9.9)
$$\frac{1-\eta_1}{1-\eta_2} \cdot \frac{\beta_{2i}}{\beta_{2i+1}} = \frac{b_{2i+2}^{\omega_{2i+1}}}{b_{2i+1}^{\eta_2-\eta_1-\omega_{2i}}}$$

(9.10)
$$\frac{1-\eta_2}{1-\eta_1} \cdot \frac{\beta_{2i+1}}{\beta_{2i+2}} = \frac{b_{2i+2}^{\eta_2-\eta_1-\omega_{2i+1}}}{b_{2i+3}^{\omega_{2i+2}}}$$

which implies that for $i \geq 1$,

(9.11)
$$\bar{f}'(b_i) = \lim_{r\to b_i^+} \bar{f}'(r) = \lim_{r\to b_i^-} \bar{f}'(r).$$

We define $\bar{h}(r)$ in the following way:

(9.12)
$$\bar{h}(r) = \begin{cases} \alpha_{2i} b_{2i+1}^{-\epsilon_{2i}} r^{1+\epsilon_{2i}} & r \in (b_{2i}, b_{2i+1}] \\ \\ \alpha_{2i+1} b_{2i+2}^{\epsilon_{2i+1}} r^{1-\epsilon_{2i+1}} & r \in (b_{2i+1}, b_{2i+2}] \end{cases}$$

where for $i = 0, 1, 2, \cdots$, we have

Assumption 4.

(9.13) $\quad \alpha_{2i+2} > \alpha_{2i} > 0$, $\quad \alpha_{2i+1} > \alpha_{2i+3} > 0$

(9.14) $\quad \lim\limits_{i\to\infty} \alpha_{2i} = 1$, $\quad \alpha_0 \geq \dfrac{99}{100}$, $\quad \lim\limits_{i\to\infty} \alpha_{2i+1} = 0$, $\quad \alpha_1 \leq \dfrac{1}{100}$

ϵ_i, b_i are to be determined later, and satisfies

Assumption 5.

(9.15) $1 < b_0 < b_1 < b_2 < \cdots$, $\lim_{i \to \infty} b_i = \infty$

(9.16) $1 > \epsilon_0 > \epsilon_1 > \epsilon_2 > \cdots$, $\lim_{i \to \infty} \epsilon_i = 0$

We also have the following equations for $i = 0, 1, 2, \cdots$,

Assumption 6.

(9.17) $\dfrac{\alpha_{2i}}{\alpha_{2i+1}} = \left(\dfrac{1 - \epsilon_{2i+1}}{1 + \epsilon_{2i}} \right) \left(\dfrac{b_{2i+2}}{b_{2i+1}} \right)^{\epsilon_{2i+1}}$

(9.18) $\dfrac{\alpha_{2i+2}}{\alpha_{2i+1}} = \left(\dfrac{1 - \epsilon_{2i+1}}{1 + \epsilon_{2i+2}} \right) \left(\dfrac{b_{2i+3}}{b_{2i+2}} \right)^{\epsilon_{2i+2}}$

which implies that for $i \geq 1$,

(9.19) $\bar{h}'(b_i) = \lim_{r \to b_i^+} \bar{h}'(r) = \lim_{r \to b_i^-} \bar{h}'(r).$

In the rest part of **Step (I)**, we will prove $Rc(\bar{f}, \bar{h}) > 0$ on (b_0, ∞) except the points b_i, $i = 1, 2, \cdots$.
(**i**). We firstly consider the interval $(b_0, b_1]$:
If we assume that

Assumption 7. $b_0^{\eta_1} \geq 7$

then

(9.20) $Rc|_{k_1}(\bar{f}, \bar{h}) \geq \dfrac{2}{r^2} \left[\beta_0^{-2} b_1^{2\omega_0} r^{2\eta_1} - (1 - \eta_1)(3 + 2\epsilon_0 - \eta_1) \right] > 0.$

If we assume that

Assumption 8. $\alpha_0 b_0^{\eta_1} > \left(\dfrac{b_1}{b_0} \right)^{\epsilon_0}$

then $\bar{h}(r) > \bar{f}(r)$. And if we further assume that

Assumption 9. $\epsilon_{2i} < \dfrac{1}{2}(\alpha_{2i}^{-1} - 1)$, $i = 0, 1, 2, \cdots$

we have

$Rc|_{k_2}(\bar{f}, \bar{h}) > \dfrac{6}{\bar{h}^2} - \dfrac{\epsilon_0(1 + \epsilon_0)}{r^2} - \dfrac{3(1 + \epsilon_0)^2}{r^2} - \dfrac{3(1 - \eta_1)(1 + \epsilon_0)}{r^2}$

(9.21) $> \dfrac{6}{r^2} \left[\alpha_0^{-2} - (1 + 2\epsilon_0)^2 \right] \geq 0.$

If we assume

Assumption 10. $\epsilon_0 < \dfrac{1}{4} \eta_1 (1 - \eta_1) b_0^{-\eta_1} b_1^{-\omega_0}$

then

$$(9.22) \qquad Rc(\vec{n}, \vec{n}) = \frac{3\eta_1(1 - \eta_1)}{r^2} - 4\frac{\epsilon_0(1 + \epsilon_0)}{r^2} > 0.$$

From (9.20), (9.21) and (9.22), we get $Rc(\bar{f}, \bar{h}) > 0$ on (b_0, b_1).
(ii). Next we consider the interval $(b_{2i}, b_{2i+1}]$, $i \geq 1$.
From Assumption 7 and (9.2), it is easy to get

$$(9.23) \qquad\qquad\qquad Rc|_{k_1} > 0$$

If we assume that

Assumption 11. $b_{2i}^{\eta_1} > \alpha_{2i-1}^{-1}\left(\dfrac{1 + \epsilon_{2i}}{1 - \epsilon_{2i-1}}\right)$, $\quad i = 1, 2, \cdots$

then $\bar{h}(r) > \bar{f}(r)$, from it and Assumption 9, we get
(9.24)

$$Rc|_{k_2} > \frac{6}{\bar{h}^2} - \frac{\epsilon_{2i}(1 + \epsilon_{2i})}{r^2} - \frac{3(1 + \epsilon_{2i})^2}{r^2} - \frac{3(1 - \eta_1)(1 + \epsilon_{2i})}{r^2} > 0.$$

Similarly, from Assumption 10, we get

$$(9.25) \qquad\qquad\qquad Rc(\vec{n}, \vec{n}) > 0$$

From (9.23), (9.24) and (9.25), we get that $Rc(\bar{f}, \bar{h}) > 0$ on (b_{2i}, b_{2i+1}), where $i \geq 1$.
(iii). Finally, we consider the interval $(b_{2i+1}, b_{2i+2}]$, $i \geq 0$.
From Assumption 7, (9.5) and (9.12), it is easy to get

$$(9.26) \qquad Rc|_{k_1} \geq \frac{2}{r^2}\left[\left(\frac{1 - \eta_2}{1 - \eta_1}\right)^2 \beta_{2i}^{-2} b_{2i+1}^{2\eta_1 + 2\omega_{2i}} - 3\right] > 0.$$

From Assumption 7, it is easy to get that $\bar{h}(r) > \bar{f}(r)$. From (9.6) and Assumption 9, we get

$$(9.27) \qquad Rc|_{k_2} > \frac{6}{\bar{h}^2} - \frac{3(1 - \epsilon_{2i+1})^2}{r^2} - 3\frac{(1 - \eta_2)(1 - \epsilon_{2i+1})}{r^2} > 0.$$

From $\bar{f}'' < 0$ and $\bar{h}'' < 0$, it is easy to get

$$(9.28) \qquad\qquad\qquad Rc(\vec{n}, \vec{n}) > 0.$$

From (9.26), (9.27) and (9.28), we get that $Rc(\bar{f}, \bar{h}) > 0$ on (b_{2i+1}, b_{2i+2}), where $i \geq 0$.
From all the above, we get $Rc(\bar{f}, \bar{h}) > 0$ on (b_0, ∞) excepts the points b_i, where $i \geq 1$.
Note $\bar{f}$ has jump discontinuities at the points b_j, $j = 1, 2, \cdots$. For $i = 0, 1, 2, \cdots$, we have

$$(9.29) \quad \tau_{2i+1} \doteq \bar{f}(b_{2i+1}) - \lim_{r \to b_{2i+1}^+} \bar{f}(r) = -\left(\frac{\eta_2 - \eta_1}{1 - \eta_1}\right)\beta_{2i+1} b_{2i+2}^{\omega_{2i+1}} b_{2i+1}^{1 - \eta_2}$$

$$(9.30) \quad \tau_{2i+2} \doteq \bar{f}(b_{2i+2}) - \lim_{r \to b_{2i+2}^+} \bar{f}(r) = \left(\frac{\eta_2 - \eta_1}{1 - \eta_1}\right)\beta_{2i+1} b_{2i+2}^{1 - \eta_2 + \omega_{2i+1}}.$$

Similarly, $\bar{h}$ has jump discontinuities at the points b_j, $j = 1, 2, \cdots$. For $i = 0, 1, 2, \cdots$, we have

$$(9.31) \quad \delta_{2i+1} \doteq \bar{h}(b_{2i+1}) - \lim_{r \to b_{2i+1}^+} \bar{h}(r) = -\alpha_{2i} b_{2i+1} \left(\frac{\epsilon_{2i+1} + \epsilon_{2i}}{1 - \epsilon_{2i+1}} \right)$$

$$(9.32) \quad \delta_{2i+2} \doteq \bar{h}(b_{2i+2}) - \lim_{r \to b_{2i+2}^+} \bar{h}(r) = \alpha_{2i+1} b_{2i+2} \left(\frac{\epsilon_{2i+2} + \epsilon_{2i+1}}{1 + \epsilon_{2i+2}} \right).$$

9.2. Step (II).
We construct $\tilde{f}$, $\tilde{h}$ on the interval $[0, b_0]$ in this step.

Define $\tilde{f}$, $\tilde{h}$ on $[0, b_0]$ as the following:

$$(9.33) \qquad \tilde{f}(r) \doteq \begin{cases} r & r \in [0, \frac{b_0}{2}] \\ r - C_1 \left(r - \frac{b_0}{2} \right)^2 & r \in (\frac{b_0}{2}, b_0] \end{cases}$$

where $C_1 = \frac{1}{b_0} \left[1 - \beta_0 b_1^{-\omega_0} b_0^{-\eta_1} (1 - \eta_1) \right] > 0$.

$$(9.34) \qquad \tilde{h}(r) \doteq \begin{cases} r & r \in [0, \frac{b_0}{2}] \\ r - C_2 \left(r - \frac{b_0}{2} \right)^2 & r \in (\frac{b_0}{2}, b_0] \end{cases}$$

where $C_2 = \frac{1}{b_0} \left[1 - \alpha_0 (1 + \epsilon_0) (\frac{b_0}{b_1})^{\epsilon_0} \right] > 0$.

Then $\tilde{f}(0) = \tilde{h}(0) = 0$, $\tilde{f}'(0) = \tilde{h}'(0) = 1$,

$$\tilde{f}'(b_0) = \beta_0 b_1^{-\omega_0} b_0^{-\eta_1} (1 - \eta_1), \quad \tilde{h}'(b_0) = \alpha_0 (1 + \epsilon_0) \left(\frac{b_0}{b_1} \right)^{\epsilon_0}.$$

On $(\frac{b_0}{2}, b_0)$, we have $\tilde{f}''(r) = -2C_1 < 0$ and $\tilde{h}''(r) = -2C_2 < 0$, hence

$$(9.35) \quad (1 - \eta) b_0^{-\eta} \leq \tilde{f}'(r) \leq 1, \quad \alpha_0 (1 + \epsilon_0) \left(\frac{b_0}{b_1} \right)^{\epsilon_0} \leq \tilde{h}'(r) \leq 1.$$

It is easy to see $\tilde{f}'(b_0) = \lim_{r \to b_0^+} \bar{f}'(r)$ and $\tilde{h}'(b_0) = \lim_{r \to b_0^+} \bar{h}'(r)$.

In the rest part of **Step (II)**, we will show that $Rc(\tilde{f}, \tilde{h}) > 0$ on $(\frac{b_0}{2}, b_0)$.

It is obvious that $Rc(\tilde{f}, \tilde{h}) = 0$ on $[0, \frac{b_0}{2})$.

Also it is easy to get that $Rc(\vec{n}, \vec{n}) > 0$ from $\tilde{f}'' < 0$ and $\tilde{h}'' < 0$ on $(\frac{b_0}{2}, b_0)$.

Next if we assume that

Assumption 12. $\alpha_0 (1 + \epsilon_0) \left(\frac{b_0}{b_1} \right)^{\epsilon_0} = \frac{2}{3} + \frac{1}{3} \beta_0 b_1^{-\omega_0} b_0^{-\eta_1} (1 - \eta_1)$

then $C_2 = \frac{1}{3} C_1$, $\tilde{h}'(r) > \tilde{f}'(r)$ on $[\frac{b_0}{2}, b_0]$. Hence $\tilde{h}(r) \geq \tilde{f}(r)$.

If we further have

$$(9.36) \qquad \left(\frac{\tilde{f}}{\tilde{h}} \right)^3 \geq \tilde{f}' \tilde{h}'$$

then it is easy to get $Rc|_{k_1} > 0$ and $Rc|_{k_2} > 0$ on $(\frac{b_0}{2}, b_0)$.

To show $\left(\frac{\tilde{f}}{\tilde{h}}\right)^3 \geq \tilde{f}'\tilde{h}'$ on $[\frac{b_0}{2}, b_0]$, we consider the function

$$\varphi_1(r) \doteq \tilde{f}^3 - \tilde{f}'\tilde{h}'\tilde{h}^3.$$

Note $\varphi_1(\frac{b_0}{2}) = 0$, we only need to show $\varphi_1'(r) \geq 0$ on $[\frac{b_0}{2}, b_0]$. It is easy to get

$$\varphi_1'(r) \geq 3\tilde{f}'[\tilde{f} - \tilde{h}'h][\tilde{f} + \tilde{h}'\tilde{h}].$$

Hence we just need to show that $\tilde{f} - \tilde{h}'\tilde{h} \geq 0$ on $[\frac{b_0}{2}, b_0]$. Define

$$\varphi_2(r) \doteq \tilde{f} - \tilde{h}'\tilde{h}.$$

Observe that $\varphi_2(\frac{b_0}{2}) = 0$, the problem reduces to show that

$$(9.37) \qquad \varphi_2'(r) = \tilde{f}' - (\tilde{h}')^2 - \tilde{h}\tilde{h}'' \geq 0 , \quad r \in \left[\frac{b_0}{2}, b_0\right].$$

Let $\varphi_3(r) \doteq \tilde{f}' - (\tilde{h}')^2 - \tilde{h}\tilde{h}''$, then

$$(9.38) \qquad\qquad \varphi_3'(r) = 6C_2\tilde{h}' - 2C_1 \leq 0.$$

Now using Assumption 12, which is equivalent to $C_2 = \frac{1}{3}C_1$, it is easy to get

$$(9.39) \qquad\qquad \varphi_3(b_0) = b_0 C_2\left(1 - \frac{1}{2}b_0 C_1\right) > 0.$$

From (9.38) and (9.39), we get $\varphi_3(r) \geq 0$. Hence (9.36) is obtained, we are done.

9.3. Step (III).
By adjusting the values of the functions $\bar{f}$, $\bar{h}$, by suitable constants on each interval $(b_i, b_{i+1}]$, we can remove the jump discontinuities, thereby obtaining C^1 functions $\hat{f}$, $\hat{h}$ by gluing $\tilde{f}$, $\tilde{h}$ with $\bar{f}$, $\bar{h}$.

The functions $\hat{f}$, $\hat{h}$ may not have the second derivatives at the points b_i.

Now we define

$$(9.40) \quad \hat{f}(r) \doteq \begin{cases} \tilde{f}(r) & r \in [0, b_0] \\ \bar{f}(r) + \sum_{l=0}^{k} \tau_l & r \in (b_k, b_{k+1}] , \quad k = 0, 1, 2, \cdots \end{cases}$$

where $\tau_0 \doteq \tilde{f}(b_0) - \lim\limits_{r \to b_0^+} \bar{f}(r) = \frac{b_0}{4}\left[3 - (3+\eta_1)\beta_0 b_1^{-\omega_0} b_0^{-\eta_1}\right]$, when $l \geq 1$,

τ_l is defined in (9.29) and (9.30). From Assumption 7 we can get that $\tau_0 \in (0, \frac{3}{4}b_0)$, and it is also easy to check that $\hat{f}$ is of class C^1 on $[0, \infty)$

Similarly, we define

$$(9.41) \quad \hat{h}(r) \doteq \begin{cases} \tilde{h}(r) & r \in [0, b_0] \\ \bar{h}(r) + \sum_{l=0}^{k} \delta_l & r \in (b_k, b_{k+1}] , k = 0, 1, 2, \cdots \end{cases}$$

where $\delta_0 \doteq \tilde{h}(b_0) - \lim\limits_{r \to b_0^+} \bar{h}(r)$, when $l \geq 1$, δ_l is defined in (9.31) and (9.32). It is easy to get that $\delta_0 \in (0, \frac{3}{4}b_0]$. And it is also easy to check that $\hat{h}$ is of class C^1 on $[0, \infty)$.

If we assume

Assumption 13. $b_{2k+1}^{1-\frac{1}{2}(\eta_2+\eta_1)} \leq b_{2k+2}^{1-\eta_2}$, $\quad \forall k \geq 0$.

We have the following claim about τ_i:

Claim 9.1.

$$(9.42) \qquad |\tau_0| \leq \left(\frac{2b_0}{b_1^{1-\eta_1-\omega_0}} \right) \min_{r \in (b_j, b_{j+1}]} \bar{f}(r) , \qquad j \geq 1$$

$$(9.43) \qquad |\tau_i| \leq \left(\frac{\eta_2 - \eta_1}{1 - \eta_2} \right) \min_{r \in (b_i, b_{i+1}]} \bar{f}(r) , \qquad i \geq 1$$

$$(9.44) \qquad |\tau_i| \leq \left(b_i^{-\frac{1}{2}(\eta_2-\eta_1)} \right) \min_{r \in (b_j, b_{j+1}]} \bar{f}(r) , \qquad j > i \geq 1.$$

Proof: (9.42) follows directly from the definition of τ_0, (9.5), (9.9) and (9.10).

(9.43) follows from (9.29), (9.30), (9.5) and (9.10).

There are five cases for (9.44), in the rest of the proof, $k \geq 0$.

(1). When $i = 2k + 1$, $j = 2k + 2$, $k \geq 0$, we have

$$\frac{|\tau_i|}{\min_{r \in (b_j, b_{j+1}]} \bar{f}(r)} = \left(\frac{\eta_2 - \eta_1}{1 - \eta_2} \right) \cdot \left(\frac{b_{2k+1}}{b_{2k+2}} \right)^{1-\eta_2}$$

using Assumption 13, (9.44) is obtained in this case.

(2). When $i = 2k + 1$, $j = 2\tilde{k}$, $\tilde{k} > k + 1$, we have

$$\frac{|\tau_i|}{\min_{r \in (b_j, b_{j+1}]} \bar{f}(r)} = \frac{\eta_2 - \eta_1}{1 - \eta_1} \cdot \frac{\beta_{2k+1}}{\beta_{2\tilde{k}-2}} \cdot \frac{b_{2k+2}^{\omega_{2k+1}} b_{2k+1}^{1-\eta_2}}{b_{2\tilde{k}}^{1-\eta_2} b_{2\tilde{k}-1}^{\eta_2-\eta_1-\omega_{2\tilde{k}-2}}}$$

$$\leq b_{2k+2}^{\eta_1-\eta_2+\omega_{2k+1}+\omega_{2\tilde{k}-1}} \leq b_{2k+1}^{-\frac{1}{2}(\eta_2-\eta_1)}.$$

Then (9.44) is obtained in this case.

(3). When $i = 2k + 1$, $j = 2\tilde{k} + 1$, $\tilde{k} > k$, we have

$$\frac{|\tau_i|}{\min_{r \in (b_j, b_{j+1}]} \bar{f}(r)} = \frac{1 - \eta_2}{1 - \eta_1} \cdot \frac{\eta_2 - \eta_1}{1 - \eta_1} \cdot \frac{\beta_{2k+1}}{\beta_{2\tilde{k}}} \cdot \frac{b_{2k+2}^{\omega_{2k+1}} b_{2k+1}^{1-\eta_2}}{b_{2\tilde{k}+1}^{1-\eta_1-\omega_{2\tilde{k}}}}$$

$$\leq b_{2k+1}^{-\frac{1}{2}(\eta_2-\eta_1)}.$$

Hence (9.44) holds in this case.

(4). When $i = 2k + 2$, $j = 2\tilde{k}$, $\tilde{k} > k + 1$, we have

$$\frac{|\tau_i|}{\min_{r \in (b_j, b_{j+1}]} \bar{f}(r)} = \frac{\eta_2 - \eta_1}{1 - \eta_1} \cdot \frac{\beta_{2k+1}}{\beta_{2\tilde{k}-2}} \cdot \frac{b_{2k+2}^{1-\eta_2+\omega_{2k+1}}}{b_{2\tilde{k}}^{1-\eta_2} b_{2\tilde{k}-1}^{\eta_2-\eta_1-\omega_{2\tilde{k}-2}}}$$

$$\leq b_{2k+2}^{-\frac{1}{2}(\eta_2-\eta_1)}.$$

(9.44) is got here.

(5). When $i = 2k + 2$, $j = 2\tilde{k} + 1$, $\tilde{k} > k$, we have

$$\frac{|\tau_i|}{\min_{r \in (b_j, b_{j+1}]} \bar{f}(r)} = \frac{1 - \eta_2}{1 - \eta_1} \cdot \frac{\eta_2 - \eta_1}{1 - \eta_1} \cdot \frac{\beta_{2k+1}}{\beta_{2\tilde{k}}} \cdot \frac{b_{2k+2}^{1-\eta_2+\omega_{2k+1}}}{b_{2\tilde{k}+1}^{1-\eta_1-\omega_{2\tilde{k}}}}$$

$$\leq b_{2k+2}^{-\frac{1}{2}(\eta_2-\eta_1)}.$$

This completes our proof of (9.44). q.e.d.

Similarly, We have the following claim about δ_i:

Claim 9.2.

(9.45) $$|\delta_0| \leq 3\frac{b_0}{b_1} \min_{r \in (b_j, b_{j+1}]} \bar{h}(r) , \quad j \geq 1$$

(9.46) $$|\delta_i| \leq 4\epsilon_{i-1} \min_{r \in (b_j, b_{j+1}]} \bar{h}(r) , \quad 1 \leq i \leq j.$$

Proof: For $i \geq 1$, we can get the following estimate:

$$\frac{\delta_0}{\min_{r \in (b_{2i}, b_{2i+1}]} \bar{h}(r)}$$

$$\leq \frac{3}{4} \cdot \frac{b_0}{\alpha_{2i} b_{2i+1}^{-\epsilon_{2i}} b_{2i}^{1+\epsilon_{2i}}}$$

$$= \frac{3}{4} \frac{1}{\alpha_{2i-1}} \cdot \frac{1 + \epsilon_{2i}}{1 - \epsilon_{2i-1}} \cdot \frac{b_0}{b_{2i-1}} \cdot \left(\frac{\alpha_{2i-1}}{\alpha_{2i-2}} \cdot \frac{1 - \epsilon_{2i-1}}{1 + \epsilon_{2i-2}}\right)^{\frac{1}{\epsilon_{2i-1}}}$$

$$\leq \frac{3}{2} \frac{1}{\alpha_0} \frac{b_0}{b_1} \leq 3\frac{b_0}{b_1}.$$

Similarly, we can get that for $i \geq 0$,

$$\frac{\delta_0}{\min_{r \in (b_{2i+1}, b_{2i+2}]} \bar{h}(r)} \leq \frac{3}{4} \cdot \frac{b_0}{\alpha_{2i+1} b_{2i+2}^{\epsilon_{2i+1}} b_{2i+1}^{1-\epsilon_{2i+1}}} = \frac{3}{4} \frac{b_0}{\alpha_{2i}} \cdot \frac{1 - \epsilon_{2i+1}}{(1 + \epsilon_{2i}) b_{2i+1}}$$

$$\leq \frac{3}{2} \frac{1}{\alpha_0} \frac{b_0}{b_1} \leq 3\frac{b_0}{b_1}.$$

By the above two inequalities, we obtain (9.45).

For $k \geq 1$,

$$\min_{r \in (b_{2k}, b_{2k+1}]} \bar{h}(r) \geq \alpha_{2k} b_{2k+1}^{-\epsilon_{2k}} b_{2k}^{1+\epsilon_{2k}} = \alpha_{2k-1} b_{2k} \Big(\frac{1+\epsilon_{2k}}{1-\epsilon_{2k-1}}\Big)$$

$$= \alpha_{2k-2} b_{2k} \Big(\frac{b_{2k-1}}{b_{2k}}\Big)^{\epsilon_{2k-1}} \cdot \Big(\frac{1+\epsilon_{2k-2}}{1-\epsilon_{2k-1}}\Big) \cdot \Big(\frac{1+\epsilon_{2k}}{1-\epsilon_{2k-1}}\Big)$$

$$\geq \alpha_{2k-2} b_{2k-1}.$$

When $1 \leq i < k$,

$$|\delta_{2i}| = \alpha_{2i-1} b_{2i} \cdot \frac{\epsilon_{2i-1} + \epsilon_{2i}}{1+\epsilon_{2i}} \leq 2\epsilon_{2i-1} \alpha_{2i-1} b_{2i}$$

$$\leq 2\epsilon_{2i-1} \alpha_{2k-2} b_{2k-1} \leq 2\epsilon_{2i-1} \min_{r \in (b_{2k}, b_{2k+1}]} \bar{h}(r).$$

And

$$|\delta_{2k}| = \alpha_{2k-1} b_{2k} \cdot \frac{\epsilon_{2k} + \epsilon_{2k-1}}{1+\epsilon_{2k}}$$

$$\leq \min_{r \in (b_{2k}, b_{2k+1}]} \bar{h}(r) \cdot \frac{1-\epsilon_{2k-1}}{1+\epsilon_{2k}} \cdot \frac{\epsilon_{2k} + \epsilon_{2k-1}}{1+\epsilon_{2k}}$$

$$\leq 2\epsilon_{2k-1} \min_{r \in (b_{2k}, b_{2k+1}]} \bar{h}(r).$$

When $0 \leq i < k$,

$$|\delta_{2i+1}| = \alpha_{2i} b_{2i+1} \cdot \frac{\epsilon_{2i+1} + \epsilon_{2i}}{1-\epsilon_{2i+1}} \leq 4\epsilon_{2i} \min_{r \in (b_{2k}, b_{2k+1}]} \bar{h}(r).$$

From all the above, we get

$$(9.47) \qquad |\delta_i| \leq 4\epsilon_{i-1} \min_{r \in (b_{2k}, b_{2k+1}]} \bar{h}(r) , \quad 1 \leq i \leq 2k.$$

For $k \geq 0$,

$$\min_{r \in (b_{2k+1}, b_{2k+2}]} \bar{h}(r) \geq \alpha_{2k+1} b_{2k+1} \Big(\frac{b_{2k+2}}{b_{2k+1}}\Big)^{\epsilon_{2k+1}} = \alpha_{2k} b_{2k+1} \Big(\frac{1+\epsilon_{2k}}{1-\epsilon_{2k+1}}\Big)$$

$$\geq \alpha_{2k} b_{2k+1}.$$

When $0 \leq i \leq k$,

$$|\delta_{2i+1}| \leq 4\epsilon_{2i} \alpha_{2k} b_{2k+1} \leq 4\epsilon_{2i} \min_{r \in (b_{2k+1}, b_{2k+2}]} \bar{h}(r).$$

When $1 \leq i \leq k$,

$$|\delta_{2i}| \leq 2\epsilon_{2i-1} \alpha_{2k} b_{2k+1} \leq 2\epsilon_{2i-1} \min_{r \in (b_{2k+1}, b_{2k+2}]} \bar{h}(r).$$

Hence we obtain that

$$(9.48) \qquad |\delta_i| \leq 4\epsilon_{i-1} \min_{r \in (b_{2k+1}, b_{2k+2}]} \bar{h}(r) , \quad 1 \leq i \leq 2k+1.$$

From (9.47) and (9.48), we (9.46).   q.e.d.

We will assume

Assumption 14. $\sum\limits_{l=0}^{\infty} \epsilon_l \doteq \delta < 1, \quad \sum\limits_{l=1}^{\infty} b_l^{-\frac{1}{2}(\eta_2-\eta_1)} \doteq \tau < 1$

where δ and τ are positive constants to be determined later.

We define $\zeta_k = \sum\limits_{l=0}^{k} \tau_l$, $\xi_k = \sum\limits_{l=0}^{k} \delta_l$, then

$$\hat{f}|_{(b_k, b_{k+1}]} = \bar{f} + \zeta_k , \quad \hat{h}|_{(b_k, b_{k+1}]} = \bar{h} + \xi_k.$$

Note that we have $Rc(\hat{f}, \hat{h}) \geq 0$ on $[0, b_0)$ from (**II**). In the rest part of (**III**), we will prove $Rc(\hat{f}, \hat{h}) > 0$ on (b_0, ∞) except at points b_j, $j = 1, 2, \cdots$.

(**i**). We firstly consider the interval $(b_0, b_1]$.

on $(b_0, b_1]$, from Assumption 7, we get

$$
\begin{aligned}
Rc|_{k_1}(\hat{f}, \hat{h}) &\geq \frac{2\left[r^2 - \bar{f}^2(1-\eta_1)^2 - 2\bar{f}(\bar{f}+\tau_0)\right]}{(\bar{f}+\tau_0)^2 r^2} \\
&\geq \frac{2\bar{f}^2\left[\left(\delta_0^{-1} b_1^{\omega_0} r^{\eta_1}\right)^2 - (1-\eta_1)^2 - 2\left(1 + \frac{3}{4}\delta_0^{-1} b_1^{\omega_0} b_0^{\eta_1}\right)\right]}{(\bar{f}+\tau_0)^2 r^2} \\
&\geq \frac{2\bar{f}^2\left[\delta_0^{-1} b_1^{\omega_0}\left(b_0^{2\eta_1} - 1 - 4b_0^{\eta_1}\right)\right]}{(\bar{f}+\tau_0)^2 r^2} > 0.
\end{aligned}
$$

From Assumption 10, we obtain that

$$
\begin{aligned}
Rc(\vec{n}, \vec{n}) &> \frac{3\eta_1(1-\eta_1)\bar{f}}{(\bar{f}+\tau_0)r^2} - \frac{4\epsilon_0(\epsilon_0+1)}{r^2} \\
&\geq \frac{2}{r^2}\left[\frac{3}{2} \cdot \frac{(1-\eta_1)\eta_1}{1 + b_0^{\eta_1} b_1^{\omega_0}} - 2\epsilon_0(\epsilon_0+1)\right] > 0.
\end{aligned}
$$

We assume that for $i \geq 0$,

Assumption 15. $\epsilon_{2i} < \omega_{2i}$

then $\hat{h}' > \hat{f}'$ on $(b_0, b_1]$. Combining with $\hat{h}(b_0) > \hat{f}(b_0)$, we get that $\hat{h} > \hat{f}$ on $(b_0, b_1]$.

From (9.6), combining with Assumption 9, we can get

$$
\begin{aligned}
Rc|_{k_2}(\hat{f}, \hat{h}) &\geq \frac{6}{\hat{h}^2} - \frac{\hat{h}''}{\hat{h}} - \frac{3(\hat{h}')^2}{\hat{h}^2} - \frac{3\hat{f}'\hat{h}'}{\hat{f}\hat{h}} \\
&= \frac{1}{\hat{h}^2}\Big\{ 6 - \Big[\frac{\bar{h}}{r} + \frac{\delta_0}{r}\Big](1 + \epsilon_0)\epsilon_0\frac{\bar{h}}{r} - 3\Big[(1 + \epsilon_0)\frac{\bar{h}}{r}\Big]^2 \\
&\qquad - 3(1 - \eta_1)\frac{\bar{f}}{\bar{f} + \tau_0}(1 + \epsilon_0)\frac{\bar{h}}{r}\Big(\frac{\bar{h}}{r} + \frac{\delta_0}{r}\Big) \Big\} \\
&\geq \frac{1}{\hat{h}^2}\Big[6 - \alpha_0(1 + \epsilon_0)\epsilon_0\Big(\frac{3}{4} + \alpha_0\Big) - 3\big(\alpha_0(1 + \epsilon_0)\big)^2 \\
&\qquad - 3(1 - \eta_1)\alpha_0(1 + \epsilon_0)\Big(\alpha_0 + \frac{3}{4}\Big)\Big] \\
&\geq \frac{1}{\hat{h}^2}[6 - 2\epsilon_0 - 3 - 6(1 - \eta_1)] > 0.
\end{aligned}
$$

So, we proved that $Rc(\hat{f}, \hat{h}) > 0$ on (b_0, b_1).

(ii). Next we consider the interval $(b_{2i}, b_{2i+1}]$, $i \geq 1$. We assume that

Assumption 16.

$$
\frac{2b_0}{b_1^{1-\eta_1-\omega_0}} + \frac{\eta_2 - \eta_1}{1 - \eta_2} + \tau < \eta_1^3,
$$

$$
b_1^{2\eta_1} > 2 + 20\Big(1 - 3\frac{b_0}{b_1} - 4\delta\Big)^{-1}
$$

then combining with Claim 9.1 and Claim 9.2, we get

(9.49)

$$
\begin{aligned}
Rc|_{k_1}(\hat{f}, \hat{h}) &\geq \frac{2\Big[1 - (1 - \eta_1)^2\big(\frac{\bar{f}}{r}\big)^2\Big]}{\big(\bar{f} + \zeta_{2i}\big)^2} - \frac{4(1 - \eta_1)(1 + \epsilon_{2i})\big(\frac{\bar{f}}{r}\big)\big(\frac{\bar{h}}{r}\big)}{\big(\bar{f} + \zeta_{2i}\big)\big(\bar{h} + \xi_{2i}\big)} \\
&= \frac{2\bar{f}^2}{\big(\bar{f} + \zeta_{2i}\big)^2 r^2}\Big[\frac{r^2}{\bar{f}^2} - (1 - \eta_1)^2 - 2(1 - \eta_1)(1 + \epsilon_{2i})\frac{\bar{h}}{\bar{h} + \xi_{2i}}\Big(1 + \frac{\zeta_{2i}}{\bar{f}}\Big)\Big] \\
&\geq \frac{2\bar{f}^2}{\big(\bar{f} + \zeta_{2i}\big)^2 r^2}\Big[b_2^{2\eta_1} - 1 - \frac{2\Big(1 + \frac{2b_0}{b_1^{1-\eta_1-\omega_0}} + \frac{\eta_2-\eta_1}{1-\eta_2} + \tau\Big)}{1 - \frac{3b_0}{b_1} - 4\delta}\Big] \\
&> 0.
\end{aligned}
$$

If we further assume that

Assumption 17. $\epsilon_0 \leq \frac{1}{10}\eta_1(1 - \eta_1)\Big(1 - 3\frac{b_0}{b_1} - 4\delta\Big)$

then we get

$$Rc(\vec{n}, \vec{n}) = \frac{3\eta_1(1 - \eta_1)\bar{f}}{(\bar{f} + \zeta_{2i})r^2} - \frac{4\epsilon_{2i}(1 + \epsilon_{2i})\bar{h}}{(\bar{h} + \xi_{2i})r^2}$$

$$\geq \frac{3\eta_1(1 - \eta_1)}{\left(1 + \frac{2b_0}{b_1^{1-\eta_1-\omega_0}} + \frac{\eta_2-\eta_1}{1-\eta_2} + \tau\right)r^2} - \frac{4\epsilon_{2i}(1 + \epsilon_{2i})}{\left(1 - \frac{3b_0}{b_1} - 4\delta\right)r^2}$$

$$\geq \frac{3}{r^2\left(1 - 3\frac{b_0}{b_1} - 4\delta\right)}\left[\frac{\eta_1(1 - \eta_1)\left(1 - 3\frac{b_0}{b_1} - 4\delta\right)}{\left(1 + \frac{2b_0}{b_1^{1-\eta_1-\omega_0}} + \frac{\eta_2-\eta_1}{1-\eta_2} + \tau\right)} - 2\epsilon_0\right]$$

$$> 0.$$

We assume that

Assumption 18. $b_1^{\eta_1} \geq \frac{100}{1-4\delta}$

From Assumptions 15 and 18, we get $\bar{h} \geq \alpha_0 b_{2i}^{\eta_1} \bar{f}$ on $(b_{2i}, b_{2i+1}]$. Also note that the following holds:

$$\hat{f} = \bar{f} + \zeta_{2i} \leq 5\bar{f}, \quad \hat{h} = \bar{h} + \xi_{2i} \geq (1 - 4\delta)\bar{h}$$

the above three inequalities imply that $\hat{h} \geq \hat{f}$ on $(b_{2i}, b_{2i+1}]$.

We further assume that

Assumption 19. $\dfrac{3b_0}{b_1} + 4\delta \leq \eta_1$

Observe that $\epsilon_0 < \frac{\eta_1^2}{1+\eta_1}$, then using Assumption 9, we have

$$Rc|_{k_2}(\hat{f}, \hat{h}) \geq \frac{1}{\hat{h}^2}\left[6 - \hat{h}\hat{h}'' - 3(\hat{h}')^2 - 3\frac{\hat{f}'}{\hat{f}}\hat{h}'\hat{h}\right]$$

$$= \frac{1}{\hat{h}^2}\left\{6 - \frac{\bar{h}^2}{r^2}\epsilon_{2i}(1 + \epsilon_{2i})\left(\frac{\hat{h}}{\bar{h}}\right) - 3\left[(1 + \epsilon_{2i})\frac{\bar{h}}{r}\right]^2\right.$$

$$\left. - 3\frac{\bar{f}(1 - \eta_1)(1 + \epsilon_{2i})}{\bar{f} + \zeta_{2i}} \cdot \frac{\bar{h}(\bar{h} + \xi_{2i})}{r^2}\right\}$$

$$\geq \frac{1}{\hat{h}^2}\left\{6 - \epsilon_{2i}(1 + \epsilon_{2i})\alpha_{2i}^2\left(1 + 3\frac{b_0}{b_1} + 4\delta\right) - 3\left[\alpha_{2i}(1 + \epsilon_{2i})\right]^2\right.$$

$$\left. 3\frac{(1 - \eta_1)(1 + \epsilon_{2i})\left(1 + 3\frac{b_0}{b_1} + 4\delta\right)}{1 - \frac{2b_0}{b_1^{1-\eta_1-\omega_0}} - \frac{\eta_2-\eta_1}{1-\eta_2} - \tau}\alpha_{2i}^2\right\}$$

$$\geq \frac{1}{\hat{h}^2}\left[3 - \frac{\eta_1^2}{1 + \eta_1}(1 + \eta_1) - 3\frac{1 - \eta_1}{1 - \eta_1^3}(1 + \eta_1)\right] > 0.$$

From all the above, we proved that $Rc(\hat{f}, \hat{h}) > 0$ on (b_{2i}, b_{2i+1}), where $i \geq 1$.

(**iii**). Finally we consider the interval (b_{2i+1}, b_{2i+2}), where $i \geq 0$. From Assumption 16, similarly as (9.49), we get

$$Rc|_{k_1} \geq \frac{2\bar{f}^2}{(\bar{f} + \zeta_{2i+1})^2 r^2}\left[\left(\frac{r^{\eta_2}}{\beta_{2i+1} b_{2i+2}^{\omega_{2i+1}}}\right)^2 - 1\right.$$

$$\left. - \frac{2}{1 - \frac{3b_0}{b_1} - 4\delta}\left(1 + \frac{2b_0}{b_1^{1-\eta_1-\omega_0}} + \frac{\eta_2 - \eta_1}{1 - \eta_2} + \tau\right)\right]$$

$$\geq \frac{2\bar{f}^2}{(\bar{f} + \zeta_{2i+1})^2 r^2}\left[\left(\frac{1 - \eta_2}{\beta_{2i}(1 - \eta_1)}\right)^2 b_{2i+1}^{2\eta_1} - 1 - \frac{4}{1 - \frac{3b_0}{b_1} - 4\delta}\right]$$

$$\geq \frac{\bar{f}^2}{(\bar{f} + \zeta_{2i+1})^2 r^2}\left[b_{2i+1}^{2\eta_1} - 2 - 8\left(1 - \frac{3b_0}{b_1} - 4\delta\right)^{-1}\right] > 0.$$

And $Rc(\vec{n}, \vec{n}) > 0$ is trivial by $\hat{f}'' < 0$ and $\hat{h}'' < 0$.
It is easy to see that we also have

(9.50) $\hat{f} = \bar{f} + \zeta_{2i+1} \leq 5\bar{f}$, $\hat{h} = \bar{h} + \xi_{2i+1} \geq (1 - 4\delta)\bar{h}$

Using (9.9) and (9.17), we have

(9.51) $\bar{h} \geq \frac{\alpha_{2i}}{\beta_{2i}} \cdot \frac{1 + \epsilon_{2i}}{1 - \epsilon_{2i+1}} \cdot \frac{1 - \eta_2}{1 - \eta_1} b_{2i+1}^{\eta_1 + \omega_{2i}} \bar{f} \geq \frac{\alpha_0}{2} b_{2i+1}^{\eta_1} \bar{f}$

from (9.50), (9.51) and Assumption 18, we can get $\hat{h} > \hat{f}$ on (b_{2i+1}, b_{2i+2}), where $i \geq 0$.

From Assumption 16 and Assumption 19, we can get

$$Rc|_{k_2}(\hat{f}, \hat{h}) \geq \frac{1}{\hat{h}^2}\left[6 - 3\left(\frac{\bar{h}}{r}\right)^2 (1 - \epsilon_{2i+1})^2\right.$$

$$\left. - 3\frac{\bar{h}^2}{r^2}(1 - \epsilon_{2i+1})(1 - \eta_2) \cdot \frac{\left(1 + \frac{\xi_{2i+1}}{h}\right)}{\left(1 + \frac{\zeta_{2i+1}}{\bar{f}}\right)}\right]$$

$$\geq \frac{1}{\hat{h}^2}\left\{6 - 3\left[\alpha_{2i}(1 + \epsilon_{2i})\right]^2 \left[1 + \frac{(1 + \eta_1)(1 - \eta_2)}{(1 - \eta_1^3)(1 - \epsilon_{2i+1})}\right]\right\}$$

$$\geq \frac{1}{\hat{h}^2}\left\{6 - 3\left[1 + \frac{1 + \eta_1}{1 + \eta_1 + \eta_1^2}\right]\right\} > 0$$

in the last inequality, we used the inequality $\dfrac{1 - \eta_2}{1 - \eta_1} < 1 - \epsilon_0$ from (9.6).

From all the above, we get $Rc(\hat{f}, \hat{h}) > 0$ on (b_0, ∞) excepts at points b_i, $i = 1, 2, \cdots$.

9.4. Step (IV).

Finally, we can remove the jump discontinuities in the functions, $\hat{f}''$, $\hat{h}''$, by modifying them by linear interpolation, in arbitrarily small neighborhoods of the points, $\{b_i\}_{i=0}^{\infty}$. Call the resulting functions f'', h'', and let

the corresponding functions, f, h, be obtained by integration with respect to r, subject to the conditions, $f(0) = h(0) = 0$, $f'(0) = h'(0) = 1$. The modification in the second derivatives can be performed on intervals whose size decreases rapidly enough to ensure the nonnegative property of $Rc|_{k_1}(f, h)$, $Rc|_{k_2}(f, h)$ and $Rc(\vec{n}, \vec{n})(f, h)$ on $[0, \infty)$.

For (M^8, g), M^8 is diffeomorphic to $\mathbb{R}^8$, $g = dr^2 + f^2 k_1 + h^2 k_2$, define two sequences $\{t_i\}_{i=0}^{\infty}$, $\{\tilde{t}_i\}_{i=0}^{\infty}$ as the following:

$$t_i = \left(b_{2i+1}^{1-\epsilon_{2i}}\right)^2 , \qquad \tilde{t}_i = \left(b_{2i+2}^{1-\epsilon_{2i+1}}\right)^2 , \qquad i = 0, 1, 2, \cdots$$

And define the scaling metrics $g_i \doteq t_i^{-1} g$ and $\tilde{g}_i \doteq \tilde{t}_i^{-1} g$, we also assume that

Assumption 20. $\lim\limits_{i \to \infty} b_i^{\epsilon_i} = 1$, $\lim\limits_{i \to \infty} b_{i+1}^{\epsilon_i^2} = 1$, $\lim\limits_{i \to \infty} b_{i+1}^{\epsilon_i} = \infty$.

It is not hard to check that we can find sequences $\{b_i\}$, $\{\alpha_i\}$, $\{\beta_i\}$, $\{\omega_i\}$, $\{\epsilon_i\}$ and η_1, η_2 satisfying the Assumptions $1 - 20$. Hence we get

$$(9.52) \qquad \left(M^8, g_i, y, \nu_i\right) \xrightarrow{d_{GH}} \left(M_\infty, \rho_\infty, y_\infty, \nu_\infty\right)$$

and define $\nu_i(A) \doteq t_i^{\frac{n}{2}} V(\sqrt{t_i})^{-1} \mu_i(A)$, where μ_i is the volume element determined by metric g_i.

M_∞ is diffeomorphic to $\mathbb{R}^5$ with metric $\rho_\infty = dr^2 + \frac{1}{4} r^2 g_{\mathbb{S}^4}$, and

$$\nu_\infty\left(B_\infty(r)\right) = r^{8-3\eta_1}.$$

On the other side, we have

$$(9.53) \qquad \left(M^8, \tilde{g}_i, y, \tilde{\nu}_i\right) \xrightarrow{d_{GH}} \left(\tilde{M}_\infty, \tilde{\rho}_\infty, y_\infty, \tilde{\nu}_\infty\right)$$

where $\tilde{\nu}_i(A) \doteq \tilde{t}_i^{\frac{n}{2}} V(\sqrt{\tilde{t}_i})^{-1} \tilde{\mu}_i(A)$, and $\tilde{u}_i$ is the volume element determined by $\tilde{g}_i$.

$\tilde{M}_\infty$ is diffeomorphic to $\overline{\mathbb{R}^+}$ with metric $\tilde{\rho}_\infty = dr^2$ and

$$\tilde{\nu}_\infty\left(B_\infty(r)\right) = r^{8-3\eta_2}.$$

From the proof of Theorem 1.6, we can get that for rotational symmetric functions on $(M_\infty, \rho_\infty, y_\infty, \nu_\infty)$ and $(\tilde{M}_\infty, \tilde{\rho}_\infty, y_\infty, \tilde{\nu}_\infty)$ respectively,

$$\Delta_{(\rho_\infty, \nu_\infty)} = \frac{\partial^2}{\partial r^2} + \frac{7 - 3\eta_1}{r} \cdot \frac{\partial}{\partial r} , \qquad \Delta_{(\tilde{\rho}_\infty, \tilde{\nu}_\infty)} = \frac{\partial^2}{\partial r^2} + \frac{7 - 3\eta_2}{r} \cdot \frac{\partial}{\partial r}.$$

Then it is not hard to get

$$(9.54) \qquad H_\infty(x_\infty, y_\infty, t) = C_H \cdot t^{-\frac{1}{2}(8-3\eta_1)} \exp\left(-\frac{d_{\rho_\infty}(x_\infty, y_\infty)^2}{4t}\right)$$

$$(9.55) \qquad \tilde{H}_\infty(x_\infty, y_\infty, t) = C_{\tilde{H}} \cdot t^{-\frac{1}{2}(8-3\eta_2)} \exp\left(-\frac{d_{\tilde{\rho}_\infty}(x_\infty, y_\infty)^2}{4t}\right)$$

where $C_H = \left(\int_0^\infty e^{-\frac{u^2}{4}} u^{7-3\eta_1} du \right)^{-1}$ and $C_{\tilde{H}} = \left(\int_0^\infty e^{-\frac{u^2}{4}} u^{7-3\eta_2} du \right)^{-1}$,
which follows from

$$\int_{M_\infty} H_\infty d\nu_\infty = 1 \ , \qquad \int_{\tilde{M}_\infty} \tilde{H}_\infty d\tilde{\nu}_\infty = 1.$$

From (4.13) and Proposition 7.7, we get

$$\lim_{i \to \infty} V(\sqrt{t_i}) H(x,y,t_i) = H_\infty(y_\infty, y_\infty, 1) = C_H$$

$$\lim_{i \to \infty} V(\sqrt{\tilde{t}_i}) H(x,y,\tilde{t}_i) = \tilde{H}_\infty(y_\infty, y_\infty, 1) = C_{\tilde{H}}.$$

But from $\eta_1 < \eta_2$, it is easy to see that $C_H < C_{\tilde{H}}$. Hence

$$\lim_{i \to \infty} V(\sqrt{t_i}) H(x,y,t_i) < \lim_{i \to \infty} V(\sqrt{\tilde{t}_i}) H(x,y,\tilde{t}_i).$$

This answers one open question raised in [Li86] negatively. That is, without maximal volume growth assumption, $\lim_{t \to \infty} V(\sqrt{t}) H(x,y,t)$ does not generally exist.

Appendix A. Rellich-type Compactness theorem

Similar with the Rellich-Kondrachov Theorem for Sobolev spaces on a fixed domain, we have Rellich-type Compactness Theorem in the Gromov-Hausdorff sense, which was used in the proof of Theorem 3.1. In this appendix we will give a complete proof of Rellich-type Compactness Theorem.

We firstly state some background knowledge needed for the proof.

Definition A.1 (Measure approximation, [KS03]). Let M_i and M_∞ be measure spaces. A net $\{\varphi_i : M_i \supset \mathcal{D}(\varphi_i) \to M_\infty\}$ of maps is called a **measure approximation** if the following are satisfied:

- Each φ_i is a measurable map from a Borel subset $\mathcal{D}(\varphi_i)$ of M_i to M_∞.
- The push-forward by φ_i of the measure on M_i weakly-* converges to the measure on M_∞, i.e., for any $f \in C_c(M_\infty)$,

(A.1) $$\lim_{i \to \infty} \int_{\mathcal{D}(\varphi_i)} f \circ \varphi_i d\nu_i = \int_{M_\infty} f d\nu_\infty$$

where $C_c(M_\infty)$ is the set of continuous functions on M_∞ with compact support.

As in [Fuk87] (also see [KS03]), there is another definition of measured Gromov-Hausdorff convergence as the following.

Definition A.2 (Measured Gromov-Hausdorff convergence). If ν_i, ν_∞ are Borel regular measures on M_i^n, M_∞, we say that $(M_i^n, y_i, \rho_i, \nu_i)$ converges to $(M_\infty, y_\infty, \rho_\infty, \nu_\infty)$ **in the measured Gromov-Hausdorff sense**, if there exists a measure approximation $\{\varphi_i : M_i \to M_\infty\}$, such

that each φ_i is an ϵ_i-Gromov-Hausdorff approximation for some $\epsilon_i \to 0$, and $\varphi_i(y_i) = y_\infty$.

Remark A.3 (Fukaya's definition VS definition of Cheeger & Colding).
If $(M_i^n, y_i, \rho_i, \nu_i)$ converges to $(M_\infty, y_\infty, \rho_\infty, \nu_\infty)$ in the measured Gromov-Hausdorff sense, from the above definition, we have

- $(M_i^n, y_i, \rho_i) \xrightarrow{d_{GH}} (M_\infty, y_\infty, \rho_\infty)$.
- In addition, for any $x_i \to x_\infty$, $(x_i \in M_i^n,\ x_\infty \in M_\infty)$, $r > 0$, we have
$$\nu_i\Big(B_i(x_i, r)\Big) \to \nu_\infty\Big(B_\infty(x_\infty, r)\Big)$$
where (M_∞, ρ_∞) is a length space with length metric ρ_∞, and
$$B_i(x_i, r) = \{z \in M_i^n|\ d_{\rho_i}(z, x_i) \le r\}\ ,$$
$$B_\infty(x_\infty, r) = \{z \in M_\infty|\ d_{\rho_\infty}(z, x_\infty) \le r\}.$$

The above two items were used to define the measured Gromov-Hausdorff convergence in [Che99](also see Definition 2.2). Hence the definition of the measured Gromov-Hausdorff convergence we chose (following [Fuk87]), implies the measured Gromov-Hausdorff convergence discussed in Cheeger and Colding's work.

However, from Proposition 2.2 in [KS03], in fact, the definition of the measured Gromov-Hausdorff convergence in the Definition 2.2 is equivalent to the one used by Cheeger and Colding.

In most parts of the paper, we used the definition of the measured Gromov-Hausdorff convergence by Cheeger and Colding as in Definition 2.2. However, to prove the following Rellich-type compactness result in the Gromov-Hausdorff sense, we will use the definition of Fukaya in the Definition A.2.

And as in [KS08], we define L^p convergence in Gromov-Hausdorff topology in the following.

Definition A.4 (L^p Convergence in G-H topology). Assume that $\{f_i\}_{i=1}^\infty$ are functions on M_i^n, f_∞ is a function on M_∞, we say $f_i \to f_\infty$ in L^p sense on $U \subset M_\infty$, if there exists $f_\infty^{(j)} \in C_c(U)$, such that

(A.2)
$$\lim_{j \to \infty} \int_U |f_\infty^{(j)} - f_\infty|^p d\nu_\infty = 0\ , \quad \lim_{j \to \infty} \overline{\lim_{i \to \infty}} \int_{U_i} |f_i - f_\infty^{(j)} \circ \varphi_i|^2 d\nu_i = 0$$

where $\varphi_i : U_i \to U$ is a measure approximation and an ϵ_i-Gromov-Hausdorff approximation for some $\epsilon_i \to 0$.

Theorem A.5 (Rellich-type Compactness Theorem). *Assume*
$$B_i(x_i, r) \subset (M_i^n, y_i, \rho_i, \nu_i)\ , \qquad B_\infty(x_\infty, r) \subset (M_\infty, y_\infty, \rho_\infty, \nu_\infty)$$

and $B_i(x_i, r) \xrightarrow{d_{GH}} B_\infty(x_\infty, r)$ *in the measured Gromov-Hausdorff sense,* u_i *is a function on* M_i^n, *and for some fixed constant* $N > 0$,

$$(A.3) \qquad\qquad \int_{B_i(x_i,r)} \left[|u_i|^2 + |\nabla u_i|^2 \right] d\nu_i \leq N.$$

Then there exists a subsequence of $\{u_i\}$ *such that* $u_i \to u_\infty$ *in* L^2 *sense on any* $K_\infty \subset\subset \mathring{B}_\infty(x_\infty, r)$, *where* $\mathring{B}_\infty(x_\infty, r)$ *denotes the interior of* $B_\infty(x_\infty, r)$.

Remark A.6. The proof of the above theorem was sketched in [Din02]. Following closely the argument in [KS08] (see Theorem 4.15 there), also compare [CM97a], we give a detailed proof here.

Proof: For $K_\infty \subset\subset \mathring{B}_\infty(x_\infty, r)$, assume $d_\infty(K_\infty, \partial B_\infty) = 100r_0 > 0$. Then there exists $i_0 > 0$, for $i > i_0$, $d_{\rho_i}(\phi_i(K_\infty), \partial B_i) = 10r_0 > 0$.

Define $K_i = \phi_i(K_\infty) \subset B_i(x_i, r)$. Take a sequence of numbers $r_j \searrow 0$, $j = 1, 2, \cdots$, and $r_j \leq 0$. Let $\{B_i(z_{jk}^i, r_j)\}_{k=1}^{N_j^i}$ be a maximal set of disjoint balls with radius r_j, centers z_{jk}^i in K_i.

First, by the volume comparison theorem,

$$\nu_i\big(B_i(z_{jk}^i, r_j)\big) \geq \nu_i\big(B_i(z_{jk}^i, r_j + 2r)\big) \cdot \left(\frac{r_j}{r_j + 2r} \right)^n$$
$$\geq C(r_j, r, n)\nu_i\big(B_i(x_i, r)\big).$$

Note

$$\sum_{k=1}^{N_j^i} \nu_i\big(B_i(z_{jk}^i, r_j)\big) \leq \nu_i\big(B_i(x_i, r)\big)$$

therefore

$$N_j^i \leq C(r_j, r, n).$$

It follows from maximality that double the balls covers K_i. We now get N_j^i disjoint subsets $S_{j1}^i, S_{j2}^i \cdots, S_{jN_j^i}^i$ which cover K_i, where

$$S_{jk}^i = B_i(z_{jk}^i, 2r_j) \backslash \left(\cup_{l=1}^{k-1} B_i(z_{jl}^i, 2r_j) \right).$$

We define a step function $\bar{u}_j^i : K_i \to \mathbb{R}$ by $\bar{u}_j^i = \bar{u}_{jk}^i$ on each S_{jk}^i, where

$$\bar{u}_{jk}^i = \frac{1}{\nu_i\big(B_i(z_{jk}^i, 2r_j)\big)} \int_{B_i(z_{jk}^i, 2r_j)} u_i d\nu_i.$$

Let $\eta(y)$ be the number of k, such that $y \in B_i(z_{jk}^i, 4r_j)$ and let $\bar{C}_i = \max_{y \in B_i(x_i, r)} \eta(y)$.

If $y \in \cap_{m=1}^{\eta(y)} B_i(z_{jk}^i, 4r_j)$, it follows that $B_i(y, 5r_j)$ contains all of the balls

$$B_i(z_{j1}^i, r_j), B_i(z_{j2}^i, r_j), \cdots, B_i(z_{j\eta(y)}^i, r_j).$$

Since these are disjoint,

$$(A.4) \qquad \sum_{m=1}^{\eta(y)} \nu_i\Big(B_i(z_{jm}^i, r_j)\Big) \le \nu_i\Big(B_i(y, 5r_j)\Big).$$

Also for each $m = 1, 2, \cdots, \eta(y)$, the doubling condition together with the triangle inequality yields

$$(A.5) \qquad \nu_i\Big(B_i(y, 5r_j)\Big) \le \nu_i\Big(B_i(z_{jm}^i, 9r_j)\Big) \le 9^n \nu_i\Big(B_i(z_{jm}^i, r_j)\Big).$$

Combining (A.4) and (A.5), we see that $\eta(y) \le 9^n = C(n)$, hence $\bar{C}_i \le C(n)$.

We have the following claim.

Claim A.7.

$$(A.6) \qquad \lim_{j \to \infty} \overline{\lim_{i \to \infty}} \int_{K_i} |u_i - \bar{u}_j^i|^2 d\nu_i = 0.$$

Proof:

$$\int_{K_i} |u_i - \bar{u}_j^i|^2 = \sum_{k=1}^{N_j^i} \int_{S_{jk}^i} |u_i - \bar{u}_{jk}^i|^2$$

$$\le \sum_{k=1}^{N_j^i} \int_{B_i(z_{jk}^i, 2r_j)} |u_i - \bar{u}_{jk}^i|^2$$

$$\le \sum_{k=1}^{N_j^i} C(n)(2r_j)^2 \int_{B_i(z_{jk}^i, 4r_j)} |\nabla u_i|^2$$

$$\le \bar{C}_i C(n) r_j^2 \int_{B_i(x_i, r)} |\nabla u_i|^2 \le C(n, N) r_j^2.$$

The conclusion follows from it, and $\lim_{j \to \infty} r_j = 0$. q.e.d.

It follows from the Cauchy-Schwarz inequality, together with the doubling condition that

$$\bar{u}_{jk}^i \le \frac{1}{\sqrt{\nu_i\left(B_i\left(z_{jk}^i, 2r_j\right)\right)}} \left(\int_{B_i\left(z_{jk}^i, 2r_j\right)} u_i^2\right)^{\frac{1}{2}}$$

$$\le \frac{N}{\sqrt{\nu_i\left(B_i(x_i, r)\right)}} \cdot \left(\frac{r_j + r}{r_j}\right)^{\frac{n}{2}}$$

$$\text{(A.7)} \qquad\qquad \le N \cdot C\left(n, r_0, r, \nu_\infty\left(B_\infty(x, r)\right)\right)$$

note that the bound on the right side is independent of i, k. Hence for fixed j, k, $\{\bar{u}_{jk}^i\}_{i=1}^\infty$ has a convergent subsequence.

There is a measure approximation $\varphi_i : B_i(x_i, r) \to B_\infty(x_\infty, r)$, such that each φ_i is an ϵ_i-approximation for some $\epsilon_i \searrow 0+$. There is a subsequence of $\{i\}$ depending on j, denoted as $\mathcal{I}_j$, such that for every $k = 1, 2, \cdots, N_j^i$,

$$z_{jk} \doteq \lim_{i\to\infty} \varphi_i(z_{jk}^i) , \quad N_j \doteq \lim_{i\to\infty} N_J^i , \quad \bar{u}_{jk} \doteq \lim_{i\to\infty} \bar{u}_{jk}^i$$

all the above limits exist, where $i \in \mathcal{I}_j$.

By (A.7), replacing $\mathcal{I}_j$ with a subset of $\mathcal{I}_j$, also denoted as $\mathcal{I}_j$, we can assume that $N_j = N_j^i$ for all $i \in \mathcal{I}_j$. We may assume that $\mathcal{I}_{j+1} \subset \mathcal{I}_j$ for every j.

Therefore, by a diagonal argument, we find a common cofinal subnet of all $\mathcal{I}_j$, and denote it by $\mathcal{I}$. Set

$$S_{jk} \doteq B_\infty(z_{jk}, 2r_j) \backslash \cup_{l=1}^{k-1} B_\infty(z_{jl}, 2r_j) , \qquad 1 \le k \le N_j.$$

Define

$$\xi[x, a, b](y) = \begin{cases} 1 & if \ \rho_\infty(x, y) \le a \\ \frac{b - \rho_\infty(x,y)}{b-a} & if \ a < \rho_\infty(x, y) < b \\ 0 & if \ \rho_\infty(x, y) \ge b. \end{cases}$$

We see that $\xi[x, a, b]$ is a Lipschitz function with Lipschitz constant $\frac{1}{b-a}$.

For any $\epsilon > 0$, $y \in K_\infty$, we define

$$\zeta_{S_{jk}}^\epsilon(y) = \xi[z_{jk}, r_j - 2\epsilon, r_j - \epsilon](y) \cdot \prod_{l=1}^{k-1} \left\{1 - \xi[z_{jl}, r_j - 2\epsilon, r_j - \epsilon]\right\}.$$

It is easy to check that

$$\lim_{\epsilon\to 0+} |\zeta_{S_{jk}}^\epsilon - \chi_{S_{jk}}|_{L^2(K_\infty)} = 0 , \quad \lim_{\epsilon\to 0+} \lim_{i\to\infty} |\zeta_{S_{jk}}^\epsilon \circ \varphi_i - \chi_{S_{jk}^i}|_{L^2(K_i)} = 0$$

for $i \in \mathcal{I}$ and any $j = 1, 2, \cdots, k = 1, 2, \cdots, N_j$.

For $\bar{u}_{jk} = \lim_{i \to \infty} \bar{u}_{jk}^i$, we define two functions by

$$\bar{u}_j(x) = \sum_{k=1}^{N_j} \chi_{S_{jk}}(x)\bar{u}_{jk} , \quad \tilde{u}_j^\epsilon(x) = \sum_{k=1}^{N_j} \zeta_{S_{jk}}^\epsilon(x)\bar{u}_{jk}.$$

Then

$$\lim_{\epsilon \to \infty} \lim_{i \to \infty} |\tilde{u}_j^\epsilon - \bar{u}_j^i|_{L^2(K_i)}$$
$$\leq \lim_{\epsilon \to \infty} \lim_{i \to \infty} \sum_{k=1}^{N_j} \left[|\bar{u}_{jk}| \cdot |\zeta_{S_{jk}}^\epsilon \circ \varphi_i - \chi_{S_{jk}^i}|_{L^2(K_i)} + \nu_i(K_i)|\bar{u}_{jk} - u_{jk}^i| \right]$$
$$= 0$$

that is $\lim_{\epsilon \to \infty} \lim_{i \to \infty} |\tilde{u}_j^\epsilon - \bar{u}_j^i|_{L^2(K_i)} = 0$.

Hence

$$|\bar{u}_j - \bar{u}_{j'}|_{L^2} \leq \lim_{\epsilon \to \infty} \left(|\bar{u}_j - \tilde{u}_j^\epsilon|_{L^2} + |\bar{u}_{j'} - \tilde{u}_{j'}^\epsilon|_{L^2} + |\tilde{u}_j^\epsilon - \tilde{u}_{j'}^\epsilon|_{L^2} \right)$$
$$\leq \lim_{\epsilon \to \infty} \lim_{i \to \infty} |\tilde{u}_j^\epsilon \circ \varphi_i - \tilde{u}_{j'}^\epsilon \circ \varphi_i|_{L^2}$$
$$\leq \lim_{i \to \infty} |\bar{u}_j^i - \bar{u}_{j'}^i|_{L^2}$$
$$\leq \lim_{i \to \infty} |\bar{u}_j^i - u_i|_{L^2} + \lim_{i \to \infty} |\bar{u}_{j'}^i - u_i|_{L^2}.$$

From Claim A.7, we get that $\{\bar{u}_j\}$ is a Cauchy sequence in $L^2(K_\infty)$, then set $u_\infty \doteq \lim_{j \to \infty} \bar{u}_j \in L^2(K_\infty)$. From the above argument, it is easy to see that $u_i \to u_\infty$ in L^2 sense on K_∞, this completes the proof of Theorem A.5. q.e.d.

References

[BKN12] I. Bakas, S. Kong, and L. Ni, *Ancient solutions of Ricci flow on spheres and generalized Hopf fibrations*, J. Reine Angew. Math. **663** (2012), 209–248, MR 2889711, Zbl 1269.53065.

[CCG+10] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni, *The Ricci flow: techniques and applications. Part III. Geometric-analytic aspects*, Mathematical Surveys and Monographs, 163. American Mathematical Society, Providence, RI, 517 pp, 2010, MR 2604955, Zbl 1216.53057.

[BBI01] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, 33. American Mathematical Society, Providence, RI, xiv+415 pp, 2001, MR 1835418, Zbl 0981.51016.

[Che83] J. Cheeger, *Spectral geometry of singular Riemannian spaces*, J. Differential Geom. **18** (1983), no. 4, 575–657, MR 0730920, Zbl 0529.58034.

[Che99] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal. **9** (1999), no. 3, 428–517, MR 1708448, Zbl 0942.58018.

[CC96] J. Cheeger and T. H. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. of Math. (2) **144** (1996), no. 1, 189–237, MR 1405949, Zbl 0865.53037.

[CC97]   J. Cheeger and T. H. Colding, *On the structure of spaces with Ricci curvature bounded below. I*, J. Differential Geom. **46** (1997), no. 3, 406–480, MR 1484888, Zbl 0902.53034.

[CC00a] J. Cheeger and T. H. Colding, *On the structure of spaces with Ricci curvature bounded below. II*, J. Differential Geom. **54** (2000), no. 1, 13–35, MR 1815410, Zbl 1027.53042.

[CC00b] J. Cheeger and T. H. Colding, *On the structure of spaces with Ricci curvature bounded below. III*, J. Differential Geom. **54** (2000), no. 1, 37–74, MR 1815411, Zbl 1027.53043.

[Che75] S. Y. Cheng, *Eigenvalue comparison theorems and its geometric applications*, Math. Z. **143** (1975), no. 3, 289–297, MR 0378001, Zbl 0329.53035.

[CY75]   S. Y. Cheng and S.-T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975), no. 3, 333–354, MR 0385749, Zbl 0312.53031.

[CM97a] T. H. Colding and W. P. Minicozzi II, *Harmonic functions on manifolds*, Ann. of Math. (2) **146** (1997), no. 3, 725–747, MR 1491451, Zbl 0928.53030.

[CM97b] T. H. Colding and W. P. Minicozzi II, *Large scale behavior of kernels of Schrödinger operators*, Amer. J. Math. **119** (1997), no. 6, 1355–1398, MR 1481818, Zbl 0897.58041.

[Dav89] E. B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, 92. Cambridge University Press, Cambridge, x+197 pp, 1989, MR 0990239, Zbl 0699.35006.

[Din02] Y. Ding, *Heat kernels and Green's functions on limit spaces*, Comm. Anal. Geom. **10** (2002), no. 3, 475–514, MR 1912256, Zbl 1018.58013.

[Dod83] J. Dodziuk, *Maximum principle for parabolic inequalities and the heat flow on open manifolds*, Indiana Univ. Math. J. **32** (1983), no. 5, 703–716, MR 0711862, Zbl 0526.58047.

[Fuk87] K. Fukaya, *Collapsing of Riemannian manifolds and eigenvalues of Laplace operator*, Invent. Math. **87** (1987), no. 3, 517–547, MR 0874035, Zbl 0589.58034.

[GT01]   D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, xiv+517 pp, 2001, MR 1814364, Zbl 1042.35002.

[GH]   A. Grigor'yan and J. Hu, *Upper bounds of heat kernels on doubling spaces*, Preprint.

[GH08]   A. Grigor'yan and J. Hu, *Off-diagonal upper estimates for the heat kernel of the Dirichlet forms on metric spaces*, Invent. Math. **174** (2008), no. 1, 81–126, MR 2430977, Zbl 1154.47034.

[GHL10] A. Grigor'yan, J. Hu, and K.-S. Lau, *Comparison inequalities for heat semigroups and heat kernels on metric measure spaces*, J. Funct. Anal. **259** (2010), no. 10, 2613–2641, MR 2679020, Zbl 1207.58021.

[Gro99] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics, 152. Birkhäuser Boston, Inc., Boston, MA, xx+585 pp, 1999, MR 1699320, Zbl 0953.53002.

[HK95]   P. Hajłasz and P. Koskela, *Sobolev meets Poincaré*, C. R. Acad. Sci. Paris Sér. I Math. **320** (1995), no. 10, 1211–1215, MR 1336257, Zbl 0837.46024.

[Kat95] T. Kato, *Perturbation theory for linear operators*, Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, xxii+619 pp., 1995, MR 1335452, Zbl 0836.47009.

[KS03] K. Kuwae and T. Shioya, *Convergence of spectral structures: a functional analytic theory and its applications to spectral geometry*, Comm. Anal. Geom. **11** (2003), no. 4, 599–673, MR 2015170, Zbl 1092.53026.

[KS08] K. Kuwae and T. Shioya, *Variational convergence over metric spaces*, Trans. Amer. Math. Soc. **360** (2008), no. 1, 35–75, MR 2341993, Zbl 1127.53034.

[Li80] P. Li, *On the Sobolev constant and the p-spectrum of a compact Riemannian manifold*, Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 4, 451–468, MR 0608289, Zbl 0466.53023.

[Li86] P. Li, *Large time behavior of the heat equation on complete manifolds with nonnegative Ricci curvature*, Ann. of Math. (2) **124** (1986), no. 1, 1–21, MR 0847950, Zbl 0613.58032.

[Li12] P. Li, *Geometric analysis*, Cambridge Studies in Advanced Mathematics, 134. Cambridge University Press, Cambridge, x+406 pp, 2012, MR 2962229, Zbl 1246.53002.

[LS84] P. Li and R. Schoen, *L^p and mean value properties of subharmonic functions on Riemannian manifolds*, Acta Math **153** (1984), no. 3-4, 279–301, MR 0766266, Zbl 0556.31005.

[LT91] P. Li and L.-F. Tam, *The heat equation and harmonic maps of complete manifolds*, Invent. Math. **105** (1991), no. 1, 1–46, MR 1109619, Zbl 0748.58006.

[LTW97] P. Li, L.-F. Tam, and J. Wang, *Sharp bounds for the Green's function and the heat kernel*, Math. Res. Lett. **4** (1997), no. 4, 589–602, MR 1470428, Zbl 0889.58074.

[LY86] P. Li and S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math **156** (1986), no. 3-4, 153–201, MR 0834612, Zbl 0611.58045.

[LM72] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications. Vol. I.*, Translated from the French by P. Kenneth. Die Grundlehren der mathematischen Wissenschaften, Band 181. Springer-Verlag, New York-Heidelberg, xvi+357 pp, 1972, MR 0350177, Zbl 0223.35039.

[Mat95] P. Mattila, *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability*, Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, Cambridge, xii+343 pp, 1995, MR 1333890, Zbl 0819.28004.

[Per97] G. Perelman, *A complete Riemannian manifold of positive Ricci curvature with Euclidean volume growth and nonunique asymptotic cone*, Comparison geometry (Berkeley, CA, 199394), Math. Sci. Res. Inst. Publ. **30** (1997), 165–166, MR 1452873, Zbl 0887.53038.

[RR93] M. Renardy and R. C. Rogers, *An introduction to partial differential equations*, Texts in Applied Mathematics, 13. Springer-Verlag, New York, xiv+428 pp, 1993, MR 1211418, Zbl 0917.35001.

[SC92a] L. Saloff-Coste, *Uniformly elliptic operators on Riemannian manifolds*, J. Differential Geom. **36** (1992), no. 2, 417–450, MR 1180389, Zbl 0735.58032.

[SC92b] L. Saloff-Coste, *A note on Poincaré, Sobolev, and Harnack inequalities*, Internat. Math. Res. Notices. **2** (1992), 27–38, MR 1150597, Zbl 0769.58054.

[SC02] L. Saloff-Coste, *Aspects of Sobolev-type inequalities*, London Mathematical Society Lecture Note Series, 289. Cambridge University Press, Cambridge, x+190 pp., 2002, MR 1872526, Zbl 0991.35002.

[SY94] R. Schoen and S.-T. Yau, *Lectures on differential geometry*, International Press, Cambridge, MA, 414 pp, 1994, MR 1333601, Zbl 0830.53001.

[Stu94] K.-T. Sturm, *Analysis on local Dirichlet spaces. I. Recurrence, conservativeness and L^p-Liouville properties*, J. Reine Angew. Math. **456** (1994), 173–196, MR 1301456, Zbl 0806.53041.

[Stu95] K.-T. Sturm, *Analysis on local Dirichlet spaces. II. Upper Gaussian estimates for the fundamental solutions of parabolic equations*, Osaka J. Math. **32** (1995), no. 2, 275–312, MR 1355744, Zbl 0854.35015.

[Stu96] K.-T. Sturm, *Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality*, J. Math. Pures Appl. (9) **75** (1996), no. 3, 273–297, MR 1387522, Zbl 0854.35016.

[Stu98] K.-T. Sturm, *Diffusion processes and heat kernels on metric spaces*, Ann. Probab. **26** (1998), no. 1, 1–55, MR 1617040, Zbl 0936.60074.

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