HYPERBOLIC ALEXANDROV-FENCHEL QUERMASSINTEGRAL INEQUALITIES II

Yuxin Ge, Guofang Wang & Jie Wu

Abstract

In this paper we first establish an optimal Sobolev-type inequality for hypersurfaces in $\mathbb{H}^n$ (see Theorem 1.1). As an application we obtain hyperbolic Alexandrov–Fenchel inequalities for curvature integrals and quermassintegrals. More precisely, we prove a geometric inequality in the hyperbolic space $\mathbb{H}^n$, which is a hyperbolic Alexandrov–Fenchel inequality,

$$\int_\Sigma \sigma_{2k} \geq C_{n-1}^{2k} \omega_{n-1} \left\{ \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{1}{k}} + \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{1}{k} \frac{n-1-2k}{n-1}} \right\}^k,$$

when Σ is a horospherical convex and $2k \leq n - 1$. Equality holds if and only if Σ is a geodesic sphere in $\mathbb{H}^n$. Here $\sigma_j = \sigma_j(\kappa)$ is the jth mean curvature and $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_{n-1})$ is the set of the principal curvatures of Σ. Also, an optimal inequality for quermassintegral in $\mathbb{H}^n$ is

$$W_{2k+1}(\Omega) \geq \frac{\omega_{n-1}}{n} \sum_{i=0}^k \frac{n-1-2k}{n-1-2k+2i} C_k^i \left(\frac{nW_1(\Omega)}{\omega_{n-1}} \right)^{\frac{n-1-2k+2i}{n-1}},$$

provided that $\Omega \subset \mathbb{H}^n$ is a domain with $\Sigma = \partial\Omega$ horospherical convex, where $2k \leq n - 1$. Equality holds if and only if Σ is a geodesic sphere in $\mathbb{H}^n$. Here $W_r(\Omega)$ is quermassintegrals in integral geometry.

1. Introduction

In this paper we first establish Sobolev type inequalities for hypersurfaces in the hyperbolic space $\mathbb{H}^n$. Let g be a Riemannian metric on a Riemannian manifold. Its kth Gauss–Bonnet curvature (or Lovelock curvature) L_k is defined by (see [13], for example)

$$(1.1) \qquad L_k := \frac{1}{2^k} \delta_{j_1 j_2 \cdots j_{2k-1} j_{2k}}^{i_1 i_2 \cdots i_{2k-1} i_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}}.$$

Here $R_{ij}{}^{kl}$ is the Riemannian curvature tensor in the local coordinates with respect to g, and the generalized Kronecker delta is defined by

$$\delta_{i_1 i_2 \ldots i_r}^{j_1 j_2 \ldots j_r} = \det \begin{pmatrix} \delta_{i_1}^{j_1} & \delta_{i_1}^{j_2} & \cdots & \delta_{i_1}^{j_r} \\ \delta_{i_2}^{j_1} & \delta_{i_2}^{j_2} & \cdots & \delta_{i_2}^{j_r} \\ \vdots & \vdots & \vdots & \vdots \\ \delta_{i_r}^{j_1} & \delta_{i_r}^{j_2} & \cdots & \delta_{i_r}^{j_r} \end{pmatrix}.$$

When $k = 1$, L_1 is just the scalar curvature R. When $k = 2$, it is the so-called (second) Gauss-Bonnet curvature

$$L_2 = \|Rm\|^2 - 4\|Ric\|^2 + R^2,$$

where Rm, Ric are the Riemannian curvature tensor and the Ricci tensor with respect to g, respectively. The Gauss–Bonnet curvature L_k is a very natural generalization of the scalar curvature. When the underlying manifold is local conformally flat, L_k equals the σ_k-scalar curvature up to a constant multiple; more precisely (cf. [14])

(1.2) $L_k = 2^k k!(n - 1 - k)(n - 2 - k) \cdots (n - 2k)\sigma_k(g).$

Here the σ_k-scalar curvature was introduced in Viaclovsky [30] by

(1.3) $$\sigma_k(g) := \sigma_k(\Lambda_g),$$

and Λ_g is the set of the eigenvalues of the Schouten tensor A_g defined by

(1.4) $$A_g = \frac{1}{n-3}\left(Ric_g - \frac{R_g}{2(n-2)}g\right).$$

Here we consider the $(n-1)$-dimensional manifold M with metric g. The σ_k-scalar curvature is also a very natural generalization of the scalar curvature R (in fact, $\sigma_1(g) = \frac{1}{2(n-2)}R$) and has been intensively studied in the fully non-linear Yamabe problem. The fully non-linear Yamabe problem for σ_k is a generalization of ordinary Yamabe problem for the scalar curvature R. In the ordinary Yamabe problem, the following functional, the so-called Yamabe functional, plays a crucial role:

(1.5) $$\mathcal{F}_1(g) = (vol(g))^{-\frac{n-3}{n-1}} \int R_g d\mu(g).$$

For a given conformal class $[g] = \{e^{-2u}g \,|\, u \in C^\infty(M)\}$, the Yamabe constant is defined by

$$Y_1([g]) = \inf_{\tilde{g} \in [g]} \mathcal{F}_1(\tilde{g}).$$

By the resolution of the Yamabe problem, Aubin and Schoen [2, 25] proved that for any metric g on M,

(1.6) $$Y_1([g]) \le Y_1([g_{\mathbb{S}^{n-1}}])$$

and

$$Y_1([g]) < Y_1([g_{\mathbb{S}^{n-1}}]) \text{ for any } (M, [g]) \text{ other than } [g_{\mathbb{S}^{n-1}}],$$

where $[g_{\mathbb{S}^{n-1}}]$ is the conformal class of the standard round metric on the sphere $\mathbb{S}^{n-1}$. From this, one can see the importance of the constant $Y_1([g_{\mathbb{S}^{n-1}}])$. In fact, one can prove that

$$(1.7) \qquad Y_1([g_{\mathbb{S}^{n-1}}]) = (n-1)(n-2)\omega_{n-1}^{\frac{2}{n-1}},$$

where ω_{n-1} is the volume of $g_{\mathbb{S}^{n-1}}$. It is easy to see that (1.6), together with (1.7), is equivalent to

$$(1.8) \qquad \int_M L_1 d\mu(g) = \int_M R_g d\mu(g) \geq (n-1)(n-2)\omega_{n-1}^{\frac{2}{n-1}} vol(g)^{\frac{n-3}{n-1}},$$

for any $g \in [g_{\mathbb{S}^{n-1}}]$, which is in fact an optimal Sobolev inequality. See [20]. As a natural generalization, we proved in [19] a generalized Sobolev inequality for σ_k-scalar curvature $\sigma_k(g)$, which states

$$(1.9) \qquad \int_M \sigma_k(g) d\mu(g) \geq \frac{C_{n-1}^k}{2^k} \omega_{n-1}^{\frac{2k}{n-1}} vol(g)^{\frac{n-1-2k}{n-1}},$$

for any $g \in \mathcal{C}_k([g_{\mathbb{S}^{n-1}}])$, where $\mathcal{C}_k([g_{\mathbb{S}^{n-1}}]) = [g_{\mathbb{S}^{n-1}}] \cap \Gamma_k^+$ and $\Gamma_k^+ = \{g \,|\, \sigma_j(g) > 0, \forall j \leq k\}$. In this paper, we denote $C_{n-1}^k = \frac{(n-1)!}{k!(n-1-k)!}$. By (1.2), inequality (1.9) can be written in the form

$$(1.10) \qquad \int_\Sigma L_k d\mu(g) \geq C_{n-1}^{2k}(2k)! \, \omega_{n-1}^{\frac{2k}{n-1}} (vol(g))^{\frac{n-1-2k}{n-1}},$$

for any $g \in \mathcal{C}_{k-1}([g_{\mathbb{S}^{n-1}}])$. We call both inequalities (1.8) and (1.10) *optimal Sobolev inequalities* and would like to investigate which classes of metrics satisfy the optimal Sobolev inequalities. (1.8) and (1.10) imply that a suitable subclass of the conformal class of the standard round metric satisfies the optimal Sobolev inequalities. From (1.6) we know in any conformal class other than the conformal class of the standard round metric that there exist many metrics which do not satisfy the optimal Sobolev inequality. Hence it is natural to ask if there are other interesting classes of metrics that satisfy the optimal Sobolev inequality. Observe that for a closed hypersurface Σ in $\mathbb{R}^n$,

$$(1.11) \qquad L_k = (2k)! \, \sigma_{2k},$$

where σ_{2k} is the $2k$-mean curvature of Σ, which is defined by

$$\sigma_j = \sigma_j(\kappa),$$

where $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_{n-1})$, κ_j $(1 \leq j \leq n-1)$ is the principal curvature of B and B is the second fundamental form of Σ induced by the

standard Euclidean metric. The classical Alexandrov–Fenchel inequality (see [27] for instance) implies for convex hypersurfaces in $\mathbb{R}^n$ that

$$(1.12) \quad \int_\Sigma L_k(g) d\mu(g) = (2k)! \int_\Sigma \sigma_{2k} d\mu(g) \geq C_{n-1}^{2k}(2k)! \, \omega_{n-1}^{\frac{2k}{n-1}} |\Sigma|^{\frac{n-1-2k}{n-1}}.$$

In this paper we use $|\Sigma|$ to denote the area of Σ with respect to the induced metric. Inequality (1.12) implies that the induced metric of any convex hypersurfaces in $\mathbb{R}^n$ satisfy the optimal Sobolev inequalities. The convexity can be weakened. See the work of Guan and Li [18], Huisken [21] and Chang and Wang [6].

In this paper we prove that the induced metric of horospherical convex hypersurfaces in $\mathbb{H}^n$ also satisfies the optimal Sobolev inequalities.

Theorem 1.1. *Let* $2k < n - 1$. *Any horospherical convex hypersurfaces* Σ *in* $\mathbb{H}^n$ *satisfies*

$$(1.13) \quad \int_\Sigma L_k d\mu(g) \geq C_{n-1}^{2k}(2k)! \, \omega_{n-1}^{\frac{2k}{n-1}} |\Sigma|^{\frac{n-1-2k}{n-1}},$$

and equality holds if and only if Σ *is a geodesic sphere.*

A hypersurface in $\mathbb{H}^n$ is *horospherical convex* if all principal curvatures are larger than or equal to 1. The horospherical convexity is a natural geometric concept. For any hypersurface in $\mathbb{H}^n$, the Gauss-Bonnet curvature L_k of the induced metric of the hypersurface can be expressed in terms of the curvature integrals by (see also Lemma 3.1 below)

$$(1.14) \quad L_k = C_{n-1}^{2k}(2k)! \sum_{j=0}^{k} (-1)^j \frac{C_k^j}{C_{n-1}^{2k-2j}} \sigma_{2k-2j}.$$

Comparing (1.12) for $\mathbb{R}^n$ with (1.13) for $\mathbb{H}^n$ and (1.11) with (1.14), we obtain the same inequality for L_k, though L_k has different expression in terms of the curvature integrals. We remark that when $2k = n-1$, (1.13) becomes an equality for any hypersurface diffeomorphic to a sphere, i.e.,

$$\int_\Sigma L_{\frac{n-1}{2}} dv(g) = (n-1)! \omega_{n-1}.$$

This follows from the Gauss–Bonnet–Chern theorem.

As a first direct application, we establish Alexandrov–Fenchel type inequalities for curvature integrals.

Theorem 1.2. *Let* $2k \leq n-1$. *Any horospherical convex hypersurface* $\Sigma \subset \mathbb{H}^n$ *satisfies*

$$(1.15) \quad \int_\Sigma \sigma_{2k} \geq C_{n-1}^{2k} \omega_{n-1} \left\{ \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{1}{k}} + \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{1}{k} \frac{n-1-2k}{n-1}} \right\}^k,$$

and equality holds if and only if Σ *is a geodesic sphere.*

When $k = 1$, Theorem 1.1 (and hence Theorem 1.2) are true even for any star-shaped and two-convex hypersurfaces in $\mathbb{H}^n$, ie., $\sigma_1 \geq 0$ and $\sigma_2 \geq 0$, which were proved by Li-Wei-Xiong in a recent work [22]. Theorem 1.1 was proved for $k = 2$ in our recent paper [15]. Due to the complication of the variational structure of $\int \sigma_k$ in the hyperbolic space, the case $k \geq 2$ is quite different from the case $k = 1$. For the case $k \geq 2$, the horospherical convexity of the hypersurface Σ plays an essential role.

At the end of this paper we show that a similar inequality holds for σ_1 and propose a conjecture for general odd σ_{2k+1}.

Another application is an optimal inequality for quermassintegrals in $\mathbb{H}^n$. For a (geodesically) convex domain $\Omega \subset \mathbb{H}^n$ with $\Sigma = \partial\Omega$, quermassintegrals are defined by

$$(1.16) \qquad W_r(\Omega) := \frac{(n-r)\omega_{r-1}\cdots\omega_0}{n\omega_{n-2}\cdots\omega_{n-r-1}} \int_{\mathcal{L}_r} \chi(L \cap \Omega)dL,$$

where $\mathcal{L}_r$ is the space of r-dimensional totally geodesic subspaces L in $\mathbb{H}^n$, ω_r is the area of the r-dimensional standard round sphere, and dL is the natural (invariant) measure on $\mathcal{L}_r$ (cf. [[24, 28]]). As in the Euclidean case we take $W_0(\Omega) = Vol(\Omega)$. With these definitions, unlike the euclidean case, the quermassintegral in $\mathbb{H}^n$ do not coincide with the mean curvature integrals, but they are closely related (cf. [28]):

$$(1.17) \qquad \frac{1}{C_{n-1}^r} \int_{\Sigma} \sigma_r = n\left(W_{r+1}(\Omega) + \frac{r}{n-r+1}W_{r-1}(\Omega)\right),$$

with

$$W_0(\Omega) = Vol(\Omega), \quad W_1(\Omega) = \frac{1}{n}|\Sigma|.$$

The relationship between W_0 and W_1, the hyperbolic isoperimetric inequality, was established by Schmidt [26] 70 years ago. When $n = 2$, the hyperbolic isoperimetric inequality is

$$L^2 \geq 4\pi A + A^2,$$

where L is the length of a curve γ in $\mathbb{H}^2$ and A is the area of the enclosed domain by γ. In general, this hyperbolic isoperimetric inequality has no explicit form. There have been many attempts to establish a relationship between $W_k(\Omega)$ in the hyperbolic space $\mathbb{H}^n$. See, for example, [24] and [29]. In [11], Gallego and Solanes proved by using integral geometry the following interesting inequality for convex domains in $\mathbb{H}^n$:

$$(1.18) \qquad W_r(\Omega) > \frac{n-r}{n-s}W_s(\Omega), \quad r > s,$$

which implies

$$(1.19) \qquad \int_{\Sigma} \sigma_k d\mu > cC_{n-1}^k|\Sigma|,$$

where $c = 1$ if $k > 1$ and $c = (n-2)/(n-1)$ if $k = 1$ and $|\Sigma|$ is the area of Σ. Here $d\mu$ is the area element of the induced metric. The constants in (1.18) and (1.19) are optimal in the sense that one cannot replace them by bigger constants. However, for fixed $|\Sigma|$ these inequalities are far from sharp.

As another application of Theorem 1.1, we have the following optimal inequalities of $W_k(\Omega)$ for general odd k in terms of $W_1 = \frac{1}{n}|\Sigma|$.

Theorem 1.3. *Let* $2k \leq n - 1$. *If* $\Omega \subset \mathbb{H}^n$ *is a domain with* $\Sigma = \partial\Omega$ *horospherical convex, then*

$$(1.20) \quad W_{2k+1}(\Omega) \geq \frac{\omega_{n-1}}{n} \sum_{i=0}^{k} \frac{n-1-2k}{n-1-2k+2i} C_k^i \left(\frac{nW_1(\Omega)}{\omega_{n-1}} \right)^{\frac{n-1-2k+2i}{n-1}},$$

where ω_{n-1} *is the area of the unit sphere* $\mathbb{S}^{n-1}$. *Equality holds if and only if* Σ *is a geodesic sphere.*

As a direct corollary, we solve an isoperimetric problem for horospherical convex surfaces with fixed W_1.

Corollary 1.4. *Let* $2k \leq n - 1$. *In the class of horospherical convex hypersurfaces in* $\mathbb{H}^n$ *with fixed* W_1, *the minimum of* W_{2k+1} *is achieved by and only by the geodesic spheres.*

Corollary 1.4 answers a question asked in the paper of Gao, Hug, and Schneider [**12**] in this case.

In order to prove Theorem 1.1, motivated by [**15**] and [**22**] (see also [**4**] and [**9**]), we consider the following functional:

$$(1.21) \qquad\qquad Q(\Sigma) := |\Sigma|^{-\frac{n-1-2k}{n-1}} \int_{\Sigma} L_k.$$

Here L_k is the Gauss–Bonnet curvature with respect to the induced metric g on Σ. This is a Yamabe type functional. One of crucial points of this paper is to show that functional Q is non-increasing under the inverse curvature flow

$$(1.22) \qquad\qquad \frac{\partial\Sigma_t}{\partial t} = \frac{n-2k}{2k} \frac{\sigma_{2k-1}}{\sigma_{2k}} \nu,$$

where ν is the outer normal of Σ_t, provided that the initial hypersurface is horospherical convex. One can show that horospherical convexity is preserved by flow (1.22). By the convergence results of Gerhardt [**16**] on the inverse curvature flow (1.22), we show that the flow approaches surfaces whose induced metrics belong to the conformal class of the standard round sphere metric. Therefore, we can use the result (1.10) to show that

$$Q(\Sigma) \geq \lim_{t\to\infty} Q(\Sigma_t) \geq C_{n-1}^{2k}(2k)! \omega_{n-1}^{\frac{2k}{n-1}}.$$

The rest of this paper is organized as follows. In Section 2, we present some basic facts about the elementary functions σ_k and recall the generalized Sobolev inequality (2.9) from [**19**]. In Section 3, we present the relationship between various geometric quantities, including the intrinsic geometric quantities $\int_\Sigma L_k$, the curvature integrals $\int_\Sigma \sigma_k$, and the quermassintegrals $W_r(\Omega)$. In Section 4, we prove the crucial monotonicity of Q and analyze its asymptotic behavior under flow (1.22). The proof of our main theorems are given in Section 5. In Section 6, we show that a similar inequality holds for σ_1 and propose a conjecture for integral integrals σ_{2k+1}.

2. Preliminaries

Let σ_k be the kth elementary symmetry function $\sigma_k : \mathbb{R}^{n-1} \to \mathbb{R}$ defined by

$$\sigma_k(\Lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k} \quad \text{for } \Lambda = (\lambda_1, \cdots, \lambda_{n-1}) \in \mathbb{R}^{n-1}.$$

For a symmetric matrix B, denote $\lambda(B) = (\lambda_1(B), \cdots, \lambda_n(B))$ the eigenvalues of B. We set

$$\sigma_k(B) := \sigma_k(\lambda(B)).$$

The Garding cone Γ_k^+ is defined as

$$\Gamma_k^+ = \{\Lambda \in \mathbb{R}^{n-1} \,|\, \sigma_j(\Lambda) > 0, \quad \forall j \leq k\}.$$

A symmetric matrix B is said to belong to Γ_k^+ if $\lambda(B) \in \Gamma_k^+$. We collect the basic facts about σ_k, which will be directly used in this paper. For other related facts, see a survey by Guan [**17**, **22**].

$$(2.1) \qquad \sigma_k(B) = \frac{1}{k!} \delta_{j_1 \cdots j_k}^{i_1 \cdots i_k} b_{i_1}^{j_1} \cdots b_{i_k}^{j_k},$$

where $B = (b_j^i)$. In the following, for simplicity of notation we denote

$$p_k = \frac{\sigma_k}{C_{n-1}^k}.$$

Lemma 2.1. *For $\Lambda \in \Gamma_k^+$, we have the following Newton–MacLaurin inequalities*

(2.2) $$\frac{p_{k-1}p_{k+1}}{p_k^2} \leq 1,$$

(2.3) $$\frac{p_1 p_{k-1}}{p_k} \geq 1.$$

Moreover, equality holds in (2.2) or (2.3) at Λ if and only if $\Lambda = c(1, 1, \cdots, 1)$.

The Newton–MacLaurin inequalities play a very important role in proving the geometric inequalities mentioned above. However, we will see that these inequalities are not precise enough to show our inequality (1.13).

Let $\mathbb{H}^n = \mathbb{R}^+ \times \mathbb{S}^{n-1}$ with the hyperbolic metric

$$\bar{g} = dr^2 + \sinh^2 r g_{\mathbb{S}^{n-1}},$$

where $g_{\mathbb{S}^{n-1}}$ is the standard round metric on the unit sphere $\mathbb{S}^{n-1}$ and $\Sigma \subset \mathbb{H}^n$ a smooth closed hypersurface in $\mathbb{H}^n$ with a unit outward normal ν. Let h be the second fundamental form of Σ, and let $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$ be the set of principal curvatures of Σ in $\mathbb{H}^n$ with respect to ν. The kth mean curvature of Σ is defined by

$$\sigma_k = \sigma_k(\kappa).$$

We now consider the curvature evolution equation

(2.4) $$\frac{d}{dt}X = F\nu,$$

where $\Sigma_t = X(t, \cdot)$ is a family of hypersurfaces in $\mathbb{H}^n$, ν is the unit outward normal to $\Sigma_t = X(t, \cdot)$, and F is a speed function which may depend on the position vector X and principal curvatures of Σ_t. However, for simplicity of notation, we will omit the subscript t if there is no confusion. One can check that [**23**] along the flow

(2.5) $$\frac{d}{dt}\int_\Sigma \sigma_k d\mu = (k+1)\int_\Sigma F\sigma_{k+1}d\mu + (n-k)\int_\Sigma F\sigma_{k-1}d\mu,$$

and thus

(2.6) $$\frac{d}{dt}\int_\Sigma p_k d\mu = \int_\Sigma \left((n-k-1)p_{k+1} + kp_{k-1}\right)Fd\mu.$$

If one compares flow (2.4) in $\mathbb{H}^n$ with a similar flow of hypersurfaces in $\mathbb{R}^n$, the last term in (2.5) is an extra term. This extra term comes from -1, the sectional curvature of $\mathbb{H}^n$, and makes the phenomenon of $\mathbb{H}^n$ quite different from the one of $\mathbb{R}^n$.

As mentioned above, we use the following inverse flow

(2.7) $$\frac{d}{dt}X = \frac{p_{2k-1}}{p_{2k}}\nu.$$

By using the result of Gerhardt [16] we obtain the following proposition.

Proposition 2.2. *If the initial hypersurface Σ is horospherical convex, then the solution for the flow (2.7) exists for all time $t > 0$ and preserves the condition of horospherical convexity. Moreover, the hypersurfaces Σ_t become more and more umbilical in the sense of*

$$|h_i{}^j - \delta_i{}^j| \leq Ce^{-\frac{t}{n-1}}, \quad t > 0,$$

i.e., the principal curvatures are uniformly bounded and converge exponentially fast to one. Here $h_i{}^j = g^{ik}h_{kj}$, where g is the induced metric and h is the second fundamental form.

Proof. For the long-time existence of the inverse curvature flow, see the work of Gerhardt [16]. The preservation of the horospherical convexity along flow (2.7) was proved in [15] with the help of a maximum principle for tensors of Andrews [1].   q.e.d.

Let g be a Riemannian metric on M^{n-1}. Denote Ric_g and R_g the Ricci tensor and the scalar curvature of g, respectively. The Schouten tensor A_g is defined by (1.4). The σ_k-scalar curvature, which was introduced by Viaclovsky [30], is defined by

$$\sigma_k(g) := \sigma_k(A_g).$$

This is a natural generalization of the scalar curvature R. In fact, $\sigma_1(g) = \frac{1}{2(n-2)}R$. Recall that M is of dimension $n - 1$. We now consider the conformal class $[g_{\mathbb{S}^{n-1}}]$ of the standard sphere $\mathbb{S}^{n-1}$ and the following functionals defined by

$$(2.8) \quad \mathcal{F}_k(g) = vol(g)^{-\frac{n-1-2k}{n-1}} \int_{\mathbb{S}^{n-1}} \sigma_k(g)\,d\mu, \quad k = 0, 1, \ldots, n-1.$$

If a metric g satisfies $\sigma_j(g) > 0$ for any $j \leq k$, we call it k-positive and denote $g \in \Gamma_k^+$. From Theorem 1.A in [19], we have the following.

Proposition 2.3. *Let $0 < k < \frac{n-1}{2}$ and $g \in [g_{\mathbb{S}^{n-1}}]$ k-positive. We have*

$$(2.9) \qquad\qquad \mathcal{F}_k(g) \geq \mathcal{F}_k(g_{\mathbb{S}^{n-1}}) = \frac{C_{n-1}^k}{2^k}\omega_{n-1}^{\frac{2k}{n-1}}.$$

Inequality (2.9) is a generalized Sobolev inequality, since when $k = 1$ inequality (2.9) is just the optimal Sobolev inequality. See, for example, [20]. For another Sobolev inequalities, see also [3] and [7].

3. Relationship between various geometric quantities

The Gauss–Bonnet curvatures L_k, and hence $\int_\Sigma L_k$, are intrinsic geometric quantities, which depend only on the induced metric g on Σ and do not depend on the embeddings of (Σ, g). Lemma 3.2 and Lemma

3.3 below imply that σ_{2k}, $\int_\Sigma \sigma_{2k}$ and W_{2k+1} are also intrinsic. σ_{2k+1}, $\int_\Sigma \sigma_{2k+1}$, and W_{2k} are extrinsic. The functionals $\int_\Sigma L_k$ are new geometric quantities for the study of the integral geometry in $\mathbb{H}^n$. In this section we present the relationship between these geometric quantities.

We first have a relation between L_k and σ_k.

Lemma 3.1. *For a hypersurface* (Σ, g) *in* $\mathbb{H}^n$, *its Gauss–Bonnet curvature* L_k *can be expressed by higher-order mean curvatures*

$$(3.1) \qquad L_k = C_{n-1}^{2k}(2k)! \sum_{i=0}^k C_k^i(-1)^i p_{2k-2i}.$$

Hence we have

$$(3.2) \qquad \begin{aligned} \int_\Sigma L_k &= C_{n-1}^{2k}(2k)! \sum_{i=0}^k C_k^i(-1)^i \int_\Sigma p_{2k-2i} \\ &= C_{n-1}^{2k}(2k)! \sum_{i=0}^k (-1)^i \frac{C_k^i}{C_{n-1}^{2k-2i}} \int_\Sigma \sigma_{2k-2i}. \end{aligned}$$

Proof. First recall the Gauss formula

$$R_{ij}{}^{sl} = (h_i^s h_j^l - h_i^l h_j^s) - (\delta_i^s \delta_j^l - \delta_i^l \delta_j^s),$$

where $h_i^j := g^{is} h_{sj}$ and h is the second fundamental form. Then substituting the Gauss formula above into (1.1) and recalling (2.1), we have by a straightforward calculation

$$\begin{aligned} L_k &= \frac{1}{2^k} \delta_{j_1 j_2 \cdots j_{2k-1} j_{2k}}^{i_1 i_2 \cdots i_{2k-1} i_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}} \\ &= \delta_{j_1 j_2 \cdots j_{2k-1} j_{2k}}^{i_1 i_2 \cdots i_{2k-1} i_{2k}} (h_{i_1}^{j_1} h_{i_2}^{j_2} - \delta_{i_1}^{j_1} \delta_{i_2}^{j_2}) \cdots (h_{i_{2k-1}}^{j_{2k-1}} h_{i_{2k}}^{j_{2k}} - \delta_{i_{2k-1}}^{j_{2k-1}} \delta_{i_{2k}}^{j_{2k}}) \\ &= \sum_{i=0}^k C_k^i(-1)^i(n-2k)(n-2k+1)\cdots(n-1-2k+2i)\big((2k-2i)!\sigma_{2k-2i}\big) \\ &= C_{n-1}^{2k}(2k)! \sum_{i=0}^k C_k^i(-1)^i p_{2k-2i}. \end{aligned}$$

Here in the second equality we use the symmetry of generalized Kronecker delta, and in the third equality we use (2.1) and the basic property of the generalized Kronecker delta

$$(3.3) \qquad \delta_{j_1 j_2 \cdots j_{p-1} j_p}^{i_1 i_2 \cdots i_{p-1} i_p} \delta_{i_1}^{j_1} = (n-p) \delta_{j_2 j_3 \cdots j_p}^{i_2 i_3 \cdots i_p},$$

which follows from the Laplace expansion of determinant.   q.e.d.

Motivated by the expression (3.1), we introduce the following notations:

$$(3.4) \qquad \widetilde{L}_k = \sum_{i=0}^k C_k^i(-1)^i p_{2k-2i}, \qquad \widetilde{N}_k = \sum_{i=0}^k C_k^i(-1)^i p_{2k-2i+1}.$$

It is clear that

$$L_k = (2k)! C_{n-1}^{2k} \widetilde{L}_k, \qquad N_k = (2k)! C_{n-1}^{2k} \widetilde{N}_k.$$

Lemma 3.2. *We have*

$$(3.5) \qquad \sigma_{2k} = C_{n-1}^{2k} p_{2k} = C_{n-1}^{2k} \left(\sum_{i=0}^{k} C_k^i \widetilde{L}_i \right),$$

and hence

$$\int_{\Sigma} \sigma_{2k} = C_{n-1}^{2k} \sum_{i=0}^{k} C_k^i \int_{\Sigma} \widetilde{L}_i = \frac{1}{(2k)!} \sum_{i=0}^{k} C_k^i \int_{\Sigma} L_i.$$

To show Theorem 1.3 below, we need the following lemma.

Lemma 3.3. *The quermassintegral W_{2k+1} can be expressed in terms of the integral of $\widetilde{L}_i$:*

$$(3.6) \qquad W_{2k+1}(\Omega) = \frac{1}{n} \sum_{i=0}^{k} C_k^i \frac{n-1-2k}{n-1-2k+2i} \int_{\Sigma} \widetilde{L}_{k-i}.$$

Proof. We use the induction argument to show (3.6). When $k = 0$, we have by (1.17) that $W_1(Q) = \frac{1}{n}|\Sigma|$. We then assume that (3.6) holds for $k-1$, i.e.,

$$W_{2k-1}(\Omega) = \frac{1}{n} \sum_{j=0}^{k-1} C_{k-1}^j \frac{n+1-2k}{n+1-2k+2j} \widetilde{L}_{k-1-j}$$

$$(3.7) \qquad = \frac{1}{n} \int_{\Sigma} \sum_{i=1}^{k} C_{k-1}^{i-1} \frac{n+1-2k}{n-1-2k+2i} \widetilde{L}_{k-i}.$$

By (1.17) and (3.5), we have

$$W_{2k+1}(\Omega) = \frac{1}{n} \int_{\Sigma} p_{2k} - \frac{2k}{n-2k+1} W_{2k-1}(\Omega)$$

$$= \frac{1}{n} \int_{\Sigma} \sum_{i=0}^{k} C_k^i \widetilde{L}_i - \frac{2k}{n-2k+1} W_{2k-1}(\Omega).$$

Substituting (3.7) into the above, one immediately obtains (3.6) for k. q.e.d.

One can also show the following relation between the quermassintegrals and the curvature integrals.

Lemma 3.4.
$$(3.8)$$
$$W_{2k+1}(\Omega) = \frac{1}{n} \sum_{j=0}^{k} (-1)^j \frac{(2k)!!(n-2k-1)!!}{(2k-2j)!!(n-2k-1+2j)!!} \frac{1}{C_{n-1}^{2k-2j}} \int_{\Sigma} \sigma_{2k-2j},$$

where

$$(2k-1)!! := (2k-1)(2k-3)\cdots 1 \quad and \quad (2k)!! := (2k)(2k-2)\cdots 2.$$

Proof. One can show this relation by a direct computation. See also [**24**] or [**29**].   q.e.d.

4. Monotonicity

In this section we prove the monotonicity of functional Q under inverse curvature flow. First, we have the variational formula for $\int \widetilde{L}_k$.

Lemma 4.1. *Along the inverse flow (2.7), we have*

$$(4.1) \quad \frac{d}{dt}\int_\Sigma \widetilde{L}_k = (n-1-2k)\int_\Sigma \widetilde{L}_k + (n-1-2k)\int_\Sigma \left(\widetilde{N}_k\frac{p_{2k-1}}{p_{2k}} - \widetilde{L}_k\right).$$

Proof. It follows from (2.6) that along the inverse flow (2.4), we have

(4.2)

$$\frac{d}{dt}\int_\Sigma \widetilde{L}_k$$

$$=\int_\Sigma \sum_{i=0}^k C_k^i(-1)^i\Big((n-1-2k+2i)p_{2k-2i+1} + 2(k-i)p_{2k-2i-1}\Big)F$$

$$=\int_\Sigma \sum_{i=0}^k C_k^i(-1)^i\big(n-1-2k+2i\big)p_{2k-2i+1}F$$

$$+\int_\Sigma \sum_{j=1}^k C_k^{j-1}(-1)^{j-1}2(k-j+1)p_{2k-2j+1}F$$

$$=\int_\Sigma \sum_{i=0}^k C_k^i(-1)^i\big(n-1-2k\big)p_{2k-2i+1}F$$

$$+\int_\Sigma \sum_{j=1}^k 2(-1)^j\Big(C_k^j j - C_k^{j-1}(k-j+1)\Big)p_{2k-2j+1}F$$

$$=(n-1-2k)\int_\Sigma \sum_{i=0}^k C_k^i(-1)^i p_{2k-2i+1}$$

$$=(n-1-2k)\int_\Sigma \widetilde{N}_k F$$

$$=(n-1-2k)\int_\Sigma \widetilde{L}_k + (n-1-2k)\int_\Sigma \left(\widetilde{N}_k F - \widetilde{L}_k\right).$$

Substituting $F = \frac{p_{2k-1}}{p_{2k}}$ into above, we get the desired result.   q.e.d.

In order to show the monotonicity of the functional Q defined in (1.21) under the inverse flow (2.7), we need to show the non-positivity of the last term in (4.1). That is,

$$(4.3) \qquad \frac{p_{2k-1}}{p_{2k}} \widetilde{N}_k - \widetilde{L}_k \leq 0.$$

When $k = 1$, (4.3) is just

$$\frac{p_1}{p_2}(p_3 - p_1) - (p_2 - 1) \leq 0,$$

which follows from the Newton–Maclaurin inequalities in Lemma 2.1. In fact, it is clear that

$$\frac{p_1}{p_2}(p_3 - p_1) - (p_2 - 1) = (\frac{p_1 p_3}{p_2} - p_2) + (1 - \frac{p_1^2}{p_2}).$$

Hence the non-positivity follows, for both terms are non-positive, by Lemma 2.1. This was used in [**22**]. When $k \geq 2$, the proof of (4.3) becomes more complicated. When $k = 2$, one needs to show the non-positivity of

(4.4)
$$\frac{p_3}{p_4}(p_5 - 2p_3 + p_1) - (p_4 - 2p_2 + 1) = \left(\frac{p_3}{p_4}p_5 - p_4\right) + 2\left(p_2 - \frac{p_3^2}{p_4}\right) + \left(\frac{p_3}{p_4}p_1 - 1\right).$$

By Lemma 2.1, the first two terms are non-positive, but the last term is non-negative. It was showed in [**15**] that (4.4) is non-positive if $\kappa \in \mathbb{R}^{n-1}$ satisfying

$$(4.5) \qquad \kappa \in \{\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_{n-1}) \in \mathbb{R}^{n-1} \,|\, \kappa_i \geq 1\}.$$

We want to show that (4.3) is true for general $k \leq \frac{1}{2}(n - 1)$. This is one of the key points of this paper. This case is more complicated than the case $k = 2$.

Proposition 4.2. *For any κ satisfying (4.5), we have*

$$(4.6) \qquad \frac{p_{2k-1}}{p_{2k}} \widetilde{N}_k - \widetilde{L}_k \leq 0.$$

Equality holds if and only if one of the following two cases holds:

$$either \quad (i)\, \kappa_i = \kappa_j \,\forall\, i, j \quad or \quad (ii)\, \exists\, i \text{ with } \kappa_i < 1 \,\&\, \kappa_j = 1 \,\forall j \neq i.$$

We sketch the proof in several steps. Before the proof, we introduce the notation of $\sum\limits_{cyc}$ to simplify notation. Precisely, given $n - 1$ numbers $(\kappa_1, \kappa_2, \cdots, \kappa_{n-1})$, we denote $\sum\limits_{cyc} f(\kappa_1, \cdots, \kappa_{n-1})$ the cyclic summation

that takes over all *different* terms of the type $f(\kappa_1, \cdots, \kappa_{n-1})$. For instance,

$$\sum_{cyc} \kappa_1 = \kappa_1 + \kappa_2 + \cdots + \kappa_{n-1}, \qquad \sum_{cyc} \kappa_1^2 \kappa_2 = \sum_{i=1}^{n-1} \left(\kappa_i^2 \sum_{j \neq i} \kappa_j \right),$$

$$\sum_{cyc} \kappa_1(\kappa_2 - \kappa_3)^2 = \sum_{i=1}^{n-1} \left(\kappa_i \sum_{\substack{1 \leq j < k \leq n-1 \\ j,k \neq i}} (\kappa_j - \kappa_k)^2 \right),$$

$$= (n-3) \sum_{cyc} \kappa_1 \kappa_2^2 - 6 \sum_{cyc} \kappa_1 \kappa_2 \kappa_3.$$

Lemma 4.3. *For any κ satisfying (4.5), we have*

$$(4.7) \qquad \widetilde{N}_k - p_1 \widetilde{L}_k \leq 0.$$

Equality holds if and only if one of the following two cases holds:

$$either \quad (i)\, \kappa_i = \kappa_j \; \forall\, i, j \quad or \quad (ii)\, \exists\, i \; with \; \kappa_i > 1 \,\&\, \kappa_j = 1 \; \forall j \neq i.$$

Proof. It is crucial to observe that (4.7) is indeed equivalent to the following inequality:

$$(4.8)$$
$$\sum_{\substack{1 \leq i_m \leq n-1, \\ i_j \neq i_l (j \neq l)}} \kappa_{i_1}(\kappa_{i_2}\kappa_{i_3}-1)(\kappa_{i_4}\kappa_{i_5}-1)\cdots(\kappa_{i_{2k-2}}\kappa_{i_{2k-1}}-1)(\kappa_{i_{2k}}-\kappa_{i_{2k+1}})^2 \geq 0,$$

where the summation takes over all the $(2k+1)$-elements permutation of $\{1, 2, \cdots, n-1\}$. For the convenience of the reader, we sketch the proof of (4.8) briefly. First, note that from (3.4) that

$$\begin{aligned}
(p_1\widetilde{L}_k - \widetilde{N}_k) &= p_1 \sum_{i=0}^{k} C_k^{k-i}(-1)^{k-i}p_{2i} - \sum_{i=0}^{k} C_k^{k-i}(-1)^{k-i}p_{2i+1} \\
&= \sum_{i=0}^{k}(-1)^{k-i}C_k^i(p_1 p_{2i} - p_{2i+1}).
\end{aligned}$$

Next, we calculate each term $p_1 p_{2i} - p_{2i+1}$ carefully. By using

$$(n-1)C_{n-1}^j = (j+1)C_{n-1}^{j+1} + (n-1)C_{n-1}^{j-1}),$$

we have

$$\begin{aligned}
\sigma_j \sigma_1 &= \left(\sum_{cyc} \kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_j} \right) \left(\sum_{cyc} \kappa_{i_{j+1}} \right) \\
&= (j+1) \left(\sum_{cyc} \kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_j} \kappa_{i_{j+1}} \right) + \sum_{cyc} \kappa_{i_1}^2 \kappa_{i_2} \cdots \kappa_{i_j},
\end{aligned}$$

and

$$
\begin{aligned}
&p_1 p_{2j} - p_{2j+1} \\
&= \frac{1}{(n-1)C_{n-1}^{2j}} \left(\sum_{cyc} \kappa_{i_1}\kappa_{i_2}\cdots\kappa_{i_{2j}} \right) \left(\sum_{cyc} \kappa_{i_{2j+1}} \right) \\
&\quad - \frac{1}{C_{n-1}^{2j+1}} \sum_{cyc} \kappa_{i_1}\kappa_{i_2}\cdots\kappa_{i_{2j+1}} \\
&= \frac{1}{(n-1)C_{n-1}^{2j}C_{n-1}^{2j+1}} \Big(C_{n-1}^{2j+1}(2j+1) \sum_{cyc} \kappa_{i_1}\kappa_{i_2}\cdots\kappa_{i_{2j}}\kappa_{i_{2j+1}} \\
&\quad + C_{n-1}^{2j+1} \sum_{cyc} \kappa_{i_1}^2 \kappa_{i_2}\cdots\kappa_{i_{2j}} - (n-1)C_{n-1}^{2j} \sum_{cyc} \kappa_{i_1}\kappa_{i_2}\cdots\kappa_{i_{2j+1}} \Big) \\
&= \frac{1}{(n-1)C_{n-1}^{2j}C_{n-1}^{2j+1}} \cdot \frac{C_{n-1}^{2j+1}}{n-2j} \sum_{cyc} \kappa_{i_1}\kappa_{i_2}\cdots\kappa_{i_{2j-1}}(\kappa_{i_{2j}} - \kappa_{i_{2j+1}})^2 \\
&= \frac{(2j)!(n-2j-2)!}{(n-1)\cdot(n-1)!} \sum_{cyc} \kappa_{i_1}\kappa_{i_2}\cdots\kappa_{i_{2j-1}}(\kappa_{i_{2j}} - \kappa_{i_{2j+1}})^2.
\end{aligned}
$$

In (4.8), the coefficient of $\kappa_1\kappa_2\cdots\kappa_{2j-1}(\kappa_{2j} - \kappa_{2j+1})^2$ is

$$
\begin{aligned}
2(-1)^{k-j}C_{k-1}^{j-1}(2j-1)!C_{n-2j-2}^{2k-2j}[2(k-j)]! &= \frac{(-1)^{k-j}}{k}C_k^j(2j)!(n-2j-2)! \\
&= (-1)^{k-j}C_k^j \frac{(2j)!(n-2j-2)!}{(n-1)\cdot(n-1)!} \cdot \frac{(n-1)\cdot(n-1)!}{k}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
0 &\le \sum_{\substack{1 \le i_m \le n-1, \\ i_j \ne i_l(j \ne l)}} \kappa_{i_1}(\kappa_{i_2}\kappa_{i_3}-1)(\kappa_{i_4}\kappa_{i_5}-1)\cdots(\kappa_{i_{2k-2}}\kappa_{i_{2k-1}}-1)\big(\kappa_{i_{2k}}-\kappa_{i_{2k+1}}\big)^2 \\
&= \frac{(n-1)\cdot(n-1)!}{k} \sum_{j=0}^{k}(-1)^{k-j}C_k^j(p_1 p_{2j} - p_{2j+1}) \\
&= \frac{(n-1)\cdot(n-1)!}{k}(p_1\widetilde{L}_k - \widetilde{N}_k).
\end{aligned}
$$

This finishes the proof.   q.e.d.

In view of (4.8), we have the following remark that will be used later.

Remark 4.4. *For any* $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$ *satisfying* $0 < \kappa_i \le 1$, $(i = 1, \cdots, n-1)$, *then* $(-1)^{k-1}\big(\widetilde{N}_k - p_1\widetilde{L}_k\big) \le 0$.

Lemma 4.5. *For any* κ *satisfying (4.5), we have*

$$
\widetilde{N}_k \ge 0, \quad \widetilde{L}_k \ge 0.
$$

Proof. They are equivalent to the following inequalities, respectively:

(4.9)
$$\sum_{\substack{1 \le i_m \le n-1, \\ i_j \ne i_l (j \ne l)}} \kappa_{i_1}(\kappa_{i_2}\kappa_{i_3}-1)(\kappa_{i_4}\kappa_{i_5}-1)\cdots$$

(4.10)
$$(\kappa_{i_{2k-2}}\kappa_{i_{2k-1}}-1)(\kappa_{i_{2k}}\kappa_{i_{2k+1}}-1) \ge 0,$$

(4.11)
$$\sum_{\substack{1 \le i_m \le n-1, \\ i_j \ne i_l (j \ne l)}} (\kappa_{i_2}\kappa_{i_3}-1)(\kappa_{i_4}\kappa_{i_5}-1)\cdots(\kappa_{i_{2k-2}}\kappa_{i_{2k-1}}-1)(\kappa_{i_{2k}}\kappa_{i_{2k+1}}-1) \ge 0.$$

where the summation takes over all the $(2k+1)$-elements permutation of $\{1, 2, \cdots, n-1\}$. The proof to show the equivalence of (4.9) and (4.11) is exactly the same as the one of (4.8). Hence we omit it here.　　q.e.d.

Remark 4.6. *For any $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$ satisfying $0 < \kappa_i \le 1, (i = 1, \cdots, n-1)$, then $(-1)^k \widetilde{N}_k \ge 0, (-1)^k \widetilde{L}_k \ge 0$.*

Making use of Lemma 4.3 and Remark 4.4, we can show the following result, which is stronger than Proposition 4.2.

Lemma 4.7. *For any κ satisfying (4.5), we have*
$$p_{2k}\widetilde{N}_k - p_{2k+1}\widetilde{L}_k \le 0.$$

Proof. According to the induction argument proved in [**15**, p.8], we only need to prove it for $n - 1 = 2k + 1$. Let $z_i = \frac{1}{\kappa_i} \le 1$, and let
$$\hat{p}_i = p_i(z_1, z_2, \cdots, z_{2k+1}).$$

It is clear that

(4.12)
$$\hat{p}_j = \frac{p_{2k+1-j}}{p_{2k+1}}.$$

By Remark 4.4, we have

(4.13)
$$(-1)^{k-1} \sum_{i=0}^{k} C_k^i (-1)^i \hat{p}_{2k-2i+1} - (-1)^{k-1}\hat{p}_1 \sum_{i=0}^{k} C_k^i(-1)^i \hat{p}_{2k-2i} \le 0,$$

which is equivalent to

(4.14)
$$(-1)^{k-1} \sum_{i=0}^{k} C_k^i (-1)^i \frac{p_{2i}}{p_{2k+1}} - (-1)^{k-1}\frac{p_{2k}}{p_{2k+1}} \sum_{i=0}^{k} C_k^i(-1)^i \frac{p_{2i+1}}{p_{2k+1}} \le 0.$$

Thus we have

(4.15)
$$\sum_{i=0}^{k} C_k^i(-1)^{k-i}p_{2i} - \frac{p_{2k}}{p_{2k+1}} \sum_{i=0}^{k} C_k^i(-1)^{k-i}p_{2i+1} \ge 0,$$

which implies $\frac{p_{2k}}{p_{2k+1}}\widetilde{N}_k - \widetilde{L}_k \leq 0$.   q.e.d.

Proof of Proposition 4.2. Then by the Newton–MacLaurin inequality $p_{2k-1}p_{2k+1} \leq p_{2k}^2$, we obtain

$$\frac{p_{2k-1}}{p_{2k}}\widetilde{N}_k - \widetilde{L}_k \leq \frac{p_{2k}}{p_{2k+1}}\widetilde{N}_k - \widetilde{L}_k \leq 0,$$

which is exactly (4.6). Here we have used Lemma 4.5.   q.e.d.

Remark 4.8. *Proposition* 4.2 *holds for* $\kappa \in \mathbb{R}^{n-1}$ *with* $\kappa_i \kappa_j \geq 1$ *for any* i, j. *This is equivalent to the condition that the sectional curvature of* Σ *is non-negative.*

Remark 4.9. *From the proof of Proposition* 4.2, *it is easy to see that* (4.6) *has an inverse inequality for* $\kappa \in \mathbb{R}^{n-1}$ *with* $0 \leq \kappa_i \leq 1$.

Now we have a monotonicity of $Q(\Sigma_t)$ defined by (1.21) under the flow (2.7).

Theorem 4.10. *Functional* Q *is non-increasing under the flow* (2.7), *provided that the initial surface is horospherical convex.*

Proof. It follows from (3.1), (3.4), and Proposition 4.2 that

$$(4.16) \qquad \frac{d}{dt}\int_\Sigma L_k \leq (n-1-2k)\int_\Sigma L_k.$$

On the other hand, by (2.6) and (2.3), we also have

$$(4.17) \qquad \frac{d}{dt}|\Sigma_t| = \int_{\Sigma_t}\frac{p_{2k-1}}{p_{2k}}(n-1)p_1 d\mu \geq (n-1)|\Sigma_t|.$$

Combining (4.16) and (4.17) together, we complete the proof.

                                                                    q.e.d.

Remark 4.11. *From the above proof, one can check that to obtain a monotonicity of* Q *it is enough to choose* $F = \frac{1}{p_1}$. *Then from* (4.1) *and* (4.7), *it holds for all* k

$$\frac{d}{dt}\int_\Sigma \widetilde{L}_k = (n-2k-1)\int_\Sigma \widetilde{L}_k + \int_\Sigma (n-2k-1)\left(\frac{1}{p_1}\widetilde{N}_k - \widetilde{L}_k\right)$$
$$\leq (n-2k-1)\int_\Sigma \widetilde{L}_k.$$

5. Proof of main theorems

Now we are ready to show our main theorems.

Proof of Theorem 1.1. First, recall the definition (1.21) of the functional Q. By (1.21), (1.13) is equivalent to

$$(5.1) \qquad Q(\Sigma) \geq C_{n-1}^{2k}(2k)!\,\omega_{n-1}^{\frac{2k}{n-1}}.$$

Let $\Sigma(t)$ be a solution of flow (2.7) obtained by the work of Gerhardt [**16**]. This flow preserves the horospherical convexity and non-increases for the functional Q. Hence, to show (5.1) we only need to show

$$(5.2) \qquad \lim_{t\to\infty} Q(\Sigma_t) \geq C_{n-1}^{2k}(2k)!\,\omega_{n-1}^{\frac{2k}{n-1}}.$$

Since Σ is a horospherical convex hypersurface in $(\mathbb{H}^n, \bar{g})$, it is written as graph of function $r(\theta)$, $\theta \in \mathbb{S}^{n-1}$. We denote $X(t)$ as graphs $r(t,\theta)$ on $\mathbb{S}^{n-1}$ with the standard metric $\hat{g}$. We set $\lambda(r) = \sinh(r)$ and we have $\lambda'(r) = \cosh(r)$. It is clear that

$$(\lambda')^2 = (\lambda)^2 + 1.$$

We define $\varphi(\theta) = \Phi(r(\theta))$. Here Φ verifies

$$\Phi' = \frac{1}{\lambda}.$$

We define another function

$$v = \sqrt{1 + |\nabla\varphi|_{\hat{g}}^2}.$$

By [**16**], we have the following results.

Lemma 5.1.
$$\lambda = O(e^{\frac{t}{n-1}}), \qquad |\nabla\varphi| + |\nabla^2\varphi| = O(e^{-\frac{t}{n-1}}).$$

From Lemma (5.1), we have the following expansions:

$$(5.3) \qquad \lambda' = \lambda(1 + \frac{1}{2}\lambda^{-2}) + O(e^{-\frac{4t}{n-1}}),$$

and

$$(5.4) \qquad \frac{1}{v} = 1 - \frac{1}{2}|\nabla\varphi|_{\hat{g}}^2 + O(e^{-\frac{4t}{n-1}}).$$

We have also

$$(5.5) \qquad \nabla\lambda = \lambda\lambda'\nabla\varphi.$$

The second fundamental form of Σ is written in an orthogonal basis (see [**10**], for example)

$$\begin{aligned}
h_i^j &= \frac{\lambda'}{v\lambda}\left(\delta_i^j - \frac{\varphi_i^j}{\lambda'} + \frac{\varphi_i\varphi_l\varphi^{jl}}{v^2\lambda'}\right) \\
&= \delta_i^j + (\frac{1}{2\lambda^2} - \frac{1}{2}|\nabla\varphi|^2)\delta_i^j - \frac{\varphi_i^j}{\lambda} + O(e^{-\frac{4t}{n-1}}),
\end{aligned}$$

where the second equality follows from (5.3) and (5.4). We set

$$(5.6) \qquad T_i^j = (\frac{1}{2\lambda^2} - \frac{1}{2}|\nabla\varphi|^2)\delta_i^j - \frac{\varphi_i^j}{\lambda};$$

then from the Gauss equations we obtain

$$\begin{aligned}
R_{ij}{}^{kl} &= -(\delta_i^k \delta_j^l - \delta_i^l \delta_j^k) + (h_i^k h_j^l - h_i^l h_j^k) \\
&= \delta_i^k T_j^l + T_i^k \delta_j^l - T_i^l \delta_j^k - \delta_i^l T_j^k + O(e^{-\frac{4t}{n-1}}).
\end{aligned}$$

It follows from (1.1) that

$$\begin{aligned}
L_k &= \tfrac{1}{2^k} \delta_{j_1 j_2 \cdots j_{2k-1} j_{2k}}^{i_1 i_2 \cdots i_{2k-1} i_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}} \\
&= 2^k \delta_{j_1 j_2 \cdots j_{2k-1} j_{2k}}^{i_1 i_2 \cdots i_{2k-1} i_{2k}} T_{i_1}^{j_1} \delta_{i_2}^{j_2} \cdots T_{i_{2k-1}}^{j_{2k-1}} \delta_{i_{2k}}^{j_{2k}} + O(e^{-\frac{(2k+2)t}{n-1}}) \\
&= 2^k (n-1-k) \cdots (n-2k) \delta_{j_1 j_3 \cdots j_{2k-1}}^{i_1 i_3 \cdots i_{2k-1}} T_{i_1}^{j_1} T_{i_3}^{j_3} \cdots T_{i_{2k-1}}^{j_{2k-1}} \\
&\quad + O(e^{-\frac{(2k+2)t}{n-1}}) \\
&= 2^k k! (n-1-k) \cdots (n-2k) \sigma_k(T) + O(e^{-\frac{(2k+2)t}{n-1}}).
\end{aligned}$$

Here in the second equality we use the fact

$$\begin{aligned}
&\delta_{j_1 j_2 \cdots j_{2k-1} j_{2k}}^{i_1 i_2 \cdots i_{2k-1} i_{2k}} T_{i_1}^{j_1} \delta_{i_2}{}^{j_2} = \delta_{j_1 j_2 \cdots j_{2k-1} j_{2k}}^{i_1 i_2 \cdots i_{2k-1} i_{2k}} \delta_{i_1}^{j_1} T_{i_2}^{j_2} \\
&= -\delta_{j_1 j_2 \cdots j_{2k-1} j_{2k}}^{i_1 i_2 \cdots i_{2k-1} i_{2k}} T_{i_1}^{j_2} \delta_{i_2}^{j_1} = -\delta_{j_1 j_2 \cdots j_{2k-1} j_{2k}}^{i_1 i_2 \cdots i_{2k-1} i_{2k}} \delta_{i_1}^{j_2} T_{i_2}^{j_1},
\end{aligned}$$

and in the third equality we use (2.1) and (3.3).
Recall $\varphi_i = \lambda_i / \lambda\lambda'$; then by (5.3) we have

$$(5.7) \qquad \varphi_{ij} = \frac{\lambda_{ij}}{\lambda^2} - \frac{2\lambda_i \lambda_j}{\lambda^3} + O(e^{-\frac{3t}{n-1}}).$$

By the definition of the Schouten tensor,

$$A_{\hat{g}} = \frac{1}{n-3}\left(Ric_{\hat{g}} - \frac{R_{\hat{g}}}{2(n-2)}\hat{g} \right) = \frac{1}{2}\hat{g}.$$

Its conformal transformation formula is well known (see, for example, [30])

$$\begin{aligned}
A_{\lambda^2 \hat{g}} &= -\frac{\nabla^2 \lambda}{\lambda} + \frac{2\nabla\lambda \otimes \nabla\lambda}{\lambda^2} - \frac{1}{2}\frac{|\nabla\lambda|^2}{\lambda^2}\hat{g} + A_{\hat{g}} \\
(5.8) \qquad &= -\frac{\nabla^2 \lambda}{\lambda} + \frac{2\nabla\lambda \otimes \nabla\lambda}{\lambda^2} - \frac{1}{2}\frac{|\nabla\lambda|^2}{\lambda^2}\hat{g} + \frac{1}{2}\hat{g}.
\end{aligned}$$

Substituting (5.5) and (5.7) into (5.6), together with (5.8), we have

$$T_i^j = ((\lambda^2 \hat{g})^{-1} A_{\lambda^2 \hat{g}})_i^j + O(e^{-\frac{4t}{n-1}}),$$

which implies

$$(5.9) \qquad L_k = 2^k k! (n-1-k) \cdots (n-2k) \sigma_k(A_{\lambda^2 \hat{g}}) + O(e^{-\frac{(2k+2)t}{n-1}}).$$

As before, $\Sigma(t)$ is a horospherical convex hypersurface. As a consequence, Σ has non-negative sectional curvature so that $T + O(e^{-\frac{4t}{n-1}})$ is

positive definite. We consider $\tilde{\lambda} := \lambda^{1-e^{-\frac{t}{n-1}}}$ and the conformal metric $\tilde{\lambda}^2\hat{g}$. We have

$$
\begin{aligned}
\tilde{\lambda}^2(\tilde{\lambda}^2\hat{g})^{-1}A_{\tilde{\lambda}^2\hat{g}} &= \frac{1}{2}e^{-\frac{t}{n-1}}I + \frac{1}{2}e^{-\frac{t}{n-1}}(1-e^{-\frac{t}{n-1}})\frac{|\nabla\lambda|^2}{\lambda^2}I \\
&\quad -e^{-\frac{t}{n-1}}(1-e^{-\frac{t}{n-1}})\hat{g}^{-1}\frac{\nabla\lambda\otimes\nabla\lambda}{\lambda^2} \\
&\quad +\lambda^2(1-e^{-\frac{t}{n-1}})(\lambda^2\hat{g})^{-1}A_{\lambda^2\hat{g}}.
\end{aligned}
$$

Recall $\frac{1}{2}e^{-\frac{t}{n-1}}I + \lambda^2(1-e^{-\frac{t}{n-1}})(\lambda^2\hat{g})^{-1}A_{\lambda^2\hat{g}} \in \Gamma^+_{n-1}$ for sufficiently large t and $\frac{1}{2}e^{-\frac{t}{n-1}}(1-e^{-\frac{t}{n-1}})\frac{|\nabla\lambda|^2}{\lambda^2}I - e^{-\frac{t}{n-1}}(1-e^{-\frac{t}{n-1}})\hat{g}^{-1}\frac{\nabla\lambda\otimes\nabla\lambda}{\lambda^2} \in \overline{\Gamma^+_k}$ for any $k \leq \frac{n-1}{2}$. Therefore, we infer $\tilde{\lambda}^2\hat{g} \in \Gamma^+_k$ for any $k \leq \frac{n-1}{2}$. The Sobolev inequality (2.9) for the σ_k operator gives

(5.10)
$$
(vol(\tilde{\lambda}^2\hat{g}))^{-\frac{n-1-2k}{n-1}}\int_{\mathbb{S}^{n-1}}\sigma_k(A_{\tilde{\lambda}^2\hat{g}})dvol_{\tilde{\lambda}^2\hat{g}} \geq \frac{(n-1)\cdots(n-k)}{2^k k!}\omega_{n-1}^{\frac{2k}{n-1}}.
$$

On the other hand, we have

(5.11)
$$
\begin{aligned}
&(vol(\tilde{\lambda}^2\hat{g}))^{-\frac{n-1-2k}{n-1}}\int_{\mathbb{S}^{n-1}}\sigma_k(A_{\tilde{\lambda}^2\hat{g}})dvol_{\tilde{\lambda}^2\hat{g}} \\
&= (1+o(1))(vol(\lambda^2\hat{g}))^{-\frac{n-1-2k}{n-1}}\int_{\mathbb{S}^{n-1}}\sigma_k(A_{\lambda^2\hat{g}})dvol_{\lambda^2\hat{g}},
\end{aligned}
$$

since

$$
\lambda^{-e^{-\frac{t}{n-1}}} = 1 + o(1).
$$

As a consequence of (5.9),(5.10) and (5.11), we deduce

$$
\lim_{t\to+\infty}(vol(\Sigma(t)))^{-\frac{n-1-2k}{n-1}}\int_{\Sigma(t)}L_k \geq (n-1)(n-2)\cdots(n-2k)\omega_{n-1}^{\frac{2k}{n-1}}.
$$

When (5.1) is an equality, then Q is constant along the flow. Then (4.17) is an equality, which implies that equality in the inequality

$$
\frac{p_{2k-1}}{p_{2k}}p_1 \geq 1,
$$

holds. Therefore, Σ is a geodesic sphere.   q.e.d.

Proof of Theorem 1.2. It follows from (3.1), (3.4), and Theorem 1.1 that when $n - 1 > 2k$

(5.12)
$$
\int_\Sigma \tilde{L}_k \geq \omega_{n-1}^{\frac{2k}{n-1}}(|\Sigma|)^{\frac{n-1-2k}{n-1}}.
$$

Using the expression (3.5) of $\int_\Sigma \sigma_k$ in terms of $\int_\Sigma \tilde{L}_j$, we get the desired result:

$$\int_\Sigma \sigma_{2k} \geq C_{n-1}^{2k} \omega_{n-1} \left\{ \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{1}{k}} + \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{1}{k}\frac{n-1-2k}{n-1}} \right\}^k.$$

By Theorem 1.1, equality holds if and only if Σ is a geodesic sphere.

When $n - 1 = 2k$, since the hypersurface Σ is convex, we know that (1.13) is an equality when $n - 1 = 2k$ by the Gauss–Bonnet–Chern theorem, even for any hypersurface diffeomorphic to a sphere. Hence, in this case, we also have all the above inequalities with equality, which in turn implies by [**22**] or [**15**] that Σ is a geodesic sphere.   q.e.d.

Proof of Theorem 1.3. When $n - 1 > 2k$, the proof follows directly from (5.12) and Lemma 3.3. When $n - 1 = 2k$, the proof follows by the same reason as in Theorem 1.2.   q.e.d.

From (1.17), it is easy to see that Theorem 1.3 implies Theorem 1.2; meanwhile, Theorem 1.2 may not directly imply Theorem 1.3, since there are negative coefficients in (3.8) above.

6. Alexandrov–Fenchel inequality for odd k

In this section, we show an Alexandrov–Fenchel inequality for σ_1, which follows from the result of Cheng and Zhou [**8**] and Theorem 1.2 (or more precisely from [**22**]).

Theorem 6.1. *Let* $n \geq 2$. *Any horospherical convex hypersurface* $\Sigma \subset \mathbb{H}^n$ *satisfies*

$$(6.1) \qquad \int_\Sigma \sigma_1 \geq (n-1)\omega_{n-1} \left\{ \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^2 + \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{2(n-2)}{n-1}} \right\}^{\frac{1}{2}},$$

where ω_{n-1} *is the area of the unit sphere* $\mathbb{S}^{n-1}$ *and* $|\Sigma|$ *is the area of* Σ. *Equality holds if and only if* Σ *is a geodesic sphere.*

Proof. Notice that the horospherical convex condition implies that the Ricci curvature of Σ is non-negative. We observe first that by a direct computation (1.4) in [**8**],

$$\int_\Sigma |H - \overline{H}|^2 \leq \frac{n-1}{n-2} \int_\Sigma |B - \frac{H}{n-1}g|^2$$

is equivalent to

$$(6.2) \qquad \int_\Sigma \sigma_2 \int_\Sigma \sigma_0 \leq \frac{n-2}{2(n-1)} \left(\int_\Sigma \sigma_1 \right)^2.$$

Then we use the optimal inequality for σ_2 proved in [**22**] (see also Theorem 1.2),

$$(6.3) \qquad \int_\Sigma \sigma_2 \geq \frac{(n-1)(n-2)}{2} \left(\omega_{n-1}^{\frac{2}{n-1}} |\Sigma|^{\frac{n-3}{n-1}} + |\Sigma| \right),$$

to obtain the desired inequality for σ_1,

$$\int_\Sigma \sigma_1 \geq (n-1)\omega_{n-1}\left\{\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^2 + \left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{2(n-2)}{n-1}}\right\}^{\frac{1}{2}}.$$

When (6.1) is an equality, in turn, (6.3) is also a equality, and then it follows from [**22**] that the hypersurface is a geodesic sphere.   q.e.d.

Motivated by Theorem 1.2 and (6.2), we would like propose the following conjecture

Conjecture 6.2. *Let $n - 1 \geq 2k + 1$. Any horospherical convex hypersurface $\Sigma \subset \mathbb{H}^n$ satisfies*

$$\int_\Sigma \sigma_{2k+1} \geq C_{n-1}^{2k+1}\omega_{n-1}\left\{\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{2}{2k+1}} + \left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{2}{2k+1}\frac{(n-2k-2)}{n-1}}\right\}^{\frac{2k+1}{2}}.$$

Equality holds if and only if Σ is a geodesic sphere.

The conjecture follows from Theorem 1.2 and the following conjecture:

$$(6.4) \qquad \frac{\left(C_{n-1}^{2k+1}\right)^2}{C_{n-1}^{2k+2}C_{n-1}^{2k}}\int_\Sigma \sigma_{2k+2}\int_\Sigma \sigma_{2k} \leq \left(\int_\Sigma \sigma_{2k+1}\right)^2.$$

Note added in proof: Conjecture 6.2 was proved in [**31**].

References

[1] B. Andrews, *Pinching estimates and motion of hypersurfaces by curvature functions*, J. reine angew. Math. **608** (2007), 17–31, MR 2339467, Zbl 1129.53044.

[2] T. Aubin, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. 55 (1976), 269–296, MR 0431287, Zbl 0336.53033.

[3] W. Beckner, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality,* Ann. of Math. **138**(1993), 213–242, MR 1230930, Zbl 0826.58042.

[4] S. Brendle, P.-K. Hung, and M.-T. Wang, *A Minkowski-type inequality for hypersurfaces in the Anti-deSitter-Schwarzschild manifold,* arXiv: 1209.0669.

[5] Y.D. Burago & V.A. Zalgaller, *Geometric Inequalities,* Springer, Berlin, 1988, MR 0936419, Zbl 0633.53002.

[6] S.-Y. A. Chang & Y.Wang, *On Aleksandrov–Fenchel inequalities for k-convex domains,* Milan J. Math., **79** (2011), no. 1, 13–38, MR 2831436, Zbl 1229.52002.

[7] S.-Y. A. Chang & P. C. Yang, *The inequality of Moser and Trudinger and applications to conformal geometry,* Comm. Pure Appl. Math. **56** (2003), 1135–1150, MR 1989228, Zbl 1049.53025.

[8] X. Cheng & D. Zhou, *Rigidity for nearly umbilical hypersurfaces in space forms,* arXiv:1208.1786.

[9] L.L. de lima & F. Girão, *An Alexandrov–Fenchel-type inequality in hyperbolic space with an application to a Penrose inequality*, arXiv:1209.0438v2.

[10] Q. Ding, *The inverse mean curvature flow in rotationally symmetric spaces*, Chinese Annals of Mathematics, Series B, 1–18 (2010), MR 2772224, Zbl 1211.53084.

[11] E. Gallego & G. Solanes, *Integral geometry and geometric inequalities in hyperbolic space*, Differential Geom. Appl. **22** (2005), 315–32, MR 2166125, Zbl 1077.52010.

[12] F. Gao, D. Hug & R. Schneider, *Intrinsic volumes and polar sets in spherical space,* Homage to Luis Santaló, Vol. 1 (Spanish), Math. Notae **41** (2001/02), 159–76 (2003), MR 2049442, Zbl 1202.52006.

[13] Y. Ge, G. Wang & J. Wu, *A new mass for asymptotically flat manifolds*, arXiv:1211.3645.

[14] Y. Ge, G. Wang & J. Wu, *The Gauss–Bonnet-Chern mass of conformally flat manifolds,*, to appear in IMRN.

[15] Y. Ge, G. Wang & J. Wu, *Hyperbolic Alexandrov–Fenchel quermassintegral inequalities I*, arXiv:1303.1714.

[16] C. Gerhardt, *Inverse curvature flows in hyperbolic space*, J. Differential Geom. **89** (2011), no. 3, 487–527, MR 2879249, Zbl 1252.53078.

[17] P. Guan, *Topics in Geometric Fully Nonlinear Equations,* Lecture Notes, http://www.math.mcgill.ca/guan/notes.html.

[18] P. Guan & J. Li, *The quermassintegral inequalities for k-convex starshaped domains,* Adv. Math. **221** (2009), 1725–1732, MR 2522433, Zbl 1170.53058.

[19] P. Guan & G. Wang, *Geometric inequalities on locally conformally flat manifolds*, Duke Math. J. **124** (2004), 177–212, MR 2072215, Zbl 1059.53034.

[20] E. Hebey, *Sobolev Spaces on Riemannian manifolds,* Lecture Notes in Mathematics 1635, Springer-Verlag, Berlin, 1996, MR 1481970, Zbl 0866.58068.

[21] G.Huisken, in preparation.

[22] H. Li, Y. Wei & C. Xiong, *A geometric inequality on hypersurface in hyperbolic space*, ArXiv:1211.4109.

[23] R. Reilly, *On the Hessian of a function and the curvatures of its graph*, Michigan Math. J. **20** (1973) 373–383, MR 0334045, Zbl 0311.53030.

[24] L. A. Santaló, *Integral Geometry and Geometric Probability*, Addison-Wesley, Reading, MA, (1976), MR 0433364, Zbl 0342.53049.

[25] R.Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Diff. Geom. **20** (1984), 479–495, MR 0788292, Zbl 0576.53028.

[26] E. Schmidt, *Die isoperimetrischen Ungleichungen auf der gewöhnlichen Kugel und für Rotationskörper im n-dimensionalen sphärischen Raum.* (German) Math. Z. **46**, (1940), 743–794, MR 0003733, Zbl 0023.38101.

[27] R. Schneider, *Convex bodies: The Brunn-Minkowski Theory*, Cambridge University Press, Cambridge, England, 1993, MR 1216521, Zbl 0798.52001.

[28] G. Solanes, *Integral geometry and the Gauss–Bonnet theorem in constant curvature spaces*, Trans. Amer. Math. Soc. **358** (2006), no. 3, 1105–1115, MR 2187647, Zbl 1082.53075.

[29] G. Solanes, *Integrals de curvatura i geometria integral a l'espai hiperbolic*, Univ. Aut. Barcelona, PhD Thesis, (2003).

[30] J. Viaclovsky, *Conformal geometry, contact geometry, and the calculus of variations,* Duke Math. J. **101** (2000), no. 2, 283–316, MR 1738176, Zbl 0990.53035.

[31] G. Wang, & C. Xia, *Isoperimetric type problems and Alexandrov-Fenchel type inequalities in the hyperbolic space,* Arxiv:1304.1674

LABORATOIRE D'ANALYSE ET DE MATHÉMATIQUES APPLIQUÉES
CNRS UMR 8050
DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ PARIS EST-CRÉTEIL VAL DE MARNE
61 AVENUE DU GÉNÉRAL DE GAULLE
94010 CRÉTEIL CEDEX, FRANCE
E-mail address: ge@u-pec.fr

ALBERT-LUDWIGS-UNIVERSITÄT FREIBURG
MATHEMATISCHES INSTITUT
ECKERSTR. 1
D-79104 FREIBURG, GERMANY
E-mail address: guofang.wang@math.uni-freiburg.de

SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA
HEFEI 230026, P. R. CHINA
AND
ALBERT-LUDWIGS-UNIVERSITÄT FREIBURG
MATHEMATISCHES INSTITUT
ECKERSTR. 1
D-79104 FREIBURG, GERMANY
E-mail address: jie.wu@math.uni-freiburg.de