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CURVATURES OF DIRECT IMAGE SHEAVES OF VECTOR BUNDLES AND APPLICATIONS

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Abstract

Let $p : \mathcal{X} \to S$ be a proper Kähler fibration and $\mathcal{E} \to \mathcal{X}$ a Hermitian holomorphic vector bundle. As motivated by the work of Berndtsson ([**Bern09**]), by using basic Hodge theory, we derive several general curvature formulas for the direct image $p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ for general Hermitian holomorphic vector bundle $\mathcal{E}$ in a simple way. A straightforward application is that, if the family $\mathcal{X} \to S$ is infinitesimally trivial and Hermitian vector bundle $\mathcal{E}$ is Nakano-negative along the base S, then the direct image $p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ is Nakano-negative. We also use these curvature formulas to study the moduli space of projectively flat vector bundles with positive first Chern classes and obtain that, if the Chern curvature of direct image $p_*(K_X \otimes E)$–of a positive projectively flat family $(E, h(t))_{t\in\mathbb{D}} \to X$–vanishes, then the curvature forms of this family are connected by holomorphic automorphisms of the pair (X, E).

1. Introduction

Let $\mathcal{X}$ be a Kähler manifold with dimension $m + n$ and S a Kähler manifold with dimension m. Let $p : \mathcal{X} \to S$ be a proper Kähler fibration. Hence, for each $s \in S$,

$$X_s := p^{-1}(\{s\})$$

is a compact Kähler manifold with dimension n. Let $(\mathcal{E}, h^{\mathcal{E}}) \to \mathcal{X}$ be a Hermitian holomorphic vector bundle. Consider the space of holomorphic $\mathcal{E}$-valued $(n, 0)$-forms on X_s,

$$E_s := H^0(X_s, \mathcal{E}_s \otimes K_{X_s}) \cong H^{n,0}(X_s, \mathcal{E}_s)$$

where $\mathcal{E}_s = \mathcal{E}|_{X_s}$. It is well-known that, if the vector bundle E is "positive" in certain sense, there is a natural holomorphic structure on

$$E = \bigcup_{s\in S} \{s\} \times E_s$$

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such that E is isomorphic to the direct image sheaf $p_*(K_{\mathcal{X}/S}\otimes\mathcal{E})$. Using the canonical isomorphism $K_{\mathcal{X}/S}|_{X_s}\cong K_{X_s}$, a *local smooth* section u of E over S can be identified as a holomorphic $\mathcal{E}$-valued $(n,0)$ form on X_s.

By the identification above, there is a natural metric on E. For any local smooth section u of E, one can define a Hermitian metric on E by

$$h(u,u)=c_n\int_{X_s}\{u,u\} \tag{1.1}$$

where $c_n=(\sqrt{-1})^{n^2}$. Here, we only use the Hermitian metric of $\mathcal{E}_s$ on each fiber X_s and we do not specify background Kähler metrics on the fibers. Berndtsson defined in [**Bern09**, Lemma 4.1], a natural Chern connection D on (E,h), and computed the curvature tensor of direct image $p_*(K_{\mathcal{X}/S}\otimes\mathcal{L})$ of (semi-)positive line bundle $\mathcal{L}\to\mathcal{X}$.

Next we would like to describe our results in this paper briefly. As motivated by the work of Berndtsson ([**Bern09**]), we compute the curvature tensor of the direct images $p_*(K_{\mathcal{X}/S}\otimes\mathcal{E})$ for arbitrary Hermitian vector bundles $\mathcal{E}\to\mathcal{X}$ by using basic Hodge theory which also simplify Berndtsson's original proofs mildly. In the following formulations, if not otherwise stated, we do not make any positivity or negativity assumption on the curvature tensors of $\mathcal{E}$ or $\mathcal{L}$, but we assume that every E_s has the same dimension.

Let (X,ω_g) be a compact Kähler manifold with complex dimension n and $F\to X$ a Hermitian vector bundle with Chern connection $\nabla=\nabla'+\nabla''$. At first, by Hodge theory on vector bundles(Lemma 2.1), we observe that if $\alpha\in\Omega^{n,0}(X,F)$, and it has no harmonic part, then $v=\nabla'^*\mathbb{G}'\alpha$ is a solution to $\nabla' v=\alpha$ where $\mathbb{G}'$ is the Green's operator with respect to ∇'. Moreover, $\nabla''v$ is a primitive $(n-1,1)$ form. We can apply this observation to the Kähler fibration $p:\mathcal{X}\to S$. Let $(t^1,\cdots,t^m)$ be local holomorphic coordinates on the base S centered at some point $s\in S$. Let $\nabla^{\mathcal{E}}=\nabla'+\nabla''$ be the Chern connection of the Hermitian vector bundle $(\mathcal{E},h^{\mathcal{E}})$ over $\mathcal{X}$ and $\nabla_X=\nabla'_X+\nabla''_X$ be the restriction of $\nabla^{\mathcal{E}}$ on the fiber $\mathcal{E}_s\to X_s$. For any *local holomorphic* section u of $E=p_*(K_{\mathcal{X}/S}\otimes\mathcal{E})$, by the identification stated above, it can be represented by a local smooth $\mathcal{E}$-valued $(n,0)$ form u on $\mathcal{X}$ with the property that

$$\nabla' u=dt^i\wedge\nu_i,\quad \nabla''u=dt^j\wedge\eta_j$$

where ν_i and η_j are forms on $\mathcal{X}$ of bidegree $(n,0)$ and $(n-1,1)$ respectively. It is easy to see that ν_i and η_j are not uniquely determined as forms on $\mathcal{X}$, but their restrictions to fibers are(see Section 2.3 for more details). It is worth pointing out that, when restricted to each fiber, $[\eta_j]$ is closely related to the Kodaira-Spencer class of the deformation $\mathcal{X}\to S$ (See Remark 2.6). We set

$$v_i=-\nabla'_X\mathbb{G}'\pi_\perp(\nu_i)$$

where $\pi_\perp = \mathbb{I} - \pi$ and $\pi : \Omega^{n,0}(X_s, \mathcal{E}_s) \to H^{n,0}(X_s, \mathcal{E}_s)$ is the orthogonal projection on each fiber. At first, we derive a curvature formula for $E = p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ by a simple method (see Theorem 3.3).

Theorem 1.1. *Let Θ^E be the Chern curvature of $E = p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$. For any local holomorphic section u of E, the curvature Θ^E has the following "negative form":*

$$\begin{aligned}(\sqrt{-1}\Theta^E u, u) &= c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}} u, u\right\} - (\Delta'_X v_i, v_j) \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j) \\ &\quad + (\eta_i, \eta_j) \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j). \qquad (1.2)\end{aligned}$$

We shall explain this curvature formula in details in the following sections, and also make a simple example in Section 4 to explain why this "negative form" is "natural".

By a decomposition for the second term on the right hand side of (1.2),

$$(\Delta'_X v_i, v_j) = (\Delta'_X v_i, \Delta'_X v_j) + (\Delta'_X v_i, v_j - \Delta'_X v_j)$$

we obtain a curvature form with significant geometric interpretations and it is related to deformation theory of vector bundles. Let

$$\alpha_i = \Theta^{\mathcal{E}}\left(\frac{\partial}{\partial t^i}\right)\Big|_{X_s} \in \Omega^{0,1}(X_s, End(\mathcal{E}_s)). \qquad (1.3)$$

(Note that, if the family is infinitesimally trivial, $[\alpha_i] \in H^{0,1}(X_s, End(\mathcal{E}_s))$ is the Kodaira-Spencer class ([**SchTo92**, Proposition 1]) of the deformation $\mathcal{E} \to \mathcal{X} \to S$ in the direction of $\frac{\partial}{\partial t^i} \in T_s S$.) We observe that

$$\Delta'_X v_i = -\sqrt{-1}\Lambda_g (\alpha_i \cup u)$$

when restricted to the fiber X_s where Λ_g is the contraction operator with respect to the Kähler metric ω on the fiber X_s.

Theorem 1.2. *The curvature Θ^E of $E = p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ has the following "geodesic form":*

$$\begin{aligned}&(\sqrt{-1}\Theta^E u, u) \\ &= c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}} u, u\right\} - (\alpha_i \cup u, \alpha_j \cup u) \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j) \\ &\quad + (\Delta'_X v_i, \Delta'_X v_j - v_j) \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j) + (\eta_i, \eta_j) \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j). \qquad (1.4)\end{aligned}$$

Rewriting each line on the right hand side of (1.4) a little bit, we reach the following special case, which is also of particular interest, since the first line in (1.4) is exactly in the geodesic form.

Corollary 1.3. *Let $(\mathcal{L}, h^{\mathcal{L}} = e^{-\varphi})$ be a Hermitian line bundle over $\mathcal{X}$ such that $(\mathcal{L}|_{X_s}, h^{\mathcal{L}}_{X_s})$ is positive on each fiber X_s. The curvature Θ^{E_k}*

of $E_k = p_*(K_{\mathcal{X}/S} \otimes \mathcal{L}^k)$ *has the form:*

$$\begin{aligned} & (\sqrt{-1}\Theta^{E_k}u, u) \\ = \; & c_n \int_{X_s} kc_{i\bar{j}}(\varphi)\{u,u\}(\sqrt{-1}dt^i \wedge d\bar{t}^j) \\ & + \frac{1}{k}\left((\Delta'_X + k)^{-1}\left(\nabla''_X \Delta'_X v_i\right), \nabla''_X \Delta'_X v_j\right) \cdot (\sqrt{-1}dt^i \wedge d\bar{t}^j) \\ & + (\eta_i, \eta_j) \cdot (\sqrt{-1}dt^i \wedge d\bar{t}^j), \end{aligned} \tag{1.5}$$

where $c_{i\bar{j}}(\varphi)$ *is given by*

$$c_{i\bar{j}}(\varphi) = \frac{\partial^2 \varphi}{\partial t^i \partial \bar{t}^j} - \left\langle \overline{\partial}_X \left(\frac{\partial \varphi}{\partial t^i}\right), \overline{\partial}_X \left(\frac{\partial \varphi}{\partial t^j}\right) \right\rangle_g . \tag{1.6}$$

Remark 1.4. 1) The curvature formula (1.5) is derived implicitly in some special cases by different authors (c.f. [**Bern09a**], [**LSYau09**], [**Sch13**].)

2) In the real parameter case,

$$c(\varphi) = \ddot{\varphi} - |\overline{\partial}_X \dot{\varphi}|^2_g.$$

When $c(\varphi) = 0$, it is the geodesic equation in the space of Kähler potentials. For this comprehensive topic, we just refer the reader to [**Semmes92**], [**Donald99**], [**Chen00**], [**PhoStu06**], [**Bern09a**], [**Bern11a**] and references therein.

3) For the vector bundle case, the authors also expect that the first line on the right hand side of (1.4), i.e.

$$c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}}u, u\right\} - (\alpha_i \cup u, \alpha_j \cup u) \cdot (\sqrt{-1}dt^i \wedge d\bar{t}^j)$$

can be written into certain geodesic form in the space of Hermitian metrics on $\mathcal{E}$ when $\mathcal{E}$ has some stability property (see formula (3.12) for the line bundle case).

4) If $p : \mathcal{X} \to S$ is the universal curve with genus $g \geq 2$, i.e. $p : \mathcal{T}_g \to \mathcal{M}_g$. If $\mathcal{L} = K_{\mathcal{T}_g/\mathcal{M}_g}$, one can deduce Wolpert's curvature formula ([**Wolp86**]) for the (dual) Weil-Petersson metric on $p_*(K^{\otimes 2}_{\mathcal{T}_g/\mathcal{M}_g})$ easily from (1.5) (see also [**Siu86**], [**LSYau09**] [**Bern11**] and [**Sch13**]).

5) When $k = 1$, one can use (1.5) to study the convex and concave property of the logarithm volume functional on a Fano manifold ([**Bern11a**], see also Theorem 4.6, Proposition 4.7). Intrinsically, it amounts to the standard $\overline{\partial}$-estimate $\|\psi\| \leq \|\overline{\partial}\psi\|$ on functions with $\int_X \psi = 0$ if the Fano manifold is polarized by its anti-canonical class.

As the first application of Theorem 1.1, we obtain

Theorem 1.5. *Let $\mathcal{X} \to S$ be infinitesimally trivial. If there exists a Hermitian metric on $\mathcal{E}$ which is Nakano-negative along the base, then $p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ is Nakano-negative.*

Next, we follow Berndtsson's ideas in his remarkable papers [**Bern09**], [**Bern09a**], [**Bern11**], [**Bern11a**] and set

$$\widetilde{u} = u - dt^i \wedge v_i.$$

By using "Berndtsson's magic formula"

$$c_n \int_{X_s} \{u, u\} = c_n \int_{X_s} \{\widetilde{u}, \widetilde{u}\},$$

we obtain

Theorem 1.6. *The curvature Θ^E of $p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ has the following "positive form":*

$$\begin{aligned} (\sqrt{-1}\Theta^E u, u) &= c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}}\widetilde{u}, \widetilde{u}\right\} \\ &+ \left(\eta_i + \nabla''_X v_i, \eta_j + \nabla''_X v_j\right) \cdot (\sqrt{-1} dt^i \wedge d\overline{t}^j). \end{aligned} \tag{1.7}$$

When $\mathcal{E}$ is a line bundle, the curvature formula (1.7) is implicitly obtained by Berndtsson in [**Bern09**], [**Bern09a**], [**Bern11**] and [**Bern11a**]. When $\mathcal{E}$ is a Nakano-positive vector bundle, a similar formulation seems to be obtained in [**MouTak08**] by using Berndtsson's idea, but v_i are not given explicitly. As it is shown, these v_i play a key role in these curvature formulas and also their applications.

Let $c_{i\overline{j}}$ be the $\mathcal{E}$-valued $(n,0)$-form coefficient of $dt^i \wedge d\overline{t}^j$ in the local expression of

$$\Theta^{\mathcal{E}}(u - dt^i \wedge v_i),$$

and $d_{i\overline{j}}$ be the $\mathcal{E}$-valued $(n,0)$-form coefficient of $dt^i \wedge d\overline{t}^j$ in the local expression of

$$\sqrt{-1}\nabla''\nabla'\widetilde{u}.$$

Theorem 1.7. *The curvature Θ^E of $E = p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ has the following "compact form":*

$$(\sqrt{-1}\Theta^E u, u) = c_n \int_{X_s} \{d_{i\overline{j}}, u\} \cdot (\sqrt{-1} dt^i \wedge d\overline{t}^j).$$

Moreover, if the family $\mathcal{X} \to S$ is infinitesimally trivial,

$$\begin{aligned} (\sqrt{-1}\Theta^E u, u) &= c_n \int_{X_s} \{d_{i\overline{j}}, u\} \cdot (\sqrt{-1} dt^i \wedge d\overline{t}^j) \\ &= c_n \int_{X_s} \{c_{i\overline{j}}, u\} \cdot (\sqrt{-1} dt^i \wedge d\overline{t}^j). \end{aligned}$$

As applications, we use it to study the degeneracy of the curvature tensor of $p_*(K_{\mathcal{X}/S}\otimes\mathcal{E})$ under the assumption that $\mathcal{X}\to S$ is infinitesimally trivial and $(\mathcal{E},h^{\mathcal{E}})\to\mathcal{X}$ is Nakano semi-positive. In this case, $c_{i\bar{j}}$ is closely related to the geometry of the family $\mathcal{E}\to\mathcal{X}\to S$. When $(\mathcal{E},h^{\mathcal{E}}=e^{-\varphi})$ is a relatively positive line bundle, $c_{i\bar{j}}$ is the same as the geodesic term $c_{i\bar{j}}(\varphi)(u)$ defined in (1.6) when the curvature degenerates. Furthermore, when $H^{n,1}(X_s,\mathcal{E}_s)=0$, we show that v_i are all holomorphic over the total space $\mathcal{X}$ and we can use it to construct holomorphic automorphisms of the family $\mathcal{E}\to\mathcal{X}\to S$ and study the moduli space of projectively flat vector bundles.

We consider a family of projectively flat vector bundles $(\mathcal{E}_s,h^{\mathcal{E}_s})_{s\in S}$ with polarization

$$\sqrt{-1}\Theta^{\mathcal{E}_s}=\omega_g\otimes h^{\mathcal{E}_s}. \tag{1.8}$$

Let W_i be the dual vector of the Kodaira-Spencer form α_i defined in (1.3), i.e. W_i is an $End(\mathcal{E}_s)$-valued $(1,0)$ vector field. Then v_i, u and the Kodaira-Spencer vectors W_i are related by

$$i_{W_i}u=-v_i \tag{1.9}$$

when the curvature of $p_*(K_{\mathcal{X}/S}\otimes\mathcal{E})$ is degenerated. In this case, W_i is an $End(\mathcal{E}_s)$-valued holomorphic vector field on the fiber. We also see that, the horizontal lift of $\frac{\partial}{\partial t^i}$,

$$V_i=\frac{\partial}{\partial t^i}-W_i$$

is a (local) $End(\mathcal{E})$-valued holomorphic vector field over the total space $\mathcal{X}$. Moreover, the Lie derivatives of the curvature tensor of $\mathcal{E}_s$ with respect to V_i are all zero, i.e.

$$\mathscr{L}_{V_i}\omega_g=0.$$

That means, if the curvature of $p_*(K_{\mathcal{X}/S}\otimes\mathcal{E})$ degenerates at some point $s\in S$, then the family $\mathcal{E}\to\mathcal{X}\to S$ moves by an infinitesimal automorphism of $\mathcal{E}$ when the base point varies.

We can formulate it into a global version. Let $\mathcal{X}=X\times\mathbb{D}$, where $\mathbb{D}$ is a unit disk. Let $\mathbb{E}_0\to X$ be a holomorphic vector bundle. If $(\mathbb{E}_0,h(t))_{t\in\mathbb{D}}\to X$ is a smooth family of projectively flat vector bundles with polarization (1.8). We denote by $\mathcal{E}$, the pullback family $p_2^*(\mathbb{E}_0)$ over $p_2:\mathcal{X}\to X$.

Theorem 1.8. *If the curvature Θ^E of $E=p_*(K_{\mathcal{X}/\mathbb{D}}\otimes\mathcal{E})$ vanishes in a small neighborhood of $0\in\mathbb{D}$, then there exists a holomorphic vector field V on X with flows $\Phi_t\in Aut_H(X,\mathbb{E}_0)$ such that*

$$\Phi_t^*(\omega_t)=\omega_0$$

for small t.

Remark 1.9. We can also use the holomorphic vector field V to study the uniqueness of Hermitian-Einstein metrics on stable bundles, the stability of the direct image $p_*(\mathcal{E})$ and the asymptotic stability of $p_*(\mathcal{E}\otimes\mathcal{L}^k)$ for large k. We shall carry it out in the sequel to this paper.

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2. Background materials

2.1. Hodge theory on vector bundles. Let (E,h) be a Hermitian holomorphic vector bundle over the compact Kähler manifold (X,ω) and $\nabla=\nabla'+\nabla''$ be the Chern connection on it. Here, we also have the relation $\nabla''=\overline{\partial}$. With respect to metrics on E and X, we set

$$\Delta''=\nabla''\nabla''^*+\nabla''^*\nabla'',$$

$$\Delta'=\nabla'\nabla'^*+\nabla'^*\nabla'.$$

Accordingly, we associate the Green operators and harmonic projections $\mathbb{G}$, $\mathbb{H}$ and $\mathbb{G}'$, $\mathbb{H}'$ in Hodge decomposition to them, respectively. More precisely,

$$\mathbb{I}=\mathbb{H}+\Delta''\circ\mathbb{G},\ \mathbb{I}=\mathbb{H}'+\Delta'\circ\mathbb{G}'.$$

For any $\varphi,\psi\in\Omega^{\bullet,\bullet}(X,E)$, there is a *sesquilinear pairing*

$$\{\varphi,\psi\}=\varphi^\alpha\wedge\overline{\psi^\beta}\langle e_\alpha,e_\beta\rangle \tag{2.1}$$

if $\varphi=\varphi^\alpha e_\alpha$ and $\psi=\psi^\beta e_\beta$ in the local frame $\{e_\alpha\}$ of E. By the metric compatible property,

$$\partial\{\varphi,\psi\}=\{\nabla'\varphi,\psi\}+(-1)^{p+q}\{\varphi,\nabla''\psi\} \tag{2.2}$$

if $\varphi\in\Omega^{p,q}(X,E)$.

Let Θ^E be the Chern curvature of (E,h). It is well-known

$$\Delta''=\Delta'+[\sqrt{-1}\Theta^E,\Lambda_g] \tag{2.3}$$

where Λ_g is the contraction operator with respect to the Kähler metric ω. The following observation plays an important role in our computations.

Lemma 2.1. *Let E be any Hermitian vector bundle over a compact Kähler manifold (X,ω). For any $\alpha\in\Omega^{n,0}(X,E)$ with no harmonic part with respect to Δ'', i.e. $\mathbb{H}(\alpha)=0$, then*

1) $\mathbb{H}'(\alpha)=0$;
2) *The $(n-1,0)$ form $v=\nabla'^*\mathbb{G}'\alpha$ is a solution to the equation*

$$\nabla' v=\alpha;$$

3) $\nabla'' v$ *is primitive.*

Proof. The first statement follows from the Bochner identity on E-valued $(n,0)$-forms. More precisely, by (2.3)

$$\Delta''\beta = \Delta'\beta$$

for any $\beta \in \Omega^{n,0}(X,E)$. Hence $\mathbb{H}'(\beta) = \mathbb{H}(\beta)$. For (2), by Hodge decomposition, we have

$$\begin{aligned} \nabla' v &= \nabla'\nabla'^*\mathbb{G}'(\alpha) \\ &= \alpha - \mathbb{H}'(\alpha) - \nabla'^*\nabla'\mathbb{G}'(\alpha) \\ &= \alpha - \mathbb{H}'(\alpha) = \alpha. \end{aligned}$$

For (3), let $L_g = \omega\wedge$. By Hodge identity $[\nabla'^*, L_g] = -\sqrt{-1}\nabla''$,

$$\begin{aligned} \omega \wedge \nabla'' v = L_g \nabla'' v &= L_g \nabla''\nabla'^*\mathbb{G}'\alpha \\ &= -L_g \nabla'^*\nabla''\mathbb{G}'\alpha \\ &= \left(-\sqrt{-1}\nabla'' - \nabla'^* L_g\right)\nabla''\mathbb{G}'\alpha \\ &= 0 \end{aligned}$$

since $L_g\nabla''\mathbb{G}'\alpha$ is an $(n+1,2)$ form. q.e.d.

The following Riemann-Hodge bilinear relation will be used frequently, and the proof of it can be found in [**Huyb05**, Corollary 1.2.36] or [**Voisin02**, Proposition 6.29].

Lemma 2.2. *If $\varphi, \psi \in \Omega^{p,q}(X,E) \subset \Omega^n(X,E)$ are primitive, then*

$$(\varphi,\psi) = \left(\sqrt{-1}\right)^{n(n-1)+(p-q)} \int_X \{\varphi,\psi\} \tag{2.4}$$

where $(\bullet,\bullet)$ is the standard inner product (norm) induced by metrics on X and E.

2.2. Positivity of vector bundles. Let $\{z^i\}_{i=1}^n$ be the local holomorphic coordinates on X and $\{e_\alpha\}_{\alpha=1}^r$ be a local frame of E. The curvature tensor $\Theta^E \in \Gamma(X, \Lambda^2 T^*X \otimes E^* \otimes E)$ has the form

$$\Theta^E = R^\gamma_{i\bar{j}\alpha} dz^i \wedge d\overline{z}^j \otimes e^\alpha \otimes e_\gamma, \tag{2.5}$$

where $R^\gamma_{i\bar{j}\alpha} = h^{\gamma\overline{\beta}} R_{i\bar{j}\alpha\overline{\beta}}$ and

$$R_{i\bar{j}\alpha\overline{\beta}} = -\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j} + h^{\gamma\overline{\delta}} \frac{\partial h_{\alpha\overline{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\overline{\beta}}}{\partial \overline{z}^j}. \tag{2.6}$$

Here and henceforth we adopt the Einstein convention for summation.

Definition 2.3. A Hermitian vector bundle (E,h) is said to be *Griffiths-positive*, if for any nonzero vectors $u=u^i\frac{\partial}{\partial z^i}$ and $v=v^\alpha e_\alpha$,

$$\sum_{i,j,\alpha,\beta} R_{i\overline{j}\alpha\overline{\beta}}u^i\overline{u}^j v^\alpha\overline{v}^\beta>0. \tag{2.7}$$

(E,h) is said to be *Nakano-positive*, if for any nonzero vector $u=u^{i\alpha}\frac{\partial}{\partial z^i}\otimes e_\alpha$,

$$\sum_{i,j,\alpha,\beta} R_{i\overline{j}\alpha\overline{\beta}}u^{i\alpha}\overline{u}^{j\beta}>0. \tag{2.8}$$

(E,h) is said to be *dual-Nakano-positive*, if for any nonzero vector $u=u^{i\alpha}\frac{\partial}{\partial z^i}\otimes e_\alpha$,

$$\sum_{i,j,\alpha,\beta} R_{i\overline{j}\alpha\overline{\beta}}u^{i\beta}\overline{u}^{j\alpha}>0. \tag{2.9}$$

It is easy to see that (E,h) is dual-Nakano-positive if and only if (E^*,h^*) is Nakano-negative. The notions of semi-positivity, negativity and semi-negativity can be defined similarly. We say E is Nakano-positive (resp. Griffiths-positive, dual-Nakano-positive, $\cdots$), if it admits a Nakano-positive (resp. Griffiths-positive, dual-Nakano-positive, $\cdots$) metric.

2.3. Direct image sheaves of vector bundles. Let $\mathcal{X}$ be a Kähler manifold with dimension $m+n$ and S a Kähler manifold with dimension m. Let $p:\mathcal{X}\to S$ be a smooth Kähler fibration. That means, for each $s\in S$,

$$X_s:=p^{-1}(\{s\})$$

is a compact Kähler manifold with dimension n. Let $(\mathcal{E},h^{\mathcal{E}})\to\mathcal{X}$ be a Hermitian holomorphic vector bundle. In the following, we adopt the setting in [**Bern09**, Section 4]. Consider the space of holomorphic $\mathcal{E}$-valued $(n,0)$-forms on X_s,

$$E_s:=H^0(X_s,\mathcal{E}_s\otimes K_{X_s})\cong H^{n,0}(X_s,\mathcal{E}_s)$$

where $\mathcal{E}_s=\mathcal{E}|_{X_s}$. Here, we assume all E_s has the same dimension. With a natural holomorphic structure,

$$E=\bigcup_{s\in S}\{s\}\times E_s$$

is isomorphic to the direct image sheaf $p_*(K_{\mathcal{X}/S}\otimes\mathcal{E})$ if E has certain positive property.

For every point $s\in S$, we can take a local holomorphic coordinate $(W;t=(t^1,\cdots,t^m))$ centered at s such that $(W;t)$ is a unit ball in $\mathbb{C}^m$, and a system of local coordinates $\mathfrak{U}=\{(U_\alpha;z_\alpha=(z_\alpha^1,\cdots,z_\alpha^n),t)\}$ of $p^{-1}(W)\subset\mathcal{X}$. We would like to drop the index α in the sequel when no confusion arises. By the canonical isomorphism $K_{\mathcal{X}/S}|_{X_s}\cong K_{X_s}$, we

make the following identification which will be used frequently in the sequel. For more details, we refer the reader to [**Bern09**, Section 4] and [**MouTak08**, Section 2].

1) a local smooth section u of E over S is an $\mathcal{E}$-valued $(n,0)$ form on X_s. In the local holomorphic coordinates on the total space, $(z,t) := (z^1, \cdots, z^n, t^1, \cdots, t^m)$ on $\mathcal{X}$, it is equivalent to the fact that $u \wedge dt^1 \wedge \cdots \wedge dt^m$ is a local section of $K_{\mathcal{X}}$. Hence, for example, if u' is an $\mathcal{E}$-valued $(n,0)$ form on $\mathcal{X}$, such that $u' \wedge dt^1 \wedge \cdots \wedge dt^m = 0$, then $u + u'$ and u are the same local smooth section of E over S. That means, if we use an $\mathcal{E}$-valued $(n,0)$ form u on $\mathcal{X}$ to represent a given local smooth section of E, in general, u is not unique. Moreover, two representatives differ by a form $dt^i \wedge \gamma_i$ on $\mathcal{X}$ where γ_i are $(n-1,0)$ forms on $\mathcal{X}$.
2) u is a local holomorphic section of E over S, if $\overline{\partial}_{\mathcal{X}} u$ restricted to the zero form on each fiber X_s, that is

$$\overline{\partial}_{\mathcal{X}} u = \sum_j dt^i \wedge \eta_i \tag{2.10}$$

where η_i are $(n-1,1)$ forms when restricted on each fiber X_s. Clearly, η^i are not uniquely determined, but their restrictions to fibers are.

By the identification above, there is a natural metric on the E induced by metrics $h^{\mathcal{E}_s}$ on $\mathcal{E}_s$. For any local smooth section u of E, we define a Hermitian metric h on E by

$$h(u,u) = c_n \int_{X_s} \{u, u\} \tag{2.11}$$

where $c_n = (\sqrt{-1})^{n^2}$.

Next, we want to define the Chern connection for the Hermitian holomorphic vector bundle $(E,h) \to S$. Let $\nabla^{\mathcal{E}} = \nabla' + \nabla''$ be the Chern connection of $(\mathcal{E}, h^{\mathcal{E}})$ over the total space $\mathcal{X}$. Therefore $\nabla'' = \overline{\partial}_{\mathcal{X}}$. For any local smooth section u of E, it is also an $\mathcal{E}$-valued $(n,0)$-form on $\mathcal{X}$. It is obvious that

$$\nabla'' u = d\overline{t}^j \wedge \tau_{\overline{j}} + dt^i \wedge \eta_i. \tag{2.12}$$

Similarly,

$$\nabla' u = dt^i \wedge \nu_i. \tag{2.13}$$

Here, $\tau_{\overline{j}}$, η_i and ν_i are local sections over $\mathcal{X}$, and again, they are not unique on $\mathcal{X}$, but there restrictions to fibers are. The following lemma is given in [**Bern09**, Lemma 4.1].

Lemma 2.4. *Let $D = D' + D''$ be the Chern connection of the Hermitian holomorphic vector bundle $(E,h) \to S$, then for any local smooth*

section u of E,

$$D''u = \tau_{\overline{j}} d\overline{t}^j, \qquad D'u = \pi(\nu_i)dt^i \tag{2.14}$$

where π is the orthogonal projection

$$\pi : \Omega^{n,0}(X_s, \mathcal{E}_s) \to H^{n,0}(X_s, \mathcal{E}_s). \tag{2.15}$$

The following result is contained in [**Bern11**, Lemma 2.1]. For the sake of completeness, we include a proof here.

Lemma 2.5. *For any local holomorphic section u of E, we can choose a representative of u, i.e. an $\mathcal{E}$-valued $(n,0)$ form on $\mathcal{X}$ such that*

$$\nabla'' u = dt^i \wedge \eta_i \tag{2.16}$$

where η_i are primitive $(n-1,1)$ forms when restricted on each fiber X_s.

Proof. Let $\widehat{u}$ be an arbitrary $(n,0)$ form on $\mathcal{X}$ which represents the local *holomorphic* section u of E, i.e.

$$\overline{\partial}_{\mathcal{X}} \widehat{u} = dt^i \wedge \widehat{\eta}_i.$$

Let ω be the $(1,1)$ form on $\mathcal{X}$ which restricted to the Kähler forms on each fiber X_s. Since $\widehat{u} \wedge \omega$ is an $(n+1,1)$ form on $\mathcal{X}$, it can be written as

$$\widehat{u} \wedge \omega = dt^i \wedge y_i$$

for some $(n,1)$ forms y_i on $\mathcal{X}$. Hence

$$dt^i \wedge \widehat{\eta}_i \wedge \omega = \overline{\partial}_{\mathcal{X}} \widehat{u} \wedge \omega = \overline{\partial}_{\mathcal{X}}(\widehat{u} \wedge \omega) = -dt^i \wedge (\overline{\partial}_{\mathcal{X}} y_i).$$

Then $\widehat{\eta}_i \wedge \omega = -\overline{\partial}_{\mathcal{X}} y_i$ when restricted on each fiber X_s. Let $\widehat{v}_i$ to be any form on $\mathcal{X}$ such that

$$\widehat{v}_i \wedge \omega = y_i.$$

If we set $u = \widehat{u} - dt^i \wedge \widehat{v}_i$, then $\overline{\partial}_{\mathcal{X}} u = dt^i \wedge \eta_i$ where

$$\eta_i = \widehat{\eta}_i + \overline{\partial}_{\mathcal{X}} \widehat{v}_i.$$

It is obvious that η_i are primitive $(n-1,1)$ forms when restricted on each fiber X_s. q.e.d.

Remark 2.6. Let $[k_i] \in H^{0,1}(X_s, T^{1,0}X_s \otimes \mathcal{E}_s)$ be the Kodaira-Spencer class in the direction of $\partial/\partial t^i$. It is shown in [**Bern09**, p.543] and also [**Bern11**, Lemma 2.2], when restricted to each fiber, η_i and $k_i \cup u$ define the same class in $H^{n-1,1}(X_s, \mathcal{E}_s)$. In particular, if $\mathcal{X} \to S$ is infinitesimally trivial, we can choose η_i to be zero.

Let Θ^E be the Chern curvature of $(E,h) \to S$. The following formula is obvious.

Lemma 2.7. *Let u be a local holomorphic section of E over S, then*

$$\overline{\partial}\partial(u,u) = (D''D'u, u) - (D'u, D'u) = (\Theta^E u, u) - (D'u, D'u). \tag{2.17}$$

To end this section, we list some notations we shall use in the sequel:

• $D = D' + D''$ the Chern connection on the Hermitian vector bundle $(E, h) \to S$;

• $\nabla^{\mathcal{E}} = \nabla' + \nabla''$ the Chern connection of $\mathcal{E}$ over the total space $\mathcal{X}$; We will also use $\overline{\partial}_{\mathcal{X}}$ for ∇'' if there is no confusion;

• To simplify notations, we will denote the Chern connection $\nabla^{\mathcal{E}}|_{X_s}$ of the Hermitian vector bundle $(\mathcal{E}_s, h^{\mathcal{E}_s}) \to X_s$ by $\nabla_X = \nabla'_X + \nabla''_X$ although it depends on $s \in S$;

• $d = \partial + \overline{\partial}$ the natural decomposition of d on the base S;

• $\{\omega_s\}_{s\in S}$ a smooth family of Kähler metrics on $\{X_s\}_{s\in S}$;

• $\mathbb{G}'$ the Green's operator for $\Delta'_X = \nabla'_X \nabla'^*_X + \nabla'^*_X \nabla'_X$;

• $\pi : \Omega^{n,0}(X_s, \mathcal{E}_s) \to H^{n,0}(X_s, \mathcal{E}_s)$ the orthogonal projection on the fiber;

• $\pi_\perp = \mathbb{I} - \pi$.

3. Curvature formulas of direct images of vector bundles

3.1. A straightforward computation. In this section, we will derive several general curvature formulas for direct image $E = p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ by using Lemma 2.1.
The following corollary is a special case of Lemma 2.1.

Corollary 3.1. *For any local section u of E with $\nabla' u = dt^i \wedge \nu_i$, we set*

$$v_i = -\nabla'^*_X \mathbb{G}' \pi_\perp (\nu_i) . \tag{3.1}$$

where ν_i is restricted on the fiber X_s. Then

1) $\nabla'_X v_i = -\pi_\perp (\nu_i)$;
2) $\nabla''_X v_i$ *is primitive.*

Before computing the curvature tensors of the direct images, we need a well-known result:

Lemma 3.2. *∂ and $\overline{\partial}$ commute with the fiber integration. More precisely,*

$$\overline{\partial} \int_{X_s} \alpha = \int_{X_s} \overline{\partial}_{\mathcal{X}} \alpha, \qquad \partial \int_{X_s} \alpha = \int_{X_s} \partial_{\mathcal{X}} \alpha$$

for any smooth $\alpha \in \Omega^{\bullet,\bullet}(\mathcal{X})$.

Note that, in this paper, we make the following conventions. Let u be a local holomorphic section of $E = p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$,

1) we always choose a representative, i.e. an $\mathcal{E}$-valued $(n,0)$ form on $\mathcal{X}$ such that

$$\nabla'' u = dt^i \wedge \eta_i, \quad \nabla' u = dt^i \wedge \nu_i$$

where η_i are primitive $(n-1,1)$ forms when restricted on each fiber X_s, and ν_i are $(n,0)$ forms when restricted on each fiber X_s.
2) v_i is fixed to be $-\nabla'^*_X \mathbb{G}' \pi_\perp(\nu_i)$, and we do not change it anymore.

Theorem 3.3. *Let* Θ^E *be the Chern curvature of* $E = p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$. *For any local holomorphic section* u *of* E, *the curvature* Θ^E *has the following "negative form":*

$$\begin{aligned}(\sqrt{-1}\Theta^E u, u) &= c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}} u, u\right\} - (\Delta'_X v_i, v_j) \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j) \\ &\quad + (\eta_i, \eta_j) \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j). \end{aligned} \tag{3.2}$$

Proof. Since u is a local holomorphic section of $E = p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$, it can be represented by a local smooth $(n,0)$ form on $\mathcal{X}$ with the property $\nabla'' u = dt^i \wedge \eta_i$ where η_i are local $(n-1,1)$ forms over $\mathcal{X}$. Moreover, when restricted to each fiber X_s, η_i are all primitive. By comparing the top degrees along the fiber direction, we conclude that

$$\int_{X_s} \nabla' \{\nabla'' u, u\} = \int_{X_s} \nabla'_X \{\nabla'' u, u\} = 0, \tag{3.3}$$

where the second identity follows from Stokes' theorem. It is equivalent to the fact that

$$\begin{aligned} -c_n\sqrt{-1} \int_{X_s} \{\nabla' \nabla'' u, u\} &= c_n \sqrt{-1}(-1)^{n+1} \int_{X_s} \{\nabla'' u, \nabla'' u\} \\ &= c_n \sqrt{-1}(-1)^{n+1} \int_{X_s} \{dt^i \wedge \eta_i, dt^j \wedge \eta_j\} \\ &= -c_n \int_{X_s} \{\eta_i, \eta_j\} \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j) \\ &= (\eta_i, \eta_j) \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j) \end{aligned}$$

where the last identity follows from Lemma 2.2 since η_i, η_j are primitive $(n-1,1)$ forms on X_s. By taking the conjugate, we see

$$\overline{-c_n\sqrt{-1} \int_{X_s} \{\nabla' \nabla'' u, u\}} = c_n \sqrt{-1} \int_{X_s} \{u, \nabla' \nabla'' u\}.$$

On the other hand, $(\eta_i, \eta_j) \cdot (\sqrt{-1} dt^i \wedge d\overline{t}^j)$ is a real $(1,1)$ form, and we obtain

$$\begin{aligned} -c_n \sqrt{-1} \int_{X_s} \{\nabla' \nabla'' u, u\} &= (\eta_i, \eta_j) \cdot (\sqrt{-1} dt^i \wedge d\overline{t}^j) \\ &= c_n \sqrt{-1} \int_{X_s} \{u, \nabla' \nabla'' u\}. \end{aligned} \tag{3.4}$$

By curvature formula (2.17),

$$\begin{aligned} (\sqrt{-1}\Theta^E u, u) &= \sqrt{-1}(D'u, D'u) - \sqrt{-1}\partial\overline{\partial}\|u\|^2 \\ &= \sqrt{-1}(D'u, D'u) - c_n \int_{X_s} \sqrt{-1}\nabla' \left\{\nabla'' u, u\right\} \\ &\quad -c_n \int_{X_s} \sqrt{-1}(-1)^n \nabla' \{u, \nabla' u\} \\ &\overset{(3.3)}{=} \sqrt{-1}(D'u, D'u) - c_n (-1)^n \int_{X_s} \left\{\sqrt{-1}\nabla' u, \nabla' u\right\} \\ &\quad -c_n \sqrt{-1} \int_{X_s} \{u, \nabla'' \nabla' u\} \\ &= \sqrt{-1}(D'u, D'u) - c_n (-1)^n \int_{X_s} \left\{\sqrt{-1}\nabla' u, \nabla' u\right\} \\ &\quad +c_n \int_{X_s} \sqrt{-1} \left\{\Theta^{\mathcal{E}} u, u\right\} + (\eta_i, \eta_j) \cdot (\sqrt{-1} dt^i \wedge d\overline{t}^j) \end{aligned}$$

where the last identity follows from (3.4) and $\Theta^{\mathcal{E}} = \nabla' \nabla'' + \nabla'' \nabla'$. By definition (Lemma 2.4), we have $D'u = \pi(\nu_i) \wedge dt^i$. From the orthogonal decomposition, $\nu_i = \pi(\nu_i) + \pi_\perp(\nu_i)$, and the fact $\nabla' u = dt^i \wedge \nu_i$, we see that

$$\begin{aligned} &\sqrt{-1}(D'u, D'u) - c_n (-1)^n \int_{X_s} \left\{\sqrt{-1}\nabla' u, \nabla' u\right\} \\ = \ &\sqrt{-1}(D'u, D'u) - c_n (-1)^n \int_{X_s} \left\{\sqrt{-1} dt^i \wedge \nu_i, dt^j \wedge \nu_j\right\} \\ = \ &-c_n (-1)^n \int_{X_s} \left\{\sqrt{-1} dt^i \wedge \pi_\perp(\nu_i), dt^j \wedge \pi_\perp(\nu_j)\right\} \\ = \ &-c_n \int_{X_s} \{\nabla'_X v_i, \nabla'_X v_j\} (\sqrt{-1} dt^i \wedge d\overline{t}^j). \end{aligned}$$

where we use the fact that $-\pi_\perp(\nu_i) = \nabla'_X v_i$ in Corollary 3.1. Since $\nabla'_X v_i$ are top $(n,0)$ forms on each fiber, and so primitive. On the other

hand, by formula (3.1), we know $\nabla'^{*}_{X} v_i = 0$. Therefore, by Riemann-Hodge bilinear relation (2.4), we obtain

$$\begin{aligned} & \sqrt{-1}(D'u, D'u) - c_n(-1)^n \int_{X_s} \left\{\sqrt{-1}\nabla' u, \nabla' u\right\} \\ = \; & -(\nabla'_X v_i, \nabla'_X v_j) \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j) \\ = \; & -(\Delta'_X v_i, v_j) \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j). \end{aligned}$$

Now the curvature formula (3.2) follows. q.e.d.

As a straightforward consequence of Theorem 3.3, we obtain

Corollary 3.4. *The curvature Θ^E has the following form:*

$$\begin{aligned} & (\sqrt{-1}\Theta^E u, u) \\ = \; & c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}} u, u\right\} - (\Delta'_X v_i, \Delta'_X v_j) \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j) \\ & + (\Delta'_X v_i, \Delta'_X v_j - v_j) \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j) + (\eta_i, \eta_j) \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j). \end{aligned} \tag{3.5}$$

In the following, we want to interpret the second term on the right hand side of (3.5) into a geometric quantity. Let

$$\alpha_i = \Theta^{\mathcal{E}}\left(\frac{\partial}{\partial t^i}\right)\Big|_{X_s} \in \Omega^{0,1}(X_s, End(\mathcal{E}_s)). \tag{3.6}$$

If the family is infinitesimally trivial, $[\alpha_i] \in H^{0,1}(X_s, End(\mathcal{E}_s))$ is the Kodaira-Spencer class ([**SchTo92**, Proposition 1]) of the deformation $\mathcal{E} \to \mathcal{X} \to S$ in the direction of $\frac{\partial}{\partial t^i} \in T_s S$. We need to point out that, all computations are restricted to the fixed fiber $X_s (= X)$. By Hodge identity $[\Lambda_g, \nabla''_X] = -\sqrt{-1}\nabla'^{*}_{X}$, $\nabla'^{*}_{X} v_i = 0$ and $\nabla''_X(\pi(\nu_i)) = 0$, we get

$$\begin{aligned} \Delta'_X v_i &= \nabla'^{*}_{X}\nabla'_X v_i = \sqrt{-1}\Lambda_g \nabla''_X \nabla'_X v_i \\ \text{(Corollary 3.1)} \quad &= -\sqrt{-1}\Lambda_g \nabla''_X \pi_\perp(\nu_i) \\ &= -\sqrt{-1}\Lambda_g \nabla''_X \nu_i. \end{aligned} \tag{3.7}$$

On the other hand, by the relation $\nabla' u = dt^i \wedge \nu_i$, it is obvious that when restricted to X_s,

$$\nu_i = \nabla'_i u,$$

where we adopt the notation that $\nabla'_i u := (\nabla' u)(\frac{\partial}{\partial t^i})$. We get

$$\begin{aligned} \Delta_X v_i &= -\sqrt{-1}\Lambda_g \nabla''_X \nu_i \\ &= -\sqrt{-1}\Lambda_g \left(\nabla''_X \nabla'_i + \nabla'_i \nabla''_X\right) u + \sqrt{-1}\Lambda_g \nabla'_i \nabla''_X u \\ &= -\sqrt{-1}\Lambda_g(\alpha_i \cup u) + \sqrt{-1}\Lambda_g \nabla'_i \nabla''_X u \\ &= -\sqrt{-1}\Lambda_g(\alpha_i \cup u). \end{aligned} \tag{3.8}$$

where the last step follows from the fact that $\sqrt{-1}\Lambda_g\nabla'_i\nabla''_X u$ is zero when restricted to X_s. In fact,

$$\nabla''_X u = \nabla'' u - d\bar{t}^\ell \wedge \frac{\partial u}{\partial \bar{t}^\ell} = dt^j \wedge \eta_j - d\bar{t}^\ell \wedge \frac{\partial u}{\partial \bar{t}^\ell}.$$

By Corollary 3.4, we obtain the following :

Theorem 3.5. *The curvature Θ^E has the following "geodesic form":*

$$\begin{aligned} & (\sqrt{-1}\Theta^E u, u) \\ = \; & c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}} u, u\right\} - (\alpha_i \cup u, \alpha_j \cup u)\cdot(\sqrt{-1}dt^i \wedge d\bar{t}^j) \\ & +(\Delta'_X v_i, \Delta'_X v_j - v_j)\cdot(\sqrt{-1}dt^i \wedge d\bar{t}^j) + (\eta_i, \eta_j)\cdot(\sqrt{-1}dt^i \wedge d\bar{t}^j). \end{aligned} \tag{3.9}$$

Next we want to explain why (3.9) is called a "geodesic form" by a little bit more computations in some special cases. Let $(\mathcal{E}, e^{-\varphi})$ be a relative positive line bundle over $\mathcal{X}$, and we set $\omega_g = \sqrt{-1}\partial_X\overline{\partial}_X\varphi$ on each fiber.

Corollary 3.6. *Let $(\mathcal{L}, h^{\mathcal{L}} = e^{-\varphi})$ be a Hermitian line bundle over $\mathcal{X}$ such that $(\mathcal{L}|_{X_s}, h^{\mathcal{L}}_{X_s})$ is positive on each fiber X_s. Then the curvature Θ^E of $E = p_*(K_{\mathcal{X}/S}\otimes\mathcal{L})$ has the form:*

$$\begin{aligned} & (\sqrt{-1}\Theta^E u, u) \\ = \; & c_n \int_{X_s} c_{i\bar{j}}(\varphi)\left\{u, u\right\}\cdot(\sqrt{-1}dt^i \wedge d\bar{t}^j) \\ & + \left((\Delta'_X + 1)^{-1}\left(\nabla''_X\Delta'_X v_i\right), \nabla''_X\Delta'_X v_j\right)\cdot(\sqrt{-1}dt^i \wedge d\bar{t}^j) \\ & +(\eta_i, \eta_j)\cdot(\sqrt{-1}dt^i \wedge d\bar{t}^j). \end{aligned} \tag{3.10}$$

where $c_{i\bar{j}}(\varphi)$ is given by

$$c_{i\bar{j}}(\varphi) = \frac{\partial^2\varphi}{\partial t^i \partial \bar{t}^j} - \left\langle \overline{\partial}_X\left(\frac{\partial\varphi}{\partial t^i}\right), \overline{\partial}_X\left(\frac{\partial\varphi}{\partial t^j}\right)\right\rangle_g. \tag{3.11}$$

Proof. By formula (3.8), we obtain

$$\begin{aligned} & c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{L}} u, u\right\} - (\Delta'_X v_i, \Delta'_X v_j)\cdot(\sqrt{-1}dt^i \wedge d\bar{t}^j) \\ = \; & c_n \int_{X_s} c_{i\bar{j}}(\varphi)\left\{u, u\right\}\cdot(\sqrt{-1}dt^i \wedge d\bar{t}^j). \end{aligned} \tag{3.12}$$

On the other hand, we see that, the second line on the right hand side of (3.5) is non-negative. In fact, by formula (2.3) and the fact $\omega_g = \sqrt{-1}\Theta^{\mathcal{L}}|_{X_s}$, we have $\Delta''_X v_i = \Delta'_X v_i - v_i$ since v_i are $(n-1, 0)$

forms. Moreover, $\nabla''_X\Delta''_X v_i + \nabla''_X v_i = \nabla''_X\Delta'_X v_i$. Therefore,

$$\begin{aligned}(\Delta'_X v_i, \Delta'_X v_j - v_j) &= (\Delta''_X v_i + v_i, \Delta''_X v_j)\\ &= (\Delta''_X v_i, \Delta''_X v_j) + (v_i, \Delta''_X v_j)\\ &= (\Delta''_X v_i, \Delta''_X v_j) + (\Delta''_X v_i, v_j) = (\Delta'' v_i, \Delta' v_j)\\ &= (\nabla'' v_i, \nabla''\Delta'_X v_j).\end{aligned}$$

Similarly, by (2.3), we have $\Delta'_X(\nabla''_X v_i) = \Delta''_X(\nabla''_X v_i)$ since $\nabla''_X v_i$ are $(n-1,1)$ forms on the fiber. Therefore,

$$\Delta'_X(\nabla''_X v_i) = \Delta''_X(\nabla''_X v_i) = \nabla''(\Delta''_X v_i) = \nabla''_X(\Delta'_X v_i - v_i)$$

which is equivalent to

$$(\Delta'_X + 1)(\nabla''_X v_i) = \nabla''_X\Delta'_X v_i,$$

or equivalently,

$$\nabla''_X v_i = (\Delta'_X + 1)^{-1}(\nabla''_X\Delta'_X v_i).$$

Hence, we obtain

$$\begin{aligned}(\Delta'_X v_i, \Delta'_X v_j - v_j) &= (\nabla'' v_i, \nabla''\Delta'_X v_j)\\ &= \left((\Delta'_X + 1)^{-1}\left(\nabla''_X\Delta'_X v_i\right), \nabla''_X\Delta'_X v_j\right).\end{aligned}$$

q.e.d.

Remark 3.7. Similar formulas are also obtained in [**LSYau09**], [**Bern11**] and [**Sch13**].

Similarly, we get the "quantization" version:

Proposition 3.8. *The curvature Θ^{E_k} of $E_k = p_*(K_{\mathcal{X}/S}\otimes\mathcal{L}^k)$ has the following form:*

$$\begin{aligned}&(\sqrt{-1}\Theta^{E_k}u, u)\\ =\ & c_n\int_{X_s} kc_{i\bar{j}}(\varphi)\{u,u\}(\sqrt{-1}dt^i\wedge d\bar{t}^j)\\ &+\frac{1}{k}\left((\Delta'_X + k)^{-1}\left(\nabla''_X\Delta'_X v_i\right), \nabla''_X\Delta'_X v_j\right)\cdot(\sqrt{-1}dt^i\wedge d\bar{t}^j)\\ &+(\eta_i,\eta_j)\cdot(\sqrt{-1}dt^i\wedge d\bar{t}^j).\end{aligned}\tag{3.13}$$

*where $c_{i\bar{j}}(\varphi)$ is defined in (*3.11*).*

Proof. By Theorem 3.3, we rewrite the curvature formula as

$$\begin{aligned}&(\sqrt{-1}\Theta^{E_k}u, u)\\ =\ & c_n\int_{X_s}\sqrt{-1}\left\{\Theta^{\mathcal{L}^k}u, u\right\} - \frac{1}{k}(\Delta'_X v_i, \Delta'_X v_j)\cdot(\sqrt{-1}dt^i\wedge d\bar{t}^j)\\ &+\frac{1}{k}(\Delta'_X v_i, \Delta'_X v_j - kv_j)\cdot(\sqrt{-1}dt^i\wedge d\bar{t}^j)\\ &+(\eta_i,\eta_j)\cdot(\sqrt{-1}dt^i\wedge d\bar{t}^j).\end{aligned}\tag{3.14}$$

As similar as the arguments in Corollary 3.6, we deduce

$$\begin{aligned} & c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{L}^k} u, u\right\} - \frac{1}{k}(\Delta'_X v_i, \Delta'_X v_j)\cdot(\sqrt{-1}dt^i\wedge d\bar{t}^j) \\ = \;& c_n \int_{X_s} k c_{i\bar{j}}(\varphi)\{u,u\}(\sqrt{-1}dt^i\wedge d\bar{t}^j) \end{aligned}$$

and

$$(\Delta'_X v_i, \Delta'_X v_j - k v_j) = \left((\Delta'_X + k)^{-1}\left(\nabla''_X \Delta'_X v_i\right), \nabla''_X \Delta'_X v_j\right).$$

Hence (3.14) follows. q.e.d.

3.2. Computations by using Berndtsson's magic formula. In this subsection, we will derive several curvature formulas for direct image sheaf $p_*(K_{\mathcal{X}/S}\otimes\mathcal{E})$ following the ideas in [**Bern09**], [**Bern09a**], [**Bern11**] and [**Bern11a**]. However, we do not make any assumption on the curvature of $\mathcal{E}$. We only make use of the following "Berndtsson's magic formula":

Lemma 3.9. *Let u be a local* **smooth** *section of E. If $\widetilde{u} = u - dt^i\wedge v_i$, then*

$$c_n\int_{X_s}\{u,u\} = c_n\int_{X_s}\{\widetilde{u},\widetilde{u}\}. \tag{3.15}$$

Proof. It follows by comparing the (n,n)-forms along the fiber X_s. q.e.d.

In the following, u shall be a local *holomorphic* section to E, i.e. $\nabla''u = dt^i\wedge\eta_i$. Moreover, we set

$$\widetilde{u} = u - dt^i\wedge v_i \tag{3.16}$$

and thus fixed. Recall that, $v_i = -\nabla'^*_X \mathbb{G}'\pi_\perp(\nu_i)$ as defined in (3.1). It is easy to see, $\nabla''v_i = \nabla''_X v_i + d\bar{t}^j\wedge\frac{\partial v_i}{\partial\bar{t}^j}$, and so

$$\nabla''\widetilde{u} = \nabla''u + dt^i\wedge\nabla''v_i = dt^i\wedge(\eta_i + \nabla''_X v_i) + dt^i\wedge d\bar{t}^j\wedge\frac{\partial v_i}{\partial\bar{t}^j} \tag{3.17}$$

and similarly,

$$\begin{aligned} \nabla'\widetilde{u} &= \nabla' u + dt^i\wedge\nabla' v_i \\ &= dt^i\wedge\nu_i + dt^i\wedge\nabla' v_i \\ &= dt^i\wedge\left(\nu_i + \nabla'_X v_i\right) + dt^i\wedge dt^k\wedge\nabla'_k v_i \\ &= dt^i\wedge\pi(\nu_i) + dt^i\wedge dt^k\wedge\nabla'_k v_i \end{aligned}$$

since $\nabla'_X v_i = -\pi_\perp(\nu_i)$. To make the above formula into a compact form, we define

$$\mu_i := \pi(\nu_i) \tag{3.18}$$

and so

$$\nabla'\widetilde{u} = dt^i \wedge \mu_i + dt^i \wedge dt^k \wedge \nabla'_k v_i. \tag{3.19}$$

Therefore,

$$\begin{aligned} \nabla''\nabla'\widetilde{u} &= \nabla''(dt^i \wedge \mu_i + dt^i \wedge dt^k \wedge \nabla'_k v_i) \\ &= -dt^i \wedge d\overline{t}^j \wedge \frac{\partial \mu_i}{\partial \overline{t}^j} + dt^i \wedge dt^k \wedge \nabla''\nabla'_k v_i. \end{aligned} \tag{3.20}$$

By curvature formula (2.17) and the magic formula (3.15), we obtain

$$\begin{aligned} & (\sqrt{-1}\Theta^E u, u) \\ = \ & \sqrt{-1}(D'u, D'u) - \sqrt{-1}\partial\overline{\partial}\|u\|^2 \\ = \ & \sqrt{-1}(D'u, D'u) - \sqrt{-1}\partial\overline{\partial}\|\widetilde{u}\|^2 \\ = \ & \sqrt{-1}(D'u, D'u) - c_n(-1)^n \int_{X_s} \left\{\sqrt{-1}\nabla'\widetilde{u}, \nabla'\widetilde{u}\right\} \\ & -c_n \int_{X_s} \sqrt{-1}\left\{\nabla'\nabla''\widetilde{u}, \widetilde{u}\right\} - c_n(-1)^{n+1} \int_{X_s} \sqrt{-1}\left\{\nabla''\widetilde{u}, \nabla''\widetilde{u}\right\} \\ & -c_n \int_{X_s} \sqrt{-1}\left\{\widetilde{u}, \nabla''\nabla'\widetilde{u}\right\}. \end{aligned} \tag{3.21}$$

Claim. The first line and second line on the right hand side of (3.21) are all zero, i.e.

$$\sqrt{-1}(D'u, D'u) - c_n(-1)^n \int_{X_s} \left\{\sqrt{-1}\nabla'\widetilde{u}, \nabla'\widetilde{u}\right\} = 0 \tag{3.22}$$

and

$$-c_n \int_{X_s} \sqrt{-1}\left\{\nabla'\nabla''\widetilde{u}, \widetilde{u}\right\} - c_n(-1)^{n+1} \int_{X_s} \sqrt{-1}\left\{\nabla''\widetilde{u}, \nabla''\widetilde{u}\right\} = 0. \tag{3.23}$$

Proof. In fact, thanks to (3.19), we have

$$\begin{aligned} & -c_n(-1)^n \int_{X_s} \left\{\sqrt{-1}\nabla'\widetilde{u}, \nabla'\widetilde{u}\right\} \\ = \ & -c_n(-1)^n \int_{X_s} \sqrt{-1}\left\{dt^i \wedge \mu_i, dt^j \wedge \mu_j\right\} \\ = \ & -c_n \int_{X_s} \{\mu_i, \mu_j\} \cdot (\sqrt{-1} dt^i \wedge d\overline{t}^j). \end{aligned}$$

On the other hand,

$$D'u = \pi(\nu_i) \wedge dt^i = \mu_i \wedge dt^i.$$

Hence

$$\sqrt{-1}(D'u, D'u) = c_n \int_{X_s} \{\mu_i, \mu_j\} \cdot (\sqrt{-1} dt^i \wedge d\overline{t}^j).$$

We complete the proof of (3.22). On the other hand,

$$\begin{aligned} & -c_n \int_{X_s} \sqrt{-1}\left\{\nabla'\nabla''\widetilde{u},\widetilde{u}\right\} - c_n(-1)^{n+1}\int_{X_s}\sqrt{-1}\left\{\nabla''\widetilde{u},\nabla''\widetilde{u}\right\} \\ = \; & -c_n\int_{X_s}\nabla'\{\nabla''\widetilde{u},\widetilde{u}\}. \end{aligned}$$

By formula (3.17), $\{\nabla''\widetilde{u},\widetilde{u}\}$ is an $(n,n+1)$ form on the total space $\mathcal{X}$, and contains factors dt^i. To make a volume form on the fiber X_s, we obtain

$$-c_n\int_{X_s}\nabla'\{\nabla''\widetilde{u},\widetilde{u}\} = -c_n\int_{X_s}\nabla'_X\{\nabla''\widetilde{u},\widetilde{u}\} = 0,$$

by Stokes' theorem. Hence, (3.23) follows. q.e.d.

By taking conjugate of the real forms, the curvature formula (3.21) can be written as

$$\begin{aligned} & (\sqrt{-1}\Theta^E u,u) \\ = \; & -c_n\int_{X_s}\sqrt{-1}\left\{\widetilde{u},\nabla''\nabla'\widetilde{u}\right\} \\ = \; & c_n\int_{X_s}\sqrt{-1}\left\{\nabla''\nabla'\widetilde{u},\widetilde{u}\right\} \\ = \; & c_n\int_{X_s}\sqrt{-1}\left\{\Theta^{\mathcal{E}}\widetilde{u},\widetilde{u}\right\} - c_n\int_{X_s}\sqrt{-1}\left\{\nabla'\nabla''\widetilde{u},\widetilde{u}\right\} \qquad (3.24) \\ \overset{(3.23)}{=} \; & c_n\int_{X_s}\sqrt{-1}\left\{\Theta^{\mathcal{E}}\widetilde{u},\widetilde{u}\right\} + c_n(-1)^{n+1}\int_{X_s}\sqrt{-1}\left\{\nabla''\widetilde{u},\nabla''\widetilde{u}\right\} \\ = \; & c_n\int_{X_s}\sqrt{-1}\left\{\Theta^{\mathcal{E}}\widetilde{u},\widetilde{u}\right\} \\ & -c_n\int_{X_s}\left\{\eta_i+\nabla''_X v_i,\eta_j+\nabla''_X v_j\right\}\cdot(\sqrt{-1}dt^i\wedge d\bar{t}^j) \end{aligned}$$

where the last identity follows from formula (3.17). Note that, since $\nabla''_X v_i$ and η_i are primitive $(n-1,1)$ forms on X_s, by Riemann-Hodge bilinear relation (Lemma 2.2),

$$-c_n\int_{X_s}\left\{\eta_i+\nabla''_X v_i,\eta_j+\nabla''_X v_j\right\} = (\eta_i+\nabla''_X v_i,\eta_j+\nabla''_X v_j). \qquad (3.25)$$

In summary, we obtain,

Theorem 3.10. *The curvature Θ^E of $p_*(K_{\mathcal{X}/S}\otimes\mathcal{E})$ has the following "positive form":*

$$\begin{aligned} (\sqrt{-1}\Theta^E u,u) \; &= \; c_n\int_{X_s}\sqrt{-1}\left\{\Theta^{\mathcal{E}}\widetilde{u},\widetilde{u}\right\} \\ &+ \; \left(\eta_i+\nabla''_X v_i,\eta_j+\nabla''_X v_j\right)\cdot(\sqrt{-1}dt^i\wedge d\bar{t}^j). \qquad (3.26) \end{aligned}$$

Let $c_{i\bar{j}}$ be the $\mathcal{E}$-valued $(n,0)$-form coefficient of $dt^i \wedge d\bar{t}^j$ in the local expression of

$$\sqrt{-1}\Theta^{\mathcal{E}}(\widetilde{u})$$

and $d_{i\bar{j}}$ be the $\mathcal{E}$-valued $(n,0)$-form coefficient of $dt^i \wedge d\bar{t}^j$ in the local expression of

$$\sqrt{-1}\nabla''\nabla'\widetilde{u}.$$

Theorem 3.11. *The curvature Θ^E of $p_*(K_{\mathcal{X}/S}\otimes\mathcal{E})$ has the following "compact form":*

(3.27)

$$(\sqrt{-1}\Theta^E u, u) = c_n \int_{X_s} \sqrt{-1}\{\nabla''\nabla'\widetilde{u}, u\} = c_n \int_{X_s} \{d_{i\bar{j}}, u\}\cdot(\sqrt{-1}dt^i\wedge d\bar{t}^j).$$

Proof. By formula (3.24),

$$(\sqrt{-1}\Theta^E u, u) = -c_n \int_{X_s} \sqrt{-1}\{\widetilde{u}, \nabla''\nabla'\widetilde{u}\} = c_n \int_{X_s} \sqrt{-1}\{\nabla''\nabla'\widetilde{u}, \widetilde{u}\}.$$

We obtain from formula (3.20) that

$$\int_{X_s} \{\nabla''\nabla'\widetilde{u}, dt^i \wedge v_i\} = 0$$

for degree reasons. Therefore

$$(\sqrt{-1}\Theta^E u, u) = c_n \int_{X_s} \sqrt{-1}\{\nabla''\nabla'\widetilde{u}, u\}.$$

By degree reasons again, we obtain the last identity in (3.27). q.e.d.

Remark 3.12. If $\mathcal{X} \to S$ is infinitesimally trivial, we can choose a representative u such that $\nabla'' u = 0$ (i.e. $\eta_i = 0$, see Remark 2.6) and so

$$\begin{aligned}
\int_{X_s} \{\nabla'\nabla''\widetilde{u}, u\} &= -\int_{X_s} \{\nabla'(dt^i \wedge v_i), u\} \\
&= -\int_{X_s} \{\nabla'_X(dt^i \wedge v_i), u\} \\
&= (-1)^n \int_X \{dt^i \wedge v_i, \nabla''_X u\} = 0.
\end{aligned}$$

Therefore

(3.28)

$$(\sqrt{-1}\Theta^E u, u) = c_n \int_{X_s} \sqrt{-1}\{\Theta^{\mathcal{E}}\widetilde{u}, u\} = c_n \int_{X_s} \{c_{i\bar{j}}, u\} \cdot (\sqrt{-1}dt^i \wedge d\bar{t}^j).$$

4. Curvature positivity and negativity for direct images of vector bundles

As applications of curvature formulas derived in Section 3, at first, we obtain

Theorem 4.1. *Let $\mathcal{X} \to S$ be an infinitesimally trivial proper holomorphic fibration. If there exists a Hermitian metric on $\mathcal{E}$ which is Nakano-negative along the base, then $p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ is Nakano-negative.*

Proof. It follows from Theorem 3.3. Here, the curvature formula (3.2) is reduced to

(4.1)
$$(\sqrt{-1}\Theta^E u, u) = c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}} u, u\right\} - (\Delta'_X v_i, v_j) \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j).$$

We set $u = u^{i\alpha} \frac{\partial}{\partial t^i} \otimes e_\alpha$. Naturally, e_α can be viewed as a local holomorphic section of $H^{n,0}(X_s, \mathcal{E}_s)$. We set

$$\Theta^{\mathcal{E}}_{i\bar{j}\alpha\bar{\beta}} = \Theta^{\mathcal{E}}\left(\frac{\partial}{\partial t^i} \otimes e_\alpha, \frac{\partial}{\partial t^j} \otimes e_\beta\right),$$

then by (4.1),

$$\Theta^E_{i\bar{j}\alpha\bar{\beta}} u^{i\alpha}\overline{u}^{j\beta} = c_n \int_{X_s} \Theta^{\mathcal{E}}_{i\bar{j}\alpha\bar{\beta}}\{u^{i\alpha}, u^{j\beta}\} - (\Delta'_X v, v) \tag{4.2}$$

where $v = -\sum_i \nabla'^*_X \mathbb{G}' \pi_\perp \left(\nu_i^{i\alpha} \otimes e_\alpha\right)$. If $\mathcal{E}$ admits a Hermitian metric which is Nakano-negative along the base, then the first term in the formula (4.2) is negative. q.e.d.

Corollary 4.2 ([**LSYang13**])**.** *If (E, h) is a Griffiths-positive vector bundle, then $E \otimes \det E$ is both Nakano positive and dual Nakano-positive.*

Proof. The Nakano-positivity is well-known ([**Demailly**], [**Bern09**]). Now we prove the dual Nakano-positivity. Let $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ be the tautological line bundle of $\mathbb{P}(E)$. Note that $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is ample, but L is not. The metric on (E, h) induces a metric on L which is negative along the base ([**Demailly**, Chapter V, formula 15.15], [**LSYang13**, (2.12)]). On the other hand, it is easy to see

$$E^* \otimes \det E^* = p_*(K_{\mathbb{P}(E)/S} \otimes L^{r+1})$$

where $p : \mathbb{P}(E) \to S$ is the projection. Hence, by Theorem 4.1, $E^* \otimes \det E^*$ is Nakano-negative, or equivalently, $E \otimes \det E$ is dual Nakano-positive. q.e.d.

Similarly, for the Nakano-positivity, it follows from Theorem 3.10 and the proof is similar to that of Theorem 4.1.

Corollary 4.3 ([**MouTak08**])**.** *$p_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ is Nakano-positive if $\mathcal{E}$ is Nakano-positive.*

Corollary 4.4 ([**Bern09**]). *Let $\mathcal{L}$ be a line bundle over $\mathcal{X}$. Then $p_*(K_{\mathcal{X}/S}\otimes\mathcal{L})$ is Nakano-positive if $\mathcal{L}$ is ample.*

Remark 4.5. Note that, in Corollary 4.3 and Corollary 4.4, the family $\mathcal{X}\to S$ are **not** necessarily infinitesimally trivial since the term related to the Kodaira-Spencer class

$$\big(\eta_i+\nabla''_X v_i,\eta_j+\nabla''_X v_j\big)\cdot(\sqrt{-1}dt^i\wedge d\bar{t}^j)$$

is nonnegative.

Let X be a compact Fano manifold and $h(t)=e^{-\varphi(t)}$ be a family of positive metrics on $L=-K_X$. Let $(z^1,\cdots,z^n)$ be the local holomorphic coordinates on X. We set the *local* volume form

$$dV_{\mathbb{C}}=(\sqrt{-1})^n(dz^1\wedge d\overline{z}^1+\cdots+dz^n\wedge d\overline{z}^n)^n.$$

It is easy to see that

$$e^{-\varphi}dV_{\mathbb{C}}$$

is a family of globally defined volume forms of X. Berndtsson in [**Bern11a**] considers the logarithm volume

$$\mathcal{F}(t)=-\log\left(\int_X e^{-\varphi}dV_{\mathbb{C}}\right) \tag{4.3}$$

and deduces that

Theorem 4.6 ([**Bern11a**]). *If $e^{-\varphi(t)}$ is a subgeodesics in the Kähler cone $\mathcal{K}_L$ of the class $c_1(L)$, i.e.*

$$c(\varphi)=\ddot{\varphi}-|\overline{\partial}_X\dot{\varphi}|^2\geq 0,$$

then $\mathcal{F}(t)$ is convex.

In fact, Theorem 4.6 can be obtained easily from Corollary 3.6, following the setting in [**Bern11a**]. To formulate it efficiently, we use complex parameter t in the unit disk $\mathbb{D}\subset\mathbb{C}$. When we consider the direct image bundle $E=p_*(K_X\otimes L)$, it is a trivial line bundle since $L=-K_X$ and $H^0(X,K_X\otimes L)\cong\mathbb{C}$. Since E is trivial, there is a constant section $u=1e_E$ of E, and it is identified as a holomorphic section u of $H^{n,0}(X,L)$,

$$u=dz^1\wedge\cdots\wedge dz^n\otimes e$$

where $e=\frac{\partial}{\partial z^1}\wedge\cdots\wedge\frac{\partial}{\partial z^n}$. Hence

$$\|u\|^2=c_n\int_X\{u,u\}=\int_X e^{-\varphi}dV_{\mathbb{C}}.$$

On the other hand, it is obvious that

$$\|u\|^2\sqrt{-1}\partial\overline{\partial}\mathcal{F}=(\sqrt{-1}\Theta^E u,u). \tag{4.4}$$

Hence, if $c(\varphi)\geq 0$, by Corollary 3.6, $\sqrt{-1}\partial\overline{\partial}\mathcal{F}$ is Hermitian semi-positive. In real parameters, it says that $\mathcal{F}$ is convex.

As a partial converse to Berndtsson's result, we have

Proposition 4.7. *Let $e^{-\varphi(t)}$ be a curve in the Kähler cone $\mathcal{K}_L$. If $\varphi(t)$ is concave in t, then so is $\mathcal{F}(t)$.*

Proof. It follows from Theorem 3.3. In fact, $\ddot{\varphi} \leq 0$ implies the first term on the right hand side of (3.2) is negative. Note that, in this case, the family is a trivial family and so the third term on the right hand side of (3.2) is zero. Therefore, $(\sqrt{-1}\Theta^E u, u) \leq 0$. By formula (4.4), we see $\mathcal{F}$ is superharmonic and in the real case, it is concave. q.e.d.

We can also see how Theorem 4.6 and Proposition 4.7 work by the following simple example. At first, we fix a positive metric $e^{-\varphi(0)}$ in $c_1(L)$ and set

$$\varphi(t) = f(t) + \varphi(0)$$

where t is a real parameter. It is obvious that

$$c(\varphi) = \ddot{\varphi} = \ddot{f}, \qquad \mathcal{F}(t) = f(t) + c.$$

Hence $\mathcal{F}$ is concave if φ is concave and vice visa.

For the general case, it is not hard to see that both Theorem 4.6 and Proposition 4.7 amount to the basic $\overline{\partial}$-estimate

$$\|\dot{\varphi}\| \leq \|\overline{\partial}_X \dot{\varphi}\| \tag{4.5}$$

if the Fano manifold X is polarized by its anti-canonical class.

5. Direct images of projectively flat vector bundles

In this section we consider an infinitesimal trivial family $\mathcal{X} \to S$ and assume that the vector bundle $(\mathcal{E}, h^{\mathcal{E}}) \to \mathcal{X}$ is Nakano semi-positive. In this case, we can choose η_i to be zero in all formulas derived in Section 3.

Let's recall that $c_{i\overline{j}}$ is the $\mathcal{E}$-valued $(n,0)$-form coefficient of $dt^i \wedge d\overline{t}^j$ in the expression

$$\Theta^{\mathcal{E}}(u - dt^i \wedge v_i).$$

There are four (linearly independent) terms in the expression of $\Theta^{\mathcal{E}}(u - dt^i \wedge v_i)$. However, if $\Theta^{\mathcal{E}}$ is Nakano semi-positive, then $c_{i\overline{j}}$ dominates the degeneracy of $\Theta^{\mathcal{E}}(u - dt^i \wedge v_i)$, i.e. $c_{i\overline{j}} = 0$ implies $\Theta^{\mathcal{E}}(u - dt^i \wedge v_i) = 0$. This is the content of the next theorem.

Theorem 5.1. *Let $(\mathcal{E}, h^{\mathcal{E}}) \to \mathcal{X}$ be Nakano semi-positive. Then*

$$(\sqrt{-1}\Theta^E u, u) = 0$$

if and only if $c_{ij} = 0$.

Proof. Note that if $(\mathcal{E}, h^{\mathcal{E}}) \to \mathcal{X}$ is Nakano semi-positive, then by formula (3.26),

$$(\sqrt{-1}\Theta^E u, u) = c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}}\widetilde{u}, \widetilde{u}\right\} + \left(\nabla''_X v_i, \nabla''_X v_j\right) \cdot (\sqrt{-1} dt^i \wedge d\overline{t}^j)$$

is a Hermitian semi-positive $(1,1)$-form. If $(\sqrt{-1}\Theta^E u, u) = 0$, we get

$$c_n \int_{X_s} \sqrt{-1}\left\{\Theta^{\mathcal{E}}\widetilde{u}, \widetilde{u}\right\} = 0$$

and so $\Theta^{\mathcal{E}}\widetilde{u} = 0$. In particular, $c_{i\bar{j}} = 0$ since $\Theta^{\mathcal{E}}\widetilde{u} = \Theta^{\mathcal{E}}(u - dt^i \wedge v_i)$.

On the other hand, by Theorem 3.11, if $c_{ij} = 0$,

$$(\sqrt{-1}\Theta^E u, u) = c_n \int_{X_s} \{c_{i\bar{j}}, u\} \cdot (\sqrt{-1} dt^i \wedge d\bar{t}^j) = 0.$$

q.e.d.

Now we continue to analyze the case $(\sqrt{-1}\Theta^E u, u) = 0$. In the following, we use an idea in [**Bern11a**].

Lemma 5.2. *If $H^{n,1}(X_s, \mathcal{E}_s) = 0$, then v_i is holomorphic on $\mathcal{X}$ for any i.*

Proof. We only need to show $\frac{\partial v_i}{\partial \bar{t}^j} = 0$ since $\nabla''_X v_i = 0$ is obvious from curvature formula (3.26) when $(\sqrt{-1}\Theta^E u, u) = 0$.

Next we claim

$$\nabla'_X\left(\frac{\partial v_i}{\partial \bar{t}^j}\right) = c_{i\bar{j}} + \frac{\partial}{\partial \bar{t}^j}\pi\left(\nu_i\right). \tag{5.1}$$

In fact,

$$\begin{aligned}
& -\nabla'\nabla''(dt^i \wedge v_i) \\
= \; & -\Theta^{\mathcal{E}}(dt^i \wedge v_i) + \nabla''\nabla'(dt^i \wedge v_i) \\
= \; & -\Theta^{\mathcal{E}}(dt^i \wedge v_i) - \nabla''(dt^i \wedge \nabla'_X v_i) - \nabla''(dt^i \wedge dt^k \wedge \nabla'_k v_i) \\
= \; & -\Theta^{\mathcal{E}}(dt^i \wedge v_i) + \nabla''\left(dt^i \wedge (\nu_i - \pi(\nu_i))\right) - \nabla''(dt^i \wedge dt^k \wedge \nabla'_k v_i) \\
= \; & -\Theta^{\mathcal{E}}(dt^i \wedge v_i) + \nabla''\nabla' u - \nabla''\left(dt^i \wedge \pi(\nu_i)\right) - \nabla''(dt^i \wedge dt^k \wedge \nabla'_k v_i) \\
= \; & -\Theta^{\mathcal{E}}(dt^i \wedge v_i) + \Theta^{\mathcal{E}} u - \nabla'\nabla'' u - \nabla''\left(dt^i \wedge \pi(\nu_i)\right) \\
& -\nabla''(dt^i \wedge dt^k \wedge \nabla'_k v_i) \\
= \; & \Theta^{\mathcal{E}}(u - dt^i \wedge v_i) - \nabla'(dt^i \wedge \eta_i) - \nabla''\left(dt^i \wedge \pi(\nu_i)\right) \\
& -\nabla''(dt^i \wedge dt^k \wedge \nabla'_k v_i).
\end{aligned}$$

By comparing the coefficients of $dt^i \wedge d\bar{t}^j$ on both sides, we get (5.1). If the curvature is zero, we also have $c_{i\bar{j}} = 0$. According to different types in Hodge decomposition, i.e. $\nabla'_X\left(\frac{\partial v_i}{\partial \bar{t}^j}\right) \in Im(\nabla'_X)$ and the holomorphic $(n,0)$ form $\frac{\partial}{\partial \bar{t}^j}\pi\left(\nu_i\right) \in Ker(\Delta''_X) = Ker(\Delta'_X)$ (when restricted on $(n,0)$ forms), we conclude from (5.1) that

$$\nabla'_X\left(\frac{\partial v_i}{\partial \bar{t}^j}\right) = \frac{\partial}{\partial \bar{t}^j}\pi\left(\nu_i\right) = 0.$$

Therefore, by Hodge relation $[\nabla''^*_X, \omega] = -\sqrt{-1}\nabla'_X$, we get $\nabla''^*_X(\omega \wedge \frac{\partial v_i}{\partial \overline{t}^j}) = 0$ and $\nabla''_X(\omega \wedge \frac{\partial v_i}{\partial \overline{t}^j}) = 0$. The cohomology assumption ensures the $(n,1)$ form $\omega \wedge \frac{\partial v_i}{\partial \overline{t}^j} = 0$ and so $\frac{\partial v_i}{\partial \overline{t}^j} = 0$. q.e.d.

In the following, we assume that $(\mathcal{E}_s, h^{\mathcal{E}_s})$ is projectively flat. Hence, the curvature tensor can be written as (c.f. [**Koba87**, p.7])

$$\sqrt{-1}\Theta^{\mathcal{E}_s} = \frac{1}{r} Ric(\det \mathcal{E}_s) \otimes h^{\mathcal{E}_s} \tag{5.2}$$

where r is the rank of $\mathcal{E}_s$. If $\det \mathcal{E}_s$ is positive, we set

$$\omega_g = \frac{1}{r} Ric(\mathcal{E}_s) = -\frac{\sqrt{-1}}{r} \partial_X \overline{\partial}_X \log \det(h^{\mathcal{E}_s}) \tag{5.3}$$

as the background Kähler metric on each fiber. Therefore,

$$\sqrt{-1}\Theta^{\mathcal{E}_s} = \omega_g \otimes h^{\mathcal{E}_s}. \tag{5.4}$$

Recall that $[\alpha_i] \in H^{0,1}(X_s, End(\mathcal{E}_s))$ is the Kodaira-Spencer class of the deformation $\mathcal{E} \to \mathcal{X} \to S$ in the direction of $\frac{\partial}{\partial t^i}$, i.e.

$$\alpha_i = \Theta^{\mathcal{E}} \left(\frac{\partial}{\partial t^i} \right) \big|_{X_s} \in \Omega^{0,1}(X_s, End(\mathcal{E}_s)). \tag{5.5}$$

Let W_i be the dual vector of α_i, i.e. W_i is an $End(\mathcal{E}_s)$-valued $(1,0)$-vector field on the fiber X_s. Then by formulas (5.4) and (5.5), we have

$$\begin{aligned} (i_{W_i}\omega) \wedge u &= \sqrt{-1}\alpha_i \wedge u = \sqrt{-1}\Theta^{\mathcal{E}} \left(\frac{\partial}{\partial t_i}, u \right) \\ &= \sqrt{-1}\nabla''_X \nu_i \end{aligned} \tag{5.6}$$

since $\nabla''_X u = 0$.

Proposition 5.3. *We have the relation*

$$i_{W_i} u = -v_i. \tag{5.7}$$

Moreover, W_i is an $End(\mathcal{E}_s)$-valued holomorphic vector field on the fiber X_s.

Proof. By formula (5.6), we obtain

$$\begin{aligned} (i_{W_i}\omega_g) \wedge u &= \sqrt{-1}\nabla''_X \nu_i = -\sqrt{-1}\nabla''_X \nabla'_X v_i \\ &= -\sqrt{-1}\Theta^{\mathcal{E}_s}(v_i) \end{aligned}$$

since $\nabla''_X v_i = 0$. On the other hand, $(i_{W_i}\omega_g) \wedge u = (i_{W_i} u) \wedge \omega_g$. Hence we obtain (5.7) by using (5.4) again. Since v_i and u are all holomorphic on each fiber, we know W_i is also holomorphic. q.e.d.

We can extend the vector field $\frac{\partial}{\partial t^i}$ to an $End(\mathcal{E}_s)$-valued vector field. We still denote it by $\frac{\partial}{\partial t^i}$. Then

$$V_i = \frac{\partial}{\partial t^i} - W_i \tag{5.8}$$

is a (local) $End(\mathcal{E})$-valued holomorphic vector field over the total space $\mathcal{X}$. Let $\mathscr{L}$ be the type $(1,0)$, $End(\mathcal{E})$-valued Lie derivative, then we have

$$\mathscr{L}_{V_i}\omega_g = 0. \tag{5.9}$$

In fact, by relation (5.5), we have

$$\mathscr{L}_{W_i}\omega_g = \nabla'_X(-\sqrt{-1}\alpha) = -\sqrt{-1}\nabla'_X\left(\Theta^{\mathcal{E}}\left(\frac{\partial}{\partial t^i}\right)|_{X_s}\right) = -\nabla'_{\frac{\partial}{\partial t^i}}\Theta^{\mathcal{E}_s}.$$

Hence, by formula (5.4), we get (5.9). That means, if the curvature Θ^E degenerates at some point $s \in S$, then the family $\mathcal{E} \to \mathcal{X} \to S$ moves by an infinitesimal automorphism of $\mathcal{E}$.

We summarize the above into a global version. Let $\mathcal{X} = X \times \mathbb{D}$, where $\mathbb{D}$ is a unit disk. Let $\mathbb{E}_0 \to X$ be a holomorphic vector bundle. If $(\mathbb{E}_0, h(t))_{t\in\mathbb{D}} \to X$ is a smooth family of projectively flat vector bundles. We assume $Ric(\det E, h(t)) > 0$ for all t and set $\omega_t = -\frac{\sqrt{-1}}{r}\partial_X\overline{\partial}_X \log\det(h(t))$ to be a smooth family of Kähler metrics on X. We also denote by $\mathcal{E}$, the pullback family $p_2^*(\mathbb{E}_0)$ over $p_2 : \mathcal{X} \to X$.

Theorem 5.4. *If the curvature Θ^E of $E = p_*(K_{\mathcal{X}/\mathbb{D}} \otimes \mathcal{E})$ vanishes in a small neighborhood of $0 \in \mathbb{D}$, then there exists a holomorphic vector field V on X with flows $\Phi_t \in Aut_H(X, \mathbb{E}_0)$ such that*

$$\Phi_t^*(\omega_t) = \omega_0$$

for small t.

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