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PROPERLY EMBEDDED, AREA-MINIMIZING SURFACES IN HYPERBOLIC 3-SPACE

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Abstract

We prove a bridge principle at infinity for area-minimizing surfaces in the hyperbolic $\mathbb{H}^3$, and we use it to prove that any open, connected, orientable surface can be properly embedded in $\mathbb{H}^3$ as an area-minimizing surface. Moreover, the embedding can be constructed in such a way that the limit sets of different ends are disjoint.

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1. Introduction

The construction of new examples of complete minimal surfaces in hyperbolic space has had a very powerful tool: the solvability of the asymptotic Plateau problem. The asymptotic Plateau problem in hyperbolic space basically asks the existence of an area-minimizing submanifold in $\mathbb{H}^{n+1}$ which is asymptotic to a given submanifold $\Gamma^{n-1} \subset \partial\mathbb{H}^{n+1}$, where $\partial\mathbb{H}^{n+1}$ represents the sphere of infinity of $\mathbb{H}^{n+1}$, which we also call *the ideal boundary of hyperbolic space.*

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Using methods from geometric measure theory, Michael Anderson [**1**] solved the asymptotic Plateau problem for absolutely area-minimizing submanifolds in any dimension and codimension.

Anderson did not impose any restriction to the topology of the solutions he gets, so we cannot get any idea about their topological properties. In this way, it becomes interesting (as in the classical Plateau problem) to find the area-minimizing solution but fixing *a priori* the topological type. In [**2**], Anderson focused on the asymptotic Plateau problem with the type of a disk and provided an existence result in dimension 3.

Moreover, in [**2**], Anderson built a special Jordan curve in $\partial\mathbb{H}^3$, such that the surface obtained as a solution to the asymptotic Dirichlet problem cannot be a plane. In fact, he built examples of genus $g > g_0$ for a particular genus g_0. In the same context, de Oliveira and Soret [**6**] demonstrated the existence of complete and stable minimal surfaces in hyperbolic 3-space for any orientable *finite* topological type. (A surface has finite topological type if it has the topology of a compact surface minus a *finite* number of points.) They also studied the isotopy type of these surfaces in some special cases. The main difference with the result of [**2**] is that Anderson begins with asymptotic data, and gives an area-minimizing surface with that particular data but without any kind of control over the topological type, while Oliveira and Soret start with a surface with boundary and build a stable embedded minimal surface in the hyperbolic space whose asymptotic (or ideal) boundary is determined essentially by the surface. In this setting, we can frame the following conjecture:

Conjecture (A. Ros)**.** Every open, connected, orientable surface can be properly and minimally embedded in $\mathbb{H}^3$.

(Recall that a connected surface is open if it is non-compact and has no boundary.) In this paper, we prove the conjecture. More precisely, we prove:

Theorem A. *Every open, connected, orientable surface can be properly embedded in* $\mathbb{H}^3$ *as an area-minimizing surface. Moreover, the embedding can be constructed in such a way that the limit sets of different ends are disjoint.*

The definition of "area-minimizing" and "uniquely area-minimizing" surfaces can be found in Section 3 (Definition 3.1). The fundamental tool in solving this problem has been the bridge principle at infinity (Section 6) which can be stated in these terms:

Theorem B (**Bridge principle at infinity**)**.** *Let* S *be an open, properly embedded, uniquely area-minimizing surface in* $\mathbb{H}^3$ *whose closure* $\overline{S} \subset \overline{\mathbb{H}^3}$ *is a smooth manifold-with-boundary. Let* Γ *be a smooth arc in* $\partial\mathbb{H}^3$ *meeting* ∂S *orthogonally and satisfying* $\Gamma \cap \partial S = \partial\Gamma$.

Consider a sequence of bridges P_n on $\partial\mathbb{H}^3$ that shrink nicely to Γ. If S is **strictly L^∞ stable** *(see Definition 4.1), then for all large enough n, there exists a strictly L^∞ stable, uniquely area-minimizing surface S_n that is properly embedded in $\mathbb{H}^3$ such that:*

1) *$\overline{S_n}$ is a smooth, embedded manifold-with-boundary in $\overline{\mathbb{H}^3}$.*
2) *$\partial S_n = (\partial S \setminus \partial P_n) \cup (\partial P_n \setminus \partial S)$.*
3) *The sequence $\overline{S_n}$ converges smoothly to $\overline{S}$ on compact subsets of $\overline{\mathbb{H}^3} \setminus \Gamma$.*
4) *The surface $\overline{S_n}$ is homeomorphic to $\overline{S} \cup P_n$.*

This bridge principle gives us some flexibility in order to construct properly embedded area-minimizing surfaces in $\mathbb{H}^3$ with arbitrary infinite topology and some kind of regularity at infinity.

Theorem C. *If S is a connected, open, orientable surface with infinite topology, then there exists a proper, area-minimizing embedding of S into $\mathbb{H}^3$ such that $\overline{S}$ is a smooth embedded manifold-with-boundary except at a single point of $\partial S \subset \partial\mathbb{H}^3$.*

Finally, we would like to point out that the same methods allow us to construct properly embedded area-minimizing surfaces so that the limit set is the whole ideal boundary $\partial\mathbb{H}^3$.

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2. Preliminaries

Throughout this paper $\mathbb{H}^{n+1}$ will represent the (n+1)-dimensional hyperbolic space. We will use the models:

(1) **Poincaré's ball model:** the open unit ball $\mathbb{B}^{n+1}$ of $\mathbb{R}^{n+1}$ endowed with Poincaré's metric $ds^2 := 4\dfrac{\sum_{i=1}^{n+1} dx_i^2}{(1-\sum_{i=1}^{n+1} x_i^2)^2}$.

(2) **Poincaré's half-space model:** the upper half-space $\{x_{n+1} > 0\} \subset \mathbb{R}^{n+1}$, endowed with the metric $ds^2 := \dfrac{1}{x_{n+1}^2}\displaystyle\sum_{i=1}^{n+1} dx_i^2$.

Let $\overline{\mathbb{H}}^{n+1}$ denote the usual compactification of $\mathbb{H}^{n+1}$. As we mentioned in the introduction, we shall denote the ideal boundary as $\partial\mathbb{H}^{n+1} := \overline{\mathbb{H}}^{n+1} \setminus \mathbb{H}^{n+1}$. Observe that $\partial\mathbb{H}^{n+1}$ is diffeomorphic to the sphere $\mathbb{S}^n$. (In the ball model, it is $\partial\mathbb{B}^{n+1}$ and in the upper half-space model, it is $\{x : x_{n+1} = 0\} \cup \{\infty\}$.)

2.1. Simple exhaustions. One of the main tools in the proofs of the theorems stated in the introduction is the existence of a particular kind of exhaustion for any open surface. In [**3**], Ferrer, Meeks, and the first author proved that every open, connected, orientable surface M has a *simple exhaustion*, i.e., a smooth, compact exhaustion $M_1 \subset M_2 \subset \cdots$ such that:

(1) M_1 is a disk.
(2) For all $n \in \mathbb{N}$, each component of $M_{n+1} \setminus \mathrm{Int}(M_n)$ has one boundary component in ∂M_n and at least one boundary component in ∂M_{n+1}.
(3) If M has infinite topology, then for all $n \in \mathbb{N}$, $M_{n+1} \setminus \mathrm{Int}(M_n)$ contains a unique nonannular component; that component is topologically a pair of pants or an annulus with a handle.
(4) If M has finite topology (with genus g and k ends), property (3) holds for $n \leq g + k$, and when $n > g + k$, all of the components of $M_{n+1} \setminus \mathrm{Int}(M_n)$ are annular.

See Figure 1.

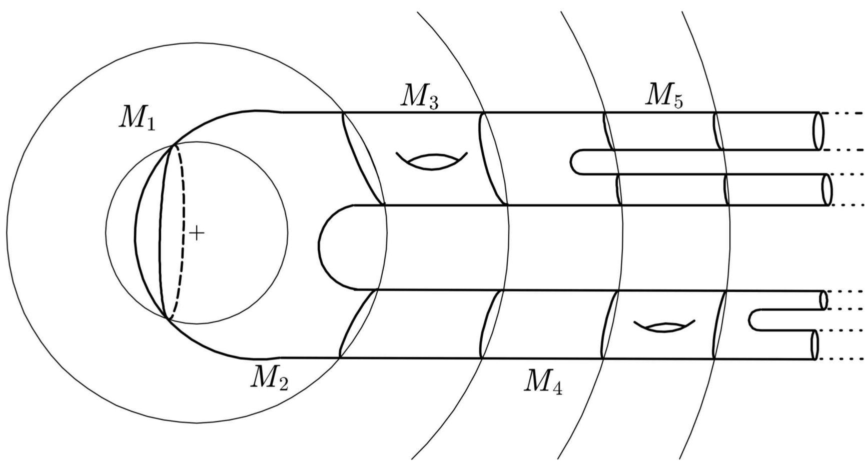

Figure 1. A simple exhaustion of M.

Remark 2.1. For the purposes of this paper, one could replace (3) in the definition of simple exhaustion by the slightly weaker condition: each component of $M_{n+1} \setminus \mathrm{Int}(M_n)$ is an annulus, a pair of pants, or an annulus with a handle. Which definition one uses does not affect any of the proofs.

2.2. Limit sets. We are also interested in the asymptotic behavior of the minimal surfaces we are going to construct. So, we need some background about the limit set of an end.

Definition 2.2. Let $\psi\colon S \to \mathbb{H}^3$ be an immersion of a surface S with possibly non-empty boundary. The **limit set** of S is $L(S) = \bigcap_{\alpha\in I} \overline{\psi(S \setminus C_\alpha)}$, where $\{C_\alpha\}_{\alpha\in I}$ is the collection of compact subdomains of S and the closure $\overline{\psi(S \setminus C_\alpha)}$ is taken in $\overline{\mathbb{H}^3}$. The **limit set** $L(E)$ **of an end** E **of** S is defined to be the intersection of the limit sets of all properly embedded subdomains of S with compact boundary which represent E.

Note that $L(S)$ is a closed set, and that if E is an end of S, then $L(E)$ is a closed subset of $L(S)$. Note also that $\psi : S \to \mathbb{H}^3$ is proper if and only if $L(S) \subset \partial\mathbb{H}^3$. (Recall that a continuous map $f : X \to Y$ between topological spaces is called *proper* provided the inverse image of every compact set is compact.) Thus if S is an open, proper submanifold of $\mathbb{H}^3$, then $L(S)$ is the equal to the set theoretic boundary $\partial S = \overline{S} \setminus S$. More generally, if S is an open, proper submanifold of an open subset U of $\mathbb{H}^3$, then $L(S) = \partial S = \overline{S} \setminus S$, where $\overline{S}$ denotes the closure of S in $\overline{\mathbb{H}^3}$.

Proposition 2.3 (Convex Hull Property)**.** *Let M be an open minimal surface in $\mathbb{H}^3$. Then M is contained in the convex hull of its limit set. In other words, if $\Sigma \subset \mathbb{H}^3$ is a totally geodesic plane and if $L(M)$ lies in the closure N of one component of $\mathbb{H}^3 \setminus \Sigma$, then M also lies in N.*

Recall that, in general, $L(M)$ may include points in $\mathbb{H}^3$ and also points in $\partial\mathbb{H}^3$.

Proof. We can assume in the upper half-space model that

$$N = \{(x, y, z) : z \geq 0 \text{ and } x^2 + y^2 + z^2 \geq 1\}.$$

The level sets of the function $(x, y, z) \mapsto x^2+y^2+z^2$ are minimal surfaces, so by the strong maximum principle, its restriction to $\overline{M}$ cannot attain its minimum at a point of M. q.e.d.

Theorem 2.4 (Strong local uniqueness theorem)**.** *Let M be an open minimal surface in $\mathbb{H}^3$ and Γ be a curve in $\partial\mathbb{H}^3$ such that $M' = M \cup \Gamma$ is a smooth, embedded submanifold (with boundary) of $\overline{\mathbb{H}^3}$. Then each point $p \in \Gamma$ has a neighborhood $U \subset \overline{\mathbb{H}^3}$ with the following property: if $S \subset \mathbb{H}^3$ is an open minimal surface with $L(S) \subset M' \cap U$, then $S \subset M \cap U$.*

Proof. We use the upper half-space model. Let v be a vector normal to $\overline{M}$ at p. Note that v is horizontal. (One easily shows, using totally geodesic barriers, that M' meets $\partial\mathbb{H}^3$ orthogonally. See, for example, [**5**].) We may assume that each line in $\mathbb{R}^3$ parallel to v intersects M' at most once. (Otherwise replace M' by $M' \cap \mathbb{B}(p, R)$ with R sufficiently small.) It follows that M' and its translates by multiples of v foliate an open subset W of $\overline{\mathbb{H}^3}$. Choose $r > 0$ small enough that $\mathbb{B}(p, r) \cap \overline{\mathbb{H}^3}$ is contained in W. Then $U := \mathbb{B}(p, r)$ has the desired property. To see that it does, suppose that S is a minimal surface in $\mathbb{H}^3$ with $L(S) \subset M' \cap U$. By the convex hull property (Proposition 2.3), $S \subset U$. The strong maximum principle then forces S to lie in M. (Consider the maximum value of $|t|$ such that S intersects $M' + tv$.) q.e.d.

3. Area-minimizing surfaces

In this section, we present some fundamental theorems about area-minimizing surfaces in hyperbolic space. Those theorems will be used repeatedly in the rest of the paper.

Definition 3.1. Suppose $S \subset \mathbb{H}^3$ is a (possibly nonorientable) compact surface with unoriented boundary. The surface S is called **area-minimizing** if S has least area among all surfaces (orientable or nonorientable) with the same boundary. For a noncompact surface S, we say that S is area-minimizing provided each compact portion of it is area-minimizing.

Now suppose that S is an open, properly embedded, area-minimizing surface in $\mathbb{H}^3$. We say that S is **uniquely area-minimizing** if it is the only area-minimizing surface with boundary ∂S.

(In the literature, "area-minimizing" as defined here is usually referred to as "area-minimizing mod 2.")

For example, the convex hull property (Proposition 2.3) implies that a totally geodesic plane is uniquely area-minimizing.

Theorem 3.2 (Boundary Regularity Theorem)**.** *Let M be an open, area-minimizing surface in $\mathbb{H}^3$. Suppose that W is an open subset of $\overline{\mathbb{H}^3}$ with the following property:*

$$L(M) \cap W \subset \Gamma$$

where Γ is a smooth, connected, properly embedded curve in $W \cap \partial\mathbb{H}^3$. Then either $L(M) \cap W$ is empty, or $L(M) \cap W = \Gamma$ and $M \cup \Gamma$ is a smooth manifold-with-boundary.

Proof. Hardt-Lin [**5**] prove that in a neighborhood U of each point of Γ, $\overline{M} \cap U$ is a union of some finite number κ of C^1 manifolds-with-boundary, the boundary being either $\Gamma \cap U$ or the empty set, and that those manifolds are disjoint except at the boundary. Their result is stated for integral currents, but their proof also works for chains mod 2 and in that case actually gives more: κ must then be 0 or 1 (because in Lemma 2.1 of their paper, if δ is sufficiently small, then κ must be 0 or 1). A priori the number κ might depend on the point, but since it is locally constant and since Γ is connected, it must in fact be constant on Γ. In case $\kappa = 1$, Tonegawa [**7**] improves the boundary regularity by showing that $M \cup \Gamma$ is C^∞. q.e.d.

Theorem 3.3 (Compactness Theorem)**.** *Let M_i be a sequence of open, area-minimizing surfaces that are properly embedded in an open subset U of $\mathbb{H}^3$. Then (after passing to a subsequence) the M_i converge smoothly with multiplicity 1 on compact subsets of U to such a surface M.*

Now suppose that $U = W \cap \mathbb{H}^3$, where W is an open subset of $\overline{\mathbb{H}^3}$. Suppose also that $L(M_i) \cap W$ is a smooth embedded curve Γ_i in $W \cap \partial\mathbb{H}^3$, and that Γ_i converges smoothly and with multiplicity m to an embedded curve Γ in $W \cap \partial\mathbb{H}^3$.

Then $M_i \cup \Gamma_i$ (which is a smooth manifold-with-boundary by Theorem 3.2) converges (in the Hausdorff topology on the space of relatively closed subsets of W) to $M \cup \Gamma$. Furthermore:

(1) *If m is odd, then $M \cup \Gamma$ is a smooth manifold-with-boundary.*
(2) *If m is even, then $\overline{M}$ is disjoint from Γ.*
(3) *If $m = 1$, then $M_i \cup \Gamma_i$ converges smoothly on compact subsets of W to $M \cup \Gamma$.*

Proof. The statement about smooth convergence on compact subsets of U is very standard. (The areas of the M_i are uniformly locally bounded, since if Ω is a bounded, open region in $\mathbb{H}^3$, then

$$\text{area}(M_i \cap \Omega) \leq \frac{1}{2}\,\text{area}(\partial\Omega). \tag{3.1}$$

Also, the curvatures of the M_i are uniformly bounded on compact subsets by standard curvature estimates for area-minimizing hypersurfaces of dimension < 8. That the multiplicity of M is 1 follows from (3.1).)

The convergence of $M_i \cup \Gamma_i$ to $M \cup \Gamma$ (as a relatively closed subset of W) follows from the convergence of M_i to M in U, the convergence of Γ_i to Γ in W, and the convex hull property. (The convex hull property ensures that points $p_i \in M_i$ cannot converge subsequentially to a point in $W_\infty \setminus \Gamma$, where $W_\infty := W \cap \partial\mathbb{H}^3$.)

Since the remaining assertions are local, we can assume (in the upper-half space model) that W is $\overline{\mathbb{H}^3} \cap \mathbb{B}$ for some open Euclidean ball centered at a point in $\partial\mathbb{H}^3$. We can also assume that Γ is connected, so that it divides $W_\infty := W \cap \partial\mathbb{H}^3$ into two components. Let p and q be a pair of points lying in different components of $W_\infty \setminus \Gamma$, and let C be a smooth curve joining p to q such that $C \setminus \{p, q\}$ lies in U. By perturbing C slightly, we may assume it intersects M transversely. Let ν be the mod 2 number of points of $M \cap C$; that number is independent of C (for C transverse to M). By the smooth convergence, M_i intersects C transversely for large i and the mod 2 number ν_i of intersection points is independent of C. The smooth convergence $M_i \to M$ also implies that $\nu_i = \nu$ for all sufficiently large i. By elementary topology that $\nu_i \cong m \,(\text{mod } 2)$ for i sufficiently large. (If this is not clear, note that C can be homotoped in W to a curve in W_∞ that intersects $\overline{M_i}$ transversely in exactly m points.)

Thus m even implies that $\nu = 0$ and m odd implies that $\nu = 1$. Assertions (1) and (2) now follow immediately from the Boundary Regularity Theorem 3.2. Assertion (3) follows from the boundary regularity estimates of Hardt-Lin and Tonegawa. q.e.d.

We remark that if $m > 1$, then the convergence of $M_i \cup \Gamma_i$ to $M \cup \Gamma$ fails to be smooth along Γ. For example, suppose in the ball model $\mathbb{B} = \mathbb{H}^3$ that Γ_i is the union of the two circles $\partial\mathbb{B} \cap \{z = \pm\epsilon_i\}$ where $\epsilon_i \to 0$. Then Γ_i converges smoothly with multiplicity $m = 2$ to the equator $\Gamma := \partial\mathbb{B} \cap \{z = 0\}$. Let M_i be an area-minimizing surface with boundary Γ_i. (Such a surface exists by a theorem of Anderson—see Theorem 3.5 below.) For small ϵ_i, one can prove that M_i is a minimal annulus that lies within Euclidean distance $O(\epsilon_i)$ from Γ. Thus $M_i \cup \Gamma_i$ converges to $M \cup \Gamma$, where M is the empty set. Note that the convergence of $M_i \cup \Gamma_i$ to Γ is not smooth.

Definition 3.4. We say that a closed set $K \subset \partial\mathbb{H}^3$ has *piecewise smooth boundary* provided

(1) K is the closure of its interior, and
(2) there is a finite set S of points such that $(\partial K) \setminus S$ is the disjoint union of a finite set of smooth curves.

Theorem 3.5 (Basic Existence Theorem)**.** *Let $K \subset \partial\mathbb{H}^3$ be a closed region with piecewise smooth boundary. Then there is an area-minimizing surface M in $\mathbb{H}^3$ such that $\partial M = \partial K$. Furthermore, if M is any area-minimizing surface in $\mathbb{H}^3$ with $\partial M = \partial K$, then*

(1) $\overline{M}$ is a smooth embedded manifold-with-boundary except at the finite set of points where ∂K is not a smooth embedded curve.
(2) There is a unique open subset $E(M, K)$ of $\mathbb{H}^3$ whose boundary in $\overline{\mathbb{H}^3}$ is $K \cup M$.

Proof. Anderson [**1**, Theorem 3] proves existence of a smooth, area-minimizing surface $M \subset H$ with the property that $\partial M = \partial K$ as flat chains mod 2 with respect to the Euclidean metric on the ball. (He states the theorem for integral currents, but exactly the same proof works for chains mod 2.) In particular, this implies that $\overline{M} \setminus M = \partial K$ as sets (i.e., in the notation of 2.2, that $L(M) = \partial K$).

The smoothness of $\overline{M}$ at the regular points of ∂K follows immediately from the Boundary Regularity Theorem 3.2.

If we identify $\mathbb{H}^3$ conformally with a ball B in $\mathbb{R}^3$, then $M \cup K$ becomes (except possibly at finitely many points) a compact, embedded, piecewise-smooth closed manifold of $\mathbb{R}^3$ contained in $\overline{B}$. By elementary topology, there is a unique open subset $E(M, K)$ of B whose boundary is $M \cup K$. q.e.d.

Lemma 3.6. *Let M be an area-minimizing surface. Let M' be a compact region in the interior of M such that M' has piecewise smooth boundary. Then M' is the unique area-minimizing surface with its boundary.*

Proof. Standard. q.e.d.

Theorem 3.7. *Let K_1 and K_2 be disjoint, closed regions in $\partial\mathbb{H}^3$ with piecewise smooth boundaries. Let M_1 and M_2 be least area surfaces with boundaries ∂K_1 and ∂K_2, and let U_i be the region enclosed by $M_i \cup K_i$. Then $\overline{U_1}$ and $\overline{U_2}$ are disjoint.*

Proof. Let $Z = \overline{U_1} \cap \overline{U_2}$. Note that Z is a compact subset of $\mathbb{H}^3$. Suppose it is nonempty. Then $U_1 \cap U_2$ is nonempty by the maximum principle (applied to M_1 and M_2). By Lemma 3.6, $U_1 \cap M_2$ is the unique least area surface with its boundary. Likewise, $U_2 \cap M_1$ is the least area surface with its boundary. But $U_1 \cap M_2$ and $U_2 \cap M_1$ have the same boundary, a contradiction. q.e.d.

Corollary 3.8. *Suppose for $i = 1, 2$ that K_i is a closed region in $\partial\mathbb{H}^3$ and that M_i is a least area surface in $\mathbb{H}^3$ with $\partial M_i = \partial K_i$. Let U_i be the region enclosed by $M_i \cup K_i$. If K_1 is contained in the interior of K_2, then $U_1 \cup M_1$ is contained in U_2.*

(This corollary is not really a corollary—but it is proved in exactly the same way as the theorem. Actually, we use the corollary but not the theorem.)

Theorem 3.9. *Let K be a closed region in $\partial\mathbb{H}^3$ with piecewise smooth boundary. Let $\mathcal{F}$ be the collection of all least area surfaces in $\mathbb{H}^3$ with boundary ∂K. Then $\mathcal{F}$ contains surfaces M_{in} and M_{out} with the following property. If $M \in \mathcal{F}$, then*

$$E(M_{\text{in}}, K) \subset E(M, K) \subset E(M_{\text{out}}, K).$$

Recall the $E(M, K)$ is the region enclosed by M and K. (We think of M_{in} and M_{out} as the innermost and outermost surfaces in the family $\mathcal{F}$.)

Proof. Let $K_1 \subset K_2 \subset \dots$ be a sequence of closed subsets of the interior of K such that each K_i has smooth boundary, such that $\cup K_i$ is the interior of K, such that $\partial K_i \to \partial K$, and such that convergence ∂K_i to ∂K is smooth except at the points where ∂K is not smooth.

Let M_i be a least area surface with boundary ∂K_i, and let M_{in} be a subsequential limit of the M_i. Then $M_{\text{in}} \in \mathcal{F}$.

Furthermore, if $M \in \mathcal{F}$, then

$$E(M_i, K_i) \subset E(M, K)$$

for all i (by the lemma), and thus $E(M_{\text{in}}, K) \subset E(M, K)$.

The assertions about M_{out} are proved in a very analogous manner.

q.e.d.

Remark 3.10. Note that M_{in} is unique, as is M_{out}. Hence if g is an isometry of $\mathbb{H}^3$ such that $g(K) = K$, then $g(M_{\text{in}}) = M_{\text{in}}$ and $g(M_{\text{out}}) = M_{\text{out}}$.

Of course $M_{\text{in}} = M_{\text{out}}$ if and only if there is only one least area surface with boundary K.

4. Strict L^∞ stability

In this section, we define strict L^∞ stability and we prove some of its basic properties. Let Ω be a Riemannian manifold that is connected but not compact.

Definition 4.1 (strict L^∞ stability)**.** Let J be a self-adjoint 2nd-order linear elliptic operator on a surface Ω. Let us say Ω is **strictly L^∞ stable** (with respect to J) if the first eigenvalue of any compact subdomain is strictly positive and if there are no nonzero bounded Jacobi fields (i.e. solutions of $Ju = 0$) on Ω.

Throughout this paper, we will use the concept of strict L^∞ stability only for minimal surfaces, and the operator J will always be the Jacobi operator. (However, the following three results hold for general manifolds Ω and operators J.)

Lemma 4.2. *Let w be a positive solution of $J w = 0$ on Ω. Then the first eigenvalue of J on every compact subdomain of Ω is strictly positive.*

The proof is standard. See, for example, Theorem 1 of [**4**].

Lemma 4.3. *Let u and w be Jacobi fields on a connected minimal hypersurface M. Suppose that u/w has a positive local maximum λ at a point p where u and w are both positive. Then $u = \lambda w$.*

Proof. By hypothesis, $u - \lambda w$ has a local maximum value 0. Thus by the strong maximum principle, $u - \lambda w$ vanishes in a neighborhood of p. By the unique continuation property for solutions of second order elliptic equations, $u - \lambda w \equiv 0$. q.e.d.

Theorem 4.4. *Suppose w is a positive solution of $Jw = 0$ such that $\lim_{p\to\partial\Omega} w(p) = \infty$. Then Ω is strictly L^∞ stable.*

Proof. We have to show that each compact subdomain is stable and that there are no nonzero bounded Jacobi fields on Ω. By Lemma 4.2, each compact subdomain is stable. Thus we need only show that there are no nonzero, bounded Jacobi fields.

Suppose $u : \Omega \to \mathbb{R}$ is a nonzero, bounded Jacobi field on Ω. We may suppose that $u > 0$ at some points. Since u/w is positive at some points and tends to 0 on $\partial\Omega$, it has a local maximum $\lambda > 0$ at some point Ω. By Lemma 4.3, $u \equiv kw$, which is impossible since u is bounded and w is unbounded. q.e.d.

Corollary 4.5. *A totally geodesic plane in* $\mathbb{H}^3$ *is strictly* L^∞ *stable.*

Proof. Without loss of generality we can assume that the plane is a hemisphere centered at the origin in the upper half-space model of $\mathbb{H}^3$. Consider the Jacobi field w that comes from dilations about 0. q.e.d.

Theorem 4.6. *Let* M *be an area-minimizing surface in* $\mathbb{H}^3$ *with* $\partial M \subset \partial\mathbb{H}^3$. *Let* p *be a regular point of* ∂M, *so that (in the upper half-space model)* $M \cup \partial M$ *is a regular manifold-with-boundary near* p.

Let u *be a bounded, nonnegative Jacobi field on* M. *Then* $\lim_{q\to p} u(q) = 0$.

Proof. Without loss of generality, $p = 0$ in the upper half-space model of $\mathbb{H}^3$. Let $p_n \in M$ be points such that $p_n \to 0$ and such that

$$u(p_n) \to \limsup_{q\to 0} u(q).$$

Suppose the supremum limit is nonzero. Then we may assume it is 1. Now make a Euclidean translation and dilation of $\mathbb{H}^3$ that moves M to M_n and that moves p_n to $(0, 0, 1)$. Let u_n be the Jacobi field on M_n corresponding to u on M. After passing to a subsequence, the M_n converge to a totally geodesic plane M^* and the u_n converge to a bounded Jacobi field u^* on M^* that attains its maximum value (1) at the point $(0, 0, 1)$. But that contradicts the strict L^∞ stability of a totally geodesic plane. q.e.d.

5. Minimal strips and skillets

In this section we define and analyze minimal strips and minimal skillets. They will be important for us because they arise as blowups in the proof of the Bridge Theorem 6.2.

Theorem 5.1. *In the upper half-space model of* $\mathbb{H}^3$, *let* K *be the strip*

$$[-1, 1] \times \mathbb{R} \times \{0\} = \{(x, y, z) : |x| \le 1, z = 0\}$$

together with the point at infinity.

Then there is a unique area-minimizing surface $M \subset \mathbb{H}^3$ *with boundary* ∂K, *and* M *has the form*

$$\{(x, y, z) : z = u(x), |x| < 1\}$$

where $u : (-1, 1) \to \mathbb{R}$ *is a smooth function such that*

$$u'' < 0,$$
$$u(x) \equiv u(-x),$$
$$\lim_{x\to\pm1} u(x) = 0.$$

Furthermore, the surface M *is strictly* L^∞ *stable.*

Definition 5.2. The surface M in Theorem 5.1 will be called the *standard minimal strip*. A surface related to M by an isometry of $\mathbb{H}^3$ will be called a *minimal strip*.

Proof of Theorem 5.1. Note that each of the planes $x = 1$ and $x = -1$ is uniquely area minimizing. However:

Claim. *For $a > 0$, let P_a be the pair of planes $x = a$ and $x = -a$. Then P_a is not area-minimizing.*

To prove the claim, note that P_a and $P_{\lambda a}$ are related by the hyperbolic isometry

$$(x, y, z) \mapsto (\lambda x, \lambda y, \lambda z).$$

Thus it suffices to prove the claim for one value of a. Let C be a solid Euclidean cylinder in $\{(x, y, z) : z > 0\}$ that is perpendicular to the planes $x = \pm a$. Note that the hyperbolic area of the two disks $P_a \cap C$ is independent of a, but that the hyperbolic area of the annular portion of C between the two planes $y = \pm a$ tends to 0 as $a \to 0$. Thus for small a, the pair $P_a \cap C$ is not area-minimizing, which implies that P_a is not area-minimizing, proving the claim.

Now suppose M is an area-minimizing surface with boundary ∂P_a. If M were not connected, it would be equal to P_a since the planes $x = a$ and $x = -a$ are each uniquely area minimizing, contradicting the claim. Thus M must be connected.

Let M_{in} and M_{out} be the innermost and outermost least area surfaces with boundary ∂K, as in Theorem 3.9. As we have just seen, M_{in} and M_{out} are connected.

Then (see Remark 3.10), M_{in} and M_{out} are both invariant under translations $(x, y, z) \mapsto (x, y + c, z)$. It follows that

$$M_{\text{in}} = \Pi^{-1} C_{\text{in}}$$

and

$$M_{\text{out}} = \Pi^{-1} C_{\text{out}},$$

where $\Pi : (x, y, z) \mapsto (x, z)$ and where C_{in} and C_{out} are smooth curves in $\{(x, z) : z > 0\}$ joining $(-1, 0)$ to $(1, 0)$.

Now if $M_{\text{in}} \neq M_{\text{out}}$, there is some $\lambda > 0$ such that λC_{in} intersects C_{out}. Thus there is a largest λ (since C_{in} and C_{out} have the same end points and have compact closures). But then λM_{in} and M_{out} violate the maximum principle.

Thus there is a unique least area surface $M = M_{\text{in}} = M_{\text{out}}$ with boundary ∂K.

Now where the tangent to the curve $C = C_{\text{in}} = C_{\text{out}}$ is not vertical, it is locally the graph of a function $z = u(x)$ that satisfies a 2nd order ODE. (Note that $(x, y) \mapsto u(x)$ is a solution of the Euler-Lagrange equation for the hyperbolic area functional. That equation is a second-order PDE, but since u is a function of x alone, the PDE reduces to an ODE.) The ODE is

$$u(x) \cdot u''(x) + 2(1 + (u'(x)^2) = 0,$$

from which we see that $u'' < 0$ and thus that C has the form

$$C = \{(0, y, u(y)) \ : \ |y| < 1\}, \quad \lim_{y \to \pm 1} u(y) = 0.$$

By Remark 3.10, M is invariant under $(x, y, z) \mapsto (-x, y, z)$ and hence the function u is even.

So, summarizing all the information that we have, we are able to deduce that $x \cdot u'(x) \leq 0$ for $-1 < x < 1$. Furthermore, we know that

$$\lim_{x \to -1} u'(x) = +\infty, \quad \text{and} \quad \lim_{x \to +1} u'(x) = -\infty.$$

Let w^* be the Jacobi field on M associated to dilations $(x, y, z) \mapsto \lambda(x, y, z)$. Note that $w^*(x, y, z)$ is independent of y:

$$w^*(x, y, z) = w^*(x, 0, z). \tag{5.1}$$

Note also that w^* is strictly positive everywhere, so compact domains in M are strictly stable. A straightforward computation gives

$$w^* = \frac{-xu' + u}{u\sqrt{1 + (u')^2}},$$

so

$$w^* \to \infty \text{ uniformly as } x \to \pm 1. \tag{5.2}$$

Now suppose that M is not L^∞ strictly stable, i.e., that M has a bounded, nonzero Jacobi field v. We may assume that v is strictly positive at some points. Let Λ be the supremum of v/w^*, and let $p_n := (x_n, y_n, z_n) \in M$ be a sequence of points such that

$$v(p_n)/w^*(p_n) \to \Lambda.$$

By (5.2), the $|x_n|$ is bounded away from 1. Thus by passing to a subsequence, we can assume that the points $(x_n, 0, z_n)$ converge to a point $p \in M$ and that the Jacobi fields $(x, y - y_n, z) \mapsto v(x, y, z)$ converge smoothly to a limit Jacobi field $\hat{v}$. Note that $\hat{v}/w^*$ attains its maximum value Λ at p. Thus the Jacobi field $\hat{v} - \Lambda \cdot w^*$ attains its maximum value, namely 0, at p. By the maximum principle, $\hat{v} - \Lambda \cdot w^*$ must be identically 0. But that is impossible since $\hat{v}$ is bounded and Λw^* is unbounded. q.e.d.

Definition 5.3 (Skillet)**.** Suppose $u : \mathbb{R} \to [0, +\infty]$ is a continuous, compactly supported function such that $u(x) = \infty$ if and only if $|x| \leq 1$ and such that $\mathcal{A} = \{(x, y) \in \mathbb{R}^2 \ : \ y \leq u(x)\}$ has a uniformly smooth boundary, with $u''(x) \geq 0$ along the boundary of $\mathcal{A}$ (see Fig. 2). Then the set $\mathcal{A}$ is called a *skillet*.

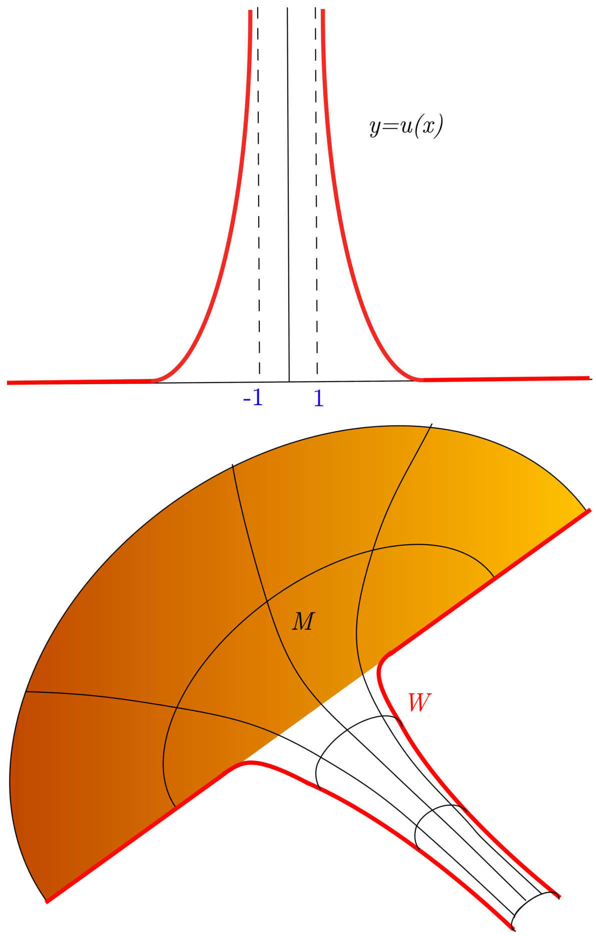

Figure 2. The boundary of a skillet and the minimal skillet M

Theorem 5.4. *Let $\mathcal{A}$ be a skillet in $\mathbb{H}^3$. Then there exists a properly embedded, uniquely area-minimizing surface M satisfying $\partial M = \partial\mathcal{A}$. The surface is a radial graph in the following sense: if $p = (0, y_p, 0)$ with $y_p < 0$, if H is the vertical halfplane $\{(x, 0, z) : z > 0\}$, and if*

$$\Pi : M \to H$$
$$\Pi(q) = \overleftrightarrow{pq} \cap H,$$

then Π is a diffeomorphism. Furthermore, M has a normal vector field ν such that $\nu \cdot (0, 1, 0)$ is everywhere strictly positive.

Definition 5.5. The minimal surface M in Theorem 5.4 is called a *minimal skillet.*

Proof. By Theorem 3.5, there exists a properly embedded, area-minimizing surface M with $\partial M = \partial \mathcal{A}$. Furthermore, $\overline{M}$ is a smooth, embedded manifold-with-boundary except at the point at infinity (where $\partial \mathcal{A}$ is not smooth).

Claim 5.6. *The surface M is asymptotic to the standard minimal strip (see Definition 5.2) as $y \to \infty$, and is asymptotic to the geodesic plane H as $|x| \to \infty$. In other words, if $M - (x, y, 0)$ is the result of translating M by $-(x, y, 0)$, then $M - (0, y, 0)$ converges smoothly to the standard minimal strip as $y \to \infty$, and $M - (x, 0, 0)$ converges smoothly to H as $|x| \to \infty$.*

This claim follows immediately from the fact that the standard minimal strip and the totally geodesic plane H are uniquely area minimizing (by Theorem 5.1 and by the convex hull property 2.3).

Claim 5.7. *The surface M lies in the region $\{y \geq 0\}$. Also, there is an $a > 0$ such that*

$$M \cap \{y > a\}$$

lies in a cylinder $x^2 + z^2 \leq r^2$. Furthermore, as $\lambda \to 0$, the surface

$$\lambda(M \cap \{x^2 + z^2 \geq r^2\})$$

converges smoothly on compact subsets of $\overline{\mathbb{H}^3} \setminus \{0\}$ to $\overline{H}$. (Here $\lambda(S)$ denotes the result of dilating S by λ about the origin.)

Proof. The first statement follows from the convex hull property (Proposition 2.3). To prove the second, note that (by Claim 5.6) we can choose $a > 0$ so that one component of $M \cap \{y > a\}$ lies in a bounded Euclidean distance from the standard minimal strip and hence lies in a cylinder $x^2 + z^2 \leq r^2$. If $M \cap \{y > a\}$ had another component Σ, then Σ would be an open minimal surface whose limit set $L(\Sigma)$ lies in the totally geodesic plane $\{(x, a, z) : z \geq 0\} \cup \{\infty\}$, contradicting the convex hull property.

We have shown that the boundary of $M \cap \{x^2 + z^2 \geq r^2\}$ coincides (except in a ball around $(0, 0, 0)$) with ∂H. Thus the boundary of $\lambda(M \cap \{x^2 + z^2 \geq r^2\})$ converges to ∂H, and the convergence is smooth away from the origin. The convergence statement of the claim now follows from the Compactness Theorem 3.3 and from the fact that H is uniquely area minimizing. q.e.d.

Claim 5.8. *Fix a point p of the form $(0, y_p, 0)$ with $y_p < 0$, and for $\lambda > 0$, let M^λ be the result of dilating M by λ about the point p. Suppose N is another area-minimizing surface such that $\partial N = \partial M$. Then M^λ is disjoint from N for $\lambda \neq 1$.*

Proof. It suffices to prove the claim for $\lambda < 1$, since the result for $\lambda > 1$ follows by switching the roles of M and N.

Since N is properly embedded in $\mathbb{H}^3$, by elementary topology we can write N as the boundary of an open region U of $\mathbb{H}^3$. We may assume that $(0,0,0)$ is not in $\overline{U}$. (Otherwise replace U by the interior of $\mathbb{H}^3 \setminus U$.)

Note that if $\lambda < 1$ and if M^λ intersects N, then there are points of $M^\lambda \cap N$ where the intersection is transverse, from which it follows that if we perturb λ slightly, M^λ still intersects N. Thus if Λ is the set of $\lambda \in (0,1)$ for which M^λ intersects N, then Λ is open.

Note that there is an $R > 0$ with the following property: if $\lambda \in \Lambda$, then $M^\lambda \cap N$ contains points in the cylinder $C = \{(x,y,z) : x^2 + z^2 \leq R^2\}$. To see this, first choose R larger than the r of Claim 5.6, from which it follows that $\overline{N} \setminus C$ is a smooth manifold-with-boundary near ∞. Now choose R even larger so that $N' := \overline{N} \setminus C$ has the strong local uniqueness property described in Theorem 2.4. If $M^\lambda \cap N$ did not have any points in C, then $L(M^\lambda \cap U)$ would be contained in N', and therefore (by Theorem 2.4), $M \cap U$ would be contained in N', a contradiction.

We claim that Λ is also relatively closed in $(0,1)$. For suppose $\lambda(i) \in \Lambda$ converges to $\lambda \in (0,1)$. By the preceding paragraph, there exist points $(x(i), y(i), z(i))$ in $M^{\lambda(i)} \cap N$ with $x(i)^2 + z(i)^2 \leq R^2$. It follows from Claim 5.6 (applied to M and to N) that $z(i)$ is bounded away from 0. (Note that ∂M^λ and ∂N are a positive Euclidean distance apart.) Also, $y(i)$ is bounded since (by Claim 5.6) M and N are both asymptotic to the standard minimal strip as $y \to \infty$, and therefore M^λ and N are a positive distance apart as $y \to \infty$. Hence, after passing to a subsequence, $(x(i), y(i), z(i))$ converges to a point $p \in M^\lambda \cap N$, proving that $M^\lambda \cap N$ is nonempty and thus that Λ is relatively closed in $(0,1)$.

Since Λ is an open and closed subset of $(0,1)$, either it is empty or else it is all of $(0,1)$. To see that it is empty, note that M^λ is disjoint for N for very small λ since, by Claim 5.7,

$$\max_{q \in M^\lambda} \operatorname{dist}_{\mathbb{R}^3}(q, T) \to 0 \quad \text{as } \lambda \to 0$$

and

$$\min_{q \in N} \operatorname{dist}_{\mathbb{R}^3}(q, T) > 0$$

where T is the union of $\{y = y_p,\, z \geq 0\}$ and $\{(0,y,0) : y \geq y_p\}$. This completes the proof of Claim 5.8. q.e.d.

Now we can complete the proof of Theorem 5.4. It follows immediately from Claim 5.8 that M is unique.

Let $H = \{(x,0,z) : z > 0\}$ and let

$$\begin{aligned} &\Pi : M \to H \\ &\Pi(q) = \overleftrightarrow{pq} \cap H. \end{aligned}$$

Applying Claim 5.8 with $N = M$, we see that each straight (Euclidean) line through $p = (0, y_p, 0)$ intersects M at most once. Thus the map

Π is a diffeomorphism from M to an open subset of Ω. It follows from Claim 5.6 that Π is proper. Hence Π is a surjective diffeomorphism.

It follows that the Jacobi field on M corresponding to dilations about $p = (0, y_p, 0)$ is everywhere positive. Letting $y_p \to -\infty$, we see that the Jacobi field on M corresponding to horizontal translations $(x, y, z) \mapsto (x, y + t, z)$ is everywhere nonnegative. By the strong maximum principle, if that Jacobi field vanished anywhere, it would vanish everywhere, which implies that M would be invariant under those translations. But that is impossible since $y \geq 0$ for $(x, y, z) \in M$. Thus the Jacobi field is everywhere positive, which implies that $\nu \cdot (0, 1, 0)$ is everywhere positive. q.e.d.

Theorem 5.9. *A minimal skillet M in $\mathbb{H}^3$ is strictly L^∞ stable.*

Proof. We will assume that the minimal skillet has been translated by $(x, y, z) \mapsto (x, y + 1, z)$, so that it lies in the region $\{(x, y, z) : z > 0 \text{ and } y > 1\}$ and is asymptotic as $x^2 + z^2 \to \infty$ to the halfplane $y = 1, z \geq 0$. As $y \to \infty$ with $x^2 + z^2$ bounded, the minimal skillet M is smoothly asymptotic to the standard minimal strip. (See Claim 5.6.)

Let w be the Jacobi field on M corresponding to dilations about 0. In other words, for $p \in M$, $w(p)$ is the (hyperbolic) length of $p^\perp$. Then because M is a radial graph about the origin (by Theorem 5.4), $w > 0$ everywhere, so compact subsets of M are strictly stable. Thus it suffices to show that M has no nonzero, bounded Jacobi fields.

Suppose to the contrary that v is a nonzero, bounded Jacobi field.

Claim 5.10. *$zw(x, y, z)$ is bounded away from 0.*

To prove the claim, note that $w(x, y, z)$ is the hyperbolic length of the vector $(x, y, z)^\perp$ at the point (x, y, z), so $zw(x, y, z)$ is the Euclidean length $|(x, y, z)^\perp|$ of $(x, y, z)^\perp$.

As $x^2 + z^2 \to \infty$ in M, $\operatorname{Tan}_{(x,y,z)} M$ converges to the plane $y = 0$, so $(x, y, z)^\perp \sim (0, y, 0)$. Also, $y \geq 1$ on M, so

$$\liminf_{x^2+z^2\to\infty} zw(x, y, z) \geq 1.$$

On sets where $x^2 + z^2$ and y are both bounded, the Euclidean length of $(x, y, z)^\perp$ is bounded away from 0 because M is a radial graph.

Thus it remains to show that the Euclidean length $|(x, y, z)^\perp|$ is bounded as $y \to \infty$ with x and z bounded. But that holds because M is asymptotic as $y \to \infty$ to the standard minimal strip (see Claim 5.6) and because the corresponding Jacobi field w^* on the standard minimal strip is bounded away from 0 (by (5.1) and (5.2)). This completes the proof of Claim 5.10.

Claim 5.11. *If $p_n = (x_n, y_n, z_n)$ is a divergent sequence in M, then $v(p_n) \to 0$.*

Proof of Claim 5.11. By passing to a subsequence, we can assume that one of the following holds:

(1) $(x_n)^2 + (z_n)^2 \to \infty$.
(2) $(x_n)^2 + (z_n)^2$ is bounded and $z_n \to 0$.
(3) $(x_n)^2 + (z_n)^2$ is bounded and z_n is bounded away from 0.

Translate M by $(-x_n, -y_n, 0)$ and then dilate by $1/z_n$ to get a surface M_n. Let v_n be the Jacobi field on M_n corresponding to v on M. By passing to a subsequence, we can assume that M_n converges smoothly to a limit surface $\hat{M}$, and that v_n converges to a bounded Jacobi field $\hat{v}$ on $\hat{M}$. In case (1), $\hat{M}$ is the vertical halfplane $\{y = 0\}$ (by Claim 5.7). In case (2), $\hat{M}$ is also a vertical halfplane, since $\partial\hat{M}$ is a line in $\partial\mathbb{H}^3$. In case (3), $\hat{M}$ is a minimal strip (see Definition 5.2) by Theorem 5.1. In all three cases, $\hat{M}$ is strictly stable. Thus $\hat{v} = 0$. Since $\hat{v}(0,0,1) = \lim v_n(0,0,1) = \lim v(p_n)$, this completes the proof of Claim 5.11. q.e.d.

Claim 5.12. *There exists a Jacobi field f on M and an $R > 0$ such that*

$$\inf_{M\cap\{x^2+z^2>R^2\}} f > 0.$$

Proof of Claim 5.12. Let $S = S_R$ be the surface obtained from $\overline{M}\cap\{x^2+z^2 > R^2\}$ by inversion in the sphere $x^2+y^2+z^2 = 1$ (where $\overline{M}$ denotes the closure of M in $\overline{\mathbb{H}^3}$). Note that if R is sufficiently large, then S is a smooth manifold-with-boundary on which y is a smooth function of x and z. Indeed, by choosing $R > 0$ sufficiently large, we can guarantee that the Euclidean unit normal to S is everywhere arbitrarily close to $(0,1,0)$. Consequently, the Jacobi field corresponding to translations in the y-direction is bounded away from 0 on S. Now let f be the corresponding Jacobi field on M. This completes the proof of Claim 5.12. q.e.d.

Now let $\lambda = \sup(v/w)$. Since we are assuming that $v > 0$ at some points, $\lambda > 0$. By Lemma 4.3, the supremum is not attained at any point of M. (Note that v cannot be a multiple of w since v is bounded and w is unbounded.) Thus if $p_n = (x_n, y_n, z_n)$ is a sequence of points in M with $v(p_n)/w(p_n) \to \lambda$, then p_n diverges in M. By Claim 5.11, $v(p_n) \to 0$. Since $\lambda > 0$, this implies that $w(p_n) \to 0$, and therefore by Claim 5.10 that $z_n \to \infty$.

It follows that by choosing $\mu < \lambda$ sufficiently close to λ, we can guarantee that

$$(v - \mu w)^+$$

is supported in $M \cap \{z > R\}$, where R is as in Claim 5.12. It follows (using Claims 5.11 and 5.12) that

$$(v - \mu w)^+/f$$

attains a positive maximum value k at some point p. Consequently,

$$(v - \mu w)/f$$

has a positive local maximum k at p, so

$$v - \mu w - kf \equiv 0$$

by Lemma 4.3. But that is impossible since $(v - \mu w)$ is negative at some points of $M \cap \{z > R\}$, whereas $f > 0$ everywhere on that set. The contradiction proves that there is no such v, and therefore that M is strictly L^∞ stable. q.e.d.

6. Bridge principle at infinity

A key tool in the construction of our minimal embeddings with arbitrary topology is a bridge principle at infinity for properly embedded area-minimizing surfaces in $\mathbb{H}^3$.

Let $M \subset \mathbb{H}^3$ be a smooth, properly embedded, open surface whose closure $\overline{M}$ is a smooth manifold-with-boundary in $\overline{\mathbb{H}^3}$. Let $\Gamma \subset \partial\mathbb{H}^3$ be a smooth embedded arc such that $\overline{M} \cap \Gamma = \partial\Gamma$ and such that Γ meets ∂M orthogonally at each of its end points. A **bridge on** M **along** Γ is the image P of a homeomorphism

$$\phi : [0,1] \times [-1,1] \longrightarrow \partial\mathbb{H}^3$$

such that $\phi(\cdot, 0)$ parametrizes Γ and $\phi(t,s) \in \overline{M}$ if and only if $t = 0$ or $t = 1$.

By the (Euclidean) **width** of P we mean

$$w(P) = \sup_{x \in P} \operatorname{dist}_{\mathbb{R}^3}(x, \partial P).$$

For the following proposition, we shall consider the *half-space model* of $\mathbb{H}^3$. In this model, the homotheties centered at points $p \in \{z = 0\}$ induce isometries of the hyperbolic space.

From now on, it will be convenient to generalize the notions of skillet and minimal skillet as follows: the image of a skillet (or minimal skillet) under any isometry of $\mathbb{H}^3$ leaving ∞ fixed will also be called a skillet (or minimal skillet).

Proposition 6.1. *Let M and Γ be as above (in the preceding four paragraphs). Then there exists a sequence of bridges $\{P_n\}_{n\in\mathbb{N}}$ on M along Γ satisfying:*

(a) The widths $w_i := w(P_i)$ tend to 0 as $i \to \infty$.

(b) The symmetric difference $(\partial P_i) \vartriangle \partial M$ is smooth, and if $x_i \in P_i$, then every sequence of i's tending to ∞ has a subsequence $i(j)$ such that

$$\left(w_{i(j)}^{-1}\right)_{\#} \left((\partial P_{i(j)}) \vartriangle \partial M - x_{i(j)}\right)$$

converges smoothly on compact sets of $\overline{\mathbb{H}^3} \setminus \{\infty\}$ to either:

(1) two parallel straight lines, or
(2) the boundary of a **skillet***.*

Recall that $A \triangle B := (A \setminus B) \cup (B \setminus A)$.

The proof of Proposition 6.1 is straightforward so we omit it. A sequence of bridges P_i that satisfies the conclusions of Proposition 6.1 is said to *shrink nicely* to Γ.

Theorem 6.2 (Bridge Theorem)**.** *Let $S \subset \mathbb{H}^3$ be an open, properly embedded, uniquely area-minimizing surface whose closure $\overline{S} \subset \overline{\mathbb{H}^3}$ is a smooth, embedded manifold-with-boundary. Let Γ be a smooth arc in $\partial\mathbb{H}^3$ meeting ∂S orthogonally and satisfying $\Gamma \cap \partial S = \partial\Gamma$. Consider a sequence of bridges P_n in $\partial\mathbb{H}^3$ that shrink nicely to Γ. If S is* **strictly L^∞ stable***, then for all large enough n, there exists a strictly L^∞ stable, uniquely area-minimizing surface S_n that is properly embedded in $\mathbb{H}^3$ and that satisfies:*

(1) $\partial S_n = \partial S \triangle \partial P_n$ (in particular, $\overline{S_n}$ is a smooth, embedded manifold-with-boundary in $\overline{\mathbb{H}^3}$).
(2) The sequence $\overline{S_n}$ converges smoothly to $\overline{S}$ on compact subsets of $\overline{\mathbb{H}^3} \setminus \Gamma$.
(3) The surface $\overline{S_n}$ is homeomorphic to $\overline{S} \cup P_n$.

Such a sequence of bridges exists by Proposition 6.1.

Proof. By Theorem 3.5, there is an area-minimizing surface S_n satisfying (1). By the Compactness Theorem 3.3, every subsequence of S_n has a further subsequence such that $\overline{S_n}$ converges smoothly on compact subsets of $\overline{\mathbb{H}^3} \setminus \Gamma$ to $\overline{Q}$, where Q is an area-minimizing surface with boundary ∂S. Since S is uniquely area-minimizing, in fact $Q = S$, which proves (2).

The key to proving the rest of the Bridge Theorem is the following:

Claim 6.3. *Let (x_n, y_n, z_n) be a sequence of points in with $z_n > 0$ and $z_n \to 0$. Translate S_n by $-(x_n, y_n, 0)$ and then dilate by $1/z_n$ to get a surface*

$$S_n' := (S_n - (x_n, y_n, 0))/z_n.$$

Then a subsequence of the S_n' converges smoothly on a compact subset of $\mathbb{H}^3$ to one of the following surfaces S': a vertical halfplane, a minimal skillet, a minimal strip (see Definition 5.2), or the empty set. In particular, S' is uniquely area-minimizing and strictly L^∞ stable.

Furthermore, if T_n is another area-minimizing surface with $\partial T_n = \partial S_n$, and if $T_n' = (T_n - (x_n, y_n, 0))/z_n$, then the corresponding subsequence of the T_n' converges to the same limit surface S'.

Proof of Claim 6.3. By the definition of nicely shrinking, after passing to a subsequence, the curves $\partial S_n'$ converge to a limit C', where C' is one of the following configurations together with the point at infinity:

(1) a straight line with multiplicity 1;
(2) a T-shaped configuration consisting of a straight line with multiplicity 1 together with a perpendicular half-line with multiplicity 2;
(3) the boundary of a skillet with multiplicity 1;
(4) two parallel lines, each with multiplicity 1;
(5) a straight line with multiplicity 2;
(6) the empty set.

The convergence is smooth except at ∞ and, in case (2), at the vertex of the T.

By the Compactness Theorem 3.3, the S'_n converge smoothly (after passing to a subsequence) to a limit surface S' whose boundary is a closed subset of C' that contains the multiplicity 1 portion of C' but none of the multiplicity 2 portion. Thus (in all these cases) $\partial S'$ is the closure of the multiplicity 1 portion of C'. In particular, $\partial S'$ is a straight line in cases (1), and (2), a skillet boundary in case (3) and a pair of parallel lines in case (4). It follows that S' is a vertical halfplane, a minimal skillet, or a minimal strip since each of those surfaces is uniquely area-minimizing.

In cases (5) and (6), $\partial S'$ is the empty set, which implies (by the convex hull property) that S' is empty.

By passing to a further subsequence, we can assume that the T'_n converge smoothly on compact subsets of $\mathbb{H}^3$ to a surface T' whose boundary is (as proved above) the closure of the multiplicity 1 portion of C'. In other words, T' and S' have the same boundary. Since in each of the cases above, S' is uniquely area-minimizing, it follows that $T' = S'$.

q.e.d.

Next we shall prove that $\overline{S_n}$ and $\overline{S} \cup P_n$ are homeomorphic. The surface $\overline{S}_n$ separates $\overline{\mathbb{H}}^3$ into two connected components, one of which contains the curve Γ which we denote by $\mathcal{Q}_n$.

For $a > 0$, we define $\mathcal{R}_a := \{(x, y, z) \in \overline{\mathbb{H}^3} \,:\, 0 \le z \le a\}$.

Claim 6.4. *There exists $a > 0$ such that $\overline{S_n} \cap \mathcal{R}_a$ does not contain any point at which the vector $\mathbf{u} := (0, 0, 1)$ is a normal vector to $\overline{S_n}$ that points into $\mathcal{Q}_n$.*

(Thus $S_n \cap \mathcal{R}_a$ might have critical points of the height function z, but at such critical points, the normal vector $(0, 0, 1)$ must point out of $\mathcal{Q}_n$, not into it.)

Proof. We proceed by contradiction. Suppose this were not the case. Thus, after passing to a subsequence, we can assume that there exists a critical point $p_n = (x_n, y_n, z_n) \in S_n$ with $\mathbf{u}$ pointing into $\mathcal{Q}_n$ at p_n and with $z_n \to 0$. Up to a subsequence, we can suppose that $\{p_n\}$ converges to some point $p_0 = (x_0, y_0, 0) \in \partial\mathbb{H}^3$.

Then, we apply the isometry $(x, y, z) \mapsto 1/z_n\,((x, y, z) - (x_0, y_0, 0))$ to S_n, p_n, Q_n, and Γ to obtain a new surface S'_n, a point $p'_n = (0, 0, 1) \in S'_n$, a region Q'_n, and a curve Γ'_n. By Claim 6.3, we can assume (by passing to a subsequence) that the surfaces S'_n converge smoothly to one of the following surfaces S': a vertical halfplane, a minimal skillet, or the standard area-minimizing strip bounded by two parallel lines. In our case, S' cannot be a vertical halfplane or a minimal skillet, because those surfaces have no points at which $\mathbf{u} = (0, 0, 1)$ is a normal vector (see Theorem 5.4), whereas $\mathbf{u}$ is normal to S' at the point $p' = (0, 0, 1)$.

Thus S' is a minimal strip. Note that the curves Γ'_n must converge to the straight line Γ' that is halfway between the two lines in $\partial S'$. It follows that the regions Ω'_n converge to the region Ω' that lies on the other side of S' from Γ'. It now follows from the description of S' in Theorem 5.1 that the vector $\mathbf{u} = (0, 0, 1)$ points into Ω' at p'. However, the smooth convergence and the choice of p_n imply that $\mathbf{u} = (0, 0, 1)$ points out of Ω' at p'. The contradiction proves the claim. q.e.d.

Claim 6.5. *The surfaces $\overline{S}_n$ and $\overline{S} \cup P_n$ are homeomorphic.*

Suppose $\overline{S}_n$ is not homeomorphic to $\overline{S} \cup P_n$. Then, because they have the same boundary, $\overline{S}_n$ and $\overline{S} \cup P_n$ cannot have the same genus. Consider the positive constant a given by Claim 6.4. The smooth convergence on compact sets implies $S_n \cap (\mathbb{H}^3 \setminus \mathcal{R}_a)$ is homeomorphic to $S \cap (\mathbb{H}^3 \setminus \mathcal{R}_a)$, so our assumption gives that $S_n \cap \mathcal{R}_a$ has nontrivial genus.

Up to a slight modification of the point of infinity in the upper half-space model of $\mathbb{H}^3$, we can assume that the function z is a **Morse function** for the surface S_n. This implies the existence of a critical point of the height function z in $\overline{S}_n \cap \mathcal{R}_a$ such that the vector $u = (0, 0, 1)$ points in the direction of the region $\mathcal{Q}_n$, which is contrary to Claim 6.4. This contradiction completes the proof of this claim.

Claim 6.6. *If n is large enough, then the surfaces S_n are uniquely area-minimizing: if T_n is any area-minimizing surface in $\mathbb{H}^3$ with $\partial T_n = \partial S_n$, then $T_n = S_n$ (for all sufficiently large n).*

Suppose the uniqueness is false. Then, up to a subsequence, we may assume that S_n and T_n are different for all n. Note that all properties we have proved for S_n also hold for T_n. In particular, T_n also converges smoothly to S on compact subsets of $\overline{\mathbb{H}^3} \setminus \Gamma$.

As S_n and T_n are asymptotic at $\partial\mathbb{H}^3$, we can find a point $p_n = (x_n, y_n, z_n) \in S_n$ that maximizes the (hyperbolic) distance to T_n. (The maximum exists because the hyperbolic distance from a point q in S_n to T_n tends to 0 as q approaches the boundary of hyperbolic space. This follows from the fact that $\overline{S_n}$ and $\overline{T_n}$ meet $\partial\mathbb{H}^3$ orthogonally along the same curve.)

By passing to a subsequence, we can assume that p_n converges to a point $p = (x, y, z)$. If $p \in S$, then the smooth convergence of T_n and S_n

to S gives rise to a nonzero Jacobi field on S that attains its maximum absolute value at p. But that is impossible by the strict L^∞ stability of S.

Thus $p \in \partial S$, so $z_n \to 0$. Translate S_n, T_n, and p_n by $-(x_n, y_n, 0)$ and then dilate by $1/z_n$ to get S'_n, T'_n, and $p' := (0,0,1) \in S'_n$ with

$$\operatorname{dist}(p', T'_n) = \max_{q \in S'_n} \operatorname{dist}(q, T'_n). \tag{6.1}$$

By Claim 6.3, we can assume (by passing to a subsequence) that S'_n and T'_n converge smoothly on compact subsets of $\mathbb{H}^3$ to the same strictly L^∞ stable limit surface S'. By (6.1), the smooth convergence of S'_n and T'_n to S' gives rise to a nonzero Jacobi field on S' that attains its maximum absolute value at the point p'. But that contradicts the strict L^∞ stability of S', thus proving Claim 6.6.

To complete the proof of the Bridge Theorem 6.2, it remains only to prove that the surface S_n is strictly L^∞ stable for all sufficiently large n. The proof is almost the same as the proof of Claim 6.6. Suppose the strict L^∞ stability fails. Then we can assume that each S_n has a nonzero bounded Jacobi field V_n. By Theorem 4.6, $V_n(p)$ tends to 0 as $p \to \partial S_n$, so $|V_n(\cdot)|$ attains its maximum at a point $p_n = (x_n, y_n, z_n)$ in S_n. We can normalize V_n so that $|V_n(p_n)| = 1$. By passing to a subsequence, we can assume that p_n converges to a point $p = (x, y, z)$.

If $z > 0$, then the V_n converge subsequentially to a Jacobi field V on S that attains its maximum absolute value of 1 at the point p. But that violates the strict L^∞ stability of S.

Thus $z = 0$. Now translate S_n by $-(x_n, y_n, 0)$ and dilate by $1/z_n$ to get a surface S'_n. By Claim 6.3, a subsequence of the S'_n converges smoothly to a strictly L^∞ stable surface S'. However, by construction, S' has a Jacobi field that attains a maximum absolute value 1 at the point $p' := (0,0,1)$, a contradiction. q.e.d.

7. Properly embedded area-minimizing surfaces in $\mathbb{H}^3$

In this section, we are going to prove the main existence results for properly embedded area minimizing surfaces with arbitrary (orientable) topology. The techniques we use are inspired by those developed by Ferrer, Meeks and the first author for the study of the Calabi-Yau problem in $\mathbb{R}^3$ (see [**3**]).

Theorem 7.1. *Let S be an open, connected, oriented surface. Then there exists a complete, proper, area-minimizing embedding $\psi : S \to \mathbb{H}^3$. Moreover, the embedding ψ can be constructed in such a way that the limit sets of different ends of S are disjoint.*

Proof. Throughout this proof we are going to use the model of the Poincaré ball. Let $\mathcal{S} = \{S_1 \subset S_2 \subset \cdots \subset S_n \subset \cdots\}$ be a simple exhaustion of S. Our purpose is to construct a sequence of properly embedded minimal surfaces $\{\Sigma_n\}_{n\in\mathbb{N}}$ and two sequences of positive real numbers $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and $\{r_n\}_{n\in\mathbb{N}}$ satisfying:

(1) $\{\varepsilon_n\} \searrow 0$ and $\{r_n\} \nearrow +\infty$;

(2) $\displaystyle\sum_{n=1}^{\infty} \varepsilon_n < 1$.

Moreover, for each $n \in \mathbb{N}$, the minimal surface Σ_n satisfies:

(I_n) Σ_n is strictly L^∞ stable and uniquely area minimizing;

(II_n) Σ_n admits a C^∞ extension $\overline{\Sigma}_n$ to $\overline{\mathbb{H}}^3$ so that $\overline{\Sigma}_n$ is diffeomorphic to S_n;

(III_n) $\Sigma_n \cap \overline{B(0,r_j)}$ is diffeomorphic to S_j, for $j = 1, \dots, n$, where $B(0,r)$ represents the hyperbolic ball centered at 0 of radius r;

(IV_n) $\Sigma_n \cap B(0,r_i)$ is a normal graph over its projection $\Sigma_{i,n} \subset \Sigma_i$, for $i < n$. Furthermore, if we write $\Sigma_n \cap B(0,r_i) = \{\exp_p(f_{i,n}(p)\cdot \nu_i(p)) \mid p \in \Sigma_{i,n}\}$, where ν_i is the Gauss map of Σ_i, then:
- $|\nabla f_{i,n}| \le \sum_{k=i+1}^{n} \varepsilon_k$ and
- $|f_{i,n}| \le \sum_{k=i+1}^{n} \varepsilon_k$, for $i = 1, \dots, n-1$.

First, we fix a sequence which satisfies $\displaystyle\sum_{n=1}^{\infty} \varepsilon_n < 1$ (for instance $\varepsilon_n = \frac{3}{\pi^2 n^2}$). The above sequences are obtained by recurrence. In order to define the first elements, we consider a totally geodesic disk in $\mathbb{H}^3$. The choice of r_1 is irrelevant.

Assume now we have defined Σ_n and r_n and satisfying items from (I_n) to (IV_n). We are going to construct the minimal surface Σ_{n+1}.

As the exhaustion $\mathcal{S}$ is simple, we know that $S_{n+1} - \mathrm{Int}(S_n)$ contains a unique nonannular component N which topologically is a pair of pants or an annulus with a handle. Label γ as the connected component of ∂N that is contained in ∂S_n. We label the connected components of $\partial \Sigma_n$, $\Gamma_1, \dots, \Gamma_k$, in such a way that γ maps to Γ_k by the homeomorphism which maps S_n into Σ_n. Then, we apply Theorem 6.2 to Σ_n in the following way.

Case 1. *N is a pair of pants.*

The curve Γ_k bounds a disk D_k in $\partial\mathbb{H}^3$ that does not intersect the other boundary curves of Σ_n. Consider an arc $\Gamma \subset D_k$ so that $\Gamma \cap \Gamma_k = \partial\Gamma$. Then we apply Theorem 6.2 to the configuration $\Sigma_n \cup \Gamma$. In this way, we construct a family $\{T_m\}_{m\in\mathbb{N}}$ of properly embedded minimal surfaces obtained from Σ_n by adding a bridge B_m^1 that "divides" Γ_k into two different curves in $\partial\mathbb{H}^3$. Note that the surfaces T_m have the same topology as S_{n+1}, for all $m \in \mathbb{N}$.

Case 2. *N is a cylinder with a handle.*

We construct the surface T_m, as in the previous case. But this time we add a second bridge B_m^2 along a curve σ joining two opposite points in ∂B_m^1 (see Figure 3). Notice that, in this way, the old annular component becomes an annulus with a handle. Again the resulting surfaces, which we still call T_m, are homeomorphic to S_{n+1}.

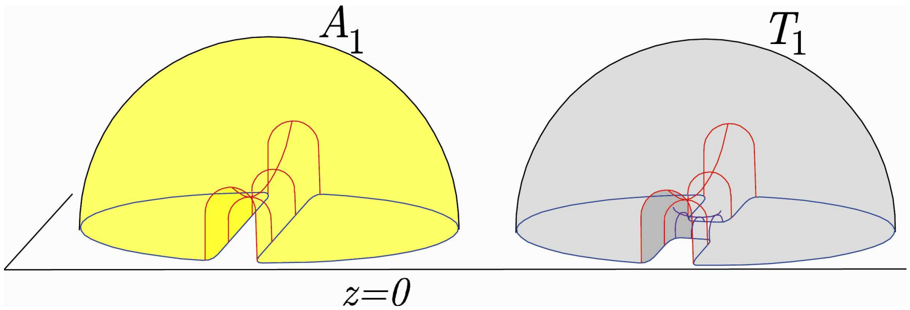

Figure 3. The surfaces A_1 and T_1

In both cases, we obtain a sequence of properly embedded, area-minimizing surfaces T_m satisfying:

(i) T_m is strictly L^∞ stable and uniquely area minimizing.
(ii) T_m admits a smooth extension to $\overline{\mathbb{H}}^3$ and $\overline{T_m}$ is diffeomorphic to S_{n+1}.
(iii) The surfaces $T_m \cap \overline{B(0,r_n)}$ are diffeomorphic to $\Sigma_n \cap \overline{B(0,r_n)}$ and converge in the C^∞ topology to $\Sigma_n \cap \overline{B(0,r_n)}$, as $m \to \infty$.

Item (iii) and property (IV$_n$) imply that $T_m \cap \overline{B(0,r_i)}$ can be expressed as a normal graph over its projection $\Sigma_{i,m} \subset \Sigma_i$, $i = 1, \dots, n$;

$$T_m \cap \overline{B(0,r_i)} = \{\exp_p (h_{m,i}(p)\, \nu_i(p)) \;\mid p \in \Sigma_{i,m}\}.$$

Since, as $m \to \infty$, the surfaces T_m converge smoothly to Σ_n in $B(0,r_n)$, and since Σ_n satisfies (IV$_n$), we have:

$$\max\{|h_{m,i}|, |\nabla h_{m,i}|\} < \sum_{k=i+1}^{n+1} \varepsilon_k \tag{7.1}$$

for m large enough.

Then, we define $\Sigma_{n+1} \stackrel{\text{def}}{=} T_m$, where m is chosen sufficiently large in order to satisfy (7.1). We choose r_{n+1} big enough to guarantee that $\Sigma_{n+1} \cap \overline{B(0,r_{n+1})}$ is diffeomorphic to S_{n+1}. It is clear that Σ_{n+1} so defined fulfills (I$_{n+1}$), ..., (IV$_{n+1}$).

Remark 7.2. Taking into account the way in which we are using the bridge principle at infinity to modify the topology of Σ_n, it is important to notice that the new boundary curves of Σ_{n+1} are contained in the disk $D_k \subset \partial\mathbb{H}^3$.

Now we have constructed our sequence of minimal surfaces $\{\Sigma_n\}_{n\in\mathbb{N}}$. Taking into account properties (IV_n), for $n \in \mathbb{N}$, and using Ascoli-Arzela's theorem, we deduce that the sequence of surfaces $\{\Sigma_n\}_{n\in\mathbb{N}}$ converges to a properly embedded minimal surface Σ in the C^m topology, for all $m \in \mathbb{N}$. Moreover, $\Sigma \cap \overline{B(0,r_i)}$ is a normal graph over is projection $\Sigma_{i,\infty} \subset \Sigma_i$, for all $i \in \mathbb{N}$, and the norm of the gradient of the graphing functions its at most 1 (see properties (IV_n)).

Finally, we check that Σ satisfies all the statements in the theorem.

• *Σ is diffeomorphic to S.* If we consider the (simple) exhaustions $\{\Sigma \cap \overline{B(0,r_n)} \mid n \in \mathbb{N}\}$ of Σ and $\{S_n \mid n \in \mathbb{N}\}$ of S, then we know that there exists a diffeomorphism $\psi_n : S_n \to \Sigma \cap \overline{B(0,r_n)}$. Furthermore, due to the way in which we have constructed Σ, we have that $\psi_n|_{S_i} = \psi_i$, for all $i < n$. Hence, we can construct a diffeomorphism $\psi : S \to \Sigma$.

If we consider on S the pull back of the metric of Σ, then ψ is the minimal embedding we are looking for.

• *Σ is area minimizing.* The limit of area-minimizing surfaces is area minimizing.

• *The limit sets of distinct ends are disjoint.* We are going to assume that Σ has at least two ends, as otherwise this property does not make sense. Two different ends of Σ, E_1 and E_2, can be represented by two disjoint components, C_1 and C_2, of $\Sigma \setminus B(0,r_n)$, for a sufficiently large $n \in \mathbb{N}$. Consider $\partial_i = C_i \cap \overline{B(0,r_n)}$, $i = 1,2$. Recall that $\Sigma \cap \overline{B(0,r_n)}$ is a graph over Σ_n. Then we label as ∂_1^n and ∂_2^n the projection over Σ_n of ∂_1 and ∂_2, respectively.

Observe that, from our method of construction, ∂_i (and ∂_i^n) is a connected curve, for $i = 1,2$. The curves ∂_1^n and ∂_2^n bound two different annular ends of Σ_n that we call A_1^n and A_2^n, respectively. For $i = 1,2$, let Γ_i^n be the ideal boundary of A_i^n:

$$\Gamma_i^n := \overline{A_i^n} \cap \partial\mathbb{H}^3.$$

The curve Γ_i^n bounds a disk $D_i^n \subset \partial\mathbb{H}^3$, $i = 1,2$, and we know that $D_1^n \cap D_2^n = \varnothing$. Taking Remark 7.2 into account, we deduce that $L(E_1) \subset D_1^n$ and $L(E_2) \subset D_2^n$. This concludes the proof. q.e.d.

We would like to finish this section by pointing out that a suitable modification of the methods allows us to construct properly embedded area-minimizing surfaces so that the limit set is the whole ideal boundary $\partial\mathbb{H}^3$.

Lemma 7.3. *If R/r is sufficiently large, then there is no open, connected, area-minimizing surface M in $\mathbb{H}^3$ such that, in the upper half-space model of $\mathbb{H}^3$,*

(i) *∂M is disjoint from $\{p \in \overline{\mathbb{H}^3} : r < |p| < R\}$, and*

(ii) *$M \cap \{|p| \le r\}$ and $M \cap \{p : |p| \ge R\}$ are both nonempty.*

Here $|p|$ denotes the Euclidean distance from p to the origin.

Proof. Suppose not. Then there is a sequence of open, connected, area-minimizing surfaces M_i in $\mathbb{H}^3$ such that

(i) ∂M_i is disjoint from $\{p \in \overline{\mathbb{H}^3} : r_i < |p| < R_i\}$,
(ii) $M \cap \{|p| \le r_i\}$ and $M \cap \{p : |p| \ge R_i\}$ are both nonempty, and
(iii) $R_i/r_i \to \infty$.

Since dilations are hyperbolic isometries, we can assume that $R_i = 1/r_i$, so that $R_i \to \infty$ and $r_i \to 0$. By the Compactness Theorem 3.3, a subsequence of the M_i converges smoothly on compact subsets of $\mathbb{H}^3$ to a properly embedded minimal surface $M \subset \mathbb{H}^3$ such that M intersects each Euclidean sphere centered at the origin and such that the limit set $L(M)$ of M is contained in $\{0, \infty\}$. By the convex hull property (Proposition 2.3), M is contained in the z-axis, which is impossible. q.e.d.

Corollary 7.4. *Let $M_1 \subset \mathbb{H}^3$ be a uniquely area-minimizing surface and let p be a point in $\overline{\mathbb{H}^3} \setminus \overline{M_1}$. Then p has a neighborhood $U \subset \overline{\mathbb{H}^3}$ with the following property: if M_2 is a uniquely area-minimizing surface that lies in U, then $M_1 \cup M_2$ is also uniquely area-minimizing.*

Proof. We can work in the half-space model of $\mathbb{H}^3$ with $p = (0, 0, 0)$. Choose $R > 0$ so that

$$M_1 \subset \{q : |q| \ge R\}.$$

Let $U = \{q : |q| < r\}$, where R/r is sufficiently large that the conclusion of Lemma 7.3 holds.

Let M_2 be a uniquely area-minimizing surface in U, and suppose that $M_1 \cup M_2$ is not uniquely area-minimizing. Then there is an area-minimizing surface M such that $\partial M = \partial M_1 \cup \partial M_2$ and such that $M \ne M_1 \cup M_2$. Since M_1 and M_2 are each uniquely area-minimizing, there must be a connected component of M that contains points in $\{q : |q| \le r\}$ and points in $\{q : |q| \ge R\}$, contradicting Lemma 7.3. q.e.d.

Proposition 7.5. *Let M be an open, connected, orientable surface. Then there exists a complete, proper, area-minimizing embedding $f\colon M \to \mathbb{H}^3$ such that the limit set is $\partial\mathbb{H}^3$.*

Proof. We want to modify the proof of Theorem 7.1 as follows: we construct a sequence $\{\Sigma_n'\}_{n\in\mathbb{N}}$ in such a way that it satisfies Properties (I_n), …, (IV_n) (see page 538) and:

> (V_n) *The Euclidean distance* from $\partial\Sigma_n$ to any point in $\partial\mathbb{H}^3$ is less than $1/n$.

To do this, once we have obtained the minimal surface Σ_n satisfying $(\mathrm{I}_n), \dots, (\mathrm{IV}_n)$, then we proceed as follows: Let $\Omega_1, \dots, \Omega_k$ be the connected components of $\partial\mathbb{H}^3 \setminus \partial\Sigma_n$. Take one of these components, Ω_i, $i \in \{1, \dots, k\}$ and consider a complete, totally geodesic disk D_i in $\mathbb{H}^3$ satisfying:

- D_i and Σ_n are disjoint;
- $\partial D_i \subset \Omega_i$;
- $\mathrm{diam}_{\mathbb{R}^3}(\partial D_i) < \frac{1}{2n}$;
- $D_i \cup \Sigma_n$ is uniquely area minimizing and strictly L^∞ stable.

Such a disk exists by Corollary 7.4. Let Γ_i be a smooth arc in Ω_i that connects $\partial\Sigma_n$ and ∂D_i and that is $\frac{1}{2n}$ close to every point in Ω_i. Then we apply Theorem 6.2 to construct a new surface by connecting Σ_n with D_i by a bridge along the arc Γ_i. Notice that the surface obtained in this way has the topology as Σ_n. We call Σ_n' the surface obtained by repeating the above procedure for all $i \in \{1, \dots, k\}$. If the width of the bridges is sufficiently small, we can guarantee that Σ_n satisfies (V_n). So, the limit surface Σ would satisfy that its limit set $L(\Sigma)$ is $\partial\mathbb{H}^3$. q.e.d.

7.1. Regularity of the boundary. Although the minimal embedding constructed in Theorem 7.1 is a limit of surfaces with smooth boundary, we cannot assert anything about the regularity at infinity of the minimal surface that we have obtained. In the case of finite topology, Oliveira and Soret [**6**] constructed minimal embeddings that extend smoothly to $\overline{\mathbb{H}}^3$. Hence, we shall center our attention on the case of open surfaces with infinite topology. If we do not care about the property that the limit sets of different ends are disjoint, then we can demonstrate the following:

Theorem 7.6. *Let S be an open surface with infinite topology. Then there exists a proper area-minimizing embedding of S into $\mathbb{H}^3$ such that the limit set in $\partial\mathbb{H}^3$ is a smooth curve except for one point. Moreover, the area-minimizing embedding extends smoothly to an embedding of S into $\overline{\mathbb{H}}^3$ except for that point.*

Proof. We will the upper half-space model of $\mathbb{H}^3$, so $\partial\mathbb{H}^3 = \{z = 0\} \cup \{\infty\}$. Let $\mathcal{S} = \{S_1 \subset S_2 \subset \cdots \subset S_n \subset \cdots\}$, a simple exhaustion for the surface S. For $n \in \mathbb{N}$, we define $X_n = \{(x, y, z) \in \overline{\mathbb{H}}^3 \ : \ 2(n-1) < x < 2n - 1\}$ and $Y_n = \{(x, y, z) \in \overline{\mathbb{H}}^3 \ : \ 2n - 1 < x < 2n\}$.

Consider a totally geodesic disk D_n contained in the region X_n given by the semi-sphere centered at $(2n - 3/2, 0, 0)$ and radius $r_n < 1/2$. Let A_n be the minimal annulus obtained by adding a bridge to D_n along a diameter of ∂D_n. Similarly, we can construct a minimal disk with a handle T_n, included in the region Y_n. First we add a bridge at infinity B to a totally geodesic disk represented by a semi-sphere centered at $(2n - 1/2, 0, 0)$ and radius $r_n < 1/2$. Later, we add a second bridge B'

along a curve in $\partial\mathbb{H}^3$ joining to opposite points of the ideal boundary of B. Notice that the surfaces A_n and T_n, $n \in \mathbb{N}$, satisfy the hypothesis of our bridge principle at infinity (Theorem 6.2).

As in the proof of Theorem 7.1, we construct our surface inductively. The first element in our sequence is the totally geodesic disk $\Sigma_1 = D_1$. The second element in the sequence, Σ_2, is obtained by joining Σ_1 with $W_2 \in \{A_2, T_2\}$ by a bridge at infinity along a curve Γ_2, which is contained in $\partial\mathbb{H}^3 \cap \{x < 4\}$. The choice of W_2 depends on the topology of $S_2 \setminus \mathrm{Int}(S_1)$. To add this bridge, we have to guarantee that $\Sigma_1 \cup W_2$ satisfies the assumptions of Theorem 6.2. It is clear that $\Sigma_1 \cup W_2$ is strictly L^∞ stable, so we only need to check that it is *uniquely area-minimizing*. By Corollary 7.4, this can be guaranteed by applying a suitable homothetical shrinking to W_2 with respect to $(5/2, 0, 0)$ or $(7/2, 0, 0)$ (depending on the nature of W_2). Observe that the ideal boundary $\partial\Sigma_2$ is a set of pairwise disjoint Jordan curves so that $\partial\mathbb{H}^3 \setminus \partial\Sigma_2$ consists of a disjoint union of disks (actually, either one or two disks) and one unbounded connected component that is not simply connected and that we shall denote $\mathcal{C}_2$.

Assume that the surface Σ_n is constructed in such a way that Σ_n is diffeomorphic to S_n and $\partial\Sigma_n$ consists of a finite set of pairwise disjoint Jordan curves and such that $\partial\mathbb{H}^3 \setminus \partial\Sigma_n$ consists of a disjoint union of finitely many topological disks together with one unbounded component $\mathcal{C}_n$. We are going to show how to construct the surface Σ_{n+1}. We know that $S_{n+1} \setminus \mathrm{Int}(S_n)$ contains exactly one non-annular connected component that we call Δ_{n+1}. Let $\sigma_{n+1} \subset \partial\Sigma_n$ be the connected component of $\partial\Sigma_n$ which corresponds to $\partial\Delta_{n+1} \cap \partial S_n$ and let $q_{n+1} = (x_{n+1}, y_{n+1}, z_{n+1})$ be the point of σ_{n+1} with the highest x-coordinate. We have that $x_{n+1} \in [m, m+1]$ for some $m \in \mathbb{N}$.

Then we are going to construct a curve $\Gamma_{n+1} \subset \mathcal{C}_n \cap \{m \leq x < 2(n+1)\}$ joining q_{n+1} and $W_{n+1} \in \{A_{n+1}, T_{n+1}\}$, where W_{n+1} depends on the topology of Δ_{n+1}. To do this, we proceed as follows. The intersection of $\{(t, 0, 0) \ : \ t \geq x_{n+1}\}$ and $\overline{\mathcal{C}_n}$ consists of a finite (disjoint) union of segments $\alpha_1 \cup \cdots \cup \alpha_l$ and a half-line r. Let α_{l+1} be the piece of r joining $\partial\Sigma_n$ and ∂W_{n+1}. For $j \in \{1, \ldots, l\}$, label β_j the arc in $\partial\Sigma_n$ that joins the end point of α_j and the initial point of α_{j+1}. Notice that, from our method of construction, the x-coordinate is non-decreasing along β_j, $j = 1, 2, \ldots, l$. Let us define

$$\gamma = \alpha_1 * \beta_1 * \alpha_2 * \cdots * \alpha_l * \beta_l * \alpha_{l+1}.$$

The curve Γ_{n+1} is a suitable perturbation of γ satisfying that $\Gamma_{n+1} \subset \mathcal{C}_n \cap \{m \leq x < 2(n+1)\}$, and that Γ_{n+1} does not touch the x-axis.

Again, up to a suitable shrinking of W_{n+1} we can assume that we are in the conditions for applying Theorem 6.2, and so we obtain Σ_{n+1} by adding a bridge along Γ_{n+1} to $\Sigma_n \cup W_{n+1}$. Observe that the bridge can be chosen so that it does not intersect the x-axis.

It is important to notice that the sequence of surfaces $\{\Sigma_n\}_{n\in\mathbb{N}}$ constructed in this way satisfies that, for all $r > 0$, the ideal boundary $\partial\Sigma_n$ intersects the region $\{x \leq r\}$ in the same set of arcs, for n sufficiently large.

It is important to note that for every r and n, $(\partial\Sigma_n) \cap \{x < r\}$ is a finite collection of arcs. Furthermore, there is an n such that

$$(\partial\Sigma_n) \cap \{x < r\} = (\partial\Sigma_k) \cap \{x < r\} \tag{7.2}$$

for all $k \geq n$.

Reasoning as in the proof of Theorem 7.1, we can guarantee that the sequence $\{\Sigma_n\}_{n\in\mathbb{N}}$ converges smoothly on compact sets to a properly embedded minimal surface Σ. From (7.2) we see that $\partial\Sigma \cap \{x \leq r\} = \partial\Sigma_n \cap \{x \leq r\}$, for $n \in \mathbb{N}$ large enough. Thus $(\partial\Sigma) \setminus \{\infty\}$ is smooth and properly embedded in $\partial\mathbb{H}^3 \setminus \{\infty\}$. q.e.d.

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