

## SUMS OF SECTIONS, SURFACE AREA MEASURES, AND THE GENERAL MINKOWSKI PROBLEM

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### Abstract

For  $2 \leq k \leq d - 1$ , the  $k$ -th mean section body,  $M_k(K)$ , of a convex body  $K$  in  $\mathbb{R}^d$ , is the Minkowski sum of all its sections by  $k$ -dimensional flats. We will show that the characterization of these mean section bodies is equivalent to the solution of the general Minkowski problem, namely that of giving the characteristic properties of those measures on the unit sphere which arise as surface area measures (of arbitrary degree) of convex bodies. This equivalence arises from an analysis of Berg's [3] solution of the Christoffel problem. We will see how the functions introduced by Berg yield an integral representation of the support function of  $M_k(K)$  in terms of the  $(d + 1 - k)$ -th surface area measure of  $K$ . Our results will be obtained using Fourier transform techniques which also yield a stability version of the fact that  $M_k(K)$  determines  $K$  uniquely.

### 1. Introduction

For a convex body  $K$  in  $\mathbb{R}^d$  and  $k = 0, 1, \dots, d$ , the *mean section body*  $M_k(K)$  is defined as the Minkowski average of all sections of  $K$  with  $k$ -dimensional (affine) flats. In terms of support functions, we have

$$(1) \quad h(M_k(K), \cdot) = \int_{A(d,k)} h(K \cap E, \cdot) \mu_k(dE),$$

where  $A(d, k)$  is the affine Grassmannian and  $\mu_k$  is the motion invariant measure on  $A(d, k)$  normalized so that the measure of all the  $k$ -flats within distance one of the origin is  $\kappa_{d-k}$ , the volume of the unit ball in  $\mathbb{R}^{d-k}$ . For standard notions in the geometry of convex bodies, including support functions, surface area measures, Steiner points, and intrinsic volumes, we refer the reader to Schneider's book [34].

In the case  $k = d$  we have  $M_d(K) = K$  ( $\kappa_0 = 1$ ). At the other extreme,  $k = 0$ , we see that  $M_0(K)$  reduces to the origin if  $\dim K < d$

and, in case  $\dim K = d$ ,

$$M_0(K) = \left\{ \int_K x \, dx \right\}$$

consists of the moment vector of  $K$ ; see Schneider [30] or [34, page 303]. We note that the moment vector is the centre of gravity of  $K$  multiplied by the volume of  $K$ . The case  $k = 1$  is also a special one. As was shown in [19], for  $\dim K = d$ ,  $M_1(K)$  is a ball whose radius is determined by the volume of  $K$ ; see [16] for an alternate proof. It is, therefore, clear that, for the very small values of  $k$  and for sufficiently low dimensional bodies  $K$ , the mean section body  $M_k(K)$  does not provide much information about the body  $K$ .

The original motivation for the introduction of mean section bodies in [19] came from stereology. Reconstruction of  $K$  from one of its mean section bodies seemed to pose many difficulties, so we focused there on questions of determination. Namely if, for some  $k \geq 2$ ,  $M_k(K_1) = M_k(K_2)$ , does it follow that  $K_1 = K_2$ ? One of the main results of [19] was to show that the mean section body  $M_2(K)$  determines  $K$  uniquely up to translations, in case  $\dim K = d$ . This was deduced from an explicit formula for the support function of  $M_2(K)$ ; see [19, Corollary 2],

$$(2) \quad h(M_2(K), u) = \frac{1}{2\pi(d-1)} \int_{S^{d-1}} \alpha(x, u) \sin \alpha(x, u) S_{d-1}(-K, dx),$$

for  $u \in S^{d-1}$ , in case the origin is the Steiner point of  $M_2(K)$ . Here,  $\alpha(x, u)$  denotes the smaller of the angles between the unit vectors  $x$ ,  $u$ , and  $S_{d-1}(-K, \cdot)$  denotes the surface area measure of the reflection of  $K$  in the origin. Also, (2) contains the correct constant which was miscalculated in [19].

In [15], it was shown that, for any translation vector  $t \in \mathbb{R}^d$  and any  $k = 1, \dots, d-1$ , equation (1) implies

$$(3) \quad M_k(K+t) = M_k(K) + \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{d-k+1}{2})}{\sqrt{\pi}\Gamma(\frac{d+1}{2})} V_{d-k}(K)t$$

(see [15, page 165]), a result which is clearly also true for  $k = 0, d$ ; here  $V_{d-k}(K)$  denotes the  $(d-k)$ -th intrinsic volume of  $K$ . It follows that, if  $K_2$  is a translate of  $K_1$  with  $\dim K_1 \geq d-k$ , then  $M_k(K_1) = M_k(K_2)$  if and only if  $K_1 = K_2$ . Consequently, this removed the translational restriction from the above uniqueness result in [19] since there we already required the bodies to be full dimensional. In [15], it was further shown that, for  $k = 3, \dots, d-1$  and centrally symmetric bodies  $K$ , the mean section body  $M_k(K)$  determines  $K$ , in case  $\dim K \geq d+2-k$ . Not surprisingly, for centrally symmetric sets  $K$ , the body  $M_k(K)$  is centrally symmetric and its Steiner point is the centre of symmetry.

More recently, in [21], we proved that, for any  $k = 2, \dots, d$ , all convex bodies  $K$ , with  $\dim K \geq d+2-k$ , are uniquely determined by  $M_k(K)$ .

This result made use of a connection between the body  $M_k(K)$  and the  $(d + 1 - k)$ -th surface area measure  $S_{d+1-k}(K, \cdot)$  of  $K$ ; namely, for arbitrary convex bodies  $K, L \subset \mathbb{R}^d$ ,

$$(4) \quad \int_{S^{d-1}} h(M_k(K), u) S_{d+1-k}(L, du) \\ = \int_{S^{d-1}} h(M_k(L), u) S_{d+1-k}(K, du).$$

This equation follows from the work of Alesker, Bernig, and Schuster [2, Corollary 7.2] on Minkowski valuations, or see [21, Theorem 2.1]. The equation (4) was used to obtain the determination of  $K$ , up to translation, by  $M_k(K)$  and then (3) can again be employed to remove the translational ambiguity.

As will be explained in the second section, mean section bodies are Minkowski valuations with special invariance properties. One of our main objectives is to find integral representations for their support functions, that is, to generalize (2) to values of  $k$  beyond 2. A number of authors have sought such representations for bodies in the range of an arbitrary Minkowski valuation. An earlier work in this direction was Schneider's paper [31]. In more recent times mean section bodies have played a role in these investigations. We mention, for example, the papers by Kiderlen [27], Schuster [36, 37], and the preprint [38] by Schuster and Wannerer. It is interesting to note that it follows from [36, Theorem 5.2] and the fact that the operator associated with  $M_2$  bodies is a Blaschke-Minkowski homomorphism, that the  $M_2$  bodies are nowhere dense in the set of all convex bodies.

In the current paper we will focus on explicit representations of the support functions of mean section bodies. We first provide an expression for the support function of  $M_k(K)$  in terms of certain Fourier transforms of  $S_{d+1-k}(K, \cdot)$ . This result, which again makes use of (4), will only pertain to the case that the Steiner point of  $M_k(K)$  is at the origin. It will provide an inversion formula, which gives the  $(d + 1 - k)$ -th surface area measure of  $K$  in terms of yet other Fourier transforms of  $S_1(M_k(K), \cdot)$ . Naturally, this yields another version of the determination of  $K$  by  $M_k(K)$ , first up to translation, but then in general by (3). Combining our Fourier transform expression with the results of Kiderlen [28] gives a stability version of this uniqueness result.

Our Fourier series expression for  $h(M_k(K), \cdot)$  in terms of  $S_{d+1-k}(K, \cdot)$  has an integral representation which is a generalization of (2) from the case  $k = 2$  to arbitrary  $k$ . This generalization employs the functions used by Berg [3] in his solution of the Christoffel problem. For each dimension  $d = 2, 3, \dots$ , he constructed a function  $g_d$  on  $(-1, 1)$  such

that

$$\int_{-1}^1 g_d(t)(1-t^2)^{(d-3)/2} dt < \infty,$$

and such that a measure  $\mu$  on  $S^{d-1}$  (with centroid at the origin) is the first surface area measure of a convex body in  $\mathbb{R}^d$  if and only if  $\int_{S^{d-1}} g_d(\langle \cdot, u \rangle) \mu(du)$  is a support function; here  $\langle u, v \rangle$  denotes the inner product of the vectors  $u, v \in S^{d-1}$ . We will show that, for  $k = 2, \dots, d$ ,

$$\int_{S^{d-1}} g_k(\langle \cdot, u \rangle) S_{d+1-k}(K, du)$$

is a multiple of  $h(M_k(K), \cdot)$ , assuming the Steiner point of  $M_k(K)$  is the origin. The connection between surface area measures and mean section bodies had been evident in [19, 15, 21] and is here made explicit. It will follow, in Theorem 4.6, from our uniqueness results, that a measure  $\mu$ , with centroid at the origin, is the  $(d+1-k)$ -th surface area measure of a convex body in  $\mathbb{R}^d$  if and only if  $\int_{S^{d-1}} g_k(\langle \cdot, u \rangle) \mu(du)$  is the support function of the  $k$ -th mean section body of a convex body in  $\mathbb{R}^d$ . Berg's result is the case  $k = d$ . The general Minkowski problem, that of characterizing the intermediate surface area measures of convex bodies, is a difficult open problem; see [1, 7, 8, 13, 32, 40]. Here we see an unexpected relationship between this question and the characterization of support functions of  $M_k$  bodies.

Section 2 provides the background for some of the techniques and results that will be used, especially those pertaining to the spherical projections and liftings studied in [17]. In section 3 we give the Fourier transform results including the stability theorem. The final section comprises the work on Berg's functions.

## 2. Preliminaries

In this section we will review the notation and expand on some of the background information provided in the introduction. We will also carry out some calculations which will be useful throughout the paper.

We will be working with convex bodies in  $d$ -dimensional Euclidean space,  $\mathbb{R}^d$ , with  $d \geq 3$  and, as already observed, we write the usual inner product as  $\langle \cdot, \cdot \rangle$ . Its unit ball will be denoted by  $B^d$  and the unit sphere by  $S^{d-1}$ . As is well known, the volume,  $\kappa_d$ , of  $B^d$  is given by

$$\kappa_d = \frac{\pi^{d/2}}{\Gamma(\frac{d+2}{2})}.$$

We will use this fact throughout the paper for the explicit formulation of constants. In various instances, we will find it convenient to postpone the calculation of (positive) dimensional constants. We will therefore use symbols such as  $c_d$  and  $c_{d,k}$  to denote numbers whose values depend only on their subscripts. These dimensional "constants" may change

their value from one part of a calculation to another even though we use the same symbol throughout.

In addition to the affine Grassmannian,  $A(d, k)$ , mentioned above, we will also consider  $G(d, k)$ , the compact Grassmannian of  $k$ -dimensional subspaces of  $\mathbb{R}^d$ . For  $L \in G(d, k)$ , the unit sphere  $S^{d-1} \cap L$  of  $L$  will be denoted by  $S^{k-1}(L)$ . For  $M \in A(d, k)$  and  $0 \leq j \leq k$ ,  $A(M, j)$  will denote the affine Grassmannian of  $j$ -flats in  $M$ . The appropriately normalized invariant measure on  $A(M, j)$  will be indicated by  $\mu_j^M$ . If  $q = 1, \dots, d$ ,  $k = d - q, \dots, d$ , and  $M \in G(d, q)$ , then, for almost all  $E \in A(d, k)$ ,  $E \cap M \in A(M, k - d + q)$ . So the characteristic invariance properties of the measures show that there is a constant  $a_{d,k,q}$  such that, for any integrable function  $f$  on  $A(d, k - d + q)$ , we have

$$(5) \quad \int_{A(d,k)} f(M \cap E) \mu_k(dE) = a_{d,k,q} \int_{A(M,d-k+q)} f(F) \mu_{k-d+q}^M(dF).$$

The value of the constant is, perhaps, most easily calculated using the Crofton formula; see [35, Theorem 5.1.1] for example. We have, for any convex body  $K \subset M \subset \mathbb{R}^d$  with  $\dim K = q$ ,

$$\begin{aligned} \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{k-d+q+1}{2})\Gamma(\frac{d+1}{2})} V_q(K) &= \int_{A(d,k)} V_{k-d+q}(K \cap E) \mu_k(dE) \\ &= a_{d,k,q} \int_{A(M,k-d+q)} V_{k-d+q}(K \cap F) \mu_{k-d+q}^M(dF) = a_{d,k,q} V_q(K). \end{aligned}$$

Thus

$$(6) \quad a_{d,k,q} = \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{k-d+q+1}{2})\Gamma(\frac{d+1}{2})}.$$

The support function of a convex body  $K$  is denoted by  $h(K, \cdot)$  and its various surface area measures by  $S_i(K, \cdot)$  for  $i = 1, \dots, d - 1$ . Although the intrinsic volumes,  $V_i(K)$  for  $i = 0, \dots, d$ , of a convex body  $K$  are independent of the ambient dimension, this is not the case for many of the other concepts we will be discussing. In particular, if  $K \subset L \in G(d, k)$  is a convex body and  $1 \leq j \leq k - 1$ , the  $j$ -th surface area measure of  $K$  calculated in  $L$  as a measure on  $S^{k-1}(L)$  will be written as  $S_j^L(K, \cdot)$  and the  $j$ -th mean section body of  $K$ , calculated in  $L$ , will be denoted by  $M_j^L(K)$ .

In the introduction, we saw, in (3), the role played by translation vectors and the fact that some of our results will be predicated on translating  $K$  so that the Steiner point of  $M_k(K)$  is at the origin. It is therefore convenient to recall, from [35], the notion of the centred support function,  $h^*(K, \cdot)$ , namely the support function of that translate,  $K^*$  of  $K$ , which has Steiner point at the origin. The additive properties of Steiner

points (see [34, Theorem 3.4.2] for example) yield

$$h^*(M_k(K), u) = \int_{A(d,k)} h^*(K \cap E, \cdot) \mu_k(dE).$$

As pointed out in [21], these translated bodies satisfy a valuation property, namely

$$M_k^*(K \cup L) + M_k^*(K \cap L) = M_k^*(K) + M_k^*(L)$$

whenever  $K, L$  and  $K \cup L$  are convex bodies in  $\mathbb{R}^d$ . It can also be seen, from the definition of the measure  $\mu_k$ , that  $M_k^*(\lambda K) = \lambda^{d+1-k} M_k^*(K)$  for  $\lambda > 0$ . Thus, in the language of [2], the mapping  $K \mapsto M_k^*(K)$  is a homogeneous, degree  $d + 1 - k$ , translation and  $SO(d)$  invariant Minkowski valuation. It is this observation that allows us to deduce (4) from the work of Alesker, Bernig, and Schuster [2].

We noted in the introduction that, for  $\dim K = d$ ,  $M_0(K)$  reduces to the moment vector of  $K$  and so, in the notation of Schneider [34, Section 5.4], we have  $M_0(K) = \{z_{d+1}(K)\}$ . In addition to the moment vector  $z_{d+1}$ , there are the intrinsic  $j$ -moment vectors  $z_j$ , for  $j = 1, \dots, d$ . They arise as coefficients in the polynomial expansion

$$z_{d+1}(K + \lambda B^d) = \sum_{i=0}^d \kappa_i z_{d+1-i}(K) \lambda^i,$$

for  $\lambda > 0$ ; see [34, (5.4.7) and (5.4.8)]. In fact, as pointed out by Schneider, for  $\dim K \geq d - i$ ,  $z_{d+1-i}(K)$  is  $V_{d-i}(K)$  times the  $(d - i)$ -th curvature measure centroid of  $K$ . These measures are supported on the boundary of  $K$  and, for sufficiently smooth bodies, are obtained as integrals of the  $(i - 1)$ -st normalized elementary symmetric function of the principal curvatures; see [34, Sections 2.5 and 4.2]. We also note that  $z_1(K)$  is the Steiner point of  $K$ .

These intrinsic moment vectors satisfy the following translational formula:

$$z_{d+1-k}(K + t) = z_{d+1-k}(K) + V_{d-k}(K)t$$

(see [34, (5.4.5)]). Not surprisingly, this is similar in nature to the translational formula (3) for the  $M_k$  bodies. The latter was used in [15] to show that, for  $k = 1, \dots, d - 1$  and  $\dim K \geq d - k$ ,

$$(7) \quad z_1(M_k(K)) = \binom{d}{k}^{-1} \frac{\kappa_{d-k} \kappa_k}{\kappa_d} z_{d+1-k}(K).$$

This result is trivially true in case  $k = d$  and follows from our remarks above in case  $k = 0$ . In case  $\dim K = d - k$ , it is clear that  $M_k(K)$  must be a multiple of the moment vector of  $K$ , calculated in the affine hull,  $\text{aff } K$ , of  $K$ . It therefore follows from (7) that, in this case,

$$M_k(K) = \binom{d}{k}^{-1} \frac{\kappa_{d-k} \kappa_k}{\kappa_d} \{z_{d+1-k}(K)\},$$

for  $k = 0, \dots, d$ .

We also mentioned in the introduction that, for  $\dim K = d$ ,  $M_1(K)$  is a ball whose radius is determined by the volume of  $K$ . In fact, it was shown in [19] that the radius of  $M_1(K)$  is

$$\frac{\kappa_{d-1}}{d\kappa_d} V_d(K).$$

As observed in [15], it follows from (7) that, for  $\dim K = d$ , we have

$$M_1(K) = \frac{\kappa_{d-1}}{d\kappa_d} \left( V_d(K)B^d + 2z_d(K) \right).$$

As observed above, the vector  $2z_d(K)$  is the surface area of  $K$  multiplied by the surface area centroid of  $K$ ; see [34, page 305]. More generally for any  $k = 1, \dots, d$  and  $\dim K = d - k + 1$ , the body  $M_k(K)$  will be a  $(d - k + 1)$ -ball whose radius is determined by  $V_{d+1-k}(K)$  and whose centre is determined by the product of the surface area and the surface area centroid of  $K$  (calculated in the affine hull of  $K$ ). If we denote by  $B_K$  the  $(d + 1 - k)$ -ball, in  $\text{aff } K$ , whose radius is  $V_{d+1-k}(K)$  and whose centre is at the surface area centroid of  $K$  multiplied by its surface area, then it follows from (1) and (5) that

$$\begin{aligned} h(M_k(K), u) &= \int_{A(d,k)} h(K \cap E, u) \mu_k(dE) \\ &= \frac{k! \kappa_k (d + 1 - k)! \kappa_{d+1-k}}{2d! \kappa_d} \int_{A(\text{aff } K, 1)} h(K \cap F, u) \mu_1^{\text{aff } K}(dF) \\ &= \frac{k! \kappa_k (d + 1 - k)! \kappa_{d+1-k}}{2d! \kappa_d} h(M_1^{\text{aff } K}(K), u) \end{aligned}$$

and so

$$(8) \quad M_k(K) = \binom{d}{k}^{-1} \frac{\kappa_k \kappa_{d-k}}{2\kappa_d} B_K.$$

We will see later that this observation is, in some sense, a forerunner of one of our main results, Theorem 3.4. For the time being, we just use it to conclude that for  $\dim K = d - k$  or  $d + 1 - k$ , the body  $M_k(K)$  only carries information about volumes, centroids, or surface area centroids. In particular, there is no information about the shape of  $K$ .

It will be convenient for our later results to find the mean section bodies of balls.

**Lemma 2.1.** *For  $k = 1, \dots, d$ ,  $r > 0$ , and  $t \in \mathbb{R}^d$ , we have*

$$\begin{aligned} M_k(rB^d + t) &= \frac{1}{2} \frac{\Gamma(\frac{d}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{d+1}{2})\Gamma(\frac{d+3-k}{2})} \pi^{(d+1-k)/2} r^{d+1-k} B^d \\ &\quad + \frac{1}{\Gamma(\frac{k+2}{2})} \pi^{k/2} r^{d-k} t. \end{aligned}$$

*Proof.* First, we calculate  $M_k(B^d)$ . We note that if  $L \in G(d, k)$  and  $x \in L^\perp \cap B^d$ , then  $(L + x) \cap B^d$  is a  $k$ -dimensional ball of radius  $\sqrt{1 - \|x\|^2}$ . Consequently,

$$\int_{L^\perp} h(B^d \cap (L + x), \cdot) dx$$

will be the support function of a ball in  $L$  of radius

$$\begin{aligned} \int_{L^\perp \cap B^d} \sqrt{1 - \|x\|^2} dx &= (d - k)\kappa_{d-k} \int_0^1 (1 - r^2)^{1/2} r^{d-1-k} dr \\ &= \frac{\pi^{(d+1-k)/2}}{2\Gamma(\frac{d+3-k}{2})}. \end{aligned}$$

As a convex body in  $\mathbb{R}^d$ , the value of its support function at an arbitrary  $u \in S^{d-1}$  is

$$\frac{\pi^{(d+1-k)/2}}{2\Gamma(\frac{d+3-k}{2})} \|u|L\| = \frac{\pi^{(d+1-k)/2}}{2\Gamma(\frac{d+3-k}{2})} V_1([o, u]|L)$$

for  $u \in S^{d-1}$ ; here  $u|L$  and  $[o, u]|L$  denote the orthogonal projections, respectively, of the unit vector  $u$  and the line segment  $[o, u]$  onto the subspace  $L$ . It follows, using the Cauchy-Kubota formulas (see [35, Theorem 6.2.2] for example) that, for  $u \in S^{d-1}$ ,

$$\begin{aligned} h(M_k(B^d), u) &= \int_{G(d,k)} \int_{L^\perp} h(B^d \cap (L + x), u) dx dL \\ &= \frac{\pi^{(d+1-k)/2}}{2\Gamma(\frac{d+3-k}{2})} \int_{G(d,k)} V_1([o, u]|L) dL \\ &= \frac{1}{2} \frac{\Gamma(\frac{d}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{d+1}{2})\Gamma(\frac{d+3-k}{2})} \pi^{(d+1-k)/2}, \end{aligned}$$

here, and in the sequel, this integration is with respect to the invariant probability measure on  $G(d, k)$ . The homogeneity properties of  $M_k$  now give

$$M_k(rB^d) = \frac{1}{2} \frac{\Gamma(\frac{d}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{d+1}{2})\Gamma(\frac{d+3-k}{2})} \pi^{(d+1-k)/2} r^{d+1-k} B^d.$$

The general result then follows from (3). q.e.d.

Next we recall some of the spherical liftings and projections investigated by Goodey, Kiderlen, and Weil in [17]. We assume  $L \in G(d, k)$  for some  $k = 1, \dots, d - 1$  and that  $m > -k$  is an integer. Then, for an integrable function  $f$  on  $S^{d-1}$ , the action of the  $m$ -weighted spherical projection operator  $\pi_{L,m}$  is

$$(\pi_{L,m}f)(u) = \int_{H^{d-k}(L,u)} f(v)\langle u, v \rangle^{k+m-1} dv, \quad u \in S^{k-1}(L).$$



Here,  $H^{d-k}(L, u) = \{v \in S^{d-1} \setminus L^\perp : \text{pr}_L v = u\}$  is the relatively open  $(d - k)$ -dimensional half sphere generated by  $L^\perp$  and  $u \in S^{k-1}(L)$  and  $\text{pr}_L v = v|L/\|v|L\|$ . The adjoint  $\pi_{L,m}^*$  is called the  $m$ -weighted spherical lifting operator. For an integrable function  $h$  on  $S^{k-1}(L)$  and  $u \in S^{d-1} \setminus L^\perp$ , we have

$$(\pi_{L,m}^* h)(u) = \|u|L\|^m h(\text{pr}_L u).$$

It should be noted that if  $K$  is a convex body in  $L \in G(d, k)$ , then  $\pi_{L,1}^*$  lifts the support function of  $K$  in  $L$  to the support function of  $K$  considered as a convex body in  $\mathbb{R}^d$ . The operators  $\pi_{L,m}$  and  $\pi_{L,m}^*$  are adjoint in the sense that

$$\int_{S^{d-1}} f(u)(\pi_{L,m}^* h)(u) \, du = \int_{S^{k-1}(L)} (\pi_{L,m} f)(v)h(v) \, dv$$

(see [17, (5.5)]). For  $1 \leq k \leq d - 1$  and  $-k < m, j < \infty$ , the mean lifted projection operator  $\pi_{m,j}^{(k)}$  acts on integrable functions  $f$  on  $S^{d-1}$  by

$$(\pi_{m,j}^{(k)} f)(u) = \int_{G(d,k)} (\pi_{L,m}^* \pi_{L,j} f)(u) \, dL, \quad u \in S^{d-1}.$$

As explained in [17], this operator also acts on measures, but, for our purposes, it suffices to consider its action on  $L^2(S^{d-1})$ . The operator intertwines the action of the rotation group  $SO(d)$  and therefore acts as a multiple of the identity on the space of spherical harmonics of a fixed degree. The injectivity of  $\pi_{m,j}^{(k)}$  as an operator on  $L^2(S^{d-1})$  then amounts to the question as to whether any of these multiples is zero. As observed in [17, page 41], it follows from the work of Kiderlen [25, Satz 3.20] or [26, page 517] that, for  $k = 2, \dots, d - 1$ ,  $\pi_{1,1-k}^{(k)}$  is injective on  $L^2(S^{d-1})$ . This observation will be important in our proof of Theorem 3.4.

It will be helpful to study the action of certain  $m$ -weighted spherical projections on spherical harmonics. For background information on spherical harmonics, we refer the reader to the book of Groemer [23]. The spaces  $H_n^d$  of  $d$ -dimensional spherical harmonics of fixed degree  $n = 0, 1, \dots$  comprise the invariant irreducible subspaces of  $L^2(S^{d-1})$  under the action of the rotation group  $SO(d)$ . Furthermore, each  $H_n^d$  is finite dimensional and is spanned by functions of the form  $P_n^d(\langle u, \cdot \rangle)$  with  $u \in S^{d-1}$ , where  $P_n^d$  is the Legendre polynomial of degree  $n$  in dimension  $d$ . We put  $N(d, n) = \dim H_n^d$ . For a fixed  $u_0 \in S^{d-1}$ , the restrictions of the functions in  $H_n^d$  to  $S^{d-2}(u_0^\perp)$  do not form an irreducible subspace of  $L^2(S^{d-2}(u_0^\perp))$  under the action of  $SO(d - 1)$  in  $u_0^\perp$ . However, there is a convenient way to represent the functions of  $H_n^d$  in terms of spherical harmonics on  $S^{d-2}(u_0^\perp)$ . This representation makes use of the associated Legendre functions  $A_{n,j}^{(d)}$  for  $j = 0, \dots, n$  and  $n = 0, 1, \dots$ .

These functions are defined by

$$A_{n,j}^{(d)}(t) = (1 - t^2)^{j/2} P_{n-j}^{d+2j}(t), \quad -1 \leq t \leq 1.$$

For  $u \in S^{d-1}$  we write  $u = tu_0 + \sqrt{1 - t^2}y$  for some  $y \in S^{d-2}(u_0^\perp)$ . Then if  $g$  is a spherical harmonic of degree  $j \leq n$  on  $S^{d-2}(u_0^\perp)$ , the function  $f$  on  $S^{d-1}$  defined by

$$f(u) = A_{n,j}^{(d)}(t)g(y)$$

is in  $H_n^d$ . In fact (see [23, Lemma 3.5.3] for example),  $H_n^d$  is spanned by the functions of the form

$$f(u) = A_{n,j}^{(d)}(t)S_{j,q}^{(d-1)}(y), \quad 0 \leq j \leq n, \quad 1 \leq q \leq N(d-1, j),$$

where the  $S_{j,q}^{(d-1)}$ , for  $q = 1, \dots, N(d-1, j)$ , form a basis of the space of spherical harmonics of degree  $j$  on  $S^{d-2}(u_0^\perp)$ . Restricting the degree  $n$  harmonic  $f$  to  $S^{d-2}(u_0^\perp)$  shows that the spherical harmonics  $S_{j,q}^{(d-1)}$  involve only those for which  $j$  has the same parity as  $n$ . The next lemma describes the action of  $\pi_{L,m}$  on the above harmonic  $f \in H_n^d$ , for  $L = u_0^\perp \in G(d, d-1)$  and an integer  $-d+1 < m \leq 1$ .

**Lemma 2.2.** *For  $u_0 \in S^{d-1}$ ,  $L = u_0^\perp \in G(d, d-1)$ , a spherical harmonic  $S_j^{(d-1)}$  of degree  $j = 0, 1, \dots$  on  $S^{d-2}(u_0^\perp)$  and an integer  $n$  of the same parity as  $j$  with  $n \geq j$ , we define  $f$  on  $S^{d-1}$  by*

$$(9) \quad f(u) = (1 - t^2)^{j/2} P_{n-j}^{d+2j}(t) S_j^{(d-1)}(y)$$

in case  $u = tu_0 + \sqrt{1 - t^2}y$  for  $y \in S^{d-2}(u_0^\perp)$ . Then, for an integer  $m$  with  $-d+1 < m \leq 1$ , we have

$$\pi_{L,m}f = \beta_{d,n,j,m} S_j^{(d-1)}$$

where, for  $j \neq m$ ,

$$\beta_{d,n,j,m} = \frac{\Gamma(\frac{n-j+1}{2})\Gamma(\frac{d+2j-1}{2})\Gamma(\frac{d+m+j-1}{2})\Gamma(\frac{n-m}{2})}{\Gamma(\frac{d+n+j-1}{2})\Gamma(\frac{d+m+n}{2})\Gamma(\frac{j-m}{2})}$$

and, noting that  $j = m$  implies either  $j = m = 0$  or  $j = m = 1$ ,

$$\beta_{d,0,0,0} = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})} \quad \text{and} \quad \beta_{d,n,0,0} = 0 \text{ if } n \neq 0,$$

$$\beta_{d,1,1,1} = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d+2}{2})} \quad \text{and} \quad \beta_{d,n,1,1} = 0 \text{ if } n \neq 1.$$

*Proof.* For the function  $f$  defined by (9) and for  $L = u_0^\perp$ , we have

$$\begin{aligned} \pi_{L,m}f(y) &= \int_{H^1(L,y)} \langle v, y \rangle^{d+m-2} f(v) dv \\ &= S_j^{(d-1)}(y) \int_{-1}^1 (1-t^2)^{(d+m+j-3)/2} P_{n-j}^{d+2j}(t) dt, \end{aligned}$$

for  $y \in S^{d-2}(L)$ . Thus

$$\pi_{L,m}f(y) = \beta_{d,n,j,m} S_j^{(d-1)}(y)$$

where

$$\beta_{d,n,j,m} = \int_{-1}^1 (1-t^2)^{(d+m+j-3)/2} P_{n-j}^{d+2j}(t) dt.$$

Integrals of this form were evaluated in [9, see equation (5.14)]. There, we used Gegenbauer polynomials  $C_n^\nu$  rather than Legendre polynomials  $P_n^d$ ; however, they are related by the simple relationship

$$P_n^d = \frac{\Gamma(d-2)\Gamma(n+1)}{\Gamma(d+n-2)} C_n^{(d-2)/2}$$

(see [9, equation (5.4)] for example).

Using the notation of [9, equation (5.14)], this gives

$$\pi_{L,m}f(y) = \beta_{d,n,j,m} S_j^{(d-1)}(y)$$

where

$$\beta_{d,n,j,m} = \frac{\Gamma(d+2j-2)\Gamma(n-j+1)}{\Gamma(d+n+j-2)} I(d+2m-2, n-m+1, n-j).$$

It follows from [9, equation (5.15)] that, in case  $n \neq m$ ,

$$\begin{aligned} \beta_{d,n,j,m} &= \left(\frac{-1}{2}\right)^{(n-j)/2} P_{n-j}^{d+2j}(0) \frac{\Gamma(\frac{1}{2})\Gamma(\frac{d+m+j-1}{2})}{\Gamma(\frac{d+m+n}{2})} \\ &\quad \times (n-m-2)(n-m-4)\cdots(j-m). \end{aligned}$$

Inserting the known value of  $P_{n-j}^{d+2j}(0)$  (see [23, Lemma 3.3.8] for example) gives the required result. For  $n = 0$  or  $1$ , and  $m = n$ , we have  $j = n$ . Direct calculations then yield

$$\beta_{d,0,0,0} = \int_{-1}^1 (1-t^2)^{(d-3)/2} dt = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})}$$

and

$$\beta_{d,1,1,1} = \int_{-1}^1 (1-t^2)^{(d-1)/2} dt = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d+2}{2})},$$

and so the proof is complete.

q.e.d.

In the case that  $f$  is a linear function, it is easy to calculate the  $m$ -weighted projection into arbitrary dimensions.

**Lemma 2.3.** *In case  $f = \langle v, \cdot \rangle$ , for some  $v \in S^{d-1}$ ,  $L \in G(d, k)$  for some  $k = 1, \dots, d - 1$  and for an integer  $m > -k$ , we have*

$$(\pi_{L,m}\langle v, \cdot \rangle)(y) = \gamma_{d,k,m}\langle v, y \rangle = \gamma_{d,k,m}\|v\|L\|\langle pr_L v, y \rangle$$

for all  $y \in S^{k-1}(L)$ , where

$$\gamma_{d,k,m} = \pi^{(d-k)/2} \frac{\Gamma(\frac{k+m+1}{2})}{\Gamma(\frac{d+m+1}{2})}.$$

*Proof.* The proof will be by induction on the dimension  $k$  starting with the case  $k = d - 1$ . For  $k = d - 1$  and  $m > -d + 1$ , we put  $L = u_0^\perp$  and note that, for  $y \in S^{d-2}(L)$ ,

$$\begin{aligned} (\pi_{L,m}\langle v, \cdot \rangle)(y) &= \int_{H^1(L,y)} \langle v, w \rangle \langle y, w \rangle^{d+m-2} dw \\ &= \int_{-1}^1 \left( t\langle v, u_0 \rangle + \sqrt{1-t^2}\langle v, y \rangle \right) (1-t^2)^{(d+m-3)/2} dt \\ &= \beta_{d,1,1,m}\langle v, y \rangle = \gamma_{d,d-1,m}\langle v, y \rangle. \end{aligned}$$

We now assume the result is true for projections into dimension  $1 < k \leq d - 1$  and consider the  $m$ -weighted spherical projection into  $L \in G(d, k - 1)$  for an integer  $m > -k + 1$ . We choose  $M \in G(d, k)$  with  $L \subset M$  and use [17, Lemma 5.4]. This, together with the induction hypothesis, implies

$$\begin{aligned} (\pi_{L,m}\langle v, \cdot \rangle)(y) &= (\pi_{L,m}^M(\pi_{M,m}\langle v, \cdot \rangle))(y) \\ &= \gamma_{d,k,m}\|v\|M\|(\pi_{L,m}^M\langle pr_M v, \cdot \rangle)(y) \\ &= \gamma_{d,k,m}\gamma_{k,k-1,m}\langle v, y \rangle, \end{aligned}$$

which gives the required result.

q.e.d.

We have seen that the projection of a linear function is just a multiple of the restriction of that function. However, Lemma 2.2 shows that, even in the case that the harmonic expansion of  $f$  on  $S^{d-1}$  has no linear harmonic, it may well be that  $\pi_{L,m}f$  has a linear harmonic on  $S^{d-1} \cap L$ . For some of our results, we will be particularly interested in this linear harmonic. So, for  $f \in L^2(S^{d-1})$ ,  $L \in G(d, k)$ , and  $m > -k$ , we denote by  $f_{L,m}^1$  the linear component of the harmonic expansion of  $\pi_{L,m}f$  in  $S^{k-1}(L)$ .

**Lemma 2.4.** *Assume  $f \in L^2(S^{d-1})$  has no linear component, that is,*

$$\int_{S^{d-1}} \langle u, v \rangle f(v) dv = 0 \quad \text{for all } u \in S^{d-1}.$$

Then, for  $1 \leq k \leq d - 1$  and  $m > -k$ , we have

$$\int_{G(d,k)} (\pi_{L,1}^* f_{L,m}^1)(u) dL = 0$$

for all  $u \in S^{d-1}$ .

*Proof.* We know that  $\pi_{1,m}^{(k)}$  is an intertwining operator and so, since  $f$  has no linear harmonic,

$$\int_{S^{d-1}} \langle u, v \rangle (\pi_{1,m}^{(k)} f)(v) dv = 0$$

for all  $u \in S^{d-1}$ . Consequently, Fubini's Theorem implies

$$\begin{aligned} & \int_{S^{d-1}} \langle u, v \rangle \int_{G(d,k)} (\pi_{L,1}^* \pi_{L,m} f)(v) dL dv \\ &= \int_{G(d,k)} \int_{S^{k-1}(L)} (\pi_{L,1} \langle u, \cdot \rangle)(w) (\pi_{L,m} f)(w) dw dL \\ &= \gamma_{d,k,1} \int_{G(d,k)} \|u\|L \int_{S^{k-1}(L)} \langle \text{pr}_L u, w \rangle (\pi_{L,m} f)(w) dw dL \\ &= \gamma_{d,k,1} \int_{G(d,k)} \|u\|L \int_{S^{k-1}(L)} \langle \text{pr}_L u, w \rangle f_{L,m}^1(w) dw dL. \end{aligned}$$

It is a consequence of the Funk-Hecke Theorem (see [23, Theorem 3.4.1] for example), that the inner integral above is  $\kappa_k f_{L,m}^1(\text{pr}_L u)$ , and so

$$0 = \int_{S^{d-1}} \langle u, v \rangle (\pi_{1,m}^{(k)} f)(v) dv = \kappa_d \int_{G(d,k)} \|u\|L f_{L,m}^1(\text{pr}_L u) dL,$$

which gives the required result.

q.e.d.

### 3. Mean section bodies and Fourier transforms

Many of the recent applications of Fourier transform techniques to the geometry of convex bodies originate with the work of Koldobsky. The reader is referred to his book [29] for background information. The early applications concerned centrally symmetric bodies and, therefore, Fourier transforms of even functions (or distributions). More recently, these techniques have been seen to be applicable to arbitrary convex bodies; see [22], for example. We will use the notation from [22] for our current applications. For a function  $f \in C^\infty(S^{d-1})$  and  $p \in \mathbb{Z}$ , we denote by  $f_p$  the homogeneous degree  $-d + p$  extension of  $f$  to  $\mathbb{R}^d \setminus \{o\}$ . The distributional Fourier transform of this is denoted by  $\hat{f}_p$ . It was noted in [22] that, for  $0 < p < d$ , the restriction of  $\hat{f}_p$  to  $S^{d-1}$  is again a smooth function. It is also easy to see that the mapping  $f \mapsto \hat{f}_p|_{S^{d-1}}$ , which we will denote by  $I_p$ , intertwines the group action of  $SO(d)$ . An application of Schur's Lemma then shows that this operator acts as a

multiple of the identity on the spaces  $H_n^d$ ,  $n = 0, 1, \dots$ , of spherical harmonics. It should, however, be noted that for even  $n$  this multiple is real, whereas for odd  $n$ , it is purely imaginary. In fact, if we denote the multiples by  $\lambda_n(d, p)$ , we have, for  $0 < p < d$ ,

$$(10) \quad \lambda_n(d, p) = \pi^{d/2} 2^p (-1)^{n/2} \frac{\Gamma(\frac{n+p}{2})}{\Gamma(\frac{n+d-p}{2})}, \quad n = 0, 1, \dots$$

(see [22], for example). Our application will make use of a composition of two of these mappings for different values of  $p$ . Thus, we will have a mapping  $I_p I_q : C^\infty(S^{d-1}) \rightarrow C^\infty(S^{d-1})$ , say, which can be defined in terms of its action, by multiplication, on the spaces  $H_n^d \subset C^\infty(S^{d-1})$ . Now, the multipliers are real for both the even and odd harmonics. We note that, for a function  $f \in L^2(S^{d-1})$  with spherical harmonic expansion  $\sum_{n=0}^\infty f_n$ , we have  $f \in C^\infty(S^{d-1})$  if and only if  $\sum_{n=0}^\infty n^m \|f_n\|_2^2$  converges for all natural numbers  $m$ . Combining this observation with (10) makes it clear that we have  $I_p I_q : C^\infty(S^{d-1}) \rightarrow C^\infty(S^{d-1})$  for  $0 < p, q < d$ .

The operators  $I_p$  can actually be extended to many values of  $p$  beyond the integers in the interval  $(0, d)$ ; in fact they exist for any  $p \in \mathbb{C}$  for which the gamma functions in (10) can be defined by analytic continuation. In this context, we will be interested in the operator  $I_{-1}$ . We will denote by  $C_0^\infty(S^{d-1})$  the space of *centred* functions in  $C^\infty(S^{d-1})$ ; these are the functions  $f$  for which

$$\int_{S^{d-1}} \langle u, v \rangle f(v) dv = 0 \quad \text{for all } u \in S^{d-1}.$$

Equivalently, they are those functions in  $C^\infty(S^{d-1})$  whose spherical harmonic expansion has no linear part. It is clear from (10) that  $I_{-1}$  can be defined on the spaces  $H_n^d$  for  $n \neq 1$ . To be specific, we will study the operator  $I_{-1} I_{k-1} : C_0^\infty(S^{d-1}) \rightarrow C_0^\infty(S^{d-1})$ , for  $k = 2, \dots, d$ , which acts on the spaces of non-linear harmonics by the product of the multiples given in (10).

For a subspace  $L \in G(d, q)$ , we will denote by  $I_p^L$ ,  $p = -1, 0, \dots, q-1$ , the corresponding operators, on  $C_0^\infty(S^{k-1}(L))$ , defined in the subspace. We note, in particular, that these operators incorporate the standard Fourier transform in  $L$ .

In order to relate these operators to mean section bodies, we will find it helpful to investigate their interaction with weighted spherical projections.

**Lemma 3.1.** *Let  $f \in C_0^\infty(S^{d-1})$ ,  $k = 3, \dots, d$ , and  $L \in G(d, d-1)$ . Then*

$$(11) \quad \pi_{L, -d+k-1} I_{-1} I_{k-1} f - 2\pi I_{-1}^L I_{k-2}^L \pi_{L, 1} f \quad \text{is linear on } S^{d-2}(L).$$

*Proof.* First we note that the statement, (11), makes sense. This follows from the fact that, since each  $I_p$  is an intertwining operator,  $I_p f$  has no linear harmonic, and because  $\pi_{L,1} f$  has no linear harmonic on  $S^{d-2}(L)$ . To see the latter, note that, for  $w \in L$  and  $f \in C_0^\infty(S^{d-1})$ , we have

$$\begin{aligned} \int_{S^{d-1} \cap L} \langle w, y \rangle \pi_{L,1} f(y) dy &= \int_{S^{d-1}} (\pi_{L,1}^* \langle w, \cdot \rangle)(u) f(u) du \\ &= \int_{S^{d-1}} \langle w, u \rangle f(u) du = 0. \end{aligned}$$

It will suffice to prove (11) for all spherical harmonics  $f$  of degree  $n \neq 1$ , and therefore for functions  $f$  of the form

$$f(u) = A_{n,j}^{(d)}(t) S_{j,q}^{(d-1)}(y), \quad 0 \leq j \leq n, \quad 1 \leq q \leq N(d-1, j),$$

where  $u = tu_0 + \sqrt{1-t^2}y$ ,  $y \in S^{d-2}(L)$ ,  $L = u_0^\perp$ , and  $n \neq 1$ . For this  $f$ , we have seen, in Lemma 2.2, that

$$\pi_{L,-d+k-1} f(y) = \beta_{d,n,j,-d+k-1} S_{j,q}^{(d-1)}(y)$$

and

$$\pi_{L,1} f(y) = \beta_{d,n,j,1} S_{j,q}^{(d-1)}(y).$$

So (11) will be proved, if we can show that

$$\begin{aligned} (12) \quad \lambda_n(d, -1) \lambda_n(d, k-1) \beta_{d,n,j,-d+k-1} \\ = 2\pi \lambda_j(d-1, -1) \lambda_j(d-1, k-2) \beta_{d,n,j,1} \end{aligned}$$

for all  $n \neq 1$  and all  $j \neq 1$  with  $n-j \geq 0$  and even. This follows from Lemma 2.2 and (10). q.e.d.

**Corollary 3.2.** *Let  $f \in C_0^\infty(S^{d-1})$ ,  $k = 3, \dots, d$ , and  $M \in G(d, d-q)$  for some  $q = 1, \dots, k-2$ . Then*

$$\pi_{M,-d-1+k} I_{-1} I_{k-1} f - (2\pi)^q I_{-1}^M I_{-q-1+k}^M \pi_{M,1} f$$

*is linear on  $S^{d-q-1}(M)$ .*

*Proof.* The proof will be by induction on  $q$ ; the case  $q = 1$  is proved in Lemma 3.1. We assume the result is true for some  $1 \leq q < k-2$  and establish it for  $q+1$ . To this end, let  $M \in G(d, d-q-1)$  and choose  $L \in G(d, d-q)$  with  $M \subset L$ . For  $f \in C_0^\infty(S^{d-1})$ , we use [17, Lemma 5.4], Lemma 2.3, and the induction hypothesis to see that

$$\begin{aligned} \pi_{M,-d-1+k} I_{-1} I_{k-1} f &= \pi_{M,-d-1+k}^L \pi_{L,-d-1+k} I_{-1} I_{k-1} f \\ &= (2\pi)^q \pi_{M,-d-1+k}^L I_{-1}^L I_{-q-1+k}^L \pi_{L,1} f, \end{aligned}$$

up to a linear function on  $S^{d-q-2}(M)$ . Using Lemma 3.1 and another application of [17, Lemma 5.4], we therefore have

$$\begin{aligned} \pi_{M,-d-1+k} I_{-1} I_{k-1} f &= (2\pi)^{q+1} I_{-1}^M I_{-q-2+k}^M \pi_{M,1}^L \pi_{L,1} f \\ &= (2\pi)^{q+1} I_{-1}^M I_{-q-2+k}^M \pi_{M,1} f, \end{aligned}$$

up to a linear function on  $S^{d-q-2}(M)$ . This completes the proof. q.e.d.

We also wish to see the relationship between mean section bodies of lower dimensional bodies and the weighted spherical projections.

**Lemma 3.3.** *Let  $C$  be a convex body in  $\mathbb{R}^d$  with  $\text{aff} C = L \in G(d, q)$  for some  $q = 1, \dots, d$ . Then, for  $k = d - q, \dots, d$  and  $u \in S^{d-1}$ , we have*

$$h(M_k(C), u) = \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{k-d+q+1}{2})\Gamma(\frac{d+1}{2})} [\pi_{L,1}^* h(M_{k-d+q}^L(C), \cdot)](u).$$

*Proof.* It follows from (5) that

$$\begin{aligned} h(M_k(C), u) &= \int_{A(d,k)} h(C \cap E, u) \mu_k(dE) \\ &= a_{d,k,q} \int_{A(L,k-d+q)} h(C \cap F, u) \mu_{k-d+q}^L(dF) \\ &= a_{d,k,q} [\pi_{L,1}^* h(M_{k-d+q}^L(C), \cdot)](u). \end{aligned}$$

The required result is now a consequence of (6). q.e.d.

Now we turn to the main result of this section.

**Theorem 3.4.** *For  $k = 2, \dots, d$  and a convex body  $K$  in  $\mathbb{R}^d$  with  $\dim K \geq d + 2 - k$ , we have*

$$(13) \quad h^*(M_k(K), \cdot) = m_{d,k} I_{-1} I_{k-1} S_{d+1-k}(-K, \cdot)$$

as distributions on  $S^{d-1}$ , where

$$m_{d,k} = -\frac{1}{2^k \pi^{(d+k)/2}} \frac{k-1}{d+1-k} \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{k}{2})}.$$

*Proof.* It will suffice to prove the result for bodies  $K$  that are sufficiently smooth that  $S_{d+1-k}(K, \cdot)$  has a density function in  $C_0^\infty(S^{d-1})$ . To see this, we first recall that the mapping  $K \mapsto M_k^*(K)$  is continuous in the Hausdorff metric on convex bodies; see the end of the proof of [21, Theorem 2.1]. Consequently, if (13) is proved for smooth bodies  $K$  then an easy approximation argument shows that, for general  $K$ ,

$$\begin{aligned} (14) \quad \int_{S^{d-1}} f(u) h^*(M_k(K), u) du \\ = m_{d,k} \int_{S^{d-1}} (I_{-1} I_{k-1} f)(u) S_{d+1-k}(-K, du) \end{aligned}$$



for all  $f \in C_0^\infty(S^{d-1})$ , as required. Furthermore, our earlier calculation of the mean section bodies of balls implies that it suffices to prove (13) with an arbitrary constant  $m_{d,k}$ .

As noted in (2), for the case  $k = 2$ , we have

$$h^*(M_2(K), u) = \frac{1}{2\pi(d-1)} \int_{S^{d-1}} \alpha(x, u) \sin \alpha(x, u) S_{d-1}(-K, dx).$$

The mapping  $f \mapsto \int_{S^{d-1}} \alpha(x, \cdot) \sin \alpha(x, \cdot) f(x) dx$  intertwines the action of  $SO(d)$  on  $C_0^\infty(S^{d-1})$ . It therefore acts as a multiple of the identity on the spaces  $H_n^d$  of spherical harmonics. It is shown in [20, Equation (5.1)] that the multiplier associated with the space  $H_n^d$  is of the form

$$(-1)^{n+1} c_d \frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+d+1}{2})\Gamma(\frac{n+d-1}{2})},$$

for  $n \neq 1$ . Comparison with (10) shows that this is a positive multiple (independent of  $n$ ) of  $-\lambda_n(d, -1)\lambda_n(d, 1)$ . It follows that (13) is proved for  $k = 2$ , in all dimensions  $d \geq 3$ .

We will now show how the proof of (13), for arbitrary  $k$ , can be reduced to the case  $k = 2$ . We let  $C$  be a convex body with  $\dim C = d + 2 - k$  and put  $L = \text{aff } C$ . Then, using [17, Theorem 6.2], we have, for an arbitrary convex body  $K$ ,

$$\begin{aligned} \int_{S^{d-1}} h(M_k(K), u) S_{d+1-k}(C, du) &= c_{d,k} \int_{S^{d-1}} h(M_k(K), u) \pi_{L, -d-1+k}^* S_{d+1-k}^L(C, \cdot)(du) \\ &= c_{d,k} \int_{S^{d+1-k}(L)} \pi_{L, -d-1+k} h(M_k(K), \cdot)(u) S_{d+1-k}^L(C, du). \end{aligned}$$

Also, by (4) and Lemma 3.3,

$$\begin{aligned} \int_{S^{d-1}} h(M_k(K), u) S_{d+1-k}(C, du) &= \int_{S^{d-1}} h(M_k(C), u) S_{d+1-k}(K, du) \\ &= c_{d,k} \int_{S^{d-1}} \pi_{L, 1}^* h(M_2^L(C), \cdot)(u) S_{d+1-k}(K, du) \\ &= c_{d,k} \int_{S^{d+1-k}(L)} h(M_2^L(C), u) (\pi_{L, 1} S_{d+1-k}(K, \cdot))(du). \end{aligned}$$

It then follows from the case  $k = 2$  and from the self-adjointness of the operators  $I_p$ , that

$$\begin{aligned} & \int_{S^{d-1}} h(M_k(K), u) S_{d+1-k}(C, du) \\ &= c_{d,k} \int_{S^{d+1-k}(L)} (I_{-1}^L I_1^L S_{d+1-k}^L(-C, \cdot))(u) (\pi_{L,1} S_{d+1-k}(K, \cdot))(du) \\ &= c_{d,k} \int_{S^{d+1-k}(L)} (I_{-1}^L I_1^L \pi_{L,1} S_{d+1-k}(K, \cdot))(u) S_{d+1-k}^L(-C, du) \\ &= c_{d,k} \int_{S^{d+1-k}(L)} (I_{-1}^L I_1^L \pi_{L,1} S_{d+1-k}(-K, \cdot))(u) S_{d+1-k}^L(C, du). \end{aligned}$$

These results are true for all convex bodies  $C \subset L$ , for which  $M_2^L(C)$  has Steiner point at the origin, and so (possibly up to a linear function)

$$\pi_{L,-d-1+k} h(M_k(K), \cdot) = c_{d,k} I_{-1}^L I_1^L \pi_{L,1} S_{d+1-k}(-K, \cdot)$$

almost everywhere in  $S^{d+1-k}(L)$ . However, each side of the equation is a continuous function and so the result holds throughout  $S^{d+1-k}(L)$ . Using Corollary 3.2, we therefore have

$$\pi_{L,-d-1+k} h(M_k(K), \cdot) = c_{d,k} \pi_{L,-d-1+k} I_{-1} I_{k-1} S_{d+1-k}(-K, \cdot),$$

again up to a linear function.

We put  $f = h^*(M_k(K), \cdot)$ ,  $g = c_{d,k} I_{-1} I_{k-1} S_{d+1-k}(-K, \cdot)$  and note that  $f$  and  $g$  have no linear harmonic on  $S^{d-1}$ . In this notation, we have proved that, for each  $L \in G(d, d + 2 - k)$ ,  $\pi_{L,-d-1+k} f$  and  $\pi_{L,-d-1+k} g$  differ by a linear function. It follows that, for each  $L \in G(d, d + 2 - k)$ , we have

$$\pi_{L,-d-1+k} f - \pi_{L,-d-1+k} g = f_{L,-d-1+k}^1 - g_{L,-d-1+k}^1.$$

Using Lemma 2.4, we have  $\pi_{1,-d-1+k}^{(d+2-k)} f = \pi_{1,-d-1+k}^{(d+2-k)} g$ . It follows from [17, page 41] that  $\pi_{1,-d-1+k}^{(d+2-k)}$  is injective and so we have the desired result. q.e.d.

We recall from (8) that, if  $K$  is the  $(d + 1 - k)$ -ball,  $B^{d+1-k}$  say, in  $L \in G(d, d + 1 - k)$ , then

$$M_k(B^{d+1-k}) = \binom{d}{k}^{-1} \frac{\kappa_k \kappa_{d-k}}{2\kappa_d} B^{d+1-k}.$$

If we denote by  $\sigma_L$  the spherical Lebesgue measure on  $S^{d-k}(L)$ , then equation (13) states

$$\|\cdot\|_L = c I_{-1} I_{k-1} \sigma_{L^\perp},$$

as functions (or distributions) on  $S^{d-1}$ . However, it is easy to check that  $\|\cdot\|_L = c I_{-1} \sigma_L$ . Thus the case  $\dim K = d + 1 - k$  of Theorem 3.4 can

be seen as equivalent to Koldobsky's [29, Lemma 3.25] orthogonality result, which can be stated as

$$I_{k-1}\sigma_{L^\perp} = c\sigma_L.$$

It was shown in [21] that bodies  $K$  with  $\dim K \geq d+2-k$  are uniquely determined by their  $k$ -th mean section body. Combining Theorem 3.4 with standard inversion formulas for Fourier transforms, we obtain a representation of the surface area measure  $S_{d+1-k}(K, \cdot)$  in terms of the first surface area measure of (a reflection of)  $M_k(K)$ . Of course, this result again gives the unique determination of  $K$  by  $M_k(K)$  (up to translation) in case  $\dim K \geq d-k+2$ , since it shows that  $S_{d+1-k}(K, \cdot)$  is determined by  $M_k(K)$ .

**Corollary 3.5.** *For  $k = 2, 3, \dots, d$  and a convex body  $K$  for which  $\dim K \geq d+2-k$ , we have*

$$S_{d+1-k}(K, \cdot) = -\frac{d-1}{(2\pi)^{2d}m_{d,k}}I_{d-1}I_{d+1-k}S_1(-M_k(K), \cdot),$$

as distributions on  $S^{d-1}$ .

*Proof.* It is well known that inversion of the Fourier transform is obtained by a further application of the Fourier transform. In the context of our homogeneous distributions  $f_p$  this amounts to the assertion that, for  $1 \leq p \leq d-1$ ,

$$(15) \quad I_{d-p}I_p = (2\pi)^d I^*,$$

where  $(I^*f)(u) = f(-u)$ . This result is clearly seen from (10).

We will denote by  $\square$  the differential operator which satisfies the distributional equation  $\square h(K, \cdot) = S_1(K, \cdot)$  for all convex bodies  $K$ ; see [3, Theorem 5.1], where the notation  $D_q^*$  is used. The eigenspaces of  $\square$  are the spaces  $H_n^d$  of spherical harmonics and the corresponding eigenvalues are

$$-(n-1)(n+d-1)/(d-1), \quad \text{for } n = 0, 1, \dots;$$

see [3, equation (18)], for example. It follows that  $\square : C_0^\infty(S^{d-1}) \rightarrow C_0^\infty(S^{d-1})$  is a bijection. Combining the value of the eigenvalues of  $\square$  with (10) shows that

$$\square I_{-1} = -\frac{1}{d-1}I_1.$$

Applying (14) to  $\square f$  gives

$$\int_{S^{d-1}} f(u) S_1(M_k(K), du) = -\frac{m_{d,k}}{d-1} \int_{S^{d-1}} (I_1 I_{k-1} f)(u) S_{d+1-k}(-K, du)$$

for all  $f \in C_0^\infty(S^{d-1})$ . Replacing the function  $f$  with  $I_{d-1}I_{d+1-k}f$  yields

$$\begin{aligned} \int_{S^{d-1}} (I_{d-1}I_{d+1-k}f)(u) S_1(M_k(K), du) \\ = -\frac{(2\pi)^{2d}m_{d,k}}{d-1} \int_{S^{d-1}} f(u) S_{d+1-k}(-K, du) \end{aligned}$$

for all  $f \in C_0^\infty(S^{d-1})$ . The self-adjointness of the  $I_p$  now gives

$$S_{d+1-k}(-K, \cdot) = -\frac{d-1}{(2\pi)^{2d}m_{d,k}} I_{d-1}I_{d+1-k}S_1(M_k(K), \cdot)$$

for all convex bodies  $K$ , as required. q.e.d.

We have already seen that mean section bodies of  $d$ -balls are again  $d$ -balls. Against this background we mention, here, a few other geometric connections between certain classes of bodies and their mean section bodies. We recall that, for  $j = 1, \dots, d-1$ , a convex body  $K$  is said to have constant  $j$ -girth if the  $j$ -th intrinsic volume  $V_j(K|u^\perp)$  of the projection of  $K$  onto the hyperplane  $u^\perp$  is constant as a function of  $u \in S^{d-1}$ ; see [14, Definition 3.3.10].

**Corollary 3.6.** *Let  $K$  be a convex body in  $\mathbb{R}^d$  with  $\dim K = d$  and let  $k = 2, 3, \dots, d$ . Then*

- (a)  $K$  is a ball if and only if  $M_k(K)$  is a ball,
- (b)  $K$  is centrally symmetric if and only if  $M_k(K)$  is centrally symmetric,
- (c)  $K$  is of constant  $(d+1-k)$ -girth if and only if  $M_k(K)$  is of constant width.

*Proof.* We first remark that these results (except (c)) are obvious in case  $k = d$  since then we have  $M_k(K) = K$ . For (c) and  $k = d$ , it is well known that constant width is the same as constant 1-girth; see, for example, [14, Theorem 3.3.13]. It is also easy to see that both (a) and (b) follow from the definition (1) because of the invariance of the measure  $\mu_k$  under the orthogonal rotations of  $O(d)$ . Here, however, we will give a unified proof of all three results based on spherical harmonic expansions and Corollary 3.5.

It follows from the injectivity of  $I_{d-1}I_{d+1-k}$  and Corollary 3.5 that the  $n$ -th spherical harmonic in the expansion of  $S_{d+1-k}(K, \cdot)$  is non-trivial if and only if the same is true of the  $n$ -th spherical harmonic of  $S_1(M_k(K), \cdot)$ . For (a), we note that balls are precisely those bodies for which one of the surface area measures has trivial  $n$ -th degree harmonics for all  $n \neq 0$ . For (b), we use the fact that a body is centrally symmetric precisely if one of its surface area measures has no non-trivial odd degree harmonics. The situation for (c) is a little more complicated. As explained in [14, Theorem 3.3.14],  $K$  has constant  $(d+1-k)$ -girth if and

only if the even part of the measure  $S_{d+1-k}(K, \cdot)$  is a multiple of spherical Lebesgue measure. In terms of spherical harmonics, this is the same as saying that the non-trivial  $n$ -th degree harmonics of  $S_{d+1-k}(K, \cdot)$  occur only for odd  $n$  and for  $n = 0$ . Consequently  $K$  has constant  $(d + 1 - k)$ -girth if and only if  $M_k(K)$  has constant 1-girth, which, as we have already observed, is equivalent to  $M_k(K)$  having constant width. q.e.d.

We remark that analogous results hold for lower dimensional bodies; these can be deduced from the above corollary together with Lemma 3.3.

Next, we will use Theorem 3.4, together with results of Kiderlen [28, Theorems 1.4 and 1.6], to obtain a stability version of the uniqueness result in [21]. A stability version for  $M_2$  bodies was first given by Hug and Schneider [24]. This was subsequently improved by Kiderlen [28]. These results were, in turn, based on Poisson integral techniques employed by Bourgain and Lindenstrauss [4, 5], Sobolev space techniques used by Campi [6], and stability estimates for the Aleksandrov-Fenchel inequalities established by Schneider [33]. Here, we will use Kiderlen's techniques to yield stability results for  $M_k$  bodies for all  $k = 2, \dots, d - 1$ .

As is usual for such results, ours will apply to bodies which are uniformly bounded from the inside and the outside by a ball. In order to quantify this, we denote by  $\mathcal{K}(R, r)$ ,  $0 < r < R$ , the convex bodies of  $\mathbb{R}^d$  which contain the ball  $rB^d$  and are contained in  $RB^d$ . We will make use of two metrics on the space of convex bodies, namely the Hausdorff metric  $\delta$ , and the  $L^2$ -metric  $\delta_2$ . We recall that each of these can be defined in terms of support functions, as follows:

$$\begin{aligned} \delta(K_1, K_2) &= \|h(K_1, \cdot) - h(K_2, \cdot)\|_\infty \\ &= \sup\{|h(K_1, u) - h(K_2, u)| : u \in S^{d-1}\}, \end{aligned}$$

$$\begin{aligned} \delta_2(K_1, K_2) &= \|h(K_1, \cdot) - h(K_2, \cdot)\|_2 \\ &= \left( \int_{S^{d-1}} |h(K_1, u) - h(K_2, u)|^2 du \right)^{1/2}. \end{aligned}$$

**Theorem 3.7.** *Assume  $K_i \in \mathcal{K}(R, r)$  for  $i = 1, 2$ . Then, for  $k = 2, \dots, d - 1$  and any  $\gamma > 0$ , there is a constant  $c = c(d, k, R, r, \gamma) > 0$  such that*

$$\delta(K_1, K_2) \leq c(\delta(M_k(K_1), M_k(K_2)))^{q_k - \gamma}$$

where

$$q_k = \begin{cases} \frac{2}{d(2d - 1)} & \text{for } k = 2 \\ 1 & \text{for } k \geq 3. \\ \frac{1}{2^{d-k-1}(d + 1)(2d + 3 - 2k)} & \end{cases}$$

*Proof.* We first note that the case  $k = 2$  is essentially Kiderlen’s result [28, page 2017].

For each convex body  $K \subset \mathbb{R}^d$  and each  $u \in S^{d-1}$ , we put  $F(K, u) = h^*(M_k(K), u)$ . It is then clear that, for each  $K$ ,  $F(K, \cdot) \in L^2(S^{d-1})$ . Furthermore, as observed in [21], the mapping  $K \mapsto F(K, \cdot)$  is continuous and intertwines the action of the rotation group  $SO(d)$ . Next, we put  $Q(K, \cdot) = S_{d+1-k}(K, \cdot)$ . Then it is an immediate consequence of Theorem 3.4 that, in the language of [28],  $Q(K, \cdot)$  is an analytic representation of  $K$  such that  $F(K, \cdot)$  depends additively on  $Q(K, \cdot)$ . In fact, Theorem 3.4 shows that if  $Q(K, \cdot)$  has spherical harmonic expansion  $Q(K, \cdot) \sim \sum_{n=0}^\infty Q_n(K, \cdot)$ , then  $F(K, \cdot)$  has spherical harmonic expansion

$$(16) \quad F(K, \cdot) \sim m_{d,k} \sum_{\substack{n=0 \\ n \neq 1}}^\infty (-1)^n \lambda_n(d, -1) \lambda_n(d, k-1) Q_n(K, \cdot).$$

Thus, in the notation of [28], we have, for  $i = 1, 2$ ,  $K_i^* \in \mathcal{K}_{\mathbf{a}, S_{d+1-k}}$  where  $\mathbf{a}$  is the sequence whose  $n$ -th entry is  $m_{d,k}(-1)^n \lambda_n(d, -1) \lambda_n(d, k-1)$  for  $n \neq 1$  and is zero otherwise.

It is a consequence of the Blaschke Selection Theorem that, for any  $R > r > 0$ , there is an  $r' > 0$  (dependent only on  $r, R$ , and the dimension  $d$ ) such that if  $K \in \mathcal{K}(R, r)$ , then  $r'B^d \subset K^*$ . Clearly, we also have  $K^* \subset 2RB^d$ . So, in fact, again in the notation of [28], we have  $K_i^* \in \mathcal{K}_{\mathbf{a}, S_{d+1-k}}(r', 2R)$  for  $i = 1, 2$ .

Moreover, it follows from Stirling’s formula and (10) that

$$|\lambda_n(d, p)| \simeq c_{d,p} n^{-(d-2p)/2}.$$

Thus, equation (16) shows that [28, (1.5) and (1.7)] are satisfied with  $\beta = d + 2 - k \geq 3$ . We note that  $M_k^*(K) = M_k^*(K^*)$  and so [28, Theorems 1.4 and 1.6] give

$$(17) \quad \delta(K_1^*, K_2^*) \leq c(\delta_2(M_k^*(K_1), M_k^*(K_2)))^{q_k-\gamma}.$$

The Steiner point of a body  $K$  yields the linear harmonic of its support function (see [23, Theorem 5.1.1] for example), and so (17) gives

$$\begin{aligned} \delta(K_1^*, K_2^*) &\leq c(\delta_2(M_k(K_1), M_k(K_2)))^{q_k-\gamma} \\ &\leq c(\delta(M_k(K_1), M_k(K_2)))^{q_k-\gamma}. \end{aligned}$$

Thus, putting  $\eta = \delta(M_k(K_1), M_k(K_2))$ , there is a translation vector  $t$  (from the Steiner point of  $K_1$  to that of  $K_2$ ) such that  $\delta(K_1 + t, K_2) \leq c\eta^{q_k-\gamma}$ . In particular

$$K_1 + t \subset K_2 + c\eta^{q_k-\gamma} B^d \subset (1 + c\eta^{q_k-\gamma}) K_2.$$

Consequently

$$\begin{aligned} M_k(K_1 + t) &\subset (1 + c\eta^{q_k - \gamma})^{d-k+1} M_k(K_2) \\ &\subset (1 + c\eta^{q_k - \gamma})^{d-k+1} (M_k(K_1) + \eta B^d). \end{aligned}$$

It therefore follows from (3) that

$$M_k(K_1) + c_{d,k} V_{d-k}(K_1)t \subset M_k(K_1) + c\eta^{q_k - \gamma} B^d.$$

Thus

$$c_{d,k} \langle t, u \rangle \leq c\eta^{q_k - \gamma} \quad \text{for all } u \in S^{d-1}.$$

So,  $\|t\| \leq c\eta^{q_k - \gamma}$  and therefore  $\delta(K_1, K_2) \leq c\eta^{q_k - \gamma}$ , as required. q.e.d.

The stability estimate can also be formulated in the spirit of the results in Hug and Schneider [24]. Namely, if  $K_1, K_2$  are convex bodies with inradius  $\geq r > 0$  and circumradius  $\leq R$ , then there is a translation  $t \in \mathbb{R}^d$  such that

$$\delta(K_1 + t, K_2) \leq c(\delta(M_k(K_1), M_k(K_2)))^{q_k - \gamma},$$

where the constants  $c, \gamma$ , and  $q_k$  are as in Theorem 3.7.

#### 4. Mean section bodies and Berg's functions

The general Minkowski problem for surface area measures asks, in each case  $i = 1, \dots, d - 1$ , for the characteristic properties of a measure  $\mu$  on  $S^{d-1}$  which guarantee that  $\mu = S_i(K, \cdot)$  for some convex body  $K$  in  $\mathbb{R}^d$ . The extreme cases  $i = 1, d - 1$  are often called the Christoffel and Minkowski problems, respectively. Minkowski showed that a measure  $\mu$  is of the form  $\mu = S_{d-1}(K, \cdot)$  for some convex body  $K$  with  $\dim K = d$  precisely when the centroid of  $\mu$  is the origin and the support of  $\mu$  does not lie in any great subsphere of  $S^{d-1}$ . We refer the reader to [34, Section 7.1] for more details and references. The case  $i = 1$  was discussed by Christoffel and eventually resolved independently by Firey [12] and Berg [3]; again we refer the reader to Schneider's book [34, Section 4.3], as well as [32] for details and references. We will analyze Berg's solution in more detail below. The intermediate cases,  $1 < i < d - 1$ , of the general Minkowski problem have proved rather intractable despite a significant amount of attention; see, for example, [1, 7, 8, 13, 32, 40]. Here we will show a connection between these problems and the characteristic properties of mean section bodies. In fact, we will show that the resolution of the general Minkowski problem is tantamount to finding characterizations of the mean section bodies.

The first step is to recall Berg's work [3] on the Christoffel problem. For each dimension  $d = 2, 3, \dots$  he introduced functions  $g_d$  on  $(-1, 1)$  such that, for each convex body  $K \subset \mathbb{R}^d$ ,

$$(18) \quad h^*(K, u) = \frac{\Gamma(\frac{d-1}{2})}{2\pi^{(d-1)/2}} \int_{S^{d-1}} g_d(\langle u, v \rangle) S_1(K, dv)$$

for all  $u \in S^{d-1}$ . He then showed the following result.

**Theorem 4.1.** (Berg, [3, Theorem 5.3]) *A measure  $\mu$  on  $S^{d-1}$ , with centroid at the origin, is the first surface area measure of a convex body if and only if*

$$\int_{S^{d-1}} g_d(\langle \cdot, v \rangle) \mu(dv)$$

*is the support function of a convex body.*

This provided a solution to the Christoffel problem since a function  $h$  on  $S^{d-1}$  is a support function if and only if its degree 1 homogeneous extension to  $\mathbb{R}^d$  is subadditive.

Berg defined his functions  $g_d$  iteratively, but gave explicit formulations for low dimensions. For example,

$$(19) \quad g_2(t) = \frac{1}{\pi} \left( (\pi - \cos^{-1} t) (1 - t^2)^{1/2} - \frac{1}{2}t \right),$$

$$(20) \quad g_3(t) = 1 + t \ln(1 - t) + \left( \frac{4}{3} - \ln 2 \right) t,$$

$$(21) \quad g_4(t) = \frac{3}{\pi} \left( (\pi - \cos^{-1} t) (1 - 2t^2)(1 - t^2)^{-1/2} + \frac{1}{2}t \right),$$

$$(22) \quad g_5(t) = 3(1 + t \ln(1 - t)) - (1 - t)^{-1} + \left( \frac{28}{5} - 3 \ln 2 \right) t.$$

Our intention, in this section, is to establish a relationship between the functions of Berg and mean section bodies. For this, we will make use of the integrability properties of the functions  $g_d$  and equation (18).

It is proved by Berg [3, Theorem 3.3 and proof of Corollary 3.9] that, for  $k \geq 2$ , the function  $g_k$  is in  $C^\infty([-1, 1))$  and that, for each  $k > 2$ , there is a number  $t_k < 1$  such that  $g_k$  decreases on  $(t_k, 1)$  with  $\lim_{t \rightarrow 1} g_k(t) = -\infty$ ; in case  $k = 2$ , we have  $g_2 \in C^\infty([-1, 1])$ . It follows that, for any  $u \in S^{d-1}$  and for any  $k$ , the function  $g_k(\langle u, \cdot \rangle)$ , defined on  $S^{d-1}$ , is measurable.

It is also shown, in Theorem 3.3 of [3], that, for each  $d = 2, 3, \dots$ ,

$$\int_{-1}^1 |g_d(t)|(1 - t^2)^{(d-3)/2} dt < \infty$$

and, in case  $d \geq 3$ ,

$$\int_{-1}^1 |g_d(t)|(1 - t^2)^{(d-4)/2} dt < \infty.$$

Consequently, for  $k = 2, \dots, d$ , we have

$$\int_{-1}^1 |g_k(t)|(1 - t^2)^{(d-3)/2} dt \leq \int_{-1}^1 |g_k(t)|(1 - t^2)^{(k-3)/2} dt < \infty.$$



It then follows from [3, Proposition 2.7] that, for any  $u \in S^{d-1}$ , the function  $g_k(\langle u, \cdot \rangle)$  is integrable on  $S^{d-1}$ . Using [3, Proposition 2.8], we deduce that  $\int_{S^{d-1}} g_k(\langle \cdot, v \rangle) \mu(dv)$  is defined almost everywhere and is integrable on  $S^{d-1}$ , for any (finite, signed) measure  $\mu$  on  $S^{d-1}$ . The next step is to prove that this  $L^1$  function is a multiple of  $I_{-1}I_{k-1}I^*\mu$ .

The following lemma will help establish this relationship.

**Lemma 4.2.** *For each  $d = 3, 4, \dots$ , and for any numbers  $\alpha, \beta \in [-1, 1)$ , we have*

$$\begin{aligned} \int_{-1}^1 g_d \left( \beta s + \alpha \sqrt{1 - \beta^2} \sqrt{1 - s^2} \right) (1 - s^2)^{(d-4)/2} ds \\ = c_d \sqrt{1 - \beta^2} g_{d-1}(\alpha) + \gamma(d, \beta) \alpha \end{aligned}$$

for some constant  $c_d$  and function  $\gamma$ .

*Proof.* For a convex body  $K \subset \mathbb{R}^d$  with  $\text{aff} K = L \in G(d, d - 1)$ , it follows from (18) and [17, Theorem 6.2] that

$$\begin{aligned} h^*(K, u) &= c_d \int_{S^{d-1}} g_d(\langle u, v \rangle) S_1(K, dv) \\ &= c_d \int_{S^{d-2}(L)} (\pi_{L,-1} g_d(\langle u, \cdot \rangle))(y) S_1^L(K, dy). \end{aligned}$$

On the other hand,

$$\begin{aligned} h^*(K, u) &= c_d \left( \pi_{L,1}^* \int_{S^{d-2}(L)} g_{d-1}(\langle \cdot, y \rangle) S_1^L(K, dy) \right) (u) \\ &= c_d \|u\| L \int_{S^{d-2}(L)} g_{d-1}(\langle \text{pr}_L u, y \rangle) S_1^L(K, dy). \end{aligned}$$

Combining these gives, for each  $L \in G(d, d - 1)$  and  $u \in S^{d-1}$ ,

$$(23) \quad \pi_{L,-1} g_d(\langle u, \cdot \rangle)(y) = c_d \|u\| L g_{d-1}(\langle \text{pr}_L u, y \rangle) + \langle w_{L,u}, y \rangle,$$

for almost all  $y \in S^{d-2}(L)$ , and for some vector  $w_{L,u} \in L$ . Each function of  $y$  in (23) is continuous at  $y \neq \text{pr}_L u$  and so the equation holds true for all  $y \neq \text{pr}_L u$ . Analogously, for fixed  $y \in S^{d-2}(L)$ , equation (23) is true for all  $u \in S^{d-1} \setminus H^1(L, y)$ . We recall that, for  $L = u_0^\perp \in G(d, d - 1)$ ,

$$\begin{aligned} \pi_{L,-1} g_d(\langle u, \cdot \rangle)(y) &= \int_{H^1(L,y)} g_d(\langle u, v \rangle) \langle y, v \rangle^{d-3} dv \\ &= \int_{-1}^1 g_d \left( \langle u, u_0 \rangle s + \sqrt{1 - \langle u, u_0 \rangle^2} \langle \text{pr}_L u, y \rangle \sqrt{1 - s^2} \right) (1 - s^2)^{(d-4)/2} ds. \end{aligned}$$

So, if  $y \in S^{d-2}(L)$  and  $u \in S^{d-1} \setminus H^1(L, y)$ ,

$$\int_{-1}^1 g_d \left( \langle u, u_0 \rangle s + \sqrt{1 - \langle u, u_0 \rangle^2} \langle \text{pr}_L u, y \rangle \sqrt{1 - s^2} \right) (1 - s^2)^{(d-4)/2} ds$$

$$(24) = c_d \|u\| L \|g_{d-1}(\langle \text{pr}_L u, y \rangle) + \langle w_{L,u}, y \rangle.$$

As functions of  $y \in S^{d-2}(L)$ , the left side and first term on the right side are constant if  $\langle \text{pr}_L u, y \rangle$  is constant. It follows that  $w_{L,u}$  must be a multiple of  $\text{pr}_L u$  and this multiple depends only on  $d$  and  $\langle u, u_0 \rangle$ .

If  $\alpha, \beta \in [-1, 1)$  and  $y \in S^{d-2}(L)$  are fixed, we can choose  $w \in S^{d-2}(L)$  such that  $\langle w, y \rangle = \alpha$  and put  $u = \beta w + \sqrt{1 - \beta^2} u_0 \in S^{d-1}$ . Then, since  $\text{pr}_L u = w \neq y$ , we get the required result from (24) applied to this  $u$  and  $y$ , since we have  $\alpha = \langle \text{pr}_L u, y \rangle$  and  $\beta = \langle u, u_0 \rangle$ . q.e.d.

We will now show that the action of the distribution  $I_{-1}I_{k-1}$  on Borel measures  $\mu$  with centroid at the origin is by integration. As was the case with the Fourier transform operators  $I_p$ , we will want to investigate the relationship between Berg's functions and the weighted spherical projections.

**Theorem 4.3.** *Assume that  $d \geq 2$  and that  $\mu$  is a Borel measure on  $S^{d-1}$  with centroid at the origin. Then, for  $k = 2, \dots, d$ , we have*

$$(25) \quad (I_{-1}I_{k-1}I^* \mu)(u) = b_{d,k} \int_{S^{d-1}} g_k(\langle u, v \rangle) \mu(dv)$$

for almost all  $u \in S^{d-1}$ , where

$$(26) \quad b_{d,k} = -\frac{\pi^{(d+1)/2} 2^{k-1}}{k-1} \frac{\Gamma(\frac{k-1}{2})}{\Gamma(\frac{d-k+2}{2})}.$$

*Proof.* The integrability properties of  $g_k$ , outlined above, show that the right hand side of (25) is defined for almost all  $u \in S^{d-1}$ . We will now establish the result by showing that both sides of the equation have the same spherical harmonic expansion; see, for example, [3, Corollary 2.13]. The Funk-Hecke Theorem, as quoted by Berg [3, Theorem 2.11], shows that the spherical harmonics are eigenfunctions of the mapping

$$f \mapsto \int_{S^{d-1}} g_k(\langle \cdot, v \rangle) f(v) dv, \quad \text{for } f \in C_0^\infty(S^{d-1}).$$

We will denote by  $\mu_n(d, k)$  the multiplier (eigenvalue) corresponding to the spherical harmonics of degree  $n$  in dimension  $d$ . Thus

$$(27) \quad \mu_n(d, k) = \int_{S^{d-1}} g_k(\langle u, v \rangle) P_n^d(\langle u, v \rangle) dv,$$

for any  $u \in S^{d-1}$ . The theorem will be proved by establishing, for  $k = 2, \dots, d$ , that

$$(28) \quad (-1)^n \lambda_n(d, -1) \lambda_n(d, k-1) = b_{d,k} \mu_n(d, k) \quad \text{for all } n \neq 1.$$

As pointed out by Berg, this result shows the equality between the two sides of the equation (25) when they are viewed as distributions. This is, of course, equivalent to the statement given in the theorem.

We will prove (28) by induction on the dimension  $d$ . First, we note that it follows from Berg's result (18), and the fact that  $M_d(K) = K$ , that (28) holds in case  $k = d$ , since

$$b_{d,d} = \frac{\Gamma(\frac{d-1}{2})}{2\pi^{(d-1)/2}m_{d,d}}.$$

It follows that (28) holds in dimension 2 since, there, we have  $k = d = 2$ .

We will now assume, inductively, that (28) is true in dimension  $d - 1$  and consider the case of dimension  $d$ . For a fixed  $L \in G(d, d - 1)$  with  $L = u_0^\perp$  and for  $n \neq 1$  we put

$$f_n(u) = A_{n,j}^{(d)}(t)S_j^{(d-1)}(y),$$

where  $u = tu_0 + \sqrt{1 - t^2}y$  with  $y = \text{pr}_L u \in S^{d-2}(L)$  and  $S_j^{(d-1)}$  is an arbitrary harmonic of degree  $j \neq 1$  on  $S^{d-2}(L)$ . Then, as we have seen,  $f_n$  is a harmonic of degree  $n$  on  $S^{d-1}$ .

Next, we note that, for  $a \in S^{d-2}(L)$  and  $u \in S^{d-1} \setminus H^1(L, a)$ , we have, by Lemma 4.2,

$$\begin{aligned} (\pi_{L,-d+k-1}g_k(\langle u, \cdot \rangle))(a) &= \int_{H^1(L,a)} g_k(\langle u, v \rangle)\langle a, v \rangle^{k-3} dv \\ &= \int_{-1}^1 g_k(\langle u, u_0 \rangle s + \langle \text{pr}_L u, a \rangle \sqrt{1 - \langle u, u_0 \rangle^2} \sqrt{1 - s^2}) (1 - s^2)^{(k-4)/2} ds \\ &= c_k \sqrt{1 - \langle u, u_0 \rangle^2} g_{k-1}(\langle \text{pr}_L u, a \rangle) + \gamma(k, \langle u, u_0 \rangle) \langle \text{pr}_L u, a \rangle \\ (29) \quad &= c_k \|u\|L \|g_{k-1}(\langle \text{pr}_L u, a \rangle) + \gamma(k, \langle u, u_0 \rangle) \langle \text{pr}_L u, a \rangle. \end{aligned}$$

Since  $H^1(L, a)$  is a set of measure zero in  $S^{d-1}$ , it follows that

$$\begin{aligned} &\int_{S^{d-1}} (\pi_{L,-d+k-1}g_k(\langle u, \cdot \rangle))(a) f_n(u) du \\ &= c_k \int_{S^{d-1}} \|u\|L \|g_{k-1}(\langle \text{pr}_L u, a \rangle) f_n(u) du \\ &\quad + \int_{S^{d-1}} \gamma(k, \langle u, u_0 \rangle) \langle \text{pr}_L u, a \rangle f_n(u) du \\ &= c_k \int_{S^{d-1}} (\pi_{L,1}^* g_{k-1}(\langle \cdot, a \rangle))(u) f_n(u) du \\ &\quad + \int_{-1}^1 \gamma(k, t) A_{n,j}^{(d)}(t) (1 - t^2)^{(d-3)/2} \int_{S^{d-2}(L)} \langle y, a \rangle S_j^{(d-1)}(y) dy dt. \end{aligned}$$

Now  $j \neq 1$ , and so we deduce that

$$\begin{aligned} \int_{S^{d-1}} (\pi_{L,-d+k-1} g_k(\langle u, \cdot \rangle))(a) f_n(u) du \\ = c_k \int_{S^{d-2}(L)} g_{k-1}(\langle v, a \rangle) (\pi_{L,1} f_n)(v) dv. \end{aligned}$$

It follows from Lemma 2.2 and equation (27) that

$$(30) \quad \begin{aligned} \int_{S^{d-1}} (\pi_{L,-d+k-1} g_k(\langle u, \cdot \rangle))(a) f_n(u) du \\ = c_k \mu_j(d-1, k-1) \beta_{d,n,j,1} S_j^{(d-1)}(a). \end{aligned}$$

On the other hand, returning to (29) and using (27), we also have

$$(31) \quad \begin{aligned} \int_{S^{d-1}} (\pi_{L,-d+k-1} g_k(\langle u, \cdot \rangle))(a) f_n(u) du \\ = \int_{S^{d-1}} f_n(u) \int_{H^1(L,a)} g_k(\langle u, v \rangle) \langle a, v \rangle^{k-3} dv du \\ = \int_{H^1(L,a)} \langle a, v \rangle^{k-3} \int_{S^{d-1}} f_n(u) g_k(\langle u, v \rangle) du dv \\ = \mu_n(d, k) \int_{H^1(L,a)} \langle a, v \rangle^{k-3} f_n(v) dv \\ = \mu_n(d, k) \beta_{d,n,j,-d+k-1} S_j^{(d-1)}(a). \end{aligned}$$

Equations (30) and (31) hold for all  $a \in S^{d-2}(L)$  and so

$$\mu_n(d, k) \beta_{d,n,j,-d+k-1} = c_k \mu_j(d-1, k-1) \beta_{d,n,j,1}$$

for all  $n \neq 1$  and all  $j \neq 1$ . By our inductive assumption

$$\mu_j(d-1, k-1) = (-1)^j b_{d-1,k-1} \lambda_j(d-1, -1) \lambda_j(d-1, k-2)$$

for all  $j \neq 1$ . Combining this with (12) and the fact that  $\beta_{d,n,j,-d+k-1} \neq 0$  gives

$$\mu_n(d, k) = c_{d,k} \lambda_n(d, -1) \lambda_n(d, k-1)$$

for all  $n \neq 1$ , and for some constant  $c_{d,k}$  dependent only on the dimensions  $d$  and  $k$ . The proof will be completed by showing that

$$(32) \quad \mu_0(d, k) = b_{d,k} \lambda_0(d, -1) \lambda_0(d, k-1),$$

with  $b_{d,k}$  as in (26). This requires us to calculate

$$\int_{S^{d-1}} g_k(\langle u, v \rangle) dv = \frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})} \int_{-1}^1 g_k(t) (1-t^2)^{(d-3)/2} dt$$

for all  $k = 2, \dots, d$ .

For the evaluation of the above integral, we will use the recurrence relation

$$(33) \quad (k-1)^2 g_{k+2}(t) = (k+1)tg'_k(t) + (k^2-1)g_k(t) + c_k t, \quad t \in (-1, 1),$$

for a given dimensional constant  $c_k$ , established by Berg [3, Theorem 3.3]. For  $k = 2, \dots, d$ , we put

$$q_{d,k} = \int_{-1}^1 g_k(t)(1-t^2)^{(d-3)/2} dt$$

and use (33) to, first, find a recursion formula for the  $q_{d,k}$ . That equation yields, for  $k = 2, \dots, d-2$ ,

$$(k-1)^2 q_{d,k+2} = (k+1) \int_{-1}^1 tg'_k(t)(1-t^2)^{(d-3)/2} dt + (k^2-1)q_{d,k}.$$

Integration by parts, together with [3, Theorem 3.3(ii)] gives, for  $k = 2, \dots, d-2$ ,

$$(34) \quad (k-1)^2 q_{d,k+2} = (k+1)(d-3)q_{d-2,k} - (k+1)(d-k-1)q_{d,k}.$$

One can calculate from (19) and (20) that

$$q_{d,2} = \frac{\sqrt{\pi}}{d-1} \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d-1}{2})} \quad \text{and} \quad q_{d,3} = \frac{2\sqrt{\pi}}{d-1} \frac{\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d-2}{2})}.$$

Combining this with the recursion formula (34) gives, for  $k = 2, \dots, d$ ,

$$q_{d,k} = \frac{(k-1)\sqrt{\pi}}{d-1} \frac{\Gamma(\frac{d-k+2}{2})}{\Gamma(\frac{d-k+1}{2})},$$

which, in turn, yields the desired result (32) and completes the proof of the theorem. q.e.d.

The main result of this section, Theorem 4.4, will provide us with a fairly explicit integral representation of the support functions of mean section bodies. For example, using (19)–(22), it will give, for bodies of

sufficiently high dimension,

$$\begin{aligned}
 h^*(M_2(K), u) &= \frac{1}{2\pi(d-1)} \int_{S^{d-1}} (\pi - \cos^{-1}\langle u, v \rangle) (1 - \langle u, v \rangle^2)^{\frac{1}{2}} S_{d-1}(K, dv), \\
 h^*(M_3(K), u) &= \frac{\Gamma(\frac{d}{2})}{(d-2)\pi^{\frac{3}{2}}\Gamma(\frac{d-1}{2})} \int_{S^{d-1}} (1 + \langle u, v \rangle \ln(1 - \langle u, v \rangle)) S_{d-2}(K, dv), \\
 h^*(M_4(K), u) &= \frac{3(d-2)}{8(d-3)\pi^2} \\
 &\quad \times \int_{S^{d-1}} (\pi - \cos^{-1}\langle u, v \rangle) (1 - 2\langle u, v \rangle^2)(1 - \langle u, v \rangle^2)^{-\frac{1}{2}} S_{d-3}(K, dv), \\
 h^*(M_5(K), u) &= \frac{2\Gamma(\frac{d}{2})}{3(d-4)\pi^{\frac{5}{2}}\Gamma(\frac{d-3}{2})} \\
 &\quad \times \int_{S^{d-1}} (3(1 + \langle u, v \rangle \ln(1 - \langle u, v \rangle)) - (1 - \langle u, v \rangle)^{-1}) S_{d-4}(K, dv).
 \end{aligned}$$

**Theorem 4.4.** *For an integer  $k = 2, \dots, d$  and a convex body  $K \subset \mathbb{R}^d$  with  $\dim K \geq d + 2 - k$ , we have*

$$(35) \quad h^*(M_k(K), u) = p_{d,k} \int_{S^{d-1}} g_k(\langle u, v \rangle) S_{d+1-k}(K, dv)$$

for all  $u \in S^{d-1}$ , where

$$(36) \quad p_{d,k} = \frac{1}{2(d+1-k)\pi^{\frac{k-1}{2}}} \frac{\Gamma(\frac{d}{2})\Gamma(\frac{k-1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{d+2-k}{2})}.$$

It follows from Berg's result (18) and from (2) that the theorem holds in the cases  $k = d$  and  $k = 2$ .

Combining Theorems 3.4 and 4.3, we see that the equation in (35) is true, for all  $k$ , for almost all  $u \in S^{d-1}$ . We will follow Berg [3], in his use of certain averaging procedures, to show that it is true for all  $u \in S^{d-1}$ , as claimed.

First, we will establish a concavity-type result for the functions  $g_k$ .

**Lemma 4.5.** *For  $k = 2, 3, \dots$ , we have*

$$g_k(t) - tg'_k(t) > 0 \quad \text{for all } -1 < t < 1.$$

*Proof.* We put

$$f_k(t) = (1 - t^2)^{(k-1)/2} (g_k(t) - tg'_k(t)).$$

Then differentiation and [3, Proposition 3.5(ii)] give, for  $-1 < t < 1$ ,

$$f'_k(t) = \frac{(k-1)^2 \kappa_{k-1}}{\kappa_k} t^2 (1 - t^2)^{(k-3)/2}.$$

So

$$f_k(t) = \frac{(k-1)^2 \kappa_{k-1}}{\kappa_k} \int_{-1}^t x^2 (1-x^2)^{(k-3)/2} dx + \alpha,$$

where  $\alpha$  is a (yet to be determined) constant of integration.

It easily follows from (33) that  $f_k(0) = g_k(0) = (k-1)/2$  for all  $k = 2, 3, \dots$ . Using this and the formulation of  $\kappa_d$  in terms of gamma functions, shows that the above constant  $\alpha$  is zero. Thus  $f_k(t) > 0$  for  $t \in (-1, 1)$  and this gives the required inequality. q.e.d.

*Proof of Theorem 4.4.* For  $0 < \rho < \pi/2$ ,  $a \in S^{d-1}$ , and a locally integrable upper semi-continuous function  $f : S^{d-1} \rightarrow \mathbb{R} \cup \{-\infty\}$ , we put

$$\mathcal{M}_f^{d,\rho}(a) = \frac{1}{(d-1)\kappa_{d-1}} \int_{S^{d-1} \cap a^\perp} f(a \cos \rho + v \sin \rho) dv$$

and

$$(37) \quad \mathcal{A}_f^{d,\rho}(a) = \frac{1}{(d-1)\kappa_{d-1}} \int_{\langle w, a \rangle \geq \cos \rho} f(w) dw = \int_0^\rho \mathcal{M}_f^{d,t}(a) \sin^{d-2} t dt.$$

We note that  $\mathcal{M}_f^{d,\rho}(a)$  is the average value of  $f$  over the points  $w \in S^{d-1}$  with  $\langle w, a \rangle = \cos \rho$ . Similarly

$$\frac{(d-1)\kappa_{d-1}}{m_d(\rho)} \mathcal{A}_f^{d,\rho}(a),$$

where

$$m_d(\rho) = (d-1)\kappa_{d-1} \int_{\cos \rho}^1 (1-t^2)^{(d-3)/2} dt$$

is the average value of  $f$  over the points  $w \in S^{d-1}$  with  $\langle w, a \rangle \geq \cos \rho$ .

For  $k = 2, \dots, d$  and a convex body  $K$  in  $\mathbb{R}^d$  with  $\dim K \geq d+2-k$ , we now define  $F_k : S^{d-1} \rightarrow \mathbb{R} \cup \{-\infty\}$  by

$$F_k(u) = \int_{S^{d-1}} g_k(\langle u, v \rangle) S_{d+1-k}(K, dv) \quad u \in S^{d-1}.$$

The upper semi-continuity and local integrability of the functions  $F_k$  follow from the corresponding properties of the  $g_k$ ; see [3, page 52]. Our objective is to prove that

$$(38) \quad \lim_{\rho \rightarrow 0} \frac{d-1}{\sin^{d-1} \rho} \mathcal{A}_f^{d,\rho}(a) = f(a) \quad \text{for all } a \in S^{d-1}$$

in both the cases  $f = h^*(M_k(K), \cdot)$  and  $f = F_k$ . This will provide a proof of Theorem 4.4 since we already have

$$\mathcal{A}_{h^*(M_k(K), \cdot)}^{d,\rho}(a) = p_{d,k} \mathcal{A}_{F_k}^{d,\rho}(a) \quad \text{for all } a \in S^{d-1}$$

because equality (35) holds for almost all  $u \in S^{d-1}$ .

It is proved in [3, pages 44 and 45] that, for  $f \in C^2(\omega)$  for some open  $\omega \subset S^{d-1}$ ,

$$(39) \quad \mathcal{M}_f^{d,\rho}(a) - f(a) \cos \rho = \frac{\sin^2 \rho}{2(d-1) \cos \rho} ((d-1)f(a) + \Delta_d^* f(a)) + \frac{1}{2} \beta(a, \rho) \sin^2 \rho,$$

where  $\beta(a, \rho) \rightarrow 0$  as  $\rho \rightarrow 0$  for each  $a \in \omega$ . Here,  $\Delta_d^*$  denotes the spherical Laplacian on  $S^{d-1}$ . We will subsequently use this equation in the case  $f = g_k(\langle u, \cdot \rangle)$ , a function with rotational symmetry. For such functions, the spherical Laplacian is easily evaluated. We have, in particular,

$$(40) \quad (\Delta_d^* g_k(\langle u, \cdot \rangle))(a) = (1 - \langle u, a \rangle^2) g_k''(\langle u, a \rangle) - (d-1) \langle u, a \rangle g_k'(\langle u, a \rangle).$$

We fix  $u \in S^{d-1}$ ; then, using (39) with  $f = g_k(\langle u, \cdot \rangle) \in C^2(S^{d-1} \setminus \{\pm u\})$  gives, for  $a \neq \pm u$ ,

$$(41) \quad \begin{aligned} \mathcal{M}_{g_k(\langle u, \cdot \rangle)}^{d,\rho}(a) - g_k(\langle u, a \rangle) \cos \rho &= \frac{\sin^2 \rho}{2(d-1) \cos \rho} ((d-1)g_k(\langle u, a \rangle) + (\Delta_d^* g_k(\langle u, \cdot \rangle))(a)) \\ &\quad + \frac{1}{2} \beta(a, \rho) \sin^2 \rho, \end{aligned}$$

where  $\beta(a, \rho) \rightarrow 0$  as  $\rho \rightarrow 0$ . Using (40) and [3, Proposition 3.5(ii)], we have

$$\begin{aligned} (\Delta_d^* g_k(\langle u, \cdot \rangle))(a) &= -(d-k) \langle u, a \rangle g_k'(\langle u, a \rangle) - (k-1) g_k(\langle u, a \rangle) \\ &\quad - \frac{(k-1)^2 \kappa_{k-1}}{\kappa_k} \langle u, a \rangle. \end{aligned}$$

Consequently, by Lemma 4.5 and (41), we have

$$(42) \quad \begin{aligned} \mathcal{M}_{g_k(\langle u, \cdot \rangle)}^{d,\rho}(a) - g_k(\langle u, a \rangle) \cos \rho &\geq - \frac{(k-1)^2 \kappa_{k-1}}{\kappa_k} \frac{\sin^2 \rho}{2(d-1) \cos \rho} \langle u, a \rangle + \frac{1}{2} \beta(a, \rho) \sin^2 \rho \end{aligned}$$

for  $a \neq \pm u$  and for  $\rho \in (0, \pi/2)$ .

As Berg points out, in his proof of [3, Theorem 4.12], there is a constant  $\alpha$  such that  $\alpha - g_k$  is positive on  $[-1, 1]$ . So, since  $S_{d+1-k}(K, \cdot)$  and the spherical Lebesgue measure on  $S^{d-2}(a^\perp)$  are finite measures, an application of Fubini's Theorem gives

$$\mathcal{M}_{F_k}^\rho(a) = \int_{S^{d-1}} \mathcal{M}_{g_k(\langle u, \cdot \rangle)}^\rho(a) S_{d+1-k}(K, du).$$



For  $k > 2$ , the measure  $S_{d+1-k}(K, \cdot)$  has no atoms; see [34, Theorem 4.6.5], for example. It therefore follows from (42) that

$$\mathcal{M}_{F_k}^{d,\rho}(a) \geq F_k(a) \cos \rho + \frac{1}{2} \binom{d}{k-1}^{-1} d\kappa_{k-1} V_{d+1-k}(K) \beta(a, \rho) \sin^2 \rho,$$

for  $k = 3, 4, \dots, d$ . We thus deduce from (37) that

$$\begin{aligned} \mathcal{A}_{F_k}^{d,\rho}(a) &\geq \int_0^\rho \sin^{d-2} t \left( F_k(a) \cos t + c_{d,k}(K) \beta(a, t) \sin^2 t \right) dt \\ &= \frac{\sin^{d-1} \rho}{d-1} F_k(a) + c_{d,k}(K) \int_0^\rho \beta(a, t) \sin^d t dt. \end{aligned}$$

Consequently

$$(43) \quad \frac{d-1}{\sin^{d-1} \rho} \mathcal{A}_{F_k}^{d,\rho}(a) \geq F_k(a) + \frac{(d-1)c_{d,k}(K)}{\sin^{d-1} \rho} \int_0^\rho \beta(a, t) \sin^d t dt.$$

It is clear (see [3, page 49]) that

$$1 \leq \frac{m_d(\rho)}{\kappa_{d-1} \sin^{d-1} \rho} \leq \frac{1}{\cos \rho}.$$

Thus

$$(44) \quad \frac{(d-1)\kappa_{d-1}}{m_d(\rho)} \leq \frac{d-1}{\sin^{d-1} \rho} \leq \frac{(d-1)\kappa_{d-1}}{m_d(\rho) \cos \rho}.$$

We now let  $a \in S^{d-1}$  and assume, first, that  $F_k(a) \geq 0$ . For any  $\alpha > F_k(a)$ , we choose a  $\delta < 1$  with  $\alpha\delta > F_k(a)$ . Then the upper semi-continuity of  $F_k$  shows that, for sufficiently small  $\rho$ , we have  $F_k(w) < \alpha\delta$  if  $\langle w, a \rangle \geq \cos \rho$  and so, for these  $\rho$ ,

$$\frac{(d-1)\kappa_{d-1}}{m_d(\rho)} \mathcal{A}_{F_k}^{d,\rho}(a) \leq \alpha\delta.$$

Combining this with (44) gives, for sufficiently small  $\rho$ ,

$$\alpha \geq \frac{\alpha\delta}{\cos \rho} \geq \frac{(d-1)\kappa_{d-1}}{m_d(\rho) \cos \rho} \mathcal{A}_{F_k}^{d,\rho}(a) \geq \frac{d-1}{\sin^{d-1} \rho} \mathcal{A}_{F_k}^{d,\rho}(a)$$

if  $\mathcal{A}_{F_k}^{d,\rho}(a) \geq 0$ ; otherwise

$$(45) \quad \alpha \geq \frac{(d-1)\kappa_{d-1}}{m_d(\rho)} \mathcal{A}_{F_k}^{d,\rho}(a) \geq \frac{d-1}{\sin^{d-1} \rho} \mathcal{A}_{F_k}^{d,\rho}(a).$$

In case  $F_k(a) < 0$ , we choose  $\alpha < 0$  with  $F_k(a) < \alpha$ . The upper semi-continuity shows that, for sufficiently small  $\rho$ , we have

$$\frac{(d-1)\kappa_{d-1}}{m_d(\rho)} \mathcal{A}_{F_k}^{d,\rho}(a) \leq \alpha < 0;$$

again this leads to the inequality (45). So we conclude that, for each  $a \in S^{d-1}$  and for each  $\alpha > F_k(a)$ , there is a  $\rho_0 \in (0, \pi/2)$  such that

$$\alpha \geq \frac{d-1}{\sin^{d-1} \rho} \mathcal{A}_{F_k}^{d,\rho}(a) \quad \text{for all } 0 < \rho < \rho_0.$$

Combining this with (43) shows that for each  $a \in S^{d-1}$  and for each  $\alpha > F_k(a)$ ,

$$\alpha \geq \frac{d-1}{\sin^{d-1} \rho} \mathcal{A}_{F_k}^{d,\rho}(a) \geq F_k(a) + \frac{(d-1)c_{d,k}(K)}{\sin^{d-1} \rho} \int_0^\rho \beta(a, t) \sin^d t \, dt,$$

for all sufficiently small  $\rho > 0$ .

Next, we choose  $\rho$  small enough to guarantee that  $|\beta(a, t)| < 1$  for all  $t \in (0, \rho)$ . Then

$$\left| \frac{1}{\sin^{d-1} \rho} \int_0^\rho \beta(a, t) \sin^d t \, dt \right| \leq \rho \sin \rho \rightarrow 0 \quad \text{as } \rho \rightarrow 0.$$

Thus

$$\alpha \geq \lim_{\rho \rightarrow 0} \frac{d-1}{\sin^{d-1} \rho} \mathcal{A}_{F_k}^{d,\rho}(a) \geq F_k(a),$$

for any  $\alpha > F_k(a)$ . It follows that (38) holds for  $f = F_k$ . However, as Berg shows [3, Proposition 4.7], this equation also holds for all subharmonic functions  $f$ , and therefore for support functions. As indicated above, this gives the desired result. q.e.d.

**Theorem 4.6.** *Let  $\mu$  be a Borel measure on  $S^{d-1}$  with centroid at the origin. Then  $\mu$  is of the form  $S_{d+1-k}(K, \cdot)$  for some convex body  $K$  in  $\mathbb{R}^d$  with  $\dim K \geq d + 2 - k$ , if and only if*

$$\int_{S^{d-1}} g_k(\langle \cdot, v \rangle) \mu(dv)$$

*is the support function of a  $k$ -th mean section body.*

*Proof.* In one direction, the result is an immediate consequence of Theorem 4.4. For the other, we assume there is body  $K$  in  $\mathbb{R}^d$  with

$$h(M_k(K), u) = \int_{S^{d-1}} g_k(\langle u, v \rangle) \mu(du) \quad \text{for all } u \in S^{d-1}.$$

It follows from Corollary 3.5 and (15) that there is a positive constant  $c_{d,k}$  such that  $\mu = c_{d,k} S_{d+1-k}(K, \cdot)$  as distributions. However, two positive measures that are equal as distributions are equal as measures. Thus  $\mu$  is the  $(d + 1 - k)$ -th surface area measure of a dilation of  $K$ . q.e.d.

**Remark 4.7.** This result shows the connection between mean section bodies and the general Minkowski problem. However, since the characterization of these bodies is unknown, the main content of the result is to show the equivalence of the two problems. We note, in

particular, that it is not sufficient for the integral to yield an arbitrary support function. If this were the case, then the sum of two intermediate surface area measures (of the same degree) would be another one. However, this is known not to be the case; see [10, 11, 18].

**Remark 4.8.** Berg’s characterization of first surface area measures corresponds to the case  $k = d$ , for which all convex bodies are  $d$ -th mean section bodies.

**Remark 4.9.** Minkowski’s characterization of  $(d - 1)$ -st surface area measures corresponds to the case  $k = 2$ . The function  $g_2$ , as Berg [3] pointed out, has properties not shared by the other  $g_k$ . In particular, it is continuous on  $[-1, 1]$ . It follows from [19, Theorem 2] that, if  $\mu$  is any discrete measure with centroid at the origin, then

$$\int_{S^{d-1}} g_2(\langle \cdot, v \rangle) \mu(dv)$$

is a support function. The continuity of  $g_2(\langle u, \cdot \rangle)$  then implies that the same is true for any positive measure with centroid at the origin (and not supported on a great subsphere).

We conclude with another consequence of Theorem 4.4. It provides an analogue of Corollary 3.2 and an extension of Lemma 4.2.

**Corollary 4.10.** *Let  $k = 3, \dots, d$ ,  $q = d - k + 2, \dots, d - 1$ ,  $H \in G(d, q)$ , and  $v \in S^{q-1}(H)$ . Then, for any  $u \in S^{q-1}(H)$  with  $u \neq v$ , we have*

$$(46) \quad [\pi_{H, -d-1+k} g_k(\langle u, \cdot \rangle)](v) = c_{d,k,q} g_{k-d+q}(\langle u, v \rangle) + c'_{d,k,q} \langle u, v \rangle.$$

*Proof.* It follows from Lemma 3.3 that, if  $K$  is a convex body with affine hull  $H \in G(d, q)$ , then

$$h^*(M_k(K), u) = c_{d,k,q} h^*(M_{k-d+q}^H(K), u) \quad \text{for all } u \in S^{q-1}(H).$$

Applying Theorem 4.4 to the left side of this equation gives, for  $p_{d,k}$  as in (36),

$$\begin{aligned} h^*(M_k(K), u) &= p_{d,k} \int_{S^{d-1}} g_k(\langle u, v \rangle) S_{d+1-k}(K, dv) \\ &= c_{d,k,q} \int_{S^{d-1}} g_k(\langle u, v \rangle) [\pi_{H, -d-1+k}^* S_{d+1-k}^H(K, \cdot)](dv) \\ &= c_{d,k,q} \int_{S^{q-1}(H)} [\pi_{H, -d-1+k} g_k(\langle u, \cdot \rangle)](v) S_{d+1-k}^H(K, dv), \end{aligned}$$

for all convex bodies  $K \subset H$ . Applying the theorem to the right side gives

$$h^*(M_{k-d+q}^H(K), u) = p_{q,k-d+q} \int_{S^{q-1}(H)} g_{k-d+q}(\langle u, v \rangle) S_{d+1-k}^H(K, dv),$$

again for all convex bodies  $K \subset H$ . It was shown by Weil [39] that differences of  $(d + 1 - k)$ -th surface area measures are dense amongst all signed measures with centroid at the origin. Consequently, for each  $u \in S^{q-1}(H)$ , there is a vector  $w_{H,u} \in S^{q-1}(H)$  such that

$$(47) \quad [\pi_{H,-d-1+k}g_k(\langle u, \cdot \rangle)](v) - c_{d,k,q}g_{k-d+q}(\langle u, v \rangle) = \langle w_{H,u}, v \rangle$$

for almost all  $v \in S^{q-1}(H)$ . For fixed  $u \in S^{q-1}(H)$ , the left hand side of this equation is continuous at all  $v \neq u$ . Consequently, (47) holds for all  $v \neq u$ . Furthermore, the left hand side of (47), as a function of  $v$ , depends only on  $\langle u, v \rangle$ . Therefore (46) is proved for all  $v \neq u$ , as required. q.e.d.

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