BLOCH'S CONJECTURE FOR CATANESE AND BARLOW SURFACES

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À la mémoire de Friedrich Hirzebruch

Abstract

Catanese surfaces are regular surfaces of general type with $p_g = 0$. They specialize to double covers of Barlow surfaces. We prove that the CH_0 group of a Catanese surface is equal to $\mathbb{Z}$, which implies the same result for the Barlow surfaces.

0. Introduction

In this paper, we establish an improved version, in the surface case, of the main theorem of [**27**] and use it in order to prove the Bloch conjecture for Catanese surfaces. We will also give a conditional application (more precisely, assuming the variational Hodge conjecture) of the same method to the Chow motive of low degree $K3$ surfaces.

Bloch's conjecture for 0-cycles on surfaces is the following:

Conjecture 0.1. *(cf. [**7**]) Let $\Gamma \in CH^2(Y \times X)$, where Y is smooth projective and X is a smooth projective surface. Assume that $[\Gamma] : H^{2,0}(X) \to H^{2,0}(Y)$ vanishes. Then*

$$\Gamma_* : CH_0(Y)_{alb} \to CH_0(X)_{alb}$$

vanishes.

Here $[\Gamma] \in H^4(Y \times X, \mathbb{Q})$ is the cohomology class of Γ and

$$CH_0(Y)_{alb} := \mathrm{Ker}\,(CH_0(Y)_{hom} \overset{alb_Y}{\to} Alb(Y)).$$

Particular cases concern the situation where Γ is the diagonal of a surface X with $p_g = 0$. Then $\Gamma_* = Id_{CH_0(X)_{alb}}$, so that the conjecture predicts $CH_0(X)_{alb} = 0$ for such a surface. This statement is known to hold for surfaces which are not of general type by [**9**], and for surfaces of general type, it is known to hold by Kimura [**17**] if the surface X is furthermore rationally dominated by a product of curves (cf. [**5**] for many such examples). Furthermore, for several other families of surfaces of general type with $q = p_g = 0$, it is known to hold either for the general

member of the family (eg. the Godeaux surfaces, cf. [**28**]), or for specific members of the family (for example the Barlow surface [**4**]).

A slightly more general situation concerns surfaces equipped with the action of a finite group G. This has been considered in the paper [**27**], where the following theorem concerning group actions on complete intersection surfaces is proved: Let X be a smooth projective variety with trivial Chow groups (i.e., the cycle class map $CH_i(X)_{\mathbb{Q}} \to H^{2n-2i}(X, \mathbb{Q})$, $n = \dim X$ is injective for all i). Let G be a finite group acting on X and let E be a G-equivariant rank $n-2$ vector bundle on X which has "enough" G-invariant sections (for example, if the group action is trivial, one asks that E is very ample). Let $\pi \in \mathbb{Q}[G]$ be a projector. Then π gives a self-correspondence Γ^{π} with $\mathbb{Q}$-coefficients (which is a projector) of the G-invariant surfaces $S_{\sigma} = V(\sigma)$, $\sigma \in H^0(X, E)^G$ (cf. Section 1). We use the notation

$$H^{2,0}(S_{\sigma})^{\pi} := \mathrm{Im}\,([\Gamma^{\pi}]^* : H^{2,0}(S_{\sigma}) \to H^{2,0}(S_{\sigma})),$$

$$CH_0(S_{\sigma})^{\pi}_{\mathbb{Q},hom} := \mathrm{Im}\,([\Gamma^{\pi}]_* : CH_0(S_{\sigma})_{\mathbb{Q},hom} \to CH_0(S_{\sigma})_{\mathbb{Q},hom}).$$

Theorem 0.2. *Assume that the smooth surfaces* $S_{\sigma} = V(\sigma)$, $\sigma \in H^0(E)^G$ *satisfy* $H^{2,0}(S_{\sigma})^{\pi} = 0$. *Then we have* $CH_0(S_{\sigma})^{\pi}_{\mathbb{Q},hom} = 0$.

Note that the Bloch conjecture has been recently proved in [**15**] and [**30**] by completely different methods for finite group actions on surfaces that in most cases do not fit at all in the above geometric setting, namely finite order symplectic automorphisms of a $K3$ surface X. For these symplectic automorphisms, one considers the cycle $\Delta_X - \frac{1}{|G|} \sum_{g \in G} \mathrm{Graph}\, g$, which acts as the identity minus the projector onto the G-invariant part; it is proved in [**30**] (the case of involutions) and [**15**] (the higher order case) that it acts as 0 on $CH_0(X)_{hom}$ (in fact on the whole of CH_0) according to Conjecture 0.1.

Theorem 0.2 is rather restrictive geometrically, due to the fact that not only do we consider 0-sets of sections of a vector bundle, but also we impose this very ampleness assumption on the vector bundle. Our first result in this paper is a relaxed version of this theorem, which works in a much more general geometric context and will be applicable in particular to the case of Catanese surfaces.

Let $\mathcal{S} \to B$ be a smooth projective morphism with two dimensional connected fibers, where B is quasi-projective. Let $\Gamma \in CH^2(\mathcal{S} \times_B \mathcal{S})_{\mathbb{Q}}$ be a codimension 2 cycle, which provides a relative 0-self correspondence of $\mathcal{S}$ over B. For $t \in B$, let $\Gamma_t := \Gamma_{|\mathcal{S}_t \times \mathcal{S}_t}$ be the restricted cycle, with cohomology class $[\Gamma_t] \in H^4(\mathcal{S}_t \times \mathcal{S}_t, \mathbb{Q})$. We have the actions

$$\Gamma_{t*} : CH_0(\mathcal{S}_t)_{\mathbb{Q}} \to CH_0(\mathcal{S}_t)_{\mathbb{Q}}, \ [\Gamma_t]^* : H^{i,0}(\mathcal{S}_t) \to H^{i,0}(\mathcal{S}_t).$$

Theorem 0.3. *Assume the following:*

(1) The fibers $\mathcal{S}_t$ *satisfy* $h^{1,0}(\mathcal{S}_t) = 0$ *and* $[\Gamma_t]^* : H^{2,0}(\mathcal{S}_t) \to H^{2,0}(\mathcal{S}_t)$ *is equal to zero.*

(2) A non-singular projective (equivalently any non-singular projective) completion $\overline{\mathcal{S} \times_B \mathcal{S}}$ of the fibered self-product $\mathcal{S} \times_B \mathcal{S}$ is rationally connected.

Then $\Gamma_{t*} : CH_0(\mathcal{S}_t)_{hom} \to CH_0(\mathcal{S}_t)_{hom}$ is nilpotent for any $t \in B$.

This statement is both weaker and stronger than Theorem 0.2 since on the one hand the conclusion only states the nilpotence of Γ_{t*}, and not its vanishing, while on the other hand the geometric context is much more flexible and the assumption on the total space of the family is much weaker.

In fact the nilpotence property is sufficient to imply the vanishing in a number of situations which we describe below. The first situation is the case where we consider a family of surfaces with $h^{2,0} = h^{1,0} = 0$. Then we get the following consequence (the Bloch conjecture for surfaces with $q = p_g = 0$ under assumption (2) below):

Corollary 0.4. *Let $\mathcal{S} \to B$ be a smooth projective morphism with two dimensional connected fibers, where B is quasi-projective. Assume the following:*

(1) The fibers $\mathcal{S}_t$ satisfy $H^{1,0}(\mathcal{S}_t) = H^{2,0}(\mathcal{S}_t) = 0$.

(2) A projective completion (or, equivalently, any projective completion) $\overline{\mathcal{S} \times_B \mathcal{S}}$ of the fibered self-product $\mathcal{S} \times_B \mathcal{S}$ is rationally connected.

Then $CH_0(\mathcal{S}_t)_{hom} = 0$ for any $t \in B$.

We refer to Section 1, Theorem 1.6 for a useful variant involving group actions, which allows one to consider many more situations (cf. [**27**] and Section 2 for examples).

Remark 0.5. The proof will show as well that we can replace assumption (2) in these statements by the following one:

(2') A (equivalently, any) smooth projective completion $\overline{\mathcal{S} \times_B \mathcal{S}}$ of the fibered self-product $\mathcal{S} \times_B \mathcal{S}$ has trivial CH_0 group.

However, it seems more natural to put a geometric assumption on the total space since this is in practice much easier to check.

In the second section of this paper, we will apply these results to prove Bloch's conjecture 0.1 for Catanese surfaces (cf. [**11**], [**25**], [**10**]). Catanese surfaces can be constructed starting from a 5×5 symmetric matrix $M(a)$, $a \in \mathbb{P}^{11}$, of linear forms on $\mathbb{P}^3$ satisfying certain conditions (cf. (3)) making their discriminant invariant under the Godeaux action (4) of $\mathbb{Z}/5\mathbb{Z}$ on $\mathbb{P}^3$. The general quintic surface $V(a)$ defined by the determinant of $M(a)$ has 20 nodes corresponding to the points $x \in \mathbb{P}^3$ where the matrix $M(a,x)$ has rank 3, and it admits a double cover $S(a)$ which is étale away from the nodes, and to which the $\mathbb{Z}/5\mathbb{Z}$-action lifts. Then the Catanese surface $\Sigma(a)$ is the quotient of $S(a)$ by this lifted action.

Catanese surfaces have a 4-dimensional moduli space. For our purpose, the geometry of the explicit 11-dimensional parameter space described in [25] is in fact more important than the structure of the moduli space.

Theorem 0.6. *Let Σ be a Catanese surface. Then $CH_0(\Sigma) = \mathbb{Z}$.*

The starting point of this work was a question asked by the authors of [10]: They needed to know that the Bloch conjecture holds for a simply connected surface with $p_g = 0$ (e.g., a Barlow surface [3]) and furthermore, they needed it for a general deformation of this surface. The Bloch conjecture was proved by Barlow [4] for some Barlow surfaces admitting an extra group action allowing to play on group theoretic arguments as in [16], but it was not known for the general Barlow surface.

Theorem 0.6 implies as well the Bloch conjecture for the Barlow surfaces, since the Barlow surfaces can be constructed as quotients of certain Catanese surfaces admitting an extra involution, namely that the determinantal equation defining $V(a)$ has to be invariant under the action of the dihedral group of order 10 (cf. [25]). The Catanese surfaces appearing in this construction of Barlow surfaces have only a 2-dimensional moduli space. It is interesting to note that we get the Bloch conjecture for the general Barlow surface via the Bloch conjecture for the general Catanese surface, but that our strategy does not work directly for the Barlow surface, which has a too small parameter space (cf. [11], [25]).

The third section of this paper applies Theorem 0.3 to prove a conditional result on the Chow motive of $K3$ surfaces which can be realized as 0-sets of sections of a vector bundle on a rationally connected variety (cf. [20], [21]). Recall that the Kuga-Satake construction (cf. [18], [13]) associates to a polarized $K3$ surface S an abelian variety $K(S)$ with the property that the Hodge structure on $H^2(S, \mathbb{Q})$ is a direct summand of the Hodge structure of $H^2(K(S), \mathbb{Q})$. The Hodge conjecture predicts that the corresponding degree 4 Hodge class on $S \times K(S)$ is algebraic. This is not known in general, but this is established for $K3$ surfaces with large Picard number (cf. [19], [23]). The next question concerns the Chow motive (as opposed to the numerical motive) of these $K3$ surfaces. The Kuga-Satake construction combined with the Bloch conjecture implies that the Chow motive of a $K3$ surface is a direct summand of the Chow motive of its Kuga-Satake variety. In this direction, we prove the following Theorem 0.7: Let X be a rationally connected variety of dimension n and let $E \to X$ be a rank $n-2$ globally generated vector bundle satisfying the following properties:

(i) The restriction map $H^0(X, E) \to H^0(z, E_{|z})$ is surjective for general $z = \{x, y\} \subset X$.

(ii) The general section σ vanishing at two general points x, y determines a smooth surface $V(\sigma)$.

We consider the case where the surfaces $S = V(\sigma)$ are algebraic $K3$ surfaces. For example, this is the case if $\det E = -K_X$, and the surfaces $S = V(\sigma)$ for general $\sigma \in H^0(X, E)$ have irregularity 0. Almost all general algebraic $K3$ surfaces of genus ≤ 20 have been described this way by Mukai (cf. [**20**], [**21**]), where X is a homogeneous variety with Picard number 1. Many more examples can be constructed starting from an X with Picard number ≥ 2.

Theorem 0.7. *Assume the variational Hodge conjecture in degree 4. Then the Chow motive of a $K3$ surface S as above is a direct summand of the Chow motive of an abelian variety.*

Remark 0.8. The variational Hodge conjecture for degree 4 Hodge classes is implied by the Lefschetz standard conjecture in degrees 2 and 4. It is used here only to conclude that the Kuga-Satake correspondence is algebraic for any S as above. Hence we could replace the variational Hodge conjecture by the Lefschetz standard conjecture or by the assumption that the cohomological motive of a general $K3$ surfaces S in our family is a direct summand of the cohomological motive of an abelian variety. The content of the theorem is that we then have the same result for the Chow motive.

As a consequence of this result, we get the following (conditional) corollaries.

Corollary 0.9. *With the same assumptions as in Theorem 0.7, let S be a member of the family of $K3$ surfaces parameterized by $\mathbb{P}(H^0(X, E))$, and let $\Gamma \in CH^2(S \times S)$ be a correspondence such that $[\Gamma]^* : H^{2,0}(S) \to H^{2,0}(S)$ is zero. Then $\Gamma_* : CH_0(S)_{hom} \to CH_0(S)_{hom}$ is nilpotent.*

Remark 0.10. Note that there is a crucial difference between Theorem 0.3 and Corollary 0.9: In Corollary 0.9, the cycle Γ is not supposed to exist on the general deformation S_t of S. (Note also that the result in Corollary 0.9 is only conditional since we need the Lefschetz standard conjecture, or at least we need to know that the Kuga-Satake correspondence is algebraic for general S_t, while Theorem 0.3 is unconditional!)

Corollary 0.11. *With the same assumptions as in Theorem 0.7, the transcendental part of the Chow motive of any member of the family of $K3$ surfaces parameterized by $\mathbb{P}(H^0(X, E))$ is indecomposable, that is, any submotive of it is either the whole motive or the 0-motive.*

1. Proof of Theorem 0.3 and some consequences

This section is devoted to the proof of Theorem 0.3 and its consequences (Corollary 0.4 or its more general form Theorem 1.6).

The proof will follow essentially the idea of [**27**]. The main novelty in the proof lies in the use of Proposition 1.3. For completeness, we also outline the arguments of [**27**], restricted to the surface case.

Consider the codimension 2-cycle

$$\Gamma \in CH^2(\mathcal{S} \times_B \mathcal{S})_{\mathbb{Q}}.$$

Assumption (1) tells us that the restricted cycle $\Gamma_t := \Gamma_{|\mathcal{S}_t \times \mathcal{S}_t}$ is cohomologous to the sum of a cycle supported on a product of (not necessarily irreducible curves) in $\mathcal{S}_t$ and of cycles pulled-back from $CH_0(\mathcal{S}_t)$ via the two projections. We deduce from this (cf. [**27**, Prop. 2.7]):

Lemma 1.1. *There exist a codimension 1 closed algebraic subset $\mathcal{C} \subset \mathcal{S}$, a codimension 2 cycle $\mathcal{Z}$ on $\mathcal{S} \times_B \mathcal{S}$ with $\mathbb{Q}$-coefficients supported on $\mathcal{C} \times_B \mathcal{C}$, and two codimension 2 cycles $\mathcal{Z}_1$, $\mathcal{Z}_2$ with $\mathbb{Q}$-coefficients on $\mathcal{S}$, such that the cycle*

$$\Gamma - \mathcal{Z} - p_1^* \mathcal{Z}_1 - p_2^* \mathcal{Z}_2$$

has its restriction to each fiber $\mathcal{S}_t \times \mathcal{S}_t$ cohomologous to 0, where p_1, $p_2 : \mathcal{S} \times_B \mathcal{S} \to \mathcal{S}$ are the two projections.

This lemma is one of the key observations in [**27**]. The existence of the data above is rather clear after a generically finite base change $B' \to B$ because it is true fiberwise. The key point is that, working with cycles with $\mathbb{Q}$-coefficients, we can descend to B and hence do not in fact need this base change, which would ruin assumption (2).

The next step consists in passing from the fiberwise cohomological equality

$$[\Gamma - \mathcal{Z} - p_1^* \mathcal{Z}_1 - p_2^* \mathcal{Z}_2]_{|\mathcal{S}_t \times \mathcal{S}_t} = 0 \text{ in } H^4(\mathcal{S}_t \times \mathcal{S}_t, \mathbb{Q})$$

to the following global vanishing:

Lemma 1.2. *(cf. [**27**, Lemma 2.12])* *There exist codimension 2 algebraic cycles $\mathcal{Z}_1'$, $\mathcal{Z}_2'$ with $\mathbb{Q}$-coefficients on $\mathcal{S}$ such that*

$$[\Gamma - \mathcal{Z} - p_1^* \mathcal{Z}_1' - p_2^* \mathcal{Z}_2'] = 0 \text{ in } H^4(\mathcal{S} \times_B \mathcal{S}, \mathbb{Q}).$$

The proof of this lemma consists in the study of the Leray spectral sequence of the fibration $p : \mathcal{S} \times_B \mathcal{S} \to B$. We know that the class $[\Gamma - \mathcal{Z} - p_1^* \mathcal{Z}_1 - p_2^* \mathcal{Z}_2]$ vanishes in the Leray quotient $H^0(B, R^4 p_* \mathbb{Q})$ of $H^4(\mathcal{S} \times_B \mathcal{S}, \mathbb{Q})$. It follows that it is of the form $p_1^* \alpha_1 + p_2^* \alpha_2$, for some rational cohomology classes α_1, α_2 on $\mathcal{S}$. One then proves that α_i can be chosen to be algebraic on $\mathcal{S}$.

The new part of the argument appears in the following proposition:

Proposition 1.3. *Under assumption (2) of Theorem* 0.3, *the following hold* :

(i) *The codimension* 2 *cycle* $\mathcal{Z}' := \Gamma - \mathcal{Z} - p_1^* \mathcal{Z}_1' - p_2^* \mathcal{Z}_2'$ *is algebraically equivalent to* 0 *on* $\mathcal{S} \times_B \mathcal{S}$.

(ii) *The restriction to the fibers* $\mathcal{S}_t \times \mathcal{S}_t$ *of the cycle* $\mathcal{Z}'$ *is a nilpotent element (with respect to the composition of self-correspondences) of* $CH^2(\mathcal{S}_t \times \mathcal{S}_t)_{\mathbb{Q}}$.

Proof. We work now with a smooth projective completion $\overline{\mathcal{S} \times_B \mathcal{S}}$. Let $D := \overline{\mathcal{S} \times_B \mathcal{S}} \setminus \mathcal{S} \times_B \mathcal{S}$ be the divisor at infinity. Let $\widetilde{D} \xrightarrow{j} \overline{\mathcal{S} \times_B \mathcal{S}}$ be a desingularization of D. The codimension 2 cycle $\mathcal{Z}'$ extends to a cycle $\overline{\mathcal{Z}'}$ over $\overline{\mathcal{S} \times_B \mathcal{S}}$. We know from Lemma 1.2 that the Hodge class $[\overline{\mathcal{Z}'}] \in H^4(\overline{\mathcal{S} \times_B \mathcal{S}}, \mathbb{Q})$ satisfies

$$[\overline{\mathcal{Z}'}]_{|\mathcal{S} \times_B \mathcal{S}} = [\mathcal{Z}'] = 0 \text{ in } H^4(\mathcal{S} \times_B \mathcal{S}, \mathbb{Q})$$

and this implies by [**31**, Prop. 3] that there is a degree 2 Hodge class α on $\widetilde{D}$ such that

$$j_* \alpha = [\overline{\mathcal{Z}'}] \text{ in } H^4(\overline{\mathcal{S} \times_B \mathcal{S}}, \mathbb{Q}).$$

By the Lefschetz theorem on $(1,1)$-classes, α is the class of a codimension 1 cycle $\mathcal{Z}''$ of $\widetilde{D}$ and we conclude that

$$[\overline{\mathcal{Z}'} - j_* \mathcal{Z}''] = 0 \text{ in } H^4(\overline{\mathcal{S} \times_B \mathcal{S}}, \mathbb{Q}).$$

We use now assumption (2) which says that the variety $\overline{\mathcal{S} \times_B \mathcal{S}}$ is rationally connected. It has then trivial CH_0, and so any codimension 2 cycle homologous to 0 on $\overline{\mathcal{S} \times_B \mathcal{S}}$ is algebraically equivalent to 0 by the following result due to Bloch and Srinivas [**8**]:

Theorem 1.4. *Let* X *be a smooth projective variety with* $CH_0(X)$ *supported on a surface* $\Sigma \subset X$. *Then codimension* 2 *cycles on* X *which are homologous to zero are algebraically equivalent to zero.*

We thus conclude that $\overline{\mathcal{Z}'} - j_* \mathcal{Z}''$ is algebraically equivalent to 0 on $\overline{\mathcal{S} \times_B \mathcal{S}}$, hence that $\mathcal{Z}' = (\overline{\mathcal{Z}'} - j_* \mathcal{Z}'')_{|\mathcal{S} \times_B \mathcal{S}}$ is algebraically equivalent to 0 on $\mathcal{S} \times_B \mathcal{S}$.

(ii) This is a direct consequence of (i), using the following nilpotence result proved independently in [**26**] and [**29**]:

Theorem 1.5. *On any smooth projective variety* X, *self-correspondences* $\Gamma \in CH^n(X \times X)_{\mathbb{Q}}$ *which are algebraically equivalent to* 0 *are nilpotent for the composition of correspondences.*

<div align="right">q.e.d.</div>

Proof of Theorem 0.3. Using the same notations as in the previous steps, we know by Proposition 1.3 that under assumptions (1) and (2), the self-correspondence

$$(1) \qquad \mathcal{Z}_t' = \Gamma_t - \mathcal{Z}_t - p_1^* \mathcal{Z}_{1,t}' - p_2^* \mathcal{Z}_{2,t}'$$

on $\mathcal{S}_t$ with $\mathbb{Q}$-coefficients is nilpotent. In particular, the morphism it induces at the level of Chow groups is nilpotent. On the other hand, recall that $\mathcal{Z}_t$ is supported on a product of curves in $\mathcal{S}_t \times \mathcal{S}_t$, hence acts trivially on $CH_0(\mathcal{S}_t)_{\mathbb{Q}}$. Obviously, both cycles $p_1^* \mathcal{Z}'_{1,t}$, $p_2^* \mathcal{Z}'_{2,t}$ act trivially on $CH_0(\mathcal{S}_t)_{\mathbb{Q},hom}$. Hence the self-correspondence $\mathcal{Z}'_t$ acts as Γ_t on $CH_0(\mathcal{S}_t)_{\mathbb{Q},hom}$, and Γ_{t*} acting on $CH_0(\mathcal{S}_t)_{\mathbb{Q},hom}$ is also nilpotent. q.e.d.

Let us now turn to our main application, namely Corollary 0.4, or a more general form of it which involves a family of surfaces S_t with an action of a finite group G and a projector $\pi \in \mathbb{Q}[G]$. Writing such a projector as $\pi = \sum_{g \in G} a_g g$, $a_g \in \mathbb{Q}$, such a projector provides a codimension 2 cycle

$$(2) \qquad \Gamma_t^{\pi} = \sum_g a_g \mathrm{Graph}\, g \in CH^2(S_t \times S_t)_{\mathbb{Q}},$$

with actions

$$\Gamma_t^{\pi *} = \sum_g a_g g^*$$

on the holomorphic forms of S, and

$$\Gamma_{t\,*}^{\pi} = \sum_g a_g g_*$$

on $CH_0(S_t)_{\mathbb{Q}}$. Denote respectively $CH_0(\mathcal{S}_t)^{\pi}_{\mathbb{Q},hom}$ the image of the projector $\Gamma_{t\,*}^{\pi}$ acting on $CH_0(S_t)_{\mathbb{Q},hom}$ and $H^{2,0}(S_t)^{\pi}$ the image of the projector $\Gamma_t^{\pi *}$ acting on $H^{2,0}(S_t)$.

Theorem 1.6. *Let $\mathcal{S} \to B$ be a smooth projective morphism with two dimensional connected fibers, where B is quasi-projective. Let G be a finite group acting in a fiberwise way on $\mathcal{S}$ and let $\pi \in \mathbb{Q}[G]$ be a projector. Assume the following:*
(1) The fibers $\mathcal{S}_t$ satisfy $H^{1,0}(\mathcal{S}_t) = 0$ and $H^{2,0}(\mathcal{S}_t)^{\pi} = 0$.
(2) A smooth projective completion (or, equivalently, any smooth projective completion) $\overline{\mathcal{S} \times_B \mathcal{S}}$ of the fibered self-product $\mathcal{S} \times_B \mathcal{S}$ is rationally connected.
Then $CH_0(\mathcal{S}_t)^{\pi}_{\mathbb{Q},hom} = 0$ for any $t \in B$.

Proof. The group G acts fiberwise on $\mathcal{S} \to B$. Thus we have the universal cycle

$$\Gamma^{\pi} \in CH^2(\mathcal{S} \times_B \mathcal{S})_{\mathbb{Q}}$$

defined as $\sum_{g \in G} a_g \mathrm{Graph}\, g$, where the graph is taken over B. Since by assumption the action of $[\Gamma^{\pi}]^*$ on $H^{2,0}(\mathcal{S}_t)$ is 0, we can apply Theorem 0.3 and conclude that Γ_{t*}^{π} is nilpotent on $CH_0(S_t)_{\mathbb{Q},hom}$. On the other hand, Γ_{t*}^{π} is a projector onto $CH_0(\mathcal{S}_t)^{\pi}_{\mathbb{Q},hom}$. The fact that it is nilpotent implies thus that it is 0, hence that $CH_0(\mathcal{S}_t)^{\pi}_{\mathbb{Q},hom} = 0$.   q.e.d.

2. Catanese and Godeaux surfaces

Our main goal in this section is to check the main assumption of Theorem 1.6, namely the rational connectedness of the fibered self-product of the universal Catanese surface, in order to prove Theorem 0.6.

We follow [11], [25]. Consider the following symmetric 5×5 matrix

$$(3) \qquad M_a = \begin{pmatrix} a_1x_1 & a_2x_2 & a_3x_3 & a_4x_4 & 0 \\ a_2x_2 & a_5x_3 & a_6x_4 & 0 & a_7x_1 \\ a_3x_3 & a_6x_4 & 0 & a_8x_1 & a_9x_2 \\ a_4x_4 & 0 & a_8x_1 & a_{10}x_2 & a_{11}x_3 \\ 0 & a_7x_1 & a_9x_2 & a_{11}x_3 & a_{12}x_4 \end{pmatrix}$$

depending on 12 parameters $a_1, \ldots, a_{12}$ and defining a symmetric bilinear (or quadratic) form $q(a,x)$ on $\mathbb{C}^5$ depending on $x \in \mathbb{P}^3$. This is a homogeneous degree 1 matrix in the variables $x_1, \ldots, x_4$, and the vanishing of its determinant gives a degree 5 surface $V(a)$ in $\mathbb{P}^3$ that generically has its singularities consisting of ordinary nodes at those points $x = (x_1, \ldots, x_4)$ where the matrix has rank only 3. We will denote by T the vector space generated by $a_1, \ldots, a_{12}$ and $B \subset \mathbb{P}(T)$ the open set of parameters a satisfying this last condition.

The surface $V(a)$ is invariant under the Godeaux action of $\mathbb{Z}/5\mathbb{Z}$ on $\mathbb{P}^3$, given by

$$g^*x_i = \zeta^i x_i, \ i = 1, \ldots, 4,$$

where g is a generator of $\mathbb{Z}/5\mathbb{Z}$ and ζ is a primitive fifth root of unity. This follows either from the explicit development of the determinant (see [25], where one monomial is incorrectly written: $x_1 x_2^2 x_4^2$ should be $x_1 x_3^2 x_4^2$) or from the following argument that we will need later on: Consider the following linear action of $\mathbb{Z}/5\mathbb{Z}$ on $\mathbb{C}^5$:

$$(4) \qquad g'(y_1, \ldots, y_5) = (\zeta y_1, \zeta^2 y_2, \ldots, \zeta^5 y_5).$$

Then one checks immediately that

$$(5) \qquad q(a,x)(g'y, g'y') = \zeta q(a, gx)(y, y'),$$

so that the discriminant of $q(a,x)$, as a function of x, is invariant under the action of g.

The Catanese surface $\Sigma(a)$ is obtained as follows: There is a natural double cover $S(a)$ of $V(a)$, which is étale away from the nodes, and parameterizes the rulings in the rank four quadric $Q(a,x)$ defined by $q(a,x)$ for $x \in V(a)$. The action of $\mathbb{Z}/5\mathbb{Z}$ on $V(a)$ lifts naturally to an action on the double cover $S(a)$, and we define $\Sigma(a)$ as the quotient of $S(a)$ by $\mathbb{Z}/5\mathbb{Z}$.

We will need later on the following alternative description of $S(a)$, which also explains the natural lift of the $\mathbb{Z}/5\mathbb{Z}$-action on $S(a)$: Note first of all that the quadrics $Q(a,x)$ pass through the point $y_0 = (0,0,1,0,0)$ of $\mathbb{P}^4$.

Lemma 2.1. *For the general point* $(a, x) \in \mathbb{P}(T) \times \mathbb{P}^3$ *such that* $x \in V(a)$, *the quadric* $Q(a, x)$ *is not singular at the point* y_0.

Proof. It suffices to exhibit one pair (a, x) satisfying this condition, and such that the surface $V(a)$ is well-defined, that is the discriminant of $q(a, x)$, seen as a function of x, is not identically 0. Indeed, the family of surfaces $V(a)$ is flat over the base near such a point, and the result for the generic pair (a, x) will then follow because the considered property is open on the total space of this family.

We choose (a, x) in such a way that the first column of the matrix (3) is 0, so that $x \in V(a)$. For example, we impose the conditions:

$$a_1 = 0, \ a_2 = 0, \ x_3 = 0, \ x_4 = 0.$$

Then the quadratic form $q(a, x)$ has for matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_7 x_1 \\ 0 & 0 & 0 & a_8 x_1 & a_9 x_2 \\ 0 & 0 & a_8 x_1 & a_{10} x_2 & 0 \\ 0 & a_7 x_1 & a_9 x_2 & 0 & 0 \end{pmatrix}$$

It is clear that y_0 is not a singular point of the corresponding quadric if $a_8 x_1 \neq 0$. On the other hand, for a satisfying $a_1 = 0$, $a_2 = 0$, and for a general point $x = (x_1, \ldots, x_4)$, the matrix of $q(a, x)$ takes the form

$$\begin{pmatrix} 0 & 0 & a_3 x_3 & a_4 x_4 & 0 \\ 0 & a_5 x_3 & a_6 x_4 & 0 & a_7 x_1 \\ a_3 x_3 & a_6 x_4 & 0 & a_8 x_1 & a_9 x_2 \\ a_4 x_4 & 0 & a_8 x_1 & a_{10} x_2 & a_{11} x_3 \\ 0 & a_7 x_1 & a_9 x_2 & a_{11} x_3 & a_{12} x_4. \end{pmatrix}$$

It is elementary to check that this matrix is generically of maximal rank.

                                                                q.e.d.

It follows from this lemma that for generic a and generic $x \in V(a)$, the two rulings of $Q(a, x)$ correspond bijectively to the two planes through y_0 contained in $Q(a, x)$. Hence for generic a, the surface $S(a)$ is birationally equivalent to the surface $S'(a)$ parameterizing planes passing through y_0 and contained in $Q(a, x)$ for some $x \in \mathbb{P}^3$ (which then lies necessarily in $V(a)$). It follows from formula (5) that $S'(a)$, which is a surface contained in the Grassmannian G_0 of planes in $\mathbb{P}^4$ passing through y_0, is invariant under the $\mathbb{Z}/5\mathbb{Z}$-action on G_0 induced by (4). Hence there is a canonical $\mathbb{Z}/5\mathbb{Z}$-action on $S'(a)$ which is compatible with the $\mathbb{Z}/5\mathbb{Z}$-action on $V(a)$, and this immediately implies that the latter lifts to an action on the surface $S(a)$ for any $a \in B$.

The Catanese surface has been defined as the quotient of $S(a)$ by $\mathbb{Z}/5\mathbb{Z}$. In the following, we are going to apply Theorem 1.6, and will thus work with the universal family $\mathcal{S} \to B$ of double covers $S(a)$, $a \in B$,

with its $\mathbb{Z}/5\mathbb{Z}$-action defined above, where $B \subset \mathbb{P}^{11}$ is the Zariski open set parameterizing smooth surfaces $S(a)$.

We prove now:

Proposition 2.2. *The universal family* $\mathcal{S} \to B$ *has the property that the fibered self-product* $\mathcal{S} \times_B \mathcal{S}$ *has a rationally connected smooth projective compactification.*

Proof. By the description given above, the family $\mathcal{S} \to B$ of surfaces $S(a)$, $a \in B$, maps birationally to an irreducible component of the following variety

$$(6) \qquad W = \{(a, x, [P]) \in \mathbb{P}(T) \times \mathbb{P}^3 \times G_0, \, q(a,x)_{|P} = 0\},$$

by the rational map that, to a general point (a, x), $x \in V(a)$, and a choice of ruling in the quadric $Q(a, x)$, associates the unique plane P passing through y_0, contained in $Q(a, x)$ and belonging to the chosen ruling. It follows that $\mathcal{S} \times_B \mathcal{S}$ maps birationally by the same map (which we will call Ψ) onto an irreducible component W_2^0 of the following variety

$$(7) \qquad W_2 := \{(a, x, y, [P], [P']) \in \mathbb{P}(T) \times \mathbb{P}^3 \times \mathbb{P}^3 \times G_0 \times G_0,$$
$$q(a, x)_{|P} = 0, \, q(a, y)_{|P'} = 0\}.$$

Let $\mathcal{E}$ be the vector bundle of rank 5 on G_0 whose fiber at a point $[P]$ parameterizing a plane $P \subset \mathbb{P}^4$ passing through y_0 is the space

$$H^0(P, \mathcal{O}_P(2) \otimes \mathcal{I}_{y_0}).$$

The family of quadrics $q(a, x)$ on $\mathbb{P}^4$ provides a 12 dimensional linear space T of sections of the bundle

$$\mathcal{F}_0 := pr_1^* \mathcal{O}_{\mathbb{P}^3}(1) \otimes pr_3^* \mathcal{E} \oplus pr_2^* \mathcal{O}_{\mathbb{P}^3}(1) \otimes pr_4^* \mathcal{E}$$

on

$$Y_0 := \mathbb{P}^3 \times \mathbb{P}^3 \times G_0 \times G_0,$$

where as usual the pr_i's denote the various projections from Y_0 to its factors. For a point $a \in T$, the corresponding section of $\mathcal{F}_0$ is equal to $(q(a, x)_{|P}, q(a, x')_{|P'})$ at the point $(x, x', [P], [P'])$ of Y_0. Formula (7) tells us that W_2 is the zero set of the corresponding universal section of the bundle

$$pr_1'^* \mathcal{O}_{\mathbb{P}(T)}(1) \otimes pr_2'^* \mathcal{F}_0$$

on $\mathbb{P}(T) \times Y_0$, where the pr_i' are now the two natural projections from $\mathbb{P}(T) \times Y_0$ to its summands. Note that W_2^0 has dimension 15, hence has the expected codimension 10, since $\dim \mathbb{P}(T) \times Y_0 = 25$.

There is now a subtlety in our situation: It is not hard to see that T generates generically the bundle $\mathcal{F}_0$ on Y_0. Hence there is a "main" component of W_2 that is also of dimension 15, and is generically fibered into $\mathbb{P}^1$'s over Y_0. This component is not equal to W_2^0 for the following reason: If one takes two general planes P, P' through y_0, and two general points x, $x' \in \mathbb{P}^3$, the conditions that $q(a, x)$ vanishes identically on P

and $q(a, x')$ vanishes identically on P' implies that the third column of the matrix $M(a)$ is identically 0, hence that the point y_0 generates in fact the kernel of both matrices $M(a, x)$ and $M(a, x')$. On the other hand, by construction of the map Ψ, generically along $\operatorname{Im} \Psi \subset W_2^0$, the point y_0 is a smooth point of the quadrics $Q(a, x)$ and $Q(a, x')$.

The following lemma describes the component W_2^0.

Lemma 2.3. *Let* $\Phi : W_2^0 \to Y_0 = \mathbb{P}^3 \times \mathbb{P}^3 \times G_0 \times G_0$ *be the restriction to* W_2^0 *of the second projection* $\mathbb{P}(T) \times Y_0 \to Y_0$.

(i) The image $\operatorname{Im} \Phi$ *is a hypersurface* $\mathcal{D}$ *which admits a rationally connected desingularization.*

(ii) The generic rank of the evaluation map restricted to $\mathcal{D}$:

$$T \otimes \mathcal{O}_{\mathcal{D}} \to \mathcal{F}_{0|\mathcal{D}}$$

is 9. (Note that the generic rank of the evaluation map $T \otimes \mathcal{O}_{Y_0} \to \mathcal{F}_0$ *is 10.)*

Proof. We already mentioned that the map $\Phi \circ \Psi$ cannot be dominating. Let us explain more precisely why, as this will provide the equation for $\mathcal{D}$: Let $[P], [P'] \in G_0$ and $x, x' \in \mathbb{P}^3$. The condition that $(x, x', P, P') \in \Phi(W_2^0) = \operatorname{Im}(\Phi \circ \Psi)$ is that for some $a \in \mathbb{P}(V)$,

$$(8) \qquad\qquad q(a, x)_{|P} = 0, \quad q(a, x')_{|P'} = 0$$

and furthermore that y_0 is a smooth point of both quadrics $Q(a, x)$ and $Q(a, x')$.

Let $e, f \in P$ be vectors such that y_0, e, f form a basis of P, and choose similarly $e', f' \in P'$ in order to get a basis of P'. Then among equations (8), we get

$$(9) \qquad q(a, x)(y_0, e) = 0, \quad q(a, x)(y_0, f) = 0,$$
$$q(a, x')(y_0, x')(y_0, e') = 0, \quad q(a, x')(y_0, f') = 0.$$

For fixed P, P', x, x', these equations are linear forms in the variables a_3, a_6, a_8, a_9 and it is not hard to see that they are linearly independent for a generic choice of P, P', x, x', so that (9) implies that $a_3 = a_6 = a_8 = a_9 = 0$. But then, looking at the matrix $M(a)$ of (3) we see that y_0 is in the kernel of $Q(a, x)$ and $Q(a, x')$. As already mentioned, the latter does not happen generically along $\operatorname{Im} \Psi$, and we deduce that $\operatorname{Im}(\Phi \circ \Psi)$ is contained in the hypersurface $\mathcal{D}$ where the four linear forms (9) in the four variables a_3, a_6, a_8, a_9 are not independent.

We claim that $\mathcal{D}$ is rationally connected. To see this we first prove that it is irreducible, which is done by restricting the equation f of $\mathcal{D}$ (which is the determinant of the (4×4) matrix whose columns are the linear forms (9) written in the basis a_3, a_6, a_8, a_9) to a subvariety Z of Y_0 of the form $Z = \mathbb{P}^3 \times \mathbb{P}^3 \times C \times C' \subset Y_0$, where C, C' are curves in G_0.

Consider the following 1 dimensional families $C \cong \mathbb{P}^1$, $C' \cong \mathbb{P}^1$ of planes passing through y_0:

$$P_t =< e_1, \lambda e_2 + \mu e_4, e_3 >, \quad t = (\lambda, \mu) \in \mathbb{P}^1 \cong C,$$
$$P'_{t'} =< e_5, \lambda' e_2 + \mu' e_4, e_3 >, \quad t' = (\lambda', \mu') \in \mathbb{P}^1 \cong C'.$$

The equations (9) restricted to the parameters t, t', x, x' give the following four combinations of a_3, a_6, a_8, a_9 depending on λ, μ, x, x':

$$a_3 x_3, \quad \lambda a_6 x_4 + \mu a_8 x_1, \quad a_9 x'_2, \quad \lambda' a_6 x'_4 + \mu' a_8 x'_1.$$

Taking the determinant of this family gives

$$f_{|Z} = x_3 x'_2 (\lambda \mu' x_4 x'_1 - \lambda' \mu x'_4 x_1).$$

The hypersurface in $Z = \mathbb{P}^3 \times \mathbb{P}^3 \times C \times C'$ defined by $f_{|Z}$ has three irreducible components, which belong respectively to the linear systems

$$|pr_1^* \mathcal{O}_{\mathbb{P}^3}(1)|, \quad |pr_2^* \mathcal{O}_{\mathbb{P}^3}(1)|,$$
$$|pr_1^* \mathcal{O}_{\mathbb{P}^3}(1) \otimes pr_2^* \mathcal{O}_{\mathbb{P}^3}(1) \otimes pr_3^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes pr_4^* \mathcal{O}_{\mathbb{P}^3}(1)|,$$

where the pr_i's are now the projections from $Z = \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$ to its factors.

As the restriction map from $\mathrm{Pic}\, Y_0$ to $\mathrm{Pic}\, Z$ is injective, we see that if $\mathcal{D}$ was reducible, it would have an irreducible component in one of the linear systems

$$|p_1^* \mathcal{O}_{\mathbb{P}^3}(1)|, \quad |p_2^* \mathcal{O}_{\mathbb{P}^3}(1)|,$$
$$|p_1^* \mathcal{O}_{\mathbb{P}^3}(1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^3}(1) \otimes p_3^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_4^* \mathcal{O}_{\mathbb{P}^3}(1)|,$$

where the p_i's are now the projections from $Y_0 = \mathbb{P}^3 \times \mathbb{P}^3 \times G_0 \times G_0$ to its factors. In particular, its restriction to either any $\mathbb{P}^3 \times \{x'\} \times \{[P]\} \times \{[P']\}$ or any $\{x\} \times \mathbb{P}^3 \times \{[P]\} \times \{[P']\}$ would have an irreducible component of degree 1.

That this is not the case is proved by considering now the case where

$$P =< e_1 + e_2, \, e_4 + e_5, \, e_3 >, \quad P' =< e_1 + e_4, \, e_2 + e_5, \, e_3 > .$$

Then the four linear forms in (9) become

$$a_3 x_3 + a_6 x_4, \quad a_8 x_1 + a_9 x_2, \quad a_3 x'_3 + a_8 x'_1, \quad a_6 x'_4 + a_9 x'_2.$$

One can immediately check that for generic choice of x', the quadratic equation in x given by this determinant has rank at least 4, and in particular is irreducible. Hence the restriction of f to $\mathbb{P}^3 \times \{x'\} \times \{[P]\} \times \{[P']\}$ has no irreducible component of degree 1, and similarly for $\{x\} \times \mathbb{P}^3 \times \{[P]\} \times \{[P']\}$.

Rational connectedness of $\mathcal{D}$ now follows from the fact that the projection on the last three summands of Y_0, restricted to $\mathcal{D}$,

$$p : \mathcal{D} \hookrightarrow Y_0 \to \mathbb{P}^3 \times G_0 \times G_0,$$

has for general fibers smooth two-dimensional quadrics, as shows the computation we just made. Hence $\mathcal{D}$ admits a rationally connected smooth projective model, for example by [**14**].

We already proved that $\mathrm{Im}\,(\Phi \circ \Psi) \subset \mathcal{D}$. We will now prove that $\mathrm{Im}\,(\Phi \circ \Psi)$ contains a Zariski open subset of $\mathcal{D}$ and also statement ii). This is done by the following argument which involves an explicit computation at a point $a_0 \in \mathbb{P}(T)$ where the surface $V(a)$ becomes very singular but has enough smooth points to conclude.

We start from the following specific matrix :

$$
(10) \qquad M_0 = \begin{pmatrix} x_1 & 0 & x_3 & 0 & 0 \\ 0 & x_3 & x_4 & 0 & 0 \\ x_3 & x_4 & 0 & x_1 & 0 \\ 0 & 0 & x_1 & x_2 & 0 \\ 0 & 0 & 0 & 0 & x_4 \end{pmatrix}
$$

We compute $\det M_0 = -x_4 x_3 x_1^3 - x_1 x_2 x_4^3 - x_4 x_2 x_3^3$. The surface $V(a_0)$ defined by $\det M_0$ is rational and smooth at the points

$$
(11) \qquad x = (1, -\tfrac{1}{2}, 1, 1), \quad x' = (1, -\tfrac{2}{9}, 2, -1).
$$

For each of the corresponding planes P and P' contained in $Q(a_0, x)$, $Q(a_0, x')$ respectively and passing through y_0, we have two choices. We choose the following:

$$
(12) \quad P := <y_0,\, e,\, f>, \quad e = e_1 + e_2 - 2e_4, \quad f = e_2 - e_4 + \sqrt{-\tfrac{1}{2}}e_5,
$$

$$
P' = <y_0,\, e',\, f>, \quad e' = 4e_1 - e_2 - 9e_4, \quad f' = e_2 + e_4 + \tfrac{4}{3}e_5.
$$

(We use here the standard basis $(e_1, \ldots, e_5)$ of $\mathbb{C}^5$, so that $y_0 = e_3$.) The vector $e - e_3$ (resp. $e' - 2e_3$) generates the kernel of $q(a_0, x)$ (resp. $q(a_0, x')$). As the quadrics $q(a_0, x)$ and $q(a_0, x')$ have rank 3 and the surface $V(a_0)$ is smooth at x and x', the points $(x, [P])$, $(x', [P'])$ determine smooth points of the surface $S(a_0)$. Even if the surface $S(a_0)$ is not smooth, so that $a_0 \notin B$, the universal family $\mathcal{S} \to B$ extends to a smooth non-proper map $p : \mathcal{S}_e \to B_e \subset \mathbb{P}(T)$ near the points $(a_0, x, [P])$ and $(a_0, x', [P'])$. Furthermore the morphism Ψ is well defined at the points (a_0, x, P), (a_0, x', P'), since the point y_0 is not a singular point of the quadric $Q(a_0, x)$ or $Q(a_0, x')$.

We claim that the map

$$
\Phi \circ \Psi : \mathcal{S}_e \times_{B_e} \mathcal{S}_e \to Y_0 = \mathbb{P}^3 \times \mathbb{P}^3 \times G_0 \times G_0
$$

has constant rank 13 near the point

$$
((a_0, x, [P]), (a_0, x', [P'])) \in \mathcal{S}_e \times_{B_e} \mathcal{S}_e.
$$

This implies that the image of $\Phi \circ \Psi$ is of dimension 13, which is the dimension of $\mathcal{D}$, so that $\mathrm{Im}\, \Phi \circ \Psi$ has to contain a Zariski open set of $\mathcal{D}$ since $\mathcal{D}$ is irreducible. This implies that $\mathrm{Im}\, \Phi = \mathcal{D}$ as desired.

To prove the claim, we observe that it suffices to show the weaker claim that the rank of

(13) $$(\Phi \circ \Psi)_* : T_{\mathcal{S}_e \times_{B_e} \mathcal{S}_e} \to T_{Y_0}$$

is equal to 13 at the given point $((a_0, x, [P]), (a_0, x', [P']))$. Indeed, as $\mathcal{S}_e \times_{B_e} \mathcal{S}_e$ is smooth at the point $((a_0, x, [P]), (a_0, x', [P']))$, the rank of the differential in (13) can only increase in a neighborhood of $((a_0, x, [P]),$ $(a_0, x', [P']))$ and on the other hand, we know that it is not of maximal rank at any point of $\mathcal{S}' \times_{B'} \mathcal{S}'$, since $\mathrm{Im}\,(\Phi \circ \Psi) \subset \mathcal{D}$. Hence it must stay constant near $((a_0, x, [P]), (a_0, x', [P']))$.

We now compute the rank of $(\Phi \circ \Psi)_*$ at the point $((a_0, x, [P]),$ $(a_0, x', [P']))$. Note that the birational map $\Psi : \mathcal{S}_e \times_{B_e} \mathcal{S}_e \to W_2$ is a local isomorphism near the point $((a_0, x, [P]), (a_0, x', [P']))$, because it is well defined at this point and its inverse too. This argument proves that not only W_2^0 but also W_2 is smooth of dimension 15 near the point $(a_0, x, x', [P], [P'])$. Hence it suffices to compute the rank of the differential

(14) $$\Phi_* : T_{W_2,(a_0,x,x',[P],[P'])} \to T_{Y_0,(x,x',[P],[P'])}$$

of the map $\Phi : W_2 \to Y_0$ at the point $(a_0, x, x', [P], [P'])$.

Recalling that W_2 is defined as the zero set of the universal section σ_{univ} of the bundle

$$pr_1'^* \mathcal{O}_{\mathbb{P}(T)}(1) \otimes pr_2'^* \mathcal{F}_0$$

on $\mathbb{P}(T) \times Y_0$, the tangent space of W_2 at the point $(a_0, x, x', [P], [P'])$ is equal to

$$\mathrm{Ker}\,(d\sigma_{univ} : T_{\mathbb{P}(T),a_0} \times T_{Y_0,(x,x',[P],[P'])} \to \mathcal{F}_{0,(x,x',[P],[P'])}).$$

Clearly the differential $d\sigma_{univ}$ restricted to $T_{\mathbb{P}(T),a_0}$ is induced by the evaluation map

(15) $$\mathrm{ev}_{(x,x',[P],[P'])} : T \to \mathcal{F}_{0,(x,x',[P],[P'])}$$

at the point $(x, x', [P], [P'])$ of Y_0. On the other hand, the differential $d\sigma_{univ}$ restricted to the tangent space $T_{Y_0,(x,x',[P],[P'])}$ is surjective because the variety $S(a_0) \times S(a_0)$ is smooth of codimension 4 and isomorphic via Φ to $V(\sigma_{univ}(a_0))$ near $(x, x', [P], [P'])$. It follows from this that the corank of the second projection

$$\Phi_* : \mathrm{Ker}\, d\sigma_{univ} = T_{W_2,(a_0,x,x',[P],[P'])} \subset T_{\mathbb{P}(T),a_0} \oplus T_{Y_0,(x,x',[P],[P'])}$$
$$\to T_{Y_0,(x,x',[P],[P'])}$$

is equal to the corank of the map $d\sigma_{univ} : T_{\mathbb{P}(T),a_0} \to \mathcal{F}_{0,(x,x',[P],[P'])})$, that is, to the corank of the evaluation map $\mathrm{ev}_{(x,x',[P],[P'])}$ of (15).

In particular, the rank of Φ_* in (14) is equal to 13 if and only if the rank of the evaluation map $\mathrm{ev}_{x,x',[P],[P']}$ is equal to 9, which is our statement (ii).

In conclusion, we proved that (i) is implied by (ii) and that (ii) itself implied by (ii) at the given point $(x, x', [P], [P'])$ of (11), (12). It just remains to prove that the rank of $\mathrm{ev}_{(x,x',[P],[P'])}$ is equal to 9 at this point, which is done by the explicit computation of the rank of the family of linear forms in the a_i given by

$$q(a,x)(y_0,e), \quad q(a,x)(y_0,f), \quad q(a,x)(e,e), \quad q(a,x)(f,f),$$
$$q(a,x)(e,f), \quad q(a,x')(y_0,e'), \quad q(a,x')(y_0,f'), \quad q(a,x')(e',e'),$$
$$q(a,x')(f',f'), \quad q(a,x')(e',f').$$

These forms are the following:

$$a_3 + a_6 - 2a_8, \quad a_6 - a_8 - \frac{1}{2}\sqrt{-\frac{1}{2}}a_9, \quad a_1 + a_5 - 2a_{10} - a_2 - 4a_4,$$

$$a_5 - \frac{a_{10}}{2} - \frac{a_{12}}{2} + 2\sqrt{-\frac{1}{2}}a_7 - 2\sqrt{-\frac{1}{2}}a_{11},$$

$$-\frac{a_2}{2} - a_4 + a_5 + \sqrt{-\frac{1}{2}}a_7 - a_{10} - 2\sqrt{-\frac{1}{2}}a_{11}$$

$$8a_3 + a_6 - 9a_8, \quad -a_6 + a_8 - \frac{8}{27}a_9,$$

$$16a_1 + 2a_5 - 18a_{10} + \frac{16}{9}a_2 + 72a_4,$$

$$2a_5 - \frac{2}{9}a_{10} - \frac{16}{9}a_{12} + \frac{8}{3}a_7 + \frac{16}{3}a_{11},$$

$$-\frac{8}{9}a_2 - 4a_4 - 2a_5 - \frac{4}{3}a_7 + 2a_{10} - 24a_{11}.$$

One can immediately check that the rank of this family is 9.   q.e.d.

It follows from Lemma 2.3 that W_2^0 (or rather a smooth projective birational model of W_2^0) is rationally connected, since a Zariski open set of W_2^0 is a $\mathbb{P}^2$-bundle over a Zariski open set $\mathcal{D}^0$ of $\mathcal{D}$ that is smooth and admits a rationally connected completion. Hence $\mathcal{S} \times_B \mathcal{S}$ is also rationally connected, as it is birationally equivalent to W_2^0.   q.e.d.

We get by application of Theorem 1.6 the following statement (cf. Theorem 0.6):

Corollary 2.4. *The Catanese surface* $\Sigma(a)$ *has* CH_0 *equal to* $\mathbb{Z}$.

Proof. Indeed, the surface $\Sigma(a)$ has $h^{1,0} = h^{2,0} = 0$ (cf. [**11**]) which means equivalently that the surface $S(a)$ introduced above has $h_{inv}^{2,0} = 0$ where "inv" means invariant under the $\mathbb{Z}/5\mathbb{Z}$-action, which by the (birational) description we gave of the family $\mathcal{S}$ of surfaces is clearly

defined on $\mathcal{S}$. The result is then a consequence of Theorem 1.6 applied to the projector π_{inv} onto the $\mathbb{Z}/5\mathbb{Z}$-invariant part, which shows that

$$CH_0(\Sigma(a))_{\mathbb{Q},hom} = CH_0(S(a))_{\mathbb{Q},hom}^{\pi_{inv}} = 0$$

and from Roitman's theorem [**24**] which says that $CH_0(\Sigma(a))$ has no torsion. q.e.d.

Corollary 2.5. *The Barlow surface $\Sigma'(b)$ has CH_0 equal to $\mathbb{Z}$.*

Proof. Indeed, the Barlow surface $\Sigma'(b)$ is a quotient of the Catanese surface $\Sigma(b)$ by an involution, hence $CH_0(\Sigma'(b)) \hookrightarrow CH_0(\Sigma(b))$ since $CH_0(\Sigma'(b))$ has no torsion by [**24**]. q.e.d.

3. On the Chow motive of complete intersection $K3$ surfaces

Let S be a $K3$ surface. The Hodge structure on $H^2(S, \mathbb{Z})$ is a weight 2 polarized Hodge structure with $h^{2,0} = 1$. In [**18**], Kuga and Satake construct an abelian variety $K(S)$ associated to this Hodge structure. Its main property is the fact that there is a natural injective morphism of weight 2 Hodge structures:

$$H^2(S, \mathbb{Z}) \to H^2(K(S), \mathbb{Z}).$$

Such a morphism of Hodge structures in turn provides, using Künneth decomposition and Poincaré duality a Hodge class

$$\alpha_S \in \mathrm{Hdg}^4(S \times K(S))$$

where $\mathrm{Hdg}^4(X)$ denotes the space of rational Hodge classes on X.

This class is not known in general to satisfy the Hodge conjecture, that is, to be the class of an algebraic cycle. This is known to hold for Kummer surfaces (see [**19**]), and for some $K3$ surfaces with Picard number 16 (see [**23**]). The deformation theory of $K3$ surfaces, and more particularly the fact that any projective $K3$ surface deforms to a Kummer surface, combined with the global invariant cycle theorem of Deligne [**12**], imply the following result (cf. [**13**], [**2**]):

Theorem 3.1. *Let S be a projective $K3$ surface. There exist a connected quasi-projective variety B, a family of projective $K3$ surfaces $\mathcal{S} \to B$, a family of abelian varieties $\mathcal{K} \to B$, (where all varieties are quasi-projective and all morphisms are smooth projective), and a Hodge class*

$$\eta \in \mathrm{Hdg}^4(\mathcal{S} \times_B \mathcal{K}),$$

such that:

1) *for some point $t_0 \in B$, $\mathcal{S}_{t_0} \cong S$;*
2) *for any point $t \in B$, $\mathcal{K}_t \cong K(S_t)$ and $\eta_t = \alpha_{S_t}$;*
3) *for some point $t_1 \in B$, the class η_{t_1} is algebraic.*

Here by "Hodge class η" we mean that the class η comes from a Hodge class on some smooth projective compactification of $\mathcal{S} \times_B \mathcal{K}$.

The existence of the family is an algebraicity statement for the Kuga-Satake construction (see [**13**]), which can then be done in family for $K3$ surfaces with given polarization, while the last item follows from the fact that the locally complete such families always contain Kummer fibers S' for which the class $\alpha_{S'}$ is known to be algebraic.

Corollary 3.2. [**2**] *The Hodge class α_S is "motivated."*

This means (cf. [**2**]) that this Hodge class can be constructed via algebraic correspondences from Hodge classes on auxiliary varieties, which are either algebraic or obtained by inverting Lefschetz operators. In particular the class α_S is algebraic if the standard Lefschetz conjecture holds.

Corollary 3.3. *Assume the variational form of the Hodge conjecture holds, or assume the Lefschetz standard conjecture in degrees 2 and 4. Then the class α_S is algebraic for any projective $K3$ surface S.*

Indeed, the variational Hodge conjecture states that in the situation of Theorem 3.1, if a Hodge class η on the total space has an algebraic restriction on one fiber, then its restriction to *any* fiber is algebraic. In our situation, it will be implied by the Lefschetz conjecture for degree 2 and degree 4 cohomology on a smooth projective compactification of $\mathcal{S} \times_B \mathcal{K}$.

From now on, we assume that the Kuga-Satake correspondence α_S is algebraic for a general projective $K3$ surface S of genus g (which means by definition that S comes equipped with an ample line bundle of self-intersection $2g - 2$). We view the class α_S as an injective morphism of Hodge structures

$$(16) \qquad\qquad \alpha_S : H^2(S, \mathbb{Q}) \to H^2(K(S), \mathbb{Q})$$

and use a polarization h of $K(S)$ to construct an inverse of α_S by the following lemma (which can be proved as well by the explicit description $K(S)$ as a complex torus and its polarization, cf. [**18**]). In the following, $H^2(S, \mathbb{Q})_{tr}$ denotes the transcendental cohomology of S, which is defined as the orthogonal complement of the Néron-Severi group of S.

Lemma 3.4. *Let $h = c_1(H) \in H^2(K(S), \mathbb{Q})$ be the class of an ample line bundle, where S is a very general $K3$ surface of genus g. Then there exists a nonzero rational number λ_g such that the endomorphism*

$$ {}^t\alpha_S \circ (h^{N-2}\cup) \circ \alpha_S : H^2(S, \mathbb{Q}) \to H^2(S, \mathbb{Q})$$

restricts to $\lambda_g Id$ on $H^2(S, \mathbb{Q})_{tr}$, where $N = \dim K(S)$ and

$$ {}^t\alpha_S : H^{2N-2}(K(S), \mathbb{Q}) \to H^2(S, \mathbb{Q})$$

is the transpose of the map α_S of (16) with respect to Poincaré duality.

Proof. The composite ${}^t\alpha_S \circ h^{N-2} \circ \alpha_S$ is an endomorphism of the Hodge structure on $H^2(S, \mathbb{Q})$. As S is very general, this Hodge structure has only a one dimensional $\mathbb{Q}$-vector space of algebraic classes, generated by the polarization, and its orthogonal is a simple Hodge structure with only the homotheties as endomorphisms. We conclude that ${}^t\alpha_S \circ h^{N-2} \circ \alpha_S$ preserves $H^2(S, \mathbb{Q})_{tr}$ and acts on it as an homothety with rational coefficient (which is thus independent of S). It just remains to show that it does not act as zero on $H^2(S, \mathbb{Q})_{tr}$. This follows from the second Hodge-Riemann bilinear relations, which say that for $\omega \in H^{2,0}(S)$, $\omega \neq 0$ we have

$$\langle \omega, {}^t\alpha_S(h^{N-2} \cup \alpha_S(\overline{\omega})) \rangle_S = \langle \alpha_S(\omega), h^{N-2} \cup \alpha_S(\overline{\omega}) \rangle_{K(S)}$$

which is > 0 because $\alpha_S(\omega) \neq 0$ in $H^{2,0}(K(S))$. Hence ${}^t\alpha_S(h^{N-2} \cup \alpha_S(\overline{\omega})) \neq 0$. q.e.d.

We now start from a rationally connected variety X of dimension n, with a vector bundle E of rank $n - 2$ on X, such that

(17) $$-K_X = \det E$$

and the following properties hold:

(18)

For general $x, y \in X$, and for general $\sigma \in H^0(X, E \otimes \mathcal{I}_x \otimes \mathcal{I}_y)$, the zero locus $V(\sigma)$ is a smooth connected surface with 0 irregularity.

The surfaces $V(\sigma)$ are then smooth $K3$ surfaces, since they have a trivial canonical bundle by (17). Let L be an ample line bundle on X, inducing a polarization of genus g on the $K3$ surface $S_\sigma := V(\sigma)$ for $\sigma \in B \subset \mathbb{P}(H^0(X, E))$.

We now prove:

Theorem 3.5. *Assume the Kuga-Satake correspondence α_S is algebraic, for the general $K3$ surface S with such a polarization. Then, for any $\sigma \in B$, the Chow motive of S_σ is a direct summand of the motive of an abelian variety.*

Let us explain the precise meaning of this statement. The algebraicity of the Kuga-Satake correspondence combined with Lemma 3.4 implies that there is a codimension 2 algebraic cycle

$$Z_S \in CH^2(S \times K(S))_{\mathbb{Q}}$$

with the property that the cycle Γ_S defined by

$$\Gamma_S = {}^t Z_S \circ h^{N-2} \circ Z_S \in CH^2(S \times S)_{\mathbb{Q}}$$

has the property that its cohomology class $[\Gamma] \in H^4(S \times S, \mathbb{Q})$ induces a nonzero homothety

(19) $$[\Gamma]_* = \lambda Id : H^2(S, \mathbb{Q})_{tr} \to H^2(S, \mathbb{Q})_{tr},$$

which can be equivalently formulated as follows: Let us introduce the cycle $\Delta_{S,tr}$, which in the case of a $K3$ surface is canonically defined, by the formula

$$(20) \qquad \Delta_{S,tr} = \Delta_S - o_S \times S - S \times o_S - \sum_{ij} \alpha_{ij} C_i \times C_j,$$

where Δ_S is the diagonal of S, o_S is the canonical 0-cycle of degree 1 on S introduced in [6], the C_i form a basis of $(\mathrm{Pic}\, S) \otimes \mathbb{Q} = \mathrm{NS}(S)_{\mathbb{Q}}$, and the α_{ij} are the coefficients of the inverse of the matrix of the intersection form of S restricted to $NS(S)$. This corrected diagonal cycle is a projector and it has the property that its action on cohomology is the orthogonal projector $H^*(S, \mathbb{Q}) \to H^2(S, \mathbb{Q})_{tr}$. Formula (19) says that we have the cohomological equality

$$(21) \qquad [\Gamma \circ \Delta_{S,tr}] = \lambda[\Delta_{S,tr}] \text{ in } H^4(S \times S, \mathbb{Q}).$$

A more precise form of Theorem 3.5 says that we can get in fact such an equality at the level of Chow groups:

Theorem 3.6. *Assume the Kuga-Satake correspondence α_S is algebraic, for a general $K3$ surface with such a polarization. Then, for any $\sigma \in B$, there is an abelian variety A_σ, and cycles $Z \in CH^2(S_\sigma \times A_\sigma)_{\mathbb{Q}}$, $Z' \in CH^{N'}(A_\sigma \times S_\sigma)_{\mathbb{Q}}$, $N' = \dim A_\sigma$, with the property that*

$$(22) \qquad Z' \circ Z \circ \Delta_{S,tr} = \lambda \Delta_{S,tr}$$

for a nonzero rational number λ.

The proof will use two preparatory lemmas. Let $B \subset \mathbb{P}(H^0(X, E))$ be the open set parameterizing smooth surfaces $S_\sigma = V(\sigma) \subset X$. Let $\pi : \mathcal{S} \to B$ be the universal family, that is:

$$(23) \qquad \mathcal{S} = \{(\sigma, x) \in B \times X, \sigma(x) = 0\}.$$

The first observation is the following:

Lemma 3.7. *Under assumption (18), the fibered self-product $\mathcal{S} \times_B \mathcal{S}$ is rationally connected (or rather, admits a smooth projective rationally connected completion).*

Proof. From (23), we deduce that

$$(24) \quad \mathcal{S} \times_B \mathcal{S} = \{(\sigma, x, y) \in B \times X \times X, \sigma(x) = \sigma(y) = 0\}.$$

Hence $\mathcal{S} \times_B \mathcal{S}$ is Zariski open in the following variety:

$$W := \{(\sigma, x, y) \in \mathbb{P}(H^0(X, E)) \times X \times X, \sigma(x) = \sigma(y) = 0\}.$$

In particular, as it is irreducible, it is Zariski open in one irreducible component W^0 of W.

Consider the projection on the two last factors:

$$(p_2, p_3) : W \to X \times X.$$

Its fibers are projective spaces, so that there is only one "main" irreducible component W^1 of W dominating $X \times X$ and it admits a smooth rationally connected completion since $X \times X$ is rationally connected.

Assumption (18) now tells us that at a general point of W^1, the first projection $p_1 : W \to B$ is smooth of relative dimension 4. It follows that W is smooth at this point which belongs to both components W^0 and W^1. Thus $W^0 = W^1$ and $\mathcal{S} \times_B \mathcal{S} \cong_{birat} W^0$ admits a smooth rationally connected completion.   q.e.d.

The next step is the following lemma:

Lemma 3.8. *Assume the Kuga-Satake correspondence α_S is algebraic for the general polarized K3 surface of genus g. Then there exist a rational number $\lambda \neq 0$, a family $\mathcal{A} \to B$ of polarized N'-dimensional abelian varieties, with relative polarization $\mathcal{L}$ and a codimension 2 cycle*

$$\mathcal{Z} \in CH^2(\mathcal{S} \times_B \mathcal{A})_{\mathbb{Q}}$$

such that for very general $t \in B$, the cycle

$$\Gamma_t := {}^t\mathcal{Z}_t \circ c_1(\mathcal{L}_t)^{N'-2} \circ \mathcal{Z}_t \circ \Delta_{tr,t}$$

satisfies:

$$(25) \qquad\qquad [\Gamma_t] = \lambda[\Delta_{tr,t}] \in H^4(S_t \times S_t, \mathbb{Q}).$$

In this formula, the term $c_1(\mathcal{L}_t)^{N'-2}$ is defined as the self-correspondence of $\mathcal{A}_t$ that consists of the cycle $c_1(\mathcal{L}_t)^{N'-2}$ supported on the diagonal of $\mathcal{A}_t$. We also recall that the codimension 2-cycle $\Delta_{tr,t} \in CH^2(S_t \times S_t)_{\mathbb{Q}}$ is the projector onto the transcendental part of the motive of S_t. The reason why the result is stated only for the very general point t is the fact that due to the possible jump of the Picard group of S_t, the generic cycle $\Delta_{tr,\overline{\eta}}$ does not specialize to the cycle $\Delta_{tr,t}$ at any closed point $t \in B$, but only at the very general one. In fact, the statement is true at any point, but the cycle $\Delta_{tr,t}$ has to be modified when the Picard group jumps.

Proof of Lemma 3.8. By our assumption, using the countability of relative Hilbert schemes and the existence of universal objects parameterized by them, there exist a generically finite cover $r : B' \to B$, a universal family of polarized abelian varieties

$$\mathcal{K} \to B', \quad \mathcal{L}_K \in \operatorname{Pic}\mathcal{K}$$

and a codimension 2 cycle $\mathcal{Z}' \in CH^2(\mathcal{S}' \times_{B'} \mathcal{K})_{\mathbb{Q}}$, where $\mathcal{S}' := \mathcal{S} \times_B B'$, with the property that

$$(26) \qquad\qquad [\mathcal{Z}_t] = \alpha_{\mathcal{S}_t} \text{ in } H^4(\mathcal{S}_t \times K(\mathcal{S}_t), \mathbb{Q})$$

for any $t \in B'$. Furthermore, by Lemma 3.4, we know that there exists a nonzero rational number λ_g such that for any $t \in B'$, we have

$$(27) \quad {}^t\alpha_{\mathcal{S}_t} \circ (h^{N-2}\cup) \circ \alpha_{\mathcal{S}_t} = \lambda_g Id : H^2(\mathcal{S}_t, \mathbb{Q})_{tr} \to H^2(\mathcal{S}_t, \mathbb{Q})_{tr},$$

where as before N is the dimension of $K(\mathcal{S}_t)$, and $h_t = c_1(\mathcal{L}_{K|\mathcal{K}_t})$. We now construct the following family of abelian varieties on B (or a Zariski open set of it)

$$\mathcal{A}_t = \prod_{t' \in r^{-1}(t)} \mathcal{K}_{t'},$$

with polarization given by

$$(28) \qquad \mathcal{L}_t = \sum_{t' \in r^{-1}(t)} pr_{t'}^*(\mathcal{L}_{K,t'}),$$

where $pr_{t'}$ is the obvious projection from $\mathcal{A}_t = \prod_{t' \in r^{-1}(t)} \mathcal{K}_{t'}$ to its factor $\mathcal{K}_{t'}$, and the following cycle $\mathcal{Z} \in CH^2(\mathcal{S} \times_B \mathcal{K})_{\mathbb{Q}}$, with fiber at $t \in B$ given by

$$(29) \qquad \mathcal{Z}_t = \sum_{t' \in r^{-1}(t)} (Id_{\mathcal{S}_t}, pr_{t'})^* \mathcal{Z}_{t'}'.$$

In the last formula, we use of course the identification

$$\mathcal{S}_{t'}' = \mathcal{S}_t, \; r(t') = t.$$

It just remains to prove formula (25). Combining (28) and (29), we get, again using the notation $h_t = c_1(\mathcal{L}_t) \in H^2(A_t, \mathbb{Q})$, $h_{t'} = c_1(\mathcal{L}_{K,t'}) \in H^2(K_{t'}, \mathbb{Q})$:

$$[\Gamma_t]^* = \big(\sum_{t' \in r^{-1}(t)} [(pr_{t'}, Id_{\mathcal{S}_t})^*({}^t\mathcal{Z}_{t'}')]^* \big) \circ \big(\sum_{t'' \in r^{-1}(t)} pr_{t''}^*(h_{t''}) \big)^{N'-2}$$

$$\cup \circ \big(\sum_{t''' \in r^{-1}(t)} (Id_{\mathcal{S}_t}, pr_{t'''})^*[\mathcal{Z}_{t'''}']^* \big) \circ \pi_{tr,t} : H^*(\mathcal{S}_t, \mathbb{Q}) \to H^*(\mathcal{S}_t, \mathbb{Q}).$$

Note that $N' = \dim \mathcal{A}_t = \sharp(r^{-1}(t)) \, N = \deg(B'/B) \, N$. We develop the product above, and observe that the only nonzero terms appearing in this development come from taking $t' = t'''$ and putting the monomial

$$pr_{t'}^*(h_{t'}))^{N-2} \cup_{t'' \neq t'} pr_{t''}^*(h_{t''}))^N$$

in the middle term. The other terms are 0 due to the projection formula. Let us explain this in the case of only two summands K_1, K_2 of dimension r with polarizations l_1, l_2 and two cycles $Z_i \in CH^2(S \times K_i)$ giving rise to a cycle of the form

$$(Id_S, pr_1)^* Z_1 + (Id_S, pr_2)^* Z_2 \in CH^2(S \times K_1 \times K_2),$$

where the pr_i's are the projections from $S \times K_1 \times K_2$ to K_i: Then we have

$$[(pr_1, Id_S)^{*t} Z_1]^* \circ (l_1 + l_2)^{2r-2} \cup \circ [(Id_S, pr_2)^* Z_2]^* = 0 :$$
$$H^2(S, \mathbb{Q})_{tr} \to H^2(S, \mathbb{Q})_{tr}$$

by the projection formula and for the same reason

$$[(pr_1, Id_S)^{*\,t}Z_1]^* \circ (l_1 + l_2)^{2r-2} \cup \circ[(Id_S, pr_1)^* Z_1]^*$$

$$= [(pr_1, Id_S)^{*\,t}Z_1]^* \circ \sum_{0 \le k \le r} \binom{2r-2}{k} (l_1^k l_2^{2r-2-k}) \cup \circ[(Id_S, pr_1)^* Z_1]^*$$

$$\underset{\substack{proj. \\ formula}}{=} \binom{2r-2}{r} \deg(l_2^r) \, [Z_1]^* \circ l_1^{r-2} \cup \circ[{}^t Z_1]^* : H^2(S, \mathbb{Q})_{tr} \to H^2(S, \mathbb{Q})_{tr}.$$

We thus get (as the degrees $\deg(h_{t'}^N)$ of the polarizations $h_{t'}$ on the abelian varieties $\mathcal{K}_{t'}$ are all equal):

$$[\Gamma_t]^* = M(\deg(h_{t'}^N))^{\deg(B'/B)-1} \Big(\sum_{t' \in r^{-1}(t)} [{}^t \mathcal{Z}_{t'}')]^* \Big) \circ h_{t'}^{N-2} \cup \circ[\mathcal{Z}_{t'}']^*) \circ \pi_{tr,t} :$$

$$H^*(\mathcal{S}_t, \mathbb{Q}) \to H^*(\mathcal{S}_t, \mathbb{Q}),$$

where M is the multinomial coefficient appearing in front of the monomial $x_1^{N-2} x_2^N \ldots x_{\deg(B'/B)}^N$ in the development of

$$(x_1 + \ldots + x_{\deg(B'/B)})^{N\deg(B'/B)-2}.$$

By (26) and (27), we conclude that

$$[\Gamma_t]^* = M(\deg(h_{t'}^N))^{\deg(B'/B)-1} \deg(B'/B) \lambda_g \pi_{tr,t} :$$
$$H^*(\mathcal{S}_t, \mathbb{Q}) \to H^*(\mathcal{S}_t, \mathbb{Q}),$$

which proves formula (25) with $\lambda = M(\deg(h_{t'}^N))^{\deg(B'/B)-1} \deg(B'/B) \lambda_g$.
q.e.d.

Proof of Theorem 3.6. Consider the cycle $\mathcal{Z} \in CH^2(\mathcal{S} \times_B \mathcal{A})_{\mathbb{Q}}$ of Lemma 3.8, and the cycle $\Gamma \in CH^2(\mathcal{S} \times_B \mathcal{S})_{\mathbb{Q}}$

(30) $$\Gamma := \Delta_{tr} \circ {}^t \mathcal{Z} \circ c_1(\mathcal{L})^{N'-2} \circ \mathcal{Z} \circ \Delta_{tr}$$

where now $c_1(\mathcal{L})$ is the class of $\mathcal{L}$ in $CH^1(\mathcal{A})$ and we denote by $c_1(\mathcal{L})^{N-2}$ the relative self-correspondence of $\mathcal{A}$ given by the cycle $c_1(\mathcal{L})^{N-2}$ supported on the relative diagonal of $\mathcal{A}$ over B (it thus induces the intersection product with $c_1(\mathcal{L})^{N-2}$ on Chow groups). Furthermore the composition of correspondences is the relative composition over B, and Δ_{tr} is the generic transcendental motive (which is canonically defined in our case, at least after restricting to a Zariski open set of B) obtained as follows: Choose a 0-cycle $o_{\overline{\eta}}$ of degree 1 on the geometric generic fiber $\mathcal{S}_{\overline{\eta}}$ and choose a basis $L_1, \ldots, L_k$ of $\operatorname{Pic} \mathcal{S}_{\overline{\eta}}$. We have then the projector $\Delta_{alg,\overline{\eta}} \in CH^2(\mathcal{S}_{\overline{\eta}})_{\mathbb{Q}}$ defined as in (20) using the fact that the intersection pairing on the group of cycles $< o_{\overline{\eta}}, \mathcal{S}_{\overline{\eta}}, L_i >$ is nondegenerate. The transcendental projector $\Delta_{tr,\overline{\eta}}$ is defined as $\Delta_{\mathcal{S}_{\overline{\eta}}} - \Delta_{alg,\overline{\eta}}$. This is a codimension 2 cycle on $\mathcal{S}_{\overline{\eta}} \times_{\overline{\mathbb{C}(B)}} \mathcal{S}_{\overline{\eta}}$ but it comes from a cycle on $\mathcal{S}'' \times_{B''} \mathcal{S}''$ for some generically finite covers $B'' \to B$, $\mathcal{S}'' = \mathcal{S} \times_B B''$, and the latter can be finally pushed-forward to a codimension 2-cycle

on $\mathcal{S} \times_B \mathcal{S}$, which one checks to be a multiple of a projector, at least over a Zariski open set of B.

The cycle Γ satisfies (25), which we rewrite as

(31) $$[\Gamma'_t] = 0 \text{ in } H^4(\mathcal{S}_t \times \mathcal{S}_t, \mathbb{Q}),$$

where $\Gamma' := \Gamma - \lambda \Delta_{tr}$.

Lemma 3.7 tells us that the fibered self-product $\mathcal{S} \times \mathcal{S}$ is rationally connected. We can thus apply Theorem 0.3 and conclude that $\Gamma'_t \in CH^2(\mathcal{S}_t \times \mathcal{S}_t)_{\mathbb{Q}}$ is nilpotent. It follows that $\Gamma_t = \lambda \Delta_{tr,t} + N_t$, where $\lambda \neq 0$ and N_t is a nilpotent cycle in $\mathcal{S}_t \times \mathcal{S}_t$ having the property that

$$\Delta_{tr,t} \circ N_t = N_t \circ \Delta_{tr,t} = N_t.$$

We conclude immediately from the standard inversion formula for $\lambda \mathbb{I} + N$, with N nilpotent and $\lambda \neq 0$, that there exists a correspondence $\Phi_t \in CH^2(\mathcal{S}_t \times \mathcal{S}_t)_{\mathbb{Q}}$ such that

$$\Phi_t \circ \Gamma_t = \Delta_{tr,t} \text{ in } CH^2(\mathcal{S}_t \times \mathcal{S}_t)_{\mathbb{Q}}.$$

Recalling now formula (30):

$$\Gamma_t := \Delta_{tr,t} \circ {}^t\mathcal{Z}_t \circ c_1(\mathcal{L}_t)^{N'-2} \circ \mathcal{Z}_t \circ \Delta_{tr,t},$$

we proved (22) with

$$Z' = \Phi_t \circ \Delta_{tr,t} \circ {}^t\mathcal{Z}_t \circ c_1(\mathcal{L}_t)^{N'-2}.$$

<div align="right">q.e.d.</div>

Let us finish by explaining the following corollaries. They all follow from Kimura's theory of finite dimensionality (cf. [**17**]) and are a strong motivation to establish the Kuga-Satake correspondence at a Chow theoretic level rather than cohomological one. We summarize Kimura's results as follows:

Theorem 3.9. *(i) Abelian varieties have a finite dimensional motive, and thus any motive which is a direct summand in the motive of an abelian variety is finite dimensional (in Kimura sense).*

(ii) For any finite dimensional motive, self-correspondences homologous to 0 are nilpotent.

Our first corollary is the following:

Corollary 3.10. *With the same assumptions as in Theorem 3.5, for any $\sigma \in B$, any self-correspondence of S_σ which is homologous to 0 is nilpotent. In particular, for any finite group action G on S_σ and any projector $\pi \in \mathbb{Q}[G]$, if $H^{2,0}(S_\sigma)^\pi = 0$, then $CH_0(S_\sigma)^\pi_{\mathbb{Q},hom} = 0$.*

Proof. The first statement follows from Theorem 3.5 and Theorem 3.9.

In the case of a finite group action, we consider as in the previous section the self-correspondence Γ^π. It is not necessarily homologous to 0, but as it acts as 0 on $H^{2,0}(S)$, its class can be written as

$$[\Gamma^\pi] = \sum_i \alpha_i [C_i] \times [C_i'] + \alpha[x \times S_\sigma] + \beta[S_\sigma \times x] \text{ in } H^4(S_\sigma \times S_\sigma, \mathbb{Q})$$

for some rational numbers α, β, α_i, curves C_i, C_j', and given point x on S_σ. Then we have

$$[\Gamma^\pi - \sum_i \alpha_i C_i \times C_i' - \alpha\, x \times S_\sigma - \beta\, S_\sigma \times x] = 0 \text{ in } H^4(S_\sigma \times S_\sigma, \mathbb{Q}),$$

from which we conclude by Kimura's theorem 3.9 that the self-correspondence

$$Z := \Gamma^\pi - \sum_i \alpha_i C_i \times C_i' - \alpha\, x \times S_\sigma - \beta\, S_\sigma \times x \in CH^2(S_\sigma \times S_\sigma)_\mathbb{Q}$$

is nilpotent. It follows that $Z_* : CH_0(S_\sigma)_{\mathbb{Q},hom} \to CH_0(S_\sigma)_{\mathbb{Q},hom}$ is nilpotent. Since it is equal to

$$\Gamma^\pi_* : CH_0(S_\sigma)_{\mathbb{Q},hom} \to CH_0(S_\sigma)_{\mathbb{Q},hom},$$

which is the projector on $CH_0(S_\sigma)^\pi_{\mathbb{Q},hom}$, we conclude that

$$CH_0(S_\sigma)^\pi_{\mathbb{Q},hom} = 0.$$

q.e.d.

Corollary 3.11. *With the same assumptions as in Theorem 3.5, the transcendental part of the Chow motive of any member of the family of K3 surfaces parameterized by $\mathbb{P}(H^0(X, E))$ is indecomposable, that is, any submotive of it is either the whole motive or the 0-motive.*

Proof. Recall that the transcendental motive of S_σ is S_σ equipped with the projector π_{tr} defined in (20). Let now $\pi \in CH^2(S_\sigma \times S_\Sigma)_\mathbb{Q}$ be a projector of the transcendental motive of S_σ, that is, $\pi \circ \pi_{tr} = \pi_{tr} \circ \pi = \pi$. Since $h^{2,0}(S_\sigma) = 1$, π_* acts either as 0 or as Id on $H^{2,0}(S_\sigma)$. In the first case, $\mathrm{Ker}\,(\pi_*)_{|H^2(S,\mathbb{Q})_{tr}}$ is a sub-Hodge structure with $(2,0)$-component equal to $H^{2,0}(S)$. Its orthogonal complement is then contained in $NS(S_\sigma)_\mathbb{Q}$, which implies that $\pi_* = 0$ on $H^2(S,\mathbb{Q})_{tr}$. In the second case, we find similarly that $\pi_* = Id$ on $H^2(S,\mathbb{Q})_{tr}$. Since $\pi = \pi_{tr} \circ \pi = \pi \circ \pi_{tr}$, it follows that π_* acts either by 0 or as π_{tr} on $H^*(S,\mathbb{Q})$. Hence the cohomology class of either π or $\pi_{tr} - \pi$ is equal to 0, from which we conclude by Theorems 3.5 and 3.9 that π or $\pi_{tr} - \pi$ is nilpotent. As both are projectors, we find that $\pi = 0$ or $\pi_{tr} = 0$ in $CH^2(S_\sigma \times S_\sigma)_\mathbb{Q}$.   q.e.d.

References

[1] Y. André, *Pour une théorie inconditionnelle des motifs*, Publications Mathématiques de l'IHÉS **83** (1996), 5–49, MR 1423019.

[2] Y. André, *On the Shafarevich and Tate conjectures for hyper-Kähler varieties*, Math. Ann. **305** (1996), no. 2, 205–248, MR 1391213.

[3] R. Barlow, *A simply connected surface of general type with $p_g = 0$*, Invent. Math. **79** (1985), no. 2, 293–301, MR 0778128.

[4] R. Barlow, *Rational equivalence of zero cycles for some more surfaces with $p_g = 0$*, Invent. Math. **79** (1985), no. 2, 303–308, MR 0778129.

[5] I. Bauer, F. Catanese, F. Grünewald, & R. Pignatelli, *Quotients of products of curves, new surfaces with $p_g = 0$ and their fundamental groups*, Amer. J. of Math. **134** (2012), 993–1049, MR 2956256.

[6] A. Beauville & C. Voisin, *On the Chow ring of a K3 surface*, J. Algebraic Geom. **13**, 3 (2004), 417–426, MR 2047674.

[7] S. Bloch. *Lectures on algebraic cycles*, Duke University Mathematics Series. IV (1980), Durham, North Carolina: Duke University, Mathematics Department, MR 0558224.

[8] S. Bloch & V. Srinivas, *Remarks on correspondences and algebraic cycles*, Amer. J. of Math. **105** (1983) 1235–1253, MR 0714776.

[9] S. Bloch, A. Kas, & D. Lieberman, *Zero cycles on surfaces with $p_g = 0$*, Compositio Math. **33** (1976), no. 2, 135–145, MR 0435073.

[10] C. Böhning, H.-C. Graf von Bothmer, L. Katzarkov, & P. Sosna, *Determinantal Barlow surfaces and phantom categories*, preprint arXiv:1210.0343.

[11] F. Catanese, *Babbage's conjecture, contact of surfaces, symmetric determinantal varieties and applications*, Invent. Math. **63** (1981), no. 3, 433–465, MR 0620679.

[12] P. Deligne, *Théorie de Hodge II*, Publ. Math. IHES **40** (1971), 5–57, MR 0498551.

[13] P. Deligne, *La conjecture de Weil pour les surfaces K3*, Inventiones Math. **15** (1972), 206–226, MR 0296076.

[14] T. Graber, J. Harris, & J. Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc., **16** (1), 57–67 (2003), MR 1937199.

[15] D. Huybrechts, *Symplectic automorphisms of K3 surfaces of arbitrary order*, Math. Res. Lett. **19** (2012), 947–951, MR 3008427.

[16] H. Inose & M. Mizukami, *Rational equivalence of 0-cycles on some surfaces of general type with $p_g = 0$*, Math. Ann. **244** (1979), no. 3, 205–217, MR 0553252.

[17] S.-I. Kimura, *Chow groups are finite dimensional, in some sense*, Math. Ann. **331** (2005), no. 1, 173–201, MR 2107443.

[18] M. Kuga & I. Satake, *Abelian varieties attached to polarized K3 surfaces*, Math. Annalen **169**, 239–242 (1967), MR 0210717.

[19] D. Morrison, *The Kuga-Satake variety of an abelian surface*, J. Algebra **92** (1985), no. 2, 454–476, MR 0778462.

[20] S. Mukai, *Polarized K3 surfaces of genus 18 and 20*, in *Complex projective geometry*, Cambridge Univ. Press, 1992, 264–276, MR 1201388.

[21] S. Mukai, *Polarized K3 surfaces of genus thirteen*, in *Moduli spaces and arithmetic geometry*, Adv. Stud. Pure Math. **45**, Math. Soc. Japan, Tokyo, (2006), 315–326, MR 2310254.

[22] D. Mumford, *Rational equivalence of 0-cycles on surfaces*, J. Math. Kyoto Univ. **9** (1968) 195–204, MR 0249428.

[23] K. Paranjape, *Abelian varieties associated to certain K3 surfaces*, Compositio Math. **68** (1988), 11–22, MR 0962501.

[24] A. Roitman, *The torsion of the group of zero-cycles modulo rational equivalence*, Ann. of Math. **111** (1980), 553–569, MR 0577137.

[25] P. Supino, *A note on Campedelli surfaces*, Geom. Dedicata **71** (1998), no. 1, 19–31, MR 1624718.

[26] V. Voevodsky, *A nilpotence theorem for cycles algebraically equivalent to zero*, Internat. Math. Res. Notices, no. **4**, 187–198, (1995), MR 1326064.

[27] C. Voisin, *The generalized Hodge and Bloch conjectures are equivalent for general complete intersections*, Annales scientifiques de l'ENS **46**, fasc. 3 (2013), 449–475, preprint arXiv:1107.2600.

[28] C. Voisin, *Sur les zéro-cycles de certaines hypersurfaces munies d'un automorphisme*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **19** (1992), no. 4, 473–492, MR 1205880.

[29] C. Voisin, *Remarks on zero-cycles of self-products of varieties*, dans *Moduli of vector bundles* (Proceedings of the Taniguchi congress on vector bundles), Maruyama Ed., Decker (1994) 265–285, MR 1397993.

[30] C. Voisin, *Symplectic involutions of K3 surfaces act trivially on CH_0*, Documenta Math. **17** (2012) 851–860, MR 3007678.

[31] C. Voisin, *Lectures on the Hodge and Grothendieck-Hodge conjectures*, Rendiconti del Seminario Matematico, **69**, no. 2, (2011), MR 2931228.

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