J. DIFFERENTIAL GEOMETRY
97 (2014) 79-89

NONABELIANIZATION OF HIGGS BUNDLES

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Abstract

We consider the integrable system on the moduli space of Higgs bundles restricted to the subvariety corresponding to representations of a surface group into certain noncompact real forms, and in doing so encounter as the fiber moduli spaces of rank 2 bundles on a spectral curve. This contrasts with the general case where the fiber is an abelian variety.

1. Introduction

The moduli space of Higgs bundles over a compact Riemann surface Σ of genus $g > 0$ for a complex group G^c has the well-known structure of a completely integrable Hamiltonian system: a proper map to an affine space, whose generic fiber is an abelian variety. For the general linear group, such a Higgs bundle consists of a vector bundle V together with a section Φ of $\operatorname{End} V \otimes K$. The coefficients of the characteristic polynomial of Φ define the map to the base, and the corresponding fiber is the Jacobian of an algebraic curve S defined by the equation $\det(xI - \Phi) = 0$. This is a covering $\pi : S \to \Sigma$ on which Φ has a single-valued eigenvalue x, a section of π^*K. Conversely, given a line bundle L on S, we obtain V as the direct image sheaf of L, and Φ as the direct image of $x : L \to L \otimes \pi^*K$. This *abelianization* process has been useful in attacking various problems relating to bundles on curves.

It is clear, however, that we could replace L by a rank r bundle E on S and obtain a Higgs bundle on Σ by the same construction. In this case, each generic eigenspace of Φ is r-dimensional and then $\det(xI - \Phi) = p(x)^r$ for some polynomial $p(x)$. What we show here is that this case occurs, with $r = 2$, very naturally when considering the Higgs bundles that correspond (by solving the gauge-theoretic Higgs bundle equations) to flat connections on Σ with holonomy in the real Lie groups $G^r = SL(m, \mathbf{H}), SO(2m, \mathbf{H})$, and $Sp(m, m)$, real forms of $G^c = SL(2m, \mathbf{C}), SO(4m, \mathbf{C})$, and $Sp(4m, \mathbf{C})$, respectively. The first two groups are often denoted by $SU^*(2m)$ and $SO^*(4m)$, but the phenomenon we are describing clearly reflects the noncommutativity of the quaternions, which justifies the former notation.

The fibers of the integrable system, even those over nonregular values, are always compact. We find here that for $SL(m,\mathbf{H})$ the fiber consists of the moduli space of semi-stable rank 2 bundles with fixed determinant on the spectral curve S. For $SO(2m,\mathbf{H})$ the fiber has many components each of which is a moduli space of semi-stable rank 2 bundles on a quotient $\bar{S}$ of the spectral curve, and for $Sp(m,m)$ it is a $\mathbf{Z}_2$-quotient of a moduli space of semi-stable rank 2 *parabolic* bundles on $\bar{S}$.

Acknowledgments. Part of this work is contained in the second author's DPhil thesis [**9**]. She acknowledges the support of Oxford University, New College, and QGM Aarhus. Both authors wish to thank ICMAT, Madrid, for support during an activity of the "Hitchin Laboratory" in March 2013, where this article originated.

2. Higgs bundles for real forms

Under suitable stability conditions, a Higgs bundle defines a solution of equations for a G-connection A, where G is the maximal compact subgroup of G^c [**6**],[**11**]. For $G = U(n)$ these are $F_A + [\Phi, \Phi^*] = 0$, and then the connection $\nabla_A + \Phi + \Phi^*$ is flat, with holonomy in $GL(n,\mathbf{C})$. To get a flat connection with holonomy in $GL(n,\mathbf{R})$, we take A to be an $SO(n)$-connection and $\Phi = \Phi^T$ using the transpose defined by the orthogonal structure on V.

For a general real form G^r of G^c, we take a U-connection where U is the maximal compact subgroup of G^r, and in the decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{m}$ take $\Phi \in H^0(\Sigma, \mathfrak{m} \otimes K)$. Much work on the study of connected components of moduli spaces of flat G^r-connections using this approach has been carried out by Bradlow, Garcia-Prada, and others. (for example, in [**4**, **5**]). For the groups in question, we have the following descriptions of the corresponding Higgs bundles:

- The group $SL(m,\mathbf{H})$ is the subgroup of $SL(2m,\mathbf{C})$ that commutes with an antilinear automorphism J of $\mathbf{C}^{2m}$ such that $J^2 = -1$. Its maximal compact subgroup is the quaternionic unitary group $Sp(m)$. Since $Sp(m)^c = Sp(2m,\mathbf{C})$, the corresponding Higgs bundle consists of a rank $2m$ symplectic vector bundle (V,ω) and the Higgs field satisfies $\Phi = \Phi^T$ for the symplectic transpose. Using ω to identify V and V^*, this means that $\Phi = \omega^{-1}\phi$ for $\phi \in H^0(\Sigma, \Lambda^2 V \otimes K)$.
- The group $SO(2m,\mathbf{H})$ is the subgroup of $GL(4m,\mathbf{C})$ that preserves a complex inner product (u,v) and commutes with an antilinear automorphism J as above for which (u,Jv) is a hermitian form. If $(u,Ju) > 0$, then $(Ju, J^2u) = -(Ju,u) < 0$, and so the form has hermitian signature $(2m,2m)$. The maximal compact subgroup is $U(2m)$ and the Higgs bundle has the form $V = W \oplus W^*$ for a rank $2m$ vector bundle W. The inner product is

defined by the natural pairing between W and W^*. The Higgs field is of the form $\Phi(w,\xi) = (\beta(\xi), \gamma(w))$, where $\beta : W^* \to W \otimes K$ and $\gamma : W \to W^* \otimes K$ are skew-symmetric.

- The group $Sp(m,m)$ is the subgroup of $GL(4m, \mathbf{C})$ that preserves a complex symplectic form ω and commutes with an antilinear automorphism J for which $\omega(u, Jv)$ is a Hermitian form of signature $(2m, 2m)$. It is the group of quaternionic matrices that are unitary with respect to an indefinite form. The maximal compact subgroup is $Sp(m) \times Sp(m)$, and the Higgs bundle is of the form $W_1 \oplus W_2$ for symplectic rank $2m$ vector bundles $(W_1, \omega_1), (W_2, \omega_2)$. The Higgs field is of the form $\Phi(v,w) = (\beta(w), \gamma(v))$, where $\beta : W_2 \to W_1 \otimes K$ and $\gamma : W_1 \to W_2 \otimes K$ with $\beta = -\gamma^T$, using the symplectic transpose.

Since $SO(2m, \mathbf{H})$ and $Sp(m,m)$ are subgroups of $SL(2m, \mathbf{H})$, we begin dealing with the first case.

3. Spectral data for $SL(m, \mathbf{H})$

As noted above, in this case we have a rank $2m$ symplectic vector bundle (V, ω) and a Higgs field that is symmetric with respect to ω. If A is a symmetric endomorphism of a symplectic vector space U of dimension $2m$, and α the corresponding element of $\Lambda^2 U^*$, then $xI - A$ is singular if and only if the exterior product $(x\omega - \alpha)^m = 0$. The Pfaffian polynomial of $xI - A$ is defined by $p(x)\omega^m = (x\omega - \alpha)^m$, and then $p(x)^2 = \det(xI - A)$. If $p(x)$ has distinct roots, then U decomposes into a sum of m two-dimensional symplectic eigenspaces of A, and in this case, and by continuity in all cases, A satisfies the matrix equation $p(A) = 0$. If A is in the Lie algebra of $SL(2m, \mathbf{C})$, then in addition $\operatorname{tr} A = 0$.

Replacing A by Φ, we have a polynomial $p(x) = x^m + a_2 x^{m-2} + \cdots + a_m$, where the coefficients $a_i \in H^0(\Sigma, K^i)$. As an $SL(2m, \mathbf{C})$-Higgs bundle, the usual spectral curve is defined by the vanishing of the characteristic polynomial, so the coefficients of $p(x)^2$, which lie in $\bigoplus_{i=2}^{2m} H^0(\Sigma, K^i)$, define a point in the base of the fibration. In our case, by a slight abuse of notation, we shall call the curve S defined by $p(x) = 0$ the spectral curve. Bertini's theorem assures us that for generic a_i the curve is nonsingular. It is a ramified m-fold cover of Σ. More precisely, we may interpret the equation $p(x) = 0$ as the vanishing of a section of $\pi^* K^m$ over the total space of the canonical bundle $\pi : K \to \Sigma$, where x is the tautological section of $\pi^* K$. The cotangent bundle of Σ is a symplectic manifold and hence has trivial canonical bundle, so $K_S \otimes \pi^* K^{-m}$ is trivial and $K_S = \pi^* K^m$. Taking degrees of both sides, this says that the genus of S is given by $g_S = m^2(g-1) + 1$.

On the spectral curve S, x is a well-defined eigenvalue of Φ, and the cokernel of $xI - \Phi$ is a rank two holomorphic vector bundle E.

It then follows, as in [**3**] (and using $p(\Phi) = 0$ instead of the Cayley–Hamilton theorem), that we can identify V with the direct image $\pi_* E$ and the Higgs field Φ as the direct image of $x : E \to E \otimes \pi^* K$ (recall that the direct image sheaf is defined for each open set $U \subset \Sigma$ by $H^0(U, \pi_* E) = H^0(\pi^{-1}(U), E)$).

If we now start with any rank 2 bundle E on S, we can obtain by the same construction a $GL(2m, \mathbf{C})$-Higgs bundle, but we need to determine the conditions on E for this to be the data for the group $SL(m, \mathbf{H})$:

Proposition 1. *Let $p(x) = x^m + a_2 x^{m-2} + \cdots + a_m$ be a section of the line bundle $\pi^* K^m$ on the cotangent bundle of Σ whose divisor is a smooth curve S, and let E be a rank 2 vector bundle on S. Then the direct image of $x : E \to E \otimes \pi^* K$ defines a semi-stable Higgs bundle on Σ for the group $SL(m, \mathbf{H})$ if and only if*

- $\Lambda^2 E \cong \pi^* K^{m-1}$,
- *E is semi-stable.*

Proof: First, we define a nondegenerate skew form on $\pi_* E$. For this, the relative duality theorem gives

$$(\pi_* E)^* \cong \pi_*(E^* \otimes K_S) \otimes K^{-1},$$

so if $V = \pi_* E$, to achieve $V \cong V^*$ we want $E^* \otimes \pi^* K^{m-1} \cong E$. But we are given $\Lambda^2 E \cong \pi^* K^{m-1}$, so this is satisfied. We should also check that the duality is provided by a skew form, and this requires a concrete expression of relative duality. At a regular value $a \in \Sigma$ of π,

$$V = \bigoplus_{y \in \pi^{-1}(a)} E_y,$$

and taking $s_y \in E_y, s' \in (E^* \otimes K_S \otimes \pi^* K^{-1})_y$, we form

$$\sum_{y \in \pi^{-1}(a)} \frac{\langle s, s' \rangle}{d\pi_y},$$

where $d\pi : K_S^{-1} \to \pi^* K^{-1}$ is the derivative, considered in $K_S \otimes \pi^* K^{-1}$. This extends for the direct image over branch points. But the isomorphism $E^* \otimes \pi^* K^{m-1} \cong E$ is skew-symmetric, showing that $V \cong V^*$ is also skew. Moreover, multiplication by x satisfies $\langle xs, s' \rangle = \langle s, xs' \rangle$, which is symmetric and defines Φ as a Higgs field satisfying $\Phi = \Phi^T$.

The semi-stability condition for Higgs bundles [**6**] is that, for each Φ-invariant subbundle $W \subset V$, $\deg W/\operatorname{rk} W \leq \deg V/\operatorname{rk} V$. But V is symplectic, so $\deg V = 0$, and then we require $\deg W \leq 0$. If $W \neq V$, the characteristic polynomial of $\Phi|_W$ properly divides that of Φ. But the characteristic polynomial of Φ is $p(x)^2$ and S is smooth, and in particular irreducible. So the characteristic polynomial for $\Phi|_W$ must be $p(x)$. Then (W, Φ) is a rank m Higgs bundle and by [**3**] is the direct image of a line bundle $L \subset E$.

From Grothendieck–Riemann–Roch, we have $(1-g)m + \deg W = (1-g_S) + \deg L$, and so $\deg W = (1-g)(m^2-m) + \deg L$. Hence the Higgs semi-stability condition is $\deg L \leq m(m-1)(g-1)$. But $\deg E = \deg \pi^* K^{m-1} = m(m-1)(2g-2)$, and so this is equivalent to the semi-stability condition $\deg L \leq \deg E/2$ for the rank 2 bundle E. □

Remarks:
1. The degree of E on S is $2m(m-1)(g-1)$, which is even, and so the moduli space of semi-stable bundles is singular, the singular locus represented by decomposable bundles $E = L \oplus (L^* \otimes \pi^* K^{m-1})$. Then relative duality gives $\pi_* E = W \oplus W^*$, and the Higgs field is $\Phi = (\phi, \phi^T)$ for a $GL(m, \mathbf{C})$-Higgs bundle (W, ϕ).
2. One may check dimensions of the moduli space here: considered as the moduli space of representations into a real form, its real dimension is $(2g-2)\dim G^r = 2(4m^2-1)(g-1)$. The complex dimension of the space of polynomials $p(x)$ is $3(g-1) + 5(g-1) + \cdots + (2m-1)(g-1) = (m^2-1)(g-1)$, and for the moduli space of stable bundles on S with fixed determinant it is $3(g_S - 1) = 3m^2(g-1)$, giving in total $(4m^2-1)(g-1)$.

4. Spectral data for $SO(2m, \mathbf{H})$

The Higgs bundle here is $V = W \oplus W^*$, where W has rank $2m$ and the Higgs field is of the form

$$\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}, \tag{1}$$

where $\beta : W^* \to W \otimes K$ and $\gamma : W \to W^* \otimes K$ are both skew-symmetric. The inclusion $SO(2m, \mathbf{H}) \subset SL(2m, \mathbf{H})$ means we should also consider this as a special case of the previous section. To do this, define a symplectic form on V by $\omega((w_1, \xi_1), (w_2, \xi_2)) = \xi_2(w_1) - \xi_1(w_2)$. Then $\omega(\Phi(w_1, \xi_1), (w_2, \xi_2)) = \xi_2(\beta\xi_1) - \gamma w_1(w_2) = -\xi_1(\beta\xi_2) + \gamma w_2(w_1)$, which is $\omega((w_1, \xi_1), \Phi(w_2, \xi_2))$, and so $\Phi = \Phi^T$.

It follows that we can use the result of the previous section to deduce that $\det(xI - \Phi) = p(x)^2$ for some polynomial of degree $2m$ and, assuming it defines a smooth curve S, Φ has generically two-dimensional eigenspaces. Globally, as before, the bundle V can be written $\pi_* E$ for a rank 2 bundle on the spectral curve S, which has genus $g_S = 4m^2(g-1) + 1$.

Suppose that $(w, \xi) \in W \oplus W^*$ is an eigenvector of Φ with eigenvalue λ. Then $\beta(\xi) = \lambda w$ and $\gamma(w) = \lambda \xi$. Hence

$$\Phi(w, -\xi) = (-\lambda w, \lambda \xi) = -\lambda(w, -\xi).$$

Thus for each two-dimensional generic eigenspace with eigenvalue λ there exists another with eigenvalue $-\lambda$. In particular, this means that $p(x) = x^{2m} + b_1 x^{2m-2} + \cdots + b_{m-1} x^2 + b_m$, where $b_i \in H^0(\Sigma, K^{2i})$ and the curve S has an involution $\sigma(x) = -x$. We now need to determine the properties of the rank 2 bundle E:

Proposition 2. *Let $p(x) = x^{2m} + b_1 x^{2m-2} + \cdots + b_{m-1}x^2 + b_m$ be a section of the line bundle $\pi^* K^{2m}$ on the cotangent bundle of Σ whose divisor is a smooth curve S, and let σ be the involution $\sigma(x) = -x$. If E is a rank 2 vector bundle on S, then the direct image of $x : E \to E \otimes \pi^* K$ defines a semi-stable Higgs bundle on Σ for the group $SO(2m, \mathbf{H})$ if and only if*

- $\Lambda^2 E \cong \pi^* K^{2m-1}$,
- *E is semi-stable,*
- *$\sigma^* E \cong E$ where the induced action on $\Lambda^2 E = \pi^* K^{2m-1}$ is trivial.*

Remark: The bundle $\pi^* K^{2m-1}$ is pulled back from Σ, and this is why it makes sense to speak of the trivial action of σ, since $\pi\sigma = \pi$. When E is stable any automorphism is a scalar, so the isomorphism gives a lifted action of the involution, well-defined modulo ± 1, and in particular the action on $\Lambda^2 E$ is well defined. In fact, in general all we require is that the action on $\Lambda^2 E$ at fixed points should be trivial.

Proof: The first part follows from Proposition 1. The bundle E is defined as the cokernel of $xI - \Phi$ in $V \otimes K$, but using the orthogonal structure on V, this means that $E^* \otimes \pi^* K$ is the kernel of $xI + \Phi$—i.e., generically the two-dimensional eigenspace of Φ with eigenvalue $-x$. We saw above how $(w, \xi) \mapsto (w, -\xi)$ gives an isomorphism between the $\pm x$ eigenspaces and so there is an isomorphism $\sigma^* E \cong E$.

Triviality of the action on $\Lambda^2 E = \pi^* K^{2m-1}$ in the statement of the proposition is a function of the isomorphism, which came to us as in Section 3 from relative duality. This gave us the following formula for the symplectic form over a regular value:

$$\sum_{y \in \pi^{-1}(a)} \frac{s \wedge s'}{d\pi_y}, \tag{2}$$

where $d\pi$ is a section of $K_S \otimes \pi^* K^{-1}$ and we use the canonical symplectic form of the cotangent bundle of Σ to identify this with $\pi^* K^{2m-1}$. Together with an isomorphism $\pi^* K^{2m-1} \cong \Lambda^2 E$, formula (2) is a well-defined scalar. Now the symplectic form on the cotangent bundle is anti-invariant under σ, which is scalar multiplication by -1 in the fibers, and the pairing symplectic form on $W \oplus W^*$ is anti-invariant under the map $(w, \xi) \mapsto (w, -\xi)$. It follows that the action on $\Lambda^2 E = \pi^* K^{2m-1}$ is trivial.

Now let $\bar{S}$ be the quotient of S by the involution, and let $p : S \to \bar{S}$ be the quotient map. Setting $z = x^2$ embeds $\bar{S}$ in the total space of K^2 with equation $z^m + b_1 z^{m-1} + \cdots + b_m = 0$. If $\bar{\pi} : K^2 \to \Sigma$ is the projection, then z is the tautological section of $\bar{\pi}^* K^2$ and $\pi : S \to \Sigma$ can be written as $\pi = \bar{\pi} \circ p$.

For an open set $U \subset \bar{S}$, $p^{-1}(U)$ is σ-invariant, and then the ± 1 eigenspaces of the action on $H^0(p^{-1}(U), E)$ decompose the direct image on $\bar{S}$ as $p_*E = E^+ \oplus E^-$. Since $(w, 0) \in W$ and $(0, \xi) \in W^*$ are the ± 1 eigenspaces of the involution, we see that $\bar{\pi}_*E^+ = W, \bar{\pi}_*E^- = W^*$.

Conversely, suppose we are given S and E as in the statement of the proposition. Then $\pi_*E = \bar{\pi}_*(E^+ \oplus E^-) = W_1 \oplus W_2$ is a symplectic vector bundle from Proposition 1. Moreover, since $\sigma(x) = -x$, $x : E \to E \otimes \pi^*K$ maps local invariant sections of E to anti-invariant ones and so the Higgs field has the off-diagonal shape of (1). To obtain $V = W \oplus W^*$, we need to show that W_1 and W_2 are Lagrangian with respect to the symplectic structure on V. Now a local section of W_1 is defined by the direct image of an invariant section s. Equation (2) evaluates the symplectic pairing of two sections s, s', but if they are both invariant, then $s \wedge s'$ is invariant. As in the discussion above, if the action on $\Lambda^2 E$ is trivial, then the denominator in (2) is anti-invariant and so the terms over y and $\sigma(y)$ cancel and so the symplectic form on V vanishes on W_1. A similar argument holds for W_2. They are therefore transverse Lagrangian subbundles, and setting $W_1 = W$, $W_2 \cong W^*$.

As a special case of the previous section, $\Phi = \Phi^T$ with respect to the symplectic form, but given its shape (1) this means that the terms β and γ are skew-symmetric. □

Proposition 2 tells us that the fiber of the integrable system for $SO(2m, \mathbf{H})$-Higgs bundles is defined by the fixed points of an involution induced by σ on the moduli space of rank 2 semi-stable bundles on S. There are several components, however. This is clear from the flat connection point of view: the maximal compact subgroup of $SO(2m, \mathbf{H})$ is $U(2m)$, and so any flat $SO(2m, \mathbf{H})$ bundle can be topologically reduced to $U(2m)$ where it has a Chern class. In the Higgs bundle description, this is the degree of the vector bundle W. As in the case of $U(m, m)$ dealt with in [**10**], we can determine this invariant by considering the action at the fixed points of σ on S.

At a fixed point a of σ there is a linear action of σ on the fiber E_a. Since the action on $\Lambda^2 E_a$ is trivial, this is scalar multiplication ± 1 and we can assign to each fixed point this number.

Proposition 3. *Suppose the action is $+1$ at M fixed points, then* $\deg W = 2M - 4m(g-1)$.

Proof: The fixed point set of σ is the intersection of the zero section of K with S. Setting $x = 0$ in the equation $x^{2m} + b_1 x^{2m-2} + \cdots + b_m = 0$, these points are the images of the $4m(g-1)$ zeros of $b_m \in H^0(\Sigma, K^{2m})$ under the zero section. The action is $+1$ at M of these points.

Choose a line bundle L on Σ of large enough degree such that the cohomology group $H^1(\Sigma, V \otimes L) = 0$. By definition of E^+, E^- we have $\dim H^0(S, E \otimes \pi^*L)^{\pm} = \dim H^0(\bar{S}, E^{\pm} \otimes \bar{\pi}^*L)$, where the superscript denotes the $\pm$ eigenspace under the action of σ. Since $V = \pi_* E$ and $H^1(\Sigma, V \otimes L) = 0$, the higher cohomology groups vanish, and applying the holomorphic Lefschetz formula [**2**], we obtain

$$\dim H^0(\bar{S}, E^+ \otimes \bar{\pi}^* L) - \dim H^0(\bar{S}, E^- \otimes \bar{\pi}^* L) = 2(M - (4m(g-1) - M)).$$

On the other hand, the sum

$$\dim H^0(\bar{S}, E^+ \otimes \bar{\pi}^* L) + \dim H^0(\bar{S}, E^- \otimes \bar{\pi}^* L) = \dim H^0(S, E \otimes \pi^* L),$$

and since V is the direct image of E, this is $\dim H^0(\Sigma, V \otimes L)$. Applying Riemann–Roch gives $4m(1 - g + \deg L)$ since V is symplectic and hence $\deg V = 0$.

Now $W = \bar{\pi}_* E^+$, so $\dim H^0(\bar{S}, E^+ \otimes \bar{\pi}^* L) = \dim H^0(\Sigma, W \otimes L) = 2m((1-g) + \deg L) + \deg W$ by Riemann–Roch, and from these three equations we obtain $\deg W = 2M - 4m(g-1)$. $\square$

Remarks:

1. Since $M \leq 4m(g-1)$, we have $|\deg W| \leq 4m(g-1)$, which is the Milnor–Wood inequality for the group $SO(2m, \mathbf{H})$.
2. In the maximal case $\deg W = 4m(g-1)$, all fixed points have action $+1$ and then the bundle E is pulled back from the curve $\bar{S}$. In this case, $\gamma : W \to W^* \otimes K$ is a homomorphism of bundles of the same degree and so is either everywhere singular or an isomorphism. But S is smooth, so b_m is not identically zero and hence $W \cong W^* \otimes K$, or setting $U = W \otimes K^{-1/2}$, $\Psi = \beta\gamma$ and we have a Higgs bundle of the same type as an $SL(m, \mathbf{H})$ bundle but with a K^2-twisted Higgs field Ψ. Moreover, the spectral data of the previous section holds if one takes the rank 2 bundle $E \otimes \bar{\pi}^* K^{-1/2}$ on $\bar{S}$. This is a case of the Cayley correspondence of [**5**].
3. For each choice of M fixed points $a_1, \ldots, a_M \in S$, using the Narasimhan–Seshadri theorem we can interpret the moduli space of invariant semi-stable rank 2 bundles on S as the moduli space of representations of the fundamental group of $\bar{S} \backslash \{p(a_1), \ldots, p(a_M)\}$ with holonomy -1 around the marked points. If M is odd, this is the (smooth and connected) moduli space of stable rank 2 bundles on $\bar{S}$ of odd degree and fixed determinant, and if M is even it is the singular moduli space of bundles of even degree. Given this, we can check dimensions as before. The curve S gives $3(g-1) + 7(g-1) + \cdots + (4m-1)(g-1) = m(2m+1)(g-1)$ complex parameters, and the moduli space of bundles on $\bar{S}$ gives $3(g_{\bar{S}} - 1) = 3m(2m-1)(g-1)$ parameters. In total, this makes $3m(2m-1)(g-1) + m(2m+1)(g-1) = m(8m-2)(g-1) = \dim SO(4m, \mathbf{C})(g-1)$.
4. In the stable case, the only other action on E is to multiply the given action by -1, which changes M to $4m(g-1) - M$ and interchanges the roles of W and W^*. Thus there are $2^{4m(g-1)-1}$ components in the fiber.

5. Spectral data for $Sp(m,m)$

For this group, the Higgs bundle is $V = W_1 \oplus W_2$ for symplectic rank $2m$ vector bundles $(W_1, \omega_1), (W_2, \omega_2)$. The Higgs field is

$$\Phi = \begin{pmatrix} 0 & \beta \\ -\beta^T & 0 \end{pmatrix}, \tag{3}$$

where $\beta^T : W_1 \to W_2 \otimes K$ is the symplectic adjoint. Since $Sp(m,m) \subset SL(2m, \mathbf{H})$, we can apply the results of Section 3, but we need a symplectic structure on V for which $\Phi = \Phi^T$. Define $\omega = (\omega_1, -\omega_2)$. Then

$$\begin{aligned} \omega(\Phi(u_1,u_2),(w_1,w_2)) &= \omega_1(\beta u_2, w_1) + \omega_2(\beta^T u_1, w_2) \\ &= \omega_1(u_2, \beta^T w_1) + \omega_2(u_1, \beta w_2), \end{aligned}$$

and this is equal to

$$\begin{aligned} -\omega_1(\beta^T w_1, u_2) - \omega_2(\beta w_2, u_1) &= -\omega(\Phi(w_1,w_2),(u_1,u_2)) \\ &= \omega((u_1,u_2), \Phi(w_1,w_2)). \end{aligned}$$

Here the only difference with the previous case is the action of the involution on E:

Proposition 4. *Let $p(x) = x^{2m} + b_1 x^{2m-2} + \cdots + b_{m-1}x^2 + b_m$ be a section of the line bundle π^*K^{2m} on the cotangent bundle of Σ whose divisor is a smooth curve S, and let σ be the involution $\sigma(x) = -x$. If E is a rank 2 vector bundle on S, then the direct image of $x : E \to E \otimes \pi^*K$ defines a semi-stable Higgs bundle on Σ for the group $Sp(m,m)$ if and only if*

- *$\Lambda^2 E \cong \pi^* K^{2m-1}$,*
- *E is semi-stable,*
- *$\sigma^* E \cong E$ where the induced action on $\Lambda^2 E = \pi^* K^{2m-1}$ is -1.*

Proof: The proof proceeds exactly as in Proposition 2 until the point where we prove that W_1 and W_2 are Lagrangian. With the opposite action on $\Lambda^2 E$, we deduce instead that W_1 and W_2 are symplectically orthogonal and hence V is the symplectic sum of W_1 and W_2. A slightly different and more detailed approach may be found in [**9**]. $\square$

Remarks:

1) Given the action of -1 on $\Lambda^2 E$, at a fixed point we have distinct $+1$ and -1 eigenspaces. Following [**1**], this defines a rank 2 bundle on the curve $\bar{S}$ with a parabolic structure at the fixed points defined by the flag given by the -1 eigenspace and the parabolic weight $1/2$. As in the previous case, the choice of action corresponds to an ordering of W_1 and W_2 so a point in the moduli space for $Sp(m,m)$ determines a point in the quotient of the moduli space of parabolic structures by interchanging the roles of the $+1$ and -1 eigenspaces.

2) Note that the two groups $SO(2m, \mathbf{H})$ and $Sp(m, m)$ correspond to the two equivariant structures on the line bundle $\Lambda^2 E \cong \pi^* K^{2m-1}$ and, as in [**1**], account for all the fixed points in the moduli space of rank 2 bundles over S.
3) We can use the parabolic aspect as a check on the dimension: the parameters for the spectral curve and bundle gives $m(8m-2)(g-1)$ as in the previous case, but there is a contribution of $1 = \dim \mathbf{P}^1$ for each of the $4m(g-1)$ parabolic points giving in total $m(8m+2)(g-1) = \dim Sp(4m, \mathbf{C})(g-1)$.
4) The thesis of Ana Peón [**8**] gives a characterization using cameral covers of the real forms that exhibit the nonabelianization phenomenon observed here.

6. Comments

1) The representation of the moduli space of flat connections as the Higgs bundle moduli space, and in particular the integrable system, depends on the choice of a complex structure on the underlying real surface Σ. Properties of the Higgs bundle can change significantly for the same representation of $\pi_1(\Sigma)$. As an example, the uniformizing representation in $PSL(2, \mathbf{R})$ of a Riemann surface has a nilpotent Higgs field for the natural complex structure but the same representation has a nonsingular spectral curve $x^2 - a = 0$ when we change the complex structure of Σ and hence its Higgs bundle moduli space. It is natural to ask which representations have smooth, or irreducible, spectral curves in *some* complex structure. The examples given here have reducible spectral curves in *any* complex structure.
2) The classical abelianization picture of a spectral curve with a line bundle over it is, in the physicists' terminology, a D-brane: a Lagrangian submanifold of the cotangent bundle of Σ together with a flat line bundle over it. However, when two such D-branes coalesce, one expects to find a flat higher rank bundle over the resulting curve. Given the stability property of the rank 2 bundle here, it follows from the Narasimhan–Seshadri theorem that this is precisely what we have. A sequence of points in the $GL(2m, \mathbf{C})$ moduli space converging to an $SL(m, \mathbf{H})$ Higgs bundle gives just such a degeneration.

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