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TWISTOR SPACES FOR HYPERKÄHLER IMPLOSIONS

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Dedicated to the memory of Friedrich Hirzebruch

Abstract

We study the geometry of the twistor space of the universal hyperkähler implosion Q for $SU(n)$. Using the description of Q as a hyperkähler quiver variety, we construct a holomorphic map from the twistor space $\mathcal{Z}_Q$ of Q to a complex vector bundle over $\mathbb{P}^1$, and an associated map of Q to the affine space $\mathcal{R}$ of the bundle's holomorphic sections. The map from Q to $\mathcal{R}$ is shown to be injective and equivariant for the action of $SU(n) \times T^{n-1} \times SU(2)$. Both maps, from Q and from $\mathcal{Z}_Q$, are described in detail for $n = 2$ and $n = 3$. We explain how the maps are built from the fundamental irreducible representations of $SU(n)$ and the hypertoric variety associated to the hyperplane arrangement given by the root planes in the Lie algebra of the maximal torus. This indicates that the constructions might extend to universal hyperkähler implosions for other compact groups.

0. Introduction

In [**4**, **5**] we introduced a hyperkähler analogue in the case of $SU(n)$ actions of Guillemin, Jeffrey, and Sjamaar's construction of *symplectic implosion* [**7**]. The aim of this paper is to find a description of hyperkähler implosion for $SU(n)$ which can be generalized to other compact groups K.

If M is a symplectic manifold with a Hamiltonian action of a compact group K with maximal torus T, the symplectic implosion M_{impl} is a stratified symplectic space (usually singular) with an action of T such that the symplectic reductions of M_{impl} by T coincide with the symplectic reductions of M by K. The implosion of the cotangent bundle $T^*K \cong K \times \mathfrak{k}^*$ acts as a universal symplectic implosion in that the implosion of a general Hamiltonian K-manifold M can be identified with the symplectic reduction by K of $M \times (T^*K)_{\mathrm{impl}}$.

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The construction in [**4**] of the universal hyperkähler implosion when $K = SU(n)$ uses quiver diagrams and gives us a stratified hyperkähler space $Q = (T^*K_{\mathbb{C}})_{\mathrm{hkimpl}}$. The hyperkähler implosion of a general hyperkähler manifold M with a Hamiltonian action of $K = SU(n)$ is then defined as the hyperkähler reduction by K of $M \times Q$. The hyperkähler strata of Q can be described in terms of open sets in complex symplectic quotients of the cotangent bundle of $K_{\mathbb{C}} = SL(n, \mathbb{C})$ by subgroups which are extensions of abelian groups by commutators of parabolic subgroups. There is an action of the maximal torus T of K, and the hyperkähler quotients by this action are the Kostant varieties, affine varieties which are closures in $\mathfrak{k}^*_{\mathbb{C}}$ of complex co-adjoint orbits. As in [**4**] we will identify Lie algebras with their duals via an invariant inner product, so that co-adjoint orbits of $K_{\mathbb{C}}$ are identified with adjoint orbits.

The universal symplectic implosion has a natural $(K \times T)$-equivariant embedding into a complex affine space, whose image is the K-sweep $K\overline{T_{\mathbb{C}}v}$ of the closure of an orbit $T_{\mathbb{C}}v$ of the complexified maximal torus $T_{\mathbb{C}}$ [**7**]. Here $\overline{T_{\mathbb{C}}v}$ is the toric variety associated to a positive Weyl chamber $\mathfrak{t}_+$ in the Lie algebra $\mathfrak{t}$ of T. It was shown in [**5**] that the hypertoric variety associated to the hyperplane arrangement given by the root planes in $\mathfrak{t}$ maps generically injectively to Q and that (for any choice of complex structure on Q) the $K_{\mathbb{C}}$-sweep $K_{\mathbb{C}}Q_T$ of its image Q_T is dense in Q. In this paper we shall construct a $(K \times T)$-equivariant embedding σ of the universal hyperkähler implosion Q for $K = SU(n)$ into a complex affine space $\mathcal{R}$ with a natural $SU(2)$-action which rotates the complex structures on $Q = (T^*K_{\mathbb{C}})_{\mathrm{hkimpl}}$. This embedding is constructed using the moment maps for the $K \times T$ action and the fundamental irreducible representations of K. Its image is the closure of the $K_{\mathbb{C}}$-sweep of the image $\sigma(Q_T)$ in $\mathcal{R}$ of the hypertoric variety associated to the hyperplane arrangement given by the root planes in $\mathfrak{t}$. Our future aim is to use this description of Q as the closure of the $K_{\mathbb{C}}$-sweep of the image of this hypertoric variety in the $(K \times T \times SU(2))$-representation $\mathcal{R}$ to extend the hyperkähler implosion construction from $K = SU(n)$ to more general compact groups K.

Any hyperkähler manifold M has a twistor space $\mathcal{Z}_M$, which is a complex manifold with additional structure from which we can recover M. As a smooth manifold $\mathcal{Z}_M$ is the product $M \times \mathbb{P}^1$ of M and the complex projective line $\mathbb{P}^1$, which is identified with the unit sphere S^2 in $\mathbb{R}^3$ in the usual way. The complex structure on $\mathcal{Z}_M$ is such that the projection $\pi\colon \mathcal{Z}_M \to \mathbb{P}^1$ is holomorphic and its fiber at any $\zeta \in \mathbb{P}^1 = S^2$ is M equipped with the complex structure determined by ζ.

We shall give a description of the twistor space for the universal hyperkähler implosion for $SU(n)$. Motivated by the embedding of the implosion into the affine space $\mathcal{R}$ described above, we shall also construct a generically injective holomorphic map from its twistor space to a vector bundle over $\mathbb{P}^1$.

The layout of the paper is as follows. In §1 we review the theory of twistor spaces for hyperkähler manifolds, and in §2 we recall the hyperkähler structure on the nilpotent cone in the Lie algebra of $K_{\mathbb{C}} = SL(n,\mathbb{C})$ obtained in [**12**]. We also describe its twistor space which can be embedded in the vector bundle $\mathcal{O}(2) \otimes \mathfrak{k}_{\mathbb{C}}$ over $\mathbb{P}^1$, where $\mathfrak{k}_{\mathbb{C}}$ is the Lie algebra of the complexification $K_{\mathbb{C}} = SL(n,\mathbb{C})$ of $K = SU(n)$. In §3 we recall the constructions of symplectic implosion from [**7**] and hyperkähler implosion for $K = SU(n)$ from [**4**, **5**]. In §4 we define a $K \times T \times SU(2)$-equivariant map σ from the universal hyperkähler implosion Q for $K = SU(n)$ to

$$\mathcal{R} = H^0(\mathbb{P}^1, (\mathcal{O}(2) \otimes (\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}})) \oplus \bigoplus_{j=1}^{n-1} \mathcal{O}(\ell_j) \otimes \wedge^j \mathbb{C}^n)$$

where $\ell_j = j(n-j)$, and an associated holomorphic map $\tilde{\sigma}$ from the twistor space $\mathcal{Z}_Q$ of Q to the vector bundle $(\mathcal{O}(2)\otimes(\mathfrak{k}_{\mathbb{C}}\oplus\mathfrak{t}_{\mathbb{C}}))\oplus\bigoplus_{j=1}^{n-1} \mathcal{O}(\ell_j) \otimes \wedge^j \mathbb{C}^n$ over $\mathbb{P}^1$. Here $SU(2)$ acts on $\mathcal{R}$ via its usual action on $\mathbb{P}^1$ and the line bundles $\mathcal{O}(\ell_j)$ over $\mathbb{P}^1$ for $j \in \mathbb{Z}$, while $K = SU(n)$ acts on $\mathcal{R}$ via its adjoint action on $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$, the trivial action on $\mathfrak{t}_{\mathbb{C}}$ and its usual action on $\wedge^j \mathbb{C}^n$. The action of T on $\mathcal{O}(2) \otimes (\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}})$ is the restriction of the K-action, but T acts on $\mathcal{O}(\ell_j) \otimes \wedge^j \mathbb{C}^n$ as multiplication by the highest weight for the irreducible representation $\wedge^j \mathbb{C}^n$ of $K = SU(n)$.

In §5 we recall the stratification given in [**4**] of Q into strata which are hyperkähler manifolds, and its refinement in [**5**] which has strata $Q_{[\sim,\mathcal{O}]}$ indexed in terms of Levi subgroups and nilpotent orbits in $K_{\mathbb{C}} = SL(n,\mathbb{C})$. In §6 we prove that the map σ defined in §4 is injective and that $\tilde{\sigma}$ is generically injective; we will see that $\tilde{\sigma}$ fails to be injective in the example $n = 2$ in §7. In §7 we describe the full structure of the twistor space of Q in terms of its embedding in the space of holomorphic sections of the vector bundle $(\mathcal{O}(2) \otimes (\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}})) \oplus \bigoplus_{j=1}^{n-1} \mathcal{O}(\ell_j) \otimes \wedge^j \mathbb{C}^n$ over $\mathbb{P}^1$, and we consider the low-dimensional examples $n = 2$ and $n = 3$ in detail. Finally, in §8 we consider how to use the description of Q as the closure of the $K_{\mathbb{C}}$-sweep of the image of a hypertoric variety in the $(K \times T \times SU(2))$-representation $\mathcal{R}$ and the corresponding description of its twistor space $\mathcal{Z}_Q$ to extend the hyperkähler implosion construction from $K = SU(n)$ to more general compact groups K.

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1. Twistor spaces

In this section we review the theory of twistor spaces for hyperkähler manifolds; for more details see [**9**].

A hyperkähler manifold M has a Riemannian metric, together with a triple of complex structures $(\mathsf{I}, \mathsf{J}, \mathsf{K})$ satisfying the quaternionic relations such that the metric is Kähler with respect to each complex structure. Thus a hyperkähler manifold has a triple (in fact, a whole two-sphere) of symplectic forms $(\omega_1, \omega_2, \omega_3)$ which are Kähler forms for the complex structures $(\mathsf{I}, \mathsf{J}, \mathsf{K})$.

If a compact group K acts on a hyperkähler manifold M preserving its hyperkähler structure with moment maps μ_1, μ_2, μ_3 for the symplectic forms $\omega_1, \omega_2, \omega_3$, then the hyperkähler quotient $\mu^{-1}(0)/K$ (where $\mu = (\mu_1, \mu_2, \mu_3)\colon M \to \mathfrak{k} \otimes \mathbb{R}^3$) inherits a hyperkähler structure from that on M.

A hyperkähler manifold M has (real) dimension $4k$ for some non-negative integer k. We can associate to M its twistor space $\mathcal{Z}_M$ which is a complex manifold of (complex) dimension $2k+1$ with some additional structure from which we can recover the hyperkähler manifold M. As a smooth manifold $\mathcal{Z}_M$ is the product $M \times S^2$ of M and the two-dimensional sphere S^2, but its complex structure at $(m, \zeta) \in M \times S^2$ is defined by $(\mathsf{I}_\zeta, \tilde{\mathsf{I}})$ where $\tilde{\mathsf{I}}$ is the usual complex structure on $S^2 \cong \mathbb{P}^1$ and if

$$\zeta = (\zeta_1, \zeta_2, \zeta_3) \in S^2 \subset \mathbb{R}^3$$

then $\mathsf{I}_\zeta = \zeta_1 \mathsf{I} + \zeta_2 \mathsf{J} + \zeta_3 \mathsf{K}$, where $(\mathsf{I}, \mathsf{J}, \mathsf{K})$ is the triple of complex structures on M as above. The twistor space $\mathcal{Z}_M$ is equipped with the following additional structure [**9**]:

1) a holomorphic projection $\pi\colon \mathcal{Z}_M \to \mathbb{P}^1$ whose fiber at $\zeta \in \mathbb{P}^1$ is M equipped with the holomorphic structure I_ζ determined by ζ;

2) a holomorphic section ω of the holomorphic vector bundle

$$\wedge^2 T_F^*(2) = \wedge^2 T_F^* \otimes \mathcal{O}(2)$$

over $\mathcal{Z}_M$ where T_F is the tangent bundle along the fibers of π, such that ω defines a holomorphic symplectic form ω_ζ on each fiber $\pi^{-1}(\zeta) \cong M$ of π;

3) a real structure (that is, an anti-holomorphic involution) τ on $\mathcal{Z}_M$ preserving this data and covering the antipodal map on $\mathbb{P}^1$.

With respect to the C^∞-identification of $\mathcal{Z}_M$ with $M \times \mathbb{P}^1$, the real structure is given by

$$\tau(m, \zeta) = (m, -1/\bar{\zeta}).$$

With respect to the fixed holomorphic section $(1/2)\partial/\partial\zeta$ of $T\mathbb{P}^1 \cong \mathcal{O}(2)$, the holomorphic symplectic form ω_ζ is given by

$$\omega_\zeta = \omega_2 + \mathbf{i}\omega_3 - 2\zeta\omega_1 - \zeta^2(\omega_2 - \mathbf{i}\omega_3),$$

where $\omega_1, \omega_2, \omega_3$ are the Kähler forms associated to the hyperkähler metric and the complex structures $\mathsf{I}, \mathsf{J}, \mathsf{K}$.

A holomorphic section of $\pi\colon \mathcal{Z}_M \to \mathbb{P}^1$ is called a twistor line, and a twistor line $\sigma\colon \mathbb{P}^1 \to \mathcal{Z}_M$ is real if $\tau\sigma(\zeta) = \sigma(-1/\bar{\zeta})$ for every $\zeta \in \mathbb{P}^1$. Each point $m \in M$ gives rise to a real twistor line $\{m\} \times \mathbb{P}^1$ with normal bundle $\mathbb{C}^{2k} \otimes \mathcal{O}(1)$, and using such twistor lines we can recover the hyperkähler manifold M from its twistor space.

If a compact group K acts on a hyperkähler manifold M preserving its hyperkähler structure with a hyperkähler moment map $\mu = (\mu_1, \mu_2, \mu_3)$, then there is an associated holomorphic map $\mathcal{Z}_\mu\colon \mathcal{Z}_M \to \mathfrak{k}^*_{\mathbb{C}} \otimes \mathcal{O}(2)$ whose restriction to each fiber of π is a complex moment map for the holomorphic symplectic form defined by ω on the fiber. The twistor space of the hyperkähler quotient $\mu^{-1}(0)/K$ is the quotient (in the sense of Kähler geometry or geometric invariant theory [**10**]) of $\mathcal{Z}_\mu^{-1}(0)$ by the complexification $K_{\mathbb{C}}$ of K.

Remark 1.1. The twistor moment map $\mathcal{Z}_\mu\colon \mathcal{Z}_M \to \mathfrak{k}^*_{\mathbb{C}} \otimes \mathcal{O}(2)$ restricts to a holomorphic section of $\mathfrak{k}^*_{\mathbb{C}} \otimes \mathcal{O}(2)$ on each twistor line $\{m\} \times \mathbb{P}^1$ in $\mathcal{Z}_M$. This gives us a map

$$\phi\colon M \to H^0(\mathbb{P}^1, \mathfrak{k}^*_{\mathbb{C}} \otimes \mathcal{O}(2)) \cong \mathfrak{k}^*_{\mathbb{C}} \otimes H^0(\mathbb{P}^1, \mathcal{O}(2))$$

whose evaluation at any $p \in \mathbb{P}^1$ can be identified (modulo choosing a basis for the one-dimensional complex vector space $\mathcal{O}(2)_p$) with the complex moment map associated to the corresponding complex structure on M. Then $\phi^{-1}(0) = \mu^{-1}(0)$ and the hyperkähler quotient of M by K is $\phi^{-1}(0)/K$.

The twistor space for a flat hyperkähler manifold $\mathbb{H}^k$ is the vector bundle $\mathcal{Z} = \mathcal{O}(1) \otimes \mathbb{C}^{2k}$ over $\mathbb{P}^1$. If we write $\mathcal{Z}$ as

$$\mathcal{Z} = \mathcal{O}(1) \otimes (W \oplus W^*)$$

where $W = \mathbb{C}^k$, then the natural pairing between W and W^* defines a constant holomorphic section ω of $\wedge^2 T_F^*(2)$. The standard hermitian structure on W gives us an identification of W with $\overline{W}^*$, and this together with the antipodal map on $\mathbb{P}^1$ gives us the real structure on $\mathcal{Z}$.

Let $\alpha_1, \ldots, \alpha_k$ be the standard coordinates on W, and let $\beta_1, \ldots, \beta_k$ be the dual coordinates on W^*. We can cover $\mathcal{Z}$ with two coordinate patches $\zeta \neq \infty$ and $\zeta \neq 0$ with coordinates

$$(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k, \zeta) \quad \text{and} \quad (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k, \tilde{\beta}_1, \ldots, \tilde{\beta}_k, \tilde{\zeta}) \tag{1.2}$$

related by the transition functions

$$\tilde{\zeta} = 1/\zeta,\ \tilde{\alpha}_j = \alpha_j/\zeta,\ \tilde{\beta}_j = \beta_j/\zeta.$$

With respect to these coordinates the real structure on $\mathcal{Z}$ is given by

$$(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k, \zeta) \mapsto (\bar{\beta}_1/\bar{\zeta}, \ldots, \bar{\beta}_k/\bar{\zeta}, -\bar{\alpha}_1/\bar{\zeta}, \ldots, -\bar{\alpha}_k/\bar{\zeta}, -1/\bar{\zeta}),$$

while ω is given by $\sum_{j=1}^k \mathrm{d}\alpha_j \wedge \mathrm{d}\beta_j/2$.

2. The nilpotent cone

In this section we recall the hyperkähler structure on the nilpotent cone in the Lie algebra of $K_{\mathbb{C}} = SL(n, \mathbb{C})$ obtained in [**12**] and describe its twistor space which can be embedded in the vector bundle $\mathcal{O}(2) \otimes \mathfrak{k}_{\mathbb{C}}$ over $\mathbb{P}^1$.

The nilpotent cone for $K_{\mathbb{C}} = SL(n, \mathbb{C})$ is identified in [**12**] with a hyperkähler quotient $M /\!\!/\!\!/ \tilde{H}$, where M is a flat hyperkähler space and $\tilde{H}$ is a product of unitary groups acting on M.

Let us choose integers $0 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_r = n$ and consider the flat hyperkähler space

(2.1)
$$M = M(\mathbf{n}) = \bigoplus_{i=1}^{r-1} \mathbb{H}^{n_i n_{i+1}} = \bigoplus_{i=1}^{r-1} \mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_{i+1}}) \oplus \mathrm{Hom}(\mathbb{C}^{n_{i+1}}, \mathbb{C}^{n_i})$$

with the hyperkähler action of $U(n_1) \times \cdots \times U(n_r)$

$$\alpha_i \mapsto g_{i+1} \alpha_i g_i^{-1}, \quad \beta_i \mapsto g_i \beta_i g_{i+1}^{-1} \qquad (i = 1, \dots r-1),$$

with $g_i \in U(n_i)$ for $i = 1, \dots, r$. Here α_i and β_i denote elements of $\mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_{i+1}})$ and $\mathrm{Hom}(\mathbb{C}^{n_{i+1}}, \mathbb{C}^{n_i})$, respectively, and right quaternion multiplication is given by

$$(\alpha_i, \beta_i)\mathbf{j} = (-\beta_i^*, \alpha_i^*). \tag{2.2}$$

We may write $(\alpha, \beta) \in M(\mathbf{n})$ as a quiver diagram:

$$0 \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} \mathbb{C}^{n_1} \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \mathbb{C}^{n_2} \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{r-2}}{\overset{\alpha_{r-2}}{\rightleftarrows}} \mathbb{C}^{n_{r-1}} \underset{\beta_{r-1}}{\overset{\alpha_{r-1}}{\rightleftarrows}} \mathbb{C}^{n_r} = \mathbb{C}^n, \tag{2.3}$$

where $\alpha_0 = \beta_0 = 0$. For brevity, we will often call such a diagram a quiver. Let $\tilde{H}$ be the subgroup, isomorphic to $\prod_{i=1}^{r-1} U(n_i)$, given by setting $g_r = 1$, and let

$$\tilde{\mu} \colon M \to \mathrm{Lie}(\tilde{H}) \otimes \mathbb{R}^3 = \mathrm{Lie}(\tilde{H}) \otimes (\mathbb{R} \oplus \mathbb{C})$$
$$\tilde{\mu}(\alpha, \beta) = \big((\alpha_i \alpha_i^* - \beta_i^* \beta_i + \beta_{i+1} \beta_{i+1}^* - \alpha_{i+1}^* \alpha_{i+1})\mathbf{i}, \alpha_i \beta_i - \beta_{i+1} \alpha_{i+1}\big)$$

be the corresponding hyperkähler moment map.

It is proved in [**12**] that when we have a full flag (that is, when $r = n$ and $n_j = j$ for each j, so that the center of $\tilde{H}$ can be identified with the maximal torus T of $K = SU(n)$), then the hyperkähler quotient $\tilde{\mu}^{-1}(0)/\tilde{H}$ of M by $\tilde{H}$ can be identified with the nilpotent cone in $\mathfrak{k}_{\mathbb{C}}$.

This hyperkähler quotient carries an $SU(n)$ action induced from the action of this group on the top space $\mathbb{C}^n$ of the quiver. In [**12**] Theorem 2.1 it is shown that, for any choice of complex structure, the complex moment map $M /\!\!/\!\!/ \tilde{H} \to \mathfrak{k}_{\mathbb{C}}$ for this action induces a bijection from $M /\!\!/\!\!/ \tilde{H}$ onto the nilpotent cone in $\mathfrak{k}_{\mathbb{C}}$. This moment map is given, with complex structure as above, by

$$(\alpha, \beta) \mapsto \alpha_{n-1} \beta_{n-1}.$$

Thus the hyperkähler moment map $\mu\colon M/\!\!/\!\!/\tilde{H} \to \mathfrak{k}\otimes\mathbb{R}^3$ provides a bijection from $M/\!\!/\!\!/\tilde{H}$ to its image in $\mathfrak{k}\otimes\mathbb{R}^3$, and this image is a $(K\times SU(2))$-invariant subset $\mathrm{Nil}(K)$ of $\mathfrak{k}\otimes\mathbb{R}^3$ such that after acting by *any* element of $SU(2)$ the projection $\mathrm{Nil}(K)\to\mathfrak{k}_{\mathbb{C}}$ given by the decomposition $\mathbb{R}^3=\mathbb{R}\oplus\mathbb{C}$ is a bijection onto the nilpotent cone in $\mathfrak{k}_{\mathbb{C}}$.

The hyperkähler structure on the nilpotent cone in $\mathfrak{k}_{\mathbb{C}}$ can in principle be determined explicitly from the bijection from $M/\!\!/\!\!/\tilde{H}$ given by $(\alpha,\beta)\mapsto\alpha_{n-1}\beta_{n-1}$, but it is not very easy to write down a lift to $\mu^{-1}(0)$ of the inverse of this bijection. However, the hyperkähler structure can be determined explicitly from the embedding of $\mathrm{Nil}(K)$ in $\mathfrak{k}\otimes\mathbb{R}^3$ as follows. The complex and complex-symplectic structures on $\mathrm{Nil}(K)$ are given by pulling back the standard complex and complex-symplectic structures on nilpotent orbits in $\mathfrak{k}_{\mathbb{C}}$ under the projections of $\mathfrak{k}\otimes\mathbb{R}^3$ onto $\mathfrak{k}\otimes\mathbb{C}$ corresponding to the different choices of complex structures, and these determine the metric.

Thus it is useful to describe as explicitly as possible the embedding of $\mathrm{Nil}(K)$ in $\mathfrak{k}\otimes\mathbb{R}^3$. If we fix a complex structure and use it to identify $\mathrm{Nil}(K)$ with the nilpotent cone in $\mathfrak{k}_{\mathbb{C}}$ and to identify $\mathfrak{k}\otimes\mathbb{R}^3$ with $\mathfrak{k}\oplus\mathfrak{k}_{\mathbb{C}}$, then the embedding is given by

$$\eta\mapsto(\Phi_n(\eta),\eta)$$

where the map Φ_n from the nilpotent cone in $\mathfrak{sl}(n,\mathbb{C})$ to $\mathfrak{su}(n)$ is the (real) moment map for the action of $K=SU(n)$ on the nilpotent cone with respect to the Kähler form determined by the standard complex structure and the hyperkähler metric. The map Φ_n is determined inductively by the properties given in Lemma 2.5, below.

Remark 2.4. Note that M has an $SU(2)$-action which commutes with the action of $\tilde{H}$ and rotates the complex structures on M. This action descends to an $SU(2)$-action on $M/\!\!/\!\!/\tilde{H}$ and hence on $\mathrm{Nil}(K)$. The induced $SU(2)$-action on $\mathrm{Nil}(K)$ extends to the action on $\mathfrak{k}\otimes\mathbb{R}^3$ given by the usual rotation action on $\mathbb{R}^3$. When $\mathrm{Nil}(K)$ is identified with the nilpotent cone in $\mathfrak{k}_{\mathbb{C}}$ by projection from $\mathfrak{k}\otimes\mathbb{R}^3$ to $\mathfrak{k}_{\mathbb{C}}$, the action of

$$\begin{pmatrix} u & v\\ -\bar{v} & \bar{u}\end{pmatrix}\in SU(2),$$

where $|u|^2+|v|^2=1$, is given by

$$\eta\mapsto u^2\eta+2uv\Phi_n(\eta)-v^2\bar{\eta}^T.$$

Lemma 2.5. *The map Φ_n from the nilpotent cone in $\mathfrak{k}_{\mathbb{C}}=\mathfrak{sl}(n,\mathbb{C})$ to $\mathfrak{k}=\mathfrak{su}(n)$ is $SU(n)$-equivariant for the adjoint action of $SU(n)$ on $\mathfrak{sl}(n,\mathbb{C})$ and $\mathfrak{su}(n)$. Furthermore, if A is a generic upper triangular $(n-1)\times(n-1)$ complex matrix with positive real eigenvalues so that the $n\times n$ matrix*

$$\begin{pmatrix} 0 & A^2\\ 0 & 0\end{pmatrix}$$

is strictly upper triangular, then

$$\Phi_n\left(\begin{pmatrix}0 & A^2\\ 0 & 0\end{pmatrix}\right) = \left(\begin{pmatrix}AB(\overline{AB})^T & 0\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 0 & (\overline{B}^{-1}\overline{A})^T B^{-1}A\end{pmatrix}\right)\mathbf{i}$$

where B is upper triangular with real positive eigenvalues and satisfies

$$\Phi_{n-1}\left(\begin{pmatrix}0 & B^{-1}A\end{pmatrix}\begin{pmatrix}AB\\ 0\end{pmatrix}\right) = (B^{-1}A(\overline{B}^{-1}\overline{A})^T - (\overline{AB})^T AB)\mathbf{i}.$$

These properties determine the continuous map Φ_n inductively.

REMARK 2.6. In order for the upper triangular matrix A to be generic in the sense of this lemma, it is enough for A to lie in the regular nilpotent orbit in $\mathfrak{k}_{\mathbb{C}}$.

Proof. First, note that any nilpotent matrix lies in the $SU(n)$-adjoint orbit of a strictly upper triangular matrix with non-negative real entries immediately above the leading diagonal, and a generic such matrix (lying in the regular nilpotent orbit in $\mathfrak{k}_{\mathbb{C}}$) can be expressed in the form

$$\begin{pmatrix}0 & A^2\\ 0 & 0\end{pmatrix}$$

where A has positive real eigenvalues. The hyperkähler quotient $M/\!\!/\!\!/\tilde{H}$ can be described as $\mu_{\tilde{H}}^{-1}(0)/\tilde{H}$ where $\mu_{\tilde{H}}$ is the hyperkähler moment map for the action of $\tilde{H}$ on M, and also as the GIT quotient of $(\mu_{\tilde{H}})_{\mathbb{C}}^{-1}(0)$ by its complexification $\tilde{H}_{\mathbb{C}}$. A quiver $(\alpha,\beta) \in M(\mathbf{n})$ as at (2.3) lies in $\mu_{\tilde{H}}^{-1}(0)/\tilde{H}$ if and only if $\beta_j\alpha_j = \alpha_{j-1}\beta_{j-1}$ for $0 < j < n$. Writing

$$\begin{pmatrix}0 & A^2\\ 0 & 0\end{pmatrix} = \alpha_{n-1}\beta_{n-1}$$

where

$$\alpha_{n-1} = \begin{pmatrix}A\\ 0\end{pmatrix} \quad \text{and} \quad \beta_{n-1} = \begin{pmatrix}0 & A\end{pmatrix},$$

so that

$$\beta_{n-1}\alpha_{n-1} = \begin{pmatrix}0 & A\end{pmatrix}\begin{pmatrix}A\\ 0\end{pmatrix}$$

is a strictly upper triangular matrix with non-negative real entries immediately above the leading diagonal, allows us inductively to find a quiver (α,β) in $(\mu_{\tilde{H}})_{\mathbb{C}}^{-1}(0)$ such that

$$\alpha_{n-1}\beta_{n-1} = \begin{pmatrix}0 & A^2\\ 0 & 0\end{pmatrix} \tag{2.7}$$

for generic A as above. In order to find the value of Φ_n on this matrix, however, we need to find a representative quiver in $\mu_{\tilde{H}}^{-1}(0)$, since

$$\Phi_n\left(\begin{pmatrix}0 & A^2\\ 0 & 0\end{pmatrix}\right) = (\alpha_{n-1}\alpha_{n-1}^* - \beta_{n-1}^*\beta_{n-1})\mathbf{i}$$

for any quiver $(\alpha,\beta) \in \mu_{\tilde{H}}^{-1}(0)$ such that (2.7) holds, and then

$$\Phi_j(\alpha_{j-1}\beta_{j-1}) = (\alpha_{j-1}\alpha_{j-1}^* - \beta_{j-1}^*\beta_{j-1})\mathbf{i}$$

for $0 < j < n$. The theory of moment maps and GIT quotients tells us that there will exist such a representative quiver in the closure of the $\tilde{H}_{\mathbb{C}}$-orbit of the quiver we have already found, and that for generic A this orbit is closed. Indeed, since $\tilde{H}_{\mathbb{C}}$ can be expressed as the product of a Borel subgroup and its maximal compact subgroup $\tilde{H}$, we can find such a representative quiver in the intersection of the Borel orbit and $\mu_{\tilde{H}}^{-1}(0)$. The maps α_{n-1} and β_{n-1} in this quiver will be given by matrices of the form

$$\begin{pmatrix} AB \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & B^{-1}A \end{pmatrix}$$

where we can assume that B is an upper triangular $(n-1) \times (n-1)$ complex matrix with positive real eigenvalues. As the quiver lies in $\mu_{\tilde{H}}^{-1}(0)$, it follows that

$$\Phi_{n-1}\left(\begin{pmatrix} 0 & B^{-1}A \end{pmatrix}\begin{pmatrix} AB \\ 0 \end{pmatrix}\right) = (B^{-1}A(\overline{B}^{-1}\overline{A})^T - (\overline{AB})^T AB)\mathbf{i}.$$

This completes the proof. q.e.d.

EXAMPLE 2.8. Consider the case when $n = 2$. If d is real and positive, then

$$\begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{d} \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{d} \end{pmatrix},$$

where

$$\begin{pmatrix} 0 & \sqrt{d} \end{pmatrix} \begin{pmatrix} \sqrt{d} \\ 0 \end{pmatrix} = 0,$$

and $\Phi_1 = 0$ so

$$\Phi_2\left(\begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} d & 0 \\ 0 & -d \end{pmatrix}\mathbf{i}.$$

$\diamondsuit$

EXAMPLE 2.9. Let $n = 3$ and let

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \alpha & \beta \\ 0 & 1/\gamma \end{pmatrix}$$

where a, c, α, γ are real and positive and $b, \beta \in \mathbb{C}$. Then, using the previous example, we have

$$\Phi_2\left(\begin{pmatrix} 0 & B^{-1}A \end{pmatrix}\begin{pmatrix} AB \\ 0 \end{pmatrix}\right) = \Phi_2\left(\begin{pmatrix} 0 & ac/\alpha\gamma \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} ac/\alpha\gamma & 0 \\ 0 & -ac/\alpha\gamma \end{pmatrix}\mathbf{i},$$

while

$$\begin{aligned}&(B^{-1}A(\overline{B}^{-1}\overline{A})^T - (\overline{AB})^T AB)\mathbf{i}\\ &= \begin{pmatrix} a^2(1/\alpha^2-\alpha^2)+|(b-c\beta\gamma)/\alpha|^2 & c\gamma(b-c\beta\gamma)/\alpha - a\alpha(a\beta+b/\gamma) \\ c\gamma\overline{(b-c\beta\gamma)}/\alpha - a\alpha\overline{(a\beta+b/\gamma)} & c^2(\gamma^2-1/\gamma^2)-|a\beta+b/\gamma|^2\end{pmatrix}\mathbf{i}.\end{aligned}$$

When these are equal we have

$$\beta = \frac{b(c\gamma^2 - a\alpha^2)}{\gamma(a^2\alpha^2+c^2\gamma^2)}$$

and

$$a^2(1/\alpha^2-\alpha^2)+\left|\frac{ab(a+c)\alpha}{a^2\alpha^2+c^2\gamma^2}\right|^2 = ac/\alpha\gamma = c^2(1/\gamma^2-\gamma^2)+\left|\frac{bc(a+c)\gamma}{a^2\alpha^2+c^2\gamma^2}\right|^2.$$

It is convenient to write $\alpha/\gamma = \delta$; then these equations are equivalent to

$$|b| = \frac{a^2\delta^2+c^2}{a+c}\sqrt{\gamma^4+\frac{a^2/\delta^2-c^2}{c^2-a^2\delta^2}} \tag{2.10}$$

and $f_{(a/c)}(\delta)=0$ where

$$f_{(a/c)}(x) = x^4 - (a/c)x^3 + (c/a)x - 1.$$

Note that this polynomial factorizes as $(x-(a/c))(x^3+(c/a))$, so it has a unique positive root $\delta = a/c$. Thus

$$\gamma = c\left(\frac{(a+c)|b|}{a^4+c^4}\right)^{1/2}$$

and $\alpha = \frac{a\gamma}{c}$ and $\beta = \frac{b(c^3-a^3)}{\gamma(a^4+c^4)} = \frac{\gamma b(c^3-a^3)}{|b|c^2(a+c)}$. Then

$$\Phi_3\left(\begin{pmatrix}0 & A^2\\ 0 & 0\end{pmatrix}\right)$$

is given by substituting these expressions for α,β,γ into the matrix

$$\begin{pmatrix} a^2\alpha^2+|a\beta+b/\gamma|^2 & (a\beta\gamma+b)c/\gamma^2 & 0\\ (a\bar\beta\gamma+\bar b)c/\gamma^2 & c^2/\gamma^2-a^2/\alpha^2 & -a(b\alpha-\beta\gamma c)/\alpha^2 \\ 0 & -a(\bar b\alpha - \bar\beta\gamma c)/\alpha^2 & |b-\beta\gamma c|^2/\alpha^2 - c^2\gamma^2\end{pmatrix}\mathbf{i}.$$

We obtain

$$\begin{pmatrix} |b|(a+c) & \frac{bc^2}{|b|} & 0\\ \frac{\bar b c^2}{|b|} & 0 & -\frac{ba^2}{|b|}\\ 0 & -\frac{\bar b a^2}{|b|} & -|b|(a+c)\end{pmatrix}\mathbf{i}$$

(cf. [**11**, §5]). ♢

Since M is a flat hyperkähler manifold, it follows as in §1 that its twistor space $\mathcal{Z}_M$ is the vector bundle

$$\mathcal{O}(1) \otimes \bigoplus_{i=1}^{r-1} (\mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_{i+1}}) \oplus \mathrm{Hom}(\mathbb{C}^{n_{i+1}}, \mathbb{C}^{n_i}))$$

over $\mathbb{P}^1$. The complex moment map for the action of $\tilde{H}$ defines a morphism $\mathcal{Z}_\mu$ from $\mathcal{Z}_M$ into the vector bundle $\mathcal{O}(2) \otimes \mathrm{Lie}\, \tilde{H}_{\mathbb{C}}$ over $\mathbb{P}^1$, and the twistor space for the hyperkähler quotient $M /\!\!/\!\!/ \tilde{H}$ is the quotient (in the sense of geometric invariant theory) by the action of the complexification $\tilde{H}_{\mathbb{C}} = \prod_{k=1}^{n-1} GL(k)$ on the zero section of this morphism. The complex moment map for the action of $K = SU(n)$ defines a $\tilde{H}_{\mathbb{C}}$-invariant morphism from $\mathcal{Z}_M$ into the vector bundle $\mathcal{O}(2) \otimes \mathfrak{k}_{\mathbb{C}}$ over $\mathbb{P}^1$, and this induces an embedding into $\mathcal{O}(2) \otimes \mathfrak{k}_{\mathbb{C}}$ of the GIT quotient which is the twistor space $\mathcal{Z}_{\mathrm{Nil}(K)}$ for $\mathrm{Nil}(K) = M /\!\!/\!\!/ \tilde{H}$. It follows from [**12**] Theorem 2.1 that this is an isomorphism of the twistor space with the closed subvariety of the vector bundle $\mathcal{O}(2) \otimes \mathfrak{k}_{\mathbb{C}}$ over $\mathbb{P}^1$ which meets each fiber in the tensor product of the corresponding fiber of $\mathcal{O}(2)$ with the nilpotent cone in $\mathfrak{k}_{\mathbb{C}}$.

The real structure on the twistor space $\mathcal{Z}_{\mathrm{Nil}(K)}$ is represented in local coordinates as at (1.2) by

$$(\alpha_{n-1}\beta_{n-1}, \zeta) \mapsto (-\overline{\beta}_{n-1}^T \overline{\alpha}_{n-1}^T / \bar{\zeta}^2, -1/\bar{\zeta})$$

or equivalently

$$(X, \zeta) \mapsto (-\bar{X}^T / \bar{\zeta}^2, -1/\bar{\zeta})$$

for X in the nilpotent cone in $\mathfrak{k}_{\mathbb{C}}$.

The remaining structure required for the twistor space $\mathcal{Z}_{\mathrm{Nil}(K)}$ is a holomorphic section ω of $\wedge^2 T_F^* \otimes \mathcal{O}(2)$ where T_F denotes the tangent bundle along the fibers of $\pi \colon \mathcal{Z}_{\mathrm{Nil}(K)} \to \mathbb{P}^1$; or rather, this is what would be required if $\mathrm{Nil}(K)$ were smooth. In fact, $\mathrm{Nil}(K)$ is singular with a stratified hyperkähler structure, where the strata are given by the (finitely many) (co-)adjoint orbits in the nilpotent cone in $\mathfrak{k}_{\mathbb{C}} \cong \mathfrak{k}_{\mathbb{C}}^*$, and ω restricts on each stratum Σ to a holomorphic section ω_Σ of $\wedge^2 T_{F,\Sigma}^* \otimes \mathcal{O}(2)$ where $T_{F,\Sigma}$ is the tangent bundle along the fibers of the restriction of π to the twistor space $\mathcal{Z}_\Sigma$.

Recall that any co-adjoint orbit $\mathcal{O}_\eta \cong K/K_\eta$ for $\eta \in \mathfrak{k}^*$ of a compact group K has a canonical K-invariant symplectic form, the Kirillov–Kostant form ω_η, which can be obtained by symplectic reduction at η from the cotangent bundle T^*K with its canonical symplectic structure. The K-invariant symplectic form ω_η is characterized by the property that the corresponding moment map for the action of K on $\mathcal{O}_\eta$ is the inclusion of $\mathcal{O}_\eta$ in $\mathfrak{k}^*$.

Similarly, a co-adjoint orbit $\mathcal{O}_\eta$ for $\eta \in \mathfrak{k}_{\mathbb{C}}^*$ of the complexification $K_{\mathbb{C}}$ of K has a canonical $K_{\mathbb{C}}$-invariant holomorphic symplectic form ω_η

which is again characterized by the property that the associated complex moment map for the action of $K_{\mathbb{C}}$ is the inclusion of $\mathcal{O}_\eta$ in $\mathfrak{k}^*_{\mathbb{C}}$.

If Σ is a stratum of $\mathrm{Nil}(K)$ given by a co-adjoint orbit in $\mathfrak{k}^*_{\mathbb{C}}$, then the holomorphic section ω_Σ of $\wedge^2 T^*_{F,\Sigma} \otimes \mathcal{O}(2)$ which restricts to a holomorphic symplectic form on each fiber of $\pi\colon \mathcal{Z}_\Sigma \to \mathbb{P}^1$ is $K_{\mathbb{C}}$-invariant for $K = SU(n)$, and the corresponding twistor moment map $\mathcal{Z}_\Sigma \to \mathfrak{k}^*_{\mathbb{C}} \otimes \mathcal{O}(2)$ is the restriction of the embedding of $\mathcal{Z}_{\mathrm{Nil}(K)}$ into $\mathfrak{k}^*_{\mathbb{C}} \otimes \mathcal{O}(2)$. Thus it follows that each ω_Σ (and hence also the holomorphic section ω of $\wedge^2 T^*_F \otimes \mathcal{O}(2)$) is given by the Kirillov–Kostant construction on the fibers of π.

The Springer resolution of the nilpotent cone in $\mathfrak{k}_{\mathbb{C}}$ is given by the complex moment map for the action of $\mathfrak{k}_{\mathbb{C}}$ on the cotangent bundle $T^*\mathcal{B}$ where $\mathcal{B}$ is the flag manifold $K_{\mathbb{C}}/B = K/T$ identified with the space of Borel subgroups of $K_{\mathbb{C}}$ [**2**]. The twistor space $\mathcal{Z}_{\mathrm{Nil}(K)}$ has a corresponding resolution of singularities

$$\tilde{\mathcal{Z}}_{\mathrm{Nil}(K)} \cong K_{\mathbb{C}} \times_B (\mathfrak{b}^0 \times \mathcal{O}(2)) \cong K \times_T (\mathfrak{t}^0 \times \mathcal{O}(2))$$

where B is a Borel subgroup of $K_{\mathbb{C}}$ containing T and $\mathfrak{b}^0$ is the annihilator of its Lie algebra in $\mathfrak{k}^*_{\mathbb{C}}$ while $\mathfrak{t}^0$ is the annihilator of the Lie algebra of T in $\mathfrak{k}^*$.

Remark 2.11. The twistor moment map $\mathcal{Z}_{\mathrm{Nil}(K)} \to \mathfrak{k}^*_{\mathbb{C}} \otimes \mathcal{O}(2)$ restricts to a holomorphic section of $\mathfrak{k}^*_{\mathbb{C}} \otimes \mathcal{O}(2)$ on each twistor line $\{m\} \times \mathbb{P}^1$ in $\mathcal{Z}_{\mathrm{Nil}(K)}$, and gives us a map

$$\phi\colon \mathrm{Nil}(K) \to H^0(\mathbb{P}^1, \mathfrak{k}^*_{\mathbb{C}} \otimes \mathcal{O}(2))$$

as in Remark 1.1. For any $p \in \mathbb{P}^1$ the composition of ϕ with the evaluation map

$$H^0(\mathbb{P}^1, \mathfrak{k}^*_{\mathbb{C}} \otimes \mathcal{O}(2)) \to \mathfrak{k}^*_{\mathbb{C}} \otimes \mathcal{O}(2)_p$$

is injective, and its image is the nilpotent cone in $\mathfrak{k}^*_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}}$ if we choose any basis for the one-dimensional complex vector space $\mathcal{O}(2)_p$ to identify $\mathfrak{k}^*_{\mathbb{C}} \otimes \mathcal{O}(2)_p$ with $\mathfrak{k}^*_{\mathbb{C}}$. If we fix the complex structure corresponding to $[1:0] \in \mathbb{P}^1$ to identify $\mathrm{Nil}(K)$ with the nilpotent cone in $\mathfrak{k}^*_{\mathbb{C}}$, then ϕ is given by

$$\phi(\eta)[u:v] = u^2\eta + 2uv\Phi_n(\eta) - v^2\bar{\eta}^T$$

when $\eta \in \mathfrak{k}_{\mathbb{C}}$ is nilpotent and $[u:v] \in \mathbb{P}^1$ and Φ_n is as at Lemma 2.5.

The map $\phi\colon \mathrm{Nil}(K) \to H^0(\mathbb{P}^1, \mathfrak{k}^*_{\mathbb{C}} \otimes \mathcal{O}(2))$ induces an embedding

$$\phi_{\mathcal{Z}}\colon \mathcal{Z}_{\mathrm{Nil}(K)} = \mathbb{P}^1 \times \mathrm{Nil}(K) \to \mathbb{P}^1 \times H^0(\mathbb{P}^1, \mathfrak{k}^*_{\mathbb{C}} \otimes \mathcal{O}(2))$$

which is *not* holomorphic, as well as the holomorphic embedding

$$\mathcal{Z}_{\mathrm{Nil}(K)} \to \mathfrak{k}^*_{\mathbb{C}} \otimes \mathcal{O}(2).$$

This map (which may also be viewed as the twistor moment map for the $K_{\mathbb{C}}$ action) is the composition of $\phi_{\mathcal{Z}}$ with the natural map

$$\mathbb{P}^1 \times H^0(\mathbb{P}^1, \mathfrak{k}^*_{\mathbb{C}} \otimes \mathcal{O}(2)) \to \mathfrak{k}^*_{\mathbb{C}} \otimes \mathcal{O}(2).$$

The real structure on $\mathcal{Z}_{\mathrm{Nil}(K)}$ extends to the real structure on $\mathbb{P}^1 \times H^0(\mathbb{P}^1, \mathfrak{k}_{\mathbb{C}}^* \otimes \mathcal{O}(2))$ determined by the standard real structure $\zeta \mapsto -1/\bar{\zeta}$ on $\mathbb{P}^1$ and the real structure $\eta \mapsto -\bar{\eta}^T$ on $\mathfrak{k}_{\mathbb{C}}$.

If X is any hyperkähler manifold on which $K = SU(n)$ acts with hyperkähler moment map $\mu_X \colon X \to \mathfrak{k}^* \otimes \mathbb{R}^3$ and corresponding

$$\phi_X \colon X \to H^0(\mathbb{P}^1, \mathfrak{k}_{\mathbb{C}}^* \otimes \mathcal{O}(2))$$

as in Remark 1.1, then we obtain a stratified hyperkähler space

$$X_{\mathrm{nil}} = (X \times \mathrm{Nil}(K)) /\!/\!/ K = \bigsqcup_{\mathcal{O}} \phi_X^{-1}(\phi(\mathcal{O}))/K$$

where the strata are indexed by the nilpotent co-adjoint orbits $\mathcal{O}$ in $\mathfrak{k}_{\mathbb{C}}^*$. Since ϕ_X and ϕ are K-equivariant and $K_{\mathbb{C}} = KP_{\mathcal{O}}$, the stratum indexed by $\mathcal{O}$ is given by

$$\phi_X^{-1}(\phi(\mathcal{O}))/K \cong \phi_X^{-1}(\phi(P_{\mathcal{O}}\eta_{\mathcal{O}}))/K_{\mathcal{O}},$$

where $\eta_{\mathcal{O}}$ is a representative of the orbit $\mathcal{O}$ in Jordan canonical form, $P_{\mathcal{O}}$ is the associated Jacobson–Morosov parabolic (see [**3**] Remark 3.8.5), and $K_{\mathcal{O}} = K \cap P_{\mathcal{O}}$.

It follows from [**4**] Theorem 7.18 that $\mathrm{Nil}(K)$ with any of its complex structures is the non-reductive GIT quotient (in the sense of [**6**]) of $K_{\mathbb{C}} \times \mathfrak{b}^0$ by the Borel $B = T_{\mathbb{C}}N$ in $K_{\mathbb{C}}$. Hence if X is a complex affine variety with respect to any of its complex structures, then X_{nil} is the complex symplectic quotient (with respect to non-reductive GIT) of X by the action of the Borel B.

3. Symplectic and hyperkähler implosion

In this section we recall the constructions of symplectic implosion from [**7**] and hyperkähler implosion for $K = SU(n)$ from [**4**, **5**].

Let M be a symplectic manifold with Hamiltonian action of a compact group K. The imploded space M_{impl} is a stratified symplectic space with a Hamiltonian action of the maximal torus T of K. It has the property that, denoting symplectic reduction at λ by $/\!/_{\lambda}^{s}$,

$$M /\!/_{\lambda}^{s} K = M_{\mathrm{impl}} /\!/_{\lambda}^{s} T \tag{3.1}$$

for all λ in the closure $\mathfrak{t}_+^*$ of a fixed positive Weyl chamber in $\mathfrak{t}^*$. In particular the implosion $(T^*K)_{\mathrm{impl}}$ of the cotangent bundle T^*K inherits a Hamiltonian $(K \times T)$-action from the Hamiltonian $(K \times K)$-action on T^*K. This example is universal in the sense that for a general M we have

$$M_{\mathrm{impl}} = (M \times (T^*K)_{\mathrm{impl}}) /\!/_0^s K.$$

Concretely, the implosion $(T^*K)_{\mathrm{impl}}$ of T^*K with respect to the right action is constructed from $K \times \mathfrak{t}_+^*$ by identifying (k_1, ξ) with (k_2, ξ) if $k_1 k_2^{-1}$ lies in the commutator subgroup of the K-stabilizer of ξ. The identifications which occur are therefore controlled by the face structure of the Weyl chamber. In particular, if ξ is in the interior of the chamber, its stabilizer is a torus and no identifications are performed. An open dense subset of $(T^*K)_{\mathrm{impl}}$, therefore, is the product of K with the interior of the Weyl chamber.

As explained in [**7**], when K is a connected, simply connected, semisimple compact Lie group, we may embed the universal symplectic implosion $(T^*K)_{\mathrm{impl}}$ in the complex affine space $E = \oplus V_\varpi$, where V_ϖ is the K-module with highest-weight ϖ and we sum over a minimal generating set Π for the monoid of dominant weights. Under this embedding, the implosion is identified with the closure $\overline{K_{\mathbb{C}} v}$, where v is the sum of the highest-weight vectors v_ϖ of the K-modules V_ϖ, and as usual $K_{\mathbb{C}}$ denotes the complexification of K. This gives an alternative, more algebraic, description of the implosion as a stratified space. For the stabilizer of v is a maximal unipotent subgroup N of $K_{\mathbb{C}}$ (that is, the commutator subgroup $[B, B]$ of the corresponding Borel subgroup B) and hence we may regard $K_{\mathbb{C}} v$ as $K_{\mathbb{C}}/N$. The lower-dimensional strata which we obtain by taking the closure are just the quotients $K_{\mathbb{C}}/[P, P]$ for standard parabolic subgroups P of $K_{\mathbb{C}}$. These standard parabolics are, of course, in bijective correspondence with the faces of the Weyl chamber, and this algebraic stratification is compatible with the symplectic stratification described above.

The whole implosion may be identified with the Geometric Invariant Theory (GIT) quotient of $K_{\mathbb{C}}$ by the non-reductive group N:

$$K_{\mathbb{C}} /\!\!/ N = \mathrm{Spec}(\mathcal{O}(K_{\mathbb{C}})^N);$$

it is often useful to view this as the canonical affine completion of the quasi-affine variety $K_{\mathbb{C}}/N$ (cf. [**6**]).

Recalling the Iwasawa decomposition $K_{\mathbb{C}} = KAN$, where $T_{\mathbb{C}} = TA$, we see that

$$\overline{K_{\mathbb{C}} v} = \overline{KAv} = K(\overline{T_{\mathbb{C}} v}), \tag{3.2}$$

so the implosion is described as the sweep under the compact group K of a toric variety $\overline{T_{\mathbb{C}} v}$. This toric variety is associated to the positive Weyl chamber $\mathfrak{t}_+$ and is in fact the subspace

$$E^N = \bigoplus_{\varpi \in \Pi} \mathbb{C} v_\varpi$$

of E spanned by the highest-weight vectors v_ϖ. The canonical affine completion $K_{\mathbb{C}} /\!\!/ N$ of K_C/N has a resolution of singularities

$$\widetilde{K_{\mathbb{C}} /\!\!/ N} = K_{\mathbb{C}} \times_B E^N \to K_{\mathbb{C}} /\!\!/ N$$

induced by the group action $K_{\mathbb{C}} \times E^N \to E$.

As explained in [**4**] one can also construct the symplectic implosion for $K = SU(n)$ using *symplectic quivers*. These are diagrams

$$0 = V_0 \stackrel{\alpha_0}{\to} V_1 \stackrel{\alpha_1}{\to} V_2 \stackrel{\alpha_2}{\to} \cdots \stackrel{\alpha_{r-2}}{\to} V_{r-1} \stackrel{\alpha_{r-1}}{\to} V_r = \mathbb{C}^n \tag{3.3}$$

where V_i is a vector space of dimension n_i. We realized the symplectic implosion as the GIT quotient of the space of full flag quivers (that is, where $r = n$ and $n_i = i$), by $\prod_{i=1}^{r-1} SL(V_i)$, where this group acts by

$$\begin{aligned} \alpha_i &\mapsto g_{i+1}\alpha_i g_i^{-1} \quad (i = 1, \dots, r-2), \\ \alpha_{r-1} &\mapsto \alpha_{r-1} g_{r-1}^{-1}. \end{aligned}$$

In [**4**] a hyperkähler analogue of the symplectic implosion was introduced for the group $K = SU(n)$. We consider quiver diagrams of the following form:

$$0 = V_0 \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} V_1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} V_2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{r-2}}{\overset{\alpha_{r-2}}{\rightleftarrows}} V_{r-1} \underset{\beta_{r-1}}{\overset{\alpha_{r-1}}{\rightleftarrows}} V_r = \mathbb{C}^n$$

where V_i is a complex vector space of complex dimension n_i and $\alpha_0 = \beta_0 = 0$. The space M of quivers for fixed dimension vector $(n_1, \dots, n_r)$ is a flat hyperkähler vector space.

As discussed earlier, there is a hyperkähler action of $U(n_1) \times \cdots \times U(n_r)$ on this space given by

$$\alpha_i \mapsto g_{i+1}\alpha_i g_i^{-1}, \quad \beta_i \mapsto g_i \beta_i g_{i+1}^{-1} \qquad (i = 1, \dots r-1),$$

with $g_i \in U(n_i)$ for $i = 1, \dots, r$. Recall that we defined $\tilde{H}$ be the subgroup, isomorphic to $U(n_1) \times \cdots \times U(n_{r-1})$, given by setting $g_r = 1$. We also let $H = SU(n_1) \times \cdots \times SU(n_{r-1}) \leqslant \tilde{H}$.

Definition 3.4. The *universal hyperkähler implosion for* $SU(n)$ is the hyperkähler quotient $Q = M /\!\!/\!\!/ H$, where M, H are as above with $r = n$ and $n_j = j$, for $j = 1, \dots, n$.

This hyperkähler quotient Q has a residual action of $(S^1)^{n-1} = \tilde{H}/H$ as well as an action of $SU(n_r) = SU(n)$. As explained in [**4**] we may identify $(S^1)^{n-1}$ with the maximal torus T of $SU(n)$. There is also an $Sp(1) = SU(2)$ action which is not hyperkähler but rotates the complex structures.

Using the standard theory relating symplectic and GIT quotients, we have a description of $Q = M /\!\!/\!\!/ H$, as the quotient (in the GIT sense) of the subvariety defined by the complex moment map equations

$$\alpha_i \beta_i - \beta_{i+1}\alpha_{i+1} = \lambda_{i+1}^{\mathbb{C}} I \qquad (0 \leqslant i \leqslant r-2) \tag{3.5}$$

where $\lambda_i^{\mathbb{C}}$ for $1 \leqslant i \leqslant r-1$ are complex scalars, by the action of

$$H_{\mathbb{C}} = \prod_{i=1}^{r-1} SL(n_i, \mathbb{C})$$

$$\alpha_i \mapsto g_{i+1}\alpha_i g_i^{-1}, \quad \beta_i \mapsto g_i \beta_i g_{i+1}^{-1} \qquad (i = 1, \dots r-2), \tag{3.6}$$

$$\alpha_{r-1} \mapsto \alpha_{r-1} g_{r-1}^{-1}, \quad \beta_{r-1} \mapsto g_{r-1}\beta_{r-1}, \tag{3.7}$$

where $g_i \in SL(n_i, \mathbb{C})$.

The element $X = \alpha_{r-1}\beta_{r-1} \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ is invariant under the action of $\prod_{i=1}^{r-1} GL(n_i, \mathbb{C})$ and transforms by conjugation under the residual $SL(n, \mathbb{C}) = SL(n_r, \mathbb{C})$ action on Q. We thus have a $T_{\mathbb{C}}$-invariant and $SL(n, \mathbb{C})$-equivariant map $Q \to \mathfrak{sl}(n, \mathbb{C})$ given by

$$(\alpha, \beta) \mapsto (X)_0 = X - \frac{1}{n}\,\mathrm{tr}(X) I_n$$

where I_n is the $n \times n$ identity matrix. This is the complex moment map for the residual $SU(n)$ action.

It is shown in [**4**] that X satisfies an equation

$$X(X + \nu_1) \dots (X + \nu_{n-1}) = 0$$

where $\nu_i = \sum_{j=i}^{r-1} \lambda_j^{\mathbb{C}}$. This generalizes the equation $X^n = 0$ in the quiver construction of the nilpotent variety in [**12**] and is a consequence of Lemma 5.9 from [**4**] which gives information about the eigenvalues of X and other endomorphisms derived from final segments of the quiver. Define

$$X_k = \alpha_{r-1}\alpha_{r-2} \dots \alpha_{r-k}\beta_{r-k} \dots \beta_{r-2}\beta_{r-1} \qquad (1 \leqslant k \leqslant r-1) \tag{3.8}$$

so that $X = X_1$. It is proved in [**4**] Lemma 5.9 that for $(\alpha, \beta) \in \mu_{\mathbb{C}}^{-1}(0)$, satisfying (3.5), we have

$$X_k X = X_{k+1} - (\lambda_{r-1}^{\mathbb{C}} + \dots + \lambda_{r-k}^{\mathbb{C}}) X_k. \tag{3.9}$$

It follows from this by induction on j that if $0 \leq j < k < r$, then

$$X_k = X_{k-j} X^j + \sum_{i=1}^{j} \sigma_i(\nu_{r-k+1}, \dots, \nu_{r-k+j}) X_{k-j} X^{j-i}$$

where σ_i denotes the ith elementary symmetric polynomial. In particular putting $j = k-1$ gives us

$$X_k = X^k + \sum_{i=1}^{k-1} \sigma_i(\nu_{r-k+1}, \dots, \nu_{r-1}) X^{k-i} = X \prod_{i=1}^{k-1} (X + \nu_{r-i}),$$

so

$$X_k = \prod_{i=0}^{k-1} (X + \sum_{j=1}^{i} \lambda_{r-j}^{\mathbb{C}}).$$

Thus we have the following lemma.

Lemma 3.10. *If* $1 \leqslant k \leqslant r-1$*, then*

$$\wedge^{r-k}(\alpha_{r-1}\alpha_{r-2}\cdots\alpha_{r-k})\wedge^{r-k}(\beta_{r-k}\cdots\beta_{r-1}) = \wedge^{r-k}\prod_{i=0}^{k-1}(X+\sum_{j=1}^{i}\lambda^{\mathbb{C}}_{r-j}).$$

Recall from (3.2) that the universal symplectic implosion is the $K_{\mathbb{C}}$-sweep of a toric variety $\overline{T_{\mathbb{C}}v}$. In [**5**] we found a hypertoric variety mapping generically injectively to the hyperkähler implosion. Hypertoric varieties are hyperkähler quotients of flat quaternionic spaces $\mathbb{H}^d$ by subtori of $(S^1)^d$; for more background see [**1**, **8**].

Definition 3.11. Let M_T be the subset of M consisting of hyperkähler quivers of the form

$$\alpha_k = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \nu^k_1 & 0 & \cdots & 0 \\ 0 & \nu^k_2 & \cdots & 0 \\ & & \cdots & \\ 0 & \cdots & 0 & \nu^k_k \end{pmatrix}$$

and

$$\beta_k = \begin{pmatrix} 0 & \mu^k_1 & 0 & \cdots & 0 \\ 0 & 0 & \mu^k_2 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 0 & \mu^k_k \end{pmatrix}$$

for some $\nu^k_i, \mu^k_i \in \mathbb{C}$. Note that this definition of M_T differs slightly from the definition in [**5**], but only by the action of an element of the Weyl group of $H \times K$.

As explained in [**5**], for quivers of this form the moment map equations for the action of H reduce to the moment map equations for the action of its maximal torus T_H which is the product over all k from 1 to $n-1$ of the standard maximal tori in $SU(k)$. We say that the quiver is *hyperkähler stable* if it has all α_i injective and all β_i surjective after a suitable rotation of complex structures. For quivers of the form above, this means that μ^k_i and ν^k_i do not both vanish for any (i,k) with $1 \leq i \leq k < n$. Two hyperkähler stable quivers of this form satisfying the hyperkähler moment map equations lie in the same orbit for the action of H if and only if they lie in the same orbit for the action of its maximal torus T_H. We therefore get a natural map ι from the hypertoric variety $M_T /\!\!/\!\!/ T_H$ to the implosion $Q = M /\!\!/\!\!/ H$, which restricts to an embedding

$$\iota\colon Q^{\mathrm{hks}}_T \to Q$$

where $Q^{\mathrm{hks}}_T = M^{\mathrm{hks}}_T /\!\!/\!\!/ T_H$ and M^{hks}_T denotes the hyperkähler stable elements of M_T. Let $Q_T = \iota(M_T /\!\!/\!\!/ T_H)$ be the image of $\iota\colon M_T /\!\!/\!\!/ T_H \to Q$.

The space M_T is hypertoric for the maximal torus $T_{\tilde{H}}$ of $\tilde{H}$, and $M_T /\!\!/\!\!/ T_H$ is hypertoric for the torus $T_{\tilde{H}}/T_H = \tilde{H}/H = (S^1)^{n-1}$, which can be identified with T as in [**5**, §3], in such a way that the induced action of $K \times T$ on Q restricts to an action of $T \times T$ on Q_T such that $(t, 1)$ and $(1, t)$ act in the same way on Q_T for any $t \in T$.

Indeed, by [**5**, Remark 3.2], $M_T /\!\!/\!\!/ T_H$ is the hypertoric variety for T associated to the hyperplane arrangement in its Lie algebra $\mathfrak{t}$ given by the root planes.

Remark 3.12. The root planes in the Lie algebra of the maximal torus $T_{U(n)}$ of $U(n)$ are the coordinate hyperplanes in $\mathrm{Lie}(T_{U(n)}) = \mathbb{R}^n$, and the corresponding hypertoric variety for $T_{U(n)}$ is $\mathbb{H}^n$. Thus we can identify $M_T /\!\!/\!\!/ T_H$ with the hypertoric variety

$$\{(w_1, \ldots, w_n) \in \mathbb{H}^n : w_1 + \cdots + w_n = 0\}$$

for $T = (S^1)^{n-1}$.

4. Toward an embedding of the universal hyperkähler implosion in a linear representation of $K \times T$

Let $\ell_j = j(n-j)$. In this section we define a $K \times T \times SU(2)$-equivariant map σ from the universal hyperkähler implosion Q for $K = SU(n)$ to

$$\mathcal{R} = H^0(\mathbb{P}^1, (\mathcal{O}(2) \otimes (\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}})) \oplus \bigoplus_{j=1}^{n-1} \mathcal{O}(\ell_j) \otimes \wedge^j \mathbb{C}^n)$$

and an associated holomorphic map $\tilde{\sigma}$ from the twistor space of Q to the vector bundle $(\mathcal{O}(2) \otimes (\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}})) \oplus \bigoplus_{j=1}^{n-1} \mathcal{O}(\ell_j) \otimes \wedge^j \mathbb{C}^n$ over $\mathbb{P}^1$; the first of these maps is proved to be injective, and the second is proved to be generically injective in §6.

As in §3 the universal symplectic implosion $(T^*K)_{\mathrm{impl}}$ for $K = SU(n)$ has a canonical embedding in a linear representation of $K \times T$ associated to its description as the non-reductive GIT quotient

$$K_{\mathbb{C}} /\!\!/ N = \mathrm{Spec}(\mathcal{O}(K_{\mathbb{C}})^N)$$

where N is a maximal unipotent subgroup of the complexification $K_{\mathbb{C}} = SL(n, \mathbb{C})$ of K (cf. [**7**]). The highest weights of the irreducible representations $\mathbb{C}^n, \wedge^2\mathbb{C}^n, \ldots, \wedge^{n-1}\mathbb{C}^n$ of K generate the monoid of dominant weights, and each $\wedge^j\mathbb{C}^n$ becomes a representation of $K \times T$ when T acts as multiplication by the inverse of the corresponding highest weight. Then $K_{\mathbb{C}} /\!\!/ N$ is embedded in the representation

$$\mathbb{C}^n \oplus \wedge^2\mathbb{C}^n \oplus \cdots \oplus \wedge^{n-1}\mathbb{C}^n$$

of $K \times T$ as the closure of the $K_{\mathbb{C}}$-orbit of

$$\sum_{j=1}^{n-1} v_j$$

where $v_j \in \wedge^j\mathbb{C}^n$ is a highest-weight vector, fixed by N.

Similarly, we expect the universal hyperkähler implosion for $K = SU(n)$ to have an embedding in a representation of $K \times T$. In this section we will define a map which will later be shown to provide such an embedding.

Let

$$0 \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} \mathbb{C} \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \mathbb{C}^2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{n-2}}{\overset{\alpha_{n-2}}{\rightleftarrows}} \mathbb{C}^{n-1} \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} \mathbb{C}^n \tag{4.1}$$

be a quiver in M which satisfies the hyperkähler moment map equations for

$$H = \prod_{k=1}^{n-1} SU(k).$$

Recalling that $\alpha_0 = \beta_0 = 0$, these equations are given by

$$\alpha_i \beta_i - \beta_{i+1}\alpha_{i+1} = \lambda_{i+1}^{\mathbb{C}} I \qquad (0 \leqslant i \leqslant n-2), \tag{4.2}$$

where $\lambda_i^{\mathbb{C}} \in \mathbb{C}$ for $1 \leqslant i \leqslant n-1$, and

$$\alpha_i\alpha_i^* - \beta_i^*\beta_i + \beta_{i+1}\beta_{i+1}^* - \alpha_{i+1}^*\alpha_{i+1} = \lambda_{i+1}^{\mathbb{R}} I \quad (0 \leqslant i \leqslant n-2), \tag{4.3}$$

where $\lambda_i^{\mathbb{R}} \in \mathbb{R}$ for $1 \leqslant i \leqslant n-1$. Then

$$(\alpha_{n-1}\beta_{n-1})_0 = \alpha_{n-1}\beta_{n-1} - \frac{1}{n}\operatorname{tr}(\alpha_{n-1}\beta_{n-1}) I_n \in \mathfrak{k}_{\mathbb{C}}$$

is invariant under the action of H, as is

$$\big((u\alpha_{n-1} + v\beta_{n-1}^*)(-v\alpha_{n-1}^* + u\beta_{n-1})\big)_0$$

for any $(u,v) \in \mathbb{C}^2$ representing an element of

$$SU(2) = \left\{ \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} : |u|^2 + |v|^2 = 1 \right\},$$

where α_{n-1}^* and β_{n-1}^* denote the adjoints of α_{n-1} and β_{n-1}. The same is true of

$$\sigma_j^{(\alpha,\beta)} = \wedge^j(\alpha_{n-1}\alpha_{n-2}\cdots\alpha_j) \in \wedge^j\mathbb{C}^n,$$

where the linear map $\wedge^j(\alpha_{n-1}\alpha_{n-2}\cdots\alpha_j)\colon \wedge^j\mathbb{C}^j \to \wedge^j\mathbb{C}^n$ is identified with the image of the standard basis element of the one-dimensional complex vector space $\wedge^j\mathbb{C}^j$, and the element

$$\sigma_j^{(u\alpha+v\beta^*, -v\alpha^*+u\beta)} = \wedge^j(u\alpha_{n-1} + v\beta_{n-1}^*)(u\alpha_{n-2} + v\beta_{n-2}^*)\cdots(u\alpha_j + v\beta_j^*)$$

of $\wedge^j\mathbb{C}^n$ is also H-invariant for any $(u,v) \in \mathbb{C}^2$. Note that

$$\mathbb{P}^1 = SU(2)/S^1$$

where

$$S^1 = \left\{ \begin{pmatrix} t & 0 \\ 0 & \bar{t} \end{pmatrix} : |t| = 1 \right\}$$

acts on $SU(2)$ by left multiplication, and that

$$(u,v) \mapsto \big((u\alpha_{n-1} + v\beta_{n-1}^*)(-v\alpha_{n-1}^* + u\beta_{n-1})\big)_0 \tag{4.4}$$

defines an element of

$$H^0(\mathbb{P}^1, \mathcal{O}(2) \otimes \mathfrak{k}_\mathbb{C})$$

whose value on a point p of $\mathbb{P}^1$ is the value at the quiver (4.1) of the corresponding complex moment map for the K-action on M (up to multiplication by a non-zero complex scalar depending on a choice of basis of the fiber $\mathcal{O}(2)_p$ of the line bundle $\mathcal{O}(2)$ at p). Similarly, the map

$$(u, v) \mapsto u^2\lambda^\mathbb{C} + uv\lambda^\mathbb{R} - v^2\bar{\lambda}^\mathbb{C} \tag{4.5}$$

defines an element of

$$H^0(\mathbb{P}^1, \mathcal{O}(2)) \otimes \mathfrak{t}_\mathbb{C}$$

whose value on a point p of $\mathbb{P}^1$ is the value at the quiver (4.1) of the corresponding complex moment map for the T-action on M (up to multiplication by a non-zero complex scalar depending on a choice of basis of the fiber $\mathcal{O}(2)_p$). Again this map is defined in an H-invariant way. Finally, if $1 \leqslant j \leqslant n-1$, then

(4.6)

$$\begin{aligned}(u, v) &\mapsto \sigma_j^{(u\alpha + v\beta^*, -v\alpha^* + u\beta)} \\ &= \wedge^j(u\alpha_{n-1} + v\beta^*_{n-1})(u\alpha_{n-2} + v\beta^*_{n-2}) \cdots (u\alpha_j + v\beta^*_j) \in \wedge^j\mathbb{C}^n\end{aligned}$$

defines an element of

$$H^0(\mathbb{P}^1, \mathcal{O}(\ell_j) \otimes \wedge^j\mathbb{C}^n).$$

Definition 4.7. Let $\ell_j = j(n-j)$ as before and let

$$\sigma\colon Q \to \mathcal{R} = H^0(\mathbb{P}^1, \mathcal{O}(2) \otimes (\mathfrak{k}_\mathbb{C} \oplus \mathfrak{t}_\mathbb{C}) \oplus \bigoplus_{j=1}^{n-1} \mathcal{O}(\ell_j) \otimes \wedge^j\mathbb{C}^n)$$

be the map defined by combining (4.4), (4.5), and (4.6), above.

Remark 4.8. The projection of σ to $H^0(\mathbb{P}^1, \mathcal{O}(2) \otimes (\mathfrak{k}_\mathbb{C} \oplus \mathfrak{t}_\mathbb{C}))$ is the map ϕ associated as in Remark 1.1 to the action of $K \times T$ on Q.

Remark 4.9. Note that σ is $(K \times T \times SU(2))$-equivariant when $K \times T \times SU(2)$ acts on $\mathcal{R}$ as described in the introduction, and $SU(2)$ acts on Q (commuting with the actions of K and T) via the inclusion $SU(2) = \mathrm{Sp}(1) \leqslant \mathbb{H}^*$. Moreover, the evaluation

$$\sigma_p\colon Q \to \mathcal{O}(2)_p \otimes (\mathfrak{k}_\mathbb{C} \oplus \mathfrak{t}_\mathbb{C}) \oplus \bigoplus_{j=1}^{n-1} \mathcal{O}(\ell_j)_p \otimes \wedge^j\mathbb{C}^n \cong \mathfrak{k}_\mathbb{C} \oplus \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{j=1}^{n-1} \wedge^j\mathbb{C}^n$$

of σ at any point p of $\mathbb{P}^1$ is a morphism of complex affine varieties with respect to the complex structure on Q determined by that point of $\mathbb{P}^1$. Furthermore, the projection of σ_p to $\mathfrak{k}_\mathbb{C}$ and to $\mathfrak{t}_\mathbb{C}$ can be identified with the complex moment map for the complex structure associated to p for the action of K and of T on Q, once we have fixed a basis element for the fiber $\mathcal{O}(2)_p$ of the line bundle $\mathcal{O}(2)$ at p. Of course, a choice of basis elements for the fibers $\mathcal{O}(\ell_j)_p$ of the line bundles $\mathcal{O}(\ell_j)$ at p for all $j \geqslant 1$ is determined canonically by a choice of basis element for the fiber $\mathcal{O}(1)_p$ of the line bundle $\mathcal{O}(1)$ at p, and so σ_p determines a map $Q \to \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{j=1}^{n-1} \wedge^j \mathbb{C}^n$ canonically up to the action of $\mathbb{C}^*$ with weight 2 on $\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}}$ and weight ℓ_j on $\wedge^j \mathbb{C}^n$.

REMARK 4.10. Note that if $u\alpha_j + v\beta_j^*$ is injective for each j, then the projection of $\sigma_{[u:v]}$ onto $E = \bigoplus_{j=1}^{n-1} \wedge^j \mathbb{C}^n$ maps the quiver into the $K_{\mathbb{C}}$-orbit of the sum

$$\sum_{j=1}^{n-1} v_j \in E^N \subseteq E$$

of highest-weight vectors v_j in the fundamental representations $\wedge^j \mathbb{C}^n$ of $K = SU(n)$ for $j = 1, \dots, n-1$. Since this condition is satisfied by generic quivers in Q, it follows that the projection of $\sigma_{[u:v]}$ onto $E = \bigoplus_{j=1}^{n-1} \wedge^j \mathbb{C}^n$ maps Q into the canonical affine completion

$$K_{\mathbb{C}} /\!\!/ N = \overline{K_{\mathbb{C}} \left(\sum_{j=1}^{n-1} v_j \right)}.$$

Indeed, recall from [**7**] that $K_{\mathbb{C}} /\!\!/ N$ is the union of finitely many $K_{\mathbb{C}}$-orbits, one for each face τ of the positive Weyl chamber $\mathfrak{t}_+$ for K, with stabilizer the commutator subgroup $[P_\tau, P_\tau]$ of the corresponding parabolic subgroup P_τ of $K_{\mathbb{C}}$ whose intersection with K is the stabiliser K_τ of τ under the (co-)adjoint action of K. It follows from Theorem 6.13 of [**4**] that if $q \in Q$, then for generic $p \in \mathbb{P}^1$ the projection of σ_p onto $E = \bigoplus_{j=1}^{n-1} \wedge^j \mathbb{C}^n$ lies in the $K_{\mathbb{C}}$-orbit in $K_{\mathbb{C}} /\!\!/ N$ with stabilizer $[P_\tau, P_\tau]$ where τ is the face of $\mathfrak{t}_+$ whose stabilizer K_τ in K is the stabilizer K_λ of the image $\lambda \in \mathfrak{k} \otimes \mathbb{R}^3$ of q under the hyperkähler moment map for the action of T on Q.

We will prove in §6 the following theorem.

Theorem 4.11. *The map $\sigma \colon Q \to \mathcal{R}$ defined at Definition* 4.7 *is injective.*

REMARK 4.12. It follows from Remark 4.9 that if $q \in Q$, then $\sigma(q)$ determines the image of q under the hyperkähler moment maps for the actions of T and K on Q. Recall that the hyperkähler reductions by the action of T on the universal hyperkähler implosion Q for $K = SU(n)$ are closures of co-adjoint orbits of $K_{\mathbb{C}} = SL(n, \mathbb{C})$. In particular, the hyperkähler reduction at level 0 is the nilpotent cone for $K_{\mathbb{C}}$, which is identified in [**12**] with the hyperkähler quotient $M /\!\!/\!\!/ \tilde{H}$, where $\tilde{H} = \prod_{k=1}^{n-1} U(k)$. This hyperkähler quotient carries an $SU(n)$ action induced from the action of this group on the top space $\mathbb{C}^n$ of the quiver. We recalled in §2 that, for any choices of complex structures, the complex moment map $M/\!\!/\!\!/\tilde{H} \to \mathfrak{k}_{\mathbb{C}}$ for this action induces a bijection from $M/\!\!/\!\!/\tilde{H}$ onto the nilpotent cone in $\mathfrak{k}_{\mathbb{C}}$, and thus the hyperkähler moment map $M/\!\!/\!\!/\tilde{H} \to \mathfrak{k}\otimes\mathbb{R}^3$ provides a bijection from $M/\!\!/\!\!/\tilde{H}$ to its image in $\mathfrak{k}\otimes\mathbb{R}^3$. Moreover, this image is a $(K\times SU(2))$-invariant subset $\mathrm{Nil}(K)$ of $\mathfrak{k}\otimes\mathbb{R}^3$ such that after acting by any element of $SU(2)$ the projection $\mathrm{Nil}(K) \to \mathfrak{k}_{\mathbb{C}}$ given by the decomposition $\mathbb{R}^3 = \mathbb{C}\oplus\mathbb{R}$ is a bijection onto the nilpotent cone in $\mathfrak{k}_{\mathbb{C}}$.

We obtain an induced map $\tilde{\sigma}$ from the twistor space $\mathcal{Z}_Q$ of Q to the vector bundle

$$\mathcal{O}(2)\otimes(\mathfrak{k}_{\mathbb{C}}\oplus\mathfrak{t}_{\mathbb{C}})\oplus\bigoplus_{j=1}^{n-1}\mathcal{O}(\ell_j)\otimes\wedge^j\mathbb{C}^n$$

over $\mathbb{P}^1$. It is the composition of the product of the identity on $\mathbb{P}^1$ and σ from Q to $\mathcal{R}$ with the natural evaluation map from $\mathbb{P}^1\times\mathcal{R}$ to

$$\mathcal{O}(2)\otimes(\mathfrak{k}_{\mathbb{C}}\oplus\mathfrak{t}_{\mathbb{C}})\oplus\bigoplus_{j-1}^{n-1}\mathcal{O}(\ell_j)\otimes\wedge^j\mathbb{C}^n.$$

As in Remark 4.9 we see that $\tilde{\sigma}$ is holomorphic and $(K\times T\times SU(2))$-equivariant. We will prove in §6 the following theorem.

Theorem 4.13. *The map*

$$\tilde{\sigma}\colon \mathcal{Z}_Q \to \mathcal{O}(2)\otimes(\mathfrak{k}_{\mathbb{C}}\oplus\mathfrak{t}_{\mathbb{C}})\oplus\bigoplus_{j=1}^{n-1}\mathcal{O}(\ell_j)\otimes\wedge^j\mathbb{C}^n$$

is generically injective; that is, its restriction to a dense Zariski-open subset of $\mathcal{Z}_Q$ is injective.

Remark 4.14. It follows from Remark 4.10 and (5.3), below, that the image of $\tilde{\sigma}$ is contained in the subvariety of $\mathcal{O}(2)\otimes(\mathfrak{k}_{\mathbb{C}}\oplus\mathfrak{t}_{\mathbb{C}})\oplus\bigoplus_{j=1}^{n-1}\mathcal{O}(\ell_j)\otimes\wedge^j\mathbb{C}^n$ whose fiber at any $p\in\mathbb{P}^1$ is identified (after choosing any basis vector for $\mathcal{O}(1)_p$ and thus for $\mathcal{O}(\ell_j)_p$ for all $j\geqslant 1$) with the product of

$$\{\,(\eta,\xi)\in\mathfrak{k}_{\mathbb{C}}\times\mathfrak{t}_{\mathbb{C}} : \eta \text{ and } \xi \text{ have the same eigenvalues}\,\}$$

and the canonical affine completion $K_{\mathbb{C}}/\!\!/N$ of $K_{\mathbb{C}}/N$. We can also impose the condition provided by Lemma 3.10. The image of σ satisfies analogous constraints.

5. Stratifying the universal hyperkähler implosion for $SU(n)$ and its twistor space

In this section we recall the stratification given in [**4**] of the universal hyperkähler implosion Q for $K = SU(n)$ into strata which are hyperkähler manifolds and its refinement in [**5**]. The refined stratification has strata $Q_{[\sim,\mathcal{O}]}$ indexed in terms of Levi subgroups and nilpotent orbits in $K_{\mathbb{C}} = SL(n,\mathbb{C})$. The latter stratification is not hyperkähler but reflects well the group structure of $K = SU(n)$. These stratifications induce corresponding stratifications of the twistor space of Q.

First of all, given a quiver we may decompose each space in the quiver into generalized eigenspaces $\ker(\alpha_i\beta_i - \tau I)^m$ of $\alpha_i\beta_i$. We showed in [**4**] using the complex moment map equations (3.5) that β_i and α_i preserve this decomposition. More precisely, we have

$$\beta_i\colon \ker(\alpha_i\beta_i - \tau I)^m \to \ker(\alpha_{i-1}\beta_{i-1} - (\lambda_i^{\mathbb{C}} + \tau)I)^m \tag{5.1}$$

and

$$\alpha_i\colon \ker(\alpha_{i-1}\beta_{i-1} - (\lambda_i^{\mathbb{C}} + \tau)I)^m \to \ker(\alpha_i\beta_i - \tau I)^m. \tag{5.2}$$

So we actually have a decomposition into subquivers. Moreover, we showed the maps (5.1) and (5.2) are bijective unless $\tau = 0$.

It follows that $\tau \neq 0$ is an eigenvalue of $\alpha_i\beta_i$ if and only if $\tau + \lambda_i^{\mathbb{C}} \neq \lambda_i^{\mathbb{C}}$ is an eigenvalue of $\alpha_{i-1}\beta_{i-1}$. Moreover, $\alpha_i\beta_i$ has zero as an eigenvalue and α_i, β_i restrict to maps between the associated generalized eigenspace with eigenvalue 0 and the generalized eigenspace for $\alpha_{i-1}\beta_{i-1}$ associated to $\lambda_i^{\mathbb{C}}$ (which could be the zero space).

One can deduce that the trace-free part X^0 of $X = \alpha_{n-1}\beta_{n-1}$ now has eigenvalues $\kappa_1, \ldots, \kappa_n$, where

$$\kappa_j = \frac{1}{n}\Big(\lambda_1^{\mathbb{C}} + 2\lambda_2^{\mathbb{C}} + \cdots + (j-1)\lambda_{j-1}^{\mathbb{C}} \\ - (n-j)\lambda_j^{\mathbb{C}} - (n-j-1)\lambda_{j+1}^{\mathbb{C}} - \cdots - \lambda_{n-1}^{\mathbb{C}}\Big).$$

In particular, if $i < j$, then

$$\kappa_j - \kappa_i = \lambda_i^{\mathbb{C}} + \lambda_{i+1}^{\mathbb{C}} + \cdots + \lambda_{j-1}^{\mathbb{C}}. \tag{5.3}$$

This shows that to understand the quiver it is important to understand when collections of λ_i sum to zero.

Now we recall that

$$T = (S^1)^{n-1} = \prod_{k=1}^{n-1} U(k)/SU(k) = \tilde{H}/H$$

acts on $Q = M /\!\!/\!\!/ H$ with hyperkähler moment map

$$\mu_{(S^1)^{n-1}}\colon Q \to \mathfrak{t} \otimes \mathbb{R}^3 = (\mathbb{R}^3)^{n-1} = (\mathbb{C} \oplus \mathbb{R})^{n-1}$$

which maps a quiver to

$$(\lambda_1, \ldots, \lambda_{n-1}) = (\lambda_1^{\mathbb{C}}, \lambda_1^{\mathbb{R}}, \ldots, \lambda_{n-1}^{\mathbb{C}}, \lambda_{n-1}^{\mathbb{R}}).$$

Definition 5.4. For each choice of $(\lambda_1, \dots, \lambda_{n-1})$ we define an equivalence relation $\sim$ on $\{1, \dots, n\}$ by declaring that if $1 \leqslant i < j \leqslant n$ then

$$i \sim j \iff \sum_{k=i}^{j-1} \lambda_k = 0 \text{ in } \mathbb{R}^3.$$

There is thus a stratification of $(\mathbb{R}^3)^{n-1} = \mathfrak{t} \otimes \mathbb{R}^3$ into strata $(\mathbb{R}^3)^{n-1}_{\sim} = (\mathfrak{t} \otimes \mathbb{R}^3)_{\sim}$, indexed by the set of equivalence relations $\sim$ on $\{1, \dots, n\}$, where

$$(\mathbb{R}^3)^{n-1}_{\sim} = \{(\lambda_1, \dots, \lambda_{n-1}) \in (\mathbb{R}^3)^{n-1} : \text{if } 1 \leqslant i < j \leqslant n \text{ then}$$
$$i \sim j \iff \sum_{k=i}^{j-1} \lambda_k = 0 \text{ in } \mathbb{R}^3\}.$$

Under the identification of T with $(S^1)^{n-1}$ using the positive simple roots as a basis for $\mathfrak{t}$, this stratification of $(\mathbb{R}^3)^{n-1} = \mathfrak{t} \otimes \mathbb{R}^3$ is induced by the stratification of $\mathfrak{t}$ associated to the root planes in $\mathfrak{t}$ (see [**4**, §3] and Remark 3.12).

We thus obtain a stratification of Q into subsets $Q_{\sim}$, which are the preimage in Q under $\mu_{(S^1)^{n-1}}$ of $(\mathbb{R}^3)^{n-1}_{\sim}$.

The choice of $\sim$ corresponds to the choice of a subgroup $K_{\sim}$ of K which is the compact real form of a Levi subgroup of $K_{\mathbb{C}}$; this subgroup $K_{\sim}$ is the centralizer of $\mu_{(S^1)^{n-1}}(\mathfrak{q}) \in \mathfrak{t} \otimes \mathbb{R}^3$ for any $\mathfrak{q} \in Q_{\sim}$.

We observe from (5.3) that if $i \sim j$, then we have equality of the eigenvalues κ_i and κ_j.

Now let Q° denote the subset of Q consisting of quivers such that

$$\sum_{k=i}^{j-1} \lambda_k = 0 \text{ in } \mathbb{R}^3 \iff \sum_{k=i}^{j-1} \lambda_k^{\mathbb{C}} = 0 \text{ in } \mathbb{C},$$

and for each equivalence relation $\sim$ on $\{1, 2, \dots, n\}$ let $Q^{\circ}_{\sim}$ denote its intersection with $Q_{\sim}$, consisting of quivers such that

$$i \sim j \iff \sum_{k=i}^{j-1} \lambda_k = 0 \text{ in } \mathbb{R}^3 \iff \sum_{k=i}^{j-1} \lambda_k^{\mathbb{C}} = 0 \text{ in } \mathbb{C}.$$

The full implosion Q is the sweep of Q° under the $SU(2)$ action.

For a quiver $\mathfrak{q}$ in Q° the equivalence relation $\sim$ for which $\mathfrak{q} \in Q_{\sim}$ is determined by the fact that we have equality of the eigenvalues κ_i and κ_j of the trace-free part X^0 of $X = \alpha_{n-1}\beta_{n-1}$ if and only if $i \sim j$. In particular, if $\mathfrak{q} \in Q^{\circ}$, then $(K_{\sim})_{\mathbb{C}}$ is the subgroup of $K_{\mathbb{C}}$ which preserves the decomposition of $\mathfrak{q}$ into the subquivers determined by the generalized eigenspaces of the compositions $\alpha_i\beta_i$.

Remark 5.5. Unfortunately, Q° is not an open subset of Q, although its intersection $Q^{\circ}_{\sim}$ with $Q_{\sim}$ is open in $Q_{\sim}$ for each $\sim$. In fact, we will show in forthcoming work that there is a desingularisation $\hat{Q}$ of Q

covered by open subsets $s\hat{Q}^\circ$ for $s \in SU(2)$ such that the image of the open subset $\hat{Q}^\circ$ of $\hat{Q}$ under $\hat{Q} \to Q$ is Q°.

Let us now return to considering the decomposition of the quiver into subquivers

$$\cdots V_i^j \underset{\beta_{i,j}}{\overset{\alpha_{i,j}}{\rightleftarrows}} V_{i+1}^j \cdots$$

determined by the generalized eigenspaces (with eigenvalues $\tau_{i+1,j}$) of the compositions $\alpha_i\beta_i$, such that

$$\alpha_{i,j}\beta_{i,j} - \beta_{i+1,j}\alpha_{i+1,j} = \lambda_{i+1}^{\mathbb{C}}$$

and $\alpha_{i,j}$ and $\beta_{i,j}$ are isomorphisms unless $\tau_{i+1,j} = 0$. If for some j we have that $\alpha_{k,j}, \beta_{k,j}$ are isomorphisms for $i+1 \leqslant k < s$ but not for $k = i, s$, then it follows that $\tau_{i+1,j} = \tau_{s+1,j} = 0$, hence $\sum_{k=i+1}^{s} \lambda_k^{\mathbb{C}} = 0$, and so since the quiver lies in $Q^\circ_\sim$ we have

$$\sum_{k=i+1}^{s} \lambda_k = 0 \in \mathbb{R}^3.$$

As explained in [**4**] and [**5**], we may contract the subquivers at edges where the maps are isomorphisms. Explicitly, if $\alpha_{i,j}$ and $\beta_{i,j}$ are isomorphisms (which will occur when the associated $\tau_{i+1,j}$ is non-zero), then we may replace

$$\cdots V_{i-1}^j \underset{\beta_{i-1,j}}{\overset{\alpha_{i-1,j}}{\rightleftarrows}} V_i^j \underset{\beta_{i,j}}{\overset{\alpha_{i,j}}{\rightleftarrows}} V_{i+1}^j \underset{\beta_{i+1,j}}{\overset{\alpha_{i+1,j}}{\rightleftarrows}} V_{i+2}^j$$

with

$$V_{i-1}^j \underset{\beta_{i-1,j}}{\overset{\alpha_{i-1,j}}{\rightleftarrows}} V_i^j \underset{(\alpha_{i,j})^{-1}\beta_{i+1,j}}{\overset{\alpha_{i+1,j}\alpha_{i,j}}{\rightleftarrows}} V_{i+2}^j,$$

and then the complex moment map equations are satisfied with

$$\alpha_{i-1,j}\beta_{i-1,j} - (\alpha_{i,j})^{-1}\beta_{i+1,j}\alpha_{i+1,j}\alpha_{i,j} = \lambda_{i-1}^{\mathbb{C}} + \lambda_i^{\mathbb{C}}.$$

If we fix an identification of V_{i+1}^j with V_i^j and apply the action of $SL(V_{i,j})$ so that $\alpha_{i,j}$ is a non-zero scalar multiple aI of the identity, then $\beta_{i,j}$ is determined by $\alpha_{i-1,j}, \alpha_{i+1,j}, \beta_{i-1,j}, \beta_{i+1,j}$ and the scalars a and $\lambda_i^{\mathbb{C}}$ via the equations (3.5) (see [**4**] for more details).

After performing such contractions, the resulting quivers satisfy the complex moment map equations with zero scalars. In other words, they satisfy the complex moment map equations for the product of the relevant $GL(V_i^j)$. In fact, because our quiver is in Q°, the full hyperkähler moment map equations for the associated product of unitary groups are satisfied, and the orbits under the action of the complex group are closed.

Remark 5.6. We are now in the situation analyzed by the third author and Kobak [**12**] in their construction of the nilpotent variety, as discussed in §2. Their results, in particular Theorem 2.1 (cf. [**4**, Proposition 5.16]), show that each contracted subquiver is the direct sum of a quiver where all α are injective and all β are surjective and a quiver in which all maps are 0. Moreover, the direct sum of the contracted subquivers is completely determined (modulo the action of the product of the GL groups) by the elements $\alpha_{n-1}\beta_{n-1}$ at the top edge of each injective/surjective subquiver. The argument of [**12**] shows these are actually nilpotent.

We also observe that because $\sim$ determines the decomposition of the original quiver into eigenspaces, the direct sum of these nilpotents is actually a nilpotent element of $(\mathfrak{k}_\sim)_\mathbb{C}$. It coincides with X^0_n, the nilpotent part in the Jordan decomposition of the trace-free part $X^0 \in \mathfrak{k}_\mathbb{C}$ of $X = \alpha_{n-1}\beta_{n-1}$. (Recall that this is the unique decomposition $X^0 = X^0_s + X^0_n$ where X^0_s and X^0_n in $\mathfrak{k}_\mathbb{C}$ satisfy $[X^0_s, X^0_n] = 0$ and X^0_s is semisimple while X^0_n is nilpotent.) Furthermore, given $\sim$, the adjoint orbit of this nilpotent element in $(\mathfrak{k}_\sim)_\mathbb{C}$ corresponds precisely to determining the dimensions of the various vector spaces in the injective/surjective subquivers (see [**5**, Remarks 5.10 and 5.11]). For example, if a quiver has all $\lambda_i = 0$, then $\sim$ has a single equivalence class, $\mathfrak{k}_\sim = \mathfrak{k}$, and the choice of $\mathcal{O}$ is just the choice of a nilpotent orbit in $\mathfrak{k}_\mathbb{C}$. At the other extreme, if no non-trivial sums are zero, the equivalence classes are singletons and $K_\sim$ is a torus. The orbit $\mathcal{O}$ must now be zero.

To each quiver in Q° we have associated an equivalence relation $\sim$ and a nilpotent orbit $\mathcal{O}$ in $(\mathfrak{k}_\sim)_\mathbb{C}$. Let $Q^\circ_{[\sim,\mathcal{O}]}$ denote the set of quivers in Q° with given $\sim$ and $\mathcal{O}$, and let $Q_{[\sim,\mathcal{O}]}$ denote the $SU(2)$ sweep of $Q^\circ_{[\sim,\mathcal{O}]}$. We may therefore stratify Q as a disjoint union

$$Q = \coprod_{\sim,\mathcal{O}} Q_{[\sim,\mathcal{O}]}$$

over all equivalence relations $\sim$ on $\{1, \dots, n\}$ and all nilpotent adjoint orbits $\mathcal{O}$ in $(\mathfrak{k}_\sim)_\mathbb{C}$.

Remark 5.7. Defining $Q_{[\sim,\mathcal{O}]}$ as the $SU(2)$ sweep of $Q^\circ_{[\sim,\mathcal{O}]}$ in this way, it is not clear that the strata $Q_{[\sim,\mathcal{O}]}$ are disjoint. Hence in [**5**] a different approach is taken in which the stratification $\{Q_{[\sim,\mathcal{O}]}\}$ of Q is initially indexed differently; the equivalence between the two viewpoints is made in [**5**, Remark 5.13].

Remark 5.8. The stratum $Q_{[\sim,\mathcal{O}]}$ in which a quiver lies is determined by the values at the quiver of the hyperkähler moment maps for the actions on Q of $K = SU(n)$ and $T = (S^1)^{n-1}$.

For the value $(\lambda_1, \dots, \lambda_{n-1})$ of $\mu_{(S^1)^{n-1}}$ determines the equivalence relation $\sim$ and also the generic choices of complex structures for which

$$\sum_{k=i}^{j-1} \lambda_k = 0 \text{ in } \mathbb{R}^3 \iff \sum_{k=i}^{j-1} \lambda_k^{\mathbb{C}} = 0 \text{ in } \mathbb{C}. \tag{5.9}$$

Moreover, for such choices of complex structures the quiver decomposes as a direct sum of subquivers determined by the generalized eigenspaces of the composition $\alpha_{n-1}\beta_{n-1}$, and this is given by the complex moment map for the action of K. It follows that the Jordan type of $\alpha_{n-1}\beta_{n-1}$ (for one of the generic choices of complex structures for which (5.9) holds) determines the nilpotent orbit $\mathcal{O}$ in $(\mathfrak{k}_\sim)_{\mathbb{C}}$.

REMARK 5.10. The stratification of Q into strata $Q_{[\sim,\mathcal{O}]}$ induces a stratification of the twistor space $\mathcal{Z}_Q$ into strata $(\mathcal{Z}_Q)_{[\sim,\mathcal{O}]}$.

Let $\sim$ be an equivalence relation on $\{1, \dots, n\}$, and let $\mathcal{O}$ be a nilpotent adjoint orbit in $(\mathfrak{k}_\sim)_{\mathbb{C}}$. In [**5**] we explained how quivers in $Q^\circ_{[\sim,\mathcal{O}]}$ may be put in standard forms using Jordan canonical form.

As above, we first decompose $\mathfrak{q}$ into a direct sum of subquivers determined by the generalized eigenspaces of the compositions $\alpha_i\beta_i$. Since $\mathfrak{q}$ lies in $Q^\circ_{[\sim,\mathcal{O}]}$ each such subquiver is the direct sum of a quiver $\mathfrak{q}^{[j]}$ of the form

$$0 \underset{\beta_0^{[j]}}{\overset{\alpha_0^{[j]}}{\rightleftarrows}} \mathbb{C}^{m_1} \underset{\beta_1^{[j]}}{\overset{\alpha_1^{[j]}}{\rightleftarrows}} \mathbb{C}^{m_2} \underset{\beta_2^{[j]}}{\overset{\alpha_2^{[j]}}{\rightleftarrows}} \cdots \underset{\beta_{n-2}^{[j]}}{\overset{\alpha_{n-2}^{[j]}}{\rightleftarrows}} \mathbb{C}^{m_{n-1}} \underset{\beta_{n-1}^{[j]}}{\overset{\alpha_{n-1}^{[j]}}{\rightleftarrows}} \mathbb{C}^{m_n} \tag{5.11}$$

where the maps $\alpha_k^{[j]}$ for $1 \leqslant k \leqslant n-1$ are injective and the maps $\beta_k^{[j]}$ for $1 \leqslant k \leqslant n-1$ are surjective, together with quivers of the form (for $1 \leqslant h \leqslant p$)

$$\mathbb{C}^{d_h} \underset{\beta_{i_h}^{(h)}}{\overset{\alpha_{i_h}^{(h)}}{\rightleftarrows}} \mathbb{C}^{d_h} \rightleftarrows \cdots \rightleftarrows \mathbb{C}^{d_h} \underset{\beta_{j_h-2}^{(h)}}{\overset{\alpha_{j_h-2}^{(h)}}{\rightleftarrows}} \mathbb{C}^{d_h}$$

in the places $i_h, i_h+1, \dots, j_h-1$, where the maps $\alpha_k^{(h)}$, $\beta_k^{(h)}$, for $i_h \leqslant k < j_h - 1$, are multiplication by complex scalars such that $\gamma_k^{(h)} = \alpha_k^{(h)} + j\beta_k^{(h)} \in \mathbb{H} \setminus \{0\}$. Moreover the combinatorial data here and the Jordan type of $\alpha_{n-1}\beta_{n-1}$ for each summand (5.11) is determined by the pair $(\sim, \mathcal{O})$.

As explained in [**5**], we may use complex linear changes of coordinates in $K_{\mathbb{C}} \times H_{\mathbb{C}} = \prod_{k=1}^n SL(k, \mathbb{C})$ to put $\alpha_{n-1}\beta_{n-1}$ into Jordan canonical form and then decompose the quiver (5.11) into a direct sum of quivers determined by the Jordan blocks of $\alpha_{n-1}^{[j]}\beta_{n-1}^{[j]}$. More precisely, $\alpha_k^{[j]}$ is a direct sum over the set B_j of Jordan blocks for $\alpha_{n-1}^{[j]}\beta_{n-1}^{[j]}$ of matrices of the form

$$(5.12)\qquad \begin{pmatrix} \xi_1^{bjk} & 0 & \cdots & 0 & 0 \\ \nu_1^{bjk} & \xi_2^{bjk} & 0 & \cdots & 0 \\ 0 & \nu_2^{bjk} & & \cdots & 0 \\ & & \cdots & & \\ 0 & \cdots & 0 & \nu_{\ell_b-n+k-1}^{bjk} & \xi_{\ell_b-n+k}^{bjk} \\ 0 & \cdots & 0 & 0 & \nu_{\ell_b-n+k}^{bjk} \end{pmatrix}$$

for some $\nu_i^{bjk}, \xi_i^{bjk} \in \mathbb{C}^*$ where ℓ_b is the size of the Jordan block $b \in B_j$, while $\beta_k^{[j]}$ is a corresponding direct sum over $b \in B_j$ of matrices of the form

$$(5.13)\qquad \begin{pmatrix} 0 & \mu_1^{bjk} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \mu_2^{bjk} & 0 & \cdots & 0 \\ & & \cdots & & & \\ 0 & 0 & \cdots & 0 & \mu_{\ell_b-n+k-1}^{bjk} & 0 \\ 0 & 0 & \cdots & 0 & 0 & \mu_{\ell_b-n+k}^{bjk} \end{pmatrix}$$

for some $\mu_i^{bjk} \in \mathbb{C}^*$, all satisfying the complex moment map equations (3.5). The quiver given by the direct sum over all the Jordan blocks $\bigcup_j B_j$ for $\alpha_{n-1}\beta_{n-1}$ has closed $(H_S)_{\mathbb{C}}$-orbit. If we allow complex linear changes of coordinates in $K_{\mathbb{C}} \times \tilde{H}_{\mathbb{C}} = SL(n,\mathbb{C}) \times \prod_{k=1}^{n-1} GL(k,\mathbb{C})$ (or equivalently allow the action of its quotient group $K_{\mathbb{C}} \times T_{\mathbb{C}}$ on $Q_{[\sim,\mathcal{O}]}$), then the quiver can be put into a more restricted form which is completely determined by $\alpha_{n-1}\beta_{n-1}$ and $(\lambda_1^{\mathbb{C}}, \dots, \lambda_{n-1}^{\mathbb{C}})$, and hence by the value of the complex moment map for the action of $K \times T$ on Q.

Remark 5.14. Let $\{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{C}^n$. When complex linear changes of coordinates in $K_{\mathbb{C}} \times H_{\mathbb{C}}$ are used to put the quiver in the standard form given by (5.12) and (5.13), then

$$\wedge^j \alpha_{n-1} \circ \cdots \circ \alpha_j$$

takes the standard basis vector for $\wedge^j \mathbb{C}^j$ to a scalar multiple of $e_1 \wedge \cdots \wedge e_j \in \wedge^j \mathbb{C}^n$.

Let $Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$ be the subset of $Q^{\circ}_{[\sim,\mathcal{O}]}$ representing quivers of the standard form described above via (5.12) and (5.13) where $\alpha_{n-1}\beta_{n-1}$ is in Jordan canonical form and the summands of the quiver corresponding to generalized eigenspaces of the compositions $\alpha_i\beta_i$ (and thus to equivalence classes for $\sim$) are ordered according to the usual ordering on the minimal elements of the equivalence classes, and the Jordan blocks for each equivalence class are ordered by size. Then in particular we have

$$Q^{\circ}_{[\sim,\mathcal{O}]} = K_{\mathbb{C}} Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$$

and the nonempty fibers of the complex moment map

$$Q^\circ_{[\sim,\mathcal{O}]} \to \mathfrak{k}_\mathbb{C} \oplus \mathfrak{t}_\mathbb{C}$$

for the action of $K \times T$ are contained in $K_\mathbb{C} \times T_\mathbb{C}$-orbits (see [**5**, §7 and in particular Lemma 7.5]).

REMARK 5.15. We can also identify $Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$ with an open subset of a hypertoric variety by replacing all the ξ entries in (5.13) with zero (see [**5**, Lemma 7.13]).

More precisely, if a quiver $\mathfrak{q}$ has α, β maps of the form given by (5.12) and (5.13), then we may obtain a new quiver which still satisfies the complex moment map equations by replacing all the ξ entries in β by zero. The resulting maps are denoted by α^T, β^T.

So if $\mathfrak{q}$ is any quiver representing a point in $Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$ whose Jordan blocks are of the form given by α_k and β_k as above, then we may form a new quiver $\mathfrak{q}^T$ from $\mathfrak{q}$ by replacing each such Jordan block with the quiver given by α_k^T and β_k^T. The new quiver now satisfies the complex moment map equations for the action of H, or equivalently for the action of the maximal torus T_H of H.

REMARK 5.16. The subgroup of $K_\mathbb{C} \times T_\mathbb{C}$ preserving the standard form must preserve the decomposition of $\mathfrak{q}$ into subquivers given by the generalized eigenspaces and hence must lie in $(K_\sim)_\mathbb{C} \times T_\mathbb{C}$.

Let P be the parabolic subgroup of $(K_\sim)_\mathbb{C}$ which is the Jacobson–Morozov parabolic of the element of the nilpotent orbit $\mathcal{O}$ for $(K_\sim)_\mathbb{C}$ given by the nilpotent component of X^0. In [**5**] we identified the group which preserves the standard form as $R_{[\sim,\mathcal{O}]} \times T_\mathbb{C}$ where $R_{[\sim,\mathcal{O}]}$ is the centralizer in P of this nilpotent element. It follows that

$$Q^\circ_{[\sim,\mathcal{O}]} \cong (K_\mathbb{C} \times T_\mathbb{C}) \times_{(R_{[\sim,\mathcal{O}]} \times T_\mathbb{C})} Q^{\circ,JCF}_{[\sim,\mathcal{O}]} \cong K_\mathbb{C} \times_{R_{[\sim,\mathcal{O}]}} Q^{\circ,JCF}_{[\sim,\mathcal{O}]}. \tag{5.17}$$

Moreover $[P,P] \cap R_{[\sim,\mathcal{O}]}$ acts trivially on $Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$. If we define $T_{[\sim,\mathcal{O}]}$ to be the intersection of $R_{[\sim,\mathcal{O}]}$ with the maximal torus T, then $(T_{[\sim,\mathcal{O}]})_\mathbb{C}/[P,P] \cap (T_{[\sim,\mathcal{O}]})_\mathbb{C}$ acts freely on $Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$.

The situation is summarized in the following theorem, which is Theorem 8.1 of [**5**].

Theorem 5.18. *For each equivalence relation $\sim$ on $\{1, \dots, n\}$ and nilpotent adjoint orbit $\mathcal{O}$ for $(K_\sim)_\mathbb{C}$, the stratum $Q_{[\sim,\mathcal{O}]}$ is the union over $s \in SU(2)$ of its open subsets $sQ^\circ_{[\sim,\mathcal{O}]}$, and*

$$Q^\circ_{[\sim,\mathcal{O}]} \cong K_\mathbb{C} \times_{R_{[\sim,\mathcal{O}]}} Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$$

where $R_{[\sim,\mathcal{O}]}$ is the centralizer in $(K_\sim)_\mathbb{C}$ of the standard representative ξ_0 in Jordan canonical form of the nilpotent orbit $\mathcal{O}$ in $(\mathfrak{k}_\sim)_\mathbb{C}$ and $Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$

can be identified with an open subset of a hypertoric variety. The image of the restriction

$$Q^{\circ}_{[\sim,\mathcal{O}]} \to \mathfrak{k}_{\mathbb{C}}$$

of the complex moment map for the action of K *on* Q *is* $K_{\mathbb{C}}((\mathfrak{t}_{\mathbb{C}})_{\sim} \oplus \mathcal{O}) \cong K_{\mathbb{C}} \times_{(K_{\sim})_{\mathbb{C}}} ((\mathfrak{t}_{\mathbb{C}})_{\sim} \oplus \mathcal{O})$ *and its fibers are single* $((T_{[\sim,\mathcal{O}]})_{\mathbb{C}} \times T_{\mathbb{C}})$*-orbits, where* $(T_{[\sim,\mathcal{O}]})_{\mathbb{C}} = T_{\mathbb{C}} \cap R_{[\sim,\mathcal{O}]}$ *and* $(T_{[\sim,\mathcal{O}]})_{\mathbb{C}}/[P,P] \cap (T_{[\sim,\mathcal{O}]})_{\mathbb{C}}$ *acts freely on* $Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$*. Here* P *is the Jacobson–Morozov parabolic of an element of the nilpotent orbit* $\mathcal{O}$ *for* $(K_{\sim})_{\mathbb{C}}$*, and* $[P,P] \cap (T_{[\sim,\mathcal{O}]})_{\mathbb{C}}$ *acts trivially on* $Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$*.*

6. Embeddings

In this section we will prove that the map σ defined in §4 is injective and the map $\tilde{\sigma}$ defined in §4 is generically injective.

First, consider the restriction of σ to the image Q_T in Q of the hypertoric variety $M_T /\!/\!/ T_H$ under the natural map $\iota\colon M_T /\!/\!/ T_H \to Q$ (see Definition 3.11).

Lemma 6.1. *The restriction of*

$$\sigma\colon Q \to \mathcal{R} = H^0(\mathbb{P}^1, (\mathcal{O}(2) \otimes (\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}})) \oplus \bigoplus_{j=1}^{n-1} \mathcal{O}(\ell_j) \otimes \wedge^j \mathbb{C}^n)$$

to $Q_T = \iota(M_T /\!/\!/ T_H)$ *is injective.*

This follows immediately from the following lemma.

Lemma 6.2. *The restriction to* $Q_T = \iota(M_T /\!/\!/ T_H)$ *of the projection*

$$\sigma_T\colon Q \to H^0(\mathbb{P}^1, (\mathcal{O}(2) \otimes \mathfrak{t}_{\mathbb{C}}) \oplus \bigoplus_{j=1}^{n-1} \mathcal{O}(\ell_j) \otimes \wedge^j \mathbb{C}^n)$$

of σ *is injective.*

Proof. By Remark 4.9 we can recover the value of the hyperkähler moment map for the action of T on $M_T /\!/\!/ T_H$ at any point from its image under $\sigma_T \circ \iota$. Since $M_T /\!/\!/ T_H$ is hypertoric, the fibers of this hyperkähler moment map are T-orbits, so it suffices to show that σ_T is injective on T-orbits in Q_T.

Recall that for any $t \in T$ the action on Q_T of $(t,1) \in K \times T$ is the same as the action of $(1,t) \in K \times T$. If $\mathfrak{q} \in Q_T$, then $\mathfrak{q} \in sQ^{\circ}_{[\sim,\mathcal{O}]}$ for some $s \in SU(2)$ and stratum $Q^{\circ}_{[\sim,\mathcal{O}]}$ with the nilpotent orbit $\mathcal{O}$ equal to the zero orbit $\{0\}$ in $\mathfrak{k}^*_{\mathbb{C}}$. Hence, in the notation of Theorem 5.18, we have

$$R_{[\sim,\mathcal{O}]} = P = (K_{\sim})_{\mathbb{C}} \quad \text{and} \quad T_{[\sim,\mathcal{O}]} = T,$$

and thus the stabilizer of $\mathfrak{q}$ in T is

$$T \cap [P, P] = T \cap [K_\sim, K_\sim] = T \cap [K_\lambda, K_\lambda]$$

where λ is the image of $\mathfrak{q}$ under the hyperkähler moment map for the action of T. Both σ_T and ι are T-equivariant, so it is enough to show that the stabilizer in T of $\sigma_T(\mathfrak{q})$ is contained in $T \cap [K_\lambda, K_\lambda]$. But it follows from Remark 4.10 that the stabilizer of $\sigma_T(\mathfrak{q})$ in K is contained in

$$K_\lambda \cap (K \cap [P_\lambda, P_\lambda]) = [K_\lambda, K_\lambda],$$

where P_λ is the standard parabolic in $K_\mathbb{C}$ whose Levi subgroup is $(K_\lambda)_\mathbb{C}$ and whose intersection with K is K_λ, so the result follows. In more detail, consider a quiver $\mathfrak{q}$ in M_T given as in Definition 3.11 by

$$\alpha_k = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \nu_1^k & 0 & \cdots & 0 \\ 0 & \nu_2^k & \cdots & 0 \\ & & \cdots & \\ 0 & \cdots & 0 & \nu_k^k \end{pmatrix}$$

and

$$\beta_k = \begin{pmatrix} 0 & \mu_1^k & 0 & \cdots & 0 \\ 0 & 0 & \mu_2^k & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 0 & \mu_k^k \end{pmatrix}$$

for some $\nu_i^k, \mu_i^k \in \mathbb{C}$. For every $(u, v) \in \mathbb{C}^2$ and every $j \in \{1, \dots, n-1\}$, its image under the composition $\sigma_T \circ \iota$ determines the element

$$\wedge^j (u\alpha_{n-1} + v\beta_{n-1}^*)(u\alpha_{n-2} + v\beta_{n-2}^*) \cdots (u\alpha_j + v\beta_j^*) \in \wedge^j \mathbb{C}^n,$$

where

$$u\alpha_k + v\beta_k^* = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ u\nu_1^k + v\bar{\mu}_1^k & 0 & \cdots & 0 \\ 0 & u\nu_2^k + v\bar{\mu}_2^k & \cdots & 0 \\ & & \cdots & \\ 0 & \cdots & 0 & u\nu_k^k + v\bar{\mu}_k^k \end{pmatrix},$$

and thus determines the product

$$\prod_{k=j}^{n-1} \prod_{i=1}^{j} (u\nu_i^k + v\bar{\mu}_i^k).$$

The action of an element t of $T \cong (S^1)^{n-1} \cong \prod_{j=1}^{n-1} U(j)/SU(j)$ represented by matrices $A_j \in U(j)$ for $j = 1, \dots, n-1$ multiplies this product by

$$\prod_{k=j}^{n-1} \det A_k.$$

The contracted quivers associated as in Remark 5.6 to a quiver of this form are all identically zero. Thus this product is non-zero (and hence $\prod_{k=j}^{n-1} \det A_k = 1$ if t stabilizes $\sigma_T(\mathfrak{q})$) precisely when $j+1$ is the smallest element of its equivalence class under $\sim$. This tells us that the stabilizer in T of $\sigma_T(\mathfrak{q})$ is contained in $[K_\lambda, K_\lambda]$ as required. q.e.d.

Remark 6.3. We note that on the subset $Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$ embedded in the hypertoric variety as in Remark 5.15, the fibers of the complex moment map for the complex structure associated to $[1:0]$ are $T_{\mathbb{C}}$-orbits. Moreover, a straightforward modification of the above proof shows that the composition $(\sigma_T)_{[1:0]}$ of σ_T with evaluation at $[1:0] \in \mathbb{P}^1$ is injective on $T_{\mathbb{C}}$-orbits in $Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$, and thus that $(\sigma_T)_{[1:0]}$ is injective on $Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$.

We now turn to showing injectivity for the map σ on the full implosion Q.

By Remark 5.8 it suffices to show that the restriction of σ to any stratum $Q_{[\sim,\mathcal{O}]}$ is injective. Indeed, we need only show that its restriction to $Q^\circ_{[\sim,\mathcal{O}]}$ is injective, since if two points lie in $Q_{[\sim,\mathcal{O}]}$, then there is some s in $SU(2)$ such that they both lie in $sQ^\circ_{[\sim,\mathcal{O}]}$, and σ is by construction $SU(2)$-equivariant. We will do this by showing that the restriction to $Q^\circ_{[\sim,\mathcal{O}]}$ of the composition $\sigma_{[1:0]}$ of σ with evaluation at $[1:0] \in \mathbb{P}^1$ is injective, and this will also show that $\tilde{\sigma}$ is generically injective.

Proposition 6.4. *For any equivalence relation $\sim$ on $\{1,\dots,n\}$ and nilpotent co-adjoint orbit $\mathcal{O}$ in $\mathfrak{k}^*_{\mathbb{C}}$, the restriction*

$$\sigma_{[1:0]} \colon Q^\circ_{[\sim,\mathcal{O}]} \to (\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}}) \otimes \mathcal{O}(2)_{[1:0]} \oplus \bigoplus_{j=1}^{n-1} \wedge^j(\mathbb{C}^n) \otimes \mathcal{O}(\ell_j)_{[1:0]}$$

to $Q^\circ_{[\sim,\mathcal{O}]}$ of the composition $\sigma_{[1:0]}$ of σ with evaluation at $[1:0] \in \mathbb{P}^1$ is injective.

Proof.

$$\sigma_{[1:0]} \colon Q^\circ_{[\sim,\mathcal{O}]} \to (\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}}) \otimes \mathcal{O}(2)_{[1:0]} \oplus \bigoplus_{j=1}^{n-1} \wedge^j(\mathbb{C}^n) \otimes \mathcal{O}(\ell_j)_{[1:0]}$$

is holomorphic and $(K \times T)$-equivariant, so it is $(K_{\mathbb{C}} \times T_{\mathbb{C}})$-equivariant. By Theorem 5.18

$$Q^\circ_{[\sim,\mathcal{O}]} \cong K_{\mathbb{C}} \times_{R_{[\sim,\mathcal{O}]}} Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$$

where $R_{[\sim,\mathcal{O}]}$ is the centralizer in $(K_\sim)_{\mathbb{C}}$ of the standard representative ξ_0 in Jordan canonical form of the nilpotent orbit $\mathcal{O}$ in $(\mathfrak{k}_\sim)_{\mathbb{C}}$. It follows immediately from the definition of $Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$ by looking at the projection of $\sigma_{[1:0]}$ to $\mathfrak{k}_{\mathbb{C}} \otimes \mathcal{O}(2)_{[1:0]}$ that if $g \in K_{\mathbb{C}}$ and $\mathfrak{q} \in Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$ and

$$g\sigma_{[1:0]}(\mathfrak{q}) \in \sigma_{[1:0]}(Q^{\circ,JCF}_{[\sim,\mathcal{O}]}), \tag{6.5}$$

then $g \in R_{[\sim,\mathcal{O}]}$.

Proposition 6.4 now follows from the following lemma.

Lemma 6.6. *The restriction of* $\sigma_{[1:0]}$ *to* $Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$ *is injective.*

Proof. As in Theorem 5.18, $Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$ can be identified with an open subset of a hypertoric variety by associating to a quiver $\mathfrak{q} \in Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$ with blocks of the form (5.12) and (5.13) a quiver $\mathfrak{q}^T$ where each ξ^{ijk}_ℓ in $\mathfrak{q}$ is replaced with zero (cf. Remark 5.15). This hypertoric variety is (up to the action of the Weyl group of $H \times K$) a subvariety of Q_T. Moreover, the projection $(\sigma_T)_{[1:0]}$ of $\sigma_{[1:0]}$ to

$$\mathfrak{t}_{\mathbb{C}} \otimes \mathcal{O}(2)_{[1:0]} \oplus \bigoplus_{j=1}^{n-1} \wedge^j(\mathbb{C}^n) \otimes \mathcal{O}(\ell_j)_{[1:0]}$$

satisfies $(\sigma_T)_{[1:0]}(\mathfrak{q}^T) = (\sigma_T)_{[1:0]}(\mathfrak{q})$, so the result follows immediately from Lemma 6.2 and Remark 6.3. q.e.d.

To complete the proof of Proposition 6.4, suppose that $\sigma_{[1:0]}(g_1\mathfrak{q}_1) = \sigma_{[1:0]}(g_2\mathfrak{q}_2)$ where g_1 and g_2 are elements of $K_{\mathbb{C}}$ and $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are elements of $Q^{\circ,JCF}_{\sim,\mathcal{O}}$. As $\sigma_{[1:0]}$ is equivariant, we have $r = g_1^{-1}g_2 \in R_{[\sim,\mathcal{O}]}$ as at (6.5), and hence $\sigma_{[1:0]}(g_1\mathfrak{q}_1) = \sigma_{[1:0]}(g_1 r\mathfrak{q}_2)$. Hence by equivariance $\sigma_{[1:0]}(\mathfrak{q}_1) = \sigma_{[1:0]}(r\mathfrak{q}_2)$ where both $r\mathfrak{q}_2$ and $\mathfrak{q}_1$ lie in $Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$, so Lemma 6.6 shows that $\mathfrak{q}_1 = r\mathfrak{q}_2$ and thus that $g_1\mathfrak{q}_1 = g_2\mathfrak{q}_2$, as desired. q.e.d.

This completes the proof of Theorem 4.11. Moreover, since the map

$$\tilde{\sigma}\colon \mathcal{Z}_Q \to \mathcal{O}(2) \otimes (\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}}) \oplus \bigoplus_{j=1}^{n-1} \mathcal{O}(\ell_j) \otimes \wedge^j \mathbb{C}^n$$

is compatible with the projections to $\mathbb{P}^1$ and is $SU(2)$-equivariant, it follows immediately from Proposition 6.4 that $\tilde{\sigma}$ is injective on the dense subset of $\mathcal{Z}_Q = \mathbb{P}^1 \times Q$ which is the union over all $(\sim, \mathcal{O})$ of the $SU(2)$-sweep of $\{[1:0]\} \times Q^\circ_{[\sim,\mathcal{O}]}$. In particular $\tilde{\sigma}$ is injective on the dense Zariski-open subset of $\mathcal{Z}_Q = \mathbb{P}^1 \times Q$ which is the $SU(2)$-sweep of $\{[1:0]\} \times Q^\circ_{[\sim,\mathcal{O}]}$ where $\sim$ and $\mathcal{O}$ are such that $i \sim j$ if and only if $i = j$ and $\mathcal{O} = \{0\}$. Therefore the proof of Theorem 4.13 is also complete.

Remark 6.7. It follows from [**5**, Remark 3.4 and §7] that the image of $\sigma\colon Q \to \mathcal{R}$ is the closure in $\mathcal{R}$ of the $K_{\mathbb{C}}$-sweep of the image $\sigma(Q_T)$ in $\mathcal{R}$ of the hypertoric variety $M_T /\!\!/\!\!/ T_H$ associated to the hyperplane arrangement in $\mathfrak{t}$ given by the root planes (see Definition 3.11, above). Here $\sigma(Q_T) = \sigma(\iota(M_T /\!\!/\!\!/ T_H))$ where the composition $\sigma \circ \iota\colon M_T /\!\!/\!\!/ T_H \to$

$\mathcal{R}$ takes a quiver of the form

$$\alpha_k = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \nu_1^k & 0 & \cdots & 0 \\ 0 & \nu_2^k & \cdots & 0 \\ & & \cdots & \\ 0 & \cdots & 0 & \nu_k^k \end{pmatrix}$$

and

$$\beta_k = \begin{pmatrix} 0 & \mu_1^k & 0 & \cdots & 0 \\ 0 & 0 & \mu_2^k & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 0 & \mu_k^k \end{pmatrix}$$

to the section $\rho_K + \rho_T + \sum_{j=1}^{n-1} \rho_j$ of $\mathcal{O}(2) \otimes (\mathfrak{k}_\mathbb{C} \oplus \mathfrak{t}_\mathbb{C}) \oplus \bigoplus_{j=1}^{n-1} \mathcal{O}(\ell_j) \otimes \wedge^j \mathbb{C}^n$ where

$$\begin{aligned} \rho_K(u,v) = \rho_T(u,v) &= \big((u\alpha_{n-1} + v\beta_{n-1}^*)(-v\alpha_{n-1}^* + u\beta_{n-1})\big)_0 \\ &= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \begin{matrix}(u\nu_1^{n-1} + v\bar{\mu}_1^{n-1}) \times \\ (-v\bar{\nu}_1^{n-1} + u\mu_1^{n-1})\end{matrix} & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & \begin{matrix}(u\nu_{n-1}^{n-1} + v\bar{\mu}_{n-1}^{n-1}) \times \\ (-v\bar{\nu}_{n-1}^{n-1} + u\mu_{n-1}^{n-1})\end{matrix} \end{pmatrix} \\ &\qquad - \sum_{i=1}^{n-1} \frac{(u\nu_i^{n-1} + v\bar{\mu}_i^{n-1})(-v\bar{\nu}_i^{n-1} + u\mu_i^{n-1})}{n} I_n \end{aligned}$$

and

$$\rho_j(u,v) = \prod_{k=j}^{n-1} \prod_{i=1}^{j} (u\nu_i^k + v\bar{\mu}_i^k) e_{j+1} \wedge \cdots \wedge e_n$$

where $e_1, \ldots, e_n$ form the standard basis for $\mathbb{C}^n$.

For any $p \in \mathbb{P}^1$ the projection to

$$\bigoplus_{j=1}^{n-1} \mathcal{O}(\ell_j)_p \otimes \wedge^j \mathbb{C}^n \cong \bigoplus_{j=1}^{n-1} \wedge^j \mathbb{C}^n$$

of the evaluation σ_p of σ at p takes Q_T to the toric variety $\overline{T_\mathbb{C} v}$ where $v \in \bigoplus_{j=1}^{n-1} \wedge^j \mathbb{C}^n$ is the sum $v = \sum_{j=1}^{n-1} v_j$ of highest-weight vectors $v_j \in \wedge^j \mathbb{C}^n$. Moreover, if $B = T_\mathbb{C} N$ is the standard Borel subgroup of $K_\mathbb{C} = SL(n, \mathbb{C})$ and $N = [B,B]$ is its unipotent radical which fixes the highest-weight vectors v_j, then this projection also takes the closure $\overline{BQ_T}$ of the B-sweep of Q_T to $\overline{Bv} = \overline{T_\mathbb{C} v}$. Thus this projection of σ_p takes

$Q = \overline{K_{\mathbb{C}}Q_T} = K\overline{BQ_T}$ to the universal symplectic implosion

$$K\overline{T_{\mathbb{C}}v} = \overline{K_{\mathbb{C}}v} \subseteq \bigoplus_{j=1}^{n-1} \wedge^j \mathbb{C}^n.$$

Furthermore, σ_p takes Q_T (respectively $\overline{BQ_T}$) birationally into the subvariety $\mathfrak{t}_{\mathbb{C}} \oplus \overline{T_{\mathbb{C}}v}$ (respectively $\mathfrak{n} \oplus \mathfrak{t}_{\mathbb{C}} \oplus \overline{T_{\mathbb{C}}v} \cong \mathfrak{n} \oplus \sigma_p(Q_T)$) of

$$\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{j=1}^{n-1} \wedge^j \mathbb{C}^n \cong \mathcal{O}(2)_p \otimes (\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}}) \oplus \bigoplus_{j=1}^{n-1} \mathcal{O}(\ell_j)_p \otimes \wedge^j \mathbb{C}^n,$$

where $\mathfrak{n}$ is the Lie algebra of N and $\mathfrak{t}_{\mathbb{C}}$ is embedded diagonally in $\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}}$, while $\mathfrak{n} \oplus \mathfrak{t}_{\mathbb{C}}$ is embedded via $(\xi, \eta) \mapsto (\xi + \eta, \eta)$.

Since $Q = \overline{K_{\mathbb{C}}Q_T} = K\overline{BQ_T}$ and $\overline{BQ_T}$ is (T)-invariant, we get a birational $K \times T$-equivariant surjection

$$K \times_T \overline{BQ_T} \to Q.$$

Similarly, the twistor space $\mathcal{Z}_Q$ of Q is the closure in $\mathbb{P}^1 \times \mathcal{R}$ of the $K_{\mathbb{C}}$-sweep of the twistor space $\mathcal{Z}_{Q_T}$ of the image Q_T in $\mathcal{R}$ of the hypertoric variety $M_T /\!/\!/ T_H$. We have

$$\mathcal{Z}_Q = \overline{K_{\mathbb{C}}\mathcal{Z}_{Q_T}} = K\overline{B\mathcal{Z}_{Q_T}},$$

and there is a birational surjection

$$K \times_T \overline{B\mathcal{Z}_{Q_T}} \to \mathcal{Z}_Q.$$

Moreover, $\tilde{\sigma}$ restricts to T-equivariant birational morphisms

$$\tilde{\sigma}|_{\mathcal{Z}_{Q_T}} : \mathcal{Z}_{Q_T} \to \mathcal{O}(2) \otimes \mathfrak{t}_{\mathbb{C}} \oplus \overline{T_{\mathbb{C}}v} \subseteq \mathcal{O}(2) \otimes (\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}}) \oplus \bigoplus_{j=1}^{n-1} \mathcal{O}(\ell_j) \otimes \wedge^j \mathbb{C}^n$$

and $\tilde{\sigma}|_{\overline{B\mathcal{Z}_{Q_T}}}$: $\overline{B\mathcal{Z}_{Q_T}} \to \mathcal{O}(2) \otimes (\mathfrak{n} \oplus \mathfrak{t}_{\mathbb{C}}) \oplus \overline{T_{\mathbb{C}}v}$. Thus the twistor space $\mathcal{Z}_Q$ is birationally equivalent to $K \times_T ((\mathcal{O}(2) \otimes \mathfrak{n}) \oplus \mathcal{Z}_{Q_T})$.

7. The twistor space of the universal hyperkähler implosion for $SU(n)$

In this section we will describe the full structure of the twistor space $\mathcal{Z}_Q$ of Q in terms of the embedding σ of Q in the space of holomorphic sections of the vector bundle $\mathcal{O}(2) \otimes (\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}}) \oplus \bigoplus_{j=1}^{n-1} \mathcal{O}(\ell_j) \otimes \wedge^j \mathbb{C}^n$ over $\mathbb{P}^1$ and consider the cases when $n = 2$ and $n = 3$ in detail. The embedding σ gives us an embedding $\sigma_{\mathcal{Z}}$ of the twistor space $\mathcal{Z}_Q = \mathbb{P}^1 \times Q$ of Q into $\mathbb{P}^1 \times \mathcal{R}$, where, as always,

$$\mathcal{R} = H^0(\mathbb{P}^1, (\mathcal{O}(2) \otimes (\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}})) \oplus \bigoplus_{j=1}^{n-1} \mathcal{O}(\ell_j) \otimes \wedge^j \mathbb{C}^n).$$

This map $\sigma_{\mathcal{Z}}$ is not holomorphic; however its composition $\tilde{\sigma}$ with the natural evaluation map from $\mathbb{P}^1 \times \mathcal{R}$ to

$$(\mathcal{O}(2) \otimes (\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}})) \oplus \bigoplus_{j=1}^{n-1} \mathcal{O}(\ell_j) \otimes \wedge^j \mathbb{C}^n$$

is holomorphic and $(K \times T \times SU(2))$-equivariant (see Remark 4.9), and $\tilde{\sigma}$ is generically injective by Theorem 4.13. Indeed, as we saw at the end of the last section, $\tilde{\sigma}$ is injective on the dense subset of $\mathcal{Z}_Q = \mathbb{P}^1 \times Q$ which is the union over all $(\sim, \mathcal{O})$ of the $SU(2)$-sweep of $\{[1:0]\} \times Q^\circ_{[\sim,\mathcal{O}]}$.

It follows that if $\mathfrak{q} \in Q$ lies in the stratum $Q_{[\sim,\mathcal{O}]}$ indexed by equivalence relation $\sim$ and nilpotent co-adjoint orbit $\mathcal{O}$, then we can find an open neighborhood $U_{\mathfrak{q}}$ of $\mathfrak{q}$ in $Q_{[\sim,\mathcal{O}]}$ and a closed subset $B_{\mathfrak{q}}$ of $\mathbb{P}_1$ of arbitrarily small area such that the restriction of $\tilde{\sigma}$ to the open subset $(\mathbb{P}^1 \setminus B_{\mathfrak{q}}) \times U_{\mathfrak{q}}$ of $\mathcal{Z}_Q$ is a holomorphic embedding. Since we can choose $B_{\mathfrak{q}}$ sufficiently small that there is some $s \in SU(2)$ for which $s(\mathbb{P}^1 \setminus B_{\mathfrak{q}})$ contains the points corresponding to the complex structures i, j, k on Q, it follows that the hypercomplex structure on Q and thus the complex structure on $\mathcal{Z}_Q$ are determined by the embedding σ.

Now consider the holomorphic section $\omega_{[\sim,\mathcal{O}]}$ of

$$\wedge^2 T^*_{F,Q_{[\sim,\mathcal{O}]}} \otimes \mathcal{O}(2)$$

where $T^*_{F,Q_{[\sim,\mathcal{O}]}}$ is the tangent bundle along the fibers of the restriction of $\pi\colon \mathcal{Z}_Q = \mathbb{P}^1 \times Q \to \mathbb{P}^1$ to $\mathbb{P}^1 \times Q_{[\sim,\mathcal{O}]}$. The reasoning above allows us to consider the restriction of $\omega_{[\sim,\mathcal{O}]}$ to $\{[1:0]\} \times Q^\circ_{[\sim,\mathcal{O}]}$. By Theorem 5.18 this can be identified with

$$K_{\mathbb{C}} \times_{(K_\sim)_{\mathbb{C}}} ((K_\sim)_{\mathbb{C}} \times_{R_{[\sim,\mathcal{O}]}} Q^{\circ,JCF}_{[\sim,\mathcal{O}]})$$

where $Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$ can, in turn, be identified with an open subset of a hypertoric variety $Q_{T,[\sim,\mathcal{O}]}$ and $K_{\mathbb{C}}/(K_\sim)_{\mathbb{C}}$ and $(K_\sim)_{\mathbb{C}}/R_{[\sim,\mathcal{O}]}$ can be identified with co-adjoint orbits in $\mathfrak{k}_{\mathbb{C}}$ and $(\mathfrak{k}_\sim)_{\mathbb{C}}$. The restriction of $\omega_{[\sim,\mathcal{O}]}$ is now obtained from the Kirillov–Kostant construction as in §2, combined with the holomorphic section of

$$\wedge^2 T^*_{F,Q_{T,[\sim,\mathcal{O}]}} \otimes \mathcal{O}(2)$$

associated to the twistor space $\mathcal{Z}_{Q_{T,[\sim,\mathcal{O}]}}$ of the hypertoric variety $Q_{T,[\sim,\mathcal{O}]}$.

Finally, as in Remark 2.11, the real structure on the twistor space $\mathcal{Z}_Q$ is determined by the embedding $\sigma_{\mathcal{Z}}$ and the real structure on $\mathbb{P}^1 \times \mathcal{R}$ determined by the real structure $\zeta \mapsto -1/\bar{\zeta}$ on $\mathbb{P}^1$, together with the real structures $\eta \mapsto -\bar{\eta}^T$ on $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{t}_{\mathbb{C}}$ and the real structures on $\wedge^j \mathbb{C}^n$ induced by the standard real structure on $\mathbb{C}^n$.

EXAMPLE 7.1. Consider the case when $n = 2$ and $K = SU(2)$. Then M is the space of quivers of the form

$$\mathbb{C} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathbb{C}^2 \tag{7.2}$$

and $Q = M /\!\!/\!\!/ SU(1) = M \cong \mathbb{C}^2 \oplus (\mathbb{C}^2)^* \cong \mathbb{H}^2$ (see [**4**, Example 8.5]). There are two equivalence relations on $\{1, 2\}$; let $\sim_1$ denote the equivalence relation with one equivalence class $\{1, 2\}$, and let $\sim_2$ denote the equivalence relation with two equivalence classes $\{1\}$ and $\{2\}$. Then $(\mathfrak{k}_{\sim_1})_{\mathbb{C}}$ has two nilpotent orbits, $\mathcal{O}_{1,0} = \{0\}$ and the orbit $\mathcal{O}_{1,1}$ of

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

while the only nilpotent orbit in $(\mathfrak{k}_{\sim_2})_{\mathbb{C}}$ is $\mathcal{O}_{2,0} = \{0\}$. The corresponding stratification of $Q = \mathbb{H}^2$ is

$$Q = Q_{[\sim_1, \mathcal{O}_{1,0}]} \sqcup Q_{[\sim_1, \mathcal{O}_{1,1}]} \sqcup Q_{[\sim_2, \mathcal{O}_{2,0}]}$$

where

$$Q_{[\sim_1, \mathcal{O}_{1,0}]} = Q^\circ_{[\sim_1, \mathcal{O}_{1,0}]} = \{(0, 0)\}.$$

Moreover $Q_{[\sim_1, \mathcal{O}_{1,1}]} = Q^\circ_{[\sim_1, \mathcal{O}_{1,1}]}$ consists of non-zero quivers of the form Eq. (7.2) satisfying the hyperkäher moment map equations for the action of $U(1)$, so by Example 2.7 they belong to the K-sweep of the set of quivers satisfying

$$\alpha = \begin{pmatrix} \sqrt{d} \\ 0 \end{pmatrix}, \quad \beta = (0 \ \sqrt{d})$$

for real and strictly positive d. The quotient of $Q_{\sim_1} = Q_{[\sim_1, \mathcal{O}_{1,0}]} \sqcup Q_{[\sim_1, \mathcal{O}_{1,1}]}$ by the action of $U(1)$ is the nilpotent cone in the Lie algebra $\mathfrak{k}_{\mathbb{C}}$ of $SL(2, \mathbb{C})$, with the quotient map given by $(\alpha, \beta) \mapsto \alpha\beta$.

Finally, $Q^\circ_{[\sim_2, \mathcal{O}_{2,0}]}$ consists of the quivers Eq. (7.2) such that the 2×2 matrix $\alpha\beta$ has distinct eigenvalues, while its sweep

$$Q_{[\sim_2, \mathcal{O}_{2,0}]} = SU(2) Q^\circ_{[\sim_2, \mathcal{O}_{2,0}]}$$

by the action of $SU(2)$ which rotates the complex structures on Q consists of the quivers Eq. (7.2) such that the 2×2 matrix

$$(u\alpha + v\beta^*)(-v\alpha^* + u\beta)$$

has distinct eigenvalues for some (and hence generic) choice of $(u, v) \in \mathbb{C}^2$.

Now consider the map

$$\sigma \colon Q = \mathbb{H}^2 \to \mathcal{R} = H^0(\mathbb{P}^1, \mathcal{O}(2) \otimes (\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}}) \oplus \mathcal{O}(1) \otimes \mathbb{C}^2).$$

The projection of σ onto

$$H^0(\mathbb{P}^1, \mathcal{O}(1) \otimes \mathbb{C}^2) \cong H^0(\mathbb{P}^1, \mathcal{O}(1)) \otimes \mathbb{C}^2 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$$

takes a quiver Eq. (7.2) to the section of $\mathcal{O}(1) \otimes \mathbb{C}^2$ over $\mathbb{P}^1$ given by

$$(u, v) \mapsto u\alpha + v\beta^*$$

and hence is bijective. Moreover, the projection onto $H^0(\mathbb{P}^1, \mathcal{O}(2) \otimes (\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}}))$ is the map induced by the hyperkähler moment maps for the $(K \times T)$-action on Q, and so σ embeds Q into $\mathcal{R}$ as the graph of this map.

Since $Q = \mathbb{H}^2$ is a flat hyperkähler manifold its twistor space is the vector bundle

$$\mathcal{Z}_Q = \mathcal{O} \otimes (\mathbb{C}^2 \oplus (\mathbb{C}^2)^*)$$

over $\mathbb{P}^1$. We can cover $\mathbb{P}^1$ with two open subsets $\{\zeta \in \mathbb{P}^1 : \zeta \neq \infty\}$ and $\{\zeta \in \mathbb{P}^1 : \zeta \neq 0\}$, and thus cover $\mathcal{Z}_Q$ with two coordinate patches where $\zeta \neq \infty$ and where $\zeta \neq 0$ with coordinates

$$(\alpha, \beta, \zeta) \text{ and } (\tilde{\alpha}, \tilde{\beta}, \tilde{\zeta})$$

for $\alpha, \tilde{\alpha} \in \mathbb{C}^2$ and $\beta, \tilde{\beta} \in (\mathbb{C}^2)^*$ related by

$$\tilde{\zeta} = 1/\zeta, \ \tilde{\alpha} = \alpha/\zeta, \ \tilde{\beta} = \beta/\zeta;$$

we have similar coordinates on $\mathcal{O}(2) \otimes (\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}}) \oplus \mathcal{O} \otimes \mathbb{C}^2$. With respect to these coordinates the map $\tilde{\sigma}\colon \mathcal{Z}_Q \to \mathcal{O}(2) \otimes (\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}}) \oplus \mathcal{O}(1) \otimes \mathbb{C}^2$ is given by

$$\tilde{\sigma}(\alpha, \beta, \zeta) = (\alpha\beta - \frac{\operatorname{tr}(\alpha\beta)}{2} I_2, \beta\alpha, \alpha, \zeta)$$

(see §4). Observe that where the coordinate α is non-zero (or equivalently defines an injective linear map $\alpha\colon \mathbb{C} \to \mathbb{C}^2$), then we can recover α, β, and ζ from $\tilde{\sigma}(\alpha, \beta, \zeta)$, but that $\tilde{\sigma}(0, \beta, \zeta) = (0, 0, 0, \zeta)$ for any β. Thus $\tilde{\sigma}$ is only generically injective on the twistor space $\mathcal{Z}_Q$.

Recall from §1 that the real structure on $\mathcal{Z}_Q$ is given in these coordinates by

$$(\alpha, \beta, \zeta) \mapsto (\bar{\beta}/\bar{\zeta}, -\bar{\alpha}/\bar{\zeta}, -1/\bar{\zeta});$$

this is induced via the embedding $\sigma_{\mathcal{Z}}$ from the real structure on $\mathbb{P}^1 \times \mathcal{R}$ determined by the standard real structures on $\mathbb{P}^1$, $\mathfrak{k}_{\mathbb{C}}$, $\mathfrak{t}_{\mathbb{C}}$, and $\mathbb{C}^2$.

Finally, observe that $Q_{[\sim_1, \mathcal{O}_{1,0}]} = Q^\circ_{[\sim_1, \mathcal{O}_{1,0}]} = \{(0,0)\}$ and that $Q_{[\sim_1, \mathcal{O}_{1,1}]} = Q^\circ_{[\sim_1, \mathcal{O}_{1,1}]}$ can be identified with the regular nilpotent orbit in $\mathfrak{k}_{\mathbb{C}}$, and the holomorphic symplectic form on each is obtained by the Kirillov–Kostant construction. Moreover,

$$Q^\circ_{[\sim_2, \mathcal{O}_{2,0}]} \cong K_{\mathbb{C}} \times_{T_{\mathbb{C}}} Q^{\circ, JCF}_{[\sim_2, \mathcal{O}_{2,0}]}$$

where $Q^{\circ, JCF}_{[\sim_2, \mathcal{O}_{2,0}]}$ can be identified with an open subset of $\mathbb{H}$; the holomorphic symplectic form on this is obtained from the Kirillov–Kostant construction on the adjoint orbit $K_{\mathbb{C}}/T_{\mathbb{C}}$ combined with the flat structure on $\mathbb{H}$. $\diamondsuit$

EXAMPLE 7.3. Finally, let us consider briefly the case when $n = 3$ and $K = SU(3)$. We have five equivalence relations $\sim_{123}, \sim_{12,3}, \sim_{13,2}, \sim_{23,1}$, and $\sim_{1,2,3}$ on $\{1, 2, 3\}$ given by the partitions

$$\{\{1,2,3\}\},\ \{\{1,2\},\{3\}\},\ \{\{1,3\},\{2\}\},\ \{\{2,3\},\{1\}\},\ \{\{1\},\{2\},\{3\}\}.$$

The Lie algebra of $(K_{\sim_{123}})_{\mathbb{C}} = K_{\mathbb{C}}$ has three nilpotent orbits $\mathcal{O}_{123,j}$ for $j = 1, 2, 3$, and the corresponding strata $Q_{[\sim_{123},\mathcal{O}_{123,j}]}$ can be described using Example 2.8. At the other extreme the Lie algebra of $(K_{\sim_{1,2,3}})_{\mathbb{C}} = T_{\mathbb{C}}$ has only the zero nilpotent orbit, and the structure of the stratum $Q_{[\sim_{1,2,3},\{0\}]}$ is similar to that of $Q_{[\sim_2,\mathcal{O}_{2,0}]}$ in Example 7.1. In between, the Lie algebras of

$$(K_{\sim_{12,3}})_{\mathbb{C}} \cong (K_{\sim_{13,2}})_{\mathbb{C}} \cong (K_{\sim_{23,1}})_{\mathbb{C}} \cong GL(2, \mathbb{C})$$

have two nilpotent orbits each, the zero orbit and the regular nilpotent orbit. These give us the six remaining strata $Q_{[\sim,\mathcal{O}]} = SU(2)Q^{\circ}_{[\sim,\mathcal{O}]}$, for each of which the open subset $Q^{\circ}_{[\sim,\mathcal{O}]}$ of $Q_{[\sim,\mathcal{O}]}$ has the form

$$Q^{\circ}_{[\sim,\mathcal{O}]} \cong K_{\mathbb{C}} \times_{GL(2,\mathbb{C})} \left(GL(2, \mathbb{C}) \times_{R_{[\sim,\mathcal{O}]}} Q^{\circ,JCF}_{[\sim,\mathcal{O}]} \right)$$

where $R_{[\sim,\mathcal{O}]}$ is the stabilizer of $\mathcal{O}$ in $GL(2, \mathbb{C})$ and $Q^{\circ,JCF}_{[\sim,\mathcal{O}]}$ can be identified with an open subset of $\mathbb{H}$. ♢

8. More general compact Lie groups

Our future aim and the main motivation for this paper is to be able to construct the hyperkähler implosion of a hyperkähler manifold M with a Hamiltonian action of any compact Lie group K. For this it suffices to construct a universal hyperkähler implosion $(T^*K_{\mathbb{C}})_{\text{hkimpl}}$ of the hyperkähler manifold $T^*K_{\mathbb{C}}$ (see [**13**]) with suitable properties. In particular, $(T^*K_{\mathbb{C}})_{\text{hkimpl}}$ should be a stratified hyperkähler space with a Hamiltonian action of $K \times T$ where T is a maximal torus of K; then we can define the hyperkähler implosion M_{hkimpl} as the hyperkähler quotient of $M \times (T^*K_{\mathbb{C}})_{\text{hkimpl}}$ by the diagonal action of K.

As Guillemin, Jeffrey and Sjamaar observed in [**7**] for symplectic implosion, it suffices to consider the case when K is semi-simple, connected and simply connected. In this case, as was noted in §3, above, we can embed the universal symplectic implosion $(T^*K)_{\text{impl}}$ in the complex affine space

$$E = \bigoplus_{\varpi \in \Pi} V_{\varpi}$$

where Π is a minimal generating set for the monoid of dominant weights; here if $\varpi \in \Pi$ then V_{ϖ} is the K-module with highest-weight ϖ and T acts on V_{ϖ} as multiplication by this highest weight. As we recalled in §3, $(T^*K)_{\text{impl}}$ is embedded in E as the closure of the $K_{\mathbb{C}}$-orbit $K_{\mathbb{C}}v$ where v is the sum $\sum_{\varpi \in \Pi} v_{\varpi}$ of highest-weight vectors $v_{\varpi} \in V_{\varpi}$, or equivalently

$(T^*K)_{\mathrm{impl}} = K(\overline{T_{\mathbb{C}}v})$ where $\overline{T_{\mathbb{C}}v}$ is the toric variety associated to the positive Weyl chamber $\mathfrak{t}_+$.

This representation $E = \bigoplus_{\varpi\in\Pi} V_\varpi$ of K gives us an identification of K with a subgroup of $\prod_{\varpi\in\Pi} SU(V_\varpi)$. Theorem 4.11 gives us an embedding σ of the universal hyperkähler implosion for $\prod_{\varpi\in\Pi} K_\varpi$, where $K_\varpi = SU(V_\varpi)$ has maximal torus T_ϖ, in

$$\prod_{\varpi\in\Pi} \mathcal{R}_\varpi = H^0(\mathbb{P}^1, \bigoplus_{\varpi\in\Pi} \mathcal{O}(2)\otimes(\mathfrak{k}_\varpi\oplus\mathfrak{t}_\varpi)_{\mathbb{C}} \oplus \bigoplus_{j=1}^{\dim V_\varpi - 1} \mathcal{O}(\ell_j)\otimes \wedge^j V_\varpi).$$

We may assume that the inclusion of K in $\prod_{\varpi\in\Pi} K_\varpi$ restricts to an inclusion of its maximal torus T in the maximal torus $\prod_{\varpi\in\Pi} T_\varpi$ of $\prod_{\varpi\in\Pi} K_\varpi$ as $K\cap\prod_{\varpi\in\Pi} T_\varpi$, and that the intersection $\mathfrak{k}\cap\prod_{\varpi\in\Pi}(\mathfrak{t}_\varpi)_+$ in $\prod_{\varpi\in\Pi}\mathfrak{k}_\varpi$ is a positive Weyl chamber $\mathfrak{t}_+$ for K. Then the hypertoric variety for T associated to the hyperplane arrangement given by the root planes in $\mathfrak{t}$ embeds in the hypertoric variety for $\prod_{\varpi\in\Pi} T_\varpi$ associated to the hyperplane arrangement given by the root planes in $\prod_{\varpi\in\Pi}\mathfrak{t}_\varpi$ and thus maps into $\prod_{\varpi\in\Pi}\mathcal{R}_\varpi$.

By analogy with the situation described in [**7**] for symplectic implosion and using Remark 6.7, we expect the twistor space $\mathcal{Z}_{(T^*K_{\mathbb{C}})_{\mathrm{hkimpl}}}$ for the universal hyperkähler implosion $(T^*K_{\mathbb{C}})_{\mathrm{hkimpl}}$ to embed in the intersection in $\mathbb{P}^1\times\prod_{\varpi\in\Pi}\mathcal{R}_\varpi$ of the corresponding twistor space for $(T^*(\prod_{\varpi\in\Pi} K_\varpi)_{\mathbb{C}})_{\mathrm{hkimpl}}$ and

$$\mathbb{P}^1\times H^0(\mathbb{P}^1, \mathcal{O}(2)\otimes(\mathfrak{k}_{\mathbb{C}}\oplus\mathfrak{t}_{\mathbb{C}}) \oplus \bigoplus_{\varpi\in\Pi}\bigoplus_{j=1}^{\dim V_\varpi - 1} \mathcal{O}(\ell_j)\otimes\wedge^j V_\varpi).$$

Moreover, we expect the image of this embedding to be the closure of the $K_{\mathbb{C}}$-sweep of the image of the twistor space of the hypertoric variety for T associated to the hyperplane arrangement given by the root planes in $\mathfrak{t}$, and that the full structure of the twistor space $\mathcal{Z}_{(T^*K_{\mathbb{C}})_{\mathrm{hkimpl}}}$ for the universal hyperkähler implosion is obtained from this embedding as in §7, above.

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