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STRONGER VERSIONS OF THE ORLICZ-PETTY PROJECTION INEQUALITY

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Abstract

We verify a conjecture of Lutwak, Yang, and Zhang about the equality case in the Orlicz-Petty projection inequality, and provide an essentially optimal stability version.

The Petty projection inequality (Theorem 1), its L_p extension, and its analytic counterparts, the Zhang-Sobolev inequality [**43**] and its L_p extension by A. Cianchi, E. Lutwak, D. Yang, and G. Zhang [**8**, **32**], are fundamental affine isoperimetric and affine analytic inequalities (see in addition, e.g., D. Alonso-Gutierrez, J. Bastero, and J. Bernués [**1**]; R.J. Gardner and G. Zhang [**14**]; C. Haberl and F.E. Schuster [**21**, **22**]; C. Haberl, F.E. Schuster and J. Xio [**23**]; E. Lutwak, D. Yang, and G. Zhang [**31**, **33**, **34**]; M. Ludwig [**27**, **28**]; M. Schmuckenschläger [**40**], F.E. Schuster, T. Wannnerer [**41**] and J. Xiao [**42**]). The notion of the projection body and its L_p extension has found its natural context in the work of E. Lutwak, D. Yang, and G. Zhang [**34**], where the authors introduced the concept of the Orlicz projection body. The fundamental result of [**34**] is the Orlicz-Petty projection inequality. The goal of this paper is to strengthen this latter inequality, extending the method of E. Lutwak, D. Yang, and G. Zhang [**34**] based on Steiner symmetrization.

When the equality case of a geometric inequality is characterized, it is a natural question how close a convex body K is to the extremals if almost inequality holds for K in the inequality. Precise answers to these questions are called stability versions of the original inequalities. Stability results for geometric estimates have important applications; see for example B. Fleury, O. Guédon and G. Paouris [**12**] for the central limit theorem on convex bodies, and D. Hug and R. Schneider [**24**] for the shape of typical cells in a Poisson hyperplane process.

Stability versions of sharp geometric inequalities have been around since the days of Minkowski; see the survey paper by H. Groemer [**17**] about developments until the early 1990s. Recently essentially optimal results were obtained by N. Fusco, F. Maggi and A. Pratelli [**13**] concerning the isoperimetric inequality, and by A. Figalli, F. Maggi and A. Pratelli ([**10**] and [**11**]) for the Brunn-Minkowski inequality; see F.

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Maggi [**35**] for a survey of their methods. In these papers, stability is understood in terms of the volume difference of normalized convex bodies. In this paper we follow J. Bourgain and J. Lindenstrauss [**5**], who used the so-called Banach-Mazur distance for their result (5) about projection bodies quoted below.

We write o to denote the origin in $\mathbb{R}^n$, $u \cdot v$ to denote the scalar product of the vectors u and v, $\mathcal{H}$ to denote the $(n-1)$-dimensional Hausdorff measure, and $[X_1, \ldots, X_k]$ to denote the convex hull of the sets $X_1, \ldots, X_k$ in $\mathbb{R}^n$. For a non-zero u in $\mathbb{R}^n$, let $u^\perp$ be the orthogonal linear $(n-1)$-subspace, and let π_u denote the orthogonal projection onto $u^\perp$. In addition, let B^n be the Euclidean unit ball, and let κ_n be its volume. For $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm. We write $A\Delta B$ to denote the symmetric difference of the sets A and B.

Throughout this article, a convex body in $\mathbb{R}^n$ is a compact convex set with non-empty interior. In addition, we write $\mathcal{K}_o^n$ to denote the set of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors. For a convex body K in $\mathbb{R}^n$, let $h_K(u) = \max_{x\in K} x \cdot u$ denote the support function of K at $u \in \mathbb{R}^n$, and let K^* be the polar of K, defined by

$$K^* = \{u \in \mathbb{R}^n : h_K(u) \leq 1\}.$$

Let S_K be the surface area measure of K on S^{n-1}. That is, if σ is an open subset of S^{n-1}, then $S_K(\sigma)$ is the $(n-1)$-dimensional Hausdorff-measure of all $x \in \partial K$, where there exists an exterior unit normal lying in σ. Minkowski's projection body ΠK is the o-symmetric convex body whose support function is

$$h_{\Pi K}(x) = \|x\| \cdot \mathcal{H}(\pi_x K) = \frac{1}{2}\int_{S^{n-1}} |x \cdot w|\, dS_K(w) \quad \text{for } x \in \mathbb{R}^n\backslash o.$$

We write $\Pi^* K$ to denote the polar of ΠK, and note that $V(\Pi^* K)V(K)^{n-1}$ is invariant under affine transformations of $\mathbb{R}^n$ (see E. Lutwak [**29**]). Petty's projection inequality can now be stated as follows.

Theorem 1 (Petty)**.** *If K is a convex body in $\mathbb{R}^n$, then*

$$V(\Pi^* K)V(K)^{n-1} \leq (\kappa_n/\kappa_{n-1})^n,$$

with equality if and only if K is an ellipsoid.

To define the Orlicz projection body introduced by E. Lutwak, D. Yang, and G. Zhang [**34**], we write $\mathcal{C}$ to denote the set of convex functions $\varphi : \mathbb{R} \to [0, \infty)$ such that $\varphi(0) = 0$, and $\varphi(-t) + \varphi(t) > 0$ for $t \neq 0$. In particular,

(1)
$$\text{every } \varphi \in \mathcal{C} \text{ is } \begin{cases} \text{either strictly monotone decreasing on } (-\infty, 0], \\ \text{or strictly monotone increasing on } [0, \infty). \end{cases}$$

Let $\varphi \in \mathcal{C}$, and let $K \in \mathcal{K}_o^n$. The corresponding Orlicz projection body $\Pi_\varphi K$ is defined in [**34**] *via* its support function such that if $x \in \mathbb{R}^n$, then
(2)

$$h_{\Pi_\varphi K}(x) = \min\left\{\lambda > 0 : \int_{S^{n-1}} \varphi\left(\frac{x \cdot w}{\lambda h_K(w)}\right) h_K(w)\, dS_K(w) \leq nV(K)\right\}.$$

Since the surface area measure of every open hemisphere is positive, (1) yields that the minimum in (2) is attained at a unique $\lambda > 0$.

An important special case is when $\varphi(t) = |t|^p$ for some $p \geq 1$. Then $\Pi_\varphi K$ is the L_p projection body $\Pi_p K$ introduced by E. Lutwak, D. Yang, and G. Zhang [**31**] (using a different normalization):

$$h_{\Pi_p K}(x)^p = \frac{1}{nV(K)} \int_{S^{n-1}} |x \cdot w|^p h_K(w)^{1-p}\, dS_K(w). \tag{3}$$

In particular, if $p = 1$, then

$$\Pi_1(K) = \frac{2}{nV(K)} \cdot \Pi K.$$

In addition, if p tends to infinity, then we may define the L_∞ polar projection body Π_∞^* to be $K \cap (-K)$.

Unlike ΠK, the Orlicz projection body $\Pi_\varphi K$ is not translation invariant for a general $\varphi \in \mathcal{C}$, and may not be o-symmetric. However, E. Lutwak, D. Yang, and G. Zhang [**34**] show that

(4) $\Pi_\varphi^* AK = A\Pi_\varphi^* K$ holds for any $A \in \mathrm{GL}(n)$, $K \in \mathcal{K}_o^n$, and $\varphi \in \mathcal{C}$.

The following Orlicz-Petty projection inequality is the main result of [**34**].

Theorem 2 (Lutwak, Yang, Zhang)**.** *Let $\varphi \in \mathcal{C}$. If $K \in \mathcal{K}_o^n$, then the volume ratio*

$$\frac{V(\Pi_\varphi^* K)}{V(K)}$$

is maximized when K is an o-symmetric ellipsoid. If φ is strictly convex, then the o-symmetric ellipsoids are the only maximizers.

If $\varphi(t) = |t|$, which is the case of the normalized classical projection body, then every ellipsoid is a maximizer in the Orlicz-Petty projection inequality (see Theorem 1). Thus, to summarize what to expect for an arbitrary $\varphi \in \mathcal{C}$, E. Lutwak, D. Yang, and G. Zhang [**34**] conjecture that every maximizer is an ellipsoid. Here we confirm this conjecture.

Theorem 3. *Let $\varphi \in \mathcal{C}$. If $K \in \mathcal{K}_o^n$ maximizes the volume ratio $V(\Pi_\varphi^* K)/V(K)$, then K is an ellipsoid.*

A natural tool for stability results of affine invariant inequalities is the Banach-Mazur distance $\delta_{\mathrm{BM}}(K, M)$ of the convex bodies K and M

defined by

$$\delta_{\rm BM}(K,M)=\min\{\lambda\geq 0:\ K-x\subset\Phi(M-y)\subset e^{\lambda}(K-x)\\ \text{for}\quad \Phi\in{\rm GL}(n),\ x,y\in\mathbb{R}^n\}.$$

In particular, if K and M are o-symmetric, then $x=y=o$ can be assumed. In addition, for a line l passing through the origin o, we write $\mathcal{K}_l$ to denote the set of o-symmetric convex bodies with axial rotational symmetry around the line l. If $K\in\mathcal{K}_l$, then

$$\delta_{\rm BM}(K,B^n)=\min\{\lambda\geq 0:\ E\subset K\subset e^{\lambda}E,\ \text{where } E\in\mathcal{K}_l \text{ is an ellipsoid}\}.$$

It follows, for example, from a theorem of F. John [**25**] that $\delta_{\rm BM}(K,B^n)\leq \ln n$ for any convex body K in $\mathbb{R}^n$.

We strengthen Theorem 3 as follows, where we set $\tilde{\varphi}(t)=\varphi(-t)+\varphi(t)$ for $\varphi\in\mathcal{C}$.

Theorem 4. *If $\varphi\in\mathcal{C}$ and $K\in\mathcal{K}_o^n$ with $\delta=\delta_{BM}(K,B^n)$, then*

$$\frac{V(\Pi^*_\varphi K)}{V(K)}\leq(1-\gamma\cdot\delta^{cn}\cdot\tilde{\varphi}(\delta^c))\cdot\frac{V(\Pi^*_\varphi B^n)}{V(B^n)},$$

where $c=840$ and $\gamma>0$ depends on n and φ.

Next we discuss what Theorem 4 yields for Petty's projection inequality.

Corollary 5. *If K is a convex body in $\mathbb{R}^n$ with $\delta=\delta_{BM}(K,B^n)$, then*

$$V(\Pi^*K)V(K)^{n-1}\leq(1-\gamma\cdot\delta^{cn})\,(\kappa_n/\kappa_{n-1})^n,$$

where $c=1680$ and $\gamma>0$ depends only on n.

The example below shows that the exponent cn for an absolute constant $c>0$ is of optimal order. G. Ambrus and the author [**2**] recently proved Corollary 5 with an exponent of the form cn^3 instead of the optimal cn.

Example Let $K=[B^n,\pm(1+\varepsilon)v)]$ for some $v\in S^{n-1}$. In this case, the Banach-Mazur distance of K from any ellipsoid is at least $\varepsilon/2$, and

$$V(\Pi^*K)V(K)^{n-1}\geq(1-\gamma_0\varepsilon^{\frac{n+1}{2}})(\kappa_n/\kappa_{n-1})^n,$$

where $\gamma_0>0$ depends only on n.

As a related result, J. Bourgain and J. Lindenstrauss [**5**] proved that if K and M are o-symmetric convex bodies in $\mathbb{R}^n$, then

$$\delta_{BM}(\Pi K,\Pi M)\geq\gamma\cdot\delta_{BM}(K,M)^{n(n+5)/2} \tag{5}$$

where $\gamma>0$ depends only on n, and they conjectured that the optimal order of the exponent is cn for an absolute constant $c>0$. The exponent in (5) has been slightly improved by S. Campi [**7**] if $n=3$, and by M. Kiderlen [**26**] for any n, but the conjecture is still wide open. Corollary 5 is in accordance with this conjecture of J. Bourgain and J. Lindenstrauss in the case when M is an ellipsoid. Actually, if K and M are not o-symmetric, then their projection bodies may coincide even if $\delta_{BM}(K,M) \neq 0$ (see R. Schneider [**38**]).

If φ is strictly convex, then E. Lutwak, D. Yang, and G. Zhang [**34**] proved that the o-symmetric ellipsoids are the only maximizers in the Orlicz-Petty projection inequality (see Theorem 2). We prove a stability version of this statement for even φ. For $K \in \mathcal{K}_o^n$, let

$$\delta^*_{\mathrm{EL}}(K) = \min\{\lambda \geq 0 : \, E \subset K \subset e^{\lambda} E \text{ for some } o\text{-symmetric ellipsoid } E\}.$$

Since $\delta^*_{\mathrm{EL}}(K)$ becomes arbitrarily large if K is translated in a way such that the origin gets close to ∂K, it is more natural to consider

$$\delta_{\mathrm{EL}}(K) = \min\{1, \delta^*_{\mathrm{EL}}(K)\}.$$

Theorem 6. *Let $\varphi \in \mathcal{C}$ be even such that $\varphi''(t)$ is continuous and positive for $t > 0$. If $K \in \mathcal{K}_o^n$ with $\delta = \delta_{\mathrm{EL}}(K)$, then*

$$\frac{V(\Pi^*_\varphi K)}{V(K)} \leq (1 - \gamma \cdot \delta^{cn} \cdot \varphi(\delta^c)) \cdot \frac{V(\Pi^*_\varphi B^n)}{V(B^n)},$$

where $c = 2520$ and $\gamma > 0$ depends only on n and φ.

Under the conditions of Theorem 6, let $K \in \mathcal{K}_o^n$ be such that $V(\Pi^*_\varphi K)/V(K)$ is very close to $V(\Pi^*_\varphi B^n)/V(B^n)$. Then Theorem 4 yields that there exists a translate K' of K such that $\delta_{\mathrm{EL}}(K')$ is small, while Theorem 6 implies that already $\delta_{\mathrm{EL}}(K)$ is small.

For the L_p projection body for $p > 1$, and for $c = 2520$, we have

$$\frac{V(\Pi^*_p K)}{V(K)} \leq \left(1 - \gamma \cdot \delta_{\mathrm{EL}}(K)^{c(n+p)}\right) \cdot \frac{V(\Pi^*_p B^n)}{V(B^n)}.$$

Here the order of the error term gets smaller and smaller as p grows. It is not surprising, because $\Pi^*_\infty(K) = K \cap (-K)$ for $K \in \mathcal{K}_o^n$, and hence $V(\Pi^*_\infty K)/V(K)$ is maximized by any o-symmetric convex body K.

Our arguments to prove Theorems 3, 4, and 6 are based on Steiner symmetrization, and are variations of the method developed in E. Lutwak, D. Yang, and G. Zhang [**34**]. The novel ideas to prove Theorems 3 and 4 are to compare shadow boundaries in two suitable independent directions, and to reduce the problem to convex bodies with axial rotational symmetry around $\mathbb{R}u$ for a $u \in S^{n-1}$. In the latter case, the shadow boundaries parallel to u and orthogonal to u are well understood, which makes it possible to perform explicit calculations.

For Theorem 4, the proof of the reduction to convex bodies with axial rotational symmetry is rather technical, so the argument for the corresponding statement, Theorem 14, is deferred to Section 5.

We note that W. Blaschke [**3**] characterized ellipsoids as the only convex bodies such that every shadow boundary is contained in some hyperplane. A stability version of this statement was proved by P.M. Gruber [**19**].

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1. Some facts about convex bodies

Unless we provide specific references, the results reviewed in this section are discussed in the monographs by T. Bonnesen, and W. Fenchel [**4**], P.M. Gruber [**20**], and R. Schneider [**39**]. We note that the L_∞-metric on the restriction of the support functions to S^{n-1} endows the space of convex bodies with the so-called Hausdorff metric. It is well-known that volume is continuous with respect to this metric, and Lemma 2.3 in E. Lutwak, D. Yang, and G. Zhang [**34**] says that the polar Orlicz projection body is also continuous for fixed $\varphi \in \mathcal{C}$.

We say that a convex body M in $\mathbb{R}^n$, $n \geq 3$, is smooth if the tangent hyperplane is unique at every boundary point, and we say that M is strictly convex if every tangent hyperplane intersects M only in one point.

Let K be a convex body in $\mathbb{R}^n$. For $v \in S^{n-1}$, let S_vK denote the Steiner symmetral of K with respect to $v^\perp$. In particular, if f, g are the concave real functions on $\pi_v K$ such that

$$K = \{y + tv : y \in \pi_v K, -g(y) \leq t \leq f(y)\},$$

then

$$S_vK = \{y + tv : y \in \pi_v K,\ |t| \leq \tfrac{f(y)+g(y)}{2}\}. \tag{6}$$

Fubini's theorem yields that $V(S_vK) = V(K)$. It is known that for any convex body K, there is a sequence of Steiner symmetrizations whose limit is a ball (of volume $V(K)$).

Next there exists a sequence of Steiner symmetrizations with respect to $(n-1)$-subspaces containing the line $\mathbb{R}v$ such that their limit is a convex body R_vK whose axis of rotational symmetry is $\mathbb{R}v$. This R_vK is the Schwarz rounding of K with respect to v. In particular, a hyperplane H intersects $\operatorname{int} K$ if and only if it intersects $\operatorname{int} R_vK$, and $\mathcal{H}(H \cap K) = \mathcal{H}(H \cap R_vK)$ in this case.

For our arguments, it is crucial to have a basic understanding of the boundaries of convex bodies. For $x \in \partial K$, let w_x be a unit exterior normal to ∂K at x. The following two well-known properties are consequences of the fact that Lipschitz functions are almost everywhere differentiable.

(i) w_x is uniquely determined at $\mathcal{H}$ almost all $x \in \partial K$.

(ii) The supporting hyperplane with exterior normal vector u intersects ∂K in a unique point for almost all $u \in S^{n-1}$.

The shadow boundary $\Xi_{u,K}$ of K with respect to a $u \in \mathbb{R}^n \backslash o$ is the family of all $x \in \partial K$ such that the line $x + \mathbb{R}u$ is tangent to K. In addition, we call the shadow boundary $\Xi_{u,K}$ thin if it contains no segment parallel to u. According to G. Ewald, D.G. Larman, and C.A. Rogers [**9**], we have

Theorem 7 (Ewald-Larman-Rogers)**.** *If K is a convex body in $\mathbb{R}^n$, then the shadow boundary $\Xi_{u,K}$ is thin for $\mathcal{H}$-almost all $u \in S^{n-1}$.*

If a connected Borel $U \subset \partial K$ is disjoint from the shadow boundary with respect to a $v \in S^{n-1}$, then for any measurable $\psi : \pi_v(U) \to \mathbb{R}$, we have

$$\int_{\pi_v(U)} \psi(y)\, dy = \int_U \psi(\pi_v x)|v \cdot w_x|\, dx. \tag{7}$$

If $K \in \mathcal{K}_o^n$, then let ϱ_K be the radial function of K on S^{n-1}, defined such that $\varrho_K(v)\, v \in \partial M$ for $v \in S^{n-1}$. It follows that

$$V(K) = \int_{S^{n-1}} \frac{\varrho_K(w)^n}{n}\, d\mathcal{H}(w). \tag{8}$$

In addition, for the polar K^* of K, and $v \in S^{n-1}$, we have

$$\varrho_{K^*}(v) = h_K(v)^{-1}. \tag{9}$$

We say that a convex body M is in isotropic position if $V(M) = 1$, the centroid of M is the origin, and there exists $L_M > 0$ such that

$$\int_M (w \cdot x)^2\, dx = L_M \quad \text{for any } w \in S^{n-1}$$

(see A. Giannopoulos [**15**], A. Giannopoulos, and V.D. Milman [**16**]; and V.D. Milman, and A. Pajor [**36**] for main properties). Any convex body K has an affine image M that is in isotropic position, and we set $L_K = L_M$. We also note that if E is an o-symmetric ellipsoid in $\mathbb{R}^n$, then for any $w \in S^{n-1}$, we have

$$\int_E (w \cdot x)^2\, dx = h_E(w)^2 V(E)^{\frac{n+2}{n}} L_{B^n}. \tag{10}$$

Let $\varphi \in \mathcal{C}$, and let $K \in \mathcal{K}_o^n$. We collect some additional properties of the Orlicz projection body. The cone volume measure V_K associated to K on S^{n-1} defined by $d\,V_K(w) = \frac{h_K(w)}{nV(K)}\, dS_K(w)$ is a probability measure whose study was initiated by M. Gromov, and V. Milman [**18**] (see, say, A. Naor [**37**] for recent applications). The definition (2) of $\Pi_\varphi K$ yields (see Lemma 2.1 in E. Lutwak, D. Yang, and G. Zhang [**34**]) that

$$x \in \Pi_\varphi^* K \text{ if and only if } \int_{S^{n-1}} \varphi\left(\frac{x \cdot w}{h_K(w)}\right) dV_K(w) \le 1. \tag{11}$$

2. Characterizing the equality case in the Orlicz-Petty projection inequality

Our method is an extension of the argument by E. Lutwak, D. Yang, and G. Zhang [**34**] to prove the Orlicz-Petty projection inequality, Theorem 2, using Steiner symmetrization. The core of the argument of [**34**] is Corollary 3.1, and here we also include a consequence of Corollary 3.1 from [**34**] for Schwarz rounding.

Lemma 8 (Lutwak, Yang, Zhang)**.** *If $\varphi \in \mathcal{C}$, $K \in \mathcal{K}_o^n$, and $v \in S^{n-1}$, then*

$$S_v \Pi_\varphi^* K \subset \Pi_\varphi^* S_v K.$$

In particular, $V(\Pi_\varphi^* S_v K) \geq V(\Pi_\varphi^* K)$ and $V(\Pi_\varphi^* R_v K) \geq V(\Pi_\varphi^* K)$.

We recall various facts from [**34**] that lead to the proof of Lemma 8, because we need them in the sequel. We note that a concave function is almost everywhere differentiable on convex sets.

Let $\varphi \in \mathcal{C}$, let $K \in \mathcal{K}_o^n$, and let v be a unit vector in $\mathbb{R}^n$. We write w_x to denote an exterior unit normal at some $x \in \partial K$. In addition, we frequently write an $x \in \mathbb{R}^n$ in the form $x = (y,t)$ if $x = y + tv$ for $y \in v^\perp$ and $t \in \mathbb{R}$. If h is a concave function on $\pi_v(\text{int}K)$, then we define

$$\langle h \rangle(z) = h(z) - z \cdot \nabla h(z) \quad \text{for } z \in \pi_v(\text{int}K) \text{ where } \nabla h(z) \text{ exists.}$$

If $\mu_1, \mu_2 > 0$, and h_1, h_2 are concave functions on $\pi_v(\text{int}K)$, then

$$\langle \mu_1 h_1 + \mu_2 h_2 \rangle = \mu_1 \langle h_1 \rangle + \mu_2 \langle h_2 \rangle.$$

Let f, g denote the concave real functions on $\pi_v K$ such that

$$K = \{(y,t) : y \in \pi_v K, -g(y) \leq t \leq f(y)\}.$$

If $x = (z, f(z)) \in \partial K$ and $\tilde{x} = (z, -g(z)) \in \partial K$ for a $z \in \pi_v(\text{int}K)$, and both f and g are differentiable at z, then

$$w_x = \left(\frac{-\nabla f(z)}{\sqrt{1 + \|\nabla f(z)\|^2}}, \frac{1}{\sqrt{1 + \|\nabla f(z)\|^2}} \right) \tag{12}$$

$$w_{\tilde{x}} = \left(\frac{-\nabla g(z)}{\sqrt{1 + \|\nabla g(z)\|^2}}, \frac{-1}{\sqrt{1 + \|\nabla g(z)\|^2}} \right). \tag{13}$$

From this, we deduce that for any $(y,t) \in \mathbb{R}^n$, we have

$$\begin{aligned} (y,t) \cdot w_x &= (-y \cdot \nabla f(z) + t) \cdot (v \cdot w_x) \\ (y,t) \cdot w_{\tilde{x}} &= (-y \cdot \nabla g(z) - t) \cdot (v \cdot w_{\tilde{x}}) \\ h_K(w_x) &= (z, f(z)) \cdot w_x = \langle f \rangle(z) \cdot (v \cdot w_x) \\ h_K(w_{\tilde{x}}) &= (z, -g(z)) \cdot w_x = -\langle g \rangle(z) \cdot (v \cdot w_{\tilde{x}}) \end{aligned} \tag{14}$$

Since for any $u \in \mathbb{R}^n$, the definitions of the cone volume measure and the surface area measure yield that

$$nV(K) \int_{S^{n-1}} \varphi\left(\frac{u \cdot w}{h_K(w)}\right) dV_K(w) = \int_{\partial K} \varphi\left(\frac{u \cdot w_x}{h_K(w_x)}\right) h_K(w_x) \, d\mathcal{H}(x),$$

we deduce from (7) and (14) the following formula, which is Lemma 3.1 in [**34**]. We note that Lemma 3.1 in [**34**] assumes that $\Xi_{v,K}$ is thin, but only uses this property to ensure that the corresponding integral over $\Xi_{v,K}$ is zero.

Lemma 9 (Lutwak, Yang, Zhang)**.** *Using the notation as above, if* $\mathcal{H}(\Xi_{v,K})=0$ *and* $(y,t)\in\mathbb{R}^n$*, then*

$$nV(K)\int_{S^{n-1}}\varphi\left(\frac{(y,t)\cdot w}{h_K(w)}\right)dV_K(w)=$$
$$\int_{\pi_vK}\varphi\left(\frac{t-y\cdot\nabla f(z)}{\langle f\rangle(z)}\right)\langle f\rangle(z)\,dz+\int_{\pi_vK}\varphi\left(\frac{-t-y\cdot\nabla g(z)}{\langle g\rangle(z)}\right)\langle g\rangle(z)\,dz.$$

We continue to use the notation of Lemma 9 and the condition $\mathcal{H}(\Xi_{v,K})=0$. If $(y,t),(y,-s)\in\partial\Pi^*_\varphi K$ for $t>-s$, then it follows from (11) that

$$\frac{1}{2}\left(\int_{S^{n-1}}\varphi\left(\frac{(y,t)\cdot w}{h_K(w)}\right)dV_K(w)+\int_{S^{n-1}}\varphi\left(\frac{(y,-s)\cdot w}{h_K(w)}\right)dV_K(w)\right)=1.$$

Therefore (6) and Lemma 9 yield that for $(y,\frac{1}{2}(t+s))\in\partial S_v\Pi^*_\varphi K$, we have

$$\begin{aligned}
&nV(K)\left[1-\int_{S^{n-1}}\varphi\left(\frac{(y,\frac{1}{2}(t+s))\cdot w}{h_{S_vK}(w)}\right)dV_{S_vK}(w)\right] &(15)\\
&=\frac{1}{2}\int_{\pi_vK}\varphi\left(\frac{t-y\cdot\nabla f(z)}{\langle f\rangle(z)}\right)\langle f\rangle(z)\,dz\\
&+\frac{1}{2}\int_{\pi_vK}\varphi\left(\frac{s-y\cdot\nabla g(z)}{\langle g\rangle(z)}\right)\langle g\rangle(z)\,dz\\
&-\int_{\pi_vK}\varphi\left(\frac{\frac{t}{2}+\frac{s}{2}-\frac{y\cdot\nabla f(z)}{2}-\frac{y\cdot\nabla g(z)}{2}}{\frac{\langle f\rangle(z)}{2}+\frac{\langle g\rangle(z)}{2}}\right)\left(\frac{\langle f\rangle(z)}{2}+\frac{\langle g\rangle(z)}{2}\right)dz &(16)\\
&-\int_{\pi_vK}\varphi\left(\frac{-\frac{t}{2}-\frac{s}{2}-\frac{y\cdot\nabla f(z)}{2}-\frac{y\cdot\nabla g(z)}{2}}{\frac{\langle f\rangle(z)}{2}+\frac{\langle g\rangle(z)}{2}}\right)\left(\frac{\langle f\rangle(z)}{2}+\frac{\langle g\rangle(z)}{2}\right)dz &(17)\\
&+\frac{1}{2}\int_{\pi_vK}\varphi\left(\frac{-t-y\cdot\nabla f(z)}{\langle f\rangle(z)}\right)\langle f\rangle(z)\,dz\\
&+\frac{1}{2}\int_{\pi_vK}\varphi\left(\frac{-s-y\cdot\nabla g(z)}{\langle g\rangle(z)}\right)\langle g\rangle(z)\,dz.
\end{aligned}$$

If $\varphi\in\mathcal{C}$, $\alpha,\beta>0$, and $a,b\in\mathbb{R}$, then the convexity of φ yields that

$$\alpha\varphi\left(\tfrac{a}{\alpha}\right)+\beta\varphi\left(\tfrac{b}{\beta}\right)\geq(\alpha+\beta)\varphi\left(\tfrac{a+b}{\alpha+\beta}\right). \tag{18}$$

If in addition $a\cdot b<0$, then we deduce from $\varphi(0)=0$ and (1) that

$$\alpha\varphi\left(\tfrac{a}{\alpha}\right)+\beta\varphi\left(\tfrac{b}{\beta}\right)>(\alpha+\beta)\varphi\left(\tfrac{a+b}{\alpha+\beta}\right). \tag{19}$$

Applying (18) in (16) and (17) shows that

$$\int_{S^{n-1}} \varphi\left(\frac{(y, \frac{1}{2}(t+s)) \cdot w}{h_{S_vK}(w)}\right) dV_{S_vK}(w) \le 1 \tag{20}$$

in (15). We conclude $(y, \frac{1}{2}(t+s)) \in \Pi^*_\varphi S_v K$ from (11), and in turn Lemma 8 in the case when $\mathcal{H}(\Xi_{v,K}) = 0$.

So far we have just copied the argument of E. Lutwak, D. Yang, and G. Zhang [**34**]. We take a different route only for analyzing the equality case in Lemma 10, using (19) instead of (18) at an appropriate place.

For a convex body K in $\mathbb{R}^n$ and $u \in \mathbb{R}^n \backslash o$, let $\Xi^+_{u,K}$ and $\Xi^-_{u,K}$ be the set of $x \in \partial K$ where all exterior unit normals have positive and negative, respectively, scalar product with u. In particular, if $\Xi_{u,K}$ is thin, then

$$\text{any } x \in \Xi_{u,K} \text{ lies in the closures of both } \Xi^+_{u,K} \text{ and } \Xi^-_{u,K}. \tag{21}$$

Lemma 10. *Let $\varphi \in \mathcal{C}$, let $K \in \mathcal{K}^n_o$, and let $u, \tilde{u} \in \partial\Pi^*_\varphi K$ and $v \in S^{n-1}$ such that u and $\tilde{u}$ are independent, both $\Xi_{u,K}$ and $\Xi_{\tilde{u},K}$ are thin, v is parallel to $u - \tilde{u}$, and $\mathcal{H}(\Xi_{v,K}) = 0$. If $V(\Pi^*_\varphi \mathcal{S}_v K) = V(\Pi^*_\varphi K)$, then $\pi_v \Xi_{u,K} = \pi_v \Xi_{\tilde{u},K}$.*

Proof. Using the notation of (15) with $u = (y, t)$ and $\tilde{u} = (y, -s)$, we write $w(z)$ and $\tilde{w}(z)$ to denote an exterior unit normal vector to ∂K at $(z, f(z))$ and $(z, g(z))$, respectively, for any $z \in \pi_v(\text{int}K)$. Since we have equality in (20), it follows from (14), (15), and (19) that $(u \cdot w(z)) \cdot (\tilde{u} \cdot \tilde{w}(z)) \ge 0$ and $(u \cdot \tilde{w}(z)) \cdot (\tilde{u} \cdot w(z)) \ge 0$ for $\mathcal{H}$-almost all $z \in \pi_v(\text{int}K)$. We conclude by continuity that if both $(z, f(z))$ and $(z, -g(z))$ are smooth points of ∂K for a $z \in \pi_v(\text{int}K)$, then

$$(u \cdot w(z)) \cdot (\tilde{u} \cdot \tilde{w}(z)) \ge 0 \text{ and } (u \cdot \tilde{w}(z)) \cdot (\tilde{u} \cdot w(z)) \ge 0. \tag{22}$$

If $(z, f(z))$ and $(z, -g(z))$ are both smooth points of ∂K for a $z \in \pi_v(\text{int}K)$, then we say that they are the double smooth twins of each other. In particular, $\mathcal{H}$-almost all points of ∂K have a double smooth twin by $\mathcal{H}(\Xi_{v,K}) = 0$.

It follows from (21) and $\mathcal{H}(\Xi_{\tilde{u},K}) = 0$ that, for any $x \in \Xi_{u,K}$, we may choose sequences $\{x_n\} \subset \Xi^+_{u,K}$ and $\{y_n\} \subset \Xi^-_{u,K}$ tending to x such that $\pi_v x_n, \pi_v y_n \notin \pi_v \Xi_{\tilde{u},K}$, and x_n and y_n have double smooth twins $\tilde{x}_n$ and $\tilde{y}_n$, respectively. Thus the sequences $\{\tilde{x}_n\}$ and $\{\tilde{y}_n\}$ tend to the same $y \in \partial K$, which readily satisfies $\pi_v y = \pi_v x$. We have $\{\tilde{x}_n\} \subset \Xi^+_{\tilde{u},K}$ and $\{\tilde{y}_n\} \subset \Xi^-_{\tilde{u},K}$ by (22), $\pi_v \tilde{x}_n = \pi_v x_n \notin \pi_v \Xi_{\tilde{u},K}$, and $\pi_v \tilde{y}_n = \pi_v y_n \notin \pi_v \Xi_{\tilde{u},K}$. Therefore $y \in \Xi_{\tilde{u},K}$. We deduce $\pi_v \Xi_{u,K} \subset \pi_v \Xi_{\tilde{u},K}$, and in turn $\pi_v \Xi_{\tilde{u},K} \subset \pi_v \Xi_{u,K}$ by an analogous argument. Q.E.D.

In our argument, we reduce the problem to convex bodies with axial rotational symmetry. Concerning their boundary structure, we use the following simple observation.

Lemma 11. *If K is a convex body in $\mathbb{R}^n$ such that the line l is an axis of rotational symmetry, and the line l_0 intersects ∂K in a segment, then either l_0 is parallel to l, or l_0 intersects l.*

Proof. For any $x \in K$, we write $\varrho(x)$ to denote the radius of the section of K by the hyperplane passing through x and orthogonal to l, where $\varrho(x) = 0$ if the section is just the point x.

Let l_0 intersect ∂K in the segment $[p, q]$, and let m be the midpoint of $[p, q]$. We write p', q', m' to denote the orthogonal projections of p, q, m respectively, onto l. It follows that

$$\begin{aligned} \varrho(m) &\geq \tfrac{1}{2}(\varrho(p) + \varrho(q)) = \tfrac{1}{2}(\|p - p'\| + \|q - q'\|) \\ &\geq \|\tfrac{1}{2}(p - p') + \tfrac{1}{2}(q - q')\| = \|m - m'\|. \end{aligned}$$

Since $m \in \partial K$, we have $\varrho(m) = \|m - m'\|$, and hence the equality case of the triangle inequality yields that $p - p'$ and $q - q'$ are parallel. Therefore l and l_0 are contained in a two-dimensional affine subspace.

Q.E.D.

Proof of Theorem 3*:* It is equivalent to show that we have strict inequality in the Orlicz-Petty projection inequality if K is not an ellipsoid. Let us assume this, and that K is in isotropic position. It is sufficient to prove that there exist a unit vector v, and a convex body M with $V(M) = 1$ such that

$$V(\Pi_\varphi^* K) \leq V(\Pi_\varphi^* M) < V(\Pi_\varphi^* \mathcal{S}_v M).$$

The idea is to reduce the problem to bodies with axial rotational symmetry because in this way we will have two shadow boundaries that are contained in some hyperplanes.

Since K is not a ball of center o, h_K is not constant; thus we may assume that for some $p \in S^{n-1}$, we have

$$h_K(p)^2 L_{B^n} \neq L_K = \int_K (p \cdot x)^2 \, dx.$$

It follows from (ii) in Section 1 that we may assume that the supporting hyperplanes with exterior normals p and $-p$ intersect K in one point.

Let K_1 be the Schwarz rounding of K with respect to $\mathbb{R}p$. In particular, $V(K_1) = V(K) = 1$, $h_{K_1}(p) = h_K(p)$, and Fubini's theorem yields

$$\int_{K_1} (p, x)^2 \, dx = \int_K (p \cdot x)^2 \, dx \neq h_{K_1}(p)^2 L_{B^n}.$$

Therefore K_1 is not an ellipsoid according to (10), and the supporting hyperplanes with exterior normals p and $-p$ intersect K_1 in one point. In particular, if $q \in S^{n-1} \cap p^\perp$, then $\Xi_{q,K_1} = q^\perp \cap \partial K_1$ is thin. We fix a $q \in S^{n-1} \cap p^\perp$. Since K_1 is not an ellipsoid, Ξ_{q,K_1} is not the relative boundary of some $(n-1)$-ellipsoid.

Case 1 Ξ_{p,K_1} is thin

In this case, Ξ_{p,K_1} is the relative boundary of some $(n-1)$-ball. Choose $t_1, s_1 > 0$ such that $u_1 = t_1 p \in \partial \Pi K_1$ and $\tilde{u}_1 = s_1 q \in \partial \Pi K_1$, and let $v_1 = (u_1 - \tilde{u}_1)$. It follows from Lemma 11 that Ξ_{v_1,K_1} contains at most two segments parallel to v_1, and hence its $\mathcal{H}$-measure is zero. We have already seen that $\Xi_{\tilde{u}_1,K_1} = \Xi_{q,K_1}$ is thin; therefore we may apply Lemma 10 to u_1, $\tilde{u}_1$, v_1. Since $\pi_{v_1}\Xi_{u_1,K_1}$ is the relative boundary of some $(n-1)$-ellipsoid, and $\pi_{v_1}\Xi_{\tilde{u}_1,K_1}$ is not, we deduce from Lemma 10 that

$$V(\Pi^* K) \le V(\Pi^* K_1) < V(\Pi^* \mathcal{S}_{v_1} K_1).$$

Case 2 Ξ_{p,K_1} is not thin

For some $\varrho, \alpha > 0$, there exists a segment of length α parallel to p such that Ξ_{p,K_1} is the Minkowski sum of the segment and the relative boundary of the $(n-1)$-ball of radius ϱ centered at o in $p^\perp$. Let K_2 be the Schwarz rounding of K_1 with respect to $\mathbb{R}q$, and hence Ξ_{p,K_2} and Ξ_{q,K_2} are both thin.

For $t \in \mathbb{R}$, let

$$H(q,t) = q^\perp + tq.$$

If $\tau \in (0,\varrho)$, then

$$\mathcal{H}(H(q,\varrho-\tau)\cap K_2) = \mathcal{H}(H(q,\varrho-\tau)\cap K_1) > \alpha\sqrt{\varrho}\kappa_{n-2}\cdot\tau^{\frac{n-2}{2}}.$$

If K_2 were an ellipsoid, then there would exist a $\gamma > 0$ depending on K_2 such that $\mathcal{H}(H(q,\varrho-\tau)\cap K_2) < \gamma \cdot \tau^{\frac{n-1}{2}}$ for $\tau \in (0,\varrho)$; therefore K_2 is not an ellipsoid. Now we choose $t_2, s_2 > 0$ such that $u_2 = t_2 q \in \partial\Pi K_2$ and $\tilde{u}_2 = s_2 p \in \partial \Pi K_2$, and let $v_2 = (u_2 - \tilde{u}_2)/\|u_2 - \tilde{u}_2\|$. An argument as above using Lemma 10 yields

$$V(\Pi^* K) \le V(\Pi^* K_1) \le V(\Pi^* K_2) < V(\Pi^* \mathcal{S}_{v_2} K_2). \quad \text{Q.E.D.}$$

3. Proof of Theorem 4

The proof is a delicate analysis of the argument of Theorem 3. For example, we need a stability version of (19).

Lemma 12. *If $\varphi \in \mathcal{C}$, $\alpha, \beta, \omega > 0$, and $a, b \in \mathbb{R}$ such that $a \cdot b < 0$, and $\frac{|a|}{\alpha}, \frac{|b|}{\beta} \ge \omega$, then*

(23)
$$\alpha\varphi\left(\tfrac{a}{\alpha}\right) + \beta\varphi\left(\tfrac{b}{\beta}\right) - (\alpha+\beta)\varphi\left(\tfrac{a+b}{\alpha+\beta}\right) \ge \tfrac{\min\{|a|,|b|\}}{\omega}\cdot(\varphi(-\omega)+\varphi(\omega)).$$

Proof. We write Ω to denote the left-hand side of (23). If $\mu \ge 1$ and $t \in \mathbb{R}$, then the convexity of φ and $\varphi(0) = 0$ yield

$$\varphi(\mu t) \ge \mu \cdot \varphi(t). \tag{24}$$

We may assume that $a \geq -b > 0$. In particular, $0 \leq \frac{a+b}{\alpha+\beta} < \frac{a}{\alpha}$, and we deduce from (24) the estimate

$$\varphi\left(\frac{a+b}{\alpha+\beta}\right) \leq \frac{\alpha(a+b)}{a(\alpha+\beta)} \cdot \varphi\left(\frac{a}{\alpha}\right).$$

It follows from this inequality and (24) that

$$\begin{aligned} \Omega &\geq \alpha\varphi\left(\frac{a}{\alpha}\right) + \beta\varphi\left(\frac{b}{\beta}\right) - \frac{\alpha(a+b)}{a} \cdot \varphi\left(\frac{a}{\alpha}\right) \\ &= \beta\varphi\left(\frac{b}{\beta}\right) + \frac{\alpha(-b)}{a} \cdot \varphi\left(\frac{a}{\alpha}\right) \geq \frac{|b|}{\omega}\varphi(-\omega) + \frac{|b|}{\omega}\varphi(\omega). \end{aligned}$$

Q.E.D.

We also need the stability version of Lemma 13 of (11). Let $c_\varphi > 0$ be defined by $\max\{\varphi(-c_\varphi), \varphi(c_\varphi)\} = 1$ for $\varphi \in \mathcal{C}$. According to Lemma 2.2 by E. Lutwak, D. Yang, and G. Zhang [**34**] stated for the Orlicz projection body, if $rB^n \subset K \subset RB^n$ for $K \in \mathcal{K}_o^n$ and $r, R > 0$, then

$$c_\varphi r B^n \subset \Pi_\varphi^* K \subset 2c_\varphi R B^n.$$

Lemma 13. *There exist $\gamma_0 \in (0,1]$ depending on n and $\varphi \in \mathcal{C}$ such that if $\eta \in [0,1)$, $x \in \mathbb{R}^n$ and K is an o-symmetric convex body, then*

$$\int_{S^{n-1}} \varphi\left(\frac{x \cdot w}{h_K(w)}\right) dV_K(w) \leq 1 - \eta \quad \textit{yields} \quad x \in (1 - \gamma_0 \cdot \eta)\, \Pi_\varphi^* K.$$

Proof. It follows from the linear covariance (4) of the polar Orlicz projection body and from John's theorem (see F. John [**25**]) that we may assume

$$B^n \subset K \subset \sqrt{n}\, B^n.$$

Thus the form of Lemma 2.2 in [**34**] above yields $\Pi_\varphi^* K \subset 2c_\varphi \sqrt{n}\, B^n$.

According to (11), there exist $y \in \partial \Pi_\varphi^* K$ and $\varepsilon \in (0,1)$ such that $x = (1-\varepsilon)y$, and hence if $w \in S^{n-1}$, then

$$\frac{|y \cdot w|}{h_K(w)} \leq 2c_\varphi \sqrt{n}.$$

Setting $\gamma_1 = \max\{\varphi'(2c_\varphi\sqrt{n}), -\varphi'(-2c_\varphi\sqrt{n})\}$, we deduce from the convexity of φ and (1) that if $t \in [-2c_\varphi\sqrt{n}, 2c_\varphi\sqrt{n}]$, then

$$\varphi((1-\varepsilon)t) \geq \varphi(t) - \gamma_1\varepsilon \cdot |t| \geq \varphi(t) - 2c_\varphi\sqrt{n} \cdot \gamma_1\varepsilon.$$

For $\gamma_2 = 2c_\varphi\sqrt{n} \cdot \gamma_1$, it follows from (11) that

$$\begin{aligned} \int_{S^{n-1}} \varphi\left(\frac{(1-\varepsilon)y \cdot w}{h_K(w)}\right) dV_K(w) \geq \int_{S^{n-1}} &\varphi\left(\frac{y \cdot w}{h_K(w)}\right) \\ &- \gamma_2\varepsilon \, dV_K(w) = 1 - \gamma_2\varepsilon. \end{aligned}$$

Therefore we may choose $\gamma_0 = \min\{1, 1/\gamma_2\}$.

Q.E.D.

An essential tool to prove Theorem 3 was the reduction to convex bodies with axial rotational symmetry such that the shadow boundaries in the directions parallel and orthogonal to the axis are thin. The core of the argument for Theorem 4 is a stability version of this reduction, Theorem 14. To state Theorem 14, we use the following terminology. We say that a convex body K in $\mathbb{R}^n$ spins around a $u \in S^{n-1}$, if K is o-symmetric, $u \in \partial K$, the axis of rotation of K is $\mathbb{R}u$, and $K \cap u^\perp = B^n \cap u^\perp$.

Theorem 14. *Let K be a convex body in $\mathbb{R}^n$, $n \geq 3$, such that $\delta_{\rm BM}(K, B^n) \geq \delta \in (0, \delta_0)$, where $\delta_0 > 0$ depends on n. Then there exist $\varepsilon \in (\delta^{24}, \delta]$ and a convex body K' spinning around a $u \in S^{n-1}$, such that K' is obtained from K by a combination of Steiner symmetrizations, linear transformations, and taking limits, and satisfies $\delta_{\rm BM}(K', B^n) \leq \varepsilon$, and*

(i) *for any o-symmetric ellipsoid E with axial rotational symmetry around $\mathbb{R}u$, one finds a ball $x + \varepsilon^2 B^n \subset {\rm int}(E \Delta K')$ where $|x \cdot u| \leq 1 - \varepsilon^2$;*
(ii) *$(1 - \varepsilon^{32})u + \varepsilon^3 v \notin K'$ for $v \in S^{n-1} \cap u^\perp$;*
(iii) *$\varepsilon^3 u + (1 - \varepsilon^7)v \notin K'$ for $v \in S^{n-1} \cap u^\perp$.*

The proof of Theorem 14, being rather technical, is deferred to Section 5.

As $\delta_{\rm BM}(K, B^n) \leq \ln n$, Theorem 4 follows from the following statement. For $\varphi \in \mathcal{C}$, if $K \in \mathcal{K}_o^n$ with $\delta_{BM}(K, B^n) \geq \delta \in (0, \delta_*)$, then

$$\frac{V(\Pi_\varphi^* K)}{V(K)} \leq (1 - \gamma \cdot \delta^{792n} \cdot \tilde{\varphi}(\delta^{840})) \frac{V(\Pi_\varphi^* B^n)}{V(B^n)} \tag{25}$$

where $\delta_*, \gamma > 0$ depend on n and φ. In the following, the implied constants in $O(\cdot)$ depend on n and φ.

We always assume that δ_* in (25), and hence δ and ε, as well, are small enough to make the argument work. In particular, $\delta_* \leq \delta_0$ where $\delta_0 > 0$ is the constant depending on n and φ of Theorem 14. It follows from the continuity of the polar Orlicz projection body that we may also assume the following. If M is a convex body spinning around a $u \in S^{n-1}$, and $\delta_{\rm BM}(M, B^n) < \delta_*$, then

$$0.9\, B^n \subset M \subset 1.1\, B^n \quad \text{and} \quad 0.9 \Pi_\varphi^* B^n \subset \Pi_\varphi^* M \subset 1.1 \Pi_\varphi^* B^n. \tag{26}$$

Let u_* and $\tilde{u}_*$ be orthogonal unit vectors in $\mathbb{R}^n$, and let $K \in \mathcal{K}_o^n$ with $\delta_{\rm BM}(K, B^n) \geq \delta \in (0, \delta_*)$. According to Theorem 14, there exist $\varepsilon \in (\delta^{24}, \delta]$ and a convex body K' spinning around u_* with $\delta_{\rm BM}(K', B^n) \leq \varepsilon$ and obtained from K by a combination of Steiner symmetrizations, linear transformations, and taking limits such that

(i) for any o-symmetric ellipsoid E with axial rotational symmetry around $\mathbb{R}u_*$, one finds a ball $x + \varepsilon^2 B^n \subset {\rm int}(E \Delta K')$ where $|x \cdot u_*| \leq 1 - \varepsilon^2$;

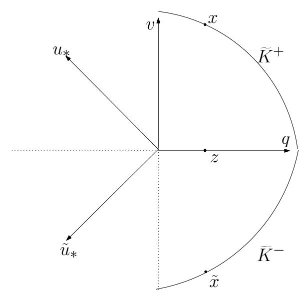

Figure 1

(ii) $(1-\varepsilon^{32})u_* + \varepsilon^3\tilde{u}_* \notin K'$;
(iii) $\varepsilon^3\, u_* + (1-\varepsilon^7)\tilde{u}_* \notin K'$.

It follows from (4) and Lemma 8 that $V(\Pi_\varphi^* K')/V(K') \geq V(\Pi_\varphi^* K)/V(K)$.

We deduce that if $\widetilde{K}$ is a smooth and strictly convex body spinning around u_* sufficiently close to K', then

(a) for any o-symmetric ellipsoid E with axial rotational symmetry around $\mathbb{R}u_*$, one finds a ball $x+\varepsilon^2\, B^n \subset \mathrm{int}(E\Delta\widetilde{K})$ where $|x\cdot u_*| \leq 1-\varepsilon^2$;
(b) $(1-\varepsilon^{32})u_* + \varepsilon^3\tilde{u}_* \notin \widetilde{K}$;
(c) $\varepsilon^3\, u_* + (1-\varepsilon^7)\tilde{u}_* \notin \widetilde{K}$;
(d) $\frac{V(\Pi_\varphi^* \widetilde{K})}{V(\widetilde{K})} \geq (1-\varepsilon^{33n}\tilde{\varphi}(\varepsilon^{35}))\cdot \frac{V(\Pi_\varphi^* K)}{V(K)}$;
(e) $\delta_{\mathrm{BM}}(\widetilde{K}, B^n) < \delta_*$.

We define $v \in S^{n-1}$ by

$$\lambda_*\, v = \varrho_{\Pi_\varphi^* \widetilde{K}}(u_*)\cdot u_* - \varrho_{\Pi_\varphi^* \widetilde{K}}(\tilde{u}_*)\cdot \tilde{u}_*$$

for some $\lambda_* > 0$. It follows from (e) and (26) that

$$\frac{1}{2} < \frac{0.9}{\sqrt{0.9^2+1.1^2}} \leq v\cdot u_* \leq \frac{1.1}{\sqrt{0.9^2+1.1^2}} < \frac{\sqrt{3}}{2}. \tag{27}$$

We plan to apply Steiner symmetrization to $\widetilde{K}$ with respect to $v^\perp$, and show that the volume of the polar Orlicz projection body increases substantially. We consider $v^\perp$ as $\mathbb{R}^{n-1}$, and set

$$v^\perp \cap B^n = B^{n-1}.$$

For $X \subset v^\perp$, the interior of X with respect to the subspace topology of $v^\perp$ is denoted by relint X.

Let q be the unit vector in the line $\mathrm{lin}\{u_*, \tilde{u}_*\}\cap v^\perp$ satisfying $q\cdot u_* < 0$ (see Figure 1). We observe that $\Xi_{u_*,\widetilde{K}} = u_*^\perp \cap \partial B^n$ and $\Xi_{\tilde{u}_*,\widetilde{K}} = \tilde{u}_*^\perp \cap \partial \widetilde{K}$, moreover,

$$\begin{aligned} E_* &= \pi_v(u_*^\perp \cap B^n), \\ \widetilde{K}_* &= \pi_v(\tilde{u}_*^\perp \cap \widetilde{K}) \end{aligned}$$

are o-symmetric, and have $\mathbb{R}q$ as their axis of rotation inside $v^\perp$. We define $\theta > 0$ by $\theta q \in \partial \widetilde{K}_*$, and the linear transform $\Phi : v^\perp \to v^\perp$ by $\Phi(\theta q) = q$, and $\Phi(y) = y$ for $y \in v^\perp \cap u_*^\perp$. Thus $\theta \in (\frac{1}{2}, \frac{\sqrt{3}}{2})$ by (27). Since $\Phi \widetilde{K}_*$ is congruent to $\tilde{u}_*^\perp \cap \widetilde{K}$, (a) yields an $(n-1)$-ball $z' + \varepsilon^2 B^{n-1} \subset \operatorname{relint} \Phi(E_*) \Delta \Phi(\widetilde{K}_*)$ where $0 \le z' \cdot q \le 1 - \varepsilon^2$. We define $z_* = \Phi^{-1} z'$, and hence

$$z_* + \tfrac{\varepsilon^2}{2} B^{n-1} \subset \operatorname{relint} E_* \Delta \widetilde{K}_*. \tag{28}$$

Since $v^\perp \cap u_*^\perp \cap E_* = v^\perp \cap u_*^\perp \cap \widetilde{K}_*$, we also deduce that

$$\varepsilon^2/2 < z_* \cdot q < \theta - (\varepsilon^2/2). \tag{29}$$

We write w_x to denote the exterior unit normal at an $x \in \partial \widetilde{K}$, and define

$$\begin{aligned} \widetilde{K}^+ &= \{x \in \partial \widetilde{K} : v \cdot w_x > 0 \text{ and } q \cdot x > 0\}; \\ \widetilde{K}^- &= \{x \in \partial \widetilde{K} : v \cdot w_x < 0 \text{ and } q \cdot x > 0\}. \end{aligned}$$

It follows from (29) that

$$z_* + \tfrac{\varepsilon^2}{2} B^{n-1} \subset \pi_v \widetilde{K}^\pm. \tag{30}$$

If $z = \pi_v x = \pi_v \tilde{x} \in z_* + \frac{\varepsilon^2}{4} B^{n-1}$ for suitable $x \in \widetilde{K}^+$ and $\tilde{x} \in \widetilde{K}^-$, then $z + \frac{\varepsilon^2}{4} B^{n-1} \subset \pi_v \widetilde{K}^\pm$ by (30). We deduce from $\widetilde{K} \subset 1.1 B^n$ (compare (26)) that $w_x \cdot v, |w_{\tilde{x}} \cdot v| > \varepsilon^2/8$, and hence (14) and (26) yield

$$0.9 \le \langle f \rangle(z), \langle g \rangle(z) < 9\varepsilon^{-2} \quad \text{for } z \in z_* + \tfrac{\varepsilon^2}{4} B^{n-1}. \tag{31}$$

Lemma 15. *If $\pi_v x = \pi_v \tilde{x} \in z_* + \frac{\varepsilon^2}{4} B^{n-1}$ for $x \in \widetilde{K}^+$ and $\tilde{x} \in \widetilde{K}^-$, then*

$$|u_* \cdot w_x|,\ |\tilde{u}_* \cdot w_{\tilde{x}}| > \varepsilon^{32}/2 \quad \textit{and} \quad (u_* \cdot w_x) \cdot (\tilde{u}_* \cdot w_{\tilde{x}}) < 0.$$

Proof. Since $\widetilde{K}$ is a smooth and strictly convex body, and has $\mathbb{R}u_*$ as its axis of rotation, we have

$$\begin{aligned} \Xi^+_{u_*, \widetilde{K}} &= \{x \in \partial K : x \cdot u_* > 0\}; \\ \Xi^+_{\tilde{u}_*, \widetilde{K}} &= \{x \in \partial K : x \cdot \tilde{u}_* > 0\}. \end{aligned}$$

It follows from $x \in \widetilde{K}^+$ and $\Xi_{u_*, \widetilde{K}} = u_*^\perp \cap S^{n-1}$ that

$$u_* \cdot w_x > 0 \text{ if and only if } \pi_v x \in \operatorname{relint} \pi_v(u_*^\perp \cap B^n) = \operatorname{relint} E_*, \tag{32}$$

and from $\tilde{x} \in \widetilde{K}^-$ that

$$\tilde{u}_* \cdot w_{\tilde{x}} > 0 \text{ if and only if } \pi_v \tilde{x} \in \operatorname{relint} \pi_v(\tilde{u}_*^\perp \cap \widetilde{K}) = \operatorname{relint} \widetilde{K}_*. \tag{33}$$

We deduce from (28), (32), and (33) that

$$(u_* \cdot w_x) \cdot (\tilde{u}_* \cdot w_{\tilde{x}}) < 0. \tag{34}$$

To have a lower estimate on $|u_* \cdot u_x|$, we observe that combining (28) with $\varepsilon^2/4 > 2\varepsilon^3$ and the fact that π_v does not increase distance yields

$$x \notin (u_*^\perp \cap \partial\widetilde{K}) + 2\varepsilon^3 B^n.$$

Thus, we conclude from (c) that $\|\pi_{u_*} x\| \leq 1 - \varepsilon^7$. It follows that $(\pi_{u_*} x) + \frac{\varepsilon^7}{2} B^n \subset \widetilde{K}$, and hence w_x is an exterior normal also to the convex hull at x of this ball and x. As $|u_* \cdot x| \leq 1$, we deduce that

$$|u_* \cdot w_x| \geq \varepsilon^7/2. \tag{35}$$

Finally, we consider $|\tilde{u}_* \cdot w_{\tilde{x}}|$. Using (28) again, we have

$$\tilde{x} \notin (\tilde{u}_*^\perp \cap \partial\widetilde{K}) + 2\varepsilon^3 B^n. \tag{36}$$

In particular, $\|\tilde{x} - u_*\| > 2\varepsilon^3$ and $\|\tilde{x} - (-u_*)\| > 2\varepsilon^3$, and hence (b) implies that $|\tilde{x} \cdot u_*| < 1 - \varepsilon^{32}$. As $\widetilde{K}$ spins around u_*, we deduce from (36) that $(\pi_{\tilde{u}_*} \tilde{x}) + \frac{\varepsilon^{32}}{2} B^n \subset \widetilde{K}$. Thus $w_{\tilde{x}}$ is an exterior normal also to the convex hull at $\tilde{x}$ of this ball and $\tilde{x}$, and hence $|\tilde{u}_* \cdot \tilde{x}| \leq 1$ yields that

$$|\tilde{u}_* \cdot w_{\tilde{x}}| \geq \varepsilon^{32}/2. \tag{37}$$

Therefore Lemma 15 is a consequence of (34), (35), and (37). Q.E.D.

We continue with the proof of Theorem 4. We use the notation of Lemma 9. In particular, we write (z, t) to denote $z + tv$ for $z \in \mathbb{R}^{n-1} = v^\perp$ and $t \in \mathbb{R}$, and f and g to denote the concave functions on $\pi_v \widetilde{K}$ such that for $z \in \mathrm{relint}\pi_v \widetilde{K}$, we have $f(z) > -g(z)$, and $(z, f(z)), (z, -g(z)) \in \partial\widetilde{K}$.

We write $\gamma_1, \gamma_2, \ldots$ to denote positive constants depending on n and φ, and we define

$$\begin{aligned}
y_* &= -\pi_v \left(\varrho_{\Pi_\varphi^* \widetilde{K}}(u_*) \cdot u_* \right) = -\pi_v \left(\varrho_{\Pi_\varphi^* \widetilde{K}}(\tilde{u}_*) \cdot \tilde{u}_* \right), \\
\Psi &= \left\{ \alpha \in S^{n-1} : \alpha \cdot v > 0 \text{ and } \pi_v \left(\varrho_{S_v \Pi * \widetilde{K}}(\alpha) \cdot \alpha \right) \in y_* + \varepsilon^{33} B^{n-1} \right\}.
\end{aligned}$$

As $0.9\Pi_\varphi^* B^n \subset S_v \Pi_\varphi^* \widetilde{K} \subset 1.1\Pi_\varphi^* B^n$ by (e) and (26), we have

$$\mathcal{H}(\Psi) > \gamma_1 \varepsilon^{33(n-1)}. \tag{38}$$

Let

$$y \in y_* + \varepsilon^{33} B^{n-1},$$

and let $(y,t),(y,-s) \in \partial\Pi^*_\varphi\widetilde{K}$ where $-s < t$, and hence $(y,\frac{t+s}{2}) \in \partial S_v\Pi^*_\varphi\widetilde{K}$. We define

$$\begin{aligned} u &= \frac{(y,t)}{\|(y,t)\|} \\ \tilde{u} &= \frac{(y,-s)}{\|(y,-s)\|} \\ \alpha &= \frac{(y,\frac{t+s}{2})}{\|(y,\frac{t+s}{2})\|} \in \Psi. \end{aligned}$$

It follows from $0.9\Pi^*_\varphi B^n \subset \Pi^*_\varphi\widetilde{K} \subset 1.1\Pi^*_\varphi B^n$ that

$$\|u-u_*\|, \|\tilde{u}-\tilde{u}_*\| < \gamma_2\varepsilon^{33}.$$

Choose δ_* small enough that $\gamma_2\varepsilon^{33} < \varepsilon^{32}/4$. We deduce from Lemma 15 that if

$$z = \pi_v x = \pi_v\tilde{x} \in z_* + (\varepsilon^2/4)\,B^{n-1}$$

for $x \in \widetilde{K}^+$ and $\tilde{x} \in \widetilde{K}^-$, then

$$|u\cdot w_x|,\ |\tilde{u}\cdot w_{\tilde{x}}| > \varepsilon^{32}/4 \ \text{ and } \ (u\cdot w_x)\cdot(\tilde{u}\cdot w_{\tilde{x}}) < 0.$$

Using now $0.9\Pi^*_\varphi B^n \subset \Pi^*_\varphi\widetilde{K} \subset 1.1\Pi^*_\varphi B^n$ and (14), we deduce

$$|t-y\cdot\nabla f(z)|,\ |s-y\cdot\nabla g(z)| > \gamma_3\varepsilon^{32} \ \text{ and } \ (t-y\cdot\nabla f(z))\cdot(s-y\cdot\nabla g(z)) < 0. \tag{39}$$

It follows from (31) and (39) that we may apply Lemma 12 with

$$a = t-y\cdot\nabla f(z),\ b = s-y\cdot\nabla g(z),\ \alpha = \langle f\rangle(z) \text{ and } \beta = \langle g\rangle(z).$$

By (31), (39), and since $\gamma_3\varepsilon^{34}/9 > \varepsilon^{35}$, we may choose $\omega = \varepsilon^{35}$ in Lemma 12, and hence (39) yields that

$$\begin{aligned} &\frac{1}{2}\int_{\pi_v\widetilde{K}}\varphi\left(\frac{t-y\cdot\nabla f(z)}{\langle f\rangle(z)}\right)\langle f\rangle(z)\,dz \\ &+\frac{1}{2}\int_{\pi_v\widetilde{K}}\varphi\left(\frac{s-y\cdot\nabla g(z)}{\langle g\rangle(z)}\right)\langle g\rangle(z)\,dz \\ &-\int_{\pi_v\widetilde{K}}\varphi\left(\frac{\frac{t}{2}+\frac{s}{2}-\frac{y\cdot\nabla f(z)}{2}-\frac{y\cdot\nabla g(z)}{2}}{\frac{\langle f\rangle(z)}{2}+\frac{\langle g\rangle(z)}{2}}\right)\left(\frac{\langle f\rangle(z)}{2}+\frac{\langle g\rangle(z)}{2}\right)\,dz \\ &\geq \gamma_4\,\varepsilon^{-3}\tilde{\varphi}(\varepsilon^{35}). \end{aligned}$$

Therefore (15), (18), and (26) lead to

$$\int_{S^{n-1}}\varphi\left(\frac{(y,\frac{1}{2}(t+s))\cdot w}{h_{S_v\widetilde{K}}(w)}\right)\,dV_{S_v\widetilde{K}}(w) \leq 1-\frac{\gamma_4\,\varepsilon^{-3}\tilde{\varphi}(\varepsilon^{35})}{nV(1.1B^n)}. \tag{40}$$

We conclude, first applying Lemma 13, then the consequence $0.9\Pi^*_\varphi B^n \subset S_v\Pi^*_\varphi\widetilde{K} \subset 1.1\Pi^*_\varphi B^n$ of (26), that if $w \in \Psi$, then

$$\begin{aligned}\varrho_{S_v\Pi^*_\varphi\widetilde{K}}(w)^n &\leq (1-\gamma_5\,\varepsilon^{-3}\tilde{\varphi}(\varepsilon^{35}))^n \cdot \varrho_{\Pi^*_\varphi S_v\widetilde{K}}(w)^n \\ &\leq \varrho_{\Pi^*_\varphi S_v\widetilde{K}}(w)^n - \gamma_6\,\varepsilon^{-3}\tilde{\varphi}(\varepsilon^{35}).\end{aligned}$$

Since $\varrho_{S_v\Pi^*_\varphi\widetilde{K}}(w) \leq \varrho_{\Pi^*_\varphi S_v\widetilde{K}}(w)$ for any $w \in S^{n-1}$ by Lemma 8, combining (8) and (38) leads to

$$\begin{aligned}V(\Pi^*_\varphi\widetilde{K}) &= V(\Pi^*_\varphi S_v\widetilde{K}) \leq V(\Pi^*_\varphi S_v\widetilde{K}) - \gamma_7\,\varepsilon^{33(n-1)-3}\tilde{\varphi}(\varepsilon^{35}) \\ &\leq (1-\gamma_8\,\varepsilon^{33n-36}\tilde{\varphi}(\varepsilon^{35}))\cdot V(\Pi^*_\varphi S_v\widetilde{K}).\end{aligned}$$

We conclude from (d) and Theorem 2 that

$$\begin{aligned}\frac{V(\Pi^*_\varphi K)}{V(K)} &\leq (1-\varepsilon^{33n}\tilde{\varphi}(\varepsilon^{35})))^{-1}(1-\gamma_8\,\varepsilon^{33n-36}\tilde{\varphi}(\varepsilon^{35}))) \cdot \frac{V(\Pi^*_\varphi\widetilde{K})}{V(\widetilde{K})} \\ &\leq (1-\gamma_9\,\varepsilon^{33n}\tilde{\varphi}(\varepsilon^{35}))) \cdot \frac{V(\Pi^*_\varphi B^n)}{V(B^n)},\end{aligned}$$

which, in turn, yields (25) by $\varepsilon \geq \delta^{24}$. Q.E.D.

4. Proof of Theorem 6

Naturally, we again need a suitable stability version of (18).

Lemma 16. *Let $\varphi \in \mathcal{C}$ be even, such that $\varphi''(t)$ is continuous and positive for $t > 0$. If $a, b, \alpha, \beta, \omega > 0$ satisfy $\omega \leq \frac{a}{\alpha}, \frac{b}{\beta} \leq \omega^{-1}$, then*

$$\begin{aligned}&\alpha\varphi\left(\tfrac{a}{\alpha}\right) + \beta\varphi\left(\tfrac{b}{\beta}\right) - (\alpha+\beta)\varphi\left(\tfrac{a+b}{\alpha+\beta}\right) \\ &\quad\geq \frac{\min\{\varphi''(t) : t \in (\omega, \omega^{-1})\}\cdot\min\{\alpha^2, \beta^2\}}{2(\alpha+\beta)} \cdot \left(\frac{a}{\alpha} - \frac{b}{\beta}\right)^2.\end{aligned}$$

Proof. The Taylor formula around $\frac{a+b}{\alpha+\beta}$ yields the estimate. Q.E.D.

Given Theorem 4, what we need to consider are translates of a convex body that are close to the unit ball.

Lemma 17. *Let $\varphi \in \mathcal{C}$ be even, such that $\varphi''(t)$ is continuous and positive for $t > 0$. There exist $\varepsilon_0, \gamma > 0$ depending on n and φ such that if $\|\theta\| \geq \varepsilon^{1/3}$ and $B^n \subset K - \theta \subset (1+\varepsilon)B^n$ for $K \in \mathcal{K}^n_0$, $\varepsilon \in (0, \varepsilon_0)$, and $\theta \in \mathbb{R}^n$, then*

$$\frac{V(\Pi^*_\varphi K)}{V(K)} < (1-\gamma\varepsilon^{\frac{2}{3}})\,\frac{V(\Pi^*_\varphi B^n)}{V(B^n)}.$$

Proof. We write σ to denote the reflection through $\theta^\perp$. Possibly after applying Schwarz rounding with respect to $v = \theta/\|\theta\|$ (compare Lemma 8), we may assume that $\mathbb{R}v$ is the axis of rotation of K. It follows that $\Pi^*_\varphi K$ also has $\mathbb{R}v$ as its axis of rotation. Since φ is even, we deduce that $\Pi^*_\varphi K$ is o-symmetric; therefore $\Pi^*_\varphi K$ is symmetric with respect to σ. We may also assume that K is smooth, and we write w_x to denote the unique exterior unit normal at $x \in \partial K$.

We write $\gamma_1, \gamma_2, \dots$ to denote positive constants depending on n and φ. In addition the implied constant in $O(\cdot)$ depends also only on n and φ. As $K \subset 3B^n$, Lemma 2.2 by E. Lutwak, D. Yang, and G. Zhang [**34**] yields

$$\Pi^*_\varphi K \subset \gamma_1 B^n. \tag{41}$$

Since $\Pi^*_\varphi K$ is o-symmetric and $\Pi^*_\varphi K \subset \gamma_1 B^n$, there exists $\gamma_2 > 0$ depending on n and φ, such that if $h_{\Pi^*_\varphi K}(u) \le \gamma_2$ for some $u \in S^{n-1}$, then $V(\Pi^*_\varphi K) < \frac{1}{2}\, V(\Pi^*_\varphi B^n)$. In particular, Lemma 17 readily holds in this case. Therefore we may assume that

$$\gamma_2\, B^n \subset \Pi^*_\varphi K. \tag{42}$$

We set $\mathbb{R}^{n-1} = v^\perp$ and $B^{n-1} = v^\perp \cap B^n$, and write the point $y + tv$ of $\mathbb{R}^n$ with $y \in \mathbb{R}^{n-1}$ and $t \in \mathbb{R}$ in the form (y,t). In addition, let f, g be the concave functions on $\pi_v K$ satisfying

$$K = \{(y,t) :\, y \in \pi_v K \text{ and } -g(y) \le t \le f(y)\}.$$

We consider

$$\begin{aligned}
&\Xi = \tfrac{3}{5}\, B^{n-1} \backslash \tfrac{1}{2}\, B^{n-1}\\
&\Psi = \{(y,t)/\|(y,t)\| \in S^{n-1} :\, y \in \tfrac{3\gamma_2}{5}\, B^{n-1},\ t > 0 \text{ and } (y,t) \in \partial \Pi^*_\varphi K\}.
\end{aligned}$$

It follows that

$$\mathcal{H}(\Psi) \ge \gamma_3. \tag{43}$$

For $y \in \frac{3\gamma_2}{5}\, B^{n-1}$ and $z \in \Xi$, let $t > 0$ such that $(y,t) \in \partial \Pi^*_\varphi K$, and hence $(y,-t) \in \partial \Pi^*_\varphi K$ since $\sigma(\Pi^*_\varphi K) = \Pi^*_\varphi K$. We plan to apply Lemma 16 with

$$a = t - y \cdot \nabla f(z),\ b = t - y \cdot \nabla g(z),\ \alpha = \langle f \rangle(z) \text{ and } \beta = \langle g \rangle(z). \tag{44}$$

Let $x, \tilde{x} \in \partial K$, and let $x', \tilde{x}' \in \partial(\theta + B^n)$ be defined in a way such that $\pi_v x = \pi_v \tilde{x} = \pi_v x' = \pi'_v \tilde{x}' = z$, $(x - \tilde{x}) \cdot v > 0$ and $(x' - \tilde{x}') \cdot v > 0$. We observe that $\sigma(\tilde{x}' - \theta) = x' - \theta$. The condition $z \in \Xi$ yields that

$$\frac{4}{5} \le v \cdot (x' - \theta) = -v \cdot (\tilde{x}' - \theta) \le \frac{\sqrt{3}}{2} < 0.9. \tag{45}$$

Since the angles between v and both (y,t) and $x'-\theta$ are at most $\gamma_4 = \arcsin\frac{3}{5}$, and $\cos 2\gamma_4 = \frac{7}{25}$, we deduce from (41) and (42) that

$$\frac{7\gamma_2}{25} \leq (y,t)\cdot(x'-\theta) = (y,-t)\cdot(\tilde{x}'-\theta) \leq \gamma_1. \tag{46}$$

To compare $x'-\theta$ and w_x, we observe that the tangent planes to $\theta+B^n$ at both x' and $\theta+w_x$ separate x and $\theta+B^n$. Since $\|x-\theta\| \leq 1+\varepsilon$, such points on $\theta+S^{n-1}$ are contained in a cap cut off by a hyperplane of distance at least $(1+\varepsilon)^{-1}$ from θ, and the diameter of the cap is at most $2\sqrt{1-(1+\varepsilon)^{-2}} < 4\varepsilon^{\frac{1}{2}}$. Therefore

$$\|w_x-(x'-\theta)\| < 4\varepsilon^{\frac{1}{2}} \text{ and } \|w_{\tilde{x}}-(\tilde{x}'-\theta)\| < 4\varepsilon^{\frac{1}{2}}. \tag{47}$$

From (14), (44), (46), and (47), we deduce that

$$\frac{a}{\alpha} = \left(1+O(\varepsilon^{\frac{1}{2}})\right)\cdot\frac{(y,t)\cdot(x'-\theta)}{h_K(w_x)}; \tag{48}$$

$$\frac{b}{\beta} = \left(1+O(\varepsilon^{\frac{1}{2}})\right)\cdot\frac{(y,t)\cdot(x'-\theta)}{h_K(w_{\tilde{x}})}. \tag{49}$$

We have $\theta\cdot w+1 \leq h_K(w) \leq \theta\cdot w+1+\varepsilon$ for any $w\in S^{n-1}$, and $\|\theta\| < 1+\varepsilon$ by $o\in\operatorname{int}K$. Therefore (45), (47), and the condition $\|\theta\| \geq \varepsilon^{1/3}$ yield

$$1+\tfrac{3}{5}\,\varepsilon^{1/3} < h_K(w_x) < 1.9 \text{ and } 0.1 < h_K(w_{\tilde{x}}) < 1-\tfrac{3}{5}\,\varepsilon^{1/3} \tag{50}$$

provided that $\varepsilon_0>0$ is suitably small. We deduce from (46), (48), (49), and (50) that there exist $\omega,\gamma_5>0$ depending on n and φ such that

$$\frac{b}{\beta}-\frac{a}{\alpha} > \gamma_5\varepsilon^{\frac{1}{3}}; \tag{51}$$

$$\omega < \frac{a}{\alpha} < \frac{b}{\beta} < \omega^{-1}. \tag{52}$$

In addition, (14), (45), (47), and (50) yield that

$$\gamma_6 < \alpha,\beta < \gamma_7. \tag{53}$$

We conclude from Lemma 16 the estimate

$$\begin{aligned}&\frac{1}{2}\int_{\pi_vK}\varphi\left(\frac{t-y\cdot\nabla f(z)}{\langle f\rangle(z)}\right)\langle f\rangle(z)\,dz+\frac{1}{2}\int_{\pi_vK}\varphi\left(\frac{t-y\cdot\nabla g(z)}{\langle g\rangle(z)}\right)\langle g\rangle(z)\,dz\\ &-\int_{\pi_vK}\varphi\left(\frac{t-\frac{y\cdot\nabla f(z)}{2}-\frac{y\cdot\nabla g(z)}{2}}{\frac{\nabla f(z)}{2}+\frac{\nabla g(z)}{2}}\right)\left(\frac{\nabla f(z)}{2}+\frac{\nabla g(z)}{2}\right)dz > \gamma_8\varepsilon^{\frac{2}{3}}.\end{aligned} \tag{54}$$

Since (54) holds for any $z\in\Xi$, and $S_v\Pi^*_\varphi K = \Pi^*_\varphi K$, we deduce from (15) and (18) that

$$\int_{S^{n-1}}\varphi\left(\frac{(y,t)\cdot w}{h_{S_vK}(w)}\right)dV_{S_vK}(w) < 1-\gamma_9\varepsilon^{\frac{2}{3}}. \tag{55}$$

Now we have (55) for all $y \in \frac{3\gamma_2}{5}\, B^{n-1}$, and hence

$$\varrho_{\Pi^*_\varphi K}(u) < (1-\gamma_{10}\varepsilon^{\frac{2}{3}})\varrho_{\Pi^*_\varphi S_v K}(u)$$

for $u \in \Psi \subset S^{n-1}$ according to Lemma 13, where $\mathcal{H}(\Psi) \geq \gamma_3$ by (43). Therefore combining Lemma 8, (8), (41), and (42) yields Lemma 17.

Q.E.D.

Theorem 6 follows from the following statement. For $\varphi \in \mathcal{C}$, there exist $\eta_0, \gamma > 0$ depending only on n and φ such that if $K \in \mathcal{K}^n_o$, $\eta \in (0, \eta_0)$, and

(56) $\qquad K \not\subset (1+\eta)E$ for any o-symmetric ellipsoid $E \subset K$,

then

$$\frac{V(\Pi^*_\varphi K)}{V(K)} \leq \left(1-\gamma\cdot\eta^{2376n}\cdot\varphi(\eta^{2520})\right)\cdot\frac{V(\Pi^*_\varphi B^n)}{V(B^n)}. \tag{57}$$

If $\delta_{BM}(K, B^n) > \eta^3/108$, then Theorem 4 yields (57). Therefore we assume that $\delta_{BM}(K, B^n) \leq \eta^3/108$. In particular, we may assume that $\theta + B^n \subset K$ for some $\theta \in \mathbb{R}^n$, and K is contained in a ball of radius $1+\frac{\eta^3}{54}$. It follows that

$$\theta + B^n \subset K \subset \theta + \left(1+\tfrac{\eta^3}{27}\right)\, B^n.$$

We deduce from (56) that $\frac{1+\|\theta\|+\frac{\eta^3}{27}}{1-\|\theta\|} > 1+\eta$, and hence $\|\theta\| > \eta/3$. Therefore we may apply Lemma 17 with $\varepsilon = \frac{\eta^3}{27}$, which, in turn, completes the proof of (57).

Q.E.D.

5. Class reduction based on Steiner symmetrization

In this section, we prove Theorem 14. Let

$$u \in S^{n-1} \quad \text{and} \quad v \in S^{n-1} \cap u^\perp.$$

Recall that a convex body K in $\mathbb{R}^n$ spins around u, if K is o-symmetric, $u \in \partial K$, the axis of rotation of K is $\mathbb{R}u$, and $K \cap u^\perp = B^n \cap u^\perp$. In this case, we call $\pm u$ the poles of K, and $\partial K \cap u^\perp \subset S^{n-1}$ the equator of K. We show that to have a stability version of the Orlicz-Petty projection inequality, we may assume that K is an o-symmetric convex body with axial rotational symmetry such that the boundary sufficiently bends near the equator and the poles.

We prepare the proof of Theorem 14 by a series of lemmas. First of all, one may assume that K is an o-symmetric convex body with axial rotational symmetry because of the following.

Lemma 18. *For any $n \geq 2$ there exists $\gamma > 0$ depending only on n, such that if K is a convex body in $\mathbb{R}^n$ such that $\delta_{\rm BM}(K, B^n) \geq \varepsilon \in (0, 1)$, then one can find an o-symmetric convex body C with axial rotational symmetry and $\delta_{\rm BM}(C, B^n) = \gamma\varepsilon^2$ that is obtained from K using Steiner symmetrizations, linear transformations, and taking limits.*

Remark: If K is o-symmetric, then $\delta_{\rm BM}(C, B^n) = \gamma\varepsilon$ is possible.

Proof. According to Theorem 1.4 in [**6**], there is an o-symmetric convex body C with axial rotational symmetry that is obtained from K using Steiner symmetrizations, linear transformations, and taking limits, and that satisfies $\delta_{\rm BM}(C_0, B^n) \geq \gamma\varepsilon^2$. We note that in Theorem 1.4, it is stated that affine transformations are needed. But translations are only used to translate K at the beginning by $-\sigma_K$ where σ_K is the centroid of K. If we perform all Steiner symmetrizations in the proof of Theorem 1.4 in [**6**] through the same hyperplanes containing the origin, then even without the translation at the beginning, the convex body C_0 will still be o-symmetric.

We may assume that $\delta_{\rm BM}(C_0, B^n) > \gamma\varepsilon^2$; otherwise we are done. Since some sequence of Steiner symmetrizations subsequently applied to C_0 converges to a Euclidean ball B_0 of volume $V(C_0)$, there is a sequence $\{C_m\}$, $m = 0, 1, 2, \ldots$ of o-symmetric convex bodies tending to B_0 such that C_m, $m > 0$, is a Schwarz rounding of C_{m-1} with respect to some $w_m \in S^{n-1}$. In particular, there is $m \geq 0$ such that $\delta_{\rm BM}(C_m, B^n) > \gamma\varepsilon^2$ and $\delta_{\rm BM}(C_{m+1}, B^n) \leq \gamma\varepsilon^2$.

For $w \in S^{n-1}$, let M_w be the Schwarz rounding of C_m with respect to $\mathbb{R}w$. Then $\delta_{\rm BM}(M_w, B^n)$ is a continuous function of w. Since $C_m = M_{w_m}$ and $C_{m+1} = M_{w_{m+1}}$, there is a $w \in S^{n-1}$ with $\delta_{\rm BM}(M_w, B^n) = \gamma\varepsilon^2$.

If K is o-symmetric, then Theorem 1.4 in [**6**] states that $\delta_{\rm BM}(C_0, B^n) \geq \gamma\varepsilon$, and hence the argument above gives $\delta_{\rm BM}(C, B^n) = \gamma\varepsilon$. Q.E.D.

In order to obtain a stability version of the Orlicz-Petty projection inequality for an o-symmetric convex body K with axial rotational symmetry, it is hard to deal with K if it is close to being flat at the poles, or close to being ruled near the equator. In these cases, we apply an extra Schwarz rounding. The precise statements are the subjects of Lemma 19 and Proposition 23. For $w \in S^{n-1}$ and $t \in \mathbb{R}$, we recall that

$$H(w, t) = w^\perp + tw.$$

The next observation considers the shape of a convex body with axial rotational symmetry near the equator.

Lemma 19. *There exist $\tau_1, \tau_2 > 0$ depending on n with the following properties. If $t \in (0, \frac{1}{3})$, the convex body K in $\mathbb{R}^n$ spins around u, and*

$$\tau_1\sqrt{t}\, u + (1 - t)v \in K,$$

then $\delta_{\rm BM}(K',B^n)\geq \tau_2 t$ *for the Schwarz rounding* K' *of* K *around* $\mathbb{R}v$.

Proof. Let E_0 be the o-symmetric ellipsoid with axial rotational symmetry around $\mathbb{R}v$ such that $v\in\partial E_0$, and $\mathcal{H}\left(E_0\cap v^\perp\right)=2\kappa_{n-1}$. For any $s\in(0,\frac{2}{3})$, we have

$$\gamma_1\sqrt{s}\,u+(1-s)v\not\in(1+\tau_2 s)E_0 \tag{58}$$

for suitable $\gamma_1>0$ and $\tau_2\in(0,1)$ depending only on n. We define τ_1 by the equation

$$(\tau_1\kappa_{n-2}/\kappa_{n-1})^{\frac{1}{n-1}}=\gamma_1\sqrt{2}.$$

Let $E\subset K'$ be an o-symmetric ellipsoid with axial rotational symmetry around $\mathbb{R}v$ such that $K'\subset\lambda E$, where $\ln\lambda=\delta_{\rm BM}(K',B^n)$. It follows from the normalization of K that $\mathcal{H}\left(K\cap v^\perp\right)\leq 2\kappa_{n-1}$; thus $E\subset E_0$.

If $\tau_1\sqrt{t}\,u+(1-t)v\in K$ for $t\in(0,\frac{1}{3})$, then $\tau_1\sqrt{t}\,u+(1-t)(u^\perp\cap B^n)\subset K$ and $\sqrt{t(2-3t)}>\sqrt{t}$ yield that

$$\tau_1\sqrt{t}\,u+(1-2t)v+\sqrt{t}(u^\perp\cap v^\perp\cap B^n)\subset K.$$

Since $H(v,1-2t)\cap K$ contains an $(n-1)$-dimensional cylinder whose height is $\tau_1\sqrt{t}$, and whose base has radius $\sqrt{t}$, we have

$$\mathcal{H}\left(H(v,1-2t)\cap K'\right)=\mathcal{H}\left(H(v,1-2t)\cap K\right)\geq\tau_1\kappa_{n-2}t^{\frac{n-1}{2}}.$$

In particular,

$$\gamma_1\sqrt{2t}+(1-2t)v=(\tau_1\kappa_{n-2}/\kappa_{n-1})^{\frac{1}{n-1}}\sqrt{t}\,u+(1-2t)v\in K'.$$

We conclude from (58) that $\lambda>1+\tau_2 2t$, and hence $\delta_{\rm BM}(K',B^n)>\tau_2 t$.

Q.E.D.

Now we consider the shape of a convex body with axial rotational symmetry near the poles. To test whether a convex body is "flat" near the poles, we will use the following statement.

Lemma 20. *There exist* $\delta_0,\tau_0,\tau\in(0,1)$ *depending on* n *with the following property. Let* $\delta\in(0,\delta_0)$, $t\in(0,\tau_0\delta)$, *and let a convex body* K *with* $\delta=\delta_{\rm BM}(K,B^n)$ *spin around* u. *If an o-symmetric ellipsoid* E *with axial rotational symmetry around* $\mathbb{R}u$ *satisfies that* $E\Delta K$ *contains no ball of the form* $x+t\,B^n$ *with* $|x\cdot u|\leq 1-t$, *then*

- *(i)* $K\subset(1+\tau t)E$;
- *(ii) assuming* $|x\cdot u|\leq 1-4t$, $x\in\partial E$ *implies* $(x+3tB^n)\cap K\neq\emptyset$, *and* $x\in\partial K$ *implies* $(x+3tB^n)\cap E\neq\emptyset$;
- *(iii)* $\theta t\in\partial E$ *where* $1+\frac{1}{2}\,\delta\leq\theta\leq 1+\tau\delta$.

Proof. We write $\gamma_1,\gamma_2,\dots$ to denote positive constants depending only on n.

For an $x\in\mathbb{R}^n$ with $|x\cdot u|\leq 1-4t$, we may assume that $x\cdot u\geq 0$. Let $v\in u^\perp$ such that $x\cdot v\geq 0$ and $x\in\text{lin}\{u,v\}$. Since $x+3tB^n$ contains $x-tu-tv+tB^n$, we deduce (ii) from the assumptions on E and K.

As K spins around u, and $\delta_{\rm BM}(K,B^n)=\delta$, we have

$$(1/2)B^n\subset(1-\gamma_1\delta)B^n\subset K\subset(1+\gamma_2\delta)B^n.$$

This combined with (ii) implies (i). In addition, we deduce from (ii) that

$$(1-\gamma_3t)K\subset\{x\in E:\,|x\cdot u|\leq 1-7t\}\subset(1+\gamma_4t)K,$$

which in turn yields that if $\theta t\in\partial E$ for $\theta>0$, then

$$\delta=\delta_{\rm BM}(K,B^n)\leq\ln\left[(1-\gamma_3t)^{-1}\cdot\theta(1-7t)^{-1}(1+\gamma_4t)\right]\leq\ln\theta+\gamma_5t.$$

Therefore, assuming $t<(2\gamma_5)^{-1}\delta$, we have $\theta\geq 1+\frac{\delta}{2}$. Q.E.D.

Corollary 21. *There exist $\delta_0,\tau_0\in(0,1)$ depending on n with the following property. Let $\delta\in(0,\delta_0)$, $t\in(0,\tau_0\delta)$, and let a convex body K with $\delta=\delta_{\rm BM}(K,B^n)$ spin around u. If an o-symmetric ellipsoid E with axial rotational symmetry around $\mathbb{R}u$ satisfies that $E\Delta K$ contains no ball of the form $x+t\,B^n$ with $|x\cdot u|\leq 1-t$, then*

$$(1-7t)u+(\sqrt{\delta}/4)v\in K.$$

Proof. By Lemma 20 (iii), we have $\theta u\in\partial E$ where $\theta>1+\frac{1}{2}\,\delta$. It follows that

$$\sqrt{1-\frac{(1-4t)^2}{\theta^2}}>\sqrt{1-\frac{1}{1+\delta}}>\sqrt{\delta}/2,$$

and hence

$$w=(1-4t)u+(\sqrt{\delta}/2)v\in E.$$

Thus, we obtain Corollary 21 from Lemma 20 (ii). Q.E.D.

If a convex body with axial rotational symmetry is “too flat” around the poles then we modify it in the following way.

Lemma 22. *If $\varepsilon\in(0,\varepsilon_0)$ for $\varepsilon_0\in(0,1)$ depending on n, and K is a convex body with $\delta_{\rm BM}(K,B^n)=\varepsilon$ spinning around u, then there exists a convex body K' that spins around u, and is obtained from K by combining linear transformations and one Schwarz rounding, such that for any o-symmetric ellipsoid E with axial rotational symmetry around $\mathbb{R}u$, one finds a ball of the form $x+4\varepsilon^2\,B^n$ in $E\Delta K'$, where $|x\cdot u|\leq 1-4\varepsilon^2$.*

Proof. In the following the implied constants in $O(\cdot)$ depend only on n, and we write $\gamma_1,\gamma_2,\dots$ to denote positive constants depending only on n. We assume that ε_0 depends only on n and is small enough to make the argument below work.

If for any o-symmetric ellipsoid E with axial rotational symmetry around $\mathbb{R}u$, one finds a ball of the form $x+\varepsilon^{3/2}\,B^n$ in $E\Delta K$ where $|(x\cdot u)|\leq 1-\varepsilon^{3/2}$, then we are done. Therefore let us assume that this is not the case, and hence there exists an o-symmetric ellipsoid E_0 with axial rotational symmetry around $\mathbb{R}u$ satisfying that $E_0 \Delta K$ contains no ball of the form $x + \varepsilon^{3/2} B^n$ with $|x \cdot u| \leq 1 - \varepsilon^{3/2}$. Let u be part of an orthonormal basis for $\mathbb{R}^n$, let Φ be the diagonal matrix that maps E_0 into B^n, and let $K_0 = \Phi K$.

By Lemma 20 (iii) applied to K and E_0, we have $\theta u \in \partial E_0$ where $1 + \frac{1}{2}\varepsilon < \theta < 1 + \gamma_1 \varepsilon$, and hence

$$(1-s)u \in \partial K_0, \quad \text{where } \tfrac{1}{4}\varepsilon < s < \gamma_2 \varepsilon.$$

In addition, Lemma 20 (i) and (ii) yield

$$\begin{aligned} K &\subset \left(1 + \gamma_3 \varepsilon^{3/2}\right) E_0, \\ (x + 3\varepsilon^{\frac{3}{2}} B^n) \cap K &\neq \emptyset \text{ for all } x \in \partial E_0 \text{ with } |x \cdot u| \leq 1 - 4\varepsilon^{3/2}. \end{aligned}$$

Thus, we deduce that

(59)

$$K_0 \subset \left(1 + \gamma_3 \varepsilon^{\frac{3}{2}}\right) B^n,$$

(60)

$$(x + 4\varepsilon^{\frac{3}{2}} B^n) \cap K \neq \emptyset \text{ for all } x \in S^{n-1} \text{ with } |x \cdot u| \leq 1 - s - 4\varepsilon^{\frac{3}{2}}.$$

Since $\frac{1}{4}\varepsilon < s < \gamma_2 \varepsilon$ implies

$$\sqrt{\left(1 + \gamma_3 \varepsilon^{\frac{3}{2}}\right)^2 - \left(1 - s - 8\varepsilon^{\frac{3}{2}}\right)^2} > \sqrt{2s} - \gamma_5 \varepsilon,$$

we deduce from (60) that

$$(1 - s - 8\varepsilon^{\frac{3}{2}})u + \left(\sqrt{2s} - \gamma_5 \varepsilon\right) v \in K_0. \tag{61}$$

We plan to apply Schwarz rounding of K_0 with respect to $\mathbb{R}u'$, where

$$u' = \sqrt{1-s}\, u + \sqrt{s}\, v.$$

It follows from $\sqrt{1-s} = 1 - \frac{1}{2}s + O(s^2)$, (59), and (61) that

$$1 - \left(\tfrac{3}{2} - \sqrt{2}\right) s - \gamma_6 \varepsilon^{\frac{3}{2}} \leq h_{K_0}(u') \leq 1 + \gamma_3 \varepsilon^{\frac{3}{2}}. \tag{62}$$

Next let

$$\varepsilon^{\frac{3}{2}}/2 < p < 2\varepsilon^{\frac{3}{2}},$$

let w be of the form $w = (1-s)\, u + t\, v$ with $w \cdot u' = h_{K_0}(u') - p$, and let $z = (h_{K_0}(u') - p)u'$. In addition, let ϱ be the radius of

$$G = H(u', h_{K_0}(u') - p) \cap (1 + \gamma_3 \varepsilon^{\frac{3}{2}}) B^n.$$

As $H(u', h_{K_0}(u') - p)$ cuts of a cap of depth at most $(\frac{3}{2} - \sqrt{2} + O(\varepsilon^{\frac{1}{2}})) \cdot s$ from $(1 + \gamma_3 \varepsilon^{\frac{3}{2}}) B^n$ by (62), and $\frac{3}{2} - \sqrt{2} = \frac{1}{2}(\sqrt{2} - 1)^2$, we have

$$\varrho \leq \left((\sqrt{2} - 1) + O(\varepsilon^{\frac{1}{2}})\right) \sqrt{s}.$$

In addition, for $y = \sqrt{1-s}\, u'$ (collinear with w and $(1-s)u$), we have

$$\|y - z\| \geq \left(\sqrt{2} - 1 - O(\varepsilon^{\frac{1}{2}})\right) s,$$

and therefore

$$\|w - z\| = \frac{\sqrt{1-s}}{\sqrt{s}}\, \|y - z\| \geq \left(\sqrt{2} - 1 - O(\varepsilon^{\frac{1}{2}})\right) \sqrt{s}.$$

Now $H(u, 1-s)$ cuts of a cap of depth

$$\varrho - \|w - z\| \leq O(\varepsilon^{\frac{1}{2}})\sqrt{s} = O(\varepsilon)$$

from G, and this cap contains $H(u', h_{K_0}(u') - p) \cap K_0$. We deduce that

$$\mathcal{H}\left(H(u', h_{K_0}(u') - p) \cap K_0\right) \leq O(\varepsilon)(\varepsilon\, \varrho)^{\frac{n-2}{2}} \leq O(\varepsilon^{\frac{1}{4}})\varepsilon^{\frac{3(n-1)}{4}}.$$

Let K_1 be the Schwarz rounding of K_0 around $\mathbb{R}u'$, and let K' be the convex body spinning around u that is the image of K_1 by a linear transformation that maps $h_{K_1}(u')u'$ into u, and $K_1 \cap u'^{\perp}$ into $B^n \cap u^{\perp}$. Thus K' satisfies

$$\mathcal{H}\left(H(u, 1 - \varepsilon^{\frac{3}{2}}) \cap K'\right) \leq O(\varepsilon^{\frac{1}{4}})\varepsilon^{\frac{3(n-1)}{4}}.$$

We conclude that $\delta_{\rm BM}(K', B^n) \geq \gamma_6 \varepsilon^{\frac{3}{2}}$ on the one hand, and

$$(1 - \varepsilon^{\frac{3}{2}})u + \gamma_7 \varepsilon^{\frac{1}{4(n-1)}} \cdot \varepsilon^{\frac{3}{4}} v \not\in K' \tag{63}$$

on the other hand.

Next we suppose that there exists some o-symmetric ellipsoid E with axial rotational symmetry around $\mathbb{R}u$, such that no ball of the form $x + 4\varepsilon^2 B^n$ with $|x \cdot u| \leq 1 - 4\varepsilon^2$ is contained in $E \Delta K'$. By Lemma 21 and $\delta_{\rm BM}(K', B^n) \geq \gamma_6 \varepsilon^{\frac{3}{2}}$, we have

$$(1 - 28\varepsilon^2)u + \gamma_8 \varepsilon^{\frac{3}{4}}\, v \in K'. \tag{64}$$

If ε_0 is small enough, then (63) contradicts (64), completing the proof of Lemma 22.

Q.E.D.

Next, strengthening Lemma 22, we are even more specific about the shape of the o-symmetric convex body with axial rotational symmetry near the poles.

Proposition 23. *If $\varepsilon \in (0, \varepsilon_0)$ for $\varepsilon_0 \in (0,1)$ depending on n, and K is a convex body spinning around u such that $\delta_{\rm BM}(K, B^n) = \varepsilon$, then there exists a convex body K' that spins around u, and is obtained from K by combining linear transformations and two Schwarz roundings, such that*

(i) *for any o-symmetric ellipsoid E with axial rotational symmetry around $\mathbb{R}u$, one finds a ball $x + 2\varepsilon^2 B^n \subset E\Delta K'$ where $|x \cdot u| \leq 1 - 2\varepsilon^2$;*

(ii) $(1 - \varepsilon^{32})u + \varepsilon^3 v \not\in K'$.

Proof. In the following the implied constants in $O(\cdot)$ depend only on n. We assume that ε_0 depends only on n and is small enough to make the argument below work.

According to Lemma 22, there exists a convex body K_0 that spins around u, and is obtained from K by combining linear transformations and a Schwarz rounding, such that for any o-symmetric ellipsoid E with axial rotational symmetry around $\mathbb{R}u$ and $E\cap u^\perp = B^n\cap u^\perp$, one finds a ball of the form $x+2\varepsilon^4 B^n$ in $E\Delta K_0$ where $|x\cdot u|\leq 1-4\varepsilon^2$. If $(1-\varepsilon^{32})u+\varepsilon^3 v\notin K_0$, then we may take $K'=K_0$. Therefore we assume that

$$(1-\varepsilon^{32})u+\varepsilon^3 v\in K_0. \tag{65}$$

To obtain K', first we apply Schwarz rounding around $\mathbb{R}u'$ for the unit vector

$$\tilde{u}=\sqrt{1-\varepsilon^{32}}\,u+\varepsilon^{16}\,v$$

to get a convex body $\widetilde{K}$. Then we set $K'=\widetilde{\Phi}\widetilde{K}$ where $\widetilde{\Phi}$ is a linear transform that maps $h_{\widetilde{K}}(\tilde{u})\tilde{u}=h_{K_0}(\tilde{u})\tilde{u}$ into u, and $\widetilde{K}\cap\tilde{u}^\perp$ into $B^n\cap u^\perp$.

Since $\delta_{\rm BM}(K_0,B^n)\leq\varepsilon$, we have

$$K_0,\widetilde{K}\subset(1+O(\varepsilon))B^n. \tag{66}$$

It follows from (65) and (66) that

$$1\leq h_{K_0}(\tilde{u})=h_{\widetilde{K}}(\tilde{u})\leq 1+O(\varepsilon). \tag{67}$$

For any $s\in(0,1)$, let $r(s)$ and $\tilde{r}(s)$ be the radii of $K\cap H(u,s)$ and $\widetilde{K}\cap H(\tilde{u},s)$, respectively. We claim that

$$\tilde{r}(s)=r(s)+O(\varepsilon^{14})\ \text{ if } s\leq 1-4\varepsilon^2. \tag{68}$$

For a fixed $s\in(0,1-4\varepsilon^2]$, let $s_1<s_2$ such that

$$[s_1,s_2]u=\pi_{\mathbb{R}u}\left[K_0\cap H(\tilde{u},s)\right].$$

Since $K_0\subset B^n+\mathbb{R}u$, it follows that

$$s-2\varepsilon^{16}<s_1<s_2<s+2\varepsilon^{16}. \tag{69}$$

Since $1-s\geq 4\varepsilon^2$ and $u\in K_0$, we deduce that

$$\|z-s\tilde{u}\|=r(s)+O(\varepsilon^{14})\ \text{ for any } z\in\partial K_0\cap H(\tilde{u},s),$$

which in turn yields (68).

Now let E be any o-symmetric ellipsoid having $\mathbb{R}u$ as an axis of rotation. For some orthogonal linear transform Φ_* that maps $\tilde{u}$ into u, we consider the o-symmetric ellipsoid $E_*=\Phi_*^{-1}\widetilde{\Phi}^{-1}E$ having again $\mathbb{R}u$ as an axis of rotation. We know that there exists x_* such that $x_*+4\varepsilon^2B^n\subset K_0\Delta E_*$ and $x_*\cdot u\leq 1-4\varepsilon^2$. It follows from (68) that for $\tilde{x}=\Phi_*x_*$ and $\widetilde{E}=\Phi_*E_*$, we have $\tilde{x}+3\varepsilon^2\subset\widetilde{K}\Delta\widetilde{E}$ and $\tilde{x}\cdot\tilde{u}\leq 1-4\varepsilon^2$. We conclude using (67) and (68) that $x+2\varepsilon^2\,B^n\subset E\Delta K'$ and $|x\cdot u|\leq 1-2\varepsilon^2$ for $x=\widetilde{\Phi}\tilde{x}$, verifying (i).

To prove (ii), let
$$\varepsilon^{32}/4 < p < 4\varepsilon^{32}.$$
If $tu \in H(\tilde{u}, h_{K_0}(\tilde{u}) - p) \cap \operatorname{int} K_0$ for $t > 0$, then $H(\tilde{u}, h_{K_0}(\tilde{u}) - p)$ cuts of a cap of depth at most $p/\varepsilon^{16} < 4\varepsilon^{16}$ from $H(u,t) \cap K_0$, and hence $H(\tilde{u}, h_{K_0}(\tilde{u}) - p) \cap K_0 \cap H(u,t)$ is an $(n-2)$-ball of radius at most $O(\varepsilon^8)$. As $K_0 \subset 2B^n$, we deduce that
$$\begin{aligned}\mathcal{H}\left(H(\tilde{u}, h_{K_0}(\tilde{u}) - p) \cap \widetilde{K}\right) &= \mathcal{H}\left(H(\tilde{u}, h_{K_0}(\tilde{u}) - p) \cap K_0\right) \\ &= O(\varepsilon^{8(n-2)}) = O(\varepsilon^{4(n-1)}),\end{aligned}$$
and thus for $\tilde{v} \in S^{n-1} \cap \tilde{u}^{\perp}$, we have
$$(h_{K_0}(\tilde{u}) - p)\tilde{u} + \gamma\varepsilon^4\tilde{v} \notin \widetilde{K},$$
where $\gamma > 0$ depends on n. We conclude, again using (67) and (68), that
$$(1-q)u + 2\gamma\varepsilon^4 v \notin K' \ \text{ for any } q \in (\varepsilon^{32}/2, 2\varepsilon^{32}),$$
which in turn yields (ii). Q.E.D.

Finally, we are in a position to prove Theorem 14.

Proof of Theorem 14*:* We assume that δ_0 (and hence δ, as well) is small enough to make the estimates below work. We write $\gamma_1, \gamma_2, \ldots$ to denote positive constants depending only on n.

According to Lemma 18 and Proposition 23, there exists a convex body K_1 spinning around u and obtained from K by a combination of Steiner symmetrizations, linear transformations and taking limits, such that for some $\eta \in (\delta^3, \delta]$, we have $\delta_{\mathrm{BM}}(K_1, B^n) \le \eta$, and

: (a) for any o-symmetric ellipsoid E with axial rotational symmetry around $\mathbb{R}u$, one finds a ball $x + 2\eta^2 B^n \subset E \Delta K_1$ where $|x \cdot u| \le 1 - 2\eta^2$;

: (b) $(1-\eta^{32})u + \eta^3 v \notin K_1$.

In particular,
$$\delta_{BM}(K_1, B^n) \ge \gamma_1 \eta^2.$$
If
$$\delta^3 + (1-\delta^7)\, v \notin K_1,$$
then we simply take $\varepsilon = \eta$ and $K' = K_1$. If
$$\delta^3 + (1-\delta^7)\, v \in K_1,$$
then let K_2 be the Schwarz rounding of K_1 around $\mathbb{R}v$, and hence $\delta_{\mathrm{BM}}(K_2, B^n) \ge \gamma_2 \eta^7$ by Lemma 19. For $\varepsilon = \delta_{\mathrm{BM}}(K_2, B^n)$, we have
$$\delta^{24} \le \delta_{\mathrm{BM}}(K_2, B^n) = \varepsilon \le \delta.$$

Since $K_1 \subset (1+\gamma_2\varepsilon)B^n$ and K_1 spins around u, if $t\in(0,\varepsilon)$, then

$$\begin{aligned}\mathcal{H}\left(K_1\cap H(v,1-t)\right) &\leq \gamma_3\varepsilon^{1/2}\mathcal{H}\left(B^n\cap u^\perp\cap H(v,1-t)\right)\\ &\leq \gamma_4\varepsilon^{1/2}t^{\frac{n-2}{2}},\\ \mathcal{H}\left(H(v,t)\cap K_1\right) &\leq (1-\gamma_5t^2)\mathcal{H}\left(H(v,0)\cap K_1\right).\end{aligned}$$

Using that $\frac{n-2}{2(n-1)}\geq\frac14$ for $n\geq 3$, we have

$$\begin{aligned}\gamma_6\varepsilon^{\frac{1}{2(n-1)}}t^{1/4}u+(1-t)\,v &\notin K_2,\\ (1-\gamma_7t^2)u+t\,v &\notin K_2.\end{aligned}$$

We transform K_2 into a convex body K' spinning around u by a linear map, which sends v into u, and $v^\perp\cap K_2$ into $u^\perp\cap B^n$. We deduce that if $t\in(0,\varepsilon/2)$, then

$$(1-t)\,u+\gamma_8\varepsilon^{\frac{1}{2(n-1)}}t^{1/4}\,v \notin K', \tag{70}$$

$$t\,u+(1-\gamma_9t^2)v \notin K'. \tag{71}$$

In (71), we choose t such that $\varepsilon^7=\gamma_9t^2$, and hence

$$\varepsilon^3\,u+(1-\varepsilon^7)\,v\notin K'.$$

We also deduce by substituting $t>0$ with $\varepsilon^3=\gamma_8\varepsilon^{\frac{1}{2(n-1)}}t^{1/4}$ in (70) that

$$(1-\varepsilon^{32})u+\varepsilon^3v\notin K'.$$

Finally suppose that for some o-symmetric ellipsoid E with axial rotational symmetry around $\mathbb{R}u$, there is no ball of the form $x+2\varepsilon^2\,B^n$ in $E\Delta K'$, where $|x\cdot u|\leq 1-2\varepsilon^2$. It follows from Corollary 21 that

$$(1-14\varepsilon^2)\,u+\gamma_{10}\varepsilon^{1/2}\,v\notin K'. \tag{72}$$

If δ_0 is small enough, then substituting $t=14\varepsilon^2$ in (70) contradicts (72). Therefore K' satisfies all requirements of Theorem 14. Q.E.D.

References

[1] D. Alonso-Gutierrez, J. Bastero & J. Bernués, *Factoring Sobolev inequalities through classes of functions*, Proc. Amer. Math. Soc., **140** (2012), 3557–3566, MR 2929024.

[2] G. Ambrus & K.J. Böröczky, *Stability results for the volume of random simplices*, arXiv:1005.5024, submitted.

[3] W. Blaschke, *Kreis und Kugel*, Veit, Leipzig 1916, Chelsea, New York 1948, 1971, de Gruyter, Berlin 1956, MR 0076364, Zbl 0070.17501.

[4] T. Bonnesen & W. Fenchel, *Theory of convex bodies*, BCS Associates, 1987, MR 0920366, Zbl 0628.52001.

[5] J. Bourgain & J. Lindenstrauss, *Projection bodies*, Geometric aspects of functional analysis (1986), Lecture Notes in Math., 1317, Springer, Berlin, 1988, 250–270. MR 0950986, Zbl 0645.52002.

[6] K.J. Böröczky, *Stability of the Blaschke-Santaló and the affine isoperimetric inequality*, Adv. Math. **225** (2010), 1914–1928, MR 2680195, Zbl 1216.52007.

[7] S. Campi, *Recovering a centred convex body from the areas of its shadows: a stability estimate*, Ann. Mat. Pura Appl. (4), **151** (1988), 289–302, MR 0964515, Zbl 0657.52001.

[8] A. Cianchi, E. Lutwak, D. Yang & G. Zhang, *Affine Moser-Trudinger and Morrey-Sobolev inequalities*, Calc. Var. Partial Differential Equations, **36** (2009), 419–436, MR 2551138, Zbl 1202.26029.

[9] G. Ewald, D.G. Larman & C.A. Rogers, *The directions of the line segments and of the r-dimensional balls on the boundary of a convex body in Euclidean space*, Mathematika, **17** (1970), 1–20, MR 0270271, Zbl 0199.57002.

[10] A. Figalli, F. Maggi & A. Pratelli, *A refined Brunn-Minkowski inequality for convex sets*, Annales de IHP (C) Non Linear Analysis, **26** (2009), 2511–2519, MR 2569906, Zbl 1192.52015.

[11] A. Figalli, F. Maggi & A. Pratelli, *A mass transportation approach to quantitative isoperimetric inequalities*, Inventiones Mathematicae, **182**, Number 1, (2010), 167–211, MR 2672283, Zbl 1196.49033.

[12] B. Fleury, O. Guédon & G. Paouris, *A stability result for mean width of L_p-centroid bodies*, Adv. Math., **214** (2007), 865–877, MR 2349721, Zbl 1132.52012.

[13] N. Fusco, F. Maggi & A. Pratelli, *The sharp quantitative isoperimetric inequality*, Ann. of Math., **168** (2008), 941–980, MR 2456887, Zbl 1187.52009.

[14] R.J. Gardner & G. Zhang, *Affine inequalities and radial mean bodies*, Amer. J. Math., **120** (1998), 505–528, MR 1623396, Zbl 0908.52001.

[15] A. Giannopoulos, *Notes on Isotropic Convex Bodies*, Preprint, http://users.uoa.gr/∼apgiannop/

[16] A. Giannopoulos & V.D. Milman, *Asymptotic convex geometry: short overview*, Different faces of geometry, Int. Math. Ser. (N. Y.), Kluwer/Plenum, New York, 2004, 87–162, MR 2102995, Zbl 1068.52011.

[17] H. Groemer, *Stability of geometric inequalities*, in P.M. Gruber, J.M. Wills, eds., *Handbook of convex geometry*, North-Holland, Amsterdam, 1993, 125–150, MR 1242978, Zbl 0789.52001.

[18] M. Gromov & V. Milman, *Generalization of the spherical isoperimetric inequality to uniformly convex Banach Spaces*, Composito Math., **62** (1987), 263–282, MR 0901393, Zbl 0623.46007.

[19] P.M. Gruber, *Stability of Blaschke's characterization of ellipsoids and Radon norms*, Discrete Comput. Geom., **17** (1997), 411–427, MR 1455690, Zbl 0887.52002.

[20] P.M. Gruber, *Convex and discrete geometry*, Springer, Berlin, 2007, MR 2335496, Zbl 1139.52001.

[21] C. Haberl & F.E. Schuster, *General L_p affine isoperimetric inequalities*, J. Differential Geom., **83** (2009), 1–26, MR 2545028, Zbl 1185.52005.

[22] C. Haberl & F.E. Schuster, *Asymmetric affine L_p Sobolev inequalities*, J. Funct. Anal., **257** (2009), no. 3, 641–658. MR 2530600, Zbl 1180.46023.

[23] C. Haberl, F.E. Schuster & J. Xiao, *An asymmetric affine Pólya-Szegő principle*, Math. Ann., **352** (2012), 517–542, MR 2885586, Zbl 1241.26014.

[24] D. Hug & R. Schneider, *Large faces in Poisson hyperplane mosaics*, Ann. Probab., **38** (2010), 1320–1344, MR 2675001, Zbl 1202.60021.

[25] F. John, *Polar correspondence with respect to a convex region*, Duke Math. J., **3** (1937), 355–369, MR 1545993, Zbl 0017.03701.

[26] M. Kiderlen, *Stability results for convex bodies in geometric tomography*, Indiana Univ. Math. J., **57** (2008), 199–2038, MR 2463960, Zbl 1161.52004.

[27] M. Ludwig, *Minkowski valuations*, Trans. Amer. Math. Soc., **357** (2005), 4191–4213, MR 2159706, Zbl 1077.52005.

[28] M. Ludwig, *Minkowski areas and valuations*, J. Differential Geom., **86** (2010), 133–161, MR 2772547, Zbl 1215.52004.

[29] E. Lutwak, *Selected affine isoperimetric inequalities*, in *P.M. Gruber, J.M. Wills, eds., Handbook of convex geometry*, North-Holland, Amsterdam, 1993, 151–176, MR 1242979, Zbl 0847.52006.

[30] E. Lutwak, *The Brunn-Minkowski-Firey theory, I. Mixed volumes and the Minkowski problem*, J. Differential Geom., **38** (1993), 131–150, MR 1231704, Zbl 0788.52007.

[31] E. Lutwak, D. Yang & G. Zhang, *L_p affine isoperimetric inequalities*, J. Differential Geom., **56** (2000), 111–132. MR 1863023, Zbl 1034.52009.

[32] E. Lutwak, D. Yang & G. Zhang, *Sharp affine L_p Sobolev inequalities*, J. Differential Geom., **62** (2002), 17–38, MR 1987375, Zbl 1073.46027.

[33] E. Lutwak, D. Yang & G. Zhang, *Optimal Sobolev norms and the L_p Minkowski problem*, Int. Math. Res. Not., 2006, Art. ID 62987, 21 pp. MR 2211138, Zbl 1110.46023.

[34] E. Lutwak, D. Yang & G. Zhang, *Orlicz projection bodies*, Adv. Math., **223** (2010), 220–242, MR 2563216, Zbl pre05643962.

[35] F. Maggi, *Some methods for studying stability in isoperimetric type problems*, Bull. Amer. Math. Soc., **45** (2008), 367–408, MR 2402947, Zbl 1146.49037.

[36] V.D. Milman & A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n–dimensional space*, in *J. Lindenstrauss, V.D. Milman, Geometric Aspects of Functional Analysis*, Lec. Notes Math. 1376, Springer, (1989), 64–104. MR 1008717, Zbl 0679.46012.

[37] A. Naor, *The Surface Measure and Cone Measure on the Sphere of l_p^n, Trans*, Amer. Math. Soc., **359** (2007), 1045–1079, MR 2262841, Zbl 1109.60006,

[38] R. Schneider, *Zu einem Problem von Shepard über die Projektionen konvexer Körper*, Math. Z., **101** (1967), 71–82, MR 0218976, Zbl 0173.24703.

[39] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge University Press, 1993, MR 1216521, Zbl 0798.52001.

[40] M. Schmuckenschläger, *Petty's projection inequality and Santalo's affine isoperimetric inequality*, Geom. Dedicata, **57** (1995), 285–295, MR 1351856, Zbl 0835.52012.

[41] F.E. Schuster & T. Wannerer, *$GL(n)$ contravariant Minkowski valuations*, Trans. Amer. Math. Soc., **364** (2012), 815–826, MR 2846354, Zbl 1246.52009.

[42] J. Xiao, *The sharp Sobolev and isoperimetric inequalities split twice*, Adv. Math., **211** (2007), 417–435, MR 2323533, Zbl 1125.26026.

[43] G. Zhang, *The affine Sobolev inequality*, J. Differential Geom., **53** (1999), 183–202, MR 1776095, Zbl 1040.53089.

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