

BOUNDED CHARACTERISTIC CLASSES AND FLAT BUNDLES

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Abstract

Let G be a connected Lie group. We show that all characteristic classes of G are bounded—when viewed in the cohomology of the classifying space of the group G with the discrete topology—if and only if the derived group of the radical of G is simply connected in its Lie group topology. We also give equivalent conditions in terms of stable commutator length and distortion.

1. Introduction

Let G be a connected Lie group, and denote by G^δ be the underlying group with the discrete topology. Let A be an abelian group endowed with a metric. The identity map $G^\delta \rightarrow G$ induces a natural ring homomorphism

$$H^*(BG, A) \rightarrow H^*(BG^\delta, A),$$

from the (singular) cohomology of the classifying space BG of G to the cohomology of BG^δ . A class α in $H^n(BG^\delta, A) = H^n(G, A)$ is *bounded* if it can be represented by a cocycle $c : G^n \rightarrow A$ with bounded image. We define a subadditive function on $H^n(G, A)$ by

$$\|\alpha\|_\infty = \inf_{[c]=\alpha} \{\sup\{|c(g_1, \dots, g_n)|; g_1, \dots, g_n \in G\}\} \in \mathbf{R} \cup \{\infty\}.$$

When $A = \mathbf{R}$ with its usual metric, it corresponds to Gromov's seminorm [12, §1.1] and was first considered by Dupont [10] in degree 2. If A is finitely generated, we use the word length with respect to a finite symmetric generating set to define a metric. On \mathbf{R}^r we consider the Euclidean metric, and when $A \cong \mathbf{Z}^r \subset \mathbf{R}^r$ is discrete we also consider the restriction of the Euclidean metric to A .

The image of the natural map $H^*(BG, \mathbf{R}) \rightarrow H^*(BG^\delta, \mathbf{R})$ is generated as a ring by bounded classes together with the image in degree two [8, Lemma 51, Theorem 54].

If a class α in the image of $H^2(BG, \mathbf{R}) \rightarrow H^2(BG^\delta, \mathbf{R})$ is bounded, then for any continuous map $\phi : \Sigma_g \rightarrow BG^\delta$ of a closed oriented surface

of genus $g \geq 1$, the characteristic number $\phi^*(\alpha)([\Sigma_g]) \in \mathbf{R}$ belongs to a bounded set $[-C, C]$ with $C = C(\alpha, g)$ depending only on α and g . Indeed,

$$|\phi^*(\alpha)([\Sigma_g])| \leq \|\alpha\|_\infty \cdot \|[\Sigma_g]\|_1 = \|\alpha\|_\infty \cdot (4g - 4).$$

(For the classical bound $C = g - 1$ on the Euler number of flat $\mathrm{GL}_2^+(\mathbf{R})$ -bundles over a surface of genus $g > 0$, see Milnor [15]. See also Gromov [12, 0.3 and 1.3] and Bucher and Monod [6], for bounds on the Euler class of flat bundles in higher dimension, as well as Smillie [17] for examples of flat manifolds M^{2n} with non-zero Euler characteristics.) If, on the contrary, there exists a sequence $(\phi_n : \Sigma_g \rightarrow BG^\delta)$ of flat G -bundles with

$$\lim_{n \rightarrow \infty} \phi_n^*(\alpha)([\Sigma_g]) = \infty,$$

then α is unbounded. (A family of maps $\phi_n : S^1 \times S^1 \rightarrow BQ^\delta$, $n \geq 1$, where Q is the quotient of the three-dimensional Heisenberg group by an infinite cyclic central subgroup, such that the sequence $\phi_n^*(\alpha)([S^1 \times S^1])$ is unbounded, where $\alpha \in H^2(BQ^\delta, \mathbf{R})$ is the image of a generator of $H^2(BQ, \mathbf{Z}) \cong \mathbf{Z}$, is given in Goldman [11].)

In the present work we give a necessary and sufficient condition on G for the classes in the image of $H^*(BG, \mathbf{R}) \rightarrow H^*(BG^\delta, \mathbf{R})$ to be bounded. Previously known sufficient conditions were: G is linear algebraic (Gromov [12, Theorem p. 23], resp. Bucher [5]), and the weaker condition that the radical R of G is linear [8, Theorem 54]; the radical of G is the largest connected, normal, solvable subgroup of G . It is well-known that R is linear if and only if the *closure* of its derived subgroup $[R, R]$ is simply connected. Hence linearity of R is stronger than simple connectedness of $[R, R]$ (see the discussion after Theorem 2.1).

Our main theorem shows that simple connectedness of $[R, R]$ is the weakest possible condition implying boundedness of characteristic classes.

Theorem 1.1. *Let G be a connected Lie group, and let R be its radical. The following conditions are equivalent.*

- 1) *All elements in the image of $H^*(BG, \mathbf{R}) \rightarrow H^*(BG^\delta, \mathbf{R})$ are bounded.*
- 2) *The derived group $[R, R]$ of the radical of G is simply connected.*

In Section 2 we recall the definitions of stable commutator length and of subgroup distortion; the equivalent conditions stated in the above theorem can also be expressed in those terms (Theorem 2.2). We also recall and discuss some background results about Borel cohomology that are closely related to Theorem 2.2; we illustrate the discussion with an example. Section 3 deals with the primary obstruction to the existence of a global section of the universal G -bundle; this is the main tool for what follows. In Section 4 we prove the existence of a non-zero lower bound on the stable commutator length of all elements in the

commutator subgroup of the universal cover of G that are central and whose stable commutator length is non-zero; this is a key point in the proof of Theorem 2.2. Section 5 is devoted to the proof of Theorem 2.2. Theorem 1.1 is obtained as a corollary of Theorem 2.2.

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2. Stable commutator length, distortion, Borel cohomology

Let G be a group and $[G, G] < G$ its commutator subgroup (derived subgroup). The commutator length $\text{cl}(z)$ of $z \in [G, G]$ is the smallest number of commutators needed to express z . The commutator length is subadditive, and the *stable commutator length* of $z \in [G, G]$ is defined as

$$\text{scl}_G(z) = \lim_{n \rightarrow \infty} \text{cl}(z^n)/n.$$

(We will drop the subscript and simply write $\text{scl}(z)$, except when we want to emphasize the reference to G .) It is subadditive on commuting elements.

Let H be a group generated by a finite symmetric set S . The word length $|h|$ of $h \in H$ is the smallest number of elements of S needed to express h . If (G, d) is a group with a left-invariant metric d , a finitely generated subgroup H of G is *distorted* in (G, d) if

$$\inf_{h \in H \setminus \{e\}} \frac{d(e, h)}{|h|} = 0.$$

Being distorted does not depend on the choice of the generating set S . An element $z \in G$ is *distorted* if the subgroup it generates is distorted in (G, d) . (Notice that torsion elements—including by convention the identity element—are *not* distorted.) In the case G is a connected Lie group and d is the geodesic metric on G induced by a left-invariant Riemannian metric, a finitely generated subgroup H of G , which is distorted with respect to d , is distorted with respect to any left-invariant Riemannian metric.

A *Borel n -cochain* on a topological group G with values in \mathbf{Z} is a map

$$c : G^n \rightarrow \mathbf{Z},$$

such that the inverse image of any subset of \mathbf{Z} is a Borel set (i.e., an element of the σ -algebra generated by the open subsets of G^n endowed

with its product topology). The usual differentials send Borel cochains to Borel coboundaries. We denote by

$$H_B^*(G, \mathbf{Z})$$

the corresponding cohomology group (see [8, §3] for details and references). A *Borel class* is an element of $H_B^*(G, \mathbf{Z})$. A Borel class is bounded if it can be represented by a Borel cocycle with bounded image.

The fundamental group $\pi_1(G)$ of a connected Lie group G embeds naturally in the universal cover \tilde{G} of G . The following theorem gives a necessary and sufficient condition in terms of integral Borel cohomology of G for $\pi_1(G)$ to be undistorted in \tilde{G} .

Theorem 2.1 ([8, Theorem 1]). *Let G be a connected Lie group. The following conditions are equivalent.*

- 1) *Each Borel cohomology class of G with \mathbf{Z} -coefficients can be represented by a Borel bounded cocycle.*
- 2) *The radical of G is linear.*
- 3) *The natural inclusion $\pi_1(G) \rightarrow \tilde{G}$ of the fundamental group of G into the universal cover of G is undistorted.*

Let us explain how each equivalent condition in the above theorem can be weakened, and how this leads to the main statement of the paper.

About integral Borel cohomology, we recall that there is a commutative square of natural morphisms

$$\begin{array}{ccc} H_B^*(G, \mathbf{Z}) & \longrightarrow & H^*(BG, \mathbf{Z}) \\ \downarrow & & \downarrow \\ H_B^*(G^\delta, \mathbf{Z}) & \longrightarrow & H^*(BG^\delta, \mathbf{Z}), \end{array}$$

where the horizontal arrows are isomorphisms [19].

It may happen that a Borel cohomology class of a Lie group that can't be represented by a Borel bounded cocycle can, when viewed as a cohomology class of the underlying discrete group, be represented by a bounded group cocycle. It is not difficult to check that this is the case with the class defined by the central extension

$$\{0\} \rightarrow \mathbf{Z} \rightarrow S^1 \times H \rightarrow G \rightarrow \{1\},$$

where G is the quotient of the product $S^1 \times H$ of the circle with the three-dimensional Heisenberg group by the central cyclic subgroup \mathbf{Z} generated by (θ, z) , where $\theta \in S^1$ has infinite order and where $z \in H$ is central non-trivial.

About the radical R of G , notice that

$$\pi_1([R, R]) \subset \pi_1(\overline{[R, R]}),$$

and hence the condition

$$\pi_1([R, R]) = \{0\}$$

is weaker than the linearity of R , which is equivalent to

$$\pi_1(\overline{[R, R]}) = \{0\}.$$

The quotient $G = (S^1 \times H)/\mathbf{Z}$ defined above, which is solvable hence coincides with its radical R has the property that

$$\{0\} = \pi_1([R, R]) \subsetneq \pi_1(\overline{[R, R]}) \cong \mathbf{Z} \times \mathbf{Z}.$$

About distortion, it may happen that a subgroup is distorted even though each of its element is undistorted. This is the case with the fundamental group of $G = (S^1 \times H)/\mathbf{Z}$ viewed as a subgroup of the universal cover of G .

The proof of the following theorem, which is the main statement of the paper, will be given in Section 5.

Theorem 2.2. *Let G be a connected Lie group, and let \tilde{G} be its universal cover. Let R be the radical of G . The following conditions are equivalent.*

- 1) *All elements in the image of $H^*(BG, \mathbf{Z}) \rightarrow H^*(BG^\delta, \mathbf{Z})$ are bounded.*
- 2) $\pi_1([R, R]) = \{0\}$.
- 3) *All elements $z \in \pi_1(G)$ are undistorted in \tilde{G} .*
- 4) *If $z \in \pi_1(G) \cap [\tilde{G}, \tilde{G}]$, then either $\text{scl}_{\tilde{G}}(z) > 0$ or z has finite order.*

In the statement of this theorem we used integer-valued group cohomology. It is well-known that all classes in the image of $H^*(BG, \mathbf{R}) \rightarrow H^*(BG^\delta, \mathbf{R})$ are bounded if and only if all classes in the image of $H^*(BG, \mathbf{Z}) \rightarrow H^*(BG^\delta, \mathbf{Z})$ are bounded; we recall the proof for the convenience of the reader. As the diagram of natural maps

$$\begin{array}{ccc} H^*(BG, \mathbf{Z}) & \longrightarrow & H^*(BG, \mathbf{R}) \\ \downarrow & & \downarrow \\ H^*(BG^\delta, \mathbf{Z}) & \longrightarrow & H^*(BG^\delta, \mathbf{R}) \end{array}$$

commutes and as $H^*(BG, \mathbf{Z}) \otimes \mathbf{R} \cong H^*(BG, \mathbf{R})$, the boundedness of integral classes implies the boundedness of the real ones. The converse also holds true because an integral class is bounded if and only if it is bounded when considered as a real class (apply [8, Lemma 29] to G^δ). As a result, Theorem 2.2 implies Theorem 1.1.

3. Primary obstruction in degree 2

Let G be a connected Lie group, and let $\pi : P \rightarrow B$ be a principal G -bundle over a connected CW -complex B . We denote by

$$o(P) \in H^2(B, \pi_1(G))$$

the primary obstruction to the existence of a section for π .

Let $z \in \pi_1(G)$ be an element of the commutator subgroup of the universal cover of G . Let $g = \text{cl}(z)$ be its commutator length, and let $x_1, \dots, x_g, y_1, \dots, y_g$ be in the universal cover such that

$$\prod_{n=1}^g [x_n, y_n] = z.$$

The $2g$ elements of G obtained by projecting $x_1, \dots, x_g, y_1, \dots, y_g$ to G define a homomorphism $\pi_1(\Sigma_g) \rightarrow G$ of the fundamental group of the closed oriented surface Σ_g of genus g to G . Let $P \rightarrow \Sigma_g$ be the associated flat G -bundle over Σ_g . By construction,

$$(1) \quad o(P)([\Sigma_{\text{cl}(z)}]) = z$$

(see Milnor [15, Lemma 2] and Goldman [11]).

Let $EG \rightarrow BG$ be the universal G -bundle. Since G is connected, BG is simply connected and therefore for any abelian group A there are natural isomorphisms

$$\begin{aligned} H^2(BG, A) &\cong \text{Hom}(H_2(BG, \mathbf{Z}), A) \\ &\cong \text{Hom}(\pi_2(BG), A) \\ &\cong \text{Hom}(\pi_1(G), A). \end{aligned}$$

In the case of $A = \pi_1(G)$, the universal class $o(EG)$ corresponds to $\text{id}_{\pi_1(G)} \in \text{Hom}(\pi_1(G), A)$.

Let $o^\delta(EG)$ denote the image of $o(EG)$ under the canonical map $H^2(BG, \pi_1(G)) \rightarrow H^2(BG^\delta, \pi_1(G))$.

Proposition 3.1. *Let G be a connected Lie group. The following conditions are equivalent.*

- 1) *The class $o^\delta(EG)$ is bounded.*
- 2) *All classes in the image of $H^2(BG, \mathbf{Z}) \rightarrow H^2(BG^\delta, \mathbf{Z})$ are bounded.*

Proof. Let $x \in H^2(BG, \mathbf{Z}) \cong \text{Hom}(\pi_1(G), \mathbf{Z})$ with corresponding homomorphism $\phi_x : \pi_1(G) \rightarrow \mathbf{Z}$. By construction, the induced coefficient homomorphism $(\phi_x)_* : H^2(BG, \pi_1(G)) \rightarrow H^2(BG, \mathbf{Z})$ maps $o(EG)$ to x . If x^δ denotes the image of x in $H^2(BG^\delta, \mathbf{Z})$, we conclude by naturality that $(\phi_x)_*(o^\delta(EG)) = x^\delta$. Because ϕ_x maps bounded sets to bounded sets, we conclude that (1) implies (2). Conversely, because $\pi_1(G)$ is a finitely generated abelian group, (2) implies that all classes

in the image of $H^2(BG, \pi_1(G)) \rightarrow H^2(BG^\delta, \pi_1(G))$ are bounded, and in particular, that the universal class $o^\delta(EG)$ is bounded. q.e.d.

If $x \in H^*(BG^\delta, A)$, we denote $x_{\mathbf{R}}$ its image in $H^*(BG^\delta, A \otimes \mathbf{R})$. The class $o_{\mathbf{R}}^\delta(EG)$ lies in the subgroup $H^2(BG^\delta, \pi_1([G, G]) \otimes \mathbf{R})$.

Proposition 3.2. *Let $p : G \rightarrow Q$ be a finite homomorphic cover of connected Lie groups. Then $p^* : H^*(BQ^\delta, \mathbf{R}) \rightarrow H^*(BG^\delta, \mathbf{R})$ is an isomorphism and maps the subgroup of bounded classes of $H^*(BQ^\delta, \mathbf{R})$ bijectively onto the subgroup of bounded classes in $H^*(BG^\delta, \mathbf{R})$. As a result, $o^\delta(EG)$ is bounded if and only if $o^\delta(EQ)$ is bounded.*

Proof. Let F be the kernel of p . Since F is a finite group, $H^n(BF, \mathbf{R}) = 0$ for $n > 0$ and therefore, $p^* : H^*(BQ^\delta, \mathbf{R}) \rightarrow H^*(BG^\delta, \mathbf{R})$ is an isomorphism. Because F is amenable, the induced map of bounded cohomology groups $p_b^* : H_b^*(Q^\delta, \mathbf{R}) \rightarrow H_b^*(G^\delta, \mathbf{R})$ is an isomorphism too. Thus p^* induces an isomorphism between the subgroup of bounded elements in $H^*(BQ^\delta, \mathbf{R})$ and those in $H^*(BG^\delta, \mathbf{R})$, proving the first part of the assertion. Let $\iota : \pi_1(G) \otimes \mathbf{R} \rightarrow \pi_1(Q) \otimes \mathbf{R}$ be the quasi-isometric isomorphism induced by the inclusion of $\pi_1(G)$ in $\pi_1(Q)$. Since

$$p^* o_{\mathbf{R}}^\delta(EQ) = \iota_* o_{\mathbf{R}}^\delta(EG),$$

one of the two is bounded if and only if the other one is. It follows that $o^\delta(EQ)$ is bounded if and only if $o^\delta(EG)$ is. q.e.d.

The map $H^*(BG, C) \rightarrow H^*(BG^\delta, C)$ is injective for every cyclic group C (see Milnor, [16]). Because $\pi_1(G)$ is a finitely generated abelian group, it follows that $H^2(BG, \pi_1(G)) \rightarrow H^2(BG^\delta, \pi_1(G))$ is injective too. Therefore, $o^\delta(EG)$ is zero if and only if G is simply connected.

Proposition 3.3. *Assume $o^\delta(EG)$ is bounded, and let $z \in \pi_1(G)$ be an element of infinite order in the commutator subgroup of the universal cover \tilde{G} of G . Then*

$$\text{scl}_{\tilde{G}}(z) \geq \frac{1}{4\|o^\delta(EG)\|_\infty} \lim_{n \rightarrow \infty} \frac{|z^n|}{n} > 0.$$

Proof. An element of infinite order in a finitely generated abelian group is undistorted; hence $\lim_{n \rightarrow \infty} |z^n|/n > 0$. For any $n \neq 0$, $g := \text{cl}(z^n) \geq 1$. Representing Σ_g as the quotient of a $4g$ -gon in the usual way and decomposing the $4g$ -gon into $4g - 2$ triangles (by coning over a vertex), Formula (1) yields

$$|z^n| \leq \|o^\delta(EG)\|_\infty (4\text{cl}(z^n) - 2).$$

The result then follows by dividing by n and taking limits. q.e.d.

Remark 3.4. Proposition 3.3 above can also be deduced from Bavard's formula [2, Proposition 3.4]

$$\text{scl}(z) = \frac{1}{4} \sup_f \frac{\lim_{n \rightarrow \infty} |f(z^n)|/n}{\|df\|_\infty},$$

which holds for any group E and any element z of the derived group $[E, E]$, and where the supremum is taken over all quasi-morphisms $f : E \rightarrow \mathbf{R}$, such that $\|df\|_\infty \neq 0$. (In the case $\{1\} \rightarrow A \rightarrow E \rightarrow G \rightarrow \{1\}$ is a central extension such that $A \cong \mathbf{Z} \subset [E, E]$ and with section σ such that the associated cocycle c_σ is bounded, then the relevant quasi-morphism to consider is the retraction $a\sigma(g) \mapsto a$.)

Because $o^\delta(EG)$ is bounded for connected semisimple Lie groups G ([8, Proposition 47, Theorem 1]), we have the following corollary.

Corollary 3.5. *Let G be a simply connected semisimple Lie group and $z \in G$ a central element of infinite order. Then $\text{scl}(z) > 0$.*

Remark 3.6. A. Borel proved in [3] that cl is bounded on connected semisimple Lie groups with finite center. Thus scl is the zero function on such groups.

As an immediate consequence of Formula (1) we also obtain the following lemma.

Lemma 3.7. *Let $\alpha \in H_2(BG^\delta, \mathbf{Z})$. Let $g(\alpha)$ denote the minimal genus of a surface representing α (cf. Hopf [13, Satz IIa] or Thom [18] or Calegari [7, 1.1.2 Example 1.4]). Then the element $o^\delta(EG)(\alpha)$ of $\pi_1(G)$, viewed as an element of the universal cover of G , lies in the commutator subgroup of \tilde{G} and*

$$\text{cl}(o^\delta(EG)(\alpha)) \leq g(\alpha).$$

For an element $x \in H_n(BG^\delta, \mathbf{R})$, we write $\|x\|_1$ for its ℓ_1 -seminorm (cf. Gromov [12]).

Proposition 3.8. *Let $\alpha \in H_2(BG^\delta, \mathbf{Z})$, and let $\alpha_{\mathbf{R}}$ denote its image in $H_2(BG^\delta, \mathbf{R})$. Then*

$$\text{scl}(o^\delta(EG)(\alpha)) \leq \frac{\|\alpha_{\mathbf{R}}\|_1}{4}.$$

Proof. Lemma 3.7 implies

$$\text{scl}(o^\delta(EG)(\alpha)) = \lim_{n \rightarrow \infty} \frac{\text{cl}(o^\delta(EG)(n\alpha))}{n} \leq \lim_{n \rightarrow \infty} \frac{g(n\alpha)}{n},$$

and from Barge-Ghys [1, Lemme 1.5] we infer that

$$\lim_{n \rightarrow \infty} \frac{g(n\alpha)}{n} = \frac{\|\alpha_{\mathbf{R}}\|_1}{4},$$

finishing the proof.

q.e.d.

4. A lower bound for the stable commutator length

Proposition 4.1. *Let G be a simply connected Lie group. There exists a constant $s > 0$ such that if $z \in [G, G]$ is central in G and $\text{scl}(z) < s$, then $\text{scl}(z) = 0$.*

Proof. Let $G = RL$ be a Levi decomposition. Let $z = r\ell$ with $r \in R$ and $\ell \in L$. Because this decomposition of z is unique and R is normal in G , ℓ is central in L and L centralizes r . First we show that $r \in [R, R]$. Case 1: R is commutative. In this case, we show that $r = e$. As L is semisimple, r lies in a direct factor of G and, because $\ell \in L = [L, L]$, r lies in $[G, G]$; thus r must be trivial.

Case 2: R is not commutative. Let $p : RL \rightarrow (R/[R, R])L$ be the projection. Then $p(z) = p(r)\ell$ and by Case 1, $p(r) = e$, which implies $r \in [R, R]$.

Because the stable commutator length vanishes for connected solvable Lie groups (cf. Bavard [2, p. 110]), we have

$$\text{scl}(r) \leq \text{scl}_R(r) = 0.$$

By subadditivity on commuting elements, it follows that $\text{scl}(z) = \text{scl}(\ell)$. But $\text{scl}(\ell) = \text{scl}_L(\ell)$. The quotient Q of L by its center is a (linear) semisimple group. Therefore, $o^\delta(EQ)$ is bounded [8, Proposition 47, Theorem 1]. Using Proposition 3.3, we obtain then an $s > 0$ such that $\text{scl}(z) \geq s$ if $\text{scl}(z) \neq 0$, completing the proof of the proposition. q.e.d.

Proposition 4.2. *Let G be a connected Lie group. The function scl is continuous on any closed abelian subgroup of $[G, G]$ and satisfies $\text{scl}(\exp(tX)) = t \cdot \text{scl}(\exp(X))$ for X in the Lie algebra of $[G, G]$ and $t \geq 0$.*

Proof. There is a compact neighborhood K of $e \in [G, G]$ such that for all $k \in K$, $\text{cl}(k) \leq N$, where N is the dimension of the Lie algebra of $[G, G]$ (Bourbaki, [4, Chap. III, Exercise 10 of §9]). Therefore, cl is bounded on compact subsets of $[G, G]$. Assume that $x \in [G, G] < G$ lies on a one-parameter subgroup $\exp(X) = x$. Consider the fractional and integer parts $1 < t = \{t\} + [t]$. As $\lim_{t \rightarrow \infty} [t]/t = 1$, and as

$$\left| \frac{\text{cl}(\exp([t]X))}{[t]} - \frac{\text{cl}(\exp(tX))}{t} \right| \leq 2 \cdot \frac{\text{cl}(\exp(\{t\}X))}{t},$$

we deduce that

$$\lim_{t \rightarrow \infty} \frac{\text{cl}(\exp(tX))}{t} = \lim_{n \rightarrow \infty} \frac{\text{cl}(x^n)}{n} = \text{scl}(x).$$

Hence for all $\exp(X) \in [G, G]$, $t \geq 0$, we have

$$\text{scl}(\exp(tX)) = t \cdot \text{scl}(\exp(X)).$$

As $\text{scl} \leq \text{cl}$, scl is also bounded on compact sets. This implies that scl is continuous at the identity. Subadditivity on commuting elements

and continuity at the identity imply continuity on any closed abelian subgroup of $[G, G]$. q.e.d.

Lemma 4.3. *Let G be a simply connected Lie group. Let $V \cong \mathbf{R}^r$ be a closed subgroup of $[G, G]$, and let $A \cong \mathbf{Z}^r$ be discrete in V and central in G . Assume $\text{scl}(a) > 0$ for all $0 \neq a \in A$. Then there is a constant $\varepsilon > 0$ such that for all $a \in A$, $\text{scl}(a) \geq \varepsilon|a|$.*

Proof. As scl is continuous on V and linear along one-parameter semi-groups, it is enough to show that scl vanishes nowhere on the unit sphere S of V . Assume there is $z \in S$ with $\text{scl}(z) = 0$. As A is cocompact in V , for each $n \in \mathbf{N}$, there is $a_n \in A \setminus \{0\}$ and $t_n > 0$ such that

$$|a_n - t_n z| < 1/n.$$

Hence

$$0 < \text{scl}(a_n) = \text{scl}(a_n - t_n z) \rightarrow 0,$$

contradicting Proposition 4.1. q.e.d.

5. Proof of the main theorem

We use the notation introduced at the beginning of Section 3.

Lemma 5.1. *Let G be a connected Lie group. Let \tilde{G} be its universal cover. Assume that $A = \pi_1(G) \cap [\tilde{G}, \tilde{G}] \cong \mathbf{Z}^r$ is discrete in a closed subgroup $V \cong \mathbf{R}^r$ of $[\tilde{G}, \tilde{G}]$. If $\text{scl}(a) > 0$ for all $a \in A \setminus \{0\}$, then $o^\delta(EG)$ is bounded.*

Proof. Let us denote $X = BG^\delta$. Let $C_2(X, \mathbf{R})$ be the space of singular 2-chains, endowed with the ℓ_1 -norm. The free \mathbf{Z} -module $Z_2(X, \mathbf{Z})$ of integral cycles contains a basis of the \mathbf{R} -vector space $Z_2(X, \mathbf{R})$ of cycles. Hence there is a unique \mathbf{R} -linear map $c : Z_2(X, \mathbf{R}) \rightarrow V$ such that for all $x \in Z_2(X, \mathbf{Z})$, $c(x) = o^\delta(EG)([x])$. Let $z \in Z_2(X, \mathbf{Q})$, $z = \sum_i r_i \sigma_i$, $r_i \in \mathbf{Q}$, σ_i a singular 2-simplex of X , $|z|_1 = \sum_i |r_i|$. Let $m \in \mathbf{N}$, such that $mr_i \in \mathbf{Z}$ for all i . Then, with $\varepsilon > 0$ as in Lemma 4.3,

$$|c(z)| = \frac{1}{m}|c(mz)| = \frac{1}{m}|o^\delta(EG)([mz])| \leq \frac{\text{scl}(o^\delta(EG)([mz]))}{m\varepsilon}$$

and because of Proposition 3.8,

$$\frac{\text{scl}(o^\delta(EG)([mz]))}{m\varepsilon} \leq \frac{\|[mz]_{\mathbf{R}}\|_1}{4m\varepsilon} \leq \frac{|z|_1}{4\varepsilon}.$$

As $Z_2(X, \mathbf{Q}) \subset Z_2(X, \mathbf{R})$ is dense, the norm of the linear map c is bounded by $(4\varepsilon)^{-1}$. Hahn-Banach's theorem provides a linear extension \hat{c} of c to all of $C_2(X, \mathbf{R})$ with the same bound on the norm. Hence

$$[\hat{c}] = o_{\mathbf{R}}^\delta(EG) \in H^2(X, A \otimes \mathbf{R}) \subset H^2(X, \pi_1(G) \otimes \mathbf{R})$$

is bounded. Hence $o^\delta(EG) \in H^2(X, \pi_1(G))$ is bounded as well. q.e.d.

We are ready for the proof of the main theorem.

Proof of Theorem 2.2. (1) implies (2):

Assume that $\pi_1([R, R]) \neq 0$. Then according to Goldman [11], and as we have seen at the beginning of Section 3, there exists a map $f : \Sigma_g \rightarrow BR^\delta$ for some closed oriented surface of genus $g \geq 1$ such that $(f^*o^\delta(ER))([\Sigma_g]) \in \pi_1([R, R])$ is not zero. Because $\pi_1([R, R])$ is torsion-free, we conclude that the image $o_{\mathbf{R}}^\delta(ER) \in H^2(BR^\delta, \pi_1(R) \otimes \mathbf{R})$ also satisfies $(f^*o_{\mathbf{R}}^\delta(ER))([\Sigma_g]) \neq 0$. It follows that $o_{\mathbf{R}}^\delta(ER)$ cannot be bounded, because the bounded cohomology of the discrete amenable group R^δ vanishes (cf. Johnson [14]). Denote by $\iota : \pi_1(R) \otimes \mathbf{R} \rightarrow \pi_1(G) \otimes \mathbf{R}$ the natural isometric embedding, and let $j : R \rightarrow G$ be the inclusion. Because

$$0 \neq \iota_*(o_{\mathbf{R}}^\delta(BR)) = j^*(o_{\mathbf{R}}^\delta(BG)),$$

we see that $o_{\mathbf{R}}^\delta(EG)$ is not bounded. It follows that $o^\delta(EG)$ cannot be bounded either.

(2) implies (3):

Assume that $z \in \pi_1(G) \subset \tilde{G}$ is distorted in \tilde{G} and hence of infinite order. Then there is an $n > 0$ such that $z^n \in \tilde{R}$, otherwise the projection of z in \tilde{G}/\tilde{R} would be a central distorted element, and this is impossible [9, Lemma 6.3]. The distorted element $z^n \in \tilde{R} \cap \pi_1(G) = \pi_1(R)$ must belong to $[\tilde{R}, \tilde{R}]$, because otherwise its projection to

$$\tilde{R}/[\tilde{R}, \tilde{R}] \subset \tilde{G}/[\tilde{R}, \tilde{R}]$$

would span a central distorted line. But a line in $\tilde{R}/[\tilde{R}, \tilde{R}] \cong \mathbf{R}^r$ that is central in $\tilde{G}/[\tilde{R}, \tilde{R}]$ is a direct factor because \tilde{G}/\tilde{R} is semisimple. Hence it cannot be distorted. We conclude that

$$z^n \in [\tilde{R}, \tilde{R}] \cap \pi_1(G) = \pi_1([R, R]),$$

contradicting (2).

(3) implies (4):

Let $\tilde{G} = \tilde{R}L$ be a Levi decomposition, and let $z = r\ell \in \pi_1(G) \cap [\tilde{G}, \tilde{G}]$ be of infinite order, $r \in \tilde{R}$ and $\ell \in L$. Note that $L = [L, L]$ and $\text{scl}(z) \geq \text{scl}(\ell) = \text{scl}_L(\ell)$. If ℓ has infinite order, Corollary 3.5 implies that $\text{scl}_L(\ell) > 0$, and we conclude that $\text{scl}(z) > 0$. If ℓ has finite order, there is an $n > 0$ such that $z^n = (r\ell)^n = r^n$, and $r^n \in \tilde{R} \cap \pi_1(G)$ actually lies in $[\tilde{R}, \tilde{R}]$, as we have seen in the course of the proof of 4.1. Proposition 19 of [8] implies that z is distorted in \tilde{R} , and hence also in \tilde{G} .

(4) implies (1):

Let K be a maximal compact subgroup of $[G, G]$. The universal cover of K is a closed subgroup $E \times S$ of $[\tilde{G}, \tilde{G}]$ where $E \cong \mathbf{R}^r$ and S is compact semisimple. Let $p : E \times S \rightarrow S$ be the projection onto the second

factor. The projection of $\pi_1(G) \cap [\tilde{G}, \tilde{G}] = \pi_1([G, G]) \subset E \times S$ in S is central and hence finite. Thanks to Proposition 3.2, we may replace G by a finite homomorphic cover and hence assume that $\pi_1(G) \cap [\tilde{G}, \tilde{G}] \subset E$. Applying Lemma 5.1 and Proposition 3.1 implies boundedness in degree 2. Boundedness in higher degree then follows from [8, Lemma 51, Theorem 54]. q.e.d.

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