TROPICAL LAMBDA LENGTHS, MEASURED LAMINATIONS AND CONVEXITY

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Abstract

This work uncovers the tropical analogue, for measured laminations, of the convex hull construction in decorated Teichmüller theory; namely, it is a study in coordinates of geometric degeneration to a point of Thurston's boundary for Teichmüller space. This may offer a paradigm for the extension of the basic cell decomposition of Riemann's moduli space to other contexts for general moduli spaces of flat connections on a surface. In any case, this discussion drastically simplifies aspects of previous related studies as is explained. Furthermore, a new class of measured laminations relative to an ideal cell decomposition of a surface is discovered in the limit. Finally, the tropical analogue of the convex hull construction in Minkowski space is formulated as an explicit algorithm that serially simplifies a triangulation with respect to a fixed lamination and has its own independent interest.

Introduction

Rich structural properties of (decorated) Teichmüller spaces [**5**, **8**] extend naturally to general cluster varieties [**1**], thus importantly including moduli spaces of flat connections on a surface for various Lie groups. These extensions include the basic coordinates, their transformation laws under mapping classes, their underlying Kähler or Poisson structure, and importantly for us here, their asymptotics under degeneration of geometric structure. In contrast, the ideal cell decomposition of decorated Teichmüller space depends on a convex hull construction in Minkowski space which has not yet found expression in the more rarefied and combinatorial terms of cluster varieties despite various attempts.

It is first of all this problem we address here by tropicalizing the classical coordinates and formulas in order to obtain a purely combinatorial formulation for the asymptotics of this convex hull construction. This in particular gives a relatively easy description of the projective laminations that are "accessible" from within a given cell in decorated Teichmüller space, i.e., there is a path contained in the cell projecting to a path in Teichmüller space itself which is asymptotic to the projective

lamination in the topology of Thurston's boundary. These accessible laminations turn out to be certain so-called balanced weighted multicurves determined by the cell, that is, by an ideal cell decomposition of the underlying surface itself. This is the higher-genus analogue of the Farey structure on the rational numbers in the classical case of elliptic curves. On the other hand, all weighted multicurves occur for some cell in decorated Teichmüller space as follows from an explicit algorithmic construction, a higher-genus analogue of continued fractions. Indeed, we think of this algorithm as a combinatorial replacement for the convex hull construction in Minkowski space at least asymptotically and hope and expect that it may have applications more generally.

More specifically and as we shall recall from [5, 8] in detail in the next section, lambda lengths $\lambda : \Delta \to \mathbb{R}_{>0}$ on the edges Δ of an ideal triangulation of a punctured surface F of finite type give global affine coordinates on the space $\widetilde{\mathcal{T}}(F)$ of decorated hyperbolic structures on F, i.e., a hyperbolic structure on the complement of a finite set of distinguished points or punctures in F together with the specification of a collection of horocycles in F, one about each puncture.

Likewise, (tangential) measures $\mu : \Delta \to \mathbb{R}$ give global linear coordinates on the space $\widetilde{\mathcal{ML}_0}(F)$ of decorated measured laminations in F introduced and studied in [4], where each puncture comes equipped with a foliated band of some width supplementing a usual measured lamination in F with compact support. In fact in the limit, $\mu = \log \lambda$, i.e., asymptotics in the sense of Thurston's boundary [13] correspond to taking logarithms. This once-mysterious appearance of the logarithms is explained by tropicalization as follows.

Given any function $\phi(a, b, \cdots) : \mathbb{R}^n_{>0} \to \mathbb{R}_{>0}$, its *tropicalization* is defined as the limit as t goes to infinity of $\frac{1}{t} \log \phi(e^{ta}, e^{tb}, \cdots)$, if it exists, taking values in the tropical semi-ring [12]. If a formula for ϕ is written by combining the coordinate functions of $\mathbb{R}^n$ using only multiplication and addition, its tropicalization is given by replacing each occurrence of multiplication by addition and each occurrence of addition by the binary maximum function with the precedence of addition over maximum. In particular, an inequality (either strict or weak) of two such expressions built only from multiplication and addition implies the corresponding (weak) inequality on their tropicalizations but not conversely.

We can nowadays say simply that [4] showed that lambda lengths tropicalize to tangential measures, a fact whose further ramifications are studied here. Just as these results provided a key explicit example and gave clues for general compactifications of cluster varieties [1], so too we hope that the discoveries of this paper might inform the tropicalization of extremal problems in related contexts, viz. extending the cell decomposition of Riemann's moduli space to general moduli spaces of flat connections.

Indeed, we study several more refined notions regarding this procedure as follows. There are two key conditions on $\lambda : \Delta \to \mathbb{R}_{>0}$:

Classical Triangle Inequality CT: For every triangle $t \in T$ with frontier edges x_i, $i = 1, 2, 3$, we have

$$\lambda(x_i) < \lambda(x_j) + \lambda(x_k) \text{ for } \{i, j, k\} = \{1, 2, 3\}.$$

Classical Face Condition CF: In the notation of Figure 1.1 for the edges nearby $e \in \Delta$, we have

$$\lambda(e)\big(\lambda(a)\lambda(b) + \lambda(c)\lambda(d)\big) \leq \lambda(f)\big((\lambda(a)\lambda(d) + \lambda(b)\lambda(c)\big),$$

with respective tropical analogues for $\mu : \Delta \to \mathbb{R}$.

Tropical Triangle Inequality TT: For every $t \in T$ with frontier edges x_i, $i = 1, 2, 3$, we have

$$\mu(x_i) \leq \max\{\mu(x_j), \mu(x_k)\} \text{ for } \{i, j, k\} = \{1, 2, 3\}.$$

Tropical Face Condition TF: In the notation of Figure 1.1 for the edges nearby $e \in \Delta$, we have

$$\mu(e) + \max\{\mu(a) + \mu(b), \mu(c) + \mu(d)\} \leq \mu(f) + \max\{\mu(a) + \mu(d), \mu(b) + \mu(c)\}.$$

Classically, CF is equivalent to minimizing the objective function (for fixed hyperbolic structure and varying ideal triangulation)

$$\sum_{t \in T} \lambda(a)\lambda(b)\lambda(c),$$

where the sum is over all triangles T complementary to Δ and where $t \in T$ has frontier edges a, b, c. This, in turn, is equivalent to a certain convexity condition [**5**, **8**] in Minkowski space, which is used to associate to Δ a corresponding cell $C(\Delta) \subset \widetilde{\mathcal{T}}(F)$ in the cell decomposition of $\widetilde{\mathcal{T}}(F)$. Furthermore, CF implies CT for positive lambda lengths.

Tropically, TF is equivalent (see Lemma 5.1) to minimizing the objective function (for fixed measure and varying ideal triangulation) given by the ordered list

$$\text{sort}_\downarrow\{\mu(a) + \mu(b) + \mu(c) : t \in T\}$$

of "perimeters" of triangles as we shall see, and in fact, there is an algorithm (see Theorem 5.2 kindly contributed by Dylan Thurston) to reduce this objective function which achieves the minimum in case the measured lamination is a weighted multicurve. Furthermore, if a global minimum also minimizes the total $\sum_{e \in \Delta} \mu(e)$, then it satisfies TT as well (see Corollary 5.4). This resolves a basic question about the existence of ideal triangulations appropriately suited to measured laminations and indeed provides an effective algorithm for finding them.

The condition TT is most interesting. First of all, TT implies each of the other three conditions (see Lemma 2.2). Secondly, as a topological condition (see Lemma 2.3), TT on tangential measure implies that every leaf is closed, i.e., the lamination is a multicurve possibly together with other puncture-parallel curves, i.e., it is a decorated weighted multicurve.

It is a basic question which collections of asymptotically short curves are accessible from within a given cell in the decomposition of $\widetilde{\mathcal{T}}(F)$. This was first answered in [**10**, **8**], which is joint work with Greg McShane, relying on rather involved estimates on the asymptotics of lambda lengths leading to a Fulton-MacPherson type treatment [**2**] of rates of their divergence. The more general result on which projectively measured laminations are accessible from within a given cell is answered here "in a trice" (see Theorem 4.2) from the facts that CF $\Rightarrow$ CT for positive lambda lengths and CT $\Rightarrow$ TT in the limit by tropicalization. Just as in [**10**], the proof is actually somewhat more involved in the non-generic case leading to the study of balanced measured laminations (in Section 3), which play the role for laminations that cyclic polygons play for geometric structures and correspond precisely to the accessible laminations. A projective lamination is balanced with respect to an ideal cell decomposition of the surface if TT holds on each triangle for any completion to an ideal triangulation.

This paper is organized as follows. In §1, we recall notation and facts about $\widetilde{\mathcal{T}}(F)$ and $\widetilde{\mathcal{ML}}_0(F)$ culminating in a theorem the author proved with Athanase Papadopoulos in 1993 that lambda lengths on decorated Teichmüller space tropicalize to tangential measures on decorated measured laminations (though this result pre-dates tropical math by several years). In §2, we systematically study TT, first; in relation to CT, CF, and TF, and second, a measured lamination satisfying TT with respect to some ideal triangulation is shown to have only closed leaves. In §3, we tropicalize two classical results [**5**, **8**] on cyclic polygons leading to the new and naturally occurring class of balanced projective laminations. These tropical results are then applied in the following section to give the measured lamination version of the earlier work with McShane on degenerations. An algorithm that plays the tropical role of the convex hull construction [**5**, **8**] is described in §5 and applied to show that for any weighted multicurve, there is an ideal triangulation so that the associated measure satisfies TT.

1. Notation and Recollections

The *decorated Teichmüller space* $\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}(F)$ of a punctured surface $F = F_g^s$ of genus g with $s \geq 1$ punctures and negative Euler characteristic is the total space of the trivial $\mathbb{R}_{>0}^s$-bundle $\pi : \widetilde{\mathcal{T}} \to \mathcal{T}$ over the usual Teichmüller space $\mathcal{T} = \mathcal{T}(F)$ of F, where the fiber over a point of $\mathcal{T}$ is the set of all possible s-tuples of horocycles, one about each puncture [**5**, **8**]. Let $MC = MC(F)$ denote the mapping class group of F, which acts naturally and π-equivariantly on $\mathcal{T}$ and $\widetilde{\mathcal{T}}$. The total space $\widetilde{\mathcal{T}}$ admitsn a MC-invariant ideal cell decomposition as well as explicit global affine coordinates which will be recalled presently.

The space $\mathcal{ML}_0 = \mathcal{ML}_0(F)$ of (possibly empty) measured laminations of compact support on F was introduced in [**13**]. The quotient

$$\mathcal{PL}_0 = \mathcal{PL}_0(F) = (\mathcal{ML}_0(F) - \mathbf{0})/\mathbb{R}_{>0}$$

under the homothetic action of $\mathbb{R}_{>0}$ on transverse measures is a PL-sphere of dimension $6g - 7 + 2s$ called *Thurston's boundary*, where $\mathbf{0}$ denotes the empty lamination, naturally compactifying $\mathcal{T} \approx \mathbb{R}^{6g-6+2s}$ to a closed ball

$$\overline{\mathcal{T}} = \overline{\mathcal{T}}(F) = \mathcal{T}(F) \sqcup \mathcal{PL}_0(F),$$

upon which MC acts naturally and continuously. Let $[\mathcal{L}] \in \mathcal{PL}_0$ denote the projective class of $\mathcal{L} \in \mathcal{ML}_0 - \{\mathbf{0}\}$.

Following [**4**], there is furthermore a vector space $\widetilde{\mathcal{ML}}_0 = \widetilde{\mathcal{ML}}_0(F)$, the space *decorated measured laminations*, together with a continuous map $\pi : \widetilde{\mathcal{ML}}_0 \to \mathcal{ML}_0$, where the fiber over a point of $\mathcal{ML}_0$ is isomorphic to $\mathbb{R}^s$. A decorated measured lamination is regarded as a measured lamination with compact support in the usual sense except that puncture-parallel leaves, called *collar curves*, are now allowed and are given a $\mathbb{R}$-valued "decoration"; if the decoration is positive, then this parameter is taken as the transverse measure of a foliated band parallel to the collar curve, and if it is negative, then it is thought of as a "deficit" of such leaves as in [**4**]. In particular, the fiber over $\mathbf{0}$ is the collection of all $\mathbb{R}$-weighted collar curves.

An *ideal triangulation* Δ of F is (the isotopy class of) a collection of arcs decomposing F into triangles with their vertices at the punctures. A *flip* on an arc a in Δ which separates distinct triangles complementary to Δ removes a from Δ and replaces it with the unique distinct edge. Let $T = T(\Delta)$ denote the set of triangles complementary to Δ in F.

REMARK 1.1. [**5**, **8**] The groupoid $Pt = Pt(F)$ whose objects are ideal triangulations and whose morphisms are flips between them is called the *Ptolemy groupoid*. The groupoid $M\Gamma = M\Gamma(F)$ whose objects are MC-orbits of ideal triangulations imbued with an enumeration of the edges, and whose morphisms are MC-orbits of pairs of such with the obvious multiplication is called the *mapping class groupoid*. Pt and $M\Gamma$ admit

simple presentations, and $M\Gamma$ contains MC as the stabilizer of any object, which is a subgroup of finite index.
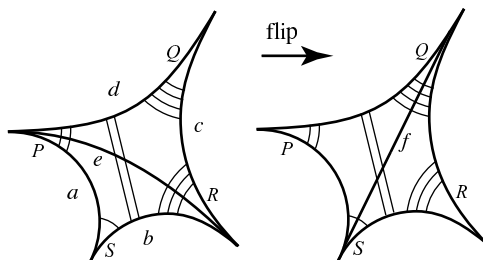

Figure 1.1. A flip.

Upon fixing an ideal triangulation Δ, there are global affine coordinates $c_L : \mathbb{R}_{>0}^{\Delta} \to \widetilde{T}$ on decorated Teichmüller space called *lambda lengths*: the lambda length of an arc α in Δ in a decorated hyperbolic surface is given by $\exp \delta/2$, where δ is the signed hyperbolic length of the geodesic homotopic to α truncated at the horocycles. The effect of a flip on lambda lengths is described by a *Ptolemy transformation* $ef = ac + bd$ in the notation of Figure 1.1, where we identify an arc with its lambda length for convenience; furthermore, the cross-ratio of the quadrilateral illustrated is given by ac/bd in lambda lengths.

There are moreover global linear coordinates $c_M : \mathbb{R}^{\Delta} \to \widetilde{\mathcal{ML}}_0$ called *(tangential) measures* on Δ regarded as a train track with stops, cf. [**4**, **9**]: if $\mathcal{L}$ is a measured lamination, then the tangential measure of an arc α in Δ is the transverse measure of $\mathcal{L}$ along α. The effect of a flip on measures is described by $e + f = \max\{a + c, b + d\}$ in the notation of Figure 1.1, where now we identify an arc with its measure for convenience.

For each ideal triangulation Δ, there is an associated branched one-submanifold called the *dual freeway* $\tau = \tau_\Delta$ as illustrated in Figure 1.2b, where there is a small triangle of τ lying inside each complementary region to Δ and one dual "long" edge meeting each arc of Δ, cf. [**4**]. The measure on the train track with stops Δ can dually be regarded as a transverse measure on the freeway τ; the values on the long edges of τ are unrestricted real numbers which extend uniquely to an $\mathbb{R}$-valued transverse measure on all of τ, which in turn uniquely determines a decorated measured lamination. Notice that in the notation of Figure 1.2 on a freeway inside a triangle, we have $\gamma = \frac{d+e-c}{2}, \delta = \frac{c+e-d}{2}, \varepsilon = \frac{c+d-e}{2}$, and inside a monogon, we have $\gamma = \delta = \frac{e}{2}, \varepsilon = c - e/2 = d - e/2$.

An *ideal cell decomposition* of F is a subset of an ideal triangulation each of whose complementary regions is simply connected. The dual
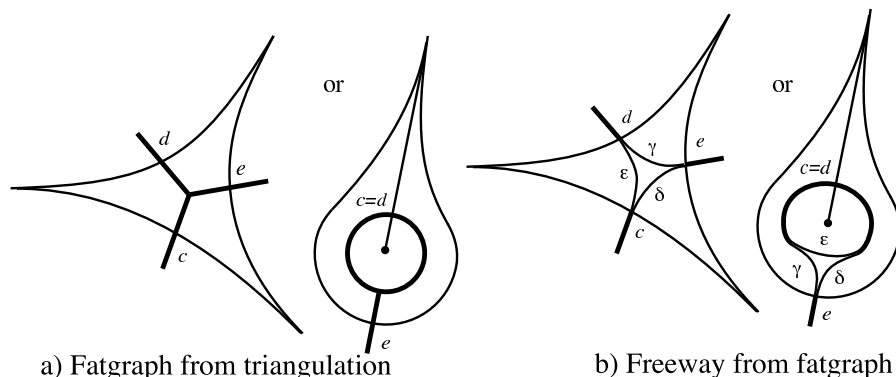
a) Fatgraph from triangulation   b) Freeway from fatgraph

Figure 1.2. Fatgraphs and freeways.

of an ideal cell decomposition is a *fatgraph* spine embedded in F as illustrated for the generic case of ideal triangulations in Figure 1.2a.

There is an MC-invariant ideal cell decomposition [**3**, **5**, **8**, **11**] of $\widetilde{\mathcal{T}}$ whose cells are in one-to-one correspondence with ideal cell decompositions of F, where the face relation is induced by inclusion. The interior of the cell $C(\Delta) \subseteq \widetilde{\mathcal{T}}$ associated to an ideal triangulation Δ is determined by the positivity $E > 0$ of a collection of expressions

$$E = \frac{a^2 + b^2 - e^2}{abe} \; + \; \frac{c^2 + d^2 - e^2}{cde}$$

in the lambda length notation of Figure 1.1 called *simplicial coordinates*, one for each edge of Δ. Note that CF is precisely equivalent to non-negativity of the simplicial coordinates, which play a key role in this paper.

Moreover, the cell $C(\Delta')$ corresponding to an ideal cell decomposition Δ' lies in the closure of $C(\Delta)$ if and only if $\Delta' \subseteq \Delta$; in this case, $C(\Delta')$ is determined by the vanishing of simplicial coordinates of edges of $\Delta - \Delta'$, and non-vanishing putative simplicial coordinates on the edges of an ideal cell decomposition Δ' uniquely determine corresponding positive lambda lengths on the edges of any ideal triangulation containing it (as will be further explained in Section 3).

In particular, the simplicial coordinates are the hyperbolic analogues of Strebel coordinates in the conformal setting, but there is no conformal analogue of lambda lengths. The explicit calculation of lambda lengths from simplicial coordinates is the "arithmetic problem" discussed at length in [**7**, **8**].

There is the following diagram summarizing the maps and spaces just defined:

$$\begin{array}{ccccccc}
 & c_L & & \log & & c_M & \\
\widetilde{\mathcal{T}} & \leftarrow & \mathbb{R}^{\Delta}_{>0} & \rightarrow & \mathbb{R}^{\Delta} & \rightarrow & \widetilde{\mathcal{ML}}_0 \\
\downarrow \pi & & & & & & \downarrow \pi \\
\mathcal{T} & & & & \mathcal{ML}_0 - \{\mathbf{0}\} & \subset & \mathcal{ML}_0 \\
\cap & & & & \downarrow [\cdot] & & \\
\overline{\mathcal{T}} & & \supset & & \mathcal{PL}_0 & &
\end{array}$$

where $\log : \mathbb{R}^{\Delta}_{>0} \to \mathbb{R}^{\Delta}$ denotes the function which in each coordinate simply takes natural logarithms, c_L denotes lambda lengths, and c_M denotes tangential measures on Δ, or equivalently transverse measures on τ.

Theorem 1.2. [**4**, **6**, **8**] *Fix an ideal triangulation Δ of F and consider a sequence $\lambda_i \in \mathbb{R}^{\Delta}_{>0}$, for $i \geq 0$, so that $\pi \circ c_L(\lambda_i)$ escapes each compactum in $\mathcal{T}$. If $\log \lambda_i$ converges projectively to $\mu \in \mathbb{R}^{\Delta}$ with corresponding decorated measured lamination $\mathcal{L}$, then $[\mathcal{L}] = [\pi \circ c_M(\log \lambda_i)]$, i.e., $\mu = \log \lambda_i$.*

Conversely, for a generic sequence λ_i escaping each compactum, there is a subsequence, henceforth still denoted λ_i so that $\pi \circ c_L(\lambda_i) \in \mathcal{T}$ converges to $[\mathcal{L}] \in \mathcal{PL}_0$ in the topology of Thurston's boundary for - Teichmüller space if and only if $\log \lambda_i$ converges projectively to μ, where $[\mathcal{L}] = [\pi \circ c_L(\mu)]$.

Furthermore, the tropicalization

$$e + f = \max\{a + c, b + d\}$$

of the Ptolemy equation $ef = ac + bd$ describes the effect of flips on measures.

REMARK 1.3. Moreover, the expression

$$\sum da \wedge db + db \wedge dc + dc \wedge da,$$

where the sum is over all triangles complementary to Δ with log lambda length coordinates a, b, c, in this clockwise cyclic order corresponding to the boundary, describes the pull-back [**6**, **8**] by $\pi \circ c_L$ of the Weil-Petersson Kähler two-form on $\mathcal{T}$ as well as half the pull-back [**4**] by $\pi \circ c_M$ of the Thurston symplectic form on $\mathcal{ML}_0$ in measures a, b, c.

2. Tropical Triangle Inequalities

Fix an ideal triangulation Δ of F and an assignment of real numbers called "weights," one to each arc in Δ, and consider the following inequalities on weights, where we shall often identify an edge with its weight for convenience.

(Classical) Triangle Inequalities CT: For each triangle complementary to Δ whose boundary has consecutive edges a, b, c, we have

$$a < b + c,$$
$$b < a + c,$$
$$c < a + b.$$

Tropical Triangle Inequalities TT: In the notation of the triangle inequalities, the maximum $\max\{a, b, c\}$ is achieved either twice or thrice, that is,

$$a \leq \max\{b, c\},$$
$$b \leq \max\{a, c\},$$
$$c \leq \max\{a, b\}.$$

(Classical) Face Conditions CF: For each quadrilateral comprised of two adjacent triangles complementary to Δ whose edges are as illustrated in Figure 1.1, non-negativity of the simplicial coordinate gives

$$\left[cd(a^2 + b^2 - e^2)\right] + \left[ab(c^2 + d^2 - e^2)\right] \geq 0,$$

i.e., we have $e^2(ab + cd) \leq (ac + bd)(ad + bc)$; or in other words, by the Ptolemy equation, we have

$$e(ab + cd) \leq f(ad + bc).$$

Tropical Face Conditions TF: In the notation of the face conditions, we have

$$2e + \max\{a + b, c + d\} \leq \max\big\{c + d + 2\max\{a, b\}, a + b + 2\max\{c, d\}\big\}$$
$$= \max\{a + c, b + d\} + \max\{a + d, b + c\},$$

where the equality follows as the tropicalization of the classical identity $ab(c^2 + d^2) = (ac + bd)(ad + bc)$; in other words, by the tropical Ptolemy equation, we have

$$e + \max\{a + b, c + d\} \leq f + \max\{a + d, b + c\}.$$

Lemma 2.1. *Fix an ideal triangulation Δ of the surface F. A collection $\lambda \in \mathbb{R}_{>0}^{\Delta}$ of lambda lengths describes a point in the interior of the cell $C(\Delta) \subseteq \tilde{\mathcal{T}}(F)$ if and only if λ minimizes the objective function $\sum_{t \in T} \lambda(a)\lambda(b)\lambda(c)$, where the sum is over all triangles $t \in T$ complementary to Δ with frontier edges a, b, c, i.e., no flip on edges of Δ decreases this objective function.*

Proof. This follows directly from the last formulation of CF.   q.e.d.

In effect, in Section 5, we shall learn that the natural tropicalization of this objective function is the ordered sequence of perimeters in decreasing order

$$\text{sort}_\downarrow \{\mu(a) + \mu(b) + \mu(c) : t \in T \text{ with frontier edges } a, b, c\},$$

for $\mu \in \mathbb{R}^\Delta$, in the sense that CF is equivalent to minimizing the former objective function for decorated hyperbolic structures and TF is equivalent to minimizing the latter for decorated measured laminations. Insofar as CF is equivalent to convexity in the convex hull construction [**5**, **8**], so too we think of $\text{sort}_\downarrow$ as measuring a kind of "tropical convexity."

Lemma 2.2. *We have the following implications:*

- TT *implies* CT, CF, *and* TF, *and these are the only implications that hold among these conditions.*

- CF *implies* CT *for positive weights.*

- TF *implies* TT *provided the sum of all the weights is minimal among such triangulations.*

Proof. The non-uniqueness of a maximum weight on edges of triangles complementary to an ideal triangulation Δ for tropical triangle inequalities shows that the classical triangle inequalities hold not only on the weights themselves, but also their squares. It follows that TT $\Rightarrow$ CT and also that TT $\Rightarrow$ CF. To see that TT $\Rightarrow$ TF in the notation of Figure 1.1, we have $e \le \max\{a, b\}$ and $e \le \max\{c, d\}$ in particular, so

$$2e + \max\{a + b, c + d\} = \max\{a + b + 2e, c + d + 2e\}$$
$$\le \max\{a + b + 2\max\{c, d\},$$
$$c + d + 2\max\{a, b\}\}$$
$$= \max\{a + c, b + d\} + \max\{a + d, b + c\},$$

as claimed, where the last equality follows as before.

To see that these are the only implications that hold among the conditions on weights, it remains to give a series of counter-examples a, b, c, d, e as usual in the notation of Figure 1.1 as follows:

4,4,1,6,5 shows TF $\not\Rightarrow$ TT; 4,3,3,4,5 shows CT $\not\Rightarrow$ TT; 3,2,3,3,4 shows CT $\not\Rightarrow$ CF; 3,1,1,1,1 shows TF $\not\Rightarrow$ CT; and 2,2,4,4,3 shows CF $\not\Rightarrow$ TF.

Extending these specifications to weights defined on an entire ideal triangulation is an exercise as likewise is the verification that these counter-examples together with the positive results determine all implications among these conditions.

For the second assertion that CF $\Rightarrow$ CT for positive weights, we argue as in [**5**, **8**] as follows. Adopt the notation of Figure 1.1 in the universal cover if the edges a, b, c, d are not distinct or if e does not have distinct endpoints. If $c + d \leq e$, then $c^2 + d^2 - e^2 \leq -2cd$. The face condition $0 \leq cd(a^2 + b^2 - e^2) + ab(c^2 + d^2 - e^2)$ then gives $0 \leq cd[(a-b)^2 - e^2]$, and we find a second triangle so that the triangle inequality fails. It follows that any failure of the triangle inequalities implies that there is a cycle of triangles of such failures, namely, there are complementary triangles $t_0, \ldots, t_n = t_0$, where the frontier of t_i consists of arcs e_{i-1}, e_i, b_i with $e_i = t_i \cap t_{i+1}$, for $i = 0, \ldots, n$, and the subscripts are taken modulo n, so that $e_j \geq b_j + e_{j-1}$, for $j = 1, \ldots, n$. Upon summing and canceling like terms, we find $0 \geq \sum_{j=1}^{n} b_j$, which is absurd for positive weights.

For the proof of the final assertion that TF implies TT under suitable conditions, suppose to the contrary that there is a triangle not satisfying TT with an edge e. Take such an edge e with maximal weight and adopt the nearby notation as in Figure 1.1. Let p, q, r, s denote the widths of the bands parallel to P, Q, R, S of a decorated measured lamination $\mu \in \mathbb{R}^\Delta$.

TF evidently implies either p or r is minimum among p, q, r, s (cf. also Lemma 5.1), and we may suppose without loss that it is p. If $\mu(b) > \mu(e)$, then we must have $\mu(a) = \mu(b)$, which implies that there are no leaves of μ in the quadrilateral near e other than those parallel to the cusps, and furthermore $p = r$, which in turn implies that $\mu(c) = \mu(d)$ as well. Similar remarks apply in the case where $\mu(c)$ or $\mu(d)$ is greater than $\mu(e)$ Thus, $\mu(e)$ must be maximal among its neighbors.

Now, since $\mu(e)$ is maximal, then in particular $\mu(e) \geq \mu(b)$, i.e., letting x denote the width of the band of leaves that crossses the two sides of the quadrilateral near e, we have $p + x + r \geq s + x + r$, that is, $p \geq s$. Since we also necessarily have $p \leq s$, we conclude $p = s$. Thus, the flip here is neutral with respect to TF.

Furthermore, $\mu(e) \geq \mu(d)$ gives $p + x + r \geq p + x + q$, that is, $r \geq q$. If this is a strict inequality, then $\mu(f) = s + x + q = p + x + q < p + x + r = \mu(e)$, so we can flip the edge e and decrease the total weight, which is contrary to hypothesis.   q.e.d.

Consider a positive measure on the frontier edges of a triangle $t \in T$ complementary to Δ satisfying TT. The corresponding measured lamination of t consists of three bands of leaves of common width a connecting each pair of frontier edges of t together with exactly one further band of width b connecting a single pair of frontier edges of t, for some $a, b \geq 0$. Of course, a positive measure on an ideal triangulation Δ of a surface F must restrict to such a special measured lamination on each triangle complementary to Δ in F, and TT imposes further global constraints as follows.

Lemma 2.3. *Suppose that $\mu : \Delta \to \mathbb{R}$ is the measure of a decorated measured lamination with respect to some ideal triangulation Δ and that μ satisfies the tropical triangle inequalities on each triangle complementary to Δ. Then each leaf of the corresponding lamination $\mathcal{L} = c(\mu)$ is a simple closed curve.*

Proof. The proof is by downward induction on the number of times that μ achieves its maximum value. If this number agrees with the number of edges of Δ, i.e., if all measures are equal, then the corresponding decorated measured lamination is a collection of equally weighted collar curves by construction.

For the induction, suppose that the lemma is established if the measure takes its maximum value $n + 1$ times, and consider a measure μ taking its maximum value n times. Let m be the maximum value of μ and say that an arc in the ideal triangulation Δ is *maximal* if its measure is exactly m. Split each band of leaves of $\mathcal{L} = c(\mu)$ crossing a maximal edge into three consecutive bands of respective widths $\frac{1}{2}(m - \varepsilon)$, ε, and $\frac{1}{2}(m - \varepsilon)$.

We claim that for sufficiently small ε, if a leaf of $\mathcal{L}$ belongs to the central band for one edge, then it will intersect maximal edges only and belong to the central band for each of them. To see this, follow a leaf from a central band. It obviously stays in the central band when crossing a triangle with three maximal edges for any value of ε. When the leaf crosses a triangle with two maximal edges and one edge of measure $m' < m$, it evidently stays in the central band if and only if $\varepsilon \le m - m'$. In particular, the leaves in central bands are thus simple closed curves for sufficiently small ε.

Let us take the maximum possible ε satisfying this condition, namely, the minimum over all triangles with exactly two maximal edges of the quantity $m - m'$, where $m' < m$ is the measure of the edge of the triangle which is not maximal. If we remove from $\mathcal{L}$ the sub-lamination consisting of all of the simple closed leaves arising from central bands to produce a measured lamination $\mathcal{L}'$, then the measure of $\mathcal{L}'$ on Δ again satisfies the tropical triangle inequalities with the maximal edges of measure $m - \varepsilon$.

On the other hand, since at least one non-maximal edge has exactly this measure by our choice of ε, the maximal value is achieved more than n times for $\mathcal{L}'$. All leaves of $\mathcal{L}'$ are therefore simple closed curves by the induction hypothesis, and so $\mathcal{L}$ has this property as well. q.e.d.

Corollary 2.4. *Suppose that $\mu : \Delta \to \mathbb{R}$ is an integral measure satisfying the tropical triangle inequalities on an ideal triangulation Δ. If μ corresponds to a simple closed curve, then its value on each edge of Δ is at most three. It follows that for each fixed ideal triangulation Δ, there are only finitely many distinct isotopy classes of simple closed*

curves whose corresponding measures on Δ satisfy the tropical triangle inequalities.

Proof. The first part follows from the proof of the previous result where the maximum possible ε for the central band (with leaves isotopic to the simple closed curve) is at least one for an integral measure, and the second part follows immediately from the first. q.e.d.

3. Cyclic polygons and balanced laminations

A tuple $a_1, \ldots, a_n$ of real numbers might satisfy the following conditions:

(Classical) Generalized Triangle Inequalities CGT: For each $i = 1, \ldots, n$, we have $a_i < \sum_{j \neq i} a_j$.

Tropical Generalized Triangle Inequalities TGT: The maximum among $\{a_i\}_1^n$ is achieved at least twice, i.e., for each $i = 1, \ldots, n$, we have $a_i \leq \max\{a_j : j \neq i\}$.

Lemma 3.1. *Fix an ideal triangulation Δ of a polygon P and consider a collection of weights on the arcs in Δ including the frontier arcs of P. Then CT on the weights of Δ implies CGT on the weights of the frontier of P, and TT likewise implies TGT on the frontier.*

Proof. Each assertion follows by induction on the number of sides of P with the vacuous basis step for P a triangle. q.e.d.

We shall say that a Euclidean polygon is *cyclic* if it inscribes in a circle and recall the following two results:

Lemma 3.2. *Consider a cyclic Euclidean quadrilateral with a fixed diagonal arc e with nearby edges a, b, c, d as depicted in Figure 1.1. Then the Euclidean lengths are related by*

$$e^2 = \frac{(ac + bd)(ad + bc)}{(ab + cd)}.$$

Proof. This is algebraically equivalent to the vanishing of the simplicial coordinate in the face condition. q.e.d.

Theorem 3.3. [**5**, **8**] *A tuple $a_i \in \mathbb{R}_{>0}$, for $i = 1, \ldots, n$, occurs as the edge lengths of a cyclic Euclidean polygon if and only if it satisfies the classical generalized triangle inequalities CGT. Furthermore, such tuples of lengths of frontier edges of polygons uniquely determine the corresponding cyclic Euclidean polygon.*

In this section, we give the tropical analogues of Lemma 3.2 in Lemma 3.4 and of Theorem 3.3 in Proposition 3.5.

If Δ' is an ideal cell decomposition and $\Delta \supseteq \Delta'$ is an ideal triangulation of some surface F, then a point of $C(\Delta') \subseteq \widetilde{\mathcal{T}}$ is completely determined by the lambda lengths of the arcs in Δ' as follows. Fix some complementary region R of Δ'. The lambda lengths on the frontier of R satisfy CGT by Lemmas 2.2 and 3.1 and so determine a cyclic Euclidean polygon according to Theorem 3.3. An edge of $\Delta - \Delta'$ triangulating R has its lambda length given by the Euclidean length of the corresponding diagonal of this cyclic polygon. In particular, lambda lengths on the frontier edges of a cyclic polygon properly embedded in a decorated surface uniquely determine the lambda lengths on all diagonals of this polygon.

We shall say that a measured lamination of a polygon is *balanced* provided that for each ideal triangulation of it, the tropical triangle inequalities hold on each complementary triangle.

Lemma 3.4. *Consider a polygon with fixed ideal triangulation Δ. If $e \in \Delta$ with nearby edges a, b, c, d as depicted in Figure 1.1, then the measures of any balanced measured lamination of the polygon are related by*

$$e = \frac{1}{2}\Big\{\max\{a+c, b+d\} + \max\{a+d, b+c\} - \max\{a+b, c+d\}\Big\}.$$

Furthermore, $a, b, c, d \geq 0$ implies $e \geq 0$.

Proof. According to Lemma 2.2, the assumed tropical triangle inequalities imply the tropical face conditions on Δ, and likewise for the ideal triangulation that arises from Δ by a flip on e. It therefore follows that

$$e \leq \frac{1}{2}\{\max\{a+c, b+d\} + \max\{a+d, b+c\} - \max\{a+b, c+d\}\},$$

$$f \leq \frac{1}{2}\{\max\{a+c, b+d\} + \max\{a+b, c+d\} - \max\{a+d, b+c\}\},$$

so the sum provides the estimate $e + f \leq \max\{a+c, b+d\}$. On the other hand, the tropicalization of the Ptolemy equation gives the equality $e + f = \max\{a+c, b+d\}$, and therefore equality must hold also in the estimates above on e and f.

To prove the final assertion about non-negativity, observe that a, b, c, d satisfies TGT by Lemma 3.1 since a, b, e and c, d, e satisfy TT by hypothesis. There are several cases depending upon which of a, b, c, d achieve the maximum $\max\{a, b, c, d\}$:

If $a = b$ is the maximum, then $e = \max\{c, d\}$; if $a = c$ or $a = d$ is the maximum, then $e = a$; if $b = c$ or $b = d$ is the maximum, then $e = b$; and if $c = d$ is the maximum, then $e = \max\{a, b\}$.   q.e.d.

Proposition 3.5. *A tuple $a_i \in \mathbb{R}_{>0}$, for $i = 1, \ldots, n$, occurs as the measure on the frontier edges of a polygon P for a balanced measured*

lamination if and only if it satisfies the TGT inequalities. Furthermore, such tuples of measures on the frontier edges of P uniquely determine the corresponding measured lamination on P.

Proof. Necessity of TGT follows from Lemma 3.1. For sufficiency and to complete the proof, we must show that there is a unique balanced measured lamination of P realizing any given tuple of putative measures on its frontier edges which satisfies TGT. The proof proceeds by induction on the number n of sides of P. The case $n = 3$, where balanced $\Leftrightarrow$ TT $\Leftrightarrow$ TGT, was discussed already in Section 1, and the case $n = 4$ follows from the previous lemma.

For the inductive step with $n \geq 5$, consider two consecutive frontier edges a, a' of P and the diagonal d of P separating them from the rest. If the putative measures a and a' are different, then $d = \max\{a, a'\}$ is uniquely determined from the balanced condition. Furthermore, suppose that a, a', a'' are consecutive frontier edges of P with d and d', respectively, the diagonals separating a, a' from a'' and separating a from a', a'', and that d'' is the diagonal of P separating a, a', a'' from the rest of the frontier. If $a = a' = a''$, then $d + d' = \max\{a + a'', a' + d''\} = a + \max\{a, d''\}$ by the tropical Ptolemy equation, and so $d < a$ implies that $d' > \max\{a, d''\} \geq a$, which violates the balanced condition. It therefore follows that $d = d' = a$ in this case. In either case, cutting on the diagonal d produces an $(n-1)$-gon to which we may apply the inductive hypothesis and thereby complete the proof of existence.

As to uniqueness, any diagonal d of P separates the frontier edges of P into two disjoint sets A, A'. The determination of measure on d in the previous paragraph is given by $d = \min\{\max A, \max A'\}$ independently of any choices as desired.   q.e.d.

REMARK 3.6. In particular, measures of a balanced decorated measured lamination on the frontier edges of a polygon properly embedded in a surface uniquely determine the measures on all diagonals of this polygon. According to Theorem 1.2, these thus computable measures describe the asymptotics of the more elusive logarithms of lambda lengths.

4. Asymptotics of cells

In this section, we give the tropical analogue of [10]; namely, we answer the question of which measured laminations are accessible, i.e., are the asymptotes of paths of geometric structures within a fixed cell $C(\Delta)$ of the cell decomposition of $\tilde{\mathcal{T}}$.

Lemma 4.1. *Fix an ideal triangulation Δ and consider a sequence $\lambda_i \in \mathbb{R}_{>0}^{\Delta}$ with $c(\lambda_i) \in C(\Delta) \subseteq \tilde{\mathcal{T}}$. If $\log \lambda_i$ converges projectively to μ, then μ satisfies TT on Δ. Conversely, if μ satisfies TT, then there is*

a sequence $\lambda_i \in C(\Delta)$ so that the $\log \lambda_i$ converges projectively to μ. In particular, if $\pi \circ c(\lambda_i)$ tends to $[\pi \circ c(\mu)]$, where $c(\lambda_i) \in C(\Delta)$, then each leaf of $\pi \circ c(\mu)$ is a simple closed curve.

Proof. To prove the first assertion, suppose to begin that $\pi \circ c(\lambda_i)$ is asymptotic to $[\pi \circ c(\mu)]$. According to Theorem 1.2, the projective class of the measure μ is given by the limiting values of logarithms of lambda lengths $\log \lambda_i$ on each edge of Δ. Furthermore, each $\lambda_i > 0$ satisfies the face conditions since $c(\lambda_i) \in C(\Delta)$ by hypothesis and hence also satisfies the triangle inequality by Lemma 2.2. Since an inequality between expressions constructed using only addition and multiplication implies its tropicalization, it follows that the tropical triangle inequalities on measures must hold as well. The last assertion then follows directly from Lemma 2.3.

For the converse, given $\mu \in \mathbb{R}^\Delta$ satisfying the tropical triangle inequalities, define the one-parameter family of lambda lengths

$$\lambda_t(e) = t^{\mu(e)}, \text{ for } e \in \Delta \text{ and } t \in \mathbb{R}_{>0}.$$

Consider an edge $e \in \Delta$ with nearby edges as illustrated in Figure 1.1. As before, since the maximum of μ is achieved either twice or thrice on $\{a, b, e\}$ and on $\{c, d, e\}$, each summand in brackets in the face condition for λ_t is strictly positive, and hence so too is their sum. It follows that $\lambda_t \in C(\Delta)$, for each $t \geq 1$, and according to Theorem 1.2, we furthermore have that $\pi \circ c(\lambda_t) \in \mathcal{T}$ converges projectively to $[\pi \circ c(\mu)] \in \mathcal{PL}_0$ as t goes to infinity, as required.   q.e.d.

Theorem 4.2. *Fix an ideal cell decomposition Δ and consider a sequence $\lambda_i \in \mathbb{R}_{>0}^\Delta$ with $c(\lambda_i) \in C(\Delta) \subseteq \widetilde{\mathcal{T}}$. If $\log \lambda_i$ converges projectively to μ with corresponding measured lamination $\mathcal{L}$, then the restriction of $\mathcal{L}$ to each component of $F - \Delta$ is balanced. Conversely, if $\mathcal{L}$ corresponding to μ is balanced on Δ, then there is a sequence $\lambda_i \in C(\Delta)$ with $\log \lambda_i$ converging projectively to μ. In particular, if $\pi \circ c(\lambda_i)$ tends to $[\mathcal{L}]$, where $c(\lambda_i) \in C(\Delta)$, then each leaf of $\mathcal{L}$ is a simple closed curve. Only finitely many simple closed curves, namely, those whose measures on Δ are bounded above by three and satisfy the tropical triangle inequalities, can occur as leaves.*

Proof. Complete Δ to an ideal triangulation Δ' making choices. The generic case of an ideal triangulation $\Delta' = \Delta$ is covered by the previous result, which furthermore shows that the balanced condition is necessary since $C(\Delta) \subseteq C(\Delta')$.

As to sufficiency, let $\mu : \Delta \to \mathbb{R}_{>0}$ denote the measure of a measured lamination $\mathcal{L}$ which is balanced on each component of $F - \Delta$. Define a one-parameter family of lambda lengths

$$\lambda_t(e) = t^{\mu(e)}, \text{ for } e \in \Delta \text{ and } t \in \mathbb{R}_{>0}.$$

Again by Lemmas 2.2 and 3.1, λ_t satisfies CGT on components of $F - \Delta$, for each t, and so uniquely determines a cyclic Euclidean polygon by Theorem 3.3 corresponding to each such component. If $e \in \Delta' - \Delta$, then let $\lambda_t(e)$ be given by the Euclidean length of the corresponding diagonal of the associated cyclic polygon. This defines λ_t on each edge of Δ' for each t.

By construction, we have $\pi \circ c(\lambda_t) \in C(\Delta')$, for all t, and by Lemma 4.1, $\pi \circ c(\lambda_t) \in C(\Delta')$ is furthermore asymptotic to a measured lamination $\mathcal{L}'$ whose measure μ' on Δ' agrees with μ on $\Delta \subseteq \Delta'$. Since the lambda lengths λ_t satisfy CT on Δ, μ' must satisfy TT on Δ'. Similarly, for any other ideal triangulation $\Delta'' \subseteq \Delta$, the analogously defined lambda lengths on Δ'' describe the same path in $\widetilde{\mathcal{T}}$, therefore have the same asymptote $\mathcal{L}'$, and likewise have measure on Δ'' satisfying TT. It follows that $\mathcal{L}'$ is balanced on each component of $F - \Delta$.

Thus, $\mathcal{L}$ and $\mathcal{L}'$ are both balanced on each component of $F - \Delta$. Since their measures μ and μ' agree on Δ by construction, we must have $\mathcal{L} = \mathcal{L}'$ by the last part of Proposition 3.5.

The final assertion follows from Corollary 2.4.   q.e.d.

5. Simplifying Triangulations

Fix any ideal triangulation Δ of a punctured surface F with complementary triangles T, and consider a (decorated) measured lamination described by a measure $\mu \in \mathbb{R}^{\Delta}$.
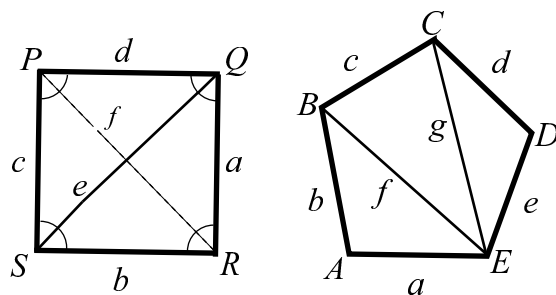

Figure 5.1. Standard notation for squares and pentagons.

One might hope to "simplify" the situation by choosing another ideal triangulation with smaller total intersection number $\sum_{X \in \Delta} \mu(X)$. The example with three bands of arcs of respective weights 7,8,2 parallel to the corners B, C, D of the pentagon in the notation of Figure 5.1 (i.e., the respective measure on $a, \ldots, g$ is given by 0,7,15,10,2,7,8) shows that there are not always flips to decrease the total intersection number.

A more delicate objective function to minimize is as follows. Given a triangle t with edges a, b, c, define the *weight* or *perimeter* of μ on t to

be $\mu(t) = \mu(a) + \mu(b) + \mu(c)$ and let

$$\mu(\Delta) = \mu(T) = \ \mathrm{sort}_\downarrow\{\mu(t) : t \in T\}$$

be the sorted list of perimeters of triangles in T ordered lexicographically in decreasing order, where we recall that T denotes the set of triangles complementary to Δ in F. Of course, the sets Δ and T uniquely determine one another, and triangle-based objective functions are perhaps more naturally apprehended as functions on T.

Lemma 5.1. *Consider a flip along an edge e of Δ triangulating a quadrilateral $\mathcal{Q}$ with corners P, Q, R, S, where e is incident on corners P and R, as illustrated on the left in Figure* 5.1, *and fix a measured lamination μ of $\mathcal{Q}$ transverse to the boundary. The following conditions are equivalent:*

i) the flip along e is non-decreasing for $\mu(\Delta)$;

ii) Q or S has the least arcs around the corners of $\mathcal{Q}$, i.e., we have

$$\min\left\{\min\{a + e - b, e + d - c\}, \ \min\{b + e - a, c + e - d\}\right\}$$

$$\leq \ \min\left\{a + b - e, c + d - e\right\};$$

iii) we have $2e + \max\{a+b, c+d\} \leq \max\{a+c, b+d\} + \max\{a+d, b+c\}$;

iv) the tropical face condition TF *holds, i.e., we have $e + \max\{a+b, c+ d\} \leq f + \max\{a + d, b + c\}$.*

Proof. $(i \Rightarrow ii)$ It is a small exercise with measures to confirm that the width of the band of leaves parallel to P (or R) is $\frac{c+d-e}{2}$ (or $\frac{a+b-e}{2}$) and furthermore the first (or second) term in the minimum on the left-hand side in part ii is twice the width of the band of leaves which are parallel to Q (or S). Thus, the minimization in part ii is indeed equivalent to the condition on corners, which is obviously a consequence of condition i.

$(ii \Rightarrow iii$ If $\min\{a + e - b, e + d - c\} \leq \min\{b + e - a, c + e - d\}$, for example, then $\min\{a+e-b, d+e-c\} \leq \min\{a+b-e, c+d-e\}$ implies, in particular, that $e \leq b$ and $e \leq c$. Thus,

$$\max\{a + c, b + d\} + \max\{a + d, b + c\} \geq a + b + 2c \geq 2e + a + b,$$
$$\max\{a + c, b + d\} + \max\{a + d, b + c\} \geq 2b + c + d \geq 2e + c + d,$$

so

$$\max\{a + c, b + d\} + \max\{a + d, b + c\} \geq \max\{2e + a + b, 2e + c + d\}$$
$$= 2e + \max\{a + b, c + d\}$$

as required. The remaining case that the band at S is thinner than the band at Q is analogous.

($iii \Leftrightarrow iv \Leftrightarrow i$) Conditions iii and iv are actually equivalent by the tropical Ptolemy equation, and iv is evidently equivalent to i by definition.
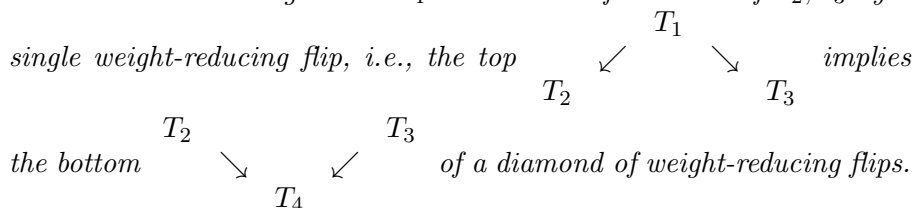
q.e.d.

It is worth emphasizing that the lemma shows in particular that the perimeter function gives the correct tropicalization TF of the convexity condition CF.

Theorem 5.2. *Fix a measured lamination μ on surface F. If $\mu(\Delta)$ is not minimal among all ideal triangulations Δ of F, then there is a (possibly empty) finite sequence of weight-neutral flips followed by a flip that reduces it.*

The proof will show that a global minimum exists in the special case that all leaves of μ are closed, but not in general. Furthermore, one can bound above the number of neutral flips required before a reducing one can be found.

Proof. To begin, we assume that μ is generic in the sense that there are no coincidences among the values taken on the edges of T, i.e., no integral equalities hold. In this case, the theorem follows from the following

Lemma 5.3 (Diamond Lemma). *Suppose that T_1 is a triangulation and each of T_2, T_3 arise from T_1 by a single weight-reducing flip. Then there is another triangulation T_4 which arises from each of T_2, T_3 by a single weight-reducing flip, i.e., the top* $\begin{array}{ccc} & T_1 & \\ \swarrow & & \searrow \\ T_2 & & T_3 \end{array}$ *implies the bottom* $\begin{array}{ccc} T_2 & & T_3 \\ \searrow & & \swarrow \\ & T_4 & \end{array}$ *of a diamond of weight-reducing flips.*

First of all, this lemma is enough to prove the theorem. Indeed, suppose that $\mu(T) > \mu(T')$. The triangulations T and T' can be connected by a sequence or path of flips. If the flip starting from T reduces the weight, then we are done. Otherwise, there is some local maximum T_1 along the path of flips. The Diamond Lemma allows us to push the weight of the path down near T_1. Taking T_1 to be the closest maximum to T along the path, then, pushing down the weight at T_1 is tantamount to moving this first maximum closer to T. This process can be repeated until the path is weight-decreasing immediately already at T as required for Theorem 5.2.

Proof of the Diamond Lemma. The argument is by cases on how the two flips from T_1 interact, and we consider the two quadrilateral neighborhoods supporting these two flips, comprising at most four triangles.

Case 1: If the two quadrilaterals are disjoint, then the two flips commute, and T_4 arises from T_1 by their composition.
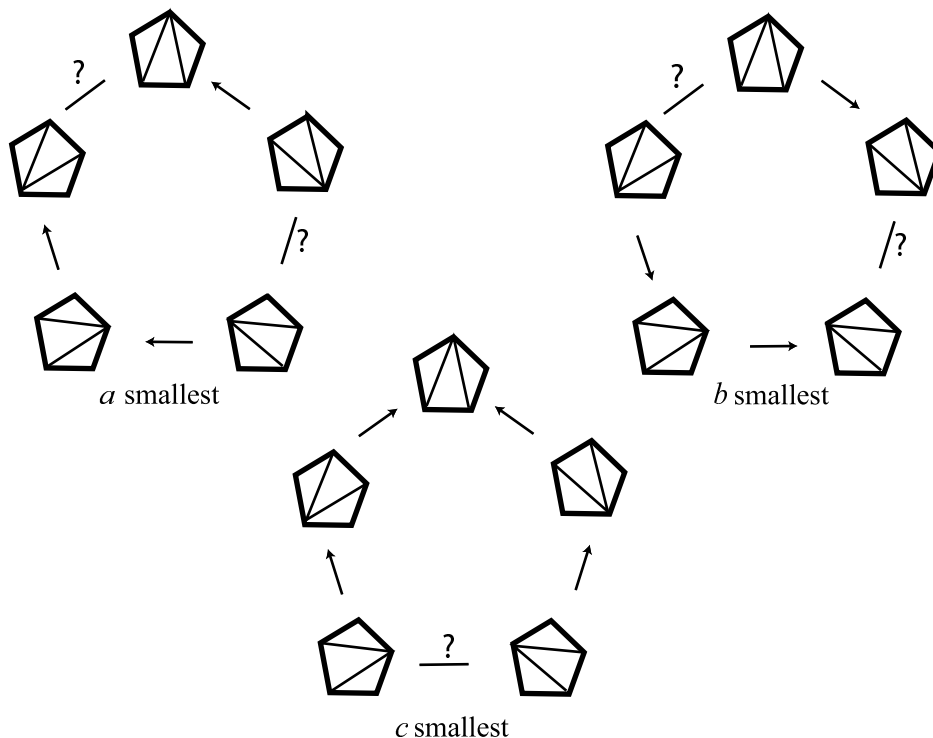

Figure 5.2. Paths in a pentagon for Case 2.

Case 2: If the two quadrilaterals overlap in one triangle, then their union is a pentagon with vertices labeled as illustrated in Figure 5.1 but here letting $a, \cdots, e$ denote the respective widths of bands of a measured lamination parallel to $A, \cdots, E$. We must check that regardless of the weights, there is a unique maximum and a unique minimum of weight among the five possible triangulations of the pentagon and all weight-nonincreasing flips, and we proceed by cases on which of a, b, c, d, e is smallest.

a is smallest: In this case, Lemma 5.1 forces the weight-decreasing flips illustrated in Figure 5.2a. No matter how the remaining arrows are completed, there can be only one maximum or minimum. A similar argument holds when e is smallest.

b is smallest: Now the forced weight-decreasing flips are illustrated in Figure 5.2b, and again, there are unique extrema. A similar argument holds when *d* is smallest.

c is smallest: In this case, almost everything is fixed as illustrated in Figure 5.2c.

Case 3: Suppose the two quadrilaterals overlap in two triangles configured as an annulus, as illustrated in Figure 5.3. We may assume without loss of generality that the measured lamination is in its standard position as illustrated on the top-right in Figure 5.3. The triangulations are then necessarily ordered in the natural way as illustrated on the bottom in the figure. Thus, there can be no local maximum, and this contradicts the initial assumptions.
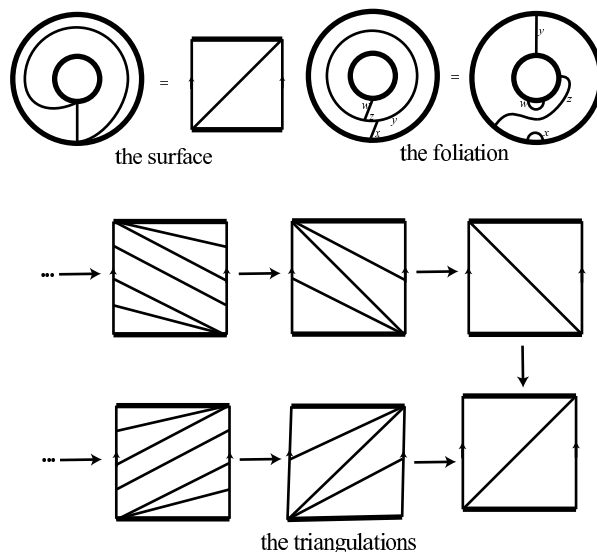

the surface the foliation

the triangulations

Figure 5.3. No local maximum for an annulus in Case 3.

Case 4: Suppose the two quadrilaterals overlap in two triangles, which is configured as a bigon as illustrated in Figure 5.4. Again, we may assume without loss of generality that the measured lamination is standard, as illustrated on the top-right in Figure 5.4. On the bottom of this figure are shown the two possible transitions from the unique possible initial triangulation each of whose edges has distinct endpoints, and arithmetic shows that the two transitions are weight-decreasing under the conditions depicted. However, it cannot be that both $x - y$ and $y - x$

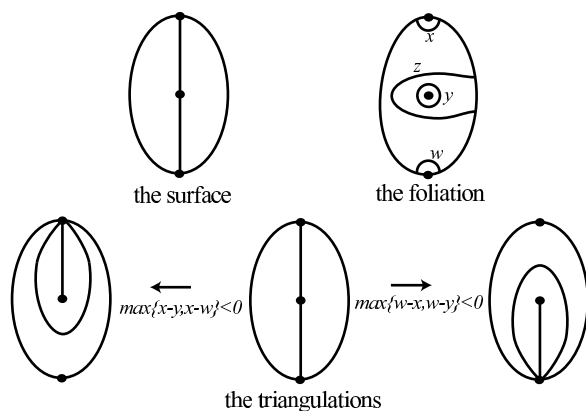are negative, so the initial triangulation is not a local maximum, again contradicting the assumptions.

Figure 5.4. No local maximum for a bigon in Case 4

This completes the proof, provided the lamination is generic. In the general case, say μ is an integral measured lamination and let $\mu' = \mu + \varepsilon$ be a generic perturbation of μ. We may perform flips trying to improve with respect to μ' as much as possible. These flips will either be weight-decreasing or -neutral. However, there is a bound in terms of the genus on how many neutral flips can be performed in sequence before returning to the same combinatorial type of triangulation, and hence there is a mapping class that leaves μ invariant. Such a sequence can therefore be removed from an overall weight-decreasing sequence of flips, so the general case follows from the generic one.   q.e.d.

Together with the third part of Lemma 2.2, this theorem has the following immediate consequence:

Corollary 5.4. *For any decorated measured lamination μ with all leaves closed in a punctured surface, there is a (not necessarily unique) ideal triangulation Δ with complementary triangles T so that μ satisfies TT on Δ. Moreover, there is an algorithm for finding Δ based on minimizing first the sorted perimeters* $\mathrm{sort}_\downarrow\{\mu(a) + \mu(b) + \mu(c) : t \in T\}$ *of triangles and then the total weight* $\sum_{e \in \Delta} \mu(e)$ *of edges.*

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