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LARGE ISOPERIMETRIC SURFACES IN INITIAL DATA SETS

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Abstract

We study the isoperimetric structure of asymptotically flat Riemannian 3-manifolds (M, g) that are $\mathcal{C}^0$-asymptotic to Schwarzschild of mass $m > 0$. Refining an argument due to H. Bray, we obtain an effective volume comparison theorem in Schwarzschild. We use it to show that isoperimetric regions exist in (M, g) for all sufficiently large volumes, and that they are close to centered coordinate spheres. This implies that the volume-preserving stable constant mean curvature spheres constructed by G. Huisken and S.-T. Yau as well as R. Ye as perturbations of large centered coordinate spheres minimize area among all competing surfaces that enclose the same volume. This confirms a conjecture of H. Bray. Our results are consistent with the uniqueness results for volume-preserving stable constant mean curvature surfaces in initial data sets obtained by G. Huisken and S.-T. Yau and strengthened by J. Qing and G. Tian. The additional hypotheses that the surfaces be spherical and far out in the asymptotic region in their results are not necessary in our work.

1. Introduction

In this paper we describe completely the large isoperimetric surfaces of asymptotically flat Riemannian 3-manifolds (M, g) that are $\mathcal{C}^0$-asymptotic to the Schwarzschild metric of mass $m > 0$. Such Riemannian manifolds arise naturally as initial data for the time-symmetric Cauchy problem for the Einstein equations in general relativity. For brevity we will refer to such (M, g) as initial data sets in the introduction.

A special case of the singularity theorem of Hawking and Penrose asserts that the future spacetime development of time-symmetric initial data for the Einstein equations that contains a closed minimal surface is causally incomplete. As a trial for cosmic censorship, R. Penrose suggested that the area of an outermost minimal surface in an initial data set should provide a lower bound for the ADM-mass of the spacetime development of the initial data set. The "Penrose inequality" has been

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established by H. Bray in [**5**] and by G. Huisken and T. Ilmanen in [**26**]. We emphasize that the outermost minimal surface is known to be outer area-minimizing (in particular, it is strongly stable). This variational feature is of essential importance in both available proofs of the Penrose inequality. A deep relation between the existence of stable minimal surfaces in initial data sets and their ADM-mass has been recognized and exploited by R. Schoen and S.-T. Yau in [**44**] in their proof of the positive energy theorem. Their work has made a profound connection between the physical concept of mass and the geometry of manifolds with non-negative scalar curvature.

It is natural to ask if other physical properties of the spacetime development of an initial data set (M, g) are captured by its geometry. Maybe they are witnessed by the existence and behavior of special surfaces in (M, g), and their behavior? The variational properties associated with constant mean curvature surfaces in (M, g) generalize the geometric properties of the horizon in a natural way.

In [**27**], G. Huisken and S.-T. Yau showed that the asymptotic region of an initial data set (M, g) that is $\mathcal{C}^4$-asymptotic to Schwarzschild of mass $m > 0$ in the sense of Definition 2.2 is foliated by strictly volume-preserving stable constant mean curvatures spheres that are perturbations of large coordinate balls. Moreover, these spheres are unique among volume-preserving stable constant mean curvature spheres in the asymptotic region that lie outside a coordinate ball of radius H^{-q}, where H denotes their constant mean curvature and where $q \in (\frac{1}{2}, 1]$. They also concluded that as the enclosed volume gets larger, these surfaces become closer and closer to round spheres whose centers converge in the limit as the volume tends to infinity to the Huisken–Yau "geometric center of mass" of (M, g). See also the announcement [**8**, p. 14]. R. Ye [**48**] has an alternative approach to proving existence of such foliations. In [**40**], J. Qing and G. Tian strengthened the uniqueness result of [**27**] by showing the following: every volume-preserving stable constant mean curvature sphere in an initial data set that is $\mathcal{C}^4$-asymptotic to Schwarzschild of mass $m > 0$ that contains a certain large coordinate ball (independent of the mean curvature of the surface) belongs to this foliation.

The assumption $m > 0$ in the results described in the preceding paragraph is necessary: the constant mean curvature surfaces of $\mathbb{R}^3$ are neither strictly volume-preserving stable nor unique. In view of the results in [**27**, **40**], and loosely speaking, positive mass has the property that it centers large, outlying volume-preserving stable constant mean curvature surfaces. Various extensions of these results that allow for weaker asymptotic conditions have been proven in [**34**], [**23**], and [**31**, **30**]. In [**22**], L.-H. Huang has shown that the "geometric center of mass" of G. Huisken and S.-T. Yau coincides with other invariantly defined notions for the center of mass.

In his thesis [**4**], H. Bray started a systematic investigation of isoperimetric surfaces in initial data sets and their relationship with mass, quasi-local mass, and the Penrose inequality. He showed that the isoperimetric surfaces of Schwarzschild are exactly round centered spheres. He deduced that the large isoperimetric surfaces in initial data sets that are compact perturbations of the exact Schwarzschild metric are also round centered spheres. Furthermore, he gave a proof of the Penrose inequality under the additional assumption that there exist connected isoperimetric surfaces enclosing any given volume in (M, g). This proof builds on H. Bray's important observation that his isoperimetric Hawking mass is monotone increasing with the volume in this case. (In fact, H. Bray pointed out that the Hawking mass is monotone along foliations through connected volume-preserving stable constant mean curvature spheres whose area is increasing, such as those constructed in [**27**, **48**].) In [**4**, p. 44], H. Bray conjectured that the volume-preserving stable constant mean curvatures surfaces of [**27**, **48**] are isoperimetric surfaces. The results in the present paper confirm this.

Theorem 1.1. *Let (M, g) be an initial data set that is $\mathcal{C}^0$-asymptotic to Schwarzschild of mass $m > 0$ in the sense of Definition* 2.2. *There exists $V_0 > 0$ such that for every $V \geq V_0$ the infimum in*

$$A_g(V) := \inf\{\mathcal{H}^2_g(\partial^*\Omega) : \Omega \textit{ is a Borel set of volume } V \textit{ that contains the horizon and has finite perimeter}\}$$

is achieved. Every minimizer has a smooth bounded representative whose boundary consists of the horizon and a connected surface that is close to a centered coordinate sphere.

In conjunction with [**27**], we immediately obtain the following corollary.

Corollary 1.2. *If the initial data set (M, g) is $\mathcal{C}^4$-asymptotic to Schwarzschild of mass $m > 0$ in the sense of Definition* 2.2*, then the boundaries of the large isoperimetric regions of Theorem* 1.1 *coincide with the volume-preserving stable constant mean curvature surfaces constructed in* [**27**]. *In particular, for every sufficiently large volume there exists a unique isoperimetric region in (M, g) of that volume. The boundaries of these regions foliate the complement of a bounded subset of (M, g).*

It follows that the isoperimetric profile $A_g(V)$ of (M, g) for large volumes V is exactly determined. This mirrors the situation in compact Riemannian manifolds whose scalar curvature assumes its maximum at a unique point p. Under those assumptions, small isoperimetric regions are known to be perturbations of geodesic balls centered at p. (This follows from [**11**]. See also [**36**, Theorem 2.2] and [**38**, Corollary 3.12].)

G. Huisken has initiated a program where the mass of an initial data set and the quasi-local mass of subsets of initial data sets are studied via isoperimetric deficits from Euclidean space. One great advantage of this approach is that only very low regularity is required of the initial data set. Theorem 1.1 identifies m as the only sensible candidate for any notion of mass that is defined in terms of $A_g(V)$ when the initial data set is $\mathcal{C}^0$-asymptotic to Schwarzschild of mass $m > 0$; cf. [**4**]. A result of X.-Q. Fan, Y. Shi, and L.-T. Tam [**15**, Corollary 2.3] subsequent to the work of G. Huisken shows that the ADM mass of an initial data set that has integrable scalar curvature and which is $\mathcal{C}^0$-asymptotic to Schwarzschild of mass $m > 0$ equals m.

In a sequel [**13**] to this paper, we generalize our main result Theorem 1.1 to arbitrary dimensions. We also show that in Corollary 1.2, it is enough to assume that (M, g) is $\mathcal{C}^2$-asymptotic to Schwarzschild of mass $m > 0$. In Appendix H of [**13**] we provide an extensive overview of the portion of the literature on isoperimetric regions on Riemannian manifolds related to our results.

Structure of this paper. In Section 2 we introduce the precise decay assumptions for initial data sets that we use in this paper, and we define what exactly we mean by isoperimetric and locally isoperimetric regions. In Section 3 we prove an effective volume comparison theorem for regions in initial data sets that are $\mathcal{C}^0$-asymptotic to Schwarzschild. In Section 4 we review the classical results on the regularity of isoperimetric regions and behavior of minimizing sequences for the isoperimetric problem that we need in this paper. The effective volume comparison theorem is applied in Section 5 to show that isoperimetric regions exist for every sufficiently large volume in initial data sets that are $\mathcal{C}^0$-asymptotic to Schwarzschild, and that these regions become close to large centered coordinate balls as their volume increases. In Section 6 we present our most general result on the behavior of isoperimetric regions in asymptotically flat initial data sets that are not assumed to be close to Schwarzschild: either such regions slide away entirely into the asymptotically flat end of the initial data set as their volume grows large, or they begin to fill up the whole initial data set. The results in this section are largely independent of the remainder of the paper. In Appendix A we collect several useful lemmas regarding integrals of polynomially decaying quantities over surfaces with quadratic area growth. In Appendix B we summarize some steps and results from H. Bray's thesis. Appendix C contains a "friendly" proof that limits of isoperimetric regions with divergent volumes in initial data sets have area-minimizing boundaries. This fact is used in the proof of Theorem 6.1.

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2. Definitions and notation

Definition 2.1. Let $m > 0$. We denote by (M_m, g_m) the complete Riemannian manifold $(\mathbb{R}^3 \setminus \{0\}, \phi_m^4 \sum_{i=1}^3 dx_i^2)$, where $\phi_m = \phi_m(x) := 1 + \frac{m}{2r}$, $r = r(x) := \sqrt{x_1^2 + x_2^2 + x_3^2}$, and where (x_1, x_2, x_3) are the coordinate functions on $\mathbb{R}^3$. (M_m, g_m) is a totally geodesic spacelike slice of the Schwarzschild spacetime of mass $m > 0$. We refer to (M_m, g_m) as the Schwarzschild metric of mass $m > 0$ for brevity, to the coordinates (x_1, x_2, x_3) as isotropic coordinates on (M_m, g_m), and to $r(x)$ as the isotropic radius of $x \in M_m$.

The conformal factor ϕ_m is harmonic on $\mathbb{R}^3 \setminus \{0\}$. It follows that the scalar curvature of g_m vanishes. The coordinate spheres $\{x \in M_m : r(x) = r\} \subset M_m$ will be denoted by S_r. Note that $S_{\frac{m}{2}}$ is a minimal surface. It is called the horizon of (M_m, g_m). The inversion $x \to \left(\frac{m}{2}\right)^2 \frac{x}{r(x)^2}$ induces a reflection symmetry of (M_m, g_m) across the horizon. The area of the isotropic coordinate sphere S_r is equal to $\phi_m^4 4\pi r^2$. Its mean curvature with respect to the unit normal $\phi_m^{-2}\partial_r$ equals $\phi_m^{-3}(1 - \frac{m}{2r})\frac{2}{r}$. The Hawking mass $m(\Sigma) := (16\pi)^{-3/2}\sqrt{\mathcal{H}^2_{g_m}(\Sigma)}\left(16\pi - \int_\Sigma H_\Sigma^2 d\mathcal{H}^2_{g_m}\right)$ which is defined on closed surfaces $\Sigma \subset M_m$ is equal to m when $\Sigma = S_r$.

Definition 2.2. An initial data set (M, g) is a connected complete Riemannian 3-manifold, possibly with compact boundary, such that there exists a bounded open set $U \subset M$ with $M \setminus U \cong_x \mathbb{R}^3 \setminus B(0, \frac{1}{2})$ and such that in the coordinates induced by $x = (x_1, x_2, x_3)$,

$$r|g_{ij} - \delta_{ij}| + r^2|\partial_k g_{ij}| + r^3|\partial^2_{kl} g_{ij}| \le C \text{ where } r := \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

If $\partial M \neq \emptyset$, we assume that ∂M is a minimal surface, and that there are no compact minimal surfaces in M besides the components of ∂M. The boundary of M is called the horizon of (M, g). Given $m > 0$ and an integer $k \ge 0$, we say that an initial data set is $\mathcal{C}^k$-asymptotic to Schwarzschild of mass $m > 0$ if

$$(1) \qquad \sum_{l=0}^{k} r^{2+l}|\partial^l (g - g_m)_{ij}| \le C \text{ where } (g_m)_{ij} = (1 + \frac{m}{2r})^4 \delta_{ij}.$$

A few remarks are in order. The decay assumptions for initial data sets here are quite weak. In particular, the ADM-mass is not defined for such initial data sets unless a further condition—namely, the integrability of the scalar curvature—is imposed.

We extend r as a smooth regular function to the entire initial data set (M, g) such that $r(U) \subset [0, 1)$, except for the case of exact Schwarzschild (M_m, g_m), where we retain the convention that $r(x)$ denotes the the isotropic radius introduced just below Definition 2.1. We use S_r to denote the surface $\{x \in M : |x| = r\}$, and B_r to denote the region $\{x \in M : |x| \leq r\}$. We will refer to S_r as the centered coordinate sphere of radius r. We will not distinguish between the end $M \setminus U$ of M and its image $\mathbb{R}^3 \setminus B(0, \frac{1}{2})$ under x. By the work of W. Meeks, L. Simon, and S.-T. Yau [**33**] (see also the discussion in [**26**, Section 4]), M is diffeomorphic to $\mathbb{R}^3$ minus a finite number of open balls whose closures are disjoint.

Given an initial data set (M, g), we fix a complete Riemannian manifold $(\hat{M}, \hat{g})$ diffeomorphic to $\mathbb{R}^3$ that contains (M, g) isometrically. We say that a Borel set $U \subset \hat{M}$ contains the horizon if $\hat{M} \setminus M \subset U$. If such a set U has locally finite perimeter, we denote its reduced boundary in $(\hat{M}, \hat{g})$ by $\partial^* U$. Note that $\partial^* U$ is supported in M, and that $\mathcal{H}^2_g(\partial^* U) = \mathcal{H}^2_{\hat{g}}(\partial^* U)$. To lighten the notation, we write $\mathcal{L}^3_g(U) := \mathcal{L}^3_{\hat{g}}(U \cap M)$ for short.

Definition 2.3. The isoperimetric area function $A_g : [0, \infty) \to [0, \infty)$ is defined by

$$A_g(V) := \inf\{\mathcal{H}^2_g(\partial^* U) : U \subset \hat{M} \text{ is a Borel containing the horizon and of finite perimeter with } \mathcal{L}^3_g(U) = V\}.$$

A Borel set $\Omega \subset \hat{M}$ containing the horizon and of finite perimeter such that $\mathcal{L}^3_g(\Omega) = V$ and $A_g(V) = \mathcal{H}^2_g(\partial^*\Omega)$ is called an isoperimetric region of (M, g) of volume V. A Borel set $\Omega \subset \hat{M}$ containing the horizon and of locally finite perimeter is called locally isoperimetric if $\mathcal{H}^2_g(B \cap \partial^*\Omega) \leq \mathcal{H}^2_g(B \cap \partial^* U)$ whenever $B \subset \hat{M}$ is a bounded open subset of $\hat{M}$ and $U \subset \hat{M}$ is a Borel set containing the horizon and of locally finite perimeter such that $\mathcal{L}^3_{\hat{g}}(\Omega \cap B) = \mathcal{L}^3_{\hat{g}}(U \cap B)$ and $\Omega \Delta U \Subset B$.

The definition of A_g, as well as that of isoperimetric and locally isoperimetric regions, is independent of the particular extension $(\hat{M}, \hat{g})$ of (M, g). Note that $A_g(0) = \mathcal{H}^2_g(\partial M)$ and that $A_g(V) > \mathcal{H}^2_g(\partial M)$ for every $V > 0$. The latter assertion follows from the assumption that the boundary of M is an outermost minimal surface. Locally isoperimetric regions arise naturally as limits of isoperimetric regions whose volumes diverge. A good example to keep in mind is a half-space in $\mathbb{R}^3$. Standard results in geometric measure theory imply that the boundary of a (locally) isoperimetric region Ω is smooth, that $\Omega \cap \partial M = \emptyset$ unless the enclosed volume $\mathcal{L}^3_g(\Omega) = \mathcal{L}^3_{\hat{g}}(\Omega \cap M)$ is 0, and that isoperimetric regions are compact. Indications of the proofs of these facts with precise references to the literature to assist the reader are given in Section 4, below.

The inequalities in the following lemma are well-known, and we recall them for convenient reference.

Lemma 2.4. *Let (M, g) be an initial data set. There exists a constant $\gamma > 0$ such that*

$$\left(\int_M |f|^{\frac{3}{2}} d\mathcal{L}^3_g \right)^{\frac{2}{3}} \leq \gamma \int_M |\nabla f| d\mathcal{L}^3_g \text{ for every } f \in \mathcal{C}^1_c(M). \tag{2}$$

If the boundary of M is empty, the constant $\gamma > 0$ can be chosen such that for any bounded Borel set $\Omega \subset M$ with finite perimeter one has that

$$\mathcal{L}^3_g(\Omega)^{\frac{2}{3}} \leq \gamma \mathcal{H}^2_g(\partial^* \Omega).$$

Proof. The Sobolev inequality stated here can be obtained exactly as in [**44**, Lemma 3.1] by combining, in a contradiction argument, the Euclidean Sobolev inequality in the form

$$\left(\int_{\mathbb{R}^3 \setminus B(0,1)} |f|^{\frac{3}{2}} d\mathcal{L}^3_\delta \right)^{\frac{2}{3}} \leq \gamma_0 \int_{\mathbb{R}^3 \setminus B(0,1)} |\nabla f| d\mathcal{L}^3_\delta \text{ for all } f \in \mathcal{C}^1_c(\mathbb{R}^3)$$

and Poincaré-type inequalities (see [**29**, §8.12] for the appropriate version with critical exponent) on precompact coordinate charts. We recall (cf. [**7**, Theorem II.2.1]) that the isoperimetric estimate for smoothly bounded compact regions Ω follows from applying this Sobolev inequality to approximations of the indicator function χ_Ω by Lipschitz functions that are one on Ω and that drop off to 0 linearly in the distance from Ω. The isoperimetric inequality for sets of finite perimeter is obtained by approximation through smooth sets. q.e.d.

3. Effective refinement of H. Bray's characterization of isoperimetric surfaces in Schwarzschild

In his thesis [**4**], H. Bray proved that large isoperimetric surfaces of compact perturbations of the Schwarzschild metric with mass $m > 0$ are centered coordinate spheres in isotropic coordinates. In this section, we refine H. Bray's work to derive an effective lower bound for the isoperimetric defect of off-centered surfaces in Schwarzschild. This bound gives us enough quantitative information to characterize large isoperimetric surfaces in manifolds that are $\mathcal{C}^0$-asymptotic to Schwarzschild of mass $m > 0$, as we will see in Section 5.

We begin with a description of the "volume-preserving" charts used by H. Bray. We refer the reader to Appendix B for an overview of related results from H. Bray's thesis that should be noted in this context.

Let $\alpha > 0$. Consider the metric cone $\alpha^{-2}ds^2 + \alpha s^2 g_{\mathbb{S}^2}$ on $(0,\infty)\times\mathbb{S}^2$. The sphere $\{c\}\times\mathbb{S}^2$ has area $\alpha 4\pi c^2$ and mean curvature $\frac{2\alpha}{c}$. One can choose $c > 0$ and $\alpha > 0$ so that the intrinsic geometry and (constant, outward) mean curvature of the sphere $\{c\}\times\mathbb{S}^2$ with respect to this cone coincide with that of the centered sphere S_r (with $r > \frac{m}{2}$) in (M_m, g_m). Using the remarks below Definition 2.1, we see that this requires that

$$\begin{aligned} c^3 &= r^3 \frac{\phi_m^7}{1 - m/(2r)} = r^3(1 + \frac{4m}{r} + O(\frac{1}{r^2})), \\ \alpha &= \phi_m^{-\frac{2}{3}}(1 - \frac{m}{2r})^{\frac{2}{3}} = 1 - \frac{2m}{3r} + O(\frac{1}{r^2}). \end{aligned}$$

Note that $\alpha \in (0,1)$ and that $\alpha \nearrow 1$ as $r \to \infty$. We emphasize that α and c are uniquely determined by r. The scalar curvature of this conical metric equals $2\frac{1-\alpha^3}{\alpha s^2}$. In particular, it is positive for $\alpha \in (0,1)$.

The volume between the sphere S_r of (isotropic) radius r and the horizon $S_{\frac{m}{2}}$ in Schwarzschild is $4\pi \int_{\frac{m}{2}}^{r} (1 + \frac{m}{2r})^6 r^2 dr = \frac{4\pi r^3}{3}(1 + \frac{9m}{2r} + O(\frac{1}{r^2}))$. The volume of the (punctured) disk $(0,c]\times\mathbb{S}^2$ in the cone metric above equals $\frac{4\pi c^3}{3} = \frac{4\pi r^3}{3}(1 + \frac{4m}{r} + O(\frac{1}{r^2}))$. We denote the difference between the Schwarzschild volume and the cone volume by V_0. Note that $V_0 = \frac{4\pi r^3}{3}\frac{m}{2r} + O(r) = \frac{4\pi c^3}{3}\frac{m}{2c} + O(c)$.

Following H. Bray, we represent the part of the Schwarzschild metric (M_m, g_m) that lies outside the centered sphere of isotropic radius r in the form $u_c^{-2}ds^2 + u_c s^2 g_{\mathbb{S}^2}$ on $[c,\infty)\times\mathbb{S}^2$ for some radial function u_c. This requires that $u_c(c) = \alpha$ and $\partial u_c|_c = 0$, and that u_c satisfies a certain second-order ordinary differential equation (to make the scalar curvature vanish). We remark that by Birkhoff's theorem and the constancy of the Hawking mass along centered spheres in Schwarzschild there is a first integral for u_c.

Finally, let $g_m^c := u_c^{-2}ds^2 + u_c s^2 g_{\mathbb{S}^2}$ be the metric on $(0,\infty)\times\mathbb{S}^2$ with $u_c(s) = \alpha$ for $s \leq c$ and $u_c(s)$ is equal to $u_c = u_c(s)$ from the preceding paragraph when $s \geq c$. To summarize, we have that u_c is $\mathcal{C}^{1,1}$, is radial, and is such that the set $[c,\infty)\times\mathbb{S}^2$ in the g_m^c metric is isometric to the exterior of a round sphere S_r of isotropic radius r in the Schwarzschild manifold of mass m, and such that $u_c(s) = \alpha$ for $s \leq c$ for some constant α, such that the boundaries $\{c\}\times\mathbb{S}^2$ and S_c correspond and such that the mean curvature of $\{c\}\times\mathbb{S}^2$ from the inside (the conical part) matches that from the outside (in Schwarzschild).

A key feature of this construction used by H. Bray is that the volume element $s^2 ds \wedge dg_{\mathbb{S}^2}$ of g_m^c is independent of c. By definition of V_0, the Schwarzschild volume between the horizon and a centered Schwarzschild sphere isometric to the sphere $\{s\}\times\mathbb{S}^2$ (with $s \geq c$) in $((0,\infty)\times\mathbb{S}^2, g_m^c)$ equals $\frac{4\pi s^3}{3} + V_0$. Thus its area equals $A_m(\frac{4\pi s^3}{3} + V_0)$, where A_m is the function that assigns to every volume (measured relative to the horizon) the area of a centered sphere in Schwarzschild that encloses that volume. On the other hand, the area of $\{s\} \times \mathbb{S}^2$ is given explicitly by $u_c(s) 4\pi s^2$. In combination this yields the following explicit expression for u_c:

$$u_c(s) := \frac{A_m(V + V_0)}{(36\pi)^{\frac{1}{3}} V^{\frac{2}{3}}} \text{ for all } s \geq c, \text{ where } V := \frac{4\pi s^3}{3}; \text{ cf. [4, p. 34].}$$

It is known (and easy to verify) that

$$\frac{\mathcal{H}^2_{g_m}(\partial B(0,r))}{(36\pi)^{\frac{1}{3}} \mathcal{L}^3_{g_m}(B(0,r) \setminus B(0, \frac{m}{2}))^{\frac{2}{3}}} = 1 - \frac{m}{r} + O\left(\frac{1}{r^2}\right),$$

and from this that

$$A_m(V) = 4\pi R^2 \left(1 - \frac{m}{R} + O\left(\frac{1}{R^2}\right)\right) \text{ where } R := \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}. \tag{3}$$

By assumption we have that $u_c(c) = \alpha = 1 - \frac{2m}{3c} + O(\frac{1}{c^2})$. For a fixed $\tau \in (1, \infty)$, we are interested in estimating $u_c(\tau c) - u_c(c)$. Note that

$$\begin{aligned} u_c(\tau c) &= \frac{A_m\left(\frac{4\pi(\tau c)^3}{3}\left(1 + \frac{m}{2\tau^3 c} + O(\frac{1}{c^2})\right)\right)}{4\pi(\tau c)^2} \\ &= \left(1 + \frac{m}{2\tau^3 c} + O\left(\frac{1}{c^2}\right)\right)^{\frac{2}{3}} \left(1 - \frac{m}{\tau c} + O\left(\frac{1}{c^2}\right)\right) \\ &= 1 - \frac{2m}{3c}\left(\frac{3}{2\tau} - \frac{1}{2\tau^3}\right) + O\left(\frac{1}{c^2}\right). \end{aligned}$$

This means that for $\tau_0 \in (1, \infty)$ fixed and $\tau \geq \tau_0$ we have that

$$u_c(\tau c) - u_c(c) = u_c(\tau c) - \alpha \geq \frac{1}{2} \frac{(\tau + \frac{1}{2})(\tau - 1)^2}{\tau^3} \frac{2m}{3c} \tag{4}$$

provided that c is sufficiently large (depending only on m and τ_0). This quantifies the fact from [**4**] that $u_c(s)$ is increasing for $s \geq c$; see Appendix B.

In the proof of the following lemma, we supply some additional details and in fact make a slightly different claim than [**4**, p. 37]:

Lemma 3.1 (Cf. [**4**, p. 37])**.** *Consider the conical part of the metric* g_m^c *given by* $\alpha^{-2} ds^2 + \alpha s^2 g_{\mathbb{S}^2}$ *on* $(0, c) \times \mathbb{S}^2$ *where* α *and* c *are such that the outward mean curvature of* $\{c\} \times \mathbb{S}^2$ *with respect to* g_m^c *is the same as that of a centered sphere* S_r *of area* $\alpha 4\pi c^2$ *in Schwarzschild with mass* m*. Then there exists* $s_0 \geq 0$ *and a smooth radial function* $w_c : (s_0, c] \to [1, \infty)$ *such that* $w_c^4 g_m^c$ *is isometric to the Schwarzschild metric interior to the mean-convex sphere* S_r*, and such that* $w_c(c) = 1$ *and* $\partial_s w_c|_c = 0$.

Proof. The scalar curvature $\mathrm{R}_{g_m^c} = 2\frac{1-\alpha^3}{\alpha s^2}$ of the conical part of the metric g_m^c is strictly positive. For the conformal metric $w_c^4 g_m^c$ to be isometric to (part) of a Schwarzschild metric, it is necessary that its scalar curvature vanishes and hence that w_c is a solution of the elliptic (Yamabe) equation $-8\Delta_{g_m^c} w_c + \mathrm{R}_{g_m^c} w_c = 0$. This equation reduces to a second-order ordinary differential equation if we are solving for radial functions. Hence we can solve this equation for s close to c with initial data $w_c(c) = 1$ and $\partial_s w_c|_c = 0$. By Birkhoff's theorem, $w_c^4 g_m^c$ is isometric to (part of) a Schwarzschild metric. To determine the mass $\hat{m}$ of this metric, we evaluate its Hawking mass on the sphere $\{c\} \times \mathbb{S}^2$. Since the initial data are chosen so that the area and mean curvature of this sphere coincide with that of an umbilic constant mean curvature sphere of a Schwarzschild metric of mass m, we obtain that $\hat{m} = m$. On every connected open sub-interval of $(0, c]$ that contains c and on which the solution w_c exists and is non-negative, we have that $\Delta_{g_m^c} w_c = \frac{1}{s^2}\partial_s(s^2\alpha^2\partial_s w_c) = \frac{1}{8} R_{g_m^c} w_c \geq 0$. Integrating up and using that $\partial_s w_c|_c = 0$, it follows that $\partial_s w_c \leq 0$ on any such interval. Moreover, we see that $w_c(s)$ is a decreasing function of s. In particular, $w_c \geq 1$ on any such interval. The constancy of the Hawking mass is equivalent to the existence of a first integral for the ordinary differential equation satisfied by w_c. We let $(s_0, c]$ be the maximally left-extended interval of existence of the solution w_c. Since the metric $w_c^4 g_m^c$ on $(s_0, c] \times \mathbb{S}^2$ is isometric to (part) of a Schwarzschild metric, it follows that $w_c \nearrow \infty$ as $s \searrow s_0$ and that we actually obtain an isometric copy of the full spatial Schwarzschild metric that lies to the mean-concave side of S_r. q.e.d.

Fix an isotropic sphere S_r in (M_m, g_m), let g_m^c be the metric on $(0, \infty) \times \mathbb{S}^2$ constructed above, and let w_c be as in Lemma 3.1, extended by 1 to $s \geq c$, so that $((s_0, \infty) \times \mathbb{S}^2, w_c^4 g_m^c)$ is isometric to (M_m, g_m). We will refer to it as the volume-preserving chart associated with S_r. Recall that the isotropic sphere S_r corresponds to the coordinate sphere $\{c\} \times \mathbb{S}^2$ in $((s_0, \infty) \times \mathbb{S}^2, w_c^4 g_m^c)$. Finally, let Σ be a surface in $((s_0, \infty) \times \mathbb{S}^2, w_c^4 g_m^c)$ homologous to the horizon that encloses the same (relative) volume as $\{c\} \times \mathbb{S}^2$. The reader should keep in mind that Σ might consist of the horizon itself (enclosing volume zero) and another surface that is the boundary of a compact set that is disjoint from the horizon. In this case the area of the horizon is counted as part of the area of Σ.

Since $w_c \geq 1$ it follows that the volume enclosed by Σ with respect to the g_m^c metric (and relative to the horizon of the Schwarzschild metric $w_c^4 g_m^c$ in the same coordinate chart) is at least that enclosed by $\{c\} \times \mathbb{S}^2$. Note that as quadratic forms, $\alpha^2 g_m^c \leq \delta := ds^2 + s^2 g_{\mathbb{S}^2} \leq u_c^{-1} g_m^c$, since $\alpha \leq u_c \leq 1$. As in [**4**], but meticulously recording the error terms in the computation, we obtain that

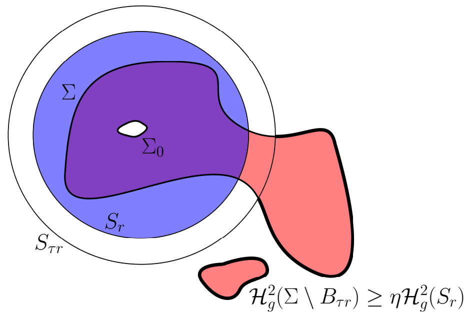

Figure 1. A large portion of the area of Σ lies outside of $B_{\tau r}$.

$$\begin{aligned}\mathcal{H}^2_{g_m}(\Sigma) = \mathcal{H}^2_{w_c^4 g^c_m}(\Sigma) \geq \mathcal{H}^2_{g^c_m}(\Sigma) &= \int_\Sigma d\mathcal{H}^2_{g^c_m} \\ &\geq \int_\Sigma u_c d\mathcal{H}^2_\delta = \int_\Sigma (u_c - \alpha) d\mathcal{H}^2_\delta + \alpha \int_\Sigma d\mathcal{H}^2_\delta \\ &\geq \int_\Sigma (u_c - \alpha) d\mathcal{H}^2_\delta + \alpha \mathcal{H}^2_\delta(\{c\} \times \mathbb{S}^2) \\ &= \int_\Sigma (u_c - \alpha) d\mathcal{H}^2_\delta + \mathcal{H}^2_{g_m}(S_r) \\ &\geq \alpha^2 \int_\Sigma (u_c - \alpha) d\mathcal{H}^2_{g^c_m} + \mathcal{H}^2_{g_m}(S_r) \qquad (5)\end{aligned}$$

The third inequality follows from the Euclidean isoperimetric inequality. The definition below is natural in view of this estimate and the expansion of u_c for $s \geq c$.

Definition 3.2 (see figure 1)**.** Let (M, g) be an initial data set that is $\mathcal{C}^0$-asymptotic to Schwarzschild of mass $m > 0$. Let Ω be a bounded Borel set with finite perimeter in (M, g) that contains the horizon. Given parameters $\tau > 1$ and $\eta \in (0, 1)$, we say that such a set Ω is (τ, η)-off-center if

1) $\mathcal{L}^3_g(\Omega)$ is so large that there exists a coordinate sphere $S_r = \partial B_r$ with $\mathcal{L}^3_g(\Omega) = \mathcal{L}^3_g(B_r)$ and $r \geq 1$, and if
2) $\mathcal{H}^2_g(\partial^* \Omega \setminus B_{\tau r}) \geq \eta \mathcal{H}^2_g(S_r)$.

Let Σ be a surface in Schwarzschild containing the horizon and enclosing volume V with it, and let $r \geq \frac{m}{2}$ be such that $\mathcal{L}^3_{g_m}(B_r \setminus B_{\frac{m}{2}}) = V$. Assume that r is large and that Σ is (τ, η)-off-center. It is easy to see that as $c \to \infty$, the isotropic sphere $S_{\tau r}$ corresponds to $\{(\tau + o(1))c\} \times \mathbb{S}^2$ in the volume-preserving chart. It follows that for r sufficiently large, a portion $\eta \mathcal{H}^2_{g_m}(S_r)$ of the area of Σ lies in the region $(\frac{1+\tau}{2} c, \infty) \times \mathbb{S}^2$ of the volume-preserving chart. We can use this information together with

(4), replacing τ by $(1+\tau)/2$, to continue the estimate (5). We obtain that

$$\begin{aligned}\mathcal{H}^2_{g_m}(\Sigma) &\geq \mathcal{H}^2_{g_m}(S_r) + \frac{\eta m}{96}\left(1-\frac{1}{\tau}\right)^2 \frac{\mathcal{H}^2_{g_m}(S_r)}{r} \\ &\geq \mathcal{H}^2_{g_m}(S_r) + \frac{\eta m \pi}{24}\left(1-\frac{1}{\tau}\right)^2 r\end{aligned}$$

for all r sufficiently large, depending only on m and τ. We have used that $2\alpha \geq 1$ and $2r \geq c$ for r sufficiently large here, and that $\mathcal{H}^2_{g_m}(S_r) \geq 4\pi r^2$.

The arguments leading to this estimate also apply if $\Sigma = \partial^*\Omega$ is the reduced boundary of a finite-perimeter Borel set Ω containing the horizon.

Proposition 3.3 (Effective volume comparison in Schwarzschild)**.** *For $m > 0$ and $(\tau, \eta) \in (1, \infty) \times (0, 1)$ there exists $V_0 > 0$ with the following property: Let $V \geq V_0$ and $r \geq \frac{m}{2}$ such that $V = \mathcal{L}^3_{g_m}(B_r \setminus B_{\frac{m}{2}})$, and let $\Omega \subset \mathbb{R}^3$ be a bounded finite-perimeter Borel set such that $B_{\frac{m}{2}} \subset \Omega$ and $\mathcal{L}^3_{g_m}(\Omega \setminus B_{\frac{m}{2}}) = V$. If Ω is (τ, η)-off-center, i.e., if $\mathcal{H}^2_{g_m}(\partial^*\Omega \setminus B_{\tau r}) \geq \eta \mathcal{H}_{g_m}(S_r)$, then*

$$\mathcal{H}^2_{g_m}(\partial^*\Omega) \geq \mathcal{H}^2_{g_m}(S_r) + \frac{\eta m \pi}{24}\left(1-\frac{1}{\tau}\right)^2 r. \tag{6}$$

This is our effective refinement of H. Bray's argument in exact Schwarzschild. The study of effective isoperimetric inequalities is classical with much recent activity; see, e.g., [**18**, **16**, **9**]. The effective volume comparison in Proposition 3.3 is not obtained from lifting an effective isoperimetric inequality from the Euclidean background. It depends on the particular form of the Schwarzschild metric in an essential way.

In the proof of the theorem below, we will appreciate that we can quantify how much an off-center surface in Schwarzschild falls short of being isoperimetric. The defect is large enough for us to carry out the comparison on arbitrary initial data sets that are $\mathcal{C}^0$-asymptotic to Schwarzschild of mass $m > 0$:

Theorem 3.4. *Let (M, g) be an initial data set that is $\mathcal{C}^0$-asymptotic to Schwarzschild of mass $m > 0$. For every tuple $(\tau, \eta) \in (1, \infty) \times (0, 1)$ and constant $\Theta > 0$, there exists a constant $V_0 > 0$ with the following property: Let Ω be a bounded finite-perimeter Borel set containing the horizon with $\mathcal{L}^3_g(\Omega) \geq V_0$ that is (τ, η)-off-center and such that $\mathcal{H}^2_g(\partial^*\Omega)^{\frac{1}{2}} \mathcal{L}^3_g(\Omega)^{-\frac{1}{3}} \leq \Theta$ and $\mathcal{H}^2_g(B_\sigma \cap \partial^*\Omega) \leq \Theta\sigma^2$ for all $\sigma \geq 1$. Then*

$$\mathcal{H}^2_g(S_r) + \frac{\eta m \pi}{300}\left(1-\frac{1}{\tau}\right)^2 r \leq \mathcal{H}^2_g(\partial^*\Omega). \tag{7}$$

Here, $S_r \subset M$ is the centered coordinate sphere that encloses g-volume $\mathcal{L}^3_g(\Omega)$ with the horizon.

Remark: The form of the constant that multiplies r in (7) is immaterial. The explicit expression is given to indicate the dependence on the parameters.

Proof. For ease of exposition, we only consider smooth regions Ω. The result for sets of finite perimeter follows by approximation. By Lemma 2.4, $\mathcal{H}^2_g(\partial\Omega) \to \infty$ as $\mathcal{L}^3_g(\Omega) \to \infty$. Note also that $\mathcal{L}^3_g(\Omega) = \frac{4\pi r^3}{3} + O(r^2)$.

We break the argument into several steps:

(a) Let $\tilde{\Omega} := \Omega \cup B_1 \subset M$. Let $\tilde{\Omega}_m := (x(\Omega \setminus B_1) \cup B(0,1)) \setminus B(0, \frac{m}{2})$ be the corresponding region in Schwarzschild.

(b) Note that $\mathcal{L}^3_g(\tilde{\Omega}) = \mathcal{L}^3_g(\Omega) + O(1)$ and $\mathcal{H}^2_g(\partial\tilde{\Omega}) = \mathcal{H}^2_g(\partial\Omega) + O(1)$. Moreover, $\tilde{\Omega}$ satisfies $\mathcal{H}^2_g(B_\sigma \cap \partial\tilde{\Omega}) \leq \tilde{\Theta}\sigma^2$ for all $\sigma \geq 1$ where $\tilde{\Theta}$ depends only on Θ and (M, g).

(c) By Corollary A.2 with $\beta = \frac{1}{2}$,

$$\mathcal{H}^2_{g_m}(\partial\tilde{\Omega}_m) \leq \mathcal{H}^2_g(\partial\tilde{\Omega}) + O(\mathcal{H}^2_g(\partial\tilde{\Omega})^{\frac{1}{4}}) \leq \mathcal{H}^2_g(\partial\Omega) + O(\mathcal{H}^2_g(\partial\Omega)^{\frac{1}{4}}).$$

(d) By Lemma A.3 with $\alpha = \frac{3}{2}$, $\mathcal{L}^3_{g_m}(\tilde{\Omega}_m) = \mathcal{L}^3_g(\Omega) + O(\mathcal{L}^3_g(\Omega)^{\frac{1}{2}})$.

(e) By Lemma A.3 with $\alpha = \frac{3}{2}$ and choice of r, $\mathcal{L}^3_{g_m}(B_r \setminus B_{\frac{m}{2}}) = \mathcal{L}^3_{g_m}(B_r \setminus B_1) + O(1) = \mathcal{L}^3_g(B_r \setminus B_1) + O(\mathcal{L}^3_g(B_r \setminus B_1)^{\frac{1}{2}}) = \mathcal{L}^3_g(\Omega) + O(\mathcal{L}^3_g(\Omega)^{\frac{1}{2}})$.

(f) By (d) and (e) and choice of r, we have that $\mathcal{L}^3_{g_m}(\tilde{\Omega}_m) = \mathcal{L}^3_{g_m}(B_r \setminus B_{\frac{m}{2}}) + O(r^{\frac{3}{2}})$. Let $\tilde{r}$ be such that $\mathcal{L}^3_{g_m}(\tilde{\Omega}_m) = \mathcal{L}^3_{g_m}(B_{\tilde{r}} \setminus B_{\frac{m}{2}})$. Then $\tilde{r} = r + O(r^{-\frac{1}{2}})$.

(g) The Schwarzschild region $\tilde{\Omega}_m \subset M_m$ is $(\frac{1+\tau}{2}, \frac{\eta}{2})$-off-center provided that $\mathcal{L}^3_g(\Omega)$ is sufficiently large. Hence

$$A_m(\mathcal{L}^3_{g_m}(\tilde{\Omega}_m)) + \frac{\eta m \pi}{192}\left(1 - \frac{1}{\tau}\right)^2 \tilde{r} \leq \mathcal{H}^2_{g_m}(\partial\tilde{\Omega}_m)$$

by (6).

(h) $\mathcal{H}^2_{g_m}(S_r) = A_m(\mathcal{L}^3_{g_m}(B_r \setminus B_{\frac{m}{2}})) \leq A_m(\mathcal{L}^3_{g_m}(\tilde{\Omega}_m)) + O(\mathcal{L}^3_g(\Omega)^{\frac{1}{6}})$ where the inequality follows by explicit computation (using (3)) from

$$\mathcal{L}^3_{g_m}(B_r \setminus B_{\frac{m}{2}}) = \mathcal{L}^3_g(\Omega) + O(\mathcal{L}^3_g(\Omega)^{\frac{1}{2}}).$$

(i) $\mathcal{H}^2_g(S_r) \leq \mathcal{H}^2_{g_m}(S_r) + O(1)$. This is obvious.

(j) $\mathcal{H}^2_g(S_r) \leq \mathcal{H}^2_g(\partial\Omega) - \frac{\eta m \pi}{200}\left(1 - \frac{1}{\tau}\right)^2 r + O(\mathcal{L}^3_g(\Omega)^{\frac{1}{6}}) + O(\mathcal{H}^2_g(\partial\Omega)^{\frac{1}{4}})$.

The conclusion follows from this since $\mathcal{H}^2_g(\partial\Omega)^{\frac{1}{2}}\mathcal{L}^3_g(\Omega)^{-\frac{1}{3}} \leq \Theta$. q.e.d.

4. Regularity of isoperimetric regions and the behavior of minimizing sequences

In this section, we review the regularity theory for minimizers of area under a volume constraint in the presence of a smooth obstacle. The results discussed here are well known and can be deduced from classical sources. For completeness and clarity, and because we have not been able to find a reference that includes our set up here completely, we supply a detailed outline of the argument along with further references, where more details on specific parts of the argument can be found.

We consider an initial data set (M, g) and its extension $(\hat{M}, \hat{g})$ to a complete boundaryless Riemannian 3-manifold, as in Section 2. Recall that the horizon ∂M, if non-empty, is the outermost minimal surface of $\hat{M}$.

Proposition 4.1. *An isoperimetric region containing the horizon has smooth, compact boundary. If this boundary intersects the horizon, then they coincide.*

Proof. We first discuss the regularity of the reduced boundary $\partial^*\Omega$ away from the coincidence set $\text{supp}(\partial^*\Omega) \cap \partial M$.

A complete proof that $\partial^*\Omega$ has constant mean curvature away from the coincidence set is given in [**12**, Proposition 2.1]. This puts the monotonicity formula at one's disposal, and standard regularity analysis (see, e.g., [**12**, Theorem 2.5], which eventually refers to the classical paper [**20**]) applies. The key points here are that there is no mass loss in the convergence (as sets of locally finite perimeter) of blow-up sequences of $\partial^*\Omega$ at a point $x \in \text{supp}(\partial^*\Omega) \setminus \partial M$, implying in conjunction with the monotonicity formula that the limiting objects are tangent cones, and that these tangent cones are area-minimizing boundaries and thus planes. In other words, the volume constraint scales away in the blow-up limit. The proof of both these points proceeds as in the case of area-minimizing boundaries (cf. [**45**]), applying, for example, the argument in [**19**, Lemma 2.1] to make effective use of the isoperimetric property of $\partial^*\Omega$. The regularity of $\partial^*\Omega$ near x then follows at once from Allard's theorem. See also, e.g., [**1**, **20**, **35**] for alternative ways of arguing this step.

That $\partial^*\Omega$ is compact follows in a standard way from the monotonicity formula (see, e.g., [**6**, Lemma 10]) and an explicit bound on $\mathcal{H}^2_g(\partial^*\Omega)$ that can be obtained from comparison; cf. [**12**, Corollary 2.4].

The regularity of $\partial^*\Omega$ along the horizon ∂M follows from [**19**], [**47**], and [**20**]; see also [**26**, Theorem 1.3] and the references provided there. Again, we outline the key points. We may assume that $\mathcal{L}^3_g(\Omega \cap M) > 0$. Using that $\partial^*\Omega$ contains regular points, one concludes that $\partial^*\Omega$ is almost minimizing in $\hat{M}$ (i.e., across the horizon) without volume constraint. It follows as above that the mean curvature of $\partial^*\Omega$ is defined and bounded along the coincidence set, that there is no mass loss in the convergence of tangent blow-up sequences at points in the coincidence set, that the limits are cones, and finally that these cones are area minimizing and thus planes. Hence $\partial^*\Omega$ is a $\mathcal{C}^{1,\alpha}$ surface near ∂M.

The next step is to argue that the constant mean curvature of $\partial^*\Omega$ away from the coincidence set, H, is non-negative. If $H < 0$, then one could take the minimal area enclosure of Ω in M and use the same argument as above to show that it is a smooth minimal surface away from the coincidence set of $\partial^*\Omega$ with the horizon, where it is a priori only $\mathcal{C}^{1,\alpha}$; cf. [**26**, Theorem 1.3 (ii)]. The minimal area enclosure of Ω is weakly mean-convex. The Harnack inequality shows that its components either coincide with components of the horizon, or are disjoint from the horizon. The latter scenario (for any component) contradicts our assumption that the horizon is the outermost minimal surface in M. We see that $H > 0$ unless $\partial\Omega = \partial M$. A first variation argument shows that $\partial^*\Omega$ is weakly mean-convex along the coincidence set. Again, we can conclude from the Harnack inequality that the coincidence set is either empty or that $\partial^*\Omega = \partial M$. q.e.d.

We also want to understand the behavior of general minimizing sequences in initial data sets. The following proposition is a slight extension of a special case of [**41**, Theorem 2.1]; see also [**12**, **4**] and the remarks below.

Proposition 4.2. *Given $V > 0$, there exists an isoperimetric region $\Omega \subset \hat{M}$ containing the horizon and a radius $r \in [0, \infty)$ such that $\mathcal{L}^3_g(\Omega) + \frac{4\pi r^3}{3} = V$ and such that $\mathcal{H}^2_g(\partial\Omega) + 4\pi r^2 = A_g(V)$. If $r > 0$ and $\mathcal{L}^3_g(\Omega) > 0$, then the mean curvature of $\partial\Omega$ equals $\frac{2}{r}$.*

Proof. By [**41**, Theorem 2.1] and a simple rescaling argument, there exists an isoperimetric region Ω containing the horizon and a sequence of finite-perimeter Borel sets Ω_i diverging to infinity such that $\Omega \cap \Omega_i = \emptyset$, $\mathcal{L}^3_g(\Omega) + \mathcal{L}^3_g(\Omega_i) = V$, and $\mathcal{H}^2_g(\partial\Omega) + \lim_{i\to\infty} \mathcal{H}^2_g(\partial^*\Omega_i) = A_g(V)$. Applying the Euclidean isoperimetric inequality with a small fudge factor that tends to 1 as $i \to \infty$ to the sets Ω_i, we see that the sets Ω_i can be replaced by coordinate balls $B(p_i, r_i)$ of the same volume and such that $p_i \to \infty$. The observation about the mean curvature of the sphere that represents the runaway volume follows from a first variation argument. q.e.d.

Proposition 4.2 leaves the possibility that part of the volume of a minimizing sequence for the isoperimetric problem slides to infinity. If this happens, the leftover isoperimetric limit is not a solution of the original problem. In Euclidean space, the situation is well understood: for example, in [**12**], it is shown that to every closed curve in $\mathbb{R}^3$ and volume V there exists a mass-minimizing integer multiplicity current that bounds the curve while enclosing oriented volume V relative to a fixed filling of the curve. A key ingredient in the proof is the exact isoperimetric inequality for $\mathbb{R}^3$. It is used to argue that runaway volume can be clipped off and kept at fixed finite distance as a ball of the same volume, not increasing the area. A delicate cut and paste argument is developed in [**4**, Sections 2.7 and 2.9] to show existence of isoperimetric regions on compact perturbations of Schwarzschild. In the proof, H. Bray uses an additional assumption ("Condition 1") in a subtle way to ensure that his isoperimetric Hawking mass is a monotone function of the volume.

For later use, we state the following simple lemma. It follows readily from explicit comparison either with small geodesic balls or with large coordinate balls:

Lemma 4.3. *Let (M, g) be an initial data set. There exists a constant $\Theta > 0$ so that for every isoperimetric region Ω containing the horizon one has that $\mathcal{H}^2_g(B_r \cap \partial\Omega) \leq \Theta r^2$ for all $r \geq 1$, and that $\mathcal{H}^2_g(\partial\Omega)^{\frac{1}{2}}\mathcal{L}^3_g(\Omega)^{-\frac{1}{3}} \leq \Theta$ provided $\mathcal{L}^3_g(\Omega) \geq 1$.*

5. Large isoperimetric regions center

Theorem 5.1. *Let (M, g) be an initial data set that is $\mathcal{C}^0$-asymptotic to Schwarzschild of mass $m > 0$. There exists a large constant $V_0 > 0$ with the following property: Let Ω be an isoperimetric region containing the horizon such that $\mathcal{L}^3_g(\Omega) = V \geq V_0$. Let $r \geq 1$ be such that $\mathcal{L}^3_g(B_r) = V$. Then $\partial\Omega$ is a smooth connected hypersurface close to the centered coordinate sphere S_r. The scale invariant $\mathcal{C}^2$-norms of functions that describe such large isoperimetric surfaces as normal graphs above the corresponding centered coordinate spheres tend to zero as the enclosed volume diverges to infinity.*

Proof. Let $\{\Omega_i\}_{i=1}^\infty$ be a sequence of isoperimetric regions containing the horizon and with $\mathcal{L}^3_g(\Omega_i) \to \infty$. In view of Lemma 4.3, fixing parameters $(\tau, \eta) \in (1, \infty) \times (0, 1)$, we can apply Theorem 3.4 to Ω_i provided i is sufficiently large.

We consider the parts of the regions Ω_i that lie in $M \setminus B_1 \cong_x \mathbb{R}^3 \setminus B(0, 1)$. We use homotheties $h_\lambda : x \to \lambda{\cdot}x$ in the Euclidean chart to scale down by a factor $\lambda_i = \left(3\mathcal{L}^3_g(\Omega_i \setminus B_1)/(4\pi)\right)^{\frac{1}{3}}$ to obtain sets $\hat{\Omega}_i \subset \mathbb{R}^3 \setminus B(0, \lambda_i^{-1})$ that are locally isoperimetric with respect to the metric $g_i := \lambda_i^{-2} h^*_{\lambda_i} g$ and such that $\mathcal{L}^3_{g_i}(\hat{\Omega}_i) = \frac{4\pi}{3}$. Note that $(\mathbb{R}^3 \setminus B(0, \lambda_i^{-1}), g_i) \to (\mathbb{R}^3 \setminus \{0\}, \sum_{j=1}^3 dx_j^2)$ in $\mathcal{C}^2_{loc}$ and that $\mathcal{L}^3_{g_i}(B(0, 1) \setminus B(0, \lambda_i^{-1})) \to \frac{4\pi}{3}$. Passing to a subsequence if necessary, we can assume that $\hat{\Omega}_i$ converges locally as a set of finite perimeter to Ω in $\mathbb{R}^3$.

We claim that $\limsup_{i\to\infty} \mathcal{L}^3_{g_i}(\hat{\Omega}_i \setminus B(0, \tau)) = 0$ for every $\tau > 1$. Suppose that not. Passing to a subsequence if necessary, it follows that for some $\tau > 1$ and $\varepsilon > 0$ we have that $\mathcal{L}^3_{g_i}(\hat{\Omega}_i \setminus B(0, \tau)) \geq \varepsilon$ for all i. The relative isoperimetric inequality, an appropriate version of which follows from (2) in a standard way, gives that $\mathcal{H}^2_{g_i}(\partial\hat{\Omega}_i \setminus B(0, \tau)) \gtrsim$

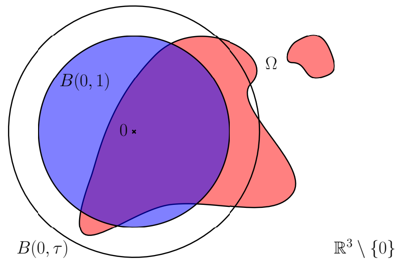

Figure 2. If the blow-down limit Ω has volume outside of $B(0,\tau)$, then the original sets Ω_i are $(\frac{1+\tau}{2},\eta)$-off-center for some $\eta > 0$.

$\mathcal{L}^3_{g_i}(\hat{\Omega}_i \setminus B(0,\tau))^{\frac{2}{3}}$. In particular, $\mathcal{H}^2_{g_i}(\partial\hat{\Omega}_i \setminus B(0,\tau)) \geq 2\eta\mathcal{H}^2_{g_i}(\partial B(0,1))$ for some $\eta > 0$ and for all i. This implies that each Ω_i is $(\frac{1+\tau}{2},\eta)$-off-center. (The reason we are passing from τ to $\frac{1+\tau}{2}$ and from 2η to η is that we have to adjust for the volume by $\mathcal{L}^3_g(\Omega_i \setminus B_1) + \mathcal{L}^3_g(B_1)$.) See Figure 2. Theorem 3.4 shows that Ω_i is not isoperimetric, contradicting our assumption. Thus $\limsup_{i\to\infty} \mathcal{L}^3_{g_i}(\hat{\Omega}_i \setminus B(0,\tau)) = 0$ for every $\tau > 1$, as desired. It follows that $\Omega = B(0,1)$.

For isoperimetric regions, convergence as sets of locally finite perimeter is equivalent to locally smooth convergence so long as the volume does not shrink away. (See, e.g., [**42**, Proposition 5].) It follows that, for i large, the boundary of Ω_i contains a component Σ_i that is close to the centered coordinate sphere S_{r_i} whose radius r_i is such that $\mathcal{L}^3_g(B_{r_i}) = \mathcal{L}^3_g(\Omega_i)$.

Let $\tilde{\Omega}_i$ be the bounded component of $M \setminus \Sigma_i$. We claim that $\Omega_i = \tilde{\Omega}_i$.

To see that $\Omega_i \subset \tilde{\Omega}_i$, note first that the components of $\partial\Omega_i$ all have the same constant mean curvature $\sim 2/r_i$. Assume that, after passing to a subsequence if necessary, every Ω_i has at least one component that is disjoint from $\tilde{\Omega}_i$. The preceding analysis shows that such components have to slide off to infinity in the preceding blow-down limit. The monotonicity formula shows that the area of these components subconverges to a positive number in the blow-down limit. It follows that $\limsup_{i\to\infty} \mathcal{H}^2_g(\Omega_i)\mathcal{L}^3_g(\Omega_i)^{-2/3} > (4\pi)(4\pi/3)^{-2/3}$. On the other hand, a comparison with large coordinate balls gives that $\limsup_{i\to\infty} \mathcal{H}^2_g(\Omega_i)\mathcal{L}^3_g(\Omega_i)^{-2/3} \leq (4\pi)(4\pi/3)^{-2/3}$. This contradiction shows that $\Omega_i \subset \tilde{\Omega}_i$.

Assume that Ω_i is properly contained in $\tilde{\Omega}_i$. The blow-down argument shows that $\Omega_i \cap B_{\mu_i} = \tilde{\Omega}_i \cap B_{\mu_i}$ where $\mu_i \geq 1$ are such that $\mu_i/r_i \to 0$ as $i \to \infty$. Consider the region obtained from $\tilde{\Omega}_i$ by pushing its outer boundary Σ_i, which is close to a large coordinate sphere, inward until the resulting region has volume $\mathcal{L}^3_g(\Omega_i)$. The boundary area of this new region is strictly less than that of Ω_i. This contradicts the assumption that Ω_i is an isoperimetric region. Thus $\Omega_i = \tilde{\Omega}_i$. q.e.d.

Theorem 5.2. *Let (M, g) be an initial data set that is $\mathcal{C}^0$-asymptotic to Schwarzschild of mass $m > 0$. There exists $V_0 > 0$ so that for every volume $V \geq V_0$ there is an isoperimetric region Ω containing the horizon with $\mathcal{L}^3_g(\Omega) = V$.*

Proof. Let $V_i \to \infty$ be a divergent sequence of volumes. Let $r_i \geq 0$ be radii and Ω_i be isoperimetric regions containing the horizon as in Proposition 4.2. Using (3), we see that $\mathcal{L}^3_g(\Omega_i) \to \infty$. From Theorem 5.1 we know that for i large, Ω_i is close to a large centered coordinate ball in (M, g). If $r_i > 0$, then the mean curvature of $\partial\Omega_i$ and hence the radius of the coordinate sphere that it is close to correspond to that of $B(0, r_i) \subset \mathbb{R}^3$, by Proposition 4.2. A configuration of two large disjoint coordinate balls in (M, g) of essentially the same radius is not isoperimetric, and far from it. Hence $r_i = 0$ for i sufficiently large, and the theorem follows. q.e.d.

6. Isoperimetric regions in initial data sets with general asymptotics

Let (M, g) be an initial data set with non-negative scalar curvature. Let $\{\Omega_i\}_{i=1}^\infty$ be a sequence of isoperimetric regions containing the horizon such that $\mathcal{L}^3_g(\Omega_i) \to \infty$. The argument in Proposition [**42**, Proposition 5] shows that the Ω_i subconverge to a locally isoperimetric region Ω. In Theorem 6.1, below, we show that the unbounded components of $\partial\Omega$ are totally geodesic and that the scalar curvature of M vanishes on them.

If we assume that the scalar curvature of M is everywhere positive, this result puts a strong limitation on the possible behavior of large isoperimetric regions; cf. Corollary 6.2.

Theorem 6.1 is the precursor of our more subtle result in [**14**], which applies to regions whose boundaries are only assumed to be volume-preserving stable constant mean curvature surfaces. The proofs of both results are based on ideas of R. Schoen and S.-T. Yau in [**44**]. On a technical level, the proof of Theorem 6.1 is quite different from that in [**14**], so we include it.

Theorem 6.1 (Cf. Theorem 1.5 in [**14**])**.** *Assumptions as in the first paragraph. Then $\partial\Omega$ has at most one unbounded component, and this component is a totally geodesic area-minimizing hypersurface. The scalar curvature of M vanishes on any unbounded component of $\partial\Omega$.*

Proof. By [**14**, Corollary 5.6], the mean curvature of $\partial\Omega_i$ tends to zero as $i \to \infty$. It follows that $\partial\Omega$ is a minimal surface. Let Σ be an unbounded component of $\partial\Omega$. We employ an argument of R. Schoen and S.-T. Yau from [**44**] to show that Σ is totally geodesic. We follow the main steps of [**44**] very closely and highlight our minor adaptations to the present context. The important difference with [**44**] is that we don't a priori know that Σ is strongly stable. That this is nevertheless the case follows from the result presented in Appendix C.

(a) By Appendix C, Σ is an area-minimizing boundary. Hence Σ is stable with respect to compactly supported variations:

$$\int_{\Sigma} \left(|h|^2 + \mathrm{Rc}_g(\nu,\nu)\right) \phi^2 d\mathcal{H}^2_g \leq \int_{\Sigma} |\nabla_\Sigma \phi|^2 d\mathcal{H}^2_g \text{ for all } \phi \in \mathcal{C}^1_c(M). \tag{8}$$

(b) This step and the next one differ slightly from [**44**]. The homothetic rescalings $\lambda^{-1}(\Sigma \setminus B_1) \subset \mathbb{R}^3 \setminus B(0,\lambda^{-1})$ subconverge as $\lambda \to \infty$ to area-minimizing boundaries in $\mathbb{R}^3 \setminus \{0\}$ (with the Euclidean metric). Such boundaries are hyperplanes. It follows from (iterations of) this argument that Σ intersects any sufficiently large coordinate sphere $S_r \subset M$ in a circle. It follows that Σ is planar outside a compact subset of (M,g). In particular, Σ is of finite topological type.

(c) Σ is a mass-minimizing integral current in (M,g). The argument is indirect. Consider a large coordinate ball B_r with mean-convex boundary S_r. From the preceding step, we know that S_r intersects Σ transversely in a smooth connected curve. By the maximum principle argument of [**46**], the mass-minimizing current in (M,g) spanning $\Sigma \cap S_r$ lies inside of B_r and is disjoint from the horizon. By [**21**], this mass-minimizing integral current is a smooth, embedded, multiplicity 1 hypersurface. Its area must be $\mathcal{H}^2_g(\Sigma \cap B_r)$. It follows that Σ is mass-minimizing with respect to current deformations, and not just among boundaries. We are grateful to Leo Rosales and to Brian White for helping us with this point.

(d) Σ has quadratic area growth: there exists a constant $\Theta > 0$ depending only on (M,g) such that $\mathcal{H}^2_g(\Sigma \cap B_r) \leq \Theta r^2$ for all $r \geq 1$ sufficiently large. This follows from the mass-minimizing property of Σ and comparison with large coordinate spheres.

(e) We have that $\int_\Sigma |\mathrm{Rc}_g| d\mathcal{H}^2_g < \infty$. This follows from Lemma A.1 and because $|\mathrm{Rc}_g| = O(r^{-3})$.

(f) Because Σ has quadratic area growth, we can use the "logarithmic cut-off trick" in (8) to obtain that $\int_\Sigma \left(|h|^2 + \mathrm{Rc}_g(\nu,\nu)\right) d\mathcal{H}^2_g < \infty$. It follows from the Gauss equation that $\int_\Sigma |\kappa| d\mathcal{H}^2_g < \infty$, where κ is the Gauss curvature of Σ.

(g) From the Gauss equation and the Cohn–Vossen inequality [**10**], one sees that

$$0 \leq \int_\Sigma \left(\mathrm{R}_g + |h|^2\right) d\mathcal{H}^2_g \leq 2\int_\Sigma \kappa d\mathcal{H}^2_g \leq 4\pi\chi(\Sigma). \tag{9}$$

(The Cohn–Vossen inequality applies because Σ is complete, has absolutely integrable Gauss curvature, and is of finite topological type. See also [**39**, p. 86] and [**17**, p. 1]). The theorem will follow if we can show that $\int_\Sigma \kappa d\mathcal{H}^2_g = 0$. We will assume for a contradiction that $\int_\Sigma \kappa d\mathcal{H}^2_g > 0$. Note that in this case (9) implies that Σ is homeomorphic to the plane $\mathbb{C}$.

(h) Since $\int_\Sigma |\kappa| d\mathcal{H}^2_g < \infty$ and $\Sigma \cong \mathbb{C}$, a theorem of A. Huber's [**24**] gives that there exists a conformal diffeomorphism $F : \mathbb{C} \to \Sigma$. (We refer the reader to [**28**] for a comprehensive discussion of the topological type and the conformal structure of complete surfaces the negative part of whose Gaussian curvature is integrable.) By results of R. Finn's [**17**] and A. Huber's [**25**], one has that $4\int_\Sigma \kappa d\mathcal{H}^2 = 4\pi - \lim_{k\to\infty} A_k^{-1} L_k^2$ (the existence of the limit is part of the conclusion) where $L_k = \mathcal{H}^1_g(F(\{z \in \mathbb{C} : |z| = k\}))$ and $A_k := \mathcal{H}^2_g(F(\{z \in \mathbb{C} : |z| \leq k\}))$. The goal is to show that $\int_\Sigma \kappa d\mathcal{H}^2_g = 0$. Since we already know that $0 \leq \int_\Sigma \kappa d\mathcal{H}^2_g$, this boils down to showing that $4\pi \leq \lim_{k\to\infty} A_k^{-1} L_k^2$.

(i) Since F is proper, $F(\{z \in \mathbb{C} : |z| = k\})$ will lie outside every given compact subset of Σ (and hence of M) provided i is sufficiently large. Hence we can view $F(\{z \in \mathbb{C} : |z| = k\})$ as a curve in Euclidean space $(\mathbb{R}^3, \delta_{ij})$ by using the coordinate system in the asymptotically flat end of (M, g). Let $\tilde\Sigma_k$ be the least (Euclidean) area integer multiplicity current spanning $F(\{z \in \mathbb{C} : |z| = k\}) \subset \mathbb{R}^3$. There are two cases: either $\tilde\Sigma_k$ leaves every compact subset of $\mathbb{R}^3$ as $k \to \infty$, in which case $M_g(\tilde\Sigma_k) = (1 + o(1)) M_\delta(\tilde\Sigma_k)$ (where M_g and M_δ denote the current mass with respect to g and δ, respectively), or there exists a radius $r_1 \geq 1$ such that $\tilde\Sigma_{k'} \cap B_{r_1} \neq \emptyset$ for a subsequence $\tilde\Sigma_{k'}$. Either way, the argument in [**44**, p. 56–57] can be followed verbatim to conclude that $4\pi \leq \lim_{k\to\infty} A_k^{-1} L_k^2$ and hence that $\int_\Sigma \kappa d\mathcal{H}^2_g = 0$. This contradicts our assumption and finishes the proof that Σ is totally geodesic (and that $\partial\Omega$ cannot have unbounded components if $\mathrm{R}_g > 0$).

It follows from the isoperimetric property and Lemma 6.3, below, that $\partial\Omega$ can only have one unbounded totally geodesic component. q.e.d.

Corollary 6.2. *Let (M, g) be an initial data set whose scalar curvature is everywhere positive. If the horizon ∂M is empty, we also assume that there are no closed minimal surfaces in M. Let $\Omega_i \subset M$ be a sequence of isoperimetric regions enclosing the horizon whose volumes tend to infinity. Then* $\limsup_{i\to\infty} \Omega_i := \bigcap_{j=1}^\infty \bigcup_{i=j}^\infty \Omega_i$ *equals either ∂M or M.*

Lemma 6.3 (Essentially [**2**, Proposition 3.1])**.** *Let (M, g) be an initial data set that satisfies the decay assumptions (10). There exist a radius $r_0 \geq 1$ and a constant $C \geq 1$ with the following property: If Σ is a complete unbounded properly embedded totally geodesic surface in M, then $\Sigma \setminus B_{r_0}$ consists of finitely many components $\Sigma_1, \ldots, \Sigma_m$. Moreover, there exists a coordinate plane $P = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \setminus B(0, 1) : ax_1 + bx_2 + cx_3 = 0\}$ and functions $f_k : P \setminus B_{r_0} \to \mathbb{R}$ such that the graph of*

f_k above $P \setminus B_{r_0}$ is contained in Σ_k and such that

$$(\log r)^{-1}|f_k| + r|\partial f_k| + r^2|\partial^2 f_k| \leq C \text{ for every } k = 1, \ldots, m.$$

Appendix A. Integral decay estimates

Our computations in this appendix take place on the part of an initial data set (M, g) that is diffeomorphic to $\mathbb{R}^3 \setminus B(0,1)$ and such that

$$r|g_{ij} - \delta_{ij}| \leq C \text{ for all } r := |x| \geq 1. \tag{10}$$

For Corollary A.3 we require in addition that

$$r^2|g_{ij} - \left(1 + \frac{m}{2r}\right)^4 \delta_{ij}| \leq C \text{ for all } r := |x| \geq 1, \tag{11}$$

i.e., that (M, g) is $\mathcal{C}^0$-asymptotic to Schwarzschild of mass $m > 0$.

Lemma A.1. *Let (M, g) be an initial data set. Let $r_0 \geq 1$. For every closed hypersurface $\Sigma \subset M$ such that $\mathcal{H}^2_g(\Sigma \cap B_r \setminus B_{r_0}) \leq \Theta r^2$ holds for all $r \geq r_0$, one has that*

$$\int_{\Sigma \setminus B_{r_0}} r^{-\gamma} d\mathcal{H}^2_g \leq \frac{\gamma}{\gamma - 2} \Theta {r_0}^{2-\gamma}$$

for every $\gamma > 2$.

Proof. The proof uses the co-area formula exactly as in [**44**, p. 52]. q.e.d.

Corollary A.2. *Let (M, g) be an initial data set. Let $r_0 \geq 1$. For every closed hypersurface $\Sigma \subset M$ such that $\mathcal{H}^2_g(\Sigma \cap B_r \setminus B_{r_0}) \leq \Theta r^2$ holds for all $r \geq r_0$, one has that*

$$\int_{\Sigma \setminus B_{r_0}} r^{-2} d\mathcal{H}^2_g \leq r_0^{-\beta} \mathcal{H}^2_g(\Sigma \setminus B_{r_0})^{\frac{\beta}{2}} \left(\frac{2\Theta}{\beta}\right)^{\frac{2-\beta}{2}}$$

for every $\beta \in (0, 2)$.

Lemma A.3. *Let (M, g) be an initial data set satisfying (11). There is a constant $C' \geq 1$ depending only on C such that for every $r_0 \geq 1$ and every bounded measurable subset $\Omega \subset M$ one has that*

$$|\mathcal{L}^3_g(\Omega \setminus B_{r_0}) - \mathcal{L}^3_{g_m}(\Omega \setminus B_{r_0})| \leq C' \left(\frac{3-\alpha}{\alpha - 1}\right)^{\frac{3-\alpha}{3}} \mathcal{L}^3_g(\Omega \setminus B_{r_0})^{\frac{\alpha}{3}} r_0^{1-\alpha}$$

for every $\alpha \in (1, 3)$.

Proof. The volume elements differ by terms $O(r^{-2})$. The estimate follows from the Hölder inequality and the fact that

$$\int_{\mathbb{R}^3 \setminus B(0, r_0)} r^{\frac{-2\alpha}{3-\alpha}} d\mathcal{L}^3_\delta = \frac{3-\alpha}{3(\alpha - 1)} {r_0}^{\frac{3(1-\alpha)}{3-\alpha}}$$

for $\alpha \in (1, 3)$. q.e.d.

Appendix B. Further results in H. Bray's thesis

B.1. $u_c(s)$ **is increasing.** For the convenience of the reader, we reproduce H. Bray's proof that the function $u_c(s)$ in Section 3 is increasing, using our notation. This fact is all that H. Bray needed to show that the isoperimetric surfaces in Schwarzschild are the centered spheres.

Lemma B.1 ([**4**, Lemma 2])**.** *Let $c > 0$ and let $g_m^c = u_c(s)^{-2} ds^2 + u_c(s) s^2 g_{\mathbb{S}^2}$ be a smooth metric on $[c, \infty) \times \mathbb{S}^2$ with $u_c(c) = \alpha < 1$ and $\partial_s u_c|_c = 0$. Assume that $([c, \infty) \times \mathbb{S}^2, g_m^c)$ is isometric to the mean concave exterior region that lies beyond an umbilic constant mean curvature sphere of area $\alpha 4\pi c^2$ in Schwarzschild of mass $m > 0$. Then $u_c(s) \in (\alpha, 1)$ for $s > c$ and u_c is increasing in s.*

Proof. The Hawking mass is a first integral for the second-order ordinary differential equation that u_c is required to satisfy so that the metric $g_m^c = u_c^{-2} ds^2 + u_c s^2 g_{\mathbb{S}^2}$ is scalar flat. Up to a positive multiplicative constant, the Hawking mass of $\{s\} \times \mathbb{S}^2$ with respect to g_m^c is given by

$$y(s) \left(1 - \frac{y(s)^4 y'(s)^2}{s^4}\right). \tag{12}$$

Here, $y(s) := \sqrt{u_c(s) s^2}$ and the prime denotes differentiation with respect to s. It follows that $(s^3 - y^3)' \geq 0$ and hence $1 - \left(\frac{c}{s}\right)^3 (1 - \alpha^{\frac{3}{2}}) \geq u_c(s)^{\frac{3}{2}}$. We see that $u_c(s) < 1$ for all $s \geq c$. Assume that there is an $s \in [c, \infty)$ such that $u_c'(s) = 0$. For such s, we have that $y'(s) = \sqrt{u_c(s)}$ and $y''(s) = \frac{s}{2\sqrt{u_c(s)}} u_c''(s)$. Differentiating the constant Hawking mass (12), we obtain that

$$y''(s) = \frac{(1 - u_c(s)^3) s^{\frac{3}{2}}}{2 u_c(s)^{\frac{5}{2}}}.$$

Since we already know that $u_c(s) < 1$, we obtain that $y''(s) > 0$ and hence $u_c''(s) > 0$ for every $s \geq c$ such that $u_c'(s) = 0$. This implies that $u_c(s)$ is increasing. q.e.d.

B.2. Isoperimetric surfaces in compact perturbations of Schwarzschild. In [**4**, Section 2.6], H. Bray shows that in an initial data set (M, g) that is identically Schwarzschild outside a compact set, the large umbilic constant mean curvature spheres are isoperimetric surfaces for the volume they enclose with the horizon of (M, g). We briefly outline H. Bray's argument:

Fix a large centered coordinate sphere S_r that lies in the Schwarzschild part of the manifold. As explained in Section 3, one can construct a manifold $((0, \infty) \times \mathbb{S}^2, g_m^c)$ by gluing the tip of a cone to the sphere S_r in Schwarzschild in such a way that both the metric and the mean curvature match. H. Bray then constructs an area non-increasing map $\phi : (M, g) \to ((0, \infty) \times \mathbb{S}^2, g_m^c)$ so that the sphere S_r in (M, g) is mapped isometrically onto $\{c\} \times \mathbb{S}^2$ in $((0, \infty) \times \mathbb{S}^2, g_m^c)$ and such that ϕ is an isometry outside of S_r (respectively $\{c\} \times \mathbb{S}^2$). The construction of ϕ on the remainder of M starts at S_r and proceeds inward incrementally. In the spherically symmetric part of (M, g), one chooses ϕ to be also spherically symmetric and such that ϕ decreases area as little as possible. This requirement leads to an ordinary differential equation for the stretching of the spheres that lie inside of S_r. The analysis of this ordinary differential equation then yields that if r is sufficiently large, a certain sphere that is still outside the compact perturbation and hence in the spherically symmetric part of (M, g) gets mapped to the tip of the cone. In particular, all of the non-Schwarzschild part of (M, g) is mapped to the vertex of the cone.

Since ϕ is area non-increasing inside of S_r, it is also volume non-increasing. This implies that any other surface Σ in (M, g) which contains at least as much volume as S_r has larger area: use ϕ to map S_r and Σ to $((0, \infty) \times \mathbb{S}^2, g_m^c)$, and use that S_r is (outer) isoperimetric in $((0, \infty) \times \mathbb{S}^2, g_m^c)$.

H. Bray's technique to identify the isoperimetric surfaces of Schwarzschild has been generalized to a certain class of rotationally symmetric manifolds in [**3**], and further in [**32**]. See also the comment before the statement of Theorem 2.6 in [**32**] for a clarification of the hypotheses in [**3**].

Appendix C. Remark on locally isoperimetric surfaces

In this appendix we show that an unbounded minimal surface is area-minimizing if it is the smooth limit of isoperimetric surfaces. This observation is used in Section 6.

The proof follows from the same (classical) techniques that establish the regularity of isoperimetric surfaces. We include details for completeness and clarity. We deliberately phrase the proof in non-technical terms to help those readers who are not experts in geometric measure theory.

The regularity of rectifiable boundaries that minimize area with respect to a volume constraint was established in [**19**] and [**20**]. Implicitly, this result is already contained in [**1**]; cf. the remarks in the introduction of [**35**]. The papers [**19**, **20**] both rely on De Giorgi's method. The ways they go about dealing with the volume constraint are very different, however. In [**19**], a perturbation vector field is used to adjust the volume of a region by a small given amount while changing the area of its boundary in a controlled way. Morally, we follow the approach of [**19**] closely in this appendix. In [**20**], it is shown that there exist open balls in an isoperimetric region as well as its complement, so that small volume can be added or deleted in a controlled way.

We refer the reader to [**35**], in particular to Proposition 3.1 therein, for the development of the regularity theory for isoperimetric surfaces in Riemannian manifolds. There, further important references to the literature can be found. We also refer the reader to the paper [**37**], which contains useful observations regarding locally isoperimetric regions and additional references.

Let (M, g) be an initial data set as in Definition 2.2. In particular, (M, g) is homogeneously regular, i.e., its curvature tensor is bounded and its injectivity radius is bounded below, cf. [**43**, remarks below Theorem 3].

Let $\Omega \subset M$ be a smooth region that minimizes area with respect to compactly supported volume-preserving deformations, i.e., for every region $\Omega' \subset M$ such that $\Omega \Delta \Omega' \Subset B$ where B is bounded and open in M and such that $\mathcal{L}^3_g(B \cap \Omega) = \mathcal{L}^3_g(B \cap \Omega')$ one has that $\mathcal{H}^2_g(B \cap \partial\Omega) \leq \mathcal{H}^2_g(B \cap \partial\Omega')$. In addition, we assume that $\partial\Omega$ is minimal and unbounded.

Using the monotonicity formula in the form [**43**, Section 5] and elementary comparison arguments, one obtains that

$$\begin{aligned} \mathcal{H}^2_g(\partial\Omega) &= \infty, \\ \limsup_{s\to\infty} s^{-2}\mathcal{H}^2_g(B_s \cap \partial\Omega) &< \infty, \\ \liminf_{s\to\infty} \frac{\mathcal{H}^2_g(B_{s+1} \cap \partial\Omega)}{\mathcal{H}^2_g(B_s \cap \partial\Omega)} &= 1. \end{aligned} \tag{13}$$

Given $s, r \geq 1$ with $s \geq r$, we let $A_{r,s} := B_s \setminus B_r$.

Proposition C.1. *Let (M, g) and $\Omega \subset M$ be as above. Then $\partial\Omega$ is area minimizing.*

Proof. Let ν be the outward unit normal field of $\partial\Omega$. Let exp denote the exponential map of (M, g). A variation of the proof of [**42**, Proposition 5] shows that the curvature of $\partial\Omega$ is bounded and that there exists $\varepsilon \in (0, \frac{1}{2})$ small such that the map $E : \partial\Omega \times (-\varepsilon, \varepsilon) \to M$ defined by $E(\sigma, t) = \exp_\sigma(t\nu(\sigma))$ is a diffeomorphism with its image. The constant $\varepsilon \in (0, \frac{1}{2})$ can be chosen so that for some $C \geq 1$, the following holds:

Let $f \in \mathcal{C}^1_c(\partial\Omega)$ be such that $0 \leq f \leq \varepsilon/2$. Let Ω_f be the compact region bounded by $\{\exp_\sigma f(\sigma) : \sigma \in \partial\Omega\}$. Let $W_f := \{x \in M : \mathrm{dist}_g(x, \mathrm{supp}(f)) < 1\}$. Then

$$\begin{aligned} C\mathcal{H}^2_g(W_f \cap \partial\Omega) \sup_{\sigma\in\partial\Omega} f(\sigma) &\geq \mathcal{L}^3_g(W_f \cap \Omega_f) - \mathcal{L}^3_g(W_f \cap \Omega) \\ &\geq \frac{1}{C}\mathcal{H}^2_g(U \cap \partial\Omega) \inf_{\sigma\in U\cap\partial\Omega} f(\sigma) \text{ for every open } U \subset \mathrm{supp}(f). \end{aligned} \tag{14}$$

Using also that $\partial\Omega$ is minimal,

$$|\mathcal{H}^2_g(W_f \cap \partial\Omega_f) - \mathcal{H}^2_g(W_f \cap \partial\Omega)| \leq C \int_{\partial\Omega} (f^2 + |\nabla f|^2). \tag{15}$$

Let $\Omega' \subset M$ be a region with $\Omega' \Delta \Omega \Subset M$. Let $r \geq 1$ be such that $\Omega' \Delta \Omega \subset B_r$. Let

$$\Delta V := \mathcal{L}^3_g(\Omega \cap B_r) - \mathcal{L}^3_g(\Omega' \cap B_r).$$

Assume that $\Delta V > 0$. (The discussion when $\Delta V < 0$ is analogous.)

Since $\mathcal{H}^2_g(\partial\Omega) = \infty$, we have that $\frac{\varepsilon}{3C}\mathcal{H}^2_g(A_{r+2,s-1} \cap \partial\Omega) \geq \Delta V$ if s is sufficiently large. By (13), there exists a sequence $s_i \to \infty$ such that the quotients of $\mathcal{H}^2_g(A_{r,s_i+1} \cap \partial\Omega)$ and $\mathcal{H}^2_g(A_{r+2,s_i-1} \cap \partial\Omega)$ are close to 1.

Given $\delta_i \in (0, \varepsilon/2)$, let $f \in \mathcal{C}^2_c(\partial\Omega)$ with $\operatorname{supp}(f) \subset A_{r+1,s_i}$ be such that $0 \leq f \leq \delta_i$ and $|\nabla f| \leq 2\delta_i$, and such that $f = \delta_i$ on A_{r+2,s_i-1}. Using (14) with $U = A_{r+2,s_i-1}$, we see that $\mathcal{L}^3_g(\Omega_f \cap A_{r,s_i+1}) - \mathcal{L}^3_g(\Omega \cap A_{r,s_i+1}) = \Delta V$ for a choice of $\delta_i \sim \frac{\Delta V}{\mathcal{H}^2_g(A_{r+1,s_i} \cap \partial\Omega)}$. From (15), we obtain that

$$|\mathcal{H}^2_g(B_{s_i+1} \cap \partial\Omega_f) - \mathcal{H}^2_g(B_{s_i+1} \cap \partial\Omega)| = O\left(\frac{(\Delta V)^2}{\mathcal{H}^2_g(A_{r+1,s_i} \cap \partial\Omega)}\right).$$

Let $\tilde\Omega \subset M$ be the region such that $\tilde\Omega \setminus B_r = \Omega_f \setminus B_r$ and such that $\tilde\Omega \cap B_r = \Omega' \cap B_r$. Then $\mathcal{L}^3_g(B_{s_i+1} \cap \tilde\Omega) = \mathcal{L}^3_g(B_{s_i+1} \cap \Omega)$. Then

$$\begin{aligned}\mathcal{H}^2_g(B_{s_i+1} \cap \partial\Omega) &\leq \mathcal{H}^2_g(B_{s_i+1} \cap \partial\tilde\Omega) \\ &= \mathcal{H}^2_g(B_{s_i+1} \cap \partial\Omega') + O\left(\frac{(\Delta V)^2}{\mathcal{H}^2_g(A_{r+1,s_i} \cap \partial\Omega)}\right).\end{aligned}$$

The last term tends to zero as $i \to \infty$. Since $\Omega \setminus B_r = \Omega' \setminus B_r$, we see that $\mathcal{H}^2_g(B_r \cap \partial\Omega) \leq \mathcal{H}^2_g(B_r \cap \partial\Omega')$. q.e.d.

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