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THE SMALE CONJECTURE FOR SEIFERT FIBERED SPACES WITH HYPERBOLIC BASE ORBIFOLD

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Abstract

Let M be a closed orientable 3-manifold admitting an $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ geometry, or equivalently a Seifert fibered space with a hyperbolic base 2-orbifold. Our main result is that the connected component of the identity map in the diffeomorphism group $\mathrm{Diff}(M)$ is either contractible or homotopy equivalent to S^1, according as the center of $\pi_1(M)$ is trivial or infinite cyclic. Apart from the remaining case of non-Haken infranilmanifolds, this completes the homeomorphism classifications of $\mathrm{Diff}(M)$ and of the space of Seifert fiberings $\mathrm{SF}(M)$ for compact orientable aspherical 3-manifolds. We also prove that when M has an $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ geometry and the base orbifold has underlying manifold the 2-sphere with three cone points, the inclusion $\mathrm{Isom}(M) \to \mathrm{Diff}(M)$ is a homotopy equivalence.

Let M be a smooth closed manifold and $\mathrm{Diff}(M)$ the space of diffeomorphisms of M with the C^∞-topology. The path component of $\mathrm{Diff}(M)$ containing the identity Id_M is denoted by $\mathrm{diff}(M)$. In this paper, we focus on the case when M is a closed orientable 3-manifold admitting an $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ geometry, or equivalently M is a Seifert fibered space with a hyperbolic base 2-orbifold. Waldhausen [**Wa**] and, for the non-Haken cases, Scott [**Sc3**] together with Boileau-Otal [**BO**] proved that for such M, an element f of $\mathrm{Diff}(M)$ belongs to $\mathrm{diff}(M)$ if and only if f is homotopic to Id_M, and consequently homotopic diffeomorphisms are isotopic. In [**So**], the second author gave a new proof based on the insulator methods of Gabai [**Ga1**]. Our main result is:

Main Theorem. *Let M be a closed orientable Seifert fibered space with a hyperbolic base 2-orbifold. Then $\mathrm{diff}(M)$ is contractible or is homotopy equivalent to S^1, according as the center of $\pi_1(M)$ is trivial or infinite cyclic.*

As we will see, combined with known results the Main Theorem reduces two longstanding conjectural pictures in the topology of compact orientable aspherical 3-manifolds to a single remaining case, namely that of non-Haken infranilmanifolds. The first conjectural picture is the

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homeomorphism classification of $\mathrm{Diff}(M)$. It is known that $\mathrm{Diff}(M)$ is an infinite-dimensional separable Fréchet manifold, so its homeomorphism type is determined by its homotopy type. Moreover, since $\mathrm{Diff}(M)$ is a topological group, any two components are homeomorphic. Therefore the homeomorphism type of $\mathrm{Diff}(M)$ is determined by the cardinality of the mapping class group $\mathrm{Mod}(M)$ and the homotopy type of $\mathrm{diff}(M)$.

Here and throughout, we denote by $k = k(M)$ the rank of the center of $\pi_1(M)$, which is 0 if M does not admit a Seifert fibering. When M is Seifert-fibered, k is 3 if M is the 3-torus, is 1 when M is the orientable circle bundle over the Klein bottle that admits a cross-section, and in all other cases is 1 or 0 according as the base 2-orbifold of M is orientable or not. By $(S^1)^k$, we mean the product of k copies of S^1, where $(S^1)^0$ means a single point.

From work of Hatcher [**Ha1**] and Ivanov [**I1**, **I2**], we know that for Haken 3-manifolds, possibly with nonempty boundary, $\mathrm{diff}(M) \simeq (S^1)^k$ except in two cases: the solid torus, for which $\mathrm{diff}(M) \simeq S^1 \times S^1$, and D^3, for which $\mathrm{diff}(M) \simeq \mathrm{SO}(3)$ [**Ha2**]. Apart from these exceptional cases, the path component $\mathrm{isom}(M)$ of Id_M in the isometry group $\mathrm{Isom}(M)$ is $(S^1)^k$, when one uses a metric on M of maximal symmetry (that is, one for which the Lie group $\mathrm{Isom}(M)$ has maximal dimension and maximal number of components), and the homotopy equivalence $(S^1)^k \to \mathrm{diff}(M)$ is simply the inclusion $\mathrm{isom}(M) \to \mathrm{diff}(M)$. For the exceptional Haken cases, $\mathrm{isom}(M) \to \mathrm{diff}(M)$ is still a homotopy equivalence. For hyperbolic M, Haken or not, Gabai [**Ga2**] proved that $\mathrm{diff}(M)$ is contractible; in this case $k = 0$ and $\mathrm{isom}(M)$ is a point, so $\mathrm{isom}(M) \to \mathrm{diff}(M)$ is again a homotopy equivalence.

Among the closed orientable aspherical 3-manifolds, there remain only the non-Haken Seifert fibered cases. It is well-known that such a manifold must have as base orbifold a 2-sphere with three cone points, and such a Seifert fibered manifold is non-Haken if and only if its first homology group is finite [**Wa1**]. They have $k = 1$ and (as we will check) $\mathrm{isom}(M) = S^1$. There are two classes:

1. The non-Haken manifolds among those of the Main Theorem.
2. The non-Haken infranilmanifolds. A nilmanifold is a 3-manifold that is a quotient of Heisenberg space by a torsion-free lattice; topologically these are the S^1-bundles over the torus with nonzero Euler class. An infranilmanifold is a finite quotient of a nilmanifold. Their base orbifolds have cone points of types $(2, 4, 4)$, $(2, 3, 6)$, or $(3, 3, 3)$.

The homotopy equivalence $S^1 \to \mathrm{diff}(M)$ in the Main Theorem is realized as the inclusion $\mathrm{isom}(M) \to \mathrm{diff}(M)$, when M has its standard geometry. Therefore, combining the previous results, we have

Theorem 1. *Let M be a compact orientable aspherical 3-manifold with a metric of maximal symmetry, other than a non-Haken infranilmanifold. Then the inclusion* $\mathrm{isom}(M) \to \mathrm{diff}(M)$ *is a homotopy equivalence.*

Since any two infinite-dimensional separable Fréchet spaces are homeomorphic, we have as a corollary to Theorem 1 the homeomorphism classification of $\mathrm{Diff}(M)$ in the compact orientable aspherical case:

Corollary 1. *Let M be a compact orientable aspherical 3-manifold, other than a non-Haken infranilmanifold. Give M a metric of maximal symmetry. Then* $\mathrm{Diff}(M)$ *is homeomorphic to* $\mathrm{Mod}(M) \times \mathrm{isom}(M) \times F$, *where F is an infinite-dimensional separable Fréchet space.*

The homotopy equivalence in Theorem 1 may be viewed as a weak form of the original Smale Conjecture, which asserts that $\mathrm{Isom}(S^3) \to \mathrm{Diff}(S^3)$ is a homotopy equivalence for the round 3-sphere. The original Smale Conjecture was proven in two stages by J. Cerf [**Cerf**] and A. Hatcher [**Ha2**]. For Haken 3-manifolds, $\mathrm{Isom}(M) \to \mathrm{Diff}(M)$ often fails to be surjective on path components, but for the "small" manifolds among those in the Main Theorem, we will obtain the strong form of the Smale Conjecture.

Theorem 9.3. *Let M be a closed orientable Seifert-fibered 3-manifold having an $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ geometry, and as base orbifold a 2-sphere with three cone points. Then the inclusion* $\mathrm{Isom}(M) \to \mathrm{Diff}(M)$ *is a homotopy equivalence.*

The same statement was proven for closed hyperbolic 3-manifolds by Gabai [**Ga2**]. It is known for some elliptic 3-manifolds but not others; see [**HKMR**].

The second conjectural picture affected by the Main Theorem concerns the space of Seifert fiberings $\mathrm{SF}(M)$, defined in Section 9. It is also a separable infinite-dimensional Fréchet manifold. For Haken 3-manifolds, possibly with boundary, Theorem 3.14 of [**HKMR**] is

Theorem 2. *Let Σ be a Seifert-fibered Haken 3-manifold. Then each component of* $\mathrm{SF}(\Sigma)$ *is contractible.*

Problem 3.47(A3) of the Kirby Problem List [**Ki**] is the conjecture that if M has either the $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ geometry, then $\mathrm{SF}(M)$ is contractible. We will prove that in Section 9:

Corollary 9.2. *Let M be a closed orientable Seifert-fibered 3-manifold with a hyperbolic base orbifold. Then* $\mathrm{SF}(M)$ *is contractible.*

Combining this with Theorem 2 yields

Corollary 2. *Let M be a compact orientable aspherical Seifert fibered space, other than a non-Haken infranilmanifold. Then each component of* $\mathrm{SF}(M)$ *is contractible.*

Since the Seifert fiberings on compact 3-manifolds are completely classified, Corollary 2 gives an effective homeomorphism classification of $\mathrm{SF}(M)$ for almost all compact aspherical 3-manifolds:

Corollary 3. *Let M be a compact orientable aspherical Seifert fibered space, other than a non-Haken infranilmanifold. Then* $\mathrm{SF}(M)$ *is homeomorphic to $E \times F$, where E is the discrete set of equivalence classes of Seifert fiberings, and F is an infinite-dimensional separable Fréchet space.*

The methods of our paper do not adapt to infranilmanifolds, since we rely heavily on the hyperbolicity of the base orbifold. But we know of no reason not to expect that all of the previous results that exclude these manifolds are actually true for them as well. Consequently, as discussed at the beginning of Section 6, we have structured the applications sections in such a way that if the Main Theorem is proven in the infranilmanifold case, then all the results listed above will be established in that case as well.

Section 1 will give a brief overview of the proof of the Main Theorem, while Sections 2 through 5 of this paper will give the details. Section 9, preceded by three sections of background results, gives the proofs of Corollary 9.2 and Theorem 9.3.

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1. Sketch of the proof of the Main Theorem

Palais [**Pa**] showed that $\mathrm{diff}(M)$ has the homotopy type of a CW-complex, so by use of the Whitehead Theorem, it suffices to show that $\pi_n(\mathrm{diff}(M))$ is isomorphic to $\pi_n(S^1)$ for all $n \in \mathbb{N}$. When M is Haken, the Main Theorem follows from the work of Hatcher [**Ha1**, **Ha2**] and Ivanov [**I1**, **I2**]. So we may assume that M is non-Haken, in which case the base orbifold is hyperbolic with the 2-sphere as its underlying space and singular locus consisting of three points. Note that in these cases, $k(M) = 1$.

Our proof of the Main Theorem incorporates many of the ideas of Gabai's proof of the Smale Conjecture for closed hyperbolic 3-manifolds [**Ga2**]. His approach draws on his rigidity theorem for hyperbolic 3-manifolds in [**Ga1**]. In place of the latter, we will use results from [**So**], in which Scott's rigidity theorem for Seifert fibered spaces [**Sc1**] was obtained as a 2-dimensional (and hence easier) version of Gabai's rigidity theorem.

The first step, carried out in Sections 2 and 3, is to consider an arbitrary Riemannian metric ν on M and show, using least-area techniques from [**So**], that the preimage $c^\natural$ in M of a fixed cone point of highest order in the base orbifold is the core circle of a *canonical (open) solid torus.* The canonical torus depends only on ν and has certain key limiting properties as ν is varied. Roughly speaking, the canonical solid tori for a convergent sequence of metrics converge to an open solid torus that contains the canonical torus for the limit metric. These properties are developed and used in the proof of Lemma 4.1.

Lemma 4.1 corresponds to the Coarse Torus Isotopy Theorem of Gabai [**Ga2**, Theorem 4.6]. Given a continuous map $f\colon S^n \to \mathrm{diff}(M)$, its output is a family of solid tori associated to the cells of a cell decomposition of an $(n+1)$-ball B^{n+1} with boundary S^n. These solid tori satisfy the following: (1) for $y \in S^n$, $f(y)(c^\natural)$ is a core of each solid torus associated to a cell that contains y, and (2) they are nested according to the corresponding nesting of the cells of B^{n+1}. The key idea of the proof is Gabai's: push forward the standard metric of M using the diffeomorphisms of f to obtain a map from S^n to the contractible space of Riemannian metrics on M, extend this map to B^{n+1}, and use the canonical solid tori associated to these metrics to get started on constructing the solid tori of the conclusion.

The final part of the proof, in Section 5, uses the nested solid tori from Lemma 4.1 to construct an extension of a representative $f\colon S^n \to \mathrm{diff}(M)$ of an element of $\pi_n(\mathrm{diff}(M))$ to a map $F\colon B^{n+1} \to \mathrm{diff}(M)$. Unlike the hyperbolic case, however, $\mathrm{diff}(M)$ is not simply connected; indeed $\pi_1(\mathrm{diff}(M)) \cong \pi_1(S^1)$ is generated by a circular isotopy that moves points vertically around the fibers. To handle $\pi_1(\mathrm{diff}(M))$, we utilize a maximal-tree argument to reduce to the case when each diffeomorphism associated by f to a point of S^n carries $c^\natural$ into a fixed solid torus neighborhood of $c^\natural$. Under this assumption, f can be seen to be homotopic to a well-defined element of $\pi_1(\mathrm{isom}(M))$.

2. Least area annuli with bounded deviation

Throughout the remainder of this paper, all 3-manifolds are assumed to be orientable.

Let M be a closed Seifert fibered space with the Seifert fibration $\sigma\colon M \longrightarrow O$ over a hyperbolic triangle orbifold $O = O(p,q,r)$, where p,q,r are integers with $2 \le p \le q \le r$ and $1/p + 1/q + 1/r < 1$. The cyclic subgroup $\langle\gamma\rangle$ of $\pi_1(M)$ generated by the element γ represented by a regular fiber of M coincides with the center $Z(\pi_1(M))$ of $\pi_1(M)$.

Let $a\colon F \longrightarrow O$ be an orbifold covering such that F is a closed hyperbolic surface and $\widehat{a}\colon \mathbb{H}^2 \longrightarrow F$ the universal covering. Consider the natural quotient epimorphism $\varphi\colon \pi_1(M) \longrightarrow \pi_1^{\mathrm{orb}}(O) = \pi_1(M)/\langle\gamma\rangle$

and the covering $p\colon X \longrightarrow M$ associated to $\varphi^{-1}(a_*(\pi_1(F))) \subset \pi_1(M)$. The Seifert S^1-fibration σ lifts to an S^1-fibration $\sigma_X\colon X \longrightarrow F$. We have also an S^1-fibration $\widehat{\sigma}\colon \widehat{X} \longrightarrow \mathbb{H}^2$ and a covering $\widehat{p}\colon \widehat{X} \longrightarrow X$ in the following commutative diagram.

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\widehat{\sigma}} & \mathbb{H}^2 \\ {\scriptstyle \widehat{p}}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{a}} \\ X & \xrightarrow{\sigma_X} & F \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle a} \\ M & \xrightarrow{\sigma} & O \end{array}$$

We regard $G := \pi_1^{\mathrm{orb}}(O)$ as an isometric properly discontinuous transformation group on $\mathbb{H}^2$, and also as the covering transformation group on $\widehat{X}$ with respect to $p \circ \widehat{p}$. Then, $\widehat{\sigma}$ is G-equivariant.

Let $\mathcal{RM}(M)$ be the space of Riemannian metrics on M with C^∞-topology. The metrics on $\widehat{X}$ and X induced from $\nu \in \mathcal{RM}(M)$ are also denoted by ν. Since the ν-lengths of the S^1-fibers $\widehat{\sigma}(x)^{-1}$ ($x \in \mathbb{H}^2$) are uniformly bounded, $\widehat{\sigma}$ is a quasi-isometry. In particular, the boundary $\partial_\infty \widehat{X}$ of $\widehat{X}$ as a Gromov hyperbolic space is naturally identified with $S^1_\infty = \partial \mathbb{H}^2$.

For a closed subset J of $\mathbb{H}^2$, let $\mathcal{N}_d(J, \mathbb{H}^2)$ denote the closed d-neighborhood $\{y \in \mathbb{H}^2 \mid \mathrm{dist}(y, J) \le d\}$ of J in $\mathbb{H}^2$. For any geodesic line $\alpha \in \mathbb{H}^2$, $A^\natural_\alpha = \widehat{\sigma}^{-1}(\alpha)$ is an open annulus properly embedded in $\widehat{X}$. For $C > 0$, we set $L_C(\alpha) = \widehat{\sigma}^{-1}(\mathcal{N}_C(\alpha, \mathbb{H}^2))$, which is a closed neighborhood of $A^\natural_\alpha$ in $\widehat{X}$. Note that $L_C(\alpha)$ does not depend on the Riemannian metric ν on $\widehat{X}$.

A (compact) annulus A_0 embedded in $\widehat{X}$ is *ν-least area* if A_0 has the least area among all immersed annuli A_0' in $\widehat{X}$ with $\partial A_0' = \partial A_0$ with respect to the metric ν on $\widehat{X}$. An open annulus A properly embedded in $\widehat{X}$ is said to be a *ν-least area annulus associated to α* if A satisfies the following conditions.

- There exists $C > 0$ with $A \subset L_C(\alpha)$ such that A is properly homotopic to $A^\natural_\alpha$ in $L_C(\alpha)$. Here we say that C is a *deviation* of A.
- A is *ν-least area*. This means that any compact non-contractible annulus in A is a ν-least area annulus in $\widehat{X}$.

The following lemma is a stronger version of Lemma 2.1 in [**So**].

Lemma 2.1. *Let K be a non-empty compact subset of $\mathcal{RM}(M)$. Then there exists a constant $C_K > 0$ such that, for any geodesic line α in $\mathbb{H}^2$ and any $\nu \in K$, there exists a ν-least area annulus in $\widehat{X}$ associated* *to α with deviation C_K. Moreover, C_K is a deviation of any ν-least area annulus in $\widehat{X}$ associated to α.*

Proof. The base orbifold of M is divided by three geodesic segments u_1, u_2, u_3 into two hyperbolic triangles having interior angles π/p, π/q, π/r. Since the Fuchsian group $\pi_1(F)$ is residually finite, we may assume that $a^{-1}(u_1 \cup u_2 \cup u_3)$ is a union of simple closed geodesics $l_1, \ldots, l_n$, if necessary replacing F by a suitable finite covering space.

We will first construct least area annuli associated to geodesic lines that project to one of the l_i. The preimage $T_i^\natural = \sigma_X^{-1}(l_i)$ is an embedded incompressible torus in X. By Freedman-Hass-Scott [**FHS**], there exists an embedded torus $T_{i,\nu}$ in X which is ν-least area among all tori homotopic to $T_i^\natural$ in X. Since K is compact, $s_K = \sup_{\nu\in K}\{\mathrm{Area}_\nu(T_i^\natural)\} < \infty$. Each component $A_{i,\nu}$ of $\widehat{p}^{-1}(T_{i,\nu})$ is a ν-least area open annulus associated to a component of $\widehat{a}^{-1}(l_i)$.

Next we obtain a uniform deviation C'_K for these least area annuli. Since $\mathrm{Area}_\nu(T_{i,\nu}) \le s_K$ for all $\nu \in K$ and $\inf_{\nu\in K}\{\inf_{x\in X}\{\mathrm{inj}_\nu(x)\}\} > 0$, the Ascoli-Arzelà Theorem implies that any sequence $\{T_{i,\nu_m}\}_{m=1}^\infty$ with $\nu_m \in K$ has a subsequence converging uniformly to a torus in X homotopic to T_i. This shows that the $A_{i,\nu}$ ($\nu \in K$) have a common deviation $C'_{K,i}$. We set $C'_K = \max_i\{C'_{K,i}\}$.

To define C_K, consider any geodesic line α in $\mathbb{H}^2$ and let $\mathcal{L}$ be the set of geodesic lines λ in $\mathbb{H}^2$ with $\widehat{a}(\lambda) \subset l_1 \cup \cdots \cup l_n$. Denote by $\mathcal{L}^\vee(\alpha)$ the subset of $\mathcal{L}$ consisting of the λ disjoint from α. For any $\lambda \in \mathcal{L}^\vee(\alpha)$, let $e(\lambda)$ be the component of $\mathbb{H}^2 \setminus \mathcal{N}_{C'_K}(\lambda)$ disjoint from α. As was shown in the proof of [**So**, Lemma 2.1], there exists a constant $C_K > 0$, independent of α, with $\mathcal{N}_{C_K}(\alpha, \mathbb{H}^2) \supset \mathbb{H}^2 \setminus \left(\bigcup_{\lambda\in\mathcal{L}^\vee(\alpha)} e(\lambda)\right)$. Figure 2.1 illustrates C_K.

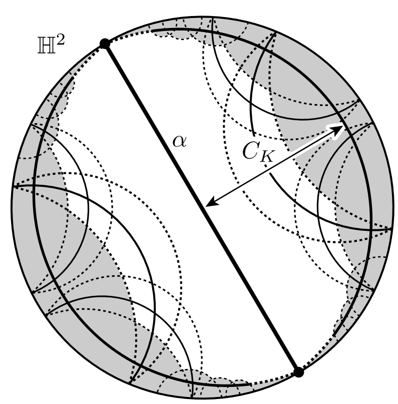

Figure 2.1. The shaded region represents $\bigcup_{\lambda\in\mathcal{L}^\vee(\alpha)} e(\lambda)$.

We are ready to construct a least area annulus A_α of deviation C_K associated to α. For any $\lambda \in \mathcal{L}^\vee(\alpha)$, take a ν-least area annulus A_λ in $\widehat{X}$

associated to λ with deviation C'_K. Let $E(\lambda)$ be the component of $\widehat{X}\setminus A_\lambda$ quasi-isometric to $e(\lambda)$ via $\widehat{\sigma}$. Let $\{J_n^+\}$, $\{J_n^-\}$ be sequences of mutually disjoint ν-least area annuli in $\widehat{X}$ associated to elements of $\mathcal{L}\setminus(\mathcal{L}^\vee(\alpha)\cup\{\alpha\})$ which converge to distinct endpoints of α in $\partial_\infty\widehat{X}=S^1_\infty$ and such that, for any n, the union $J_n^+\cup J_n^-$ excises from $\widehat{X}\setminus\bigcup_{\lambda\in\mathcal{L}^\vee(\alpha)}E(\lambda)$ a solid torus $V_n(\alpha)$ with $V_n(\alpha)\subset V_{n+1}(\alpha)$ and $\widehat{X}\setminus\bigcup_{\lambda\in\mathcal{L}^\vee(\alpha)}E(\lambda)=V_\infty(\alpha)$, where $V_\infty(\alpha)=\bigcup_n V_n(\alpha)$. Since the boundary of $V_n(\alpha)$ has non-negative mean curvature, by [**FHS**] there exists a properly embedded ν-least area annulus A_n in $V_n(\alpha)$ connecting simple non-contractible loops $d_n^\pm$ in $J_n^\pm$, as seen in Figure 2.2. As in the proof of [**So**, Lemma 2.1], one

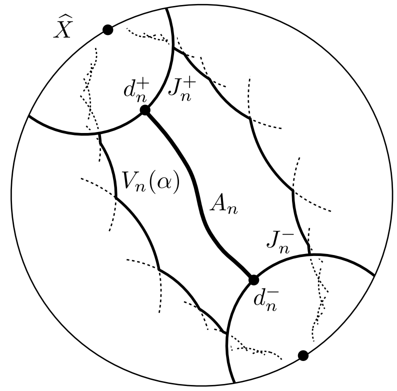

Figure 2.2

can show that $\{A_n\}$ has a subsequence converging locally uniformly to a ν-least area annulus A_α associated to α. Since $A_n\subset V_\infty(\alpha)$, we have $A_\alpha\subset V_\infty(\alpha)\subset L_{C_K}(\alpha)$. In particular, C_K is a deviation of A_α.

Now let A' be any ν-least area annulus associated to α. For any n, let $\lambda_1^{(n)},\dots,\lambda_k^{(n)}$ be the elements of $\mathcal{L}^\vee(\alpha)$ such that $A_{\lambda_i^{(n)}}$ meets $V_n(\alpha)$ non-trivially. Choose $m\in\mathbb{N}$ with $m>n$ so that $J_m^+\cup J_m^-$ is disjoint from $A_{\lambda_1^{(n)}}\cup\cdots\cup A_{\lambda_k^{(n)}}$. For $\tau\in\{+,-\}$, A' contains a non-contractible simple loop l^τ contained in the component of $\widehat{X}\setminus J_m^\tau$ disjoint from $A_{\lambda_1^{(n)}}\cup\cdots\cup A_{\lambda_k^{(n)}}$. Since the sub-annulus A'_0 of A' with $\partial A'_0=l^+\cup l^-$ is ν-least area, $A'_0\cap(A_{\lambda_1^{(n)}}\cup\cdots\cup A_{\lambda_k^{(n)}})=\emptyset$. This shows that $A'_n=A'_0\cap V_n(\alpha)$ is an annulus properly embedded in $V_n(\alpha)$ and connecting non-contractible simple loops in J_n^+ and J_n^-. Since $A'=\bigcup_n A'_n$, A' is contained in $V_\infty(\alpha)\subset L_{C_K}(\alpha)$. We conclude that C_K is a common deviation for all ν-least area annuli associated to α. q.e.d.

Lemma 2.2. *For any $\nu\in K$ and any geodesic line α in $\mathbb{H}^2$, let $\mathcal{A}_\nu(\alpha)$ be the set of all ν-least area annuli in $\widehat{X}$ associated to α. Then one of the following alternatives holds.*

(i) $\mathcal{A}_\nu(\alpha)$ *consists of a single element* $A^{\text{out}}_{\alpha[0]}\ (= A^{\text{out}}_{\alpha[1]})$.

(ii) $\mathcal{A}_\nu(\alpha)$ *contains two elements* $A^{\text{out}}_{\alpha[0]}$, $A^{\text{out}}_{\alpha[1]}$ *with* $A^{\text{out}}_{\alpha[0]} \cap A^{\text{out}}_{\alpha[1]} = \emptyset$ *such that any other elements* A *of* $\mathcal{A}_\nu(\alpha)$ *lie between* $A^{\text{out}}_{\alpha[0]}$ *and* $A^{\text{out}}_{\alpha[1]}$, *that is,* A *is contained in the component* U *of* $\widehat{X} \setminus A^{\text{out}}_{\alpha[0]} \cup A^{\text{out}}_{\alpha[1]}$ *with* $U \subset L_{C_K}(\alpha)$.

The open annuli $A^{\text{out}}_{\alpha[k]}$ given in Lemma 2.2 are called the *outermost elements* of $\mathcal{A}_\nu(\alpha)$.

Proof. We continue to use the notation of Lemma 2.1. In particular, there is a region $V_\infty(\alpha) \subset L_{C_K}(\alpha)$ for which any $A \in A_\nu(\alpha)$ is contained in $V_\infty(\alpha)$, and for any $n \in \mathbb{N}$, $A \cap V_n(\alpha)$ is a ν-least area annulus bounding non-contractible simple loops in J^+_n and J^-_n.

The closure $\partial_0 V_n(\alpha)$ of $\partial V_n(\alpha) \setminus (J^+_n \cup J^-_n)$ in $\widehat{X}$ consists of two annuli. We claim that some neighborhood of these annuli is disjoint from $\bigcup \mathcal{A}_\nu(\alpha)$. If not, then there would exist a sequence $\{A_m\}$ in $\mathcal{A}_\nu(\alpha)$ converging to an element A_∞ in $\mathcal{A}_\nu(\alpha)$ with $A_\infty \cap \partial_0 V_n(\alpha) \neq \emptyset$. Then, some $A_{\lambda^{(n)}_i}$ given in the proof of Lemma 2.1 and A_∞ would have a tangent point but no transverse points. A fundamental fact in minimal surface theory implies that $A_\infty = A_{\lambda^{(n)}_i}$. This contradicts the fact that $A_\infty \subset V_\infty(\alpha)$, establishing the claim.

By the claim, there exist sub-annuli Q^τ_n of $V_n(\alpha) \cap J^\tau_n$ for $\tau \in \{+,-\}$ such that $\mathrm{Int} Q^\tau_n$ contains $\left(\bigcup \mathcal{A}_\nu(\alpha)\right) \cap J^\tau$. We then have mutually disjoint ν-least area annuli $A_{n,0}$ and $A_{n,1}$ in $V_n(\alpha)$ with $\partial A_{n,0} \cup \partial A_{n,1} = \partial Q^+_n \cup \partial Q^-_n$ such that the union $A_{n,0} \cup A_{n,1} \cup Q^+_n \cup Q^-_n$ bounds a solid torus W_n in $V_n(\alpha)$ with $\left(\bigcup \mathcal{A}_\nu(\alpha)\right) \cap V_n(\alpha) \subset W_n \setminus (A_{n,0} \cup A_{n,1})$. Passing if necessary to subsequences, we may assume that both $\{A_{n,0}\}$ and $\{A_{n,1}\}$ converge locally uniformly to elements $A^{\text{out}}_{\alpha[0]}, A^{\text{out}}_{\alpha[1]} \in \mathcal{A}_\nu(\alpha)$ respectively. Since $A_{n,0} \cap A_{n,1} = \emptyset$ for all $n \in \mathbb{N}$, if $A^{\text{out}}_{\alpha[0]} \cap A^{\text{out}}_{\alpha[1]} \neq \emptyset$, then any elements of the intersection are tangent points but not transverse points. This implies that $A^{\text{out}}_{\alpha[0]} = A^{\text{out}}_{\alpha[1]}$ and hence $\mathcal{A}_\nu(\alpha)$ is the single element set $\{A^{\text{out}}_{\alpha[0]}\}$. In the case of $A^{\text{out}}_{\alpha[0]} \cap A^{\text{out}}_{\alpha[1]} = \emptyset$, since $\left(\bigcup \mathcal{A}_\nu(\alpha)\right) \cap V_n(\alpha) \subset W_n \setminus (A_{n,0} \cup A_{n,1})$ for any $n \in \mathbb{N}$, any elements of $\mathcal{A}_\nu(\alpha) \setminus \{A^{\text{out}}_{\alpha[0]}, A^{\text{out}}_{\alpha[1]}\}$ lie between $A^{\text{out}}_{\alpha[0]}$ and $A^{\text{out}}_{\alpha[1]}$ in $\widehat{X}$. q.e.d.

3. Canonical solid tori

For any geodesic line α in $\mathbb{H}^2$ and $\nu \in \mathcal{RM}(M)$, let $A^{\text{out}}_{\alpha[0]}$, $A^{\text{out}}_{\alpha[1]}$ be the outermost annuli in $\mathcal{A}_\nu(\alpha)$. In this section we will use these annuli to construct solid tori in M. These tori are canonical in that they depend only on the choice of Riemannian metric ν.

In the base orbifold $O = O(p,q,r)$, where $2 \leq p \leq q \leq r$, fix once and for all a singular point $\overline{x}_0$ that corresponds to the fixed point of an elliptic element of $G = \pi_1^{\mathrm{orb}}(O)$ of order r. Fix $x_0 \in (a \circ \widehat{a})^{-1}(\overline{x}_0)$ and write the orbit Gx_0 as $\{x_i\}_{i\in\Gamma}$, where Γ is an index set containing 0. For any $i, j \in \Gamma$ with $i \neq j$, let $\alpha_{i:j} = \alpha_{j:i}$ denote the perpendicular bisector line of the geodesic segment in $\mathbb{H}^2$ connecting x_i with x_j. For $\ell = 0, 1$, we write $A^{\mathrm{out}}_{i:j[\ell]}$ for $A^{\mathrm{out}}_{\alpha_{i:j}[\ell]}$.

Let $H_{i\prec j[k]}$ be the component of $\widehat{X} \setminus A^{\mathrm{out}}_{i:j[k]}$ quasi-isometric to the component of $\mathbb{H}^2 \setminus \alpha_{i:j}$ containing x_i via $\widehat{\sigma}$. If $H_{i\prec j[0]} \subset H_{i\prec j[1]}$, then we set $H^{\mathrm{inn}}_{i\prec j} = H_{i\prec j[0]}$. Otherwise set $H^{\mathrm{inn}}_{i\prec j} = H_{i\prec j[1]}$. In particular, our definition implies that $H^{\mathrm{inn}}_{i\prec j} \cap H^{\mathrm{inn}}_{j\prec i} = \emptyset$.

A simple loop c in an open solid torus U is a *core* if $U \setminus c$ is homeomorphic to $(D^\circ \setminus \{\mathbf{0}\}) \times S^1$, where D° is the open unit disk in $\mathbb{R}^2$ centered at the origin $\mathbf{0}$. A *core* of a solid torus V is a core of $\mathrm{Int}V$.

As in the proof of [**So**, Lemma 3.1], one can show that, for any $\nu \in \mathcal{RM}(M)$ and any $i \in \Gamma$, just one component of the intersection $\bigcap_{j\in\Gamma\setminus\{i\}} H^{\mathrm{inn}}_{i\prec j}$ is an open solid torus $\widehat{U}_{i,\nu}$ such that a core of $\widehat{U}_{i,\nu}$ is also a core of $\widehat{X}$, and all other components are open 3-balls.

Since G acts on both $\mathbb{H}^2$ and $\widehat{X}_\nu$ isometrically, the uniqueness of the outermost annuli implies that

$$g(A^{\mathrm{out}}_{\alpha[0]} \cup A^{\mathrm{out}}_{\alpha[1]}) = A^{\mathrm{out}}_{g(\alpha)[0]} \cup A^{\mathrm{out}}_{g(\alpha)[1]}$$

for any $g \in G$. Consequently, if $x_i = g(x_0)$ for $g \in G$, $\widehat{U}_{i,\nu} = g(\widehat{U}_{0,\nu})$. From our construction of $\widehat{U}_{i,\nu}$, we know that the stabilizer $\mathrm{stab}_G(\widehat{U}_{i,\nu})$ of $U_{i,\nu}$ in G is isomorphic to the stabilizer $\mathrm{stab}_G(x_i)$ for the action of G on $\mathbb{H}^2$. Since $\mathrm{stab}_G(x_i) \cong \mathbf{Z}_r$, $U_\nu = p \circ \widehat{p}(\widehat{U}_{i,\nu})$ is an open solid torus in M and the restriction $q_i \colon \widehat{U}_{i,\nu} \longrightarrow U_\nu$ of $p \circ \widehat{p}_i$ on $\widehat{U}_{i,\nu}$ is an r-fold cyclic covering. This U_ν is called the *ν-canonical solid torus.*

Since M is a Seifert fibered space with hyperbolic base orbifold, there exists a metric on M modeled on either $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{SL}_2}(\mathbb{R})$; see [**Th**, **Sc2**] for details. Fix such a metric, which we will call the *base metric* on M and denote by $\nu^\natural$.

We show that, for any geodesic α in $\mathbb{H}^2$, $A^\natural_\alpha = \widehat{\sigma}^{-1}(\alpha)$ is the unique $\nu^\natural$-least area annulus associated to α. For suppose that A is any $\nu^\natural$-least area annulus associated to α. If $A \neq \widehat{\sigma}^{-1}(\alpha)$, then $\widehat{\sigma}(A) \setminus \alpha$ would be non-empty. Hence we have a $\gamma \in \mathrm{Isom}(\mathbb{H}^2)$ such that $\alpha \cap \gamma(\alpha) = \emptyset$ but $\widehat{\sigma}(A) \cap \gamma(\widehat{\sigma}(A))$ is a non-empty compact set. Then there exists a isometric transformation $\widehat{\gamma}$ on $\widehat{X}_{\nu^\natural}$ covering γ such that $A \cap \widehat{\gamma}(A)$ is a non-empty compact set. This contradicts that both A and $\widehat{\gamma}(A)$ are $\nu^\natural$-least area; see for example [**FHS**, Lemma 1.3]. This shows that $A = A^\natural_\alpha$.

Since $A^\natural_{i,j} = \widehat{\sigma}^{-1}(\alpha_{i,j})$ is the unique $\nu^\natural$-least area annulus associated to $\alpha_{i,j}$, we have $A^{\mathrm{out}}_{i,j[0]} = A^{\mathrm{out}}_{i,j[1]}$ in $\widehat{X}_{\nu^\natural}$. Therefore $\widehat{c}^\natural = \widehat{\sigma}^{-1}(x_i)$ is a geodesic core of $\widehat{U}_{i,\nu^\natural}$ and $c^\natural = q_i(\widehat{c}^\natural)$ is a geodesic core of $U_{\nu^\natural}$.

4. Two key lemmas

The two lemmas in this section correspond respectively to the Coarse Torus Isotopy Theorem and the Local Contractibility Theorem of Gabai [**Ga2**, Theorems 4.6 and 6.3].

To set notation, denote by B^{n+1} the unit $(n+1)$-ball in $\mathbb{R}^{n+1}$ centered at the origin $\mathbf{0}$, and by $S^n = \partial B^{n+1}$ the unit sphere with basepoint $y_0 = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$. We always suppose that S^n and B^{n+1} have the Riemannian metrics induced from the standard Euclidean metric on $\mathbb{R}^{n+1}$.

For any cell-decomposition Δ of B^{n+1}, the set of i-cells in Δ will be denoted by $\Delta^{(i)}$ and the union $\Delta^{(0)} \cup \Delta^{(1)} \cup \cdots \cup \Delta^{(i)}$ by $\Delta^{[i]}$. For a subset Δ_0 of Δ, $|\Delta_0| := \bigcup_{\sigma\in\Delta_0} \sigma$ is the *underlying space* of Δ_0. For two solid tori W, V, the relation $W \Subset V$ means that $W \subset \mathrm{Int}V$ and W and V have a common core. Similarly, $c \Subset V$ means that c is a core of V.

Suppose that $f\colon K \longrightarrow \mathrm{diff}(M)$ is a continuous map. For $y \in K$, write f_y for the diffeomorphism $f(y)$, and for any $L \subset K$, write f_L for $f|_L$.

Lemma 4.1. *Let $f\colon S^n \longrightarrow \mathrm{diff}(M)$ be continuous. Then there exist a cell-decomposition Δ of B^{n+1} and a map V on Δ satisfying the following conditions.*

- (i) *For any $\sigma \in \Delta$, $V_\sigma := V(\sigma)$ is a solid torus in M such that if κ is a face of σ, then $V_\kappa \Subset V_\sigma$.*
- (ii) *For any $y \in \sigma \cap S^n$, $f_y(c^\natural) \Subset V_\sigma$.*

Proof. Let $\nu_S\colon S^n \longrightarrow \mathcal{RM}(M)$ be the continuous map defined by the push forward metrics $\nu_S(y) = (f_y)_*(\nu^\natural)$ $(y \in S^n)$. Since $\mathcal{RM}(M)$ is contractible, ν_S extends to a continuous map $\nu\colon B^{n+1} \longrightarrow \mathcal{RM}(M)$.

We first examine the limiting behavior of canonical solid tori. Suppose that $\{y_m\}$ is a sequence in B^{n+1}. Passing if necessary to a subsequence, we assume that $\{y_m\}$ converges to a point $y_\infty \in B^{n+1}$. For any $j \in \Gamma \setminus \{i\}$, let $A^{\mathrm{out}}_{i:j,m}$ be the outermost $\nu(y_m)$-least area annulus in $\widehat{X}$ with $A^{\mathrm{out}}_{i:j,m} = \mathrm{Fr}(H^{\mathrm{inn}}_{i\prec j})$. By Lemma 2.1, again passing if necessary to a subsequence, we may assume that these annuli $A^{\mathrm{out}}_{i:j,m}$ converge locally uniformly to $\nu(y_\infty)$-least area annuli $A_{i:j,\infty}$ in $\widehat{X}$ associated to $\alpha_{i:j}$; see [**HS**, Lemma 3.3], [**Ga1**, Lemma 3.3], and also the proof of [**So**, Theorem 0.2]. The $A_{i:j,\infty}$ may not be outermost $\nu(y_\infty)$-least area annuli. But as in the proof of [**So**, Lemma 3.1], $\bigcap_{j\in\Gamma\setminus\{i\}} H_{i\prec j}$ contains a unique open solid torus component $\widehat{U}$ to which the open solid tori $\widehat{U}_{i,\nu(y_m)}$ converge locally uniformly as embeddings from the standard open solid torus $D^\circ \times S^1$, where $H_{i\prec j}$ is the component of $\widehat{X} \setminus A_{i:j,\infty}$ containing $H^{\mathrm{inn}}_{i\prec j}$. Since each $\widehat{U}_{i,\nu(y_m)}$ is G-equivariant, $\widehat{U}$ is also G-equivariant. Thus $U = p \circ \widehat{p}(\widehat{U})$ is an embedded open solid torus in M containing $U_{\nu(y_\infty)}$.

Now, for any $y \in B^{n+1}$, fix a solid torus $V_{y,n+1} \Subset U_{\nu(y)}$. For any $y \in S^n$, since $f_y \colon M_{\nu^\natural} \longrightarrow M_{(f_y)_*(\nu^\natural)}$ is isometric, we may take $V_{y,n+1}$ so that $f_y(c^\natural) \Subset V_{y,n+1}$.

We claim that there exists $\delta_{y,n+1} > 0$ such that $V_{y,n+1} \subset U_{\nu(z)}$ if $\mathrm{dist}(y,z) < \delta_{y,n+1}$. If not, then we would have a sequence $\{z_m\}$ in B^{n+1} with $\mathrm{dist}(y, z_m) < 1/m$ and $V_{y,n+1} \not\subset U_{\nu(z_m)}$. Passing if necessary to a subsequence, we may as above assume that the $U_{\nu(z_m)}$ converge locally uniformly to an open solid torus U with $U \supset U_{\nu(y)}$. Since $V_{y,n+1}$ is a compact subset of $U_{\nu(y)} \subset U$, $V_{y,n+1}$ would be contained in $U_{\nu(z_m)}$ for all sufficiently large m, a contradiction.

Let $B^\circ_{n+1}(y)$ denote the open $\delta_{y,n+1}$-neighborhood of y in B^{n+1}. We choose the $\delta_{y,n+1}$ so that $B^\circ_{n+1}(y) \cap S^n = \emptyset$ if $y \in \mathrm{Int}B^{n+1}$. Moreover, since $f_y(c^\natural)$ moves continuously on $y \in S^n$, we may choose the $\delta_{y,n+1} > 0$ so that $f_z(c^\natural) \Subset V_{y,n+1}$ for any $z \in B^\circ_{n+1}(y) \cap S^n$.

Fix a finite collection $\{B^\circ_{n+1}(y_1), \ldots, B^\circ_{n+1}(y_k)\}$ that covers B^{n+1}. Let Δ^*_{n+1} be a piecewise smooth cell decomposition on B^{n+1} such that any $(n+1)$-cell σ of Δ^*_{n+1} is contained in at least one of the $B^\circ(y_i)$. Then, put $V^*_\sigma = V_{y_i,n+1}$ for some y_i with $B^\circ_{n+1}(y_i) \supset \sigma$.

Next, we will define a subdivision Δ^*_n of $\Delta^{*[n]}_{n+1}$. Let z be any element of B^{n+1}. As above, there exists $\delta_{z,n} > 0$ and a solid torus $V_{z,n}$ satisfying $V_{y_i,n+1} \Subset V_{z,n} \subset U_{\nu(w)}$ for any $w \in B^\circ_n(z)$ and any y_i $(i \in \{1, \ldots, k\})$ with $z \in B^\circ_{n+1}(y_i)$. For any element τ of $\Delta^{*(n)}_{n+1}$, there exists a finite subset $\{z_1, \ldots, z_l\}$ of τ such that $\{B^\circ_n(z_1), \ldots, B^\circ_n(z_l)\}$ covers τ. Then we take a cell decomposition $\Delta^*(\tau)$ of τ such that each n-cell of $\Delta^*(\tau)$ is contained in at least one of the $B^\circ_n(z_i)$ $(i = 1, \ldots, l)$. We set $\Delta^*_n = \bigcup_{\tau \in \Delta^{*(n)}_{n+1}} \Delta^*(\tau)$.

If $\sigma \in \Delta^*(\tau)^{(n)} \subset \Delta^{*(n)}_n$, then we set $V^*_\sigma = V_{z_j,n}$ for some z_j with $B^\circ_n(z_j) \supset \sigma$. If σ is contained in a face of $\sigma' \in \Delta^{*(n+1)}_{n+1}$, then τ is the face. It follows that $V^*_{\sigma'} = V_{y_i,n+1} \Subset V_{z_j,n} = V^*_\sigma$.

Repeating this process on descending skeleta, we define cell complexes $\Delta^*_{n-1}, \ldots, \Delta^*_0$ and extend the domain of the function V^* to $\Delta^{*(n-1)}_{n-1} \cup \cdots \cup \Delta^{*(0)}_0$ so that Δ^*_i is a subdivision of $\Delta^{*[i]}_{i+1}$ and $V^*_{\sigma'} \Subset V^*_\sigma$ whenever $\sigma \in \Delta^{*(i)}_i$ is in a face of $\sigma' \in \Delta^{*(i+1)}_{i+1}$. The union

$$\Delta^* = \Delta^{*(n+1)}_{n+1} \cup \Delta^{*(n)}_n \cup \cdots \cup \Delta^{*(0)}_0$$

is a cell decomposition on B^{n+1}.

Now form the double $d\Delta^*$ of Δ^* along $\Delta^*|_{S^n}$, obtaining a cell decomposition on $dB^{n+1} = S^{n+1}$. Let $(d\Delta^*)^*$ be the dual cell decomposition of $d\Delta^*$. The set Δ of all non-empty $\sigma \cap B^{n+1}$ and $\sigma \cap S^n$ for $\sigma \in (d\Delta^*)^*$ defines a cell decomposition on B^{n+1}. We define the map V satisfying conditions (i) and (ii) of this lemma as follows:

- If $\sigma \cap S^n = \emptyset$, then $V_\sigma = V^*_\tau$ for $\tau \in \Delta^{*(n+1-i)}$ dual to σ.

- If $\sigma \cap S^n \neq \emptyset$ and $\sigma \not\subset S^n$, $V_\sigma = V^*_\tau$ for $\tau \in \Delta^{*(n+1-i)}$ dual to the double $d\sigma$ of σ.
- If $\sigma \subset S^n$, then V_σ is a solid torus in $\mathrm{Int}V_{\sigma'}$ obtained by slightly shrinking $V_{\sigma'}$, where σ' is the cell of Δ with $\sigma' \not\subset S^n$ and $\sigma = \sigma' \cap S^n$.

This completes the proof. q.e.d.

Let W, V be solid tori in M with $c^\natural \Subset W \Subset V$. One can choose a Seifert fibration $\mathcal{F}$ on M so that W is a union of fibers and $c^\natural$ is an exceptional fiber of order r. The restriction $\mathcal{F}_N$ of $\mathcal{F}$ on $N = M \setminus \mathrm{Int}W$ defines a Seifert fibration over a disk with two exceptional fibers.

Let $\mathrm{Emb}(W, \mathrm{Int}V)$ be the space of embeddings of W into $\mathrm{Int}V$ with the C^∞-topology, and $\mathrm{emb}(W, \mathrm{Int}V)$ the arcwise connected component containing the inclusion $i\colon W \subset \mathrm{Int}V$. According to Lemma 5.1 and Remark 5.2 of [**Ga2**],

$$\mathrm{emb}(W, \mathrm{Int}V) \simeq \mathrm{diff}(W) \simeq \mathrm{diff}(\partial W) \simeq S^1 \times S^1,$$

where $S^1 \times S^1$ represents a free action on ∂W preserving the fibration $\mathcal{F}|_{\partial W}$. The S^1-action from the left factor preserves each fiber of $\mathcal{F}|_{\partial W}$ as a set, and the one from the right factor preserves some simple loop in ∂W meeting each fiber of $\mathcal{F}|_{\partial W}$ transversely in a single point. The left factor action extends to a fiber-preserving S^1-action on M, which defines a continuous map $\varphi\colon S^1 \longrightarrow \mathrm{diff}(M)$ with $\varphi_{y_0} = \mathrm{Id}_M$.

For any $m \in \mathbf{Z}$, we define $\varphi^m\colon S^1 \longrightarrow \mathrm{diff}(M)$ as follows.

- $(\varphi^0)_y = \mathrm{Id}_M$ for any $y \in S^1$.
- For any $m > 0$ (resp. $m < 0$), $(\varphi^m)_y\colon M \longrightarrow M$ $(y \in S^1)$ is the composition of $|m|$ copies of φ_y (resp. $(\varphi_y)^{-1}$).

Let $\mathbf{Z}_V$ be the subgroup of $\pi_1(\mathrm{emb}(W, \mathrm{Int}V))$ generated by the left factor S^1-action.

Lemma 4.2. *Suppose that $f\colon S^n \longrightarrow \mathrm{diff}(M)$ is a continuous map with $f_{y_0} = \mathrm{Id}_M$ and $f_y(c^\natural) \Subset V$ for any y in S^n.*

(i) *If $n = 1$, then f is homotopic rel. y_0 to φ^m for some $m \in \mathbf{Z}$. Moreover, if f is contractible in $\mathrm{diff}(M)$, then f extends to a continuous map $F\colon B^2 \longrightarrow \mathrm{diff}(M)$ with $F_z(c^\natural) \Subset V$ for any $z \in B^2$.*

(ii) *If $n \neq 1$, then f extends to a continuous map $F\colon B^{n+1} \longrightarrow \mathrm{diff}(M)$ with $F_z(c^\natural) \Subset V$ for any $z \in B^{n+1}$.*

Proof. Let W be a solid torus with $c^\natural \Subset W \Subset V$, sufficiently slim so that $f_y(W) \Subset V$ for any $y \in S^n$. When $n = 0$, it is not hard to construct a homotopy $F\colon [0,1] \longrightarrow \mathrm{diff}(M)$ such that $F_0 = f_{y_0}$, $F_1 = f_{y_1}$, and $F_t(c^\natural) \Subset V$ for any $t \in [0,1]$, where $S^0 = \{y_0, y_1\}$. In fact, there exists an extension $F_{[0,1/2] \cup \{1\}}$ of f_{y_0} and f_{y_1} with $F_t(c^\natural) \Subset V$ for any $t \in [0, 1/2]$ and $F_{1/2}|_W = F_1|_W$. Since the Seifert fibration on $N = M \setminus \mathrm{Int}W$ has a base orbifold with a disk as its underlying space and with two exceptional fibers, N has a unique essential annulus up to ambient isotopy. This implies that $F_{1/2}|_N$ is isotopic to $F_1|_N$, and consequently there is an extension $F_{[0,1]}$ of $F_{[0,1/2]\cup\{1\}}$ with $F_{[0,1]}(c^\natural) \subset \mathrm{Int}V$.

Suppose now that $n \geq 1$. As in the proof of [**Ga2**, p. 146, Claim] (using the Palais-Cerf covering isotopy theorem), there exists a continuous map $K\colon S^n \times [0,1] \longrightarrow \mathrm{diff}(M)$ satisfying the following conditions.

- $K_{(y,0)} = f_y$ for any $y \in S^n$ and $K_{(y_0,t)} = \mathrm{Id}_M$ for any $t \in [0,1]$.
- $K_{(y,t)}(W) \Subset V$ for any $(y,t) \in S^n \times [0,1]$.
- $K_{(y,1)}$ $(y \in S^n)$ fixes W as a set. Moreover, when $n = 1$, $K_{(y,1)}$ $(y \in S^1)$ defines an S^1-action on ∂W preserving $\mathcal{F}|_{\partial W}$.

Consider first the case of $n = 1$. If the element of $\pi_1(\mathrm{emb}(W, \mathrm{Int}V))$ represented by $K_{(y,1)}$ $(y \in S^1)$ were not contained in $\mathbf{Z}_V$, then the restriction of $K_{(y,1)}|_N$ to a basepoint $n_0 \in \partial N$ for $y \in S^1$ would not lie in the subgroup of $\pi_1(N, n_0)$ generated by a nonsingular fiber, contradicting the fact that the restriction of a circular homotopy to any basepoint must represent a central element of the fundamental group. So we may choose the homotopy K to satisfy $K_{(y,1)}|_W = \varphi_y^m|_W$ $(y \in S^1)$ for some $m \in \mathbf{Z}$.

From Hatcher [**Ha1**], the subspace of $\mathrm{diff}(M)$ consisting of diffeomorphisms g with $g|_W = \mathrm{Id}_M|_W$ is contractible. Since $K_{(y_0,1)} \circ \varphi_{y_0}^{-m} = \mathrm{Id}_M$, it follows that $K_{(y,1)} \circ (\varphi^{-m})_y$ $(y \in S^1)$ is contractible in $\mathrm{diff}(M)$ and hence f is homotopic to φ^m rel. y_0 in $\mathrm{diff}(M)$. This proves the first part of (i).

Assume now that f is contractible, and fix a basepoint x_0 in M. The trace homomorphism

$$\alpha\colon \pi_1(\mathrm{diff}(M)) \longrightarrow Z(\pi_1(M)) \cong \mathbf{Z}$$

is defined by putting, for any $g\colon S^1 \longrightarrow \mathrm{diff}(M)$ with $g_{y_0} = \mathrm{Id}_M$, $\alpha([g])$ equal to the element represented by the loop $g_y(x_0)$ $(y \in S^1)$ in M. In particular, α maps the class represented by φ^m to $m \in \mathbf{Z}$. Since f is contractible, $m = 0$. Regard B^2 as obtained from $S^1 \times [0,1]$ by shrinking $S^1 \times \{1\}$ to a point. Since $(\varphi^0)_y = \mathrm{Id}_M$ for any $y \in S^1$, K induces a continuous map $F\colon B^2 \longrightarrow \mathrm{diff}(M)$ with $F|_{S^1} = f$, $F(\mathbf{0}) = \mathrm{Id}_M$, and $F_z(W) \Subset V$ for any $z \in B^2$. This proves the remainder of (i).

Suppose now that $n > 1$. Since $\pi_n(\mathrm{emb}(W, \mathrm{Int}V)) = \{0\}$, we may apply the argument in part (i) to $K_{(y,1)}$ itself instead of to $K_{(y,1)} \circ (\varphi^{-m})_y$, obtaining an extension $F\colon B^{n+1} \longrightarrow \mathrm{diff}(M)$ of f as in (ii). q.e.d.

5. Proof of the Main Theorem

As noted in Section 1, we may assume that M is non-Haken, and it suffices to prove that $\pi_n(\mathrm{diff}(M)) \cong \pi_n(S^1)$ for all $n \geq 1$. We first examine $n = 1$.

Lemma 5.1. *Any continuous map* $f\colon S^1 \longrightarrow \mathrm{diff}(M)$ *with* $f_{y_0} = \mathrm{Id}_M$ *is homotopic to* φ^m *rel.* y_0 *for some* $m \in \mathbf{Z}$.

Proof. Fix a cell decomposition Δ of B^2 and a map V of Δ satisfying conditions (i) and (ii) of Lemma 4.1. Select a maximal tree T in $\Delta^{(1)}$ such that the complement $\Delta^{(1)} \setminus T$ consists of elements $\sigma_1, \ldots, \sigma_k$ with $y_0 \in \sigma_k \subset S^1$, $S^1 \setminus \sigma_k \subset |T|$ and such that, for any $i = 1, \ldots, k$, there exists $\tau_i \in \Delta^{(2)}$ with $\sigma_i \subset \partial\tau_i \subset |T_i| := |T| \cup \sigma_1 \cup \cdots \cup \sigma_i$; see Figure 5.1 (a).

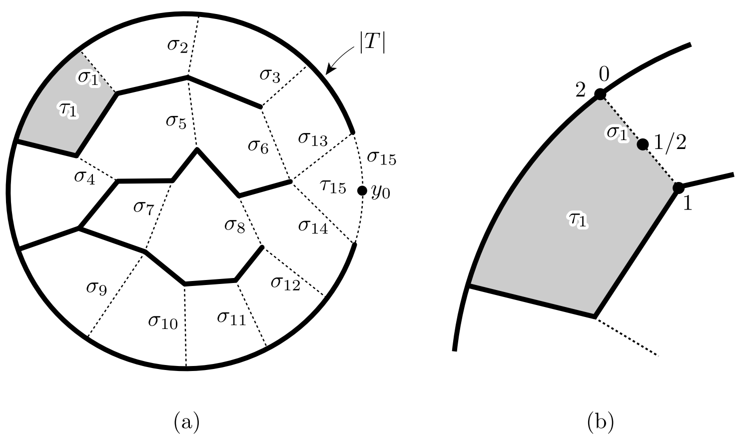

Figure 5.1

For each vertex v of $T|_{S^1}$, we have $f_v \in \mathrm{diff}(M)$ with $f_v(c^\natural) \Subset V_v$, and for each edge σ of $T|_{S^1}$, we have $f_y(c^\natural) \Subset V_\sigma$ for all $y \in \sigma$. Consider an edge σ in T having one endpoint v in S^1 and the other endpoint w in the interior of B^2. Since $V_v \Subset V_\sigma$ and $V_w \Subset V_\sigma$, we can obtain by isotopy extension a map $F_\sigma\colon \sigma \to \mathrm{diff}(M)$ with $F_v = f_v$, $F_t(c^\natural) \Subset V_\sigma$ for $t \in \sigma$, and $F_w(c^\natural) \Subset V_w$. Inducting on the distance from $|T| \cap S^1$, we have $F_{|T|}\colon |T| \to \mathrm{diff}(M)$ such that $F_v(c^\natural) \Subset V_v$ for each vertex of T and $F_t(c^\natural) \Subset V_\sigma$ for each t in each edge σ of T.

Now parameterize σ_1 and $\partial\tau_1 \setminus \mathrm{Int}\sigma_1$ respectively by $[0,1]$ and $[1,2]$ so that '$0 = 2$' in $\partial\tau_1$, as in Figure 5.1 (b). We have $F_0(c^\natural) \Subset V_{\sigma_1}$ and $F_1(c^\natural) \Subset V_{\sigma_1}$, and it follows that there is an extension of F_1 to $F_{[1/2,1]}$, such that $F_{1/2} = F_0$ and $F_t(c^\natural) \Subset V_{\sigma_1} \Subset V_{\tau_1}$ for any $t \in [1/2, 1]$.

Applying Lemma 4.2 (i) to $F_0^{-1} \circ F_t$ $(1/2 \leq t \leq 2)$ and $V := F_0^{-1}(V_{\tau_1})$, we have $j \in \mathbf{Z}$ such that the loop product of $(\varphi^j)_{2t}$ $(t \in [0, 1/2])$ and

$F_0^{-1} \circ F_t$ ($t \in [1/2, 2]$) is contractible in $\mathrm{diff}(M)$, where the domain S^1 of φ^j is supposed to be the quotient space obtained from $[0,1]$ by identifying 0 with 1 and regarding the point $0\,(=1)$ as the basepoint y_0 of S^1. Thus the extension $F_{[0,2]}$ of $F_{[1/2,2]}$ defined by $F_t = F_0 \circ (\varphi^j)_{2t}$ ($0 \le t \le 1/2$) is contractible in $\mathrm{diff}(M)$ and satisfies $F_t(c^\natural) \Subset V_{\sigma_1}$ for any $t \in [0,1]$.

So far, $f_{|T| \cap S^1}$ has been extended to $F_{|T_j|}$ satisfying the following conditions.

(a) $F_t(c^\natural) \Subset V_{\sigma_i}$ whenever $t \in \sigma_i$ for $i = 1, \ldots, j$.
(b) For any simple loop λ in $|T_j|$, the restriction F_λ is contractible in $\mathrm{diff}(M)$.

Repeating the argument, we obtain an extension $F_{|T_{k-1}|}$ satisfying (a) and (b). Using f on σ_k, we extend $F_{|T_{k-1}|}$ to $F_{|T_k|}$ satisfying (a).

By the condition (b) for $j = k-1$, for any simple loop λ in $|T_{k-1}|$, F_λ is contractible. Therefore the original f is homotopic rel. y_0 to the loop $F_{\partial\tau_k}$. Since $F_t(c^\natural) \Subset V_{\sigma_i} \Subset V_{\tau_k}$ for each $t \in \sigma_i \subset \partial\tau_k$, Lemma 4.2(i) shows that $F_{\partial\tau_k}$ is homotopic rel. y_0 to φ^m for some $m \in \mathbf{Z}$. q.e.d.

Proof of the Main Theorem. In Lemma 4.2 we defined the trace homomorphism $\alpha\colon \pi_1(\mathrm{diff}(M)) \longrightarrow Z(\pi_1(M))$. Lemma 5.1 shows that α is an isomorphism, that is, $\pi_1(\mathrm{diff}(M)) \cong \mathbf{Z}$. Moreover, the S^1-action which moves each point vertically in its fiber defines a map $S^1 \to \mathrm{diff}(M)$ which induces an isomorphism on fundamental groups, so it remains to show that $\pi_n(\mathrm{diff}(M)) = 0$ for $n > 1$.

Suppose that $n > 1$ and let $f\colon S^n \longrightarrow \mathrm{diff}(M)$ be any continuous map with $f_{y_0} = \mathrm{Id}_M$. Let Δ be a cell decomposition on B^{n+1} and V a map of Δ satisfying the conditions of Lemma 4.1. Let T_0 be a maximal subcomplex of Δ such that $|T_0|$ is simply connected and $S^n \subset |T_0| \subset S^n \cup |\Delta^{(1)}|$. We set $\Delta^{(1)} \setminus T_0 = \{\sigma_1, \ldots, \sigma_k\}$ and $|T_i| = |T_0| \cup \sigma_1 \cup \cdots \cup \sigma_i$ for $i = 1, \ldots, k$. As in the proof of Lemma 5.1, we can extend f to $F_{|T_0|}$ satisfying the conditions (a) and (b) in the proof of Lemma 5.1.

Next we will extend $F_{|T_0|}$ to σ_1 so that $F_{|T_1|}$ satisfies (a) and (b). Let v, w be the endpoints of σ_1. Fix an arc α in $|T_0|$ from w to v. As in the proof of Lemma 5.1, parameterize σ and α as $[0,1]$ and $[1,2]$ so that $v = 0 = 2$, and extend $F_{|T_0|}$ to $[1/2, 1]$ so that $F_0 = F_{1/2}$. Since $F_0(c^\natural) \Subset V_v \Subset V_{\sigma_1}$, Lemma 5.1 implies that $F_{[1/2,2]}$ is homotopic relative to $\{1/2, 2\}$ to a path in $\mathrm{diff}(M)$ with $F_t(c^\natural) \Subset V_{\sigma_1}$ at each time. Using the reverse of this path on $[0, 1/2]$ gives an extension of $F_{|T_0|}$ to $F_{|T_1|}$ such that $F_{\sigma_1 \cup \alpha}$ is a null-homotopic loop. Since the restriction of $F_{|T_0|}$ to any loop in $|T_0|$ is contractible, this implies that F_{λ_1} is also contractible for any loop λ_1 in $|T_1|$.

Repeating this process on σ_i ($i = 2, \ldots, k$), we obtain an extension $F_{|T_k|} = F_{|\Delta^{(1)}| \cup S^n}$ satisfying (a) and (b). In particular, its restriction to the boundary of any 2-cell in Δ is null-homotopic. So Lemma 4.2(i) implies that $F_{|\Delta^{(1)}|\cup S^n}$ extends to $F_{|\Delta^{(2)}|\cup S^n}$, satisfying $F_z(c^\natural) \Subset V_\tau$ for any z in each $\tau \in \Delta^{(2)}$. Then, by applying Lemma 4.2 (ii) repeatedly on the higher skeleta of Δ, one can extend $F_{|\Delta^{(2)}|\cup S^n}$ to all of $|\Delta^{(n+1)}| = B^{n+1}$. It follows that $f\colon S^n \longrightarrow \mathrm{diff}(M)$ is contractible and hence $\pi_n(\mathrm{diff}(M)) = 0$. q.e.d.

6. Deforming homotopy equivalences to diffeomorphisms

The fiber-preserving diffeomorphisms of Seifert-fibered 3-manifolds are well-understood; see for example Section 1 of Neumann and Raymond [**NR**]. Apart from a few simple exceptions, Seifert fiberings of Seifert-fibered 3-manifolds with infinite fundamental group are unique up to isotopy (see Lemma 2.1 and Corollary 2.3 of [**Oh**]), and consequently any diffeomorphism is isotopic to a fiber-preserving one.

It is also true that when M is a closed Seifert-fibered 3-manifold and $\pi_1(M)$ is infinite, any homotopy equivalence from M to M is homotopic to a diffeomorphism. This is certainly well-known in the Haken case, by Waldhausen's celebrated results [**Wa**]. For the non-Haken cases, it was proven in [**So**] when the base orbifold is hyperbolic. Although we do not actually need the non-Haken infranilmanifold case for our work here, it is appropriate to include a proof in order that all of our applications will also extend if our Main Theorem can be established in the infranilmanifold case (the only explicit invocation of the Main Theorem is in the proof of Theorem 9.1). Consequently we have included Proposition 6.1 below, which includes all non-Haken cases.

Although we are not aware of a published proof of Proposition 6.1 for the infranilmanifold case, we remark that it can be established using the work of J. Hass and P. Scott in [**HS1**]. (Fix a homotopy equivalence $g\colon M \to M$ and an immersion $j\colon T \to M$ that satisfies the 1-line 4-plane property, which exists by [**Sc1**]. Starting with the immersions j and gj, the argument of Theorem 5.2 in [**HS1**] adapts to produce the required diffeomorphism h, the key point being that the equivariance of the isomorphism in Theorem 4.3 of [**HS1**] implies that h and g induce the same outer automorphism on $\pi_1(M)$.) In addition, K.-B. Lee has shown us a proof of Proposition 6.1 using the theory of Seifert fiberings. Acknowledging those precedents, we will include here an elementary and nearly self-contained argument. It requires some notational preliminaries, but they are needed for our later work anyway.

For the remainder of this section, we assume that M is Seifert-fibered over an orbifold O which is the 2-sphere with exactly three cone points, and that $\pi_1(M)$ is infinite. To set notation, we recall a standard description of a Seifert-fibered structure on M. Remove from O the interiors of three disjoint disks, each containing one of the cone points, to obtain a disk-with-two-holes F. Then $\pi_1(F) = \langle Q_1, Q_2, Q_3 \mid Q_1Q_2Q_3 = 1\rangle$, with the three boundary circles representing the Q_i. Form $F \times S^1$, with $\pi_1(F\times S^1) = \pi_1(F)\times\langle T\rangle$. To the boundary tori, use fiber-preserving diffeomorphisms to attach suitably Seifert-fibered solid tori, each containing an exceptional fiber, so that the meridian curves represent $Q_i^{\alpha_i}T^{\beta_i}$, $1 \leq i \leq 3$. The pairs of relatively prime integers (α_i, β_i) with $\alpha_i \geq 2$ are called the (unnormalized) Seifert invariants. Different choices of β_i can yield the same (up to orientation-preserving diffeomorphism) topological fibering, but all choices are congruent modulo α_i.

From the construction, we obtain the presentation

$$\pi_1(M) = \langle q_1, q_2, q_3, t \mid tq_it^{-1} = q_i,\ q_i^{\alpha_i}t^{\beta_i} = 1,\ 1 \leq i \leq 3,\ q_1q_2q_3 = 1\rangle,$$

where the principal fiber represents the element t which generates the center C of $\pi_1(M)$. Putting $t = 1$ gives the quotient

$$\pi_1^{orb}(O) = \langle q_1, q_2, q_3 \mid q_i^{\alpha_i} = 1,\ 1 \leq i \leq 3,\ q_1q_2q_3 = 1\rangle.$$

Since M is aspherical, our next result implies that any homotopy equivalence from M to M is homotopic to a diffeomorphism.

Proposition 6.1. *Suppose that M is Seifert-fibered over an orbifold O which has three cone points and the 2-sphere as its underlying manifold, and that $\pi_1(M)$ is infinite. Let θ be an automorphism of $\pi_1(M)$. Then there exists an orientation-preserving fiber-preserving diffeomorphism of M whose induced automorphism on $\pi_1(M)$ equals θ in* $\mathrm{Out}(\pi_1(M))$.

Proof. Since C is the center of $\pi_1(M)$, there is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C & \longrightarrow & \pi & \longrightarrow & \pi_1^{orb}(O) & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle \theta|_C} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \overline{\theta}} & & \downarrow \\ 1 & \longrightarrow & C & \longrightarrow & \pi & \longrightarrow & \pi_1^{orb}(O) & \longrightarrow & 1 \end{array}$$

where the vertical maps are automorphisms. Theorem 5.8.3 of [**ZVC**], stated in our language, says that there is an orbifold diffeomorphism $g^{orb}\colon O \to O$ that induces $\overline{\theta}$ on $\pi_1^{orb}(O)$. We may assume that $g^{orb}(F) = F$, and we write $g\colon F \to F$ for the restriction of g^{orb}.

Since g is a diffeomorphism, we have $g_\#(Q_i) = \Gamma_i Q^{\epsilon}_{\sigma(i)}\Gamma_i^{-1}$ for some elements $\Gamma_i \in \pi_1(F)$, some permutation σ of $\{1, 2, 3\}$, and $\epsilon = 1$ or $\epsilon = -1$ according as g preserves or reverses orientation. Since $\overline{\theta} = g_\#^{orb}$, we can write $\theta(q_i) = \gamma_i q^{\epsilon}_{\sigma(i)}\gamma_i^{-1}t^{n_i}$ for some integers n_i, where γ_i is obtained from Γ_i by replacing each Q_i by q_i.

We claim that $n_1 + n_2 + n_3 = 0$. We have in $\pi_1(F)$ that

$$1 = g_\#(Q_1Q_2Q_3) = \Gamma_1 Q^{\epsilon}_{\sigma(1)}\Gamma_1^{-1}\ \Gamma_2 Q^{\epsilon}_{\sigma(2)}\Gamma_2^{-1}\ \Gamma_3 Q^{\epsilon}_{\sigma(3)}\Gamma_3^{-1}.$$

Since the latter word is trivial in $\pi_1(F)$, it is freely equivalent to a product of conjugates of $Q_1Q_2Q_3$ and $(Q_1Q_2Q_3)^{-1}$. Therefore the corresponding element $\gamma_1 q_1^\epsilon \gamma_1^{-1}\gamma_2 q_{\sigma(2)}^\epsilon \gamma_2^{-1}\gamma_3 q_{\sigma(3)}^\epsilon \gamma_3^{-1}$ in $\pi_1(M)$ is freely equivalent to a product of conjugates of $q_1q_2q_3$ and $(q_1q_2q_3)^{-1}$. Since the relation $q_1q_2q_3 = 1$ holds in $\pi_1(M)$, this word is trivial in $\pi_1(M)$ and we have

$$1 = \theta(q_1q_2q_3) = \gamma_1 q_1^\epsilon \gamma_1^{-1}\gamma_2 q_{\sigma(2)}^\epsilon \gamma_2^{-1}\gamma_3 q_{\sigma(3)}^\epsilon \gamma_3^{-1} t^{n_1+n_2+n_3} = t^{n_1+n_2+n_3}.$$

Since C is infinite, this shows that $n_1 + n_2 + n_3 = 0$.

Assume for now that $\theta(t) = t$. We have

$$t^{-\beta_i} = \theta(t^{-\beta_i}) = \theta(q_i^{\alpha_i}) = \gamma_i q_{\sigma(i)}^{\epsilon\alpha_i} \gamma_i^{-1} t^{n_i\alpha_i}.$$

This implies that $Q_{\sigma(i)}^{\alpha_i} = 1$ in $\pi_1^{orb}(O)$, so $\alpha_{\sigma(i)}$ divides α_i. Since this is true for all i, we have $\alpha_{\sigma(i)} = \alpha_i$. Therefore

$$t^{-\beta_i} = \gamma_i t^{-\epsilon\beta_{\sigma(i)}} \gamma_i^{-1} t^{n_i\alpha_i} = t^{-\epsilon\beta_{\sigma(i)} + n_i\alpha_i},$$

so $\epsilon\beta_{\sigma(i)} - \beta_i = n_i\alpha_i$.

Suppose for contradiction that $\epsilon = -1$. Then $\beta_{\sigma(i)} + \beta_i = -n_i\alpha_i$, and since $\alpha_{\sigma(i)} = \alpha_i$ we have $\beta_{\sigma(i)}/\alpha_{\sigma(i)} + \beta_i/\alpha_i = -n_i$. Summing this for $1 \le i \le 3$ and using $n_1 + n_2 + n_3 = 0$ gives $\sum \frac{\beta_i}{\alpha_i} = 0$ (if we already knew that θ arose from a fiber-preserving diffeomorphism, then this would amount to the fact that when a Seifert-fibered 3-manifold has an orientation-reversing fiber-preserving diffeomorphism, the Euler number of its Seifert fibration is 0). If all $\alpha_i = 2$, this is impossible, so we assume that $\alpha_1 \le \alpha_2 \le \alpha_3$ with $\alpha_3 \ge 3$. Since $\beta_{\sigma(3)}/\alpha_{\sigma(3)} + \beta_3/\alpha_3$ is an integer, $\sigma(3) \ne 3$ and we may assume that $\sigma(3) = 2$ and $\alpha_2 = \alpha_3$. But then,

$$-\frac{\beta_1}{\alpha_1} = \frac{\beta_2}{\alpha_2} + \frac{\beta_3}{\alpha_3}$$

would be an integer, a contradiction.

Let T_1, T_2, and T_3 be the boundary tori of $F \times S^1$, and fix disjoint vertical annuli A_1 and A_2 connecting T_3 to T_1 and T_2 respectively. Since $n_1+n_2+n_3 = 0$, there is a product j of fiber-preserving Dehn twists in a neighborhood of $A_1 \cup A_2$ such that $j_\#(Q_{\sigma(i)}) = Q_{\sigma(i)}T^{n_i}$ for each i. Let $h = j \circ (g \times 1_{S^1})$, a fiber-preserving diffeomorphism of $F \times S^1$. In $\pi_1(F \times S^1)$ we have $h_\#(T) = T$ and $h_\#(Q_i) = \Gamma_i Q_{\sigma(i)} \Gamma_i^{-1} T^{n_i}$. Using $\beta_{\sigma(i)} - \beta_i = n_i\alpha_i$, we have $h(Q_i^{\alpha_i}T^{\beta_i}) = \Gamma_i Q_{\sigma(i)}^{\alpha_{\sigma(i)}} \Gamma_i^{-1} T^{n_i\alpha_i} T^{\beta_i} = \Gamma_i Q_{\sigma(i)}^{\alpha_{\sigma(i)}} T^{\beta_{\sigma(i)}} \Gamma_i^{-1}$. That is, h takes meridian curves in the boundaries of the fibered solid tori of $\overline{M - F \times S^1}$ to meridian curves. Therefore h extends to a fiber-preserving diffeomorphism of M inducing θ. Since ϵ is 1, g and therefore h are orientation-preserving.

Suppose now that $\theta(t) = t^{-1}$. There is an orientation-preserving fiber-preserving diffeomorphism τ of M that reverses the direction of the fiber;

on O it induces a reflection through a circle containing the three cone points, and on each of the three fibered solid tori it is a hyperelliptic involution. Since $\tau_{\#}\theta(t) = t$, the previous case gives an orientation-preserving fiber-preserving diffeomorphism h such that $\tau_{\#}\theta = h_{\#}$ and hence $\theta = (\tau^{-1} \circ h)_{\#}$ in $\mathrm{Out}(\pi_1(M))$. q.e.d.

The following immediate corollary can also be proven by consideration of Euler numbers.

Corollary 6.2. *Suppose that M is Seifert-fibered over an orbifold O which has three cone points and the 2-sphere as its underlying manifold, and that $\pi_1(M)$ is infinite. Then every diffeomorphism of M is orientation-preserving.*

Proof. Since M is aspherical, two diffeomorphisms are homotopic if and only if they induce the same outer automorphism of $\pi_1(M)$. By Proposition 6.1, every homotopy class contains an orientation-preserving diffeomorphism, and the corollary follows since M is closed. q.e.d.

7. Isometries

Throughout this section we continue to assume that M is Seifert-fibered over an orbifold O which is the 2-sphere with exactly three cone points, and that $\pi_1(M)$ is infinite. We also continue to use the notation set up in the previous section. In this section we will analyze the isometry groups of these M.

It is known that M admits an $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}_2}(\mathbb{R})$, Nil, or Euclidean geometry such that the fibers of M are geodesics. Our reference for Seifert-fibered 3-manifolds and their geometries is [**Sc2**]. Every isometry of M is fiber-preserving: In all cases except the Euclidean geometry, every isometry of the universal cover $\widetilde{M}$ preserves the $\mathbb{R}$-fibers, so this is immediate. For the Euclidean geometry, the induced automorphism of any isometry of M must preserve the center of $\pi_1(M)$, so it takes the central element t represented by the principal fiber to either t or t^{-1} in $\pi_1(M)$. This implies that the lifted isometry preserves the $\mathbb{R}$-fibers of $\widetilde{M}$.

Proposition 7.1. *Give M its standard $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}_2}(\mathbb{R})$,* Nil*, or Euclidean geometry. If θ is any automorphism of $\pi_1(M)$, then there exists an isometry of M whose induced automorphism on $\pi_1(M)$ equals θ in* $\mathrm{Out}(\pi_1(M))$.

Proof. From Proposition 6.1, there exists a fiber-preserving diffeomorphism $f \colon M \to M$ with $f_{\#} = \theta$.

In the $\mathbb{E}^3$-case, let $\mathcal{T}(M)$ be the Teichmüller space of Euclidean structures on M with unit volume. For the other cases, $\mathcal{T}(M)$ will denote the Teichmüller space of all geometric structures on M. For $\sigma \in \mathcal{T}(M)$, let l_σ denote the length of a regular fiber of M_σ.

If M has an $\mathbb{H}\times\mathbb{R}$, $\mathbb{E}^3$, or Nil geometry, then by [**Oh**, Theorems 2.4, 2.6, 2.7] $\mathcal{T}(M)$ is homeomorphic to $\mathbb{R}$, which corresponds to the parameter $\log(l_\sigma)$ for $\sigma \in \mathcal{T}(M)$. (The statement of Theorem 2.4 in [**Oh**] contains a misprint: the exponent for the closed orientable case we use here should be $3-4\chi(X)+2k$. We remark that $\mathcal{T}(M)$ was also found for all of these cases by R. Kulkarni, K.-B. Lee, and F. Raymond [**KLR**] by a different method, although in the $\mathbb{E}^3$-case $\mathcal{T}(M)$ is given there as $\mathbb{R}^2$ since the volume is not normalized to be 1.) Since $f\colon M_\sigma \to M_{f_*(\sigma)}$ is isometric, $l_\sigma = l_{f_*(\sigma)}$ and hence $\sigma = f_*(\sigma)$ in $\mathcal{T}(M)$. It follows that f is isotopic to an isometry.

If M has an $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ geometry, then by [**Oh**, Theorem 2.5] (or [**KLR**]), $\mathcal{T}(M)$ is a single point. Again, f is isotopic to an isometry. q.e.d.

The quotient orbifold O has a unique hyperbolic structure when $\sum 1/\alpha_i < 1$, and a unique Euclidean structure up to scaling when $\sum 1/\alpha_i = 1$. An isometry of M induces an isometry of O, so the map $\mathrm{Isom}(M) \to \mathrm{Diff}^{orb}(O)$ taking each isometry f to its induced diffeomorphism $\overline{f}$ has its image in $\mathrm{Isom}(O)$.

We will need some specific isometries.

Lemma 7.2. *Give M its standard $\mathbb{H}^2\times\mathbb{R}$, $\widetilde{\mathrm{SL}_2}(\mathbb{R})$,* Nil*, or Euclidean geometry.*

(i) There is an isometric involution of M that preserves each exceptional fiber, reverses the direction of the fibers, and induces an orientation-reversing reflection on O.

(ii) Suppose that the Seifert invariants (α_j,β_j) and (α_k,β_k) of two exceptional fibers of M satisfy $\alpha_j=\alpha_k$ and $\beta_j \equiv \beta_k \bmod \alpha_j$. Then there is an isometric involution of M that interchanges these exceptional fibers, preserves the fiber direction, and on O induces an orientation-preserving isometry that interchanges the cone points corresponding to these two exceptional fibers.

Proof. For (i), consider an orientation-reversing reflection on O whose induced automorphism θ on $\pi_1^{orb}(O)$ is $\theta(q_1)=q_1^{-1}$, $\theta(q_2)=q_2^{-1}$, and $\theta(q_3)=q_2q_1q_3^{-1}q_2^{-1}q_1^{-1}$. This extends to an automorphism of $\pi_1(M)$ by putting $\theta(t)=t^{-1}$. Applying Proposition 7.1 gives an isometry as in (i) inducing θ.

For part (ii), we have by assumption that $\beta_k-\beta_j = n\alpha_j$ for some integer n. We proceed as in part (i), using an automorphism θ such that $\theta(t)=t$, $\theta(q_j)=q_kt^n$, $\theta(q_k)=q_jt^{-n}$, and for the remaining q_i, $\theta(q_i)$ is determined by the relation $\theta(q_1q_2q_3)=1$. q.e.d.

For $s\in\mathbb{R}$, let $\varphi(s)\colon M\to M$ be induced by translation by sL in the $\mathbb{R}$-fibers of $\widetilde{M}$, where L is the length of the principal fiber of M. Each $\varphi(s)=\varphi(s+1)$, so we regard $\varphi\colon S^1\to\mathrm{Isom}(M)$ as a circular isotopy of M. These are *vertical,* that is, they take each fiber of M to itself. We denote vertical maps of M by a subscript v, so the vertical isometries form a subgroup $\mathrm{Isom}_v(M)$. Corollary 6.2 yields immediately

Lemma 7.3. *No vertical diffeomorphism of M can reverse the fiber direction. Consequently,* $\mathrm{Isom}_v(M) = S^1$.

The isometry group $\mathrm{Isom}(O)$ is finite of the form $C_2 \times G$, where the C_2-factor is generated by an orientation-reversing reflection that fixes the cone points, and G is orientation-preserving and is either trivial, C_2, or D_3 according as the orders α_1, α_2, and α_3 of its cone points are distinct, exactly two are equal, or all three are equal. Note that $\mathrm{Isom}(O) \to \mathrm{Out}(\pi_1^{orb}(O))$ is injective.

Proposition 7.4. *The homomorphism* $\mathrm{Isom}(M) \to \mathrm{Out}(\pi_1(M))$ *is surjective, with kernel* $\mathrm{Isom}_v(M)$*. Consequently,* $\mathrm{Isom}(M)$ *is homeomorphic to* $\mathrm{Out}(\pi_1(M)) \times S^1$.

Proof. The surjectivity is from Proposition 7.1. An element f of the kernel must induce the identity outer automorphism on $\pi_1^{orb}(O)$, so $\overline{f}$ is the identity on O and therefore f is vertical. q.e.d.

8. Fiber-preserving diffeomorphisms

For a Seifert-fibered 3-manifold M, the fiber-preserving diffeomorphisms form a subgroup $\mathrm{Diff}_f(M)$ of $\mathrm{Diff}(M)$. From Theorem 2.2 of [**HKMR**], $\mathrm{Diff}_f(M)$ is a separable Fréchet manifold, so it is homotopy equivalent to a CW-complex.

Each element of $\mathrm{Diff}_f(M)$ induces an orbifold diffeomorphism of the base orbifold O, and by Theorem 3.9 of [**HKMR**], the map $\mathrm{Diff}_f(M) \to \mathrm{Diff}^{orb}(O)$ is a fibration over its image, with its fiber the vertical diffeomorphisms $\mathrm{Diff}_v(M)$.

We will need a description of the connected component of the identity, $\mathrm{diff}_v(M)$. Provided that M has an orientable base orbifold, it has a circular vertical isotopy that rotates each nonsingular fiber one full turn, such as the φ in the special case of Section 7.

Lemma 8.1. *Let M be an orientable Seifert-fibered* 3*-manifold with orientable base orbifold. Any circular vertical isotopy* $\varphi\colon S^1 \to \mathrm{diff}_v(M)$ *that rotates each nonsingular fiber one full turn defines a homotopy equivalence* $S^1 \simeq \mathrm{diff}_v(M)$.

Proof. Fix a basepoint m_0 in a nonsingular fiber. Restriction to m_0 defines a map (actually a fibration) $e\colon \mathrm{diff}_v(M) \to S^1$. The composition $S^1 \xrightarrow{\varphi} \mathrm{diff}_v(M) \xrightarrow{e} S^1$ is a homeomorphism, so $\varphi_\#\colon \pi_1(S^1) \to \pi_1(\mathrm{diff}_v(M))$ is injective.

Now, consider a parameterized family $f\colon (S^q, s_0) \to (\mathrm{diff}_v(M), \mathrm{Id}_M)$, for $q \geq 1$. To complete the proof that φ is a homotopy equivalence, we show that f is null-homotopic, when $q > 1$, or homotopic to a power of φ, when $q = 1$. Multiplying f by a power of φ, when $q = 1$, we may assume that $S^q \xrightarrow{f} \mathrm{diff}_v(M) \xrightarrow{e} S^1$ is null-homotopic.

Let F be the surface obtained from the base orbifold by removing the interiors of disjoint disk neighborhoods of the cone points, or if there are no cone points, by removing the interior of some disk. Consider the restriction of f to a parameterized family $g\colon S^q \longrightarrow \mathrm{diff}_v(F \times S^1)$ of vertical diffeomorphisms of $F \times S^1$. Since $S^q \xrightarrow{f} \mathrm{diff}_v(M) \xrightarrow{e} S^1$ is null-homotopic, we can lift g to $\tilde{g}\colon S^q \longrightarrow \mathrm{diff}_v(F \times \mathbb{R})$ such that $\tilde{g}(s_0) = \mathrm{Id}_{F\times\mathbb{R}}$. Note that for any $s \in S^q$, $\tilde{g}(s)$ is equivariant. This means that if we write $\tilde{g}(s)(x,t) = (x, \omega_s(x,t))$ for $(x,t) \in F \times \mathbb{R}$ and regard S^1 as $\mathbb{R}/\mathbb{Z}$, then $\omega_s(x, t+1) = \omega_s(x,t) + 1$. The homotopy $\tilde{g}_u\colon S^q \longrightarrow \mathrm{diff}_v(F \times \mathbb{R})$ ($u \in [0,1]$) defined by

$$\tilde{g}_u(s)(x,t) = (x, (1-u)\omega_s(x,t) + ut)$$

satisfies $\tilde{g}_0(s) = \tilde{g}(s)$, $\tilde{g}_1(s) = \mathrm{Id}_{F\times\mathbb{R}}$ for any $s \in S^q$ and $\tilde{g}_u(s_0) = \mathrm{Id}_{F\times\mathbb{R}}$ for any $u \in [0,1]$. Moreover, from the construction of $\tilde{g}_u$, each $\tilde{g}_u(s)$ is equivariant. Thus $\tilde{g}_u$ covers a homotopy $g_u\colon S^q \longrightarrow \mathrm{diff}_v(F \times S^1)$ between g and $\mathrm{Id}_{F\times S^1}$, which is naturally extended to a homotopy $f_u\colon S^q \longrightarrow \mathrm{diff}_v(M)$ between f and Id_M. This shows that f is contractible in $\mathrm{diff}_v(M)$. q.e.d.

We remark that when M has nonorientable base orbifold, there is no circular isotopy such as φ, and $\mathrm{diff}_v(M)$ is contractible, but we will not need this information.

Lemma 8.2. *Suppose that M is a Seifert-fibered 3-manifold with its base orbifold a 2-sphere with three cone points, and that $\pi_1(M)$ is infinite. Then $\mathrm{diff}_v(M) = \mathrm{Diff}_v(M)$.*

Proof. We must show that any vertical diffeomorphism j of M is vertically isotopic to the identity. By Lemma 7.3, j cannot reverse the fiber direction. By vertical isotopy, we can make j the identity on an exceptional fiber F_0, and then the identity on a fibered solid torus neighborhood V of F_0. In $N = M \setminus \mathrm{int}(V)$, there is a vertical annulus that separates N into two solid tori T_1 and T_2, each intersecting V in a vertical annulus. By a vertical isotopy fixing V, we can make j the identity on T_1. It is now the identity on ∂T_2, so by vertical isotopy we can make it the identity on T_2 as well. q.e.d.

Proposition 8.3. *Suppose that M is a Seifert-fibered 3-manifold with its base orbifold a 2-sphere with three cone points, and that $\pi_1(M)$ is infinite. Give M its standard $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}_2}(\mathbb{R})$, Nil, or Euclidean geometry. In the sequence*

$$\mathrm{Isom}(M) \to \mathrm{Diff}_f(M) \to \mathrm{Diff}(M) \to \mathrm{Out}(\pi_1(M)),$$

each of the three maps is bijective on path components.

Proof. By Proposition 7.1, the composition of all four maps is surjective, and hence so is $\mathrm{Diff}(M) \to \mathrm{Out}(\pi_1(M))$. By results of Scott [**Sc3**] and Boileau-Otal [**BO**], any diffeomorphism of M that is homotopic to the identity is isotopic to the identity, so $\mathrm{Diff}(M) \to \mathrm{Out}(\pi_1(M))$ is injective on path components. This proves the lemma for the third map, and that the second map is surjective on path components.

As usual, let F be the surface obtained from the base orbifold by removing the interiors of disjoint disk neighborhoods of the cone points. Consider a fiber-preserving diffeomorphism f of M that is isotopic to the identity. By fundamental group considerations, f cannot reverse the direction of the fiber, and must preserve each exceptional fiber. So by fiber-preserving isotopy we may assume that f is the identity on $\overline{M - F \times S^1}$. Every orientation-preserving diffeomorphism of F that preserves each boundary component is isotopic to the identity, allowing us to change f to be the identity in the F-coordinate of $F \times S^1$. Since f is now vertical, Lemma 8.2 shows that f is vertically isotopic to the identity. We conclude that the second map is bijective and the first map is surjective on path components.

By Proposition 7.4, $\mathrm{Isom}(M) \to \mathrm{Out}(\pi_1(M))$ is injective on path components, and hence so is the first map. This completes the proof. q.e.d.

9. The space of Seifert fiberings and the Smale Conjecture

Let M be a Seifert-fibered 3-manifold. Two (smooth) Seifert fiberings of M are considered equivalent if there is a diffeomorphism of M that takes fibers of one to fibers of the other. The coset space $\mathrm{Diff}(M)/\mathrm{Diff}_f(M)$ is the space of Seifert fiberings equivalent to the given one. Since fiberings equivalent under $\mathrm{Diff}(M)$ produce conjugate subgroups for $\mathrm{Diff}_f(M)$, the homeomorphism type of $\mathrm{Diff}(M)/\mathrm{Diff}_f(M)$ is independent of the particular fibering within its equivalence class. Taking the disjoint union of copies of $\mathrm{Diff}(M)/\mathrm{Diff}_f(M)$, one for each equivalence class of Seifert fibering, we obtain the space $\mathrm{SF}(M)$ of Seifert fiberings of M. By Theorem 3.12 of [**HKMR**], $\mathrm{SF}(M)$ is a separable Fréchet manifold locally modeled on an infinite-dimensional separable Fréchet space, and consequently it has the homotopy type of a CW-complex.

In this section, we will prove that when M is a closed orientable Seifert fibered 3-manifold with a hyperbolic base 2-orbifold, $\mathrm{SF}(M)$ is contractible. If in addition the base orbifold is a 2-sphere with three cone points, and M has its standard $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ geometry, then the inclusion $\mathrm{Isom}(M) \to \mathrm{Diff}(M)$ is a homotopy equivalence. Both of these facts rely upon the following result:

Theorem 9.1. *Let M be a closed orientable Seifert-fibered 3-manifold with a hyperbolic base orbifold. Then the inclusion $\mathrm{Diff}_f(M) \to \mathrm{Diff}(M)$ is a homotopy equivalence.*

Proof. When M is Haken, this is Theorem 3.13 of [**HKMR**], so we need only consider the case when the base orbifold is a 2-sphere with three cone points. By Proposition 8.3, the inclusion is a bijection on path components, so it remains to prove that $\mathrm{diff}_f(M) \to \mathrm{diff}(M)$ is a homotopy equivalence.

According to Theorem 3.9 of [**HKMR**], the induced map $\mathrm{Diff}_f(M) \to \mathrm{Diff}^{orb}(O)$ is a fibration over its image, and consequently the restriction $\mathrm{diff}_f(M) \to \mathrm{diff}^{orb}(O)$ is a fibration. The fiber is $\mathrm{Diff}_v(M) \cap \mathrm{diff}_f(M)$, which must be $\mathrm{diff}_v(M)$ by Lemma 8.2. Moreover, $\mathrm{diff}^{\mathrm{orb}}(O)$ is contractible, since it is essentially $\mathrm{diff}(S^2 \setminus \{\text{three points}\})$, and it follows that the inclusion $\mathrm{diff}_v(M) \to \mathrm{diff}_f(M)$ is a homotopy equivalence.

Consider the composition $S^1 \xrightarrow{\varphi} \mathrm{diff}_v(M) \to \mathrm{diff}_f(M) \to \mathrm{diff}(M)$. The first map is the homotopy equivalence of Lemma 8.1, and we have just seen that the second map is a homotopy equivalence. By the Main Theorem, the entire composition is a homotopy equivalence, and hence so is the third map. q.e.d.

The quotient map $\mathrm{Diff}(M) \to \mathrm{Diff}(M)/\mathrm{Diff}_f(M)$ is a fibration, by Theorem 3.12 of [**HKMR**]. Therefore Theorem 9.1 yields

Corollary 9.2. *Let M be a closed orientable Seifert-fibered 3-manifold with a hyperbolic base orbifold. Then $\mathrm{SF}(M)$ is contractible.*

As another consequence of Theorem 9.1, we have the Smale Conjecture for our class of non-Haken manifolds:

Theorem 9.3. *Let M be a closed orientable Seifert-fibered 3-manifold having an $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ geometry, and base orbifold a 2-sphere with three cone points. Then the inclusion $\mathrm{Isom}(M) \to \mathrm{Diff}(M)$ is a homotopy equivalence.*

Proof. According to Theorem 9.1, it suffices to show that the inclusion $\mathrm{Isom}(M) \to \mathrm{Diff}_f(M)$ is a homotopy equivalence.

As already noted, Theorem 3.9 of [**HKMR**] shows that the induced map $\mathrm{Diff}_f(M) \to \mathrm{Diff}^{orb}(O)$ is a fibration over its image, which we will denote by $\mathrm{Diff}_0^{orb}(O)$. This gives the second row of the diagram

$$\begin{array}{ccccc} \mathrm{Isom}_v(M) & \longrightarrow & \mathrm{Isom}(M) & \longrightarrow & \mathrm{Isom}_0(O) \\ \alpha\downarrow & & \downarrow & & \beta\downarrow \\ \mathrm{Diff}_v(M) & \longrightarrow & \mathrm{Diff}_f(M) & \longrightarrow & \mathrm{Diff}_0^{orb}(O) \end{array}$$

In the first row, $\mathrm{Isom}_0(O)$ is the image of $\mathrm{Isom}(M) \to \mathrm{Isom}(O)$. The second map is a homomorphism with kernel $\mathrm{Isom}_v(M)$, so the first row is also a fibration. The inclusion α is a homotopy equivalence by Lemmas 7.3, 8.1, and 8.2.

We claim that the inclusion β is also a homotopy equivalence, from which it follows that the middle vertical map is as well. Each nonidentity element of $\mathrm{Isom}_0(O)$ is nonisotopic to the identity on $\mathrm{diff}(S^2 \setminus \{\text{three points}\})$, so β is injective on path components. Let $f \in \mathrm{Diff}_f(M)$ induce $\overline{f}$ on O. By Proposition 8.3, f is isotopic through fiber-preserving diffeomorphisms to an isometry, so $\overline{f}$ is orbifold-isotopic to an isometry of O. That is, β is surjective on path components. Finally, the components of $\mathrm{Diff}^{orb}(O)$ are contractible, and the components of $\mathrm{Isom}_0(O)$ are points, so β is a homotopy equivalence and the proof is complete. q.e.d.

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