THE GEOMETRIC CAUCHY PROBLEM FOR SURFACES WITH LORENTZIAN HARMONIC GAUSS MAPS

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Abstract

The geometric Cauchy problem for a class of surfaces in a pseudo-Riemannian manifold of dimension 3 is to find the surface which contains a given curve with a prescribed tangent bundle along the curve. We consider this problem for constant negative Gauss curvature surfaces (pseudospherical surfaces) in Euclidean 3-space, and for timelike constant non-zero mean curvature (CMC) surfaces in the Lorentz-Minkowski 3-space. We prove that there is a unique solution if the prescribed curve is non-characteristic, and for characteristic initial curves (asymptotic curves for pseudo-spherical surfaces and null curves for timelike CMC) it is necessary and sufficient for similar data to be prescribed along an additional characteristic curve that intersects the first. The proofs also give a means of constructing all solutions using loop group techniques. The method used is the infinite dimensional d'Alembert type representation for surfaces associated with Lorentzian harmonic maps (1-1 wave maps) into symmetric spaces, developed since the 1990's. Explicit formulae for the potentials in terms of the prescribed data are given, and some applications are considered.

1. Introduction

The *geometric Cauchy problem* for a class of surfaces immersed in a manifold N is to find all surfaces of this class which contain some specified curve and with the surface tangent bundle prescribed along this curve. This is nothing other than the classical Björling problem for minimal surfaces addressed to other surface classes.

For minimal surfaces there is a unique solution given by a simple formula, because these surfaces have the Weierstrass representation in terms of holomorphic functions, and the prescribed data is sufficient to determine these holomorphic functions along the curve. The solution is then given by analytic extension. The geometric Cauchy problem has recently been studied in several situations involving holomorphic representations of surface classes – see, for example, [**15**, **16**, **2**, **11**,

14, **22**, **4**, **9**, **18**, **6**]. The solution of this problem is clearly a useful tool, both for proving general local properties of the surfaces and for constructing interesting examples.

The associated partial differential equations (PDE) in the works referred to above are elliptic, and the solutions therefore real analytic. It is interesting to see what can be done with hyperbolic equations. Aledo, Gálvez and Mira have recently shown that the geometric Cauchy problem can be solved for flat surfaces in the 3-sphere [**3**], which are associated to the homogeneous wave equation, a hyperbolic problem. The situation is quite different from the elliptic case, since the solutions are not real analytic in general. Nevertheless, for non-characteristic data, a unique solution is given in [**3**] using a d'Alembert type construction. Another example associated to the wave equation is that of timelike maximal surfaces in Lorentz-Minkowski 3-space $\mathbb{L}^3$, treated using split-holomorphic extensions in [**10**].

In this article we aim to address the geometric Cauchy problem for surfaces associated to harmonic maps from a Lorentzian surface into a Riemannian symmetric space. The associated PDE for the specific surfaces we will discuss are the sine-Gordon equation, the hyperbolic cosh/sinh-Gordon equations and the Liouville equation. There are no classical d'Alembert type solutions to these; however Lorentzian harmonic maps have a loop group representation [**21**, **25**]. From loop group techniques a kind of infinite dimensional d'Alembert solution can be found, whereby all solutions are given in terms of two functions, each of one variable only [**26**, **12**]. This type of solution was also derived in the late 1970's by Krichever for the sine-Gordon equation [**19**].

Examples of surfaces associated to Lorentzian harmonic maps include constant Gauss curvature -1 surfaces in $\mathbb{E}^3$ (pseudospherical surfaces), timelike constant mean curvature (CMC) surfaces in $\mathbb{L}^3$, and spacelike constant positive Gauss curvature surfaces in $\mathbb{L}^3$. Specifically, we treat the first two of these cases; since the main tool is the d'Alembert type solution for Lorentz harmonic maps, one expects that the approach can be adapted to other such problems.

We treat both pseudospherical surfaces and timelike CMC surfaces because the problem is not identical for the two cases. Firstly, the Gauss map of a timelike CMC surface is Lorentzian harmonic with respect to the first fundamental form of the surface, while for pseudospherical surfaces the Gauss map is harmonic with respect to the Lorentzian metric given by the second fundamental form. Consequently the routes to solving the problems are slightly different. Secondly, the group involved in the construction is compact in the pseudospherical case and non-compact in the timelike CMC case. The non-compact case is interesting because the loop group decomposition used is not global, an issue which will be discussed in future work.

1.1. Results of this article. By *uniqueness* of the solution of the geometric Cauchy problem, we will always mean the following: given two solutions $f : M \to N$ and $\tilde{f} : \tilde{M} \to N$, then, at any point $p = \gamma(t_0)$ on the initial curve γ, with $f(z_0) = \tilde{f}(\tilde{z}_0) = p$, there are neighbourhoods U of z_0 and $\tilde{U}$ of $\tilde{z}_0$, and an isometry $\phi : U \to \tilde{U}$ such that $f = \tilde{f} \circ \phi$.

In Section 3 we solve the geometric Cauchy problem for timelike CMC surfaces. In Theorem 3.1 we prove existence and uniqueness of the solution for timelike or spacelike initial curves, and in Theorem 3.2 we give a simple formula for the potentials used to construct the solution in terms of the initial data. If the initial curve is a null curve, we prove in Theorem 3.3 that it is necessary and sufficient to specify similar geometric Cauchy data on an additional null curve, which intersects the first, to obtain a unique solution. The potentials are also given explicitly.

As an application, we use in Section 4 the solution of the geometric Cauchy problem to find the potentials for timelike CMC surfaces of revolution.

In Section 5 we solve the geometric Cauchy problem for pseudospherical surfaces. Given an initial curve f_0 and prescribed surface normal N_0 along the curve, the characteristic case is distinguished this time not by the curve being null, but by the vanishing of the inner product $\langle f_0', N_0' \rangle$, which dictates that the curve must be an asymptotic curve of any solution surface.

Theorem 5.3 states that, if $\langle f_0', N_0' \rangle$ does not vanish, then there is a unique regular solution to the problem, provided that, in addition, f_0' and N_0' are either everywhere parallel or nowhere parallel. Explicit formulae for the potentials are given. The characteristic case, where $\langle f_0', N_0' \rangle$ is everywhere zero is treated in Section 5.2.2, and here one needs to specify an additional complex function to obtain a unique solution.

As a consequence of Theorem 5.3, Corollary 5.5 states that, given a space curve with non-vanishing curvature, with the additional condition that the torsion is either zero everywhere or never zero, then there is a unique pseudospherical surface which contains this curve as a geodesic. This geodesic is, of course, a principal curve if and only if the curve is a plane curve. Some examples are computed numerically (Figures 1, 3, 4 and 5).

Remark 1.1. Generally, we do not discuss the PDE associated to the geometric problems here. The non-characteristic Cauchy problem for the sine-Gordon equation has been well studied within the class of rapidly decreasing functions; with that type of initial data, it was solved by Ablowitz et al [1] using inverse scattering. We point out here that the construction in Section 5.2 can be used to prove global existence and uniqueness for the solution to the sine-Gordon equation with arbitrary C^∞ Cauchy data along a non-characteristic curve, and the solutions
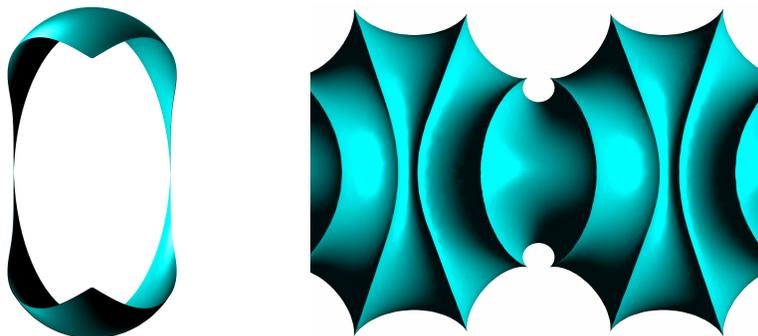
Figure 1. Two views of the unique pseudospherical surface that contains the ellipse $x^2 + (y/2)^2 = 1$ as a geodesic principal curve. The ellipse wraps around the smooth central band of the image on the right.

can be computed numerically by solving an ordinary differential equation and performing an LU decomposition. The global existence of the solution follows from the global Birkhoff decomposition proved in [**7**]. For the PDE associated to the timelike CMC surface the solution is unique for non-characteristic data, but only proved here to exist on a (large) open set containing the initial curve.

Notation: Throughout this article we follow the convention that, if $\hat{X}$ denotes any object which depends on the loop parameter λ, then dropping the hat means we evaluate at $\lambda = 1$, that is $X := \hat{X}\big|_{\lambda=1}$.

2. The infinite dimensional d'Alembert solution for timelike CMC surfaces

We first summarize the method given by Dorfmeister, Inoguchi and Toda [**12**] for constructing all timelike CMC surfaces from pairs of functions of one variable (called potentials) via a loop group splitting. The essential idea is the same as that used earlier by Toda [**26**] for pseudospherical surfaces in $\mathbb{E}^3$. The idea of using the loop parameter and the Sym formula to obtain the surface can be traced back to Sym [**24**].

2.1. The $\mathrm{SL}(2, \mathbb{R})$ frame for a Lorentz conformal immersion. We denote by $\mathbb{L}^3$ the Lorentz-Minkowski space with metric of signature

$(-++)$. Take the following basis for the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$:

$$e_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

With the inner product $\langle X, Y \rangle = \frac{1}{2} \operatorname{trace}(XY)$, the vectors e_0, e_1, e_2 form an orthonormal basis with $\langle e_0, e_0 \rangle = -1$, and we use this to identify $\mathbb{L}^3 \cong \mathfrak{sl}(2,\mathbb{R})$.

Let M be a simply connected 2-manifold.

Definition 2.1. *A conformal immersion $f : M \to \mathbb{L}^3$ is said to be* timelike *if the metric on M induced by f has signature $(-,+)$.*

In the following, a timelike immersion will always be understood as a conformal timelike immersion.

Let $f : M \to \mathbb{L}^3$ be a timelike immersion. The metric induced by f determines a Lorentz conformal structure on M. For any null coordinate system (x, y) on M, we define a function $\omega : M \to \mathbb{R}$ by the condition that the induced metric is given by

$$(2.1) \qquad \mathrm{d}s^2 = \varepsilon e^\omega \, \mathrm{d}x \, \mathrm{d}y, \qquad \varepsilon = \pm 1.$$

Let N be a unit normal field for the immersion f, and define a *coordinate frame* for f to be a map $F : M \to \mathrm{SL}(2,\mathbb{R})$ which satisfies

$$
\begin{aligned}
f_x &= \frac{\varepsilon_1}{2} e^{\omega/2} \operatorname{Ad}_F(e_0 + e_1), \\
(2.2) \qquad f_y &= \frac{\varepsilon_2}{2} e^{\omega/2} \operatorname{Ad}_F(-e_0 + e_1), \\
N &= \operatorname{Ad}_F(e_2),
\end{aligned}
$$

where $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. In this case (2.1) holds with $\varepsilon = \varepsilon_1 \varepsilon_2$. Conversely, since M is simply connected, we can always construct a coordinate frame for a timelike conformal immersion f. For a regular surface we can choose coordinates and a coordinate frame such that $\varepsilon_1 = \varepsilon_2 = \varepsilon = 1$, but we prefer a set-up that can also be used for surfaces which fail to be regular at some points.

The Maurer-Cartan form α for the frame F is defined by

$$\alpha = F^{-1}\mathrm{d}F = U\mathrm{d}x + V\mathrm{d}y.$$

With the choices made above, one easily computes

$$U = \frac{1}{4}\begin{pmatrix} -\omega_x & -4\varepsilon_1 Q e^{-\frac{\omega}{2}} \\ 2\varepsilon_1 H e^{\frac{\omega}{2}} & \omega_x \end{pmatrix}, \quad V = \frac{1}{4}\begin{pmatrix} \omega_y & -2\varepsilon_2 H e^{\frac{\omega}{2}} \\ 4\varepsilon_2 R e^{-\frac{\omega}{2}} & -\omega_y \end{pmatrix},$$

where $H = 2\varepsilon e^{-\omega}\langle f_{xy}, N \rangle$ is the mean curvature and $Q = \langle f_{xx}, N \rangle$ and $R = \langle f_{yy}, N \rangle$. The quadratic differentials $Q\mathrm{d}x^2$ and $R\mathrm{d}y^2$ are independent of the choice of null coordinates (provided the null directions are not interchanged), and are called the *Hopf differentials* of M.

2.2. The extended coordinate frame for a timelike CMC surface. The integrability condition $d\alpha + \alpha \wedge \alpha = 0$, also known as the Maurer-Cartan equation for the $\mathfrak{sl}(2,\mathbb{R})$-valued 1-form α, is equivalent to the Gauss-Codazzi equations for the surface:

$$(2.3) \qquad \varepsilon\omega_{xy} + \frac{1}{2}H^2 e^\omega - 2QRe^{-\omega} = 0,$$

$$(2.4) \qquad H_x = 2\varepsilon e^{-\omega}Q_y, \qquad H_y = 2\varepsilon e^{-\omega}R_x.$$

From these it is clear that the mean curvature H is constant if and only if Q and R depend only on x and y respectively, that is,

$$H \text{ constant} \iff Q_y = R_x = 0.$$

It follows easily that these conditions on R and Q do not depend on the choice of Lorentz null coordinates.

For any $\lambda \in \mathbb{R} \setminus \{0\}$, set $Q_\lambda = \lambda Q$ and $R_\lambda = \lambda^{-1}R$. Replacing Q and R by these in the compatibility conditions (2.3) and (2.4), equation (2.3) is unaffected, whilst equations (2.4) are satisfied for all λ if and only if the mean curvature H is constant. If this is the case, then the compatibility conditions are also unchanged if we multiply the $(2,1)$ and $(1,2)$ components of U and V respectively by λ and λ^{-1}. Set $\hat{\alpha} = \hat{U}dx + \hat{V}dy$, with

$$(2.5) \qquad \begin{aligned} \hat{U} &= \frac{1}{4}\begin{pmatrix} -\omega_x & -4\varepsilon_1 Qe^{-\frac{\omega}{2}}\lambda \\ 2\varepsilon_1 He^{\frac{\omega}{2}}\lambda & \omega_x \end{pmatrix}, \\ \hat{V} &= \frac{1}{4}\begin{pmatrix} \omega_y & -2\varepsilon_2 He^{\frac{\omega}{2}}\lambda^{-1} \\ 4\varepsilon_2 Re^{-\frac{\omega}{2}}\lambda^{-1} & -\omega_y \end{pmatrix}. \end{aligned}$$

Then we have demonstrated the following:

Lemma 2.2. *The mean curvature H is constant if and only if $d\hat{\alpha} + \hat{\alpha} \wedge \hat{\alpha} = 0$ for all $\lambda \in \mathbb{R} \setminus \{0\}$.*

By a *timelike CMC* surface we mean a timelike conformal immersion $f : M \to \mathbb{L}^3$ with constant mean curvature; if H is the constant value of the mean curvature we call it a timelike CMC H surface.

Let $\Lambda\mathrm{SL}(2,\mathbb{C})$ denote the group of (smooth) loops with values in $\mathrm{SL}(2,\mathbb{C})$, with a suitable topology. The H^s topology with $s > 1/2$ is sufficient for our purposes. Define the involution ρ on $\Lambda\mathrm{SL}(2,\mathbb{C})$ by

$$(\rho\gamma)(\lambda) = \overline{\gamma(\bar{\lambda})}.$$

Denoting by $\Lambda\mathrm{SL}(2,\mathbb{C})_\rho$ the fixed point subgroup, we note that any loop $\gamma \in \Lambda\mathrm{SL}(2,\mathbb{C})_\rho$ which extends holomorphically to some neighbourhood of $\mathbb{S}^1$, is $\mathrm{SL}(2,\mathbb{R})$-valued for real values of λ.

Consider another involution σ on $\Lambda\mathrm{SL}(2,\mathbb{C})$ given by

$$(\sigma\gamma)(\lambda) = \mathrm{Ad}_{e_2}\gamma(-\lambda).$$

It is easy to see that σ and ρ commute, and we denote by $\Lambda\mathrm{SL}(2,\mathbb{C})_\sigma$ the subgroup of $\Lambda\mathrm{SL}(2,\mathbb{C})$ of loops fixed by σ and by $\Lambda\mathrm{SL}(2,\mathbb{C})_{\sigma\rho}$ the subgroup fixed by both σ and ρ. To indicate that $\Lambda\mathrm{SL}(2,\mathbb{C})_{\sigma\rho}$ is a real form of $\Lambda\mathrm{SL}(2,\mathbb{C})_\sigma$ we will from now on use the shorthand notation

$$\mathcal{G}^{\mathbb{C}} = \Lambda\mathrm{SL}(2,\mathbb{C})_\sigma, \quad \mathcal{G} = \Lambda\mathrm{SL}(2,\mathbb{C})_{\sigma\rho}.$$

We use the same symbols to denote the infinitesimal versions of ρ and σ on the Lie algebra $\Lambda\mathfrak{sl}(2,\mathbb{C})$. Thus the Lie algebra of $\mathcal{G}$ is $Lie(\mathcal{G}) = \Lambda\mathfrak{sl}(2,\mathbb{C})_{\sigma\rho}$, the subalgebra of fixed points of σ and ρ in $\Lambda\mathfrak{sl}(2,\mathbb{C})$, whilst $Lie(\mathcal{G}^{\mathbb{C}}) = \Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma$ is the Lie algebra of $\mathcal{G}^{\mathbb{C}}$. Elements of $\mathcal{G}$, $\mathcal{G}^{\mathbb{C}}$ and their Lie algebras all have the property that diagonal and off-diagonal components are even and odd functions of λ respectively, due to the twisting introduced by σ.

Let $f : M \to \mathbb{L}^3$ be a timelike CMC surface, F a coordinate frame for f, and $(x_0, y_0) \in M$ a given point. As a consequence of Lemma 2.2, we can integrate the equation $\hat{\alpha} = \hat{F}^{-1}\mathrm{d}\hat{F}$ with the initial condition $\hat{F}(x_0, y_0) = F(x_0, y_0)$ to obtain a map $\hat{F} : M \to \mathcal{G}$ which we call an *extended coordinate frame* for f. Note that $\hat{F}\big|_{\lambda=1} = F$.

From the extended coordinate frame, we can easily reconstruct our surface f from the so-called *Sym formula*. Define a map $\mathcal{S} : \Lambda\mathrm{SL}(2,\mathbb{C}) \to \Lambda\mathfrak{sl}(2,\mathbb{C})$ by the formula, for any $\hat{G} \in \Lambda\mathrm{SL}(2,\mathbb{C})$,

$$\mathcal{S}(\hat{G}) = 2\lambda\partial_\lambda\hat{G}\hat{G}^{-1} - \mathrm{Ad}_{\hat{G}}(e_2).$$

For any λ_0 in the Riemann sphere $\hat{\mathbb{C}}$ at which the holomorphic extension of the loop $\hat{G}$ is defined, we define $\mathcal{S}_{\lambda_0}(\hat{G}) = \mathcal{S}(\hat{G})\big|_{\lambda=\lambda_0}$.

Lemma 2.3. *Let $H \in \mathbb{R}\setminus\{0\}$ and let f be a timelike CMC H surface with extended frame $\hat{F}$ as described above, with $f(p) = 0$ for some point $p \in M$. Then f is recovered by the* Sym *formula*

$$f(z) = \frac{1}{2H}\left\{\mathcal{S}_1(\hat{F}(z)) - \mathcal{S}_1(\hat{F}(p))\right\}.$$

For other values of $\lambda \in \mathbb{R}\setminus\{0\}$, $f^\lambda = \frac{1}{2H}\mathcal{S}_\lambda(\hat{F}) : M \to \mathbb{L}^3$ is also a timelike CMC H surface, with the same null coordinate system and metric, but with Hopf differentials $(Q_\lambda, R_\lambda) = (\lambda Q, \lambda^{-1}R)$.

Proof. To verify the formula for $f(z)$, set $\tilde{f}(z) = \frac{1}{2H}(\mathcal{S}_1(\hat{F}(z)) - \mathcal{S}_1(\hat{F}(p)))$. It is readily verified that $\tilde{f}_x = f_x$ and $\tilde{f}_y = f_y$. Since $\tilde{f}(p) = f(p) = 0$, we see that $\tilde{f}(z) = f(z)$ for all $z \in M$.

To verify that f^λ is timelike CMC H for other values of λ, we note $f^\lambda_x = \lambda\frac{\varepsilon_1 e^{\omega/2}}{2}\,\mathrm{Ad}_{\hat{F}}(e_0 + e_1)\big|_\lambda$ and $f^\lambda_y = \lambda^{-1}\frac{\varepsilon_2 e^{\omega/2}}{2}\,\mathrm{Ad}_{\hat{F}}(-e_0 + e_1)\big|_\lambda$, and we set $N^\lambda = \mathrm{Ad}_{\hat{F}}(e_2)\big|_\lambda$. It follows now easily that $H = 2\varepsilon e^{-\omega}\langle f^\lambda_{xy}, N^\lambda\rangle$.

q.e.d.

2.3. The loop group characterization of timelike CMC surfaces. The next proposition identifies the essential properties of the matrices $\hat{U}$ and $\hat{V}$ that we need in (2.5) in order to characterize timelike CMC surfaces as loop group maps.

Definition 2.4. *Let M be a simply connected subset of $\mathbb{R}^2$, and let (x,y) denote the standard coordinates. An* admissible frame *on M is a smooth map $\hat{F} : M \to \mathcal{G}$ such that the Maurer-Cartan form of $\hat{F}$ has the form*

$$(2.6) \qquad \hat{F}^{-1}\mathrm{d}\hat{F} = \lambda\, A_1\, \mathrm{d}x + \alpha_0 + \lambda^{-1} A_{-1}\, \mathrm{d}y,$$

where the $\mathfrak{sl}(2,\mathbb{R})$-valued 1-form α_0 is constant in λ. The admissible frame $\hat{F}$ is said to be regular *if $[A_1]_{21} \neq 0$ and $[A_{-1}]_{12} \neq 0$.*

Due to the loop group twisting, a regular admissible frame can be written $\hat{F}^{-1}\mathrm{d}\hat{F} = \hat{U}\mathrm{d}x + \hat{V}\mathrm{d}y$, with

$$\hat{U} = \begin{pmatrix} a_1 & b_1\lambda \\ c_1\lambda & -a_1 \end{pmatrix} \quad \text{and} \quad \hat{V} = \begin{pmatrix} a_2 & b_2\lambda^{-1} \\ c_2\lambda^{-1} & -a_2 \end{pmatrix},$$

where c_1 and b_2 are non-zero, and we use this notation in the next proposition.

Proposition 2.5. *Let $\hat{F} : M \to \mathcal{G}$ be a regular admissible frame and $H \neq 0$. Set $\varepsilon_1 = \mathrm{sign}(c_1)$, $\varepsilon_2 = -\mathrm{sign}(b_2)$ and $\varepsilon = \varepsilon_1\varepsilon_2$. Define a Lorentz metric on M by*

$$\mathrm{d}s^2 = \varepsilon e^\omega \mathrm{d}x\, \mathrm{d}y, \quad \varepsilon e^\omega = -\frac{4c_1 b_2}{H^2}.$$

Set

$$f^\lambda = \frac{1}{2H}\mathcal{S}_\lambda(\hat{F}) : M \to \mathbb{L}^3 \qquad (\lambda \in \mathbb{R} \setminus \{0\}).$$

Then, with respect to the choice of unit normal $N^\lambda = \mathrm{Ad}_{\hat{F}}\, e_2$ and the given metric, the surface f^λ is a timelike CMC H surface. Set

$$\rho = \left|\frac{b_2}{c_1}\right|^{\frac{1}{4}}, \qquad T = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix},$$

and set $\hat{F}_C = \hat{F}T : M \to \mathcal{G}$. Then $\hat{F}_C$ is the extended frame for the surface $f = f^1$, with respect to the coordinate frame defined by

$$f_x = \varepsilon_1\frac{1}{2}e^{\omega/2}\,\mathrm{Ad}_{F_C}(e_0 + e_1), \quad f_y = \varepsilon_2\frac{1}{2}e^{\omega/2}\,\mathrm{Ad}_{F_C}(-e_0 + e_1),$$
$$N = \mathrm{Ad}_{F_C}\, e_2 = \mathrm{Ad}_F\, e_2.$$

Proof. Since T is diagonal and constant in λ, we have $\mathcal{S}(\hat{F}_C) = \mathcal{S}(\hat{F})$ and $\hat{F}_C^{-1}\mathrm{d}\hat{F}_C = \hat{U}_C \mathrm{d}x + \hat{V}_C \mathrm{d}y$, where

$$\hat{U}_C = \begin{pmatrix} \rho^{-1}\rho_x + a_1 & \rho^{-2}b_1\lambda \\ \rho^2 c_1 \lambda & -\rho^{-1}\rho_x - a_1 \end{pmatrix},$$

$$\hat{V}_C = \begin{pmatrix} \rho^{-1}\rho_y + a_2 & \rho^{-2}b_2\lambda^{-1} \\ \rho^2 c_2 \lambda^{-1} & -\rho^{-1}\rho_y - a_2 \end{pmatrix}.$$

Differentiating $f = \frac{1}{2H}\mathcal{S}_1(\hat{F}_C)$, we use $\hat{U}_C$ and $\hat{V}_C$ to compute

$$f_x = \frac{c_1\rho^2}{H}\,\mathrm{Ad}_{F_C}(e_0 + e_1), \qquad f_y = -\frac{b_2\rho^{-2}}{H}\,\mathrm{Ad}_{F_C}(-e_0 + e_1).$$

It follows that f is conformally immersed with conformal factor $\varepsilon e^\omega = -\frac{4c_1 b_2}{H^2}$, and F_C is the coordinate frame given at equation (2.2), with ε_i as defined here. Therefore $\hat{F}_C$ has precisely the form of an extended coordinate frame for a timelike CMC H surface and, by Lemma 2.2, the result follows. q.e.d.

2.4. The d'Alembert construction of timelike CMC surfaces.
The point of Lemma 2.3 and Proposition 2.5 is that the problem of finding a conformal timelike CMC $H \neq 0$ immersion $M \to \mathbb{L}^3$ is equivalent to finding an admissible frame. In this section, we explain how to construct an admissible frame from two pairs of real functions.

Let $\Lambda^\pm \mathrm{SL}(2,\mathbb{C})_\sigma$ denote the subgroup of $\mathcal{G}^{\mathbb{C}}$ consisting of loops which extend holomorphically to $\mathbb{D}^\pm$, where $\mathbb{D}^+$ is the unit disc and $\mathbb{D}^- = \mathbb{S}^2 \setminus \{\mathbb{D}^+ \cup \mathbb{S}^1\}$, the exterior disc in the Riemann sphere. Set $0^+ := 0$ and $0^- := \infty$ and define

$$\mathcal{G}^\pm = \mathcal{G} \cap \Lambda^\pm \mathrm{SL}(2,\mathbb{C})_\sigma, \quad \mathcal{G}^\pm_* = \{\gamma \in \mathcal{G}^\pm \mid \gamma(0^\pm) = I\}.$$

We define the complex versions $\mathcal{G}^{\mathbb{C}\pm}$ analogously by substituting $\mathcal{G}^{\mathbb{C}}$ for $\mathcal{G}$ in the above definitions.

The Birkhoff decomposition for $\Lambda \mathrm{SL}(2,\mathbb{C})$ [23] restricts to a decomposition of either of the subgroups $\mathcal{G}^{\mathbb{C}}$ or $\mathcal{G}$; see [8], Proposition 1, for a general statement for fixed-point subgroups with respect to finite order automorphisms of the type used here.

Theorem 2.6 (The Birkhoff decomposition).

1) *The sets $\mathcal{B}_L = \mathcal{G}^- \cdot \mathcal{G}^+$ and $\mathcal{B}_R = \mathcal{G}^+ \cdot \mathcal{G}^-$ are both open and dense in $\mathcal{G}$. The multiplication maps*

$$\mathcal{G}^-_* \times \mathcal{G}^+ \to \mathcal{B}_L \quad \text{and} \quad \mathcal{G}^+_* \times \mathcal{G}^- \to \mathcal{B}_R$$

are both real analytic diffeomorphisms.
2) *The analogue holds substituting $\mathcal{G}^{\mathbb{C}}$, $\mathcal{G}^{\mathbb{C}\pm}$ and $\mathcal{G}^{\mathbb{C}\pm}_*$ for $\mathcal{G}$, $\mathcal{G}^\pm$ and $\mathcal{G}^\pm_*$, respectively, writing $\mathcal{B}_L^{\mathbb{C}} = \mathcal{G}^{\mathbb{C}-} \cdot \mathcal{G}^{\mathbb{C}+}$ and $\mathcal{B}_R^{\mathbb{C}} = \mathcal{G}^{\mathbb{C}+} \cdot \mathcal{G}^{\mathbb{C}-}$.*

In particular, any element $\hat{F}$ in the *big cell* $\mathcal{B} = \mathcal{B}_L \cap \mathcal{B}_R$ has exactly two decompositions

$$(2.7) \qquad \hat{F} = \hat{F}_- \hat{H}_+ = \hat{F}_+ \hat{H}_-, \quad \text{where} \quad \hat{F}_\pm \in \mathcal{G}_*^\pm, \quad \text{and} \quad \hat{H}_\pm \in \mathcal{G}^\pm,$$

and the factors $\hat{F}_\pm$ and $\hat{H}_\pm$ depend real analytically on $\hat{F}$.

Definition 2.7. *Let I_x and I_y be two real intervals, with coordinates x and y respectively. A* potential pair *(χ, ψ) is a pair of smooth $Lie(\mathcal{G})$-valued 1-forms on I_x and I_y respectively with Fourier expansions in λ as follows:*

$$\chi = \sum_{j=-\infty}^{1} \chi_i \lambda^i \mathrm{d}x, \qquad \psi = \sum_{j=-1}^{\infty} \psi_i \lambda^i \mathrm{d}y.$$

The potential pair is called regular *if $[\chi_1]_{21} \neq 0$ and $[\psi_{-1}]_{12} \neq 0$.*

The following theorem is a straightforward consequence of Theorem 2.6.

Theorem 2.8.   1) *Let M be a simply connected subset of $\mathbb{R}^2$ and $\hat{F} : M \to \mathcal{B} \subset \mathcal{G}$ an admissible frame. The pointwise (on M) Birkhoff decomposition (2.7) of $\hat{F}$ results in a potential pair $(\hat{F}_+^{-1}\mathrm{d}\hat{F}_+ , \hat{F}_-^{-1}\mathrm{d}\hat{F}_-)$, of the form*

$$\hat{F}_+^{-1}\mathrm{d}\hat{F}_+ = \chi_1 \lambda \, \mathrm{d}x, \qquad \hat{F}_-^{-1}\mathrm{d}\hat{F}_- = \psi_{-1}\lambda^{-1} \, \mathrm{d}y.$$

2) *Conversely, given any potential pair, (χ, ψ), define $\hat{F}_+ : I_x \to \mathcal{G}$ and $\hat{F}_- : I_y \to \mathcal{G}$ by integrating the differential equations*

$$\hat{F}_+^{-1}\mathrm{d}\hat{F}_+ = \chi, \quad \hat{F}_+(x_0) = I,$$
$$\hat{F}_-^{-1}\mathrm{d}\hat{F}_- = \psi, \quad \hat{F}_-(y_0) = I.$$

Define $\hat{\Phi} = \hat{F}_+^{-1}\hat{F}_- : I_x \times I_y \to \mathcal{G}$, and set $M = \hat{\Phi}^{-1}(\mathcal{B}_L)$. Pointwise on M, perform the Birkhoff decomposition $\hat{\Phi} = \hat{H}_-\hat{H}_+$, where $\hat{H}_- : M \to \mathcal{G}_*^-$ and $\hat{H}_+ : M \to \mathcal{G}^+$. Then $\hat{F} = \hat{F}_-\hat{H}_+^{-1}$ is an admissible frame.

3) *In both items (1) and (2), the admissible frame is regular if and only if the corresponding potential pair is regular. Moreover, with notation as in Definitions 2.4 and 2.7, we have $\mathrm{sign}[A_1]_{21} = \mathrm{sign}[\chi_1]_{21}$ and $\mathrm{sign}[A_{-1}]_{12} = \mathrm{sign}[\psi_{-1}]_{12}$. In fact, we have*

$$\hat{F}^{-1}\mathrm{d}\hat{F} = \lambda\chi_1\mathrm{d}x + \alpha_0 + \lambda^{-1}\hat{H}_+\big|_{\lambda=0}\psi_{-1}\hat{H}_+^{-1}\big|_{\lambda=0}\mathrm{d}y,$$

where α_0 is constant in λ.

Note that $\hat{F}$ in item (3) above is not in general an extended coordinate frame for the timelike CMC surface constructed from $\hat{F}$ via the Sym

formula. With notation as in Proposition 2.5, $\hat{F}_C = \hat{F}T$ is an extended coordinate frame for this surface, and

$$\hat{F}_C^{-1}\mathrm{d}\hat{F}_C = \lambda T^{-1}\chi_1 T\mathrm{d}x + \alpha_0 + T^{-1}\mathrm{d}T + \lambda^{-1}T^{-1}\hat{H}_+(0)\psi_{-1}\hat{H}_+(0)^{-1}T\mathrm{d}y.$$

3. The geometric Cauchy problem

In this section we show how to construct a unique timelike conformally immersed CMC surface from appropriate data along a curve. We treat two cases; first the case where the curve is non-characteristic for the PDE (here meaning not a null curve), where there is a unique solution of the geometric Cauchy problem, and next the case of a null curve, where one instead needs two curves to obtain a unique solution. We do not consider curves of "mixed type".

3.1. The case of non-characteristic curves.
Given a timelike immersion $f : M \to \mathbb{L}^3$ and a local null coordinate system (x, y) on M, the associated *Lorentz isothermal coordinate system* is defined by

$$u = \frac{1}{2}\left(x - y\right), \qquad v = \frac{1}{2}\left(x + y\right).$$

Note that, even though the null directions are well-defined, the directions determined by ∂_u and ∂_v are not, as they depend on the choice of null coordinates. With respect to the associated isothermal coordinates, a conformal metric $\mathrm{d}s^2 = e^\omega \mathrm{d}x\,\mathrm{d}y$ is of the form

$$\mathrm{d}s^2 = e^\omega(-\mathrm{d}u^2 + \mathrm{d}v^2).$$

The first problem which we shall solve is one where the following data is given:

Geometric Cauchy data: An open interval $J \subset \mathbb{R}$ containing 0, a regular smooth curve $f_0 : J \to \mathbb{L}^3$, which is either timelike or spacelike, and a regular smooth vector field $V : J \to \mathbb{L}^3$ along f_0, with the condition that the vector fields $\frac{\mathrm{d}f_0}{\mathrm{d}t}(t)$ and $V(t)$ are everywhere orthogonal and

$$\langle V(t), V(t)\rangle = -\left\langle \frac{\mathrm{d}f_0}{\mathrm{d}t}(t), \frac{\mathrm{d}f_0}{\mathrm{d}t}(t)\right\rangle \qquad (t \in J).$$

Note that our assumptions on V are equivalent to prescribing a family of timelike tangent planes along the curve $f_0(t)$.

Non-characteristic geometric Cauchy problem: Find a timelike CMC H-surface which contains the curve f_0 and is tangent along this curve to the plane spanned by $\mathrm{d}f_0/\mathrm{d}t$ and V.

3.2. Existence and uniqueness.
We state and prove the existence and uniqueness of the solution for the case that f_0 is timelike. A similar result, with the obvious changes in statement and proof, holds for the case that f_0 is spacelike.

Theorem 3.1. *Assume that geometric Cauchy data as above are given, with f_0 timelike, and let $H \neq 0$. Set $J^- = \{y \in \mathbb{R} \mid -y \in J\}$. Let $M = J \times J^-$, with coordinates (x, y), and set*

$$\Delta = \{(x, -x) \mid x \in J\} \subset J \times J^-.$$

Then:

1) *There is an open subset $W \subset M$, which contains Δ, and a unique timelike CMC H immersion $f : W \to \mathbb{L}^3$, with Lorentz isothermal coordinates $(u, v) = (\frac{1}{2}(x - y), \frac{1}{2}(x + y))$, satisfying*

$$f(u, 0) = f_0(u) \qquad and \qquad \frac{\partial f}{\partial v}(u, 0) = V(u).$$

2) *The surface so constructed is the unique, in the sense defined in Section 1.1, solution to the non-characteristic geometric Cauchy problem with the orientation given by $\frac{\mathrm{d}f_0}{\mathrm{d}t} \wedge V$.*

Proof. **Item 1:** We first show that there is a local solution in a neighbourhood of any point in Δ, and then that any two solutions agree at points where they are both defined.

Local existence: Fix a point $t_0 \in J$. Without loss of generality, we may assume that $f'(t_0)$ is a multiple of e_0 and $V(t_0)$ a multiple of e_2. We seek a solution such that t corresponds to the coordinate u along $v = 0$. Define the function $\omega_0 : J \to \mathbb{R}$ by $e^{\omega_0(t)} = \langle V(t), V(t) \rangle$, and the map $F_0 : J \to \mathrm{SL}(2, \mathbb{R})$ by

$$\frac{\mathrm{d}f_0}{\mathrm{d}t} = e^{\omega_0/2} \mathrm{Ad}_{F_0}(e_0), \quad V = e^{\omega_0/2} \mathrm{Ad}_{F_0}(e_1), \quad F_0(t_0) = I.$$

Assume that f is a solution with a coordinate frame F, satisfying $F(u, 0) = F_0(u)$, and an extended coordinate frame $\hat{F}$ constructed as in Section 2.1. Along Δ, we then have

$$\hat{F}^{-1}\hat{F}_u = \hat{U} - \hat{V}$$
$$= \frac{1}{4} \begin{pmatrix} -\omega_x - \omega_y & -4Qe^{-\omega/2}\lambda + 2He^{\omega/2}\lambda^{-1} \\ 2He^{\omega/2}\lambda - 4Re^{-\omega/2}\lambda^{-1} & \omega_x + \omega_y \end{pmatrix}.$$

At $\lambda = 1$, this should agree with

$$F_0^{-1}(F_0)_u = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

where a, b and c are known functions on Δ. Hence we have $Q - R = -\frac{1}{4}e^{\omega/2}(b - c)$ and $Q + R = \langle f_{xx} + f_{yy}, N \rangle = \langle f_{uu} + 2f_{xy}, N \rangle = \langle f_{uu}, N \rangle + He^\omega$, which give us the formulae

$$(3.1) \qquad Q = -\frac{1}{8}e^{\omega/2}(b - c) + \frac{1}{2}\left\langle \frac{\mathrm{d}^2 f}{\mathrm{d}u^2}, N \right\rangle + \frac{1}{2}He^\omega,$$

$$R = \frac{1}{8}e^{\omega/2}(b - c) + \frac{1}{2}\left\langle \frac{\mathrm{d}^2 f}{\mathrm{d}u^2}, N \right\rangle + \frac{1}{2}He^\omega.$$

These are all known functions along Δ, the normal N being given by $\mathrm{Ad}_{F_0}(e_2)$, and the diagonal components of $\hat{F}^{-1}\hat{F}_u$ along Δ are given by a and $-a$ respectively.

Thus, by defining Q and R along Δ by (3.1), we obtain a map $\hat{F}_0 : \Delta \to \mathcal{G}$ with $\hat{F}_0(x_0, x_0) = I$ and $\hat{F}_0(x, -x)\big|_{\lambda=1} = F_0(x)$. There is an open interval J' containing x_0 with $\hat{F}_0(\Delta') \subset \mathcal{B}$, where $\Delta' = \{(x,y) \in \Delta \mid x \in J'\}$. Performing the left and right normalized Birkhoff decompositions on Δ' gives $\hat{F}_0 = \hat{F}^0_- \hat{G}^0_+ = \hat{F}^0_+ G^0_-$. It follows from the construction of $\hat{F}_0$ that

$$(\hat{F}^0_-)^{-1}\mathrm{d}\hat{F}^0_- = \psi_{-1}\lambda^{-1}\mathrm{d}u, \qquad (\hat{F}^0_+)^{-1}\mathrm{d}\hat{F}^0_+ = \chi_1\lambda^1\mathrm{d}u,$$

where ψ_{-1} and χ_1 do not depend on λ. Hence $\hat{F}_-(y) = \hat{F}^0_-(-y)$, and $\hat{F}_+(x) = \hat{F}^0_+(x)$, correspond to a potential pair $\psi = \hat{F}^{-1}_-\mathrm{d}\hat{F}_-$ and $\chi = \hat{F}^{-1}_+\mathrm{d}\hat{F}_+$, on $I'_x = \{x \in \mathbb{R} \mid (x,-x) \in J'\}$ and $I'_y = \{y \in \mathbb{R} \mid (-y, y) \in J'\}$, respectively.

As in the second part of Theorem 2.8, define $\hat{\Phi} : I'_x \times I'_y \to \mathcal{G}$ by the expression $\hat{\Phi}(x,y) := \hat{F}_+(x)^{-1}\hat{F}_-(y)$; since $\hat{\Phi}(x_0, -x_0) = I$, there is an open set $W \subset M$, with $(x_0, -x_0) \in W$, such that $\hat{\Phi}(W) \subset \mathcal{B}_L$. Performing a left normalized Birkhoff splitting of $\hat{\Phi}$ on W gives $\hat{\Phi} = \hat{H}_- \hat{H}_+$, and we have an admissible frame $\hat{F} = \hat{F}_- \hat{H}^{-1}_+ : W \to \mathcal{G}$.

By construction, this frame agrees with $\hat{F}_0(u)$ along $W \cap \Delta$, and is therefore regular along this set; hence, taking W sufficiently small, regular on W. From the Sym formula we thus conclude that there is a solution defined in a neighbourhood of $(x_0, -x_0)$.

Global existence and uniqueness for item 1: We have shown that we can cover Δ with open sets W on each of which there is a solution of the geometric Cauchy problem.

Suppose $f : W \to \mathbb{L}^3$ and $\tilde{f} : \tilde{W} \to \mathbb{L}^3$ are two local solutions, with $W \cap \tilde{W} \cap \Delta \neq \emptyset$. Let $(x_0, -x_0) = (u_0, 0)$ be a point in this set and $U \subset W \cap \tilde{W}$ a contractible neighbourhood of this point. Without loss of generality, we can assume that $\frac{\mathrm{d}f_0}{\mathrm{d}x}(x_0) = \alpha^2 e_0$ and $V(x_0) = \alpha^2 e_1$. For each surface we have a unique extended frame, respectively denoted by $\hat{F}$ and $\tilde{\hat{F}}$, with $\hat{F}(x_0, -x_0) = \tilde{\hat{F}}(x_0, -x_0) = I$. Since these are in the big cell $\mathcal{B}$ in a neighbourhood $U_1 \subset U$ of $(x_0, -x_0)$, we may perform (normalized) left and right Birkhoff decompositions

$$\hat{F} = \hat{F}_- \hat{G}_+ = \hat{F}_+ \hat{G}_-, \quad \tilde{\hat{F}} = \tilde{\hat{F}}_- \tilde{\hat{G}}_+ = \tilde{\hat{F}}_+ \tilde{\hat{G}}_-.$$

It follows from the first part of the argument in the existence proof, that $\hat{F}_- = \tilde{\hat{F}}_-$ and $\hat{F}_+ = \tilde{\hat{F}}_+$ in $U_1 \cap \Delta$, since these are completely determined from the geometric Cauchy data along Δ. Consequently $\hat{F}_-(-y) = \tilde{\hat{F}}_-(-y)$, for $(y, -y) \in U_1$, and $\hat{F}_+(x) = \tilde{\hat{F}}_+(x)$ for $(x, -x) \in U_1$. Thus the normalized potentials for the two surfaces are identical

here, and the surfaces therefore are identical on U_1. Hence we have a well-defined solution on some open set containing Δ. It also follows from the argument just given that the solution is unique.

Item 2: Now suppose that $\tilde{f} : \tilde{M} \to \mathbb{L}^3$ is an arbitrary solution of the geometric Cauchy problem. To show that it agrees with the solution in item 1, it is enough to show that, locally, there exists a choice of null coordinates (x, y), with corresponding isothermal coordinates (u, v), such that the curve f_0 is given by $v = 0$. It then automatically follows that $f_v(u, 0) = V(u)$, because in such coordinates we can write $V = af_u + bf_v$; but then a is zero because f_u is tangent to the curve, to which V is assumed to be orthogonal. It then follows from the condition $\langle V, V \rangle = -\langle df_0/dt, df_0/dt \rangle$ that $b = \pm 1$. Our assumption on the orientation of the solution implies that $b = 1$.

To show that the required coordinates exist, observe that, since the curve f_0 is timelike, it can, in local null coordinates (x, y) for $\tilde{M}$, be expressed as a graph $y = h(x)$, where $h'(x) < 0$. But then the coordinates $(\tilde{x}, \tilde{y}) = (-h(x), y)$ are also null, with the same orientation, and the curve is given by $\tilde{v} = (\tilde{x} + \tilde{y})/2 = 0$.   q.e.d.

3.2.1. The boundary potential pair. The construction given in the proof of Theorem 3.1 is not very practical, as one needs to perform a Birkhoff decomposition to get the potential pair (χ, ψ). Below we show how to obtain an alternative potential pair directly from the geometric Cauchy data. Again we describe only the case when f_0 is timelike; the analogue holds for a spacelike initial curve, using $\hat{F}_+(x) = \hat{F}_0(x)$ and $\hat{F}_-(y) = \hat{F}_0(y)$, in lieu of the definitions below.

Theorem 3.2. *Assume that the geometric Cauchy data are given, and let $H \neq 0$, J^-, $M = J \times J^-$ and Δ be as in Theorem 3.1. Let $\hat{F}_0 : J \to \mathcal{G}$ be the extended frame along Δ constructed as before. Set $\hat{F}_+(x) = \hat{F}_0(x)$ and $\hat{F}_-(y) = \hat{F}_0(-y)$. Then*

1) $(\chi, \psi) = (\hat{F}_+^{-1} d\hat{F}_+, \hat{F}_-^{-1} d\hat{F}_-)$ *is a potential pair on $J \times J^-$, and is regular on an open set containing Δ.*
2) *Set $\hat{\Phi} = \hat{F}_+^{-1} \hat{F}_- : J \times J^- \to \mathcal{G}$, and $M^\circ = \hat{\Phi}^{-1}(\mathcal{B}_L)$. The surface $f : M^\circ \to \mathbb{L}^3$ obtained from the corresponding admissible frame $\hat{F}$ from Theorem 2.8 is a solution of the given geometric Cauchy problem.*

Proof. Item (1) is clear from the construction. To prove (2), we note that $\hat{F}$ is obtained by Birkhoff decomposing $\hat{F}_+^{-1} \hat{F}_- = \hat{H}_- \hat{H}_+$, and then setting $\hat{F} = \hat{F}_- \hat{H}_+^{-1}$. Along $\Delta = \{v = x + y = 0\}$, we have $u = x = -y$, so that $\hat{F}_+(x)^{-1} \hat{F}_-(y) = \hat{F}_0(x)^{-1} \hat{F}_0(-y) = I$, and thus $H_+ = I$ along Δ. Hence we have $\hat{F}(x, -x) = \hat{F}_-(-x) = \hat{F}_0(u)$ along Δ, which shows that f is a solution of the geometric Cauchy problem.   q.e.d.

3.3. The geometric Cauchy problem with null initial curve.

Next we consider the case where the initial curve f_0 is a null curve.

Null geometric Cauchy data: An open interval $J \subset \mathbb{R}$ containing 0, a regular smooth curve $f_0 : J \to \mathbb{L}^3$, which is everywhere null, that is $\left\langle \frac{\mathrm{d}f_0}{\mathrm{d}x}, \frac{\mathrm{d}f_0}{\mathrm{d}x} \right\rangle = 0$, and a regular smooth *null* vector field $V : J \to \mathbb{L}^3$, such that $\left\langle \frac{\mathrm{d}f_0}{\mathrm{d}x}, V \right\rangle > 0$ on J.

Note that the vector field V carries more information than just specifying a family of timelike tangent planes along the curve, as $\frac{\mathrm{d}f_0}{\mathrm{d}x}$ and V determine the conformal factor of the metric along the curve. However, even with this information, we do not have a unique solution in the null case.

Theorem 3.3. *Let $I_y \subset \mathbb{R}$ be an interval containing 0 and $\alpha, \beta : I_y \to \mathbb{R}$ two smooth functions, with $\alpha(0) \neq 0$. Let $H \neq 0$. Given null geometric Cauchy data on an interval I_x, together with the functions α and β, then there is an open set $U \subset I_x \times I_y$, containing the set $I_x \times \{0\}$, and a unique timelike CMC H surface $f : U \to \mathbb{L}^3$ with null coordinates (x, y) such that*

$$(3.2) \qquad f(x, 0) = f_0(x) \qquad and \qquad \frac{\partial f}{\partial y}(x, 0) = V(x).$$

Conversely, given any local solution of the geometric Cauchy problem satisfying (3.2), there is a unique pair of such functions α and β from which the solution is constructed.

Proof. Without loss of generality we may assume that $\frac{\mathrm{d}f_0}{\mathrm{d}x}(0) = \frac{1}{2}(e_0 + e_1)$ and $V(0) = \frac{1}{2}(-e_0 + e_1)$. Define a function $\omega_0 : I_x \to \mathbb{R}$ by

$$\frac{e^{\omega_0}}{2} = \left\langle \frac{\mathrm{d}f_0}{\mathrm{d}x}, V \right\rangle,$$

and $F_0 : I_x \to \mathrm{SL}(2, \mathbb{R})$ by

$$\frac{\mathrm{d}f_0}{\mathrm{d}x} = \frac{1}{2} e^{\omega_0/2} \mathrm{Ad}_{F_0}(e_0 + e_1), \quad V = \frac{1}{2} e^{\omega_0/2} \mathrm{Ad}_{F_0}(-e_0 + e_1), \quad F_0(0) = I.$$

Let us write $F_0^{-1}(F_0)_x = ae_0 + be_1 + ce_2$ and set $\chi = (\lambda(ae_0 + be_1) + ce_2)\mathrm{d}x$. We integrate $\hat{F}_0^{-1}\mathrm{d}\hat{F}_0 = \chi$ with initial condition $\hat{F}_0(0) = I$ to obtain a map $\hat{F}_0 : I_x \to \mathcal{G}$. Clearly, if f is any solution to the geometric Cauchy problem satisfying (3.2), and $\hat{F}$ is the extended coordinate frame with $\hat{F}(0, 0) = I$, and coordinates chosen so that $\varepsilon_1 = \varepsilon_2 = 1$, then $\hat{F}(x, 0) = \hat{F}_0(x)$. Thus, setting

$$\psi = \begin{pmatrix} 0 & \alpha\lambda^{-1} \\ \beta\lambda^{-1} & 0 \end{pmatrix} \mathrm{d}y,$$

then (χ, ψ) is a regular potential pair, and by Theorem 2.8, this pair corresponds to an admissible frame $\hat{F}$. By construction, $\hat{F}(x,0) = \hat{F}_0(x)$, and the corresponding surface solves the geometric Cauchy problem.

Finally, since ψ is a *normalized* potential, it is uniquely determined by an extended frame for the solutions of the geometric Cauchy problem. Thus, for any choice of ψ, there is a unique solution of the geometric Cauchy problem.   q.e.d.

Remark 3.4. In the above proof, we have $\hat{F}(x,0) = \hat{F}_0(x)$ and $\hat{F}(0,y) = \hat{F}_-(y)$, where $\hat{F}_-^{-1}d\hat{F}_- = \psi$, and $\hat{F}_-(0) = I$. Thus it is necessary and sufficient to specify the geometric Cauchy data along *two* (intersecting) null curves in order to obtain a unique solution to the problem in the null case.

Remark 3.5. We can write down the general solution quite explicitly: without loss of generality, we can assume that

$$\frac{\mathrm{d}f_0}{\mathrm{d}x}(x) \;=\; \frac{1}{2}s(x)(e_0 + \cos(\theta(x))e_1 + \sin(\theta(x))e_2),$$

$$V(x) \;=\; \frac{1}{2}t(x)(-e_0 + \cos(\theta(x))e_1 + \sin(\theta(x))e_2),$$

where s, t and θ are positive real-valued functions with $t(0) = s(0) = 1$. Then $e^\omega = st$ and one finds

$$F_0 = \begin{pmatrix} r\cos(\theta/2) & -r^{-1}\sin(\theta/2) \\ r\sin(\theta/2) & r^{-1}\cos(\theta/2) \end{pmatrix},$$

where $r = \sqrt{t/s}$. A simple calculation now gives

$$\chi = \begin{pmatrix} \frac{r_x}{r} & -\frac{\theta_x}{2r^2}\lambda \\ \frac{r^2\theta_x}{2}\lambda & -\frac{r_x}{r} \end{pmatrix} \mathrm{d}x.$$

A similar geometric expression can be given for ψ in terms of data along I_y.

4. Surfaces of revolution

Timelike surfaces of revolution in $\mathbb{L}^3$ come in three types, according to the causal character of the axis which is fixed by the revolution. The differential equations defining the possible profile curves for the three cases are given by R López [20]. These are nonlinear and solutions have only been found by numerical methods.

Below we will consider the problem of finding potential pairs for surfaces of revolution. Knowledge of these potentials can be used to construct new examples of timelike CMC surfaces, as has been done for the case of CMC surfaces in $\mathbb{R}^3$ (see, for example, [13, 17]).

We will work out the potentials for the null axis and timelike axis cases. The case with a spacelike axis can be done in a similar way.

4.1. Rotations in $\mathbb{L}^3$. First we describe the $\mathrm{SL}(2,\mathbb{R})$ matrices corresponding to rotations about the three types of axes. Without loss of generality, we can take the timelike axis given by e_0, the spacelike by e_1, and the null axis by $e_0 + e_1$. It is easy to see that the rotation matrices, up to sign, are given respectively by

$$T(t) = \begin{pmatrix} \cos \frac{1}{2}t & \sin \frac{1}{2}t \\ -\sin \frac{1}{2}t & \cos \frac{1}{2}t \end{pmatrix}, \quad S(t) = \begin{pmatrix} \cosh \frac{1}{2}t & \sinh \frac{1}{2}t \\ \sinh \frac{1}{2}t & \cosh \frac{1}{2}t \end{pmatrix},$$

$$L(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

4.2. Constructing the surfaces via the geometric Cauchy problem. To find the potential pairs corresponding to all surfaces of revolution with a given type of axis, it is enough to solve the geometric Cauchy problem along a "circle" generated by the corresponding rotation, with all possible choices of prescribed tangent plane. Since the surface is to be invariant under the relevant rotation, the vector field V for the geometric Cauchy problem is determined at a single point on the circle.

4.2.1. Timelike axis. Let us consider the timelike axis in the direction of e_0. Given a circle of radius ρ centered on this axis, we may assume that this is given by

$$f_0(t) = \rho(\sin t\, e_1 + \cos t\, e_2) = \rho \operatorname{Ad}_{T(t)}(e_2).$$

Since $f_0'(0) = \rho e_1$, we may assume that the vector field $V(t)$ for the geometric Cauchy problem satisfies $V(0) = \rho e_0$, so that $V(t) = \rho \operatorname{Ad}_{T(t)}(e_0) = \rho e_0$. Hence

$$e^{\omega_0} = \left\langle \frac{\mathrm{d}f_0}{\mathrm{d}t}, \frac{\mathrm{d}f_0}{\mathrm{d}t} \right\rangle = \rho^2.$$

From the equations

$$\frac{\mathrm{d}f_0}{\mathrm{d}t} = e^{\omega_0/2} \operatorname{Ad}_{F_0}(e_1), \quad V(t) = e^{\omega_0/2} \operatorname{Ad}_{F_0}(e_0), \quad F_0(0) = I,$$

we see that $F_0(t) = T(t)$. For a coordinate frame $F(u,v)$, in isothermal coordinates, for the solution of the geometric Cauchy problem, we should have $F(0,v) = F_0(v)$, so that

$$\frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = U + V = \frac{1}{4} \begin{pmatrix} -\omega_x + \omega_y & -4Qe^{-\omega/2} - 2He^{\omega/2} \\ 2He^{\omega/2} + 4Re^{-\omega/2} & \omega_x - \omega_y \end{pmatrix},$$

where $\omega(0, v) = \omega_0(v)$. Solving this, with $e^{\omega/2} = \rho$, we obtain $Q = R = -\frac{1}{2}\rho(1 + \rho H)$. Therefore, we set

$$
\hat{A} = \begin{pmatrix} 0 & -Q\rho^{-1}\lambda - \frac{1}{2}H\rho\lambda^{-1} \\ \frac{1}{2}H\rho\lambda + R\rho^{-1}\lambda^{-1} & 0 \end{pmatrix}
$$

$$
= \frac{1}{2}\begin{pmatrix} 0 & (1 + \rho H)\lambda - H\rho\lambda^{-1} \\ H\rho\lambda - (1 + \rho H)\lambda^{-1} & 0 \end{pmatrix},
$$

and define $\hat{F}_0$ by integrating $\hat{F}_0^{-1}\mathrm{d}\hat{F}_0 = \hat{A}\mathrm{d}v$ with $\hat{F}_0(0) = I$. As discussed in Section 2.4, the boundary potentials are given by

$$
\chi = \hat{A}\mathrm{d}x, \qquad \psi = \hat{A}\mathrm{d}y.
$$

Since $\hat{A}$ is constant in x and y, these integrate to $\hat{F}_+(x) = \exp(\hat{A}x)$ and $\hat{F}_-(y) = \exp(\hat{A}y)$, respectively. Setting $\hat{\Phi} = \hat{F}_+(x)^{-1}\hat{F}_-(y)$ and performing a normalized Birkhoff decomposition of $\hat{\Phi}$ into $\hat{H}_-\hat{H}_+$, the extended frame for the solution is given by $\hat{F} = \hat{F}_+\hat{H}_-$.

Figure 2. Partial plots of the typical timelike CMC surfaces of revolution with timelike axis, computed from the geometric Cauchy data on a circle of radius ρ. Left: $\rho H = -1$. Center $\rho H = -1/2$. Right: $\rho H = 1$.

Example 4.1. In general, the Birkhoff decomposition cannot be written down explicitly. The typical solutions have been computed numerically here (Figure 2). However, the special case when $\rho H = -1/2$, which is a transition point of the topological type of the solution, can be worked out explicitly. We have $\hat{A} = -\frac{1}{4}(\lambda + \lambda^{-1})e_0$ and in this case,

$$
\hat{\Phi}(x, y) = \exp\left(\frac{1}{4}(\lambda + \lambda^{-1})(x - y)e_0\right)
$$

$$
= \exp\left(\frac{\lambda^{-1}}{4}(x - y)e_0\right)\exp\left(\frac{\lambda}{4}(x - y)e_0\right) = \hat{H}_-\hat{H}_+.
$$

Hence, we have the extended frame

$$
\hat{F}(x, y) = \hat{F}_-(y)\hat{H}_+^{-1}(x, y) = \begin{pmatrix} \cos\left(\frac{\lambda x + \lambda^{-1}y}{4}\right) & \sin\left(\frac{\lambda x + \lambda^{-1}y}{4}\right) \\ -\sin\left(\frac{\lambda x + \lambda^{-1}y}{4}\right) & \cos\left(\frac{\lambda x + \lambda^{-1}y}{4}\right) \end{pmatrix}.
$$

Then we can compute $\mathcal{S}(\hat{F}) = \frac{1}{2}((y-x)e_0 - \sin(x+y)e_1 - \cos(x+y)e_2)$, and since we are in the case $H = -\frac{1}{2\rho}$, the surface is given by

$$\begin{aligned} f(x,y) &= \frac{1}{2H}\mathcal{S}_1 F(x,y) = \rho\frac{1}{2}((x-y)e_0 + \sin(x+y)e_1 - \cos(x_y)e_2) \\ &= \rho[u, \sin(v), \cos(v)], \end{aligned}$$

a right circular cylinder in $\mathbb{R}^3$ of radius $\rho = -\dfrac{1}{2H}$.

4.2.2. Null axis. Consider now the null axis $e_0 + e_1$, and a curve $f_0(t)$ which we can assume is a null curve of the form

$$f_0(t) = \mathrm{Ad}_{L(t)}(ae_0 + be_1 - ce_2), \quad a, \ b, \ c \text{ constant}, \ c > 0.$$

Note that we cannot apply an isometry of $\mathbb{L}^3$ to simplify f_0 and V as we did in the proof of Theorem 3.3, because this would move the aixs of the rotation.

Since $f_0'(t) = \mathrm{Ad}_{L(t)}(c(e_0 + e_1) + (-a+b)e_2)$, we see that f_0 is null if and only if $a = b$, so that $f_0'(t) = \mathrm{Ad}_{L(t)}(c(e_0+e_1)) = c(e_0+e_1)$. To find a timelike CMC surface of revolution around the null axis containing this curve, we specify a null vector field $V(t)$ along this curve, invariant under $L(t)$. We can assume that it is given by

$$V(t) = \mathrm{Ad}_{L(t)}(Ae_0 + Be_1 + Ce_2), \quad A, \ B, \ C \text{ constant}, \ B - A \geq 0.$$

where we require $-A^2 + B^2 + C^2 = 0$. As before, set

$$\frac{e^{\omega_0}}{2} = \left\langle \frac{\mathrm{d}f_0}{\mathrm{d}t}, V \right\rangle = c(B-A).$$

Hence $\dfrac{e^{\omega_0/2}}{2} = \sqrt{\dfrac{c(B-A)}{2}}$. Next we find a map F_0 along the curve satisfying

$$f_0'(t) = \frac{e^{\omega_0/2}}{2}\mathrm{Ad}_{F_0(t)}(e_0+e_1), \quad V(t) = \frac{e^{\omega_0/2}}{2}\mathrm{Ad}_{F_0(t)}(-e_0+e_1).$$

Writing $F_0(t) = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, we see that

$$c(e_0+e_1) = \frac{e^{\omega/2}}{2}\mathrm{Ad}_{F_0(t)}(e_0+e_1) = 2\sqrt{\frac{c(B-A)}{2}}\begin{pmatrix} yw & -y^2 \\ w^2 & -yw \end{pmatrix}.$$

Thus, $y = 0$ and $x^{-1} = w = \pm(2c/(B-A))^{1/4}$. We may choose the plus sign here, and, after scaling the coordinate of the curve, we may also assume that $2c = B - A$, so that $x = w = 1$. Finally, from the two expressions for $V(t)$, we obtain $z = t + C/(2c)$. A translation of the coordinate t, allows us to take $C = 0$, so that

$$F_0(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

which gives,

$$F_0^{-1}(F_0)_t = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

As in Theorem 3.3, we multiply by λ and exponentiate to obtain $\hat{F}_+(x)$, and all solutions are obtained by choosing two arbitrary functions $\alpha(y)$ and $\beta(y)$, with $\alpha(0) \neq 0$, and the potential pair:

$$\chi = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \mathrm{d}x, \quad \psi = \lambda^{-1} \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \mathrm{d}y.$$

Example 4.2. Taking $\alpha = 1$ and $\beta = 0$, gives $\hat{F}_-(y) = \begin{pmatrix} 1 & \lambda^{-1}y \\ 0 & 1 \end{pmatrix}$.

In the (left) normalized Birkhoff decomposition $\hat{F}_+^{-1}\hat{F}_- = \hat{H}_-\hat{H}_+$, it follows easily that

$$\hat{H}_- = \begin{pmatrix} 1 & \frac{y\lambda^{-1}}{1-xy} \\ 0 & 1 \end{pmatrix}.$$

Hence

$$\hat{F} = \hat{F}_+\hat{H}_- = \begin{pmatrix} 1 & \frac{y\lambda^{-1}}{1-xy} \\ \lambda x & \frac{1}{1-xy} \end{pmatrix}.$$

From the Sym formula, (choosing $H = 1/2$) we get the surface

$$f(x,y) = \frac{1}{1-xy}(2(x+y)e_0 + 2(x-y)e_1 - (3xy+1)e_2).$$

5. Pseudospherical surfaces in Euclidean 3-space

5.1. The loop group formulation. Let $D \subset \mathbb{R}^2$ be a simply connected domain. An immersion $f : D \to \mathbb{E}^3$ is said to be a *pseudospherical surface* if it has constant sectional curvature -1. Following Bobenko [5], let x and y be asymptotic coordinates for f, not necessarily arc length coordinates. Let ϕ denote the oriented angle between f_x and f_y. The first and second fundamental forms of f are given by

$$I = |f_x|\mathrm{d}x^2 + 2\cos(\phi)\mathrm{d}x\mathrm{d}y + |f_y|\mathrm{d}y^2, \quad II = 2|f_x||f_y|\sin(\phi)\mathrm{d}x\mathrm{d}y,$$

and the equations of Gauss and Codazzi-Mainardi reduce to

$$\phi_{xy} - |f_x||f_y|\sin(\phi) = 0, \quad \partial_y|f_x| = \partial_x|f_y| = 0.$$

Set $\theta = \phi/2$ and

$$(5.1) \quad E_1 = \frac{1}{2\cos(\theta)}\left(\frac{f_x}{|f_x|} + \frac{f_y}{|f_y|}\right), \quad E_2 = -\frac{1}{2\sin(\theta)}\left(\frac{f_x}{|f_x|} - \frac{f_y}{|f_y|}\right).$$

It is easy to see that E_1 and E_2 are unit principal vector fields for the surface.

To obtain a loop group formulation, we identify $\mathbb{E}^3$ with the Lie algebra $\mathfrak{su}(2)$ with the orthonormal basis

$$e_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_2 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

A *Darboux frame* for f is a map $F : D \to \mathrm{SU}(2)$ adapted to the principal directions in the sense that

$$(5.2) \qquad E_1 = \mathrm{Ad}_F\, e_1, \quad E_2 = \mathrm{Ad}_F\, e_2, \quad N = \mathrm{Ad}_F\, e_3.$$

We assume that coordinates are chosen so that $F(0,0) = I$. Let $U = F^{-1}F_x$ and $V = F^{-1}F_y$. A simple calculation shows that

$$(5.3) \quad U = \frac{i}{2} \begin{pmatrix} -\theta_x & |f_x|e^{-i\theta} \\ |f_x|e^{i\theta} & \theta_x \end{pmatrix}, \quad V = \frac{i}{2} \begin{pmatrix} \theta_y & -|f_y|e^{i\theta} \\ -|f_y|e^{-i\theta} & -\theta_y \end{pmatrix}.$$

Let us introduce the matrices
$$(5.4)$$
$$\hat{U} = \frac{i}{2} \begin{pmatrix} -\theta_x & |f_x|e^{-i\theta}\lambda \\ |f_x|e^{i\theta}\lambda & \theta_x \end{pmatrix}, \quad \hat{V} = \frac{i}{2} \begin{pmatrix} \theta_y & -|f_y|e^{i\theta}\lambda^{-1} \\ -|f_y|e^{-i\theta}\lambda^{-1} & -\theta_y \end{pmatrix},$$

where $\lambda \in \mathbb{C} \setminus \{0\}$. The equations of Gauss and Codazzi-Mainardi imply that $-\hat{U}_x + \hat{V}_y + [\hat{U}, \hat{V}] = 0$, and integrating $\hat{F}^{-1}\mathrm{d}\hat{F} = \hat{U}\mathrm{d}x + \hat{V}\mathrm{d}y$ with $\hat{F}(0,0) = I$, we obtain an *extended Darboux frame*, that is, a map $\hat{F} : D \to \Lambda\mathrm{SL}(2,\mathbb{C})_{\sigma\rho}$ with $\hat{F}\big|_{\lambda=1} = F$. Here σ is the twisting involution defined previously, and ρ is defined by

$$(\rho\gamma)(\lambda) = \left(\overline{\gamma(\bar{\lambda})}^T\right)^{-1}.$$

From $\hat{F}$ we obtain from the *Sym formula* a family of pseudospherical surfaces

$$(5.5) \qquad f^\lambda = \lambda\frac{\partial\hat{F}}{\partial\lambda}\hat{F}^{-1} \qquad (\lambda \in \mathbb{R} \setminus \{0\}).$$

It follows easily that f^1 coincides with f up to a rigid motion of $\mathbb{E}^3$.

Definition 5.1. *Let M be a simply connected subset of $\mathbb{R}^2$, and let (x,y) denote the standard coordinates. An* admissible frame *is a smooth map $\hat{F} : M \to \Lambda\mathrm{SL}(2,\mathbb{C})_{\sigma\rho}$ such that the Maurer-Cartan form of $\hat{F}$ has the form*

$$(5.6) \qquad \hat{F}^{-1}\mathrm{d}\hat{F} = \lambda\,A_1\,\mathrm{d}x + \alpha_0 + \lambda^{-1}A_{-1}\,\mathrm{d}y,$$

where the coefficients A_1 and A_{-1} and the $\mathfrak{su}(2)$-valued 1-form α_0 are constant in λ. The admissible frame $\hat{F}$ is said to be weakly regular *if $[A_1]_{12} \neq 0$ and $[A_{-1}]_{12} \neq 0$. The frame is said to be* regular *if it is weakly regular and $\mathrm{Arg}([A_1]_{12}) - \mathrm{Arg}([A_{-1}]_{12}) \neq k\pi$ for any integer k.*

Note that the reality condition given by ρ means that the matrices A_i are in $\mathfrak{su}(2)$. Hence, for a weakly regular frame, all the off-diagonal components of $A_{\pm 1}$ are non-zero. Given a weakly regular admissible frame $\hat{F}$, it is straightforward to verify (as in the proof of Proposition 2.5) that by multiplying $\hat{F}$ on the right with a matrix-valued function of the form $T = \mathrm{diag}(e^{i\mu}, e^{-i\mu})$, we can bring the Maurer-Cartan form $\hat{F}^{-1}\mathrm{d}\hat{F}$ into the form of (5.4), with $2\theta \in [0, \pi)$. If the frame is *regular*, then one also has that $2\theta \in (0, \pi)$. In this case the analogue of Proposition 2.5 holds, in the sense that regular admissible frames correspond precisely to pseudospherical surfaces.

Definition 5.2. *Let I_x and I_y be two real intervals, with coordinates x and y, respectively. A potential pair (χ, ψ) is a pair of smooth $\Lambda\mathfrak{sl}(2, \mathbb{C})_{\sigma\rho}$-valued 1-forms on I_x and I_y respectively with Fourier expansions in λ as follows:*

$$\chi = \sum_{j=-\infty}^{1} \chi_i \lambda^i \mathrm{d}x, \qquad \psi = \sum_{j=-1}^{\infty} \psi_i \lambda^i \mathrm{d}y.$$

The potential pair is called weakly regular *if $[\chi_1]_{12} \neq 0$ and $[\psi_{-1}]_{12} \neq 0$.*

Note that, again, if the pair is weakly regular then one also has $[\chi_1]_{21} \neq 0 \neq [\psi_{-1}]_{21}$. It is straightforward to verify that the analogue of Theorem 2.8 holds in this situation, with the only essential difference being that, as is shown in [**7**], the big cell is the whole group for $\Lambda\mathrm{SL}(2, \mathbb{C})_{\sigma\rho}$ so that $\Phi^{-1}(\mathcal{B}_L)$ is the whole of M. However, a weakly regular potential pair only produces a weakly regular admissible frame; there is no guarantee that the corresponding pseudospherical surface is everywhere regular.

5.2. Solution of the geometric Cauchy problem. As with timelike CMC surfaces, we will find that the problem splits into two quite different situations, that of non-characteristic curves (non-asymptotic curves) and characteristic curves (asymptotic curves). The results are broadly similar to the timelike CMC case, but with important differences.

5.2.1. The non-characteristic case. Recall that a curve f_0 on a surface with unit normal N is asymptotic if and only $\langle f_0', N' \rangle = 0$. For the first case that we consider we assume that the following is given:
Non-characteristic Geometric Cauchy data: An open interval $J \subset \mathbb{R}$ containing 0, a regular smooth map $f_0 : J \to \mathbb{E}^3$, and a regular smooth vector field $N_0 : J \to \mathbb{E}^3$, which is everywhere orthogonal to f_0', and such that
$$\langle f_0', N_0' \rangle \neq 0.$$

The Geometric Cauchy problem: Find a pseudospherical surface

$f : D \to \mathbb{E}^3$, where D is an open subset of $\mathbb{R}^2$ containing J, such that $f|_J = f_0$ and the normal to the surface along J is given by N_0.

The non-characteristic problem splits further into two subcases; however these are *not* analogous to the split into timelike/spacelike cases encountered earlier. The two cases are when f_0' and N_0' are everywhere parallel, and when f_0' and N_0' are nowhere parallel, in other words when f_0 is everywhere or is never tangent to a principal direction.

Given a pseudospherical surface $f : D \subset \mathbb{R}^2 \to \mathbb{E}^3$, with coordinates and Darboux frame as described above, let us define new coordinates by

$$u = \frac{1}{2}(x - y), \quad v = \frac{1}{2}(x + y).$$

Then

$$(5.7) \quad F^{-1}F_u = U - V = \frac{i}{2}\begin{pmatrix} -\theta_v & |f_x|e^{-i\theta} + |f_y|e^{i\theta} \\ |f_x|e^{i\theta} + |f_y|e^{-i\theta} & \theta_v \end{pmatrix}.$$

where, by (5.3), we have

$$(5.8) \qquad\qquad \theta_v = \langle (E_2)_u, E_1 \rangle.$$

Let $\alpha = |f_x| - |f_y|$ and $\beta = |f_x| + |f_y|$. From (5.1) we have

$$f_u = f_x - f_y = \alpha \cos(\theta)E_1 - \beta \sin(\theta)E_2$$

and from (5.3) we have

$$N_u = \mathrm{Ad}_F([U - V, e_3]) = -\alpha \sin(\theta)E_1 - \beta \cos(\theta)E_2.$$

Hence we have

$$\langle f_u, N_u \rangle = \frac{(\beta^2 - \alpha^2)}{2}\sin(\phi),$$

$$|N_u|^2 - |f_u|^2 = (\beta^2 - \alpha^2)\cos(\phi),$$

$$|N_u|^2 + |f_u|^2 = \beta^2 + \alpha^2.$$

It follows that

$$|f_u|^2|N_u|^2 - \langle f_u, N_u \rangle^2 = 2\alpha^2\beta^2.$$

As $\beta > 0$, we see that $\alpha = 0$ if and only if f_u and N_u are parallel, that is, if and only if the curve $v = 0$ is a principal curve on the surface. If this is the case, then

$$(5.9) \qquad\qquad \beta = 2|f_x| = 2|f_y| = \sqrt{|f_u|^2 + |N_u|^2},$$

$$(5.10) \quad \sin(\phi) = \frac{2\langle f_u, N_u \rangle}{|f_u|^2 + |N_u|^2}, \quad \cos(\phi) = \frac{|N_u|^2 - |f_u|^2}{|f_u|^2 + |N_u|^2}.$$

From the expressions $E_2 = -(\beta \sin(\theta))^{-1} f_u$ and $E_1 = E_2 \times N$ we obtain

$$(5.11) \qquad\qquad \theta_v = \langle (E_2)_u, E_1 \rangle = \frac{1}{|f_u|^2}\langle f_{uu}, f_u \times N \rangle.$$

Thus, we see that the matrices U and V in (5.3) may be obtained along the curve $v = 0$ entirely from knowledge of $f(u, 0)$ and $N(u, 0)$.

On the other hand, if f_u and N_u are never parallel along $v = 0$, equivalently, if α is non-zero everywhere along this curve, then we may assume that $\alpha > 0$. Then we have

$$E_1 = \frac{1}{\alpha}(\cos(\theta)f_u - \sin(\theta)N_u), \quad E_2 = -\frac{1}{\beta}(\sin(\theta)f_u + \cos(\theta)N_u),$$

and subsequently obtain

$$\tan(\phi) = \frac{2\langle f_u, N_u \rangle}{|N_u|^2 - |f_u|^2},$$

$$\alpha^2 = \frac{1}{2}\left(|f_u|^2(1 + \cos^{-1}(\phi)) + |N_u|^2(1 - \cos^{-1}(\phi))\right),$$

$$\beta^2 = \frac{1}{2}\left(|f_u|^2(1 - \cos^{-1}(\phi)) + |N_u|^2(1 + \cos^{-1}(\phi))\right).$$

Let us also assume that $\langle f_u, N_u \rangle \neq 0$; by changing the sign of N if necessary, we may assume that $\langle f_u, N_u \rangle > 0$. With this assumption ϕ is a well defined function with values in $(0, \pi)$, satisfying

(5.12) $$\cos(\phi) = \frac{Z}{\sqrt{1 + Z^2}}, \quad \sin(\phi) = \frac{1}{\sqrt{1 + Z^2}}, \quad Z = \frac{|N_u|^2 - |f_u|^2}{2\langle f_u, N_u \rangle}.$$

Noting that $\cos(\phi)$ has the same sign as $(|N_u|^2 - |f_u|^2)$ and recalling that $\alpha > 0$ and $\beta > 0$, we obtain

(5.13)
$$\alpha = \frac{1}{\sqrt{2}}\left(|f_u|^2 + |N_u|^2 - \sqrt{4\langle f_u, N_u \rangle^2 + \left(|N_u|^2 - |f_|^2\right)^2}\right)^{1/2},$$

$$\beta = \frac{1}{\sqrt{2}}\left(|f_u|^2 + |N_u|^2 + \sqrt{4\langle f_u, N_u \rangle^2 + (|N_u|^2 - |f_u|^2)^2}\right)^{1/2}.$$

As before, we have $\theta_v = \langle (E_2)_u, E_1 \rangle$. From the expressions above for E_1 and E_2 and from $\theta_u = -Z_u/2(1 + Z^2)$, we obtain

(5.14) $$\theta_v = \frac{\alpha Z_u}{2\beta(Z^2 + 1)} - \frac{Y}{\alpha\beta},$$

where

$$Y = \sin(\theta)\cos(\theta)\left(\langle f_{uu}, f_u \rangle - \langle N_u, N_{uu} \rangle\right)$$

$$+ \cos^2(\theta)\langle f_u, N_{uu} \rangle - \sin^2(\theta)\langle N_u, f_{uu} \rangle.$$

It follows that, as in the case $\alpha = 0$, we can construct the matrices U and V in (5.3) from knowledge of $f(u, 0)$ and $N(u, 0)$. The following result now follows easily.

Theorem 5.3. *Let the non-characteristic geometric Cauchy data f_0 and N_0 be given, where f_0' and N_0' are either everywhere parallel, or nowhere parallel. In the first case set:*

$$\alpha(t) = 0, \quad \beta(t) = \sqrt{|f_0'|^2 + |N_0'|^2}, \quad \theta_v(t) = \frac{1}{|f_0'|^2}\langle f_0'', f_0' \times N_0\rangle,$$

$$\cos(\theta(t)) = \frac{|N_0'|}{\sqrt{|f_0'|^2 + |N_0'|^2}}, \quad \sin(\theta(t)) = \frac{|f_0'|}{\sqrt{|f_0'|^2 + |N_0'|^2}}.$$

In the second case, substitute $f_0(t)$, $N_0(t)$ and t for f, N and u in the expressions at (5.12), (5.13) and (5.14), to find the expressions for $\alpha(t)$, $\beta(t)$, $\theta(t) := \phi(t)/2$ and $\theta_v(t)$. In either case define $\hat{A}(t)$ to be the loop algebra valued function:

$$\frac{i}{2}\begin{pmatrix} -\theta_v & \frac{1}{2}(\beta + \alpha)e^{-i\theta}\lambda + \frac{1}{2}(\beta - \alpha)e^{i\theta}\lambda^{-1} \\ \frac{1}{2}(\beta + \alpha)e^{i\theta}\lambda + \frac{1}{2}(\beta - \alpha)e^{-i\theta}\lambda^{-1} & \theta_v \end{pmatrix}.$$

Then

1) *The pair of 1-forms $(\chi, \psi) := (\hat{A}(x)\mathrm{d}x, -\hat{A}(-y)\mathrm{d}y)$ is a weakly regular potential pair on $J \times J^-$, where $J^- = \{y \in \mathbb{R} | -y \in J\}$.*
2) *The pseudospherical surface obtained from (χ, ψ) is regular on an open set containing $\Delta := \{(x, -x)\} \subset J \times J^-$. It is the unique solution of the given geometric Cauchy problem.*

The uniqueness in item (2) should be understood in the sense described in Section 1.1.

The proof of this theorem is essentially identical to the proof of the corresponding result for timelike CMC surfaces, see Theorems 3.1 and 3.2. The only difference here is that the weakly regular solution is defined on the whole of $J \times J^-$, as the big cell is the whole group. By construction, $\phi \in (0, \pi)$ along Δ, which implies regularity on an open set containing Δ.

Remark 5.4. The expression (5.14) for θ_v implies that the solution for the case $\alpha \neq 0$, with the choice of coordinates we seek, will not exist at points where α does vanish, unless we also have the condition that $\frac{Y}{\alpha\beta}$ is bounded. Since $\alpha\beta = 2(|f_u|^2|N_u|^2 - \langle f_u, N_u\rangle^2)$, whilst Y contains higher order derivatives, one would have to impose a condition on the second derivatives of f and N in order to extend the theorem to the situation where α is non-zero at some points but zero at others. Conditions on the second derivatives of f and N also arise in the expression for α_u, which needs to be smooth when α vanishes.

5.2.2. The characteristic case. Now we consider the case of an asymptotic curve, that is $\langle f_0', N_0'\rangle = 0$ along J.

Characteristic Geometric Cauchy data: An open interval $J \subset \mathbb{R}$ containing 0, a regular smooth map $f_0 : J \to \mathbb{E}^3$, and a regular smooth

vector field $N_0 : J \to \mathbb{E}^3$, which is everywhere orthogonal to f_0', and such that f_0' and N_0' are also orthogonal.

$$\langle f_0', N_0' \rangle = 0.$$

Since the curve is necessarily an asymptotic curve of any solution to the geometric Cauchy problem, we are looking for a solution such that $f(x, 0) = f_0(x)$.

Consider a pseudospherical surface f parameterized by asymptotic coordinates (x, y). By multiplying a Darboux frame for f on the right by the matrix function $\mathrm{diag}(e^{-i\theta/2}, e^{i\theta/2})$, we obtain a new frame F satisfying

$$\frac{f_x}{|f_x|} = \mathrm{Ad}_F(e_1), \quad \frac{1}{\sin(2\theta)}\left(-\cos(2\theta)\frac{f_x}{|f_x|} + \frac{f_y}{|f_y|}\right) = \mathrm{Ad}_F(e_2).$$

A simple calculation shows that

$$F^{-1}\mathrm{d}F = \frac{i}{2}\begin{pmatrix} -\phi_x & |f_x| \\ |f_x| & \phi_x \end{pmatrix}\mathrm{d}x + \frac{i}{2}\begin{pmatrix} 0 & -|f_y|e^{i\phi} \\ -|f_y|e^{-i\phi} & 0 \end{pmatrix}\mathrm{d}y.$$

Hence, if

$$\hat{U} = \frac{i}{2}\begin{pmatrix} -\phi_x & |f_x|\lambda \\ |f_x|\lambda & \phi_x \end{pmatrix}, \quad \hat{V} = \frac{i}{2}\begin{pmatrix} 0 & -|f_y|e^{i\phi}\lambda^{-1} \\ -|f_y|e^{-i\phi}\lambda^{-1} & 0 \end{pmatrix},$$

we may integrate $\hat{F}^{-1}\mathrm{d}\hat{F} = \hat{U}\mathrm{d}x + \hat{V}\mathrm{d}y$, with $\hat{F}(0, 0) = I$. Clearly, $\hat{F}$ will be an extended Darboux frame for f up to multiplication from the right by a diagonal matrix (which has no effect on the Sym formula).

To return to the geometric Cauchy problem, assume that characteristic geometric Cauchy data f_0 and N_0 are given. Solving $f_0'/|f_0'| = \mathrm{Ad}_{F_0}(e_1)$, $N_0 = \mathrm{Ad}_{F_0}(e_3)$ with $F_0(0) = I$ (after a suitable isometry of $\mathbb{E}^3$), we see from the condition that $\langle f_0', N_0' \rangle = 0$ that

$$F_0^{-1}\mathrm{d}F_0 = \begin{pmatrix} ia & ib \\ ib & -ia \end{pmatrix}\mathrm{d}x,$$

where a and b are two real-valued functions on J. Thus, we define the potential

$$\chi = \begin{pmatrix} ia & ib\lambda \\ ib\lambda & -ia \end{pmatrix}\mathrm{d}x.$$

To find a solution, we may now freely choose a potential of the form

$$\psi = \begin{pmatrix} 0 & \alpha\lambda^{-1} \\ -\bar{\alpha}\lambda^{-1} & 0 \end{pmatrix}\mathrm{d}y,$$

where $\alpha : I_y \to \mathbb{C}$ is a non-vanishing function. It is now not difficult to see that the analogue of Theorem 3.3 holds in this case also, where in this case the additional data required is a non-vanishing function

$\alpha : I_y \to \mathbb{C}$. The potential pair (χ, ψ) given above defines a solution to the problem, where χ is constructed from the Cauchy data, and ψ is determined by α, and conversely every solution corresponds to such a function α.

5.3. Applications. As a consequence of Theorem 5.3 we have:

Corollary 5.5. *Let $\gamma : J \to \mathbb{E}^3$ be a regular, constant-speed curve with curvature κ, and torsion τ. Suppose that κ is nonvanishing and that either γ is a plane curve or τ is nonvanishing. Then there exists a unique regular pseudospherical surface which contains γ as a geodesic and this surface can be constructed by Theorem 5.3.*

Proof. This follows by applying Theorem 5.3, choosing f_0 to be γ and N_0 to be the principal unit normal to the curve.   q.e.d.

Note that a geodesic curve γ is a principal curve if and only if it is a plane curve.
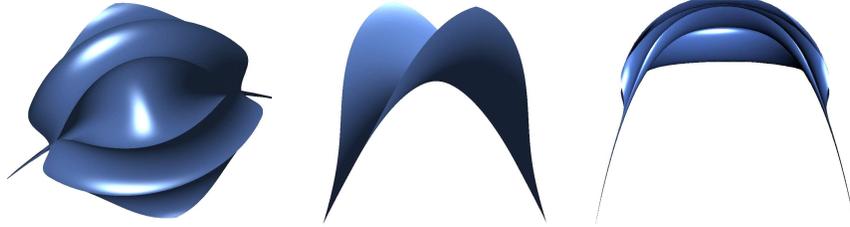
Figure 3. Left, center: two plots of a part of the unique constant Gauss curvature -1 surface which contains the parabola $y = x^2$ as a geodesic principal curve. Right: the pseudospherical surface that contains the catenary $y = \cosh(x)$ as a geodesic principal curve.

Example 5.6. We show how to construct the unique pseudospherical surface which contains the parabola $y = x^2$ as a geodesic. Take the curve $f_0(t) = (t, t^2, 0)$ and $N_0(t) = \frac{1}{\sqrt{1+4t^2}}(2t, -1, 0)$; since N_0 is the principal normal to f_0, f_0 will necessarily be a geodesic on the resulting pseudospherical surface. Note that $N_0'(t) = (1 + 4t^2)^{-3/2}(2, 4t, 0)$ is parallel to f_0', and f_0 will also be a principal curve on the surface. The solution will be well defined along the entire parabola because $\langle f_0'(t), N_0'(t) \rangle = (2 + 8t^2)(1 + 4t^2)^{-3/2} > 0$ for all t. From the formulae in Theorem 5.3 we have

$$|f_x| = |f_y| = \frac{\sqrt{(1 + 4t^2)^3 + 4}}{2(1 + 4t^2)}, \quad \theta_v = 0$$

$$e^{i\theta} = \frac{2}{\sqrt{4 + (1 + 4t^2)^3}} + i\sqrt{\frac{(1 + 4t^2)^3}{4 + (1 + 4t^2)^3}}.$$

The surface, together with the surface similarly generated by the cate-nary $f_0(t) = (t, \cosh(t), 0)$, can be seen in Figure 3.
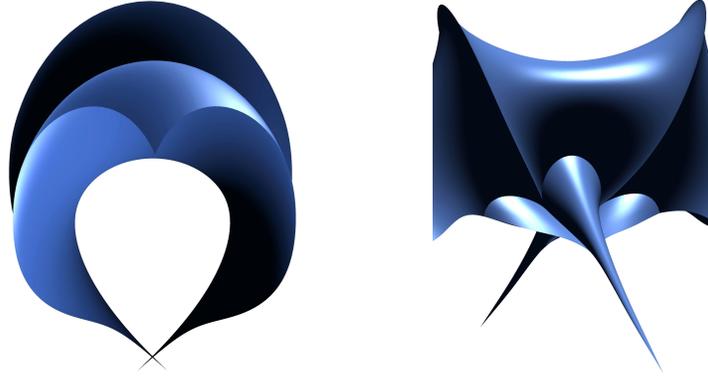

Figure 4. Two views of the unique pseudospherical sur-face that contains the cubic $y^2 = x^2(x+1)$ as a geodesic principal curve. The cusp lines converge at infinity.
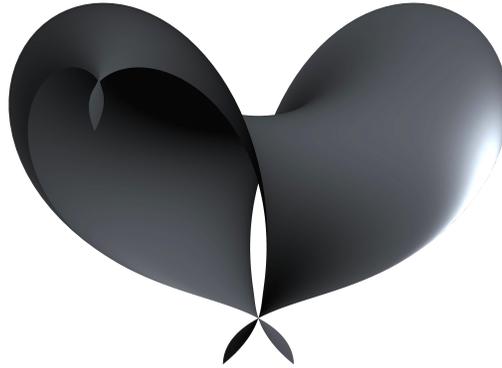
Figure 5. The surface generated by Bernoulli's Lem-niscate $(x^2 + y^2)^2 = x^2 - y^2$. The cusp lines meet at the center of the figure eight, where the curvature of the plane curve is zero (compare Figure 4).

In the same way, taking the parameterizations

$$f_0(t) = (t^2 - 1, t(t^2 - 1), 0),$$
$$f_0(t) = (\cos(t)/(1 + \sin(t)^2), \sin(2t)/(2(1 + \sin(t)^2)), 0),$$
$$f_0(t) = (\sin(t), 2\cos(t), 0),$$

of the cubic, Bernoulli's lemniscate, and an ellipse, we obtain the sur-faces shown in Figures 4, 5 and 1.

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