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COMPLETE CLASSIFICATION OF COMPACT FOUR-MANIFOLDS WITH POSITIVE ISOTROPIC CURVATURE

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Abstract

In this paper, we completely classify all compact 4-manifolds with positive isotropic curvature. We show that they are diffeomorphic to $\mathbb{S}^4$ or $\mathbb{RP}^4$ or quotients of $\mathbb{S}^3 \times \mathbb{R}$ by a cocompact fixed point free subgroup of the isometry group of the standard metric of $\mathbb{S}^3 \times \mathbb{R}$, or a connected sum of them.

1. Introduction

Let M be an n-dimensional Riemannian manifold. Recall that its curvature operator at $p \in M$ is the self adjoint linear endomorphism $\mathcal{R} : \wedge^2 T_pM \to \wedge^2 T_pM$ defined by

$$< \mathcal{R}(X \wedge Y), U \wedge V > = < Rm(X,Y)V, U >, \quad \text{for} \quad X, Y, U, V \in T_pM.$$

Here $< , >$ is the Riemannian metric and Rm is the Riemann curvature tensor on M. The Riemannian metric $< , >$ can be extended either to a complex bilinear form $(,)$ or a Hermitian inner product $<< , >>$ on $T_pM \otimes \mathbb{C}$. We extend the curvature operator to a complex linear map on $\wedge^2 T_pM \otimes \mathbb{C}$, also denoted by $\mathcal{R}$. Then, to every two-plane $\sigma \subset T_pM \otimes \mathbb{C}$, we can define the complex sectional curvature $K_{\mathbb{C}}(\sigma)$ by

$$K_{\mathbb{C}}(\sigma) = << \mathcal{R}(Z \wedge W), Z \wedge W >>$$

where $\{Z, W\}$ is a unitary basis of σ with respect to $<< , >>$. We say that M has positive isotropic curvature (PIC for short) if $K_{\mathbb{C}}(\sigma) > 0$ whenever $\sigma \subset T_pM \otimes \mathbb{C}$ is a totally isotropic two-plane for any $p \in M$. Here σ is totally isotropic if $(Z, Z) = 0$ for any $Z \in \sigma$. To clarify the meaning of positive isotropic curvature, we have the following diagram for the relative strength of the positivity for various notions of curvatures:

$$\begin{array}{ccccc} & \mathcal{R} > 0 \Rightarrow & K_{\mathbb{C}} > 0 \Rightarrow & K > 0 \Rightarrow & Ric > 0 \Rightarrow \quad R > 0 \\ & \Uparrow & \Downarrow & & \\ \text{pointwise } 1/4\text{-pinching} & & \text{PIC} \Rightarrow & R > 0 & \end{array}$$

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Here, K is the sectional curvature, i.e., the restriction of $K_{\mathbb{C}}$ on real 2-planes in $T_pM \otimes \mathbb{C}$; *Ric* is the Ricci curvature, and R is the scalar curvature on M. The pointwise 1/4-pinching condition means that, for any $p \in M$, K is positive and we have

$$1 \leq \frac{\max\{K(\sigma) : \text{2-plane}\, \sigma \subset T_pM\}}{\min\{K(\sigma) : \text{2-plane}\, \sigma \subset T_pM\}} < 4.$$

The notion of positive isotropic curvature was introduced in the paper of Micallef and Moore [**19**] in 1988 where they discovered that it can be used to control the stability of minimal surfaces just as the notion of positive sectional curvature can be used to control the stability of geodesics. Then, by using minimal surface theory, they proved

Theorem (Micallef-Moore). *Let M be a compact simply connected n-dimensional manifold with positive isotropic curvature where $n \geq 4$. Then M is homeomorphic to a sphere.*

In view of the above diagram, for $n \geq 4$, if M is a compact simply connected n-dimensional manifold with positive curvature operator or pointwise 1/4-pinching, then M is homeomorphic to a sphere. The latter generalizes the famous sphere theorem of Berger and Klingenberg. It is spectacular that, by using the Ricci flow, Böhm-Wilking [**5**] and Brendle-Schoen [**2**] have recently proved, respectively, that a compact n-dimensional simply connected manifold with positive curvature operator or pointwise 1/4-pinching is indeed diffeomorphic to the round sphere $\mathbb{S}^n$. More recently, Brendle [**1**] has further generalized the works of Böhm-Wilking [**5**] and Brendle-Schoen [**2**] and proved the following beautiful result: if M is a compact manifold with the property that $M \times \mathbb{R}$ has positive isotropic curvature, then M is diffeomorphic to a spherical space form.

In 1997, in a seminal paper [**14**], Hamilton initiated the study of positive isotropic curvature by Ricci flow. In dimension 4, he first proved that the condition of positive isotropic curvature is preserved under Ricci flow. Then, under the assumption that there are no essential incompressible space forms in the manifold, he developed a theory of Ricci flow with surgery to exploit the development of singularities in the Ricci flow to recover the topology of the manifold. Here an incompressible space form N in a four-manifold M is a smooth submanifold diffeomorphic to a spherical space form S^3/Γ such that the inclusion induces an injection from $\pi_1(N)$ to $\pi_1(M)$. It is essential unless $\Gamma = 1$ or $\Gamma = \mathbb{Z}_2$ and the normal bundle is unorientable. Hamilton's paper contained some unjustified statements which were later supplemented by the paper of Chen and Zhu [**8**]. Their main result is

Theorem (Hamilton). *Let M be a compact four manifold with no essential incompressible space form. Then M admits a metric with positive isotropic curvature if and only if it is diffeomorphic to $\mathbb{S}^4, \mathbb{RP}^4, \mathbb{S}^3 \times \mathbb{S}^1, \mathbb{S}^3 \widetilde{\times} \mathbb{S}^1$ (this is the quotient of $\mathbb{S}^3 \times \mathbb{S}^1$ by $\mathbb{Z}_2$ which acts by a reflection and the antipodal map on the first and second factor respectively), or a connected sum of them.*

Clearly, each of the manifolds $\mathbb{S}^4, \mathbb{RP}^4, \mathbb{S}^3 \times \mathbb{S}^1, \mathbb{S}^3 \widetilde{\times} \mathbb{S}^1$ listed in the above theorem admits a metric with positive isotropic curvature. A theorem of Micallef and Wang [**20**] guarantees that the connected sum of compact manifolds with positive isotropic curvature also admits such a metric. Another useful observation is that the condition of no essential incompressible space form is automatically satisfied if $\pi_1(M)$ is torsion free, i.e., contains no nontrivial element of finite order. Indeed, Γ in the above definition of essential incompressible space form must be trivial. So, if the fundamental group of a compact Riemannian four-manifold M with positive isotropic curvature contains a normal torsion free subgroup of finite index, then a finite cover of M is diffeomorphic to $\mathbb{S}^4, \mathbb{S}^3 \times \mathbb{S}^1$, or a connected sum of them. This shows the intimate connection between the topology and the fundamental group of a compact Riemannian manifold with positive isotropic curvature, at least in dimension 4.

For dimension greater than 4, it has been proved recently by Brendle and Schoen [**2**] (see also [**21**]) that the condition of positive isotropic curvature is preserved under Ricci flow, although there is yet no generalization of the curvature pinching estimates which are crucial in Hamilton's analysis of [**14**]. Another interesting result for a higher dimensional Riemannian manifold with positive isotropic curvature is the result of Fraser [**10**], who proved that $\mathbb{Z} \oplus \mathbb{Z}$ cannot occur as a subgroup of the fundamental group of such manifold when its dimension is greater than 4. We remark that Brendle and Schoen [**3**] recently extended Fraser's theorem to the case $n = 4$.

The following conjecture on the fundamental group of a compact Riemannian manifold with positive isotropic curvature was proposed by Gromov [**11**]; see also [**10**] and [**4**].

Conjecture (Gromov). *For $n \geq 4$, let M be an n-dimensional compact Riemannian manifold with positive isotropic curvature. Then the fundamental group of M contains a free subgroup of finite index.*

Recently, Schoen [**26**] (see also [**3**]) proposed a stronger conjecture:

Conjecture (Schoen). *For $n \geq 4$, let M be an n-dimensional compact Riemannian manifold with positive isotropic curvature. Then a finite cover of M is diffeomorphic to $\mathbb{S}^n, \mathbb{S}^{n-1} \times \mathbb{S}^1$, or a connected sum of them.*

Clearly, the latter conjecture implies the former. The purpose of this paper is to prove the conjecture of Schoen when $n = 4$. Indeed, we obtain a more precise result. In particular, we know exactly what are the fundamental groups of such manifolds. Our main result is

Main Theorem. *Let M be a compact 4-dimensional manifold. Then it admits a metric with positive isotropic curvature if and only if it is diffeomorphic to $\mathbb{S}^4$, $\mathbb{RP}^4$, $\mathbb{S}^3 \times \mathbb{R}/G$ or a connected sum of them. Here G is a cocompact fixed point free discrete subgroup of the isometry group of the standard metric on $\mathbb{S}^3 \times \mathbb{R}$.*

We give two immediate corollaries of our Main Theorem.

Corollary 1. *The conjecture of Schoen is true for $n = 4$.*

Proof. There is nothing to prove if M is diffeomorphic to $\mathbb{S}^4$ or $\mathbb{RP}^4$. So we may assume that M is diffeomorphic to $m\mathbb{RP}^4 \# \mathbb{S}^3 \times \mathbb{R}/G_1 \# \cdots \# \mathbb{S}^3 \times \mathbb{R}/G_k$ for some nonnegative integer m and positive integer k. The fundamental group of M is given by

$$\underbrace{\mathbb{Z}_2 * \cdots * \mathbb{Z}_2}_{m\,\text{times}} * G_1 * \cdots * G_k.$$

Now a cocompact fixed point free discrete subgroup G of the isometry group of $\mathbb{S}^3 \times \mathbb{R}$ is always virtually infinite cyclic. This is because, by the cocompactness of the action of G on $\mathbb{S}^3 \times \mathbb{R}$, G always contains an element g which acts as translation on the second factor, and the infinite cyclic subgroup generated by g must have finite index as it also acts cocompactly on $\mathbb{S}^3 \times \mathbb{R}$. Thus $\pi_1(M)$ is the free products of finite and virtually infinite cyclic groups. It is known that such a group always contains a normal free subgroup of finite index. In particular, $\pi_1(M)$ contains a torsion free normal subgroup of finite index. By the remark after the statement of Hamilton's Theorem, the conclusion in the conjecture of Schoen holds. q.e.d.

The second corollary concerns the classification of compact conformally flat Riemannian four-manifolds with positive scalar curvature. We start with a digression of the geometry of Riemannian four-manifold M. In this case, the bundle $\wedge^2 TM$ has a decomposition into the direct sum of its self-dual and anti-self-dual parts:

$$\wedge^2 TM = \wedge^2_+ TM \oplus \wedge^2_- TM.$$

The curvature operator can then be decomposed as

$$\mathcal{R} = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$$

where $A = W_+ + \frac{R}{12}$, $B = \overset{\circ}{Ric}$, $C = W_- + \frac{R}{12}$. Here $W_\pm$ are the self-dual and anti-self-dual Weyl curvature tensors respectively, while $\overset{\circ}{Ric}$ is the trace free part of the Ricci curvature tensor. Denote the eigenvalues of the matrices A, C by $a_1 \leq a_2 \leq a_3$, $c_1 \leq c_2 \leq c_3$ respectively. It is known that the condition of positive isotropic curvature is equivalent to the conditions $a_1 + a_2 > 0$ and $c_1 + c_2 > 0$. From this, it is clear that a compact conformally flat Riemannian four-manifold with positive scalar curvature always has positive isotropic curvature.

Now it had been observed by Izeki [**16**] that a compact conformally flat Riemannian four-manifold M with positive scalar curvature always has a finite cover which is diffeomorphic to $\mathbb{S}^4, \mathbb{S}^3 \times \mathbb{S}^1$, or a connected sum of them. The reason is this. Let M be such a manifold; then by a result of Schoen and Yau [**25**], $\pi_1(M)$ is a Kleinian group. In particular, it is a finitely generated subgroup of a linear group, namely $SO(5,1)$. By Selberg's Lemma, $\pi_1(M)$ contains a torsion free normal subgroup of finite index. Since such a manifold always has positive isotropic curvature, we can again apply the above remark after the statement of Hamilton's Theorem to conclude that M has a finite cover which is diffeomorphic to $\mathbb{S}^4, \mathbb{S}^3 \times \mathbb{S}^1$, or a connected sum of them.

Our Main Theorem gives a more precise classification of such manifolds.

Corollary 2. *A compact four-manifold admits a metric of positive isotropic curvature if and only if it admits a conformally flat metric of positive scalar curvature.*

Proof. The manifolds $\mathbb{S}^4$, $\mathbb{RP}^4$, $\mathbb{S}^3 \times \mathbb{R}/G$ listed in the Main Theorem clearly admit conformally flat metrics of positive scalar curvature, and we only have to invoke the fact that connected sum of conformally flat Riemannian manifolds with positive scalar curvature also admits such a metric. q.e.d.

We remark that Corollary 2 does not hold for dimension $n > 4$. The following example is taken from [**20**]. For any Riemann surface Σ_g of genus $g \geq 2$ and even $n > 4$, the manifold $M = \Sigma_g \times \mathbb{S}^{n-2}$ admits a conformally flat metric of positive scalar curvature; however, M cannot admit a metric with positive isotropic curvature.

The proof of our Main Theorem naturally divides into two parts. The first part is analytical and the second part topological.

Our argument in the first part is based on the celebrated Hamilton-Perelman theory [**14**] [**23**] on the Ricci flow with surgery. To understand the topology of a compact four-manifold with positive isotropic curvature, we take it as initial data and evolve it by the Ricci flow. It is easy to see that the solution will blow up in finite time. By applying Hamilton's curvature pinching estimates obtained in [**14**], we can get a complete understanding on the part around the singularities of the solution. Then we can perform Hamilton's surgery procedure to cut off the part around the singularities. After the surgery, due to the possible existence of essential incompressible space forms, we will get a closed (possibly disconnected) orbifold with positive isotropic curvature. After studying Ricci flow on orbifold and obtaining a detailed singularity analysis for orbifold Ricci flow, we can use the orbifold as initial data to run the Ricci flow and to do surgeries again. By repeating this procedure and extending the arguments in the previous paper [**8**] of the first and the third authors to the orbifold case, we will be able to show that, after a finite number of surgeries and discarding a finite number of pieces which are diffeomorphic to spherical orbifolds $\mathbb{S}^4/\Delta$ (here Δ denotes a finite subgroup of the orthogonal group $O(5)$) with at most isolated orbifold singularities, the solution becomes extinct. As a result, we prove that the initial manifold is diffeomorphic to an orbifold connected sum (see below or the precise definition given in Section 2) of spherical orbifolds.

The second part concerns the recovery of the topology of the manifold from the orbifold connected sum. First of all, by an algebraic lemma, we know that a spherical orbifold $\mathbb{S}^4/\Delta$ has either zero, one, or two orbifold singularities. A spherical orbifold with no orbifold singularity is simply $\mathbb{S}^4$ or $\mathbb{RP}^4$, while a spherical orbifold with one or two orbifold singularities is, after removing an open neighborhood from each of its orbifold singularities, diffeomorphic to a smooth cap or a cylinder respectively. Here a cylinder $C(\Gamma)$ is given by $\mathbb{S}^3/\Gamma \times [-1,1]$ for some finite fixed point free subgroup Γ of $SO(4)$, while a smooth cap C_Γ^σ is given as the quotient of $\mathbb{S}^3/\Gamma \times [-1,1]$ by a group of order two generated by $\hat{\sigma} : (x,s) \mapsto (\sigma(x), -s)$ where σ is a fixed point free isometric involution on $\mathbb{S}^3/\Gamma$. Now, the orbifold connected sum of spherical orbifolds is formed in two steps. In the first step, to undo the surgeries in the Ricci flow which create orbifold singularities, we glue copies of the $C(\Gamma)$'s and C_Γ^σ's along their diffeomorphic boundaries with suitable identifying maps to form a number of closed (compact) manifolds. It is not hard to see that, up to diffeomorphisms, they are essentially of two types: the self-gluing of the two ends of a cylinder $C(\Gamma)$ and the gluing of two smooth caps, C_Γ^σ and $C_\Gamma^{\sigma'}$, with diffeomorphic boundaries by suitable diffeomorphisms on $\mathbb{S}^3/\Gamma$. Since we know that any diffeomorphism on a three-dimensional spherical space form is isotopic to an isometry (see [**18**]), the resulting closed manifolds can be equipped with metrics which are locally isometric to $\mathbb{S}^3 \times \mathbb{R}$. Now, the second step in the formation of the orbifold connected sum is to undo the surgeries in the Ricci flow which do not create singularities. It consists of two types of operations. The first is taking the usual connected sums of the above closed manifolds with $\mathbb{S}^4$'s or $\mathbb{RP}^4$'s and the second is adding handles to the resulting manifold. Since the latter operations are equivalent to taking the usual connected sums with $\mathbb{S}^3 \times \mathbb{S}^1$ or $\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1$, our Main Theorem is proved.

A natural question is whether our Main Theorem and its proof can be extended to dimensions greater than 4. We believe that the analytic part of our proof will go through once Hamilton's curvature pinching estimates in [**14**] can be extended to higher dimensions. Assuming that this has been done, most of the argument in the topological part of our proof will also go through. This will allow us to show that a compact Riemannian n-dimensional manifold M with positive isotropic curvature is homeomorphic to $\mathbb{S}^n$, $\mathbb{RP}^n$, $\mathbb{S}^{n-1} \times \mathbb{R}/G$, or a connected sum of them. Here we only know that G acts differentiably on $\mathbb{S}^{n-1} \times \mathbb{R}$. The differences are due to the possible existence of (exotic) diffeomorphisms on a spherical space form $\mathbb{S}^{n-1}/\Gamma$ which are not isotopic to isometries. By the same argument as in our proof of Corollary 1, this result still implies a weaker form of the conjecture of Schoen; namely, M has a finite cover which is homeomorphic to $\mathbb{S}^n$, $\mathbb{S}^{n-1} \times \mathbb{S}^1$, or a connected sum of them. In particular, the fundamental group of M is virtually free.

Our paper is organized as follows. In Section 2, after defining the notion of orbifold connected sum; we state a main result of this paper, Theorem 2.1, which says that any compact 4-orbifold with positive isotropic curvature and with at most isolated singularities is diffeomorphic to an orbifold connected sum of spherical orbifolds $\mathbb{S}^4/\Gamma$. At the end of the section, we give some natural examples of compact 4-manifolds with positive isotropic curvature with emphasis on their constructions as orbifold connected sums of spherical orbifolds. The proof of Theorem 2.1, by Ricci flow with surgery, is given in Sections 3 and 4. In section 3, we first set up our Ricci flow with initial data given by a compact 4-orbifold with positive isotropic curvature and with at most isolated singularities. After introducing some notations and terminologies that will be used throughout this paper, we give a complete study of the canonical neighborhood structure of any ancient κ-orbifold solution. The final result is summarized in Theorem 3.10. This study is crucial for our understanding of the structure near the singularities of our Ricci flow. In Section 4, we make a detailed study of the necessary modification needed to justify the continuation of our Ricci flow on orbifold via surgeries. This section depends crucially on the work of the first and the third authors in the manifold case [**8**]. At the end, this allows us to construct a solution to the Ricci flow with surgery which becomes extinct, and it is then a simple matter to express our initial 4-orbifold as an orbifold connected sum of spherical orbifolds. In Section 5, we prove the algebraic lemma alluded to above and then use it to recover the topology of the orbifold connected sum in Theorem 2.1. This proves the Main Theorem. Finally, Section 6 is an appendix that gives the proof of a technical geometric lemma which is needed in this paper.

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2. Orbifold connected sums

We give a generalization of the construction of connected sum of manifolds to orbifolds with at most isolated singularities as follows. For an orbifold X and $x \in X$, we use Γ_x to denote the local uniformization group at x, i.e., there is an open neighborhood $B_x \ni x$ with smooth boundary which is a quotient $\tilde{B}/\Gamma_x$, where $\tilde{B}$ is diffeomorphic to $\mathbb{R}^n$ and Γ_x is a finite subgroup of linear transformations fixing the origin. Now, let $X_1, \dots, X_p$ be p $n-$dimensional orbifolds with at most isolated orbifold singularities and $x_1, x_1', x_2, x_2', \dots, x_q'$ be $2q$ distinct points (not necessarily singular) on $X_1, \dots, X_p$ such that for each pair (x_j, x_j'), Γ_{x_j} is conjugate to $\Gamma_{x_j'}$ as linear subgroups. Assume that for each $j = 1, 2, \dots, q$, we are given a diffeomorphism f_j between ∂B_{x_j} and $\partial B_{x_j'}$, then for each j, we can remove B_{x_j} and $B_{x_j'}$ from the orbifolds and identify their boundaries by using the diffeomorphism f_j to obtain a new manifold $\#_f(X_1, \dots, X_p)$ where $f = (f_1, \dots, f_q)$. We call it the orbifold connected sum of $X_1, \dots, X_p$ (by the identifying or gluing map f). Here we emphasize that the diffeomorphism type of the resulting orbifold depends only on the isotopic class of f. We will apply the preceding construction only to orbifolds of dimension 4 in this paper.

The following is the main result of this paper.

Theorem 2.1. *Let (M^4, g) be a compact 4-dimensional manifold or orbifold with at most isolated singularities with positive isotropic curvature. Then M^4 is diffeomorphic to an orbifold connected sum of a finite number of spherical 4-orbifolds $X_1 = \mathbb{S}^4/\Gamma_1, \dots, X_l = \mathbb{S}^4/\Gamma_l$, where each Γ_i is a finite subgroup of the isometry group, $O(5)$, of the standard metric on $\mathbb{S}^4$ so that the quotient orbifold X_i has at most isolated singularities.*

Theorem 2.1 will be proved by Ricci flow with surgery given in Sections 3 and 4. Here, we will give some natural examples of compact four-manifolds of positive isotropic curvature with emphasis on their constructions as orbifold connected sums of spherical orbifolds.

In dimension 4, except for $\mathbb{S}^4$ and $\mathbb{RP}^4$, the best known examples of manifolds with positive isotropic curvature are $\mathbb{S}^3/\Gamma \times \mathbb{S}^1$, where Γ is a fixed point free finite subgroup of $SO(4)$. Note that Γ can also act isometrically on $\mathbb{S}^4$ by fixing the extra direction. The quotient space $\mathbb{S}^4/\Gamma$ is then an orbifold with exactly two singularities, say P and P'; those local uniformization groups are just Γ itself. Now if we perform an orbifold connected sum on $\mathbb{S}^4/\Gamma$ with itself by using the identity map as the identifying map, it gives $\mathbb{S}^3/\Gamma \times \mathbb{S}^1$. If we take an identifying map f (in Diff($\mathbb{S}^3/\Gamma$)) in a nontrivial isotopic class, then the connected sum may give some twisted product of $\mathbb{S}^3/\Gamma$ and $\mathbb{S}^1$. We denote the manifold by $\mathbb{S}^3/\Gamma\times_f\mathbb{S}^1$. Note that the manifold $\mathbb{S}^3/\Gamma\times_f\mathbb{S}^1$ admits a metric of positive isotropic curvature. This is because, by [**18**], every diffeomorphism of a three-dimensional spherical space form $\mathbb{S}^3/\Gamma$ is isotopic to an isometry of its standard metric; thus $\mathbb{S}^3/\Gamma\times_f\mathbb{S}^1$ can be equipped with a metric locally isometric to $\mathbb{S}^3/\Gamma \times \mathbb{R}$, and hence a metric with positive isotropic curvature. Another immediate consequence is that, for fixed Γ, there are only a finite number of diffeomorphism classes of $\mathbb{S}^3/\Gamma\times_f\mathbb{S}^1$. As a simple example, if we take $\Gamma = \{1\}$ and f as an orientation reversing diffeomorphism on $\mathbb{S}^3$, the resulting manifold is $\mathbb{S}^3\tilde{\times}\mathbb{S}^1$, which is the only unoriented $\mathbb{S}^3$ bundle over $\mathbb{S}^1$.

If $\mathbb{S}^3/\Gamma$ admits a fixed point free isometry σ satisfying $\sigma^2 = 1$, then we can define a reflection $\hat{\sigma}$ on the 4-manifold $\mathbb{S}^3/\Gamma \times \mathbb{R}$ by $\hat{\sigma}(x, s) = (\sigma(x), -s)$, where $x \in \mathbb{S}^3/\Gamma, s \in \mathbb{R}$. The quotient $(\mathbb{S}^3/\Gamma \times \mathbb{R})/\{1, \hat{\sigma}\}$ is a smooth four-manifold with a neck-like end $\mathbb{S}^3/\Gamma \times \mathbb{R}$. We denote this manifold by C^{σ}_{Γ}. If we think of the sphere $\mathbb{S}^4$ as the compactification of $\mathbb{S}^3/\Gamma \times \mathbb{R}$ by adding two points (north and south poles) at infinities of $\mathbb{S}^3/\Gamma \times \mathbb{R}$, we can regard Γ and $\hat{\sigma}$ as isometries of the standard $\mathbb{S}^4$ in a natural manner. So C^{σ}_{Γ} is diffeomorphic to the smooth manifold obtained by removing a neighborhood of the unique singularity from $\mathbb{S}^4/\{\Gamma, \hat{\sigma}\}$. We call C^{σ}_{Γ} a smooth cap.

Given two smooth caps C^{σ}_{Γ} and $C^{\sigma'}_{\Gamma'}$, if Γ is conjugate to Γ' (i.e. there is an isometry γ of $\mathbb{S}^3$ such that $\Gamma = \gamma\Gamma'\gamma^{-1}$ and this is equivalent to saying that $\mathbb{S}^3/\Gamma$ is diffeomorphic to $\mathbb{S}^3/\Gamma'$), we can glue C^{σ}_{Γ} and $C^{\sigma'}_{\Gamma'}$ along their boundaries by a diffeomorphism $f : \partial C^{\sigma}_{\Gamma} \to \partial C^{\sigma'}_{\Gamma'}$. Then we get a smooth manifold and we denote it by $C^{\sigma}_{\Gamma} \cup_f C^{\sigma'}_{\Gamma'}$. By the same argument as above, it also admits a metric with positive isotropic curvature. Clearly, $C^{\sigma}_{\Gamma} \cup_f C^{\sigma'}_{\Gamma'}$ is an orbifold connected sum of $\mathbb{S}^4/\{\Gamma, \hat{\sigma}\}$ and $\mathbb{S}^4/\{\Gamma', \hat{\sigma'}\}$. Note that, in contrast to the previous case, the diffeomorphism type of the manifold $C^{\sigma}_{\Gamma} \cup_f C^{\sigma'}_{\Gamma'}$ is independent of the choice of f and we may assume, without loss of generality, that $\Gamma = \Gamma'$. As a simple example, we can take $\Gamma = \Gamma' = \{1\}$ and $\sigma = \sigma' =$ the antipodal map on the sphere $\mathbb{S}^3$; then $C^{\sigma}_{\Gamma} \cup_f C^{\sigma'}_{\Gamma'}$ is $\mathbb{RP}^4\#\mathbb{RP}^4$; which can also be identified as the quotient of $\mathbb{S}^3 \times \mathbb{S}^1$ by $\mathbb{Z}_2$, which acts by the antipodal map and a reflection on the first and second factor respectively.

Our Main Theorem says that compact four-manifolds with positive isotropic curvature are given by the connected sums of the examples listed above (see the proof of the Main Theorem in Section 5). The fundamental group of $\mathbb{S}^3/\Gamma\times_f\mathbb{S}^1$ is the semidirect product $\Gamma \rtimes_{\alpha} \mathbb{Z}$. Here $\alpha : \mathbb{Z} \to \mathrm{Aut}(\Gamma)$ is given by $\alpha(1) = f_*$ where $f_* : \Gamma \to \Gamma$ is the induced automorphism of f on $\pi_1(\mathbb{S}^3/\Gamma) \cong \Gamma$. For the manifold $C^{\sigma}_{\Gamma} \cup_f C^{\sigma'}_{\Gamma}$, we first lift the isometric involutions σ and σ' on $\mathbb{S}^3/\Gamma$ to isometries on $\mathbb{S}^3$ (denoted by the same notations) and let Γ_1 and Γ_2 be the finite groups of $O(4)$ generated by $< \Gamma, \sigma >$ and $< \Gamma, \sigma' >$ respectively. Then the fundamental group of $C_\Gamma^\sigma \cup_f C_\Gamma^{\sigma'}$ is the amalgamated product $\Gamma_1 *_\Gamma \Gamma_2$.

3. Ancient κ-solutions on orbifolds

In this section, we will start our proof of Theorem 2.1. The method is to use Ricci flow with surgery to deform the initial metric (M, g). Note that even if (M, g) is a manifold, our surgeries will in general introduce isolated orbifold singularities. Thus, it is important to develop our theory on such orbifolds.

Let (M^4, g_0) be a compact 4-dimensional orbifold with at most isolated singularities with positive isotropic curvature. We consider the following Ricci flow equation:

$$\frac{\partial g}{\partial t} = -2Ric, \qquad g\,|_{t=0} = g_0. \tag{3.1}$$

Since the implicit function theorem or the De Turck trick can still be applied on orbifolds, we have the short time solution $g(\cdot, t)$ of (3.1) (see [**15**], [**12**], [**9**]). Recall that as in the introduction, in dimension 4, the curvature operator has the following decomposition:

$$\mathcal{R} = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix},$$

and we denote the eigenvalues of matrices A, C, and $\sqrt{BB^t}$ by $a_1 \leq a_2 \leq a_3$, $c_1 \leq c_2 \leq c_3$, $b_1 \leq b_2 \leq b_3$ respectively. Since the maximum principle can still be applied on orbifolds, the positivity of the isotropic curvature and the improved pinching estimates of Hamilton are still preserved under the Ricci flow. So we have

Lemma 3.1. *(theorem B1.1 and theorem B2.3 of* [**14**]*)*
*There exist positive constants ρ, $\Lambda, P < +\infty$ depending only on the initial metric, such that the solution to the Ricci flow (*3.1*) satisfies*

$$\begin{aligned} & a_1 + \rho > 0 \text{ and } c_1 + \rho > 0, \\ & \max\{a_3, b_3, c_3\} \leq \Lambda(a_1 + \rho) \max\{a_3, b_3, c_3\} \leq \Lambda(c_1 + \rho), \\ & \frac{b_3}{\sqrt{(a_1+\rho)(c_1+\rho)}} \leq 1 + \frac{\Lambda e^{Pt}}{\max\{\log\sqrt{(a_1+\rho)(c_1+\rho)}, 2\}} \end{aligned} \tag{3.2}$$

As a result, any blowing up limit will satisfy the following restricted isotropic curvature pinching condition:

$$a_3 \leq \Lambda a_1, \quad c_3 \leq \Lambda c_1, \quad b_3^2 \leq a_1 c_1. \tag{3.3}$$

We can also define the same notion of κ-noncollapsed for a scale r_0 for solutions to the Ricci flow on orbifolds; namely, for any space time point (x_0, t_0), the condition that $|Rm|(x, t) \leq r_0^{-2}$, for all $t \in [t_0 - r_0^2, t_0]$

and $x \in B_t(x_0, r_0)$, implies $Vol_{t_0}(B_{t_0}(x_0, r_0)) \geq \kappa r_0^4$. Since integration by parts and log-Sobolev inequality still hold on closed orbifolds, we can apply the same argument as in [**22**] (theorem 4.1 of [**22**] or see lemma 2.6.1 and theorem 3.3.3 of [**6**] for the details) to show

Lemma 3.2. *For any $T > 0$, there is a κ depending on T and the initial orbifold metric, such that the smooth solution to the Ricci flow which exists for $[0, T)$ is κ-noncollapsed for scales less than $\sqrt{T}$.*

Since the scalar curvature is strictly positive, it follows from the standard maximum principle and the evolution equation of the scalar curvature that the solution must blow up at finite time. As in the smooth case [**8**], we will show in Section 4 that the geometric structure at any point with suitably large curvature is close to an ancient $\kappa-$orbifold solution (defined below). So it is important to investigate the structures of any ancient $\kappa-$orbifold solutions. This is the main goal in this section.

For the convenience of discussion, we fix some terminologies and notations here.

In this paper, a (topological) neck is defined to be a manifold diffeomorphic to $\mathbb{S}^3/\Gamma \times \mathbb{R}$ where Γ is a finite fixed point free subgroup of isometries of $\mathbb{S}^3$. A smooth cap is either C^σ_Γ (defined in Section 2) or the 4-ball $\mathbb{B}^4$. We would also like to define two types of orbifold caps. The orbifold cap of Type I is obtained by collapsing the boundary $\mathbb{S}^3/\Gamma \times \{0\}$ of $\mathbb{S}^3/\Gamma \times [0, 1)$ to a point. We denote it by C_Γ. By extending the action of Γ to an isometric action on $\mathbb{S}^4$ by fixing the extra direction, it is clear that C_Γ can also be obtained by removing a neighborhood of one singularity from the spherical orbifold $\mathbb{S}^4/\Gamma$. To define the orbifold cap of Type II, we first consider the quotient of $\mathbb{S}^4$ by the group generated by the isometry $(x_1, x_2, \dots, x_5) \to (x_1, -x_2, \dots, -x_5)$ on $\mathbb{S}^4$ which has exactly two fixed points at $(1, 0, 0, 0, 0)$ and $(-1, 0, 0, 0, 0)$ with local uniformization group $\mathbb{Z}_2$. This quotient is a spherical orbifold denoted by $\mathbb{S}^4/(x, \pm x')$. The orbifold cap of Type II, denoted by $\mathbb{S}^4/(x, \pm x')\backslash\bar{\mathbb{B}}^4$, is then obtained by removing a neighborhood of a smooth point from the spherical orbifold $\mathbb{S}^4/(x, \pm x')$.

Roughly speaking, we will show below that either the ancient $\kappa-$orbifold solution is diffeomorphic to a global quotient $\mathbb{S}^4/\Gamma$ or else it has local structures of necks, smooth caps, or orbifold caps of Type I or II described above. We start with the definition of the ancient $\kappa-$orbifold solution.

Definition 3.3. We say a solution to the Ricci flow is an **ancient $\kappa-$orbifold solution** if it is a smooth complete nonflat solution to the Ricci flow on a four-orbifold with at most isolated singularities satisfying the following three conditions:

(i) the solution exists on the ancient time interval $t \in (-\infty, 0]$,

(ii) it has positive isotropic curvature and bounded curvature, and satisfies the restricted isotropic curvature pinching condition,

$$a_3 \leq \Lambda a_1, \quad c_3 \leq \Lambda c_1, \quad b_3^2 \leq a_1 c_1, \tag{3.4}$$

(iii) κ-noncollapsed on all scales for some $\kappa > 0$.

The canonical neighborhood structure of the ancient κ-orbifold solutions are described in the following two subsections.

3.1. Ancient κ-orbifold solution with null eigenvector in curvature operator.

Theorem 3.4. *Let (X, g_t) be an ancient $\kappa-$orbifold solution defined in Definition 3.3 such that the curvature operator has a nontrivial null eigenvector somewhere. Then we have*

(i) if X is a smooth manifold, then either $X = (\mathbb{S}^3/\Gamma)\times\mathbb{R}$, or $X = C^{\sigma}_{\Gamma'}$ for some fixed point free isometric subgroup Γ or Γ' of $\mathbb{S}^3$ and σ is an fixed point free isometry on $\mathbb{S}^3/\Gamma'$ with $\sigma^2 = 1$;

(ii) if X has singularities, then X is diffeomorphic to $\mathbb{S}^4/(x, \pm x')\setminus\bar{\mathbb{B}}$. In particular, X has exactly two singularities.

Proof. Suppose the curvature operator has a nontrivial null eigenvector somewhere. Then the null eigenvectors exist everywhere in space time by Hamilton's strong maximum principle [**13**].

Case 1: X is a smooth manifold.

In this case, it is known from lemma 3.2 in [**8**] that the universal cover of X is $\mathbb{S}^3\times\mathbb{R}$. Let Γ be the group of deck transformations. We claim that the second components (acting on $\mathbb{R}$ isometrically) of Γ must contain no translation. It is the same as saying that X cannot be compact. For this, we first note that for the Ricci flow on $\mathbb{S}^3\times\mathbb{R}$, the flat $\mathbb{R}$ factor does not change while the spherical factor simply expands when time goes to $-\infty$. If X is compact, the above solution of the Ricci flow descends to a solution on X which contradicts the κ-noncollapsing assumption in the definition of ancient κ-orbifold solution. This proves our claim. Now, let $\Gamma = \Gamma^0 \cup \Gamma^1$ where the second components of Γ^0 and Γ^1 act on $\mathbb{R}$ as an identity or reflection respectively. If Γ^1 is empty, $X = (\mathbb{S}^3/\Gamma) \times \mathbb{R}$, where Γ acts on $\mathbb{S}^3$ isometrically with no fixed point. If Γ^1 is not empty, take $\sigma \in \Gamma^1$; we have $\sigma^2 \in \Gamma^0$ and $\sigma\Gamma^0 = \Gamma^1$. It is clear that X is obtained by taking the quotient of $(\mathbb{S}^3/\Gamma^0) \times \mathbb{R}$ by the group of order 2 generated by σ. Hence $X = C^{\sigma}_{\Gamma^0}$ in the notation of Section 2.

We remark that the $(\mathbb{S}^3/\Gamma) \times \mathbb{R}$ has two ends, but $C^{\sigma}_{\Gamma^0}$ has only one end.

Case 2: X is an orbifold with nonempty isolated singularities.

We first note that the complete metric space X is modeled on the geometric space $\mathbb{S}^3 \times \mathbb{R}$ via the group $ISO(\mathbb{S}^3 \times \mathbb{R})$, where $ISO(\mathbb{S}^3 \times \mathbb{R})$ is the group of isometries of $\mathbb{S}^3 \times \mathbb{R}$. In other words, X is an $(\mathbb{S}^3 \times \mathbb{R}, ISO(\mathbb{S}^3 \times \mathbb{R}))$-orbifold. By [**27**] or theorem 13.3.10 in [**24**], X must be isometric to a global quotient $\mathbb{S}^3 \times \mathbb{R}/\Gamma$ of $\mathbb{S}^3 \times \mathbb{R}$, where Γ is a discrete subgroup of standard isometries of $\mathbb{S}^3 \times \mathbb{R}$. We proceed to analyze Γ in our situation.

For an (isolated) fixed point $z \in \mathbb{S}^3 \times \mathbb{R}$, denote by $\Gamma_z = \{\gamma(z) = z, \gamma \in \Gamma\}$ the isotropy group of Γ at z. Let $\Gamma_z = \Gamma_z^0 \cup \Gamma_z^1$ where the $\mathbb{R}$ components of Γ_z^0 and Γ_z^1 act on $\mathbb{R}$ as identity or reflection respectively. We write $z = (o, 0)$ where $0 \in \mathbb{R}$ and $o \in \mathbb{S}^3$. Since $(o, 0)$ is an isolated fixed point of each $\gamma \in \Gamma_z$, this implies that $\Gamma_z^0 = \{1\}$, otherwise a nontrivial element of Γ_z^0 will fix the whole $\{o\} \times \mathbb{R}$. Also $\Gamma_z = \Gamma_z^1 =< \sigma_z >$ is generated by an isometric involution σ_z whose $\mathbb{S}^3$ component fixes the axis through o and acts antipodally on its orthogonal complement.

Note that, by the same argument as in Case 1, the $\mathbb{R}$ components of Γ must contain no translation. This implies that all fixed points of Γ have the same $\mathbb{R}$ coordinates because if we would have two elements in Γ whose $\mathbb{R}$ components reflect about points with different $\mathbb{R}$ coordinates, this will produce an element in Γ with nontrivial translation on the $\mathbb{R}$ factor. We may assume these fixed points $z, w, \ldots$ etc. lie in $\mathbb{S}^3 \times \{0\}$. We denote the generators of their isotropy groups by $\sigma_z, \sigma_w, \ldots$ etc. We call the orthogonal complement of the fixed axis of each σ_z its associated equator. Recall that σ_z acts by the antipodal map on its associated equator. If $\sigma_x \neq \sigma_y$, since the action of $\sigma_z \sigma_w$ on $\mathbb{R}$ is trivial, we have $\sigma_z \sigma_w =$ identity on the great circle C given by the intersection of their associated equators. This implies that $\sigma_z \sigma_w$ fixes every point of $\mathbb{R} \times C$, which is a contradiction. We conclude that there are exactly two fixed points A, B of Γ whose $\mathbb{S}^3$ components lie antipodally on $\mathbb{S}^3$.

Now, let Γ_0 be the subgroup of Γ generated by Γ_A and Γ_B. Clearly, Γ_0 is a normal subgroup of Γ. We claim that the action of $G = \Gamma/\Gamma_0$ on $\mathbb{S}^3 \times \mathbb{R}/\Gamma_0$ has no fixed point. Indeed, if there is some $g \in \Gamma$ and $x \in \mathbb{S}^3 \times \mathbb{R}$ such that $g\Gamma_0(x) = \Gamma_0(x)$, this will imply $gx = \gamma x$ for some $\gamma \in \Gamma_0$. Hence $\gamma^{-1} g \in \Gamma_x \subset \Gamma_0$ and $g \in \Gamma_0$, and our claim is proved. Let $G = \Gamma/\Gamma_0$. We have $G = 1$. If not, we can pick $1 \neq g \in G$; then g is a fixed point free isometry of $(\mathbb{R} \times \mathbb{S}^3)/\Gamma_0$. Since we must have $g(A) = B$ and $g(B) = A$, and g sends geodesics connecting A to B to geodesics connecting B to A, this implies that g acts on $\mathbb{RP}^2$ (as quotient of the orthogonal complement of the axis joining A and B by Γ_0) without fix points. This is impossible. So we have showed that G is trivial.

In summary, we have proved that $X = (\mathbb{S}^3 \times \mathbb{R})/\Gamma_0$. By using our notation in Section 2, X is diffeomorphic to $\mathbb{S}^4/(x, \pm x') \backslash \bar{\mathbb{B}}^4$. We note that in this case X has only one end, which is diffeomorphic to $\mathbb{S}^3 \times \mathbb{R}$. q.e.d.

3.2. Ancient κ-orbifold solution with positive curvature operator. In this section, we study ancient κ-orbifold solutions with positive curvature operator. Together with the result in section 3.1, this completes our analysis of the canonical neighborhood structure of any ancient κ-orbifold solution.

We first recall the manifold case of the result obtained in [**8**].

Theorem 3.5. *(theorem 3.8 in* [**8**]*) For every $\epsilon > 0$, one can find positive constants $C_1 = C_1(\epsilon)$, $C_2 = C_2(\epsilon)$ such that for each point (x,t) in every four-dimensional ancient κ-**manifold** solution (for some $\kappa > 0$) with restricted isotropic curvature pinching and with positive curvature operator, there is a number r with $0 < r < C_1(R(x,t))^{-\frac{1}{2}}$, so that for some open set B with $B_t(x,r) \subset B \subset B_t(x,2r)$, we have one of the following three cases:*

*(a) B is an **evolving ϵ-neck**, in the sense that it is the time slice at time t of the parabolic region $\{(x',t')|x' \in B, t' \in [t-\epsilon^{-2}R(x,t)^{-1},t]\}$ which is, after scaling with factor $R(x,t)$ and shifting the time t to 0, ϵ-close (in $C^{[\epsilon^{-1}]}$ topology) to the subset $(\mathbb{I}\times\mathbb{S}^3)\times[-\epsilon^{-2},0]$ of the evolving round cylinder $\mathbb{R}\times\mathbb{S}^3$, having scalar curvature one and length $2\epsilon^{-1}$ for $\mathbb{I}$ at time zero, or*

*(b) B is an **evolving ϵ-cap**, in the sense that it is the time slice at the time t of an evolving metric on open $\mathbb{B}^4$ or $\mathbb{RP}^4 \setminus \overline{\mathbb{B}^4}$ such that the region outside some suitable compact subset of $\mathbb{B}^4$ or $\mathbb{RP}^4 \setminus \overline{\mathbb{B}^4}$ is an evolving ϵ-neck, or*

(c) B is a compact manifold (without boundary) with positive curvature operator (thus it is diffeomorphic to $\mathbb{S}^4$ or $\mathbb{RP}^4$).

Furthermore, the scalar curvature of the ancient κ-solution in B at time t is between $C_2^{-1}R(x,t)$ and $C_2R(x,t)$.

Compared to the manifold case, the key difficulty in analyzing the local structure of ancient κ-**orbifold** solution is the possible collapsing of the solution in the presence of orbifold singularities with big local uniformization groups so that one might not be able to choose the constants C_1, C_2 independent of the noncollapsing constant κ. We will solve this problem below by lifting the ancient κ-orbifold solution to its universal cover where the above result in the manifold case can be applied and we get uniformity for the constants C_1, C_2 in our present case. Here, first of all, we need to generalize the concept of ε-neck and ε-cap to orbifold solutions with at most isolated singularities. The point is to allow a suitable isometry group to act on the usual necks and caps.

Definition 3.6. Fix $\varepsilon > 0$ and a space time point (x,t). Let $B \subset X$ be a space open subset containing x: (i) we call B an **evolving ε-neck** around (x,t) if it is the time slice at time t of the parabolic region $\{(x',t')|x' \in B, t' \in [t-\varepsilon^{-2}R(x,t)^{-1},t]\}$ which satisfies the following: there is a diffeomorphism $\varphi : \mathbb{I}\times(\mathbb{S}^3/\Gamma) \to B$ such that, after pulling back the solution $(\varphi)^*g(\cdot,\cdot)$ to $\mathbb{I}\times\mathbb{S}^3/\Gamma$, scaling with factor $R(x,t)$ and shifting the time t to 0, the solution is ε-close (in $C^{[\varepsilon^{-1}]}$ topology) to the subset $(\mathbb{I} \times \mathbb{S}^3/\Gamma) \times [-\varepsilon^{-2}, 0]$ of the evolving round cylinder $\mathbb{R} \times \mathbb{S}^3/\Gamma$, having scalar curvature one and length $2\varepsilon^{-1}$ for $\mathbb{I}$ at time zero.

(ii) We call B an **evolving ε-cap** if it is the time slice at time t of an evolving metric on either one of the following spaces: an open smooth cap $\mathbb{B}^4$ or C_Γ^σ, an open orbifold cap C_Γ of type 1 or $\mathbb{S}^4/(x, \pm x') \setminus \bar{\mathbb{B}}^4$ of type 2, such that the region outside some suitable compact subset of the space is an evolving ε-neck around some point in the sense of (i).

Let us start with the following elliptic type curvature estimate for our ancient κ-orbifold solution. The idea of the proof is to find out a global uniformization space which is not collapsed and investigate the isometric group action on it.

Proposition 3.7. *There is a universal positive function $\omega : [0, \infty) \to [0, \infty)$ such that for any ancient κ-orbifold solution on 4-orbifold X, we have*

$$R(x,t) \le R(y,t)\omega(R(y,t)d_t(x,y)^2)$$

for any $x, y \in X, t \in (-\infty, 0]$.

Proof. This proposition for the case that X is a smooth manifold has been established in [**8**] (see theorem 3.5 and proposition 3.3 in [**8**]). Thus we assume that X has at least one (orbifold) singularity below.

Case 1: Curvature operator has zero (eigenvalue) somewhere. Then by Section 3.1, the scalar curvature is constant. So the proposition holds trivially in this case.

Case 2: X is compact with positive curvature operator. By the work of Hamilton, if we continue to evolve the metric, the metric will become rounder and rounder. On the other hand, by our κ-noncollapsing assumption, and the compactness theorem of [**17**], we can extract a convergent subsequence to get a limit which is compact and round. From this, we know that the orbifold X is diffeomorphic to a compact orbifold with positive constant sectional curvature and with at most isolated singularities. By [**27**] or theorem 13.3.10 in [**24**], there is a finite subgroup $G \subset O(5)$ of isometries of $\mathbb{S}^4$ such that $\mathbb{S}^4/G$ is diffeomorphic to X. Let $\pi : \mathbb{S}^4 \to X$ be the naturally defined smooth map, and $\tilde{g}(\cdot, t) = \pi^* g(\cdot, t)$ be the induced G invariant solution of Ricci flow on smooth manifold $\mathbb{S}^4$. Now we check the κ-noncollapsing of $\tilde{g}$ on $\mathbb{S}^4$. Fix a positive number r, and suppose $\tilde{R}(\cdot, t) \le r^{-2}$ on $\tilde{B}_{t_0}(\tilde{x}, r)$ for all $t \in [t_0 - r^2, t_0]$. Let $x = \pi(\tilde{x}) \in X$, γ be a geodesic in X of length $\le r$ with $x = \gamma(0)$. Then γ has a lift to a geodesic $\tilde{\gamma}$ (which may not be unique) in $\mathbb{S}^4$ with $\tilde{\gamma}(0) = \tilde{x}$, and $L(\tilde{\gamma}) = L(\gamma)$. This implies that $\pi : \tilde{B}_{t_0}(\tilde{x}, r) \to B_{t_0}(x, r)$ is surjective. As a result, the curvature of X is also bounded by r^{-2} on $B_{t_0}(x, r) \times [t_0 - r^2, t_0]$. Hence $vol_{t_0}(\tilde{B}(\tilde{x}, r)) \ge vol_{t_0}(B(x, r)) \ge \kappa r^4$ by the κ-noncollapsing assumption on X. Thus, we have showed that the solution $\tilde{g}$ is an ancient κ-solution on the smooth manifold $\mathbb{S}^4$. By [**8**] (theorem 3.5 and proposition 3.3 in [**8**]), $\tilde{g}(\cdot, t)$ is κ_0-noncollapsed for some universal constant κ_0. Furthermore, there is a universal positive function ω such that

$$\tilde{R}(\tilde{x},t) \leq \tilde{R}(\tilde{y},t)\omega(\tilde{R}(\tilde{y},t)\tilde{d}_t(\tilde{x},\tilde{y})^2) \tag{3.5}$$

for the scalar curvature of the induced Ricci flow $\tilde{g}(\cdot,t)$ at any two points $\tilde{x},\tilde{y} \in \mathbb{S}^4, t \in (0,\infty]$. Now, for any pair of points $x, y \in X$, we take a minimal geodesic γ connecting x, y in X. Then γ can be lifted to a geodesic $\tilde{\gamma} \subset \mathbb{S}^4$ connecting two points $\tilde{x} \in \pi^{-1}(x), \tilde{y} = \pi^{-1}(y)$. Since we have $\tilde{d}(\tilde{x},\tilde{y}) \leq L(\tilde{\gamma}) = d(x,y)$, $R(x,t) = R(\tilde{x},t)$, and $R(y,t) = \tilde{R}(\tilde{y},t)$, by (3.5), we get

$$R(x,t) \leq R(y,t)\omega(R(y,t)d_t(x,y)^2)$$

as desired.

Case 3: X is noncompact with positive curvature operator. First, we define a Busemann function φ at time -1 in the following way:

$$\varphi(x) = \sup_{\gamma} \lim_{s\to+\infty}(s - d_{-1}(x,\gamma(s)))$$

where the supremum is taken over all geodesic rays γ originating from some fixed point on X. It is well-known that φ is convex (with respect to the metric at time -1), proper, and has Lipschitz constant ≤ 1. Deforming φ by the heat equation (where the Laplacian is taken with respect to the metric at time t),

$$\frac{\partial u}{\partial t} = \triangle u$$

with $u|_{t=-1} = \varphi$. By a straightforward computation, we have

$$\frac{\partial}{\partial t}u_{ij} = \triangle u_{ij} + 2R_{ikjl}u_{kl} - R_{ik}u_{kj} - R_{jk}u_{ki}$$

where $u_{ij} = \nabla^2_{ij}u$ is the Hessian of u. Since the curvature operator is positive, by maximum principle, we know that the condition $\nabla^2 u \geq 0$ is preserved. Moreover, we have $\nabla^2 u > 0$ at $t = 0$ because of the following. The kernel of $\nabla^2 u$ is a parallel distribution by the strong maximum principle of Hamilton [**13**]. If the kernel is nontrivial, then either the space splits into a product $\mathbb{R}\times\Sigma$ locally or it admits a linear function ($\nabla^2 u = 0$). But both cases contradict the strict positivity of the curvature operator.

We fix the time at $t = 0$. Note that u is still a proper function. Thus, by strict convexity of u, we know that u has a unique critical point, which is the minimal point. Now, if P is an (isolated) singular point on X, we claim that P must be a critical point of u. Indeed, let $\pi : \tilde{U} \to U$, $U = \tilde{U}/\Gamma$ be the local uniformization near P. Then $\tilde{u} = u \circ \pi$ is Γ invariant, and we have $d\gamma(\nabla\tilde{u})(P) = \nabla\tilde{u}(P)$ for any $\gamma \in \Gamma$. Since Γ has an isolated fixed point at P, we have $\sum_{\gamma\in\Gamma} d\gamma(\nabla\tilde{u})(P) = 0$ and

$\nabla\tilde{u}(P) = 0$ consequently. As a result, X has exactly one singularity (at P).

Next, we let $\xi = \frac{\nabla u}{|\nabla u|}$ be the vector field which is singular only at P. Consider the map $\Phi : C_p X = Cone(\mathbb{S}^3/\Gamma) \to X$ defined by

$$\Phi(v, s) = \alpha_v(s)$$

where $\alpha_v(s)$ is the integral curve of ξ with $\alpha_v(0) = P$ and $\alpha_v'(0) = v$. Clearly, Φ is a global orbifold diffeomorphism. We define $\tilde{\Phi} : \mathbb{R}^4 = Cone(\mathbb{S}^3) \to X$ by

$$\tilde{\Phi} = \Phi \cdot \pi$$

where $\pi : Cone(\mathbb{S}^3) \to Cone(\mathbb{S}^3/\Gamma)$ is the natural projection map. Define

$$\tilde{g}(\cdot, t) = \tilde{\Phi}^* g(\cdot, t).$$

Then $\tilde{g}(\cdot, t)$ is a smooth complete ancient κ solution on the smooth manifold $\mathbb{R}^4$ with positive curvature operator and restricted isotropic pinching condition. Moreover by [**8**] again, $\tilde{g}(\cdot, t)$ is κ_0-noncollapsed for some universal constant κ_0, and the same argument as in Case 2 completes the proof. q.e.d.

From the proof of the proposition, we get

Corollary 3.8. *Let g_t be an ancient κ-orbifold solution on a complete* ***noncompact*** *4-orbifold X with* ***positive*** *curvature operator and nonempty isolated singularities; then there is exactly one singularity O and there is a finite group of isometries $\Gamma \subset O(4)$ of the standard $\mathbb{R}^4$, such that O is the only fixed point for any element of Γ, and X is diffeomorphic to $\mathbb{R}^4/\Gamma$ as orbifold.*

Now we can state the orbifold analogue of Theorem 3.5 in the noncompact case.

Theorem 3.9. *For every $\varepsilon > 0$, one can find positive constants $C_1 = C_1(\varepsilon)$, $C_2 = C_2(\varepsilon)$ such that for each point (x, t) in every complete* ***noncompact*** *four-dimensional ancient κ-**orbifold** solution with* ***positive*** *curvature operator and nonempty isolated singularities, there is a positive number r with $\frac{1}{C_1}(R(x,t))^{-\frac{1}{2}} < r < C_1(R(x,t))^{-\frac{1}{2}}$, so that for some open subset B with $B_t(x, r) \subset B \subset B_t(x, 2r)$, one of the following cases occurs:*

*(a) B is an **evolving** ε-**neck** around (x, t),*

*(b) B is an **evolving** ε-**cap** of Type I.*

Moreover, the scalar curvature in B at time t is between $C_2^{-1}R(x, t)$ and $C_2 R(x, t)$.

Proof. We denote the unique orbifold singularity by O. By Corollary 3.8, X is diffeomorphic to $\tilde{X}/\Gamma$, where $\tilde{X}$ is diffeomorphic to $\mathbb{R}^4$ and $\Gamma \subset O(4)$ fixes the origin, denoted also by O. Let $\tilde{g}$ be the pulled back solution on $\tilde{X}$, which is a Γ-invariant solution on $\tilde{X}$. Note that the solution $\tilde{g}$ is also κ-noncollapsed and therefore κ_0-noncollapsed for some universal $\kappa_0 > 0$ by theorem 3.5 in [**8**]. Fix time $t = 0$. Now by the proof of theorem 3.8 in [**8**], there is a point $x_0 \in \tilde{X}$, such that for any given small $\eta > 0$, there is a constant $D(\eta) > 0$ depending only on η such that any $(x, 0)$ satisfying $R(x_0, 0)d_0(x, x_0)^2 \geq D(\eta)$ admits an evolving η-neck around it. We scale the solution so that $R(x_0, 0) = 1$. In the following, we describe the canonical parametrization of necks which was given by Hamilton in section C of [**14**]. We will use Hamilton's canonical parametrization to parametrize all the points outside a ball of radius $\sqrt{D(\eta)} + 1$ centered at x_0 by a canonical diffeomorphism Φ from $\mathbb{S}^3 \times \mathbb{I}$, where $\mathbb{I} \subset \mathbb{R}$ is an interval.

For any $z \in \tilde{X}$ with $d_0(z, x_0)^2 \geq D(\eta)$, there is a unique constant mean curvature hypersurface $S_z \in \tilde{X}$ passing through z. Thus, such hypersurfaces foliate $\tilde{X}$ outside the ball of radius $\sqrt{D(\eta)}+1$ centered at x_0. Each $(S_z, \tilde{g})$ can be parametrized by a harmonic diffeomorphism from the standard sphere $(\mathbb{S}^3, \bar{g})$, which is unique up to precomposing with a rotation of $(\mathbb{S}^3, \bar{g})$ since the (induced) metrics $\tilde{g}$ and the standard metric $\bar{g}$ are very close. Now, to construct Hamilton's canonical parametrization, we first choose the coordinate function s of the $\mathbb{R}$ factor as follows: if $r(s)$ is defined by

$$vol(\mathbb{S}^3 \times \{s\}, \tilde{g}) = vol(\mathbb{S}^3, \bar{g})r(s)^3,$$

then we require the function s to satisfy

$$vol(\mathbb{S}^3 \times [s_1, s_2], \tilde{g}) = vol(\mathbb{S}^3, \bar{g}) \int_{s_1}^{s_2} r(s)^3 ds.$$

Next, we require the above harmonic diffeomorphisms to satisfy the following condition. Let W be the unit vector field which is $\tilde{g}$ orthonormal to the sphere $\mathbb{S}^3 \times \{s\}$; then for any infinitesimal rotation $\bar{V}$ on $(\mathbb{S}^3 \times \{s\}, \bar{g})$, we have

$$\int_{S^3 \times \{s\}} \bar{g}(\bar{V}, W)\, \mathrm{dvol}_{\bar{g}} = 0. \tag{3.6}$$

The above parametrization $\Phi : \mathbb{S}^3 \times (A, B) \to \tilde{X}$ can be extended on one end so that it covers all points outside a ball of radius $\sqrt{D(\eta)} + 1$ centered at x_0. Without loss of generality, we assume that as $s \to B$, the points on the manifold $\tilde{X}$ diverge to infinity. Note that it was shown by Hamilton that this parametrization of a neck is unique up to precomposing with an isometry on $\mathbb{S}^3 \times (A, B)$.

Let $\hat{g} = \Phi^*\tilde{g}$. Take any $\gamma \in \Gamma$ and let $\hat{\gamma}$ be the induced diffeomorphism on $\mathbb{S}^3 \times (A, B)$ via Φ, i.e., $\hat{\gamma} = \Phi^{-1}\gamma\Phi$. Clearly, $\Phi\hat{\gamma}$ still satisfies the conditions in the above definition of Hamilton's canonical parametrization. Hence, $\hat{\gamma}$ acts as an isometry on $\mathbb{S}^3 \times (A, B)$. In other words, the group $\hat{\Gamma} = \Phi^{-1}\Gamma\Phi$ acts isometrically on $\mathbb{S}^3 \times (A, B)$ equipped with the standard metric. We claim that the $\mathbb{R}$ factors of $\hat{\Gamma}$ do not contain translation. Indeed, suppose there is a $\hat{\gamma} \in \hat{\Gamma}$ such that its $\mathbb{R}$ factor is a translation $s \mapsto s + L$ with $L > 0$ (if $L < 0$, we consider $\hat{\gamma}^{-1}$); then a point in a finite region will be mapped to points near infinity by $\hat{\gamma}^m$ for large m. Since $\hat{\gamma}^m$ are isometries, and $\tilde{X}$ splits off a line at infinity, we conclude that the curvature operator is not strictly positive in a finite region. This is a contradiction with our assumption. By similar argument, we see that the $\mathbb{R}$ factors of $\hat{\Gamma}$ also contain no reflection. As a result, $\hat{\Gamma}$ only acts on the $\mathbb{S}^3$ factor in $\mathbb{S}^3 \times (A, B)$. This implies that the above neck parametrization Φ on $\tilde{X}$ descends to a neck parametrization on X, $\phi : \mathbb{S}^3/\Gamma \times (A, B) \to X$.

We can also show that the point O has distance $\leq \sqrt{D(\eta)}+1$ from x_0. Indeed, if $d_0(x_0, O) \geq \sqrt{D(\eta)}+1$, then O is covered by the parametrization $\Phi : \mathbb{S}^3 \times (A, B) \to \tilde{X}$. Let $O = \Phi(\bar{x}, \bar{s})$, $\bar{x} \in \mathbb{S}^3$, $\bar{s} \in (A, B)$. Since the group $\hat{\Gamma}$ only acts on the $\mathbb{S}^3$ factor, we conclude that $\hat{\Gamma}$ fixes every point on $\{\bar{x}\} \times (A, B)$. This is a contradiction.

Note that as $\eta \to 0$, after normalization, the metric $\hat{g}$ will converge in C^∞_{loc} topology to the standard one. This implies that, for any $\varepsilon > 0$, there is an $\tilde{\varepsilon} > 0$ such that if $\eta < \tilde{\varepsilon}$ then for any point $P \in \mathbb{S}^3/\Gamma \times (A, B)$, the (descended) metric $\hat{g}$ on $\mathbb{S}^3/\Gamma \times (A, B)$ around P is ε close to the standard one after scaling with the factor $\hat{R}(P)$.

Now we are ready to prove the theorem. For the given $\varepsilon > 0$ in the theorem, there is an $\tilde{\varepsilon} > 0$ defined as above. For any point $x \in X$ with $d_0(O, x) \geq 2\sqrt{D(\frac{1}{2}\tilde{\varepsilon})} + 1$, a suitable portion $\mathbb{S}^3/\Gamma \times (A', B')$ of $\mathbb{S}^3/\Gamma \times (A, B)$ in the above parametrization will give an ε-neck neighborhood of x. Let $\tilde{x} \in \tilde{X}$ satisfy $d_0(\tilde{x}, O) = 10\sqrt{D(\frac{1}{2}\tilde{\varepsilon})}$, and denote the constant mean curvature hypersurface passing through $\tilde{x}$ by Σ. By theorem G1.1 in [**14**], Σ bounds an open set Ω, which is diffeomorphic to a ball $\mathbb{B}^4$, in $\tilde{X}$. Ω is Γ-invariant, and Ω/Γ contains an ε-neck as its end. The curvature estimate on Ω/Γ follows from the elliptic type curvature estimate in Proposition 3.7. Thus we only need to show that Ω/Γ is diffeomorphic to the orbifold cap C_Γ of Type I.

Let $\varphi : X \to \mathbb{R}$ be the Busemann function at time $t = 0$ on X constructed around the singular point O. Let u_δ be a family of strictly convex smooth perturbation of φ as in Proposition 3.7, such that $u_0 = \varphi$. By considering the integral curves of u_δ as in Proposition 3.7, one can show that the sublevel sets $u_\delta^{-1}(-\infty, c]$ of u_δ are diffeomorphic to C_Γ for large c.

On the other hand, we let f be the $\mathbb{R}$ coordinate of the parametrization $\phi : \mathbb{S}^3/\Gamma \times (A, B) \to X$. By a geometric argument, one can show easily that $\nabla\varphi$ is almost parallel (with error controlled by ε) to ∇f, and so does ∇u_δ for small δ. By blending the function u_δ and a multiple of f by a bump function, we get a function ψ, whose gradient curves give a diffeomorphism between $f^{-1}(-\infty, c']$ and $u_\delta^{-1}(-\infty, c]$ for some large c and c' by Morse theory. In particular, this shows that Ω/Γ is diffeomorphic to C_Γ. The proof of the theorem is completed. q.e.d.

3.3. Summary. We can now collect all the results obtained in this section to give the following canonical neighborhood decomposition theorem for an ancient κ-orbifold solution.

Theorem 3.10. *For every $\varepsilon > 0$, one can find positive constants $C_1 = C_1(\varepsilon)$, $C_2 = C_2(\varepsilon)$ such that, for every four-dimensional ancient κ-**orbifold** solution (X, g_t) and for each point (x,t), there is a number r with $\frac{1}{C_1}(R(x,t))^{-\frac{1}{2}} < r < C_1(R(x,t))^{-\frac{1}{2}}$ so that for some open subset B with $B_t(x,r) \subset B \subset B_t(x,2r)$, we have one of the following cases:*

*(a) B is an **evolving ε-neck** around (x,t),*

*(b) B is an **evolving ε-cap**,*

(c) X is diffeomorphic to a closed spherical orbifold $\mathbb{S}^4/\Gamma$ with at most isolated singularities.

Moreover, the scalar curvature of B in case (a) or (b) at time t lies between $C_2^{-1}R(x,t)$ and $C_2R(x,t)$.

Proof. By Theorem 3.4 and Theorem 3.9, we only need to consider the case when X is compact with positive curvature operator. In this case, we continue to evolve the metric by Ricci flow. Since the scalar curvature is strictly positive, the solution will blow up in finite time. By using the κ-noncollapsing of the solution as in [**22**] and the compactness theorem in [**17**], we can scale the solution in space time around a sequence of points and extract a convergent subsequence. Moreover, the limit is still an orbifold with at most isolated singularities by [**17**]. By the pinching estimate of Hamilton [**13**], the Riemannian metric in the limit orbifold has constant positive sectional curvature. So it is a global quotient of the four-sphere. q.e.d.

4. Surgical solutions

The complete description of the canonical neighborhood structure of an ancient κ-orbifold solution given in the last section will allow us to perform surgeries to the Ricci flow solution of a compact four-orbifold with positive isotropic curvature and at most isolated singularities as explained in this section.

4.1. Surgery at the first singular time. Since the scalar curvature at the initial time is strictly positive, it follows from the maximum principle and the evolution equation of the scalar curvature that the curvature must blow up at some finite time $0 < T < \infty$. As the canonical neighborhood structures of ancient κ-orbifold solutions have been completely described in the last section, by combining with a technical geometric lemma (Proposition 6.1 in the appendix), we can prove a similar singularity structure theorem for the Ricci flow solution before time T as in the manifold case (see theorem 4.1 in [**8**]).

Theorem 4.1. *Given small $\varepsilon > 0$, there is $r = r(T) > 0$ depending on ε, T and the initial metric such that for any point (x_0, t_0) with $Q = R(x_0, t_0) \geq r^{-2}$, the solution in the parabolic region $\{(x,t) \in X \times [0,T) | \; d^2_{t_0}(x, x_0) < \varepsilon^{-2}Q^{-1}, t_0 - \varepsilon^{-2}Q^{-1} < t \leq t_0\}$ is, after scaling by the factor Q, ε-close (in $C^{[\varepsilon^{-1}]}$-topology) to the corresponding subset of some ancient κ-orbifold solution.*

Proof. First of all, we may assume that the orbifold is not diffeomorphic to a spherical orbifold $\mathbb{S}^4/\Gamma$; otherwise we are in case (c) of Theorem 3.10.

Then we can argue by contradiction as in the manifold case [**8**] as follows. We choose a point (x_0, t_0) almost critically violating the conclusion of the theorem. Scale the solution around (x_0, t_0) with factor $R(x_0, t_0)$ and shift the time t_0 to 0. The key point of the proof is to bound the curvature. Note that the κ-noncollapsing condition (Lemma 3.2) and the compactness theorem [**17**] still hold for κ-noncollapsed Ricci flow solutions on orbifolds with isolated singularities. By the canonical neighborhood decomposition theorem for ancient κ-orbifold solutions, i.e., Theorem 3.10, we can show that the curvature is bounded in geodesic balls centered at x_0 with bounded radius with respect to the normalized distance. The boundedness of curvature on the limit space then follows from Proposition 6.1 in the appendix. Thus, we have all the ingredients we need to mimic the same proof in the manifold case [**8**] to show that we can extract a convergent subsequence which converges to an ancient κ-orbifold solution. This is a contradiction. q.e.d.

We denote by Ω the open set of points where curvature remains bounded as $t \to T$. Denote by $\bar{g}$ the limit of g_t on Ω as $t \to T$.

Fix $0 < \delta << \varepsilon$ small. Let $\rho = \rho(T) = \delta r(T)$ and $\Omega_\rho = \{x \in X \mid \bar{R} \leq \rho^{-2}\}$ where $r(T)$ is given in Theorem 4.1. If Ω_ρ is empty, then by Theorem 4.1 and Theorem 3.10, X is either diffeomorphic to a spherical orbifold $\mathbb{S}^4/\Gamma$ with at most isolated singularities, or X is covered by ε-necks and ε-caps. In the latter case, if there is no cap, then X is covered by ε-necks and is then diffeomorphic to $\mathbb{S}^3/\Gamma \times \mathbb{S}^1$ or $\mathbb{S}^3/\Gamma \times_f \mathbb{S}^1$; if there are caps, namely C^σ_Γ, C_Γ, $\mathbb{S}^4/(x, \pm x') \setminus \bar{\mathbb{B}}^4$, $\mathbb{B}^4$, then X is diffeomorphic to either the smooth manifolds $\mathbb{S}^4$, $\mathbb{RP}^4$, $C^\sigma_\Gamma \cup_f C^{\sigma'}_{\Gamma'}$, or one of the orbifolds $C^\sigma_\Gamma \cup_f C_{\Gamma'}$, $C_\Gamma \cup_f C_{\Gamma'}$, $\mathbb{S}^4/(x, \pm x')$, $\mathbb{S}^4/(x, \pm x')\#\mathbb{RP}^4$, $\mathbb{S}^4/(x, \pm x')\#\mathbb{S}^4/(x, \pm x')$. So we conclude that if Ω_ρ is empty, then X is diffeomorphic to a spherical orbifold $\mathbb{S}^4/\Gamma$ with at most isolated singularities or a connected sum of two spherical orbifolds $\mathbb{S}^4/\Gamma_1$ and $\mathbb{S}^4/\Gamma_2$ with at most isolated singularities.

On the other hand, if the solution of the Ricci flow achieves positive curvature operator everywhere at some time, it follows from the proof of Theorem 3.10 that X is diffeomorphic to a spherical orbifold with at most isolated singularities.

In the following, we will consider Ricci flow with surgery. The solution after the surgeries may be decomposed into several connected components. We will stop the Ricci flow on a component if either Ω_ρ is empty or the solution achieves positive curvature operator everywhere on that component. We will say that the (possibly disconnected) solution of the Ricci flow (with surgery) becomes extinct if either one of the above two cases occurs for every component of the solution. Then, we can recover the topology of our initial four-orbifold as an orbifold connected sum in Section 4.3. The purpose of the remainder of this and the next subsections is thus to construct a Ricci flow with surgery on X which becomes extinct.

We may then assume that $\Omega_\rho \neq \phi$ and so any point outside Ω_ρ has an ε-neck or ε-cap neighborhood. We are interested in those ε-horn H (consisting of ε-necks) where one of the ends is in Ω_ρ while the curvature becomes unbounded near the other end. We will perform surgeries on these horns. First of all, we need the existence of finer (than ε) necks in the ε-horn H. The reason for finding finer necks to perform surgeries is to control quantitatively the accumulation of errors caused by the surgeries.

Proposition 4.2. *For the arbitrarily given small $0 < \delta << \varepsilon$, there is a $0 < h < \delta\rho$ depending only on δ and ε, and independent of the noncollapsing parameter κ such that if H is an ε-horn whose finite end is in Ω_ρ and x is a point on H with scalar curvature $\geq h^{-2}$, then there is a δ-neck around x.*

The argument is a bit different from lemma 5.2 in [**8**]. The reason is that the canonical neighborhoods in [**8**] are universally noncollapsed, but in the present situation we do not know it beforehand.

Proof. There is a fixed point free finite group of isometries $\Gamma \in O(4)$ so that we can apply Hamilton's parametrization to parametrize the whole H, $\Phi_\Gamma : (\mathbb{S}^3/\Gamma) \times (A, B) \to H$, where Φ_Γ is a diffeomorphism. Denote by $\Phi : \mathbb{S}^3 \times (A, B) \to H$ the natural lifting of Φ_Γ. Without loss of generality, we assume that $\Phi(\mathbb{S}^3 \times \{s\})$ has nonempty intersection with Ω_ρ as $s \to A$, and the curvature becomes unbounded as $s \to B$. To prove the proposition, we argue by contradiction. Suppose $x_j \in H$ is a sequence of points with $\bar{R}(x_j) \geq h_j^{-2} \to \infty$ but x_j has no δ-neck neighborhood. We pull back the solution to $\mathbb{S}^3 \times (A, B)$ and pick a lifting $\tilde{x}_j$ of x_j for each j. Then we scale the solution by a factor $\bar{R}(x_j)$ around $\tilde{x}_j$ and shift the time T to 0. Note that the rescaled solution on $\mathbb{S}^3 \times (A, B)$ is smooth (without orbifold singularities) and uniformly noncollapsed. We can apply the same argument of step 2 in theorem 4.1 in [**8**] to show that the curvature is bounded in any fixed finite ball around the point $\tilde{x}_j$ for the rescaled solution; otherwise we get a piece of non-flat nonnegatively curved metric cone as a blow up limit, which contradicts Hamilton's strong maximum principle (see [**13**]). This implies that the two ends of $\mathbb{S}^3 \times (A, B)$ are very far away from the points $\tilde{x}_j$ (in the normalized distance). We can then extract (around $(\tilde{x}_j, T)$) a convergent subsequence so that the limit splits off a line by the Toponogov splitting theorem. By the restricted isotropic curvature pinching estimate (3.4), the limit is the standard $\mathbb{S}^3 \times \mathbb{R}$. Since the solution is Γ invariant, it descends to H and gives a δ-neck around x_j for j large enough. This is a contradiction. q.e.d.

Now we give a brief description of the original surgery of Hamilton along a usual δ-neck (N, g) with scalar curvature h^{-2} at the center $\bar{x}$. We assume the parametrization $\Phi : \mathbb{S}^3 \times (A, B) \to N$ satisfies that the metric $h^{-2}\Phi^* g$ near $\Phi^{-1}(\bar{x}) \in \mathbb{S}^3 \times (A, B)$ is δ' close to the standard metric on $\mathbb{S}^3 \times \mathbb{R}$ normalized with scalar curvature 1. Here δ' is a constant depending on δ which satisfies $\lim_{\delta \to 0} \delta' = 0$.

We assume that the center $\bar{x}$ has $\mathbb{R}$ coordinate $s = 0$. The surgery is to cut open the neck at the sphere $\{s = 0\}$ and then glue back two caps $(\mathbb{B}^4, \tilde{g})$ on both sides of the opening. We describe the metric $\tilde{g}$ on the left hand cap as follows; here the cap $\mathbb{B}^4$ is identified with $\mathbb{S}^3 \times (0, 4]$ with the end $\mathbb{S}^3 \times \{4\}$ collapsed to a point:

$$\tilde{g} = \begin{cases} g, \quad s = 0, \\ e^{-2f} g, \quad s \in [0, 2], \\ \varphi e^{-2f} g + (1 - \varphi) e^{-2f} h^2 g_0, \quad s \in [2, 3], \\ h^2 e^{-2f} g_0, \quad s \in [3, 4], \end{cases}$$

where f is some suitably chosen smooth nondecreasing convex function, φ is a smooth bump function with $\varphi = 1$ for $s \leq 2$ and $\varphi = 0$ for $s \geq 3$, and g_0 is the standard metric on $\mathbb{S}^3 \times \mathbb{R}$. The description of the metric on the right hand cap is similar.

To apply the above construction to our situation, we observe that the parametrization of our horn H, $\Phi_\Gamma : (\mathbb{S}^3/\Gamma) \times (A, B) \to H$, comes from a Γ-equivariant parametrization $\mathbb{S}^3 \times (A, B) \to N$ of a usual neck N. The above construction then descends to our situation, which gives us a surgery by gluing back two orbifold caps C_Γ after we cut open the horn H. We call the above procedure a **δ-cutoff surgery**.

Now the proof that the pinching estimates of Hamilton are preserved after surgeries can be carried through without changing a word.

Lemma 4.3. *(Hamilton* [**14**] *D3.1, Preservation of the pinching condition)*

*There is a universal positive constant δ_0 satisfying the following condition: For any $\tilde{T}$, there is a constant $h_0 > 0$ depending on the initial metric and $\tilde{T}$ such that if we perform the above δ-cutoff surgery at a δ-neck of radius h at time $T \leq \tilde{T}$ with $\delta < \delta_0$ and $h^{-2} \geq h_0^{-2}$, then after the surgery, the pinching condition (*3.2*) still holds at all points at time T.*

4.2. Ricci flow with surgery which becomes extinct. We can define the notion of Ricci flow with surgery in the same way as in [**8**] by replacing manifolds by orbifolds with at most isolated singularities. As in [**8**], the solutions to the Ricci flow with surgery in this paper are obtained by performing concrete surgeries. We cut open a neck in a horn and glue back two caps. As a result, all the connected components of the solution after surgeries are still closed orbifolds with at most isolated singularities. Note that each cross section of the neck in a horn is diffeomorphic to $\mathbb{S}^3/\Gamma$. If Γ is trivial, we glue back a usual cap B^4. If Γ is nontrivial, we glue back an orbifold cap C_Γ which will produce a new orbifold singularity coming from the tip of the cap.

As we said in the last subsection, our goal is to produce a Ricci flow solution with surgery on our initial four-orbifold which becomes extinct. To achieve this, we have to make sure that the following two properties of the solution are preserved after suitable surgeries are performed. Note that we have already shown these properties for the solution of the Ricci flow before the first singular time.

Pinching condition: *There exist positive constants $\rho, \Lambda, P < +\infty$ such that there hold*

$$a_1 + \rho > 0 \ \textit{and} \ c_1 + \rho > 0,$$

$$\max\{a_3, b_3, c_3\} \leq \Lambda(a_1 + \rho) \ \textit{and} \ \max\{a_3, b_3, c_3\} \leq \Lambda(c_1 + \rho),$$

and

$$\frac{b_3}{\sqrt{(a_1+\rho)(c_1+\rho)}} \leq 1 + \frac{\Lambda e^{Pt}}{\max\{\log\sqrt{(a_1+\rho)(c_1+\rho)}, 2\}},$$

everywhere.

Canonical neighborhood condition (with accuracy ε): *Let g_t be a solution to the Ricci flow with surgery starting with (*3.1*). For the given $\varepsilon > 0$, there exist two constants $C_1(\varepsilon)$, $C_2(\varepsilon)$ and a non-increasing positive function $r(t)$ on $[0, +\infty)$ with the following properties. For every point (x, t) where the scalar curvature $R(x, t)$ is at least $r^{-2}(t)$, there is an open set B, $B_t(x, \sigma) \subset B \subset B_t(x, \sigma)$, with $0 < \sigma < C_1(\varepsilon)R(x, t)^{-\frac{1}{2}}$ such that one of the following cases occurs:*

(a) B is a strong ε-neck,

(b) B is an ε-cap,

(c) at time t, X is diffeomorphic to a closed spherical orbifold $\mathbb{S}^4/\Gamma$ with at most isolated singularities.

Moreover, for (a) and (b), the scalar curvature in B at time t is between $C_2^{-1}R(x,t)$ and $C_2R(x,t)$, and satisfies the gradient estimate

$$|\nabla R| < \eta R^{\frac{3}{2}} \text{ and } |\frac{\partial R}{\partial t}| < \eta R^2,$$

for some universal constant η.

Here, we give the precise definitions of ε-cap and strong ε-neck. We say that an open set B on an orbifold is an **ε-neck** if there is a diffeomorphism $\varphi : \mathbb{I} \times (\mathbb{S}^3/\Gamma) \to B$ such that the pulled back metric $(\varphi)^*g$, after scaling with suitable factor, is ε-close (in $C^{[\varepsilon^{-1}]}$ topology) to the standard metric on $\mathbb{I} \times (\mathbb{S}^3/\Gamma)$ with scalar curvature 1 and $\mathbb{I} = (-\varepsilon^{-1}, \varepsilon^{-1})$. An open set B on an orbifold is an **ε-cap** if B is diffeomorphic to either a smooth cap $\mathbb{B}^4$, C_Γ^σ, or an orbifold cap of Type I or Type II, i.e., C_Γ or $\mathbb{S}^4/(x, \pm x')\backslash\bar{\mathbb{B}}^4$, and if the region around the end of B is an ε-neck. A **strong ε-neck** B at (x,t) is the time slice at time t of the solution of the Ricci flow in the parabolic region $\{(x',t')|x' \in B, t' \in [t - R(x,t)^{-1}, t]\}$, which also has the property that there is a diffeomorphism $\varphi : \mathbb{I} \times (\mathbb{S}^3/\Gamma) \to B$ such that the pulled back solution $(\varphi)^*g(\cdot,\cdot)$, after scaling with the factor $R(x,t)$ and shifting the time t to 0, is ε-close (in $C^{[\varepsilon^{-1}]}$ topology) to the subset $(\mathbb{I}\times\mathbb{S}^3/\Gamma)\times[-1,0]$ of the evolving round cylinder $\mathbb{R} \times (\mathbb{S}^3/\Gamma)$ having scalar curvature one and length $2\varepsilon^{-1}$ for $\mathbb{I}$ at time zero.

Besides the two conditions above, the κ-noncollapsing condition defined in the last section is also crucial. For convenience, we first recall its definition here. Let κ be a positive constant. We say that the solution is **κ-noncollapsed on the scales less than** ρ if it satisfies the following property: if

$$|Rm(\cdot,\cdot)| \le r^{-2}$$

on $P(x_0,t_0,r,-r^2) = \{(x',t') \mid x' \in B_{t'}(x_0,r), t' \in [t_0 - r^2, t_0]\}$ and $r < \rho$, then we have

$$Vol_{t_0}(B_{t_0}(x_0,r)) \ge \kappa r^4.$$

Since we are dealing with a solution with surgeries, the condition $|Rm(x,t)| \le r^{-2}$ is only imposed on the region of the parabolic neighborhood $P(x_0,t_0,r,-r^2)$ where the solution is defined.

The importance of the κ-noncollapsing condition is (at least) twofold. First of all, the κ-noncollapsing condition allows us to take limits of surgical orbifold solutions of the Ricci flow; this is used to verify the canonical neighborhood condition after surgeries. Secondly, in the end, the κ-noncollapsing condition allows us to show that the number of surgeries performed is finite before the solution becomes extinct.

To justify the preservation of the above conditions of the solution of the Ricci flow of our initial four-orbifold after surgeries, we must perform our surgeries carefully. For this, we need the following extension of the result of Proposition 4.2 which allows us to find finer necks to perform surgeries for surgical orbifold solutions.

Proposition 4.4. *Suppose that we have a solution to the Ricci flow with surgery on* $(0, T)$ *satisfying the a priori conditions as above and the solution becomes singular as* $t \to T$. *Then, for the* ε *given in the canonical neighborhood condition and for an arbitrarily given small* δ *with* $0 < \delta << \varepsilon$, *there is a constant* h *with* $0 < h < \delta\rho(T) = \delta^2 r(T)$ *(*$r(T)$ *as given in the canonical neighborhood condition) and depending only on* δ, ε, *and* $r(T)$ *such that the following holds: If, at the time* T, *a point* x *on an* ε*-horn* H *whose finite end is in* $\Omega_{\rho(T)}$ *has curvature* $\geq h^{-2}$, *then there is a* δ*-neck around it.*

Proof. We observe that the canonical neighborhoods of the points in the ε-horn H which are far from the end are all strong ε-necks. Thus the solution around any point $\bar{x}$ on H with $R(\bar{x}, T) \geq h^{-2}$ has existed for a previous time interval $(T - R(\bar{x}, T)^{-1}, T)$. Suppose the proposition is not true. We use Hamilton's parametrization as in Proposition 4.2 to pull back the solution to $\mathbb{S}^3 \times (A, B)$. By the same argument there, we can extract a convergent subsequence from the parabolic scalings around suitable points $\bar{x}$ with $R(\bar{x}, T) \geq h^{-2} \to \infty$. The limit solution is just the standard solution on $\mathbb{S}^3 \times \mathbb{R}$ which exists at least on the time interval $(-1, 0]$ after shifting the time to 0. Moreover, the solution on all points (on the original space) at normalized time $-1 + \frac{1}{100}$ still has strong ε-neck neighborhoods and the scalar curvature is ≤ 1 as $h^{-1} \to \infty$. So we can actually extract a subsequence so that the limit solution is defined at least on $[-2, 0]$. Since this solution on $\mathbb{S}^3 \times (A, B)$ is Γ-invariant, this gives a δ-neck, as h^{-1} is very large. This is a contradiction. q.e.d.

We can now discuss the preservation of the pinching condition, the canonical neighborhood condition with accuracy ε, and the uniform κ-noncollapsing condition of the solution of our Ricci flow after suitable surgeries are performed. First of all, the condition for the preservation of the pinching condition after surgeries is already given in Lemma 4.3. Our proof of the justification of the other two conditions after suitable surgeries will follow the same strategy as in [**8**]. In Lemma 4.5 below, we will prove the uniform κ-noncollapsing condition under the assumption that the canonical neighborhood condition holds with accuracy ε for some parameter $\tilde{r}$ which may be very small. This will then be used to justify the canonical neighborhood condition itself (see Theorem 4.6).

The key point of Lemma 4.5 is that even if we perform δ-cutoff surgeries with sufficiently fine δ, which depends on $\tilde{r}$, the noncollapsing constant κ we obtained is uniform and independent of $\tilde{r}$. In lemma 5.5 of [**8**], the same estimate was deduced when the initial space is a compact four-manifold with no essential space form (and with positive isotropy curvature). The fact that the canonical neighborhoods in [**8**] are not collapsed played a crucial role in the proof there. In the current context, a priori, the canonical neighborhoods may be sufficiently collapsed. Therefore, we need a different argument. Our idea is the following. Suppose we want to check the uniform κ-noncollapsing at a space time point (x_0, t_0). When the scale at (x_0, t_0) is not too small compared with the canonical neighborhood parameter $\tilde{r}$, we observe that if a surgery occurred near the time t_0, then it is performed far away from x_0. The argument of Perelman's Jacobian comparison theorem can be modified to apply as in the previous case in [**8**] to show the uniform κ-noncollapsing. When the scale at (x_0, t_0) is small compared with the canonical neighborhood parameter $\tilde{r}$, we first show that the space has a canonical geometric neck near x_0 and it can be extended to form a long geometric tube so that the other end of the tube has a neck of big scale. After showing that the neck with big scale is uniform κ-noncollapsing, we get a control on the order of the fundamental group of the neck which shows that the original neck with small scale is also uniform κ-noncollapsing.

Lemma 4.5. *Given a compact four-orbifold M with positive isotropic curvature and with at most isolated singularities, a small $\varepsilon > 0$, and a positive integer l, recall that δ_0 is the universal constant appearing in Lemma 4.3. Suppose that we have already constructed the sequences $\tilde{\delta}_j > 0$, $\tilde{r}_j > 0$, $\kappa_j > 0$, for $0 \le j \le l-1$, with the following properties: for any solution of the Ricci flow with surgery on $[0,T)$ with $T \in [l\varepsilon^2, (l+1)\varepsilon^2]$ and with the four-orbifold M as initial data, obtained by $\delta(t)$-cutoff surgeries at different times t's where $\delta(t)$ satisfies $\delta(t) \le \delta_0$ and $\delta(t) \le \tilde{\delta}_j$ for $t \in [j\varepsilon^2, (j+1)\varepsilon^2]$ for all $0 \le j \le l-1$, we have*

(i) the pinching condition holds on $[0,T)$;

(ii) the canonical neighborhood condition (with accuracy ε) holds with parameter $\tilde{r}_j > 0$ on each $[j\varepsilon^2, (j+1)\varepsilon^2]$ for all $0 \le j \le l-1$;

(iii) the κ_j-noncollapsing condition for all scales less than ε holds on $[j\varepsilon^2, (j+1)\varepsilon^2]$ for all $0 \le j \le l-1$.

Then there exists a $\kappa_l = \kappa_l(\kappa_{l-1}, \tilde{r}_{l-1}, \varepsilon) > 0$, and for any $\tilde{r} > 0$, there exists $\tilde{\delta}_l = \tilde{\delta}_l(\kappa_{l-1}, \tilde{r}, \varepsilon) > 0$ such that if we have a solution of the Ricci flow with $\delta(t)-$cutoff surgery on $[0, T')$ for some $T' \in [l\varepsilon^2, (l+1)\varepsilon^2]$ which satisfies the following:

(a) the canonical neighborhood condition (with accuracy ε) with parameter $\tilde{r}$ on $[l\varepsilon^2, T)$

(b) the condition that for each $t \in [l\varepsilon^2, T)$ and on each connected component of the solution, there is a point x on it such that $R(x,t) \le \tilde{r}^{-2}$;

(c) the condition that $\delta(t) \le \tilde{\delta}_j$ on $[j\varepsilon^2, (j+1)\varepsilon^2]$ for all $0 \le j \le l-1$, and $\delta(t) \le \tilde{\delta}_l$ on $[(l-1)\varepsilon^2, T')$,

then the solution is κ_l-noncollapsed on $[(l-1)\varepsilon^2, T')$ for all scales less than ε.

Proof. Take any $r_0 < \varepsilon$, $t_0 \in (l\varepsilon^2, T)$ and x_0 in the solution of the Ricci flow at time t_0. Suppose $R(\cdot,\cdot) \leq r_0^{-2}$ on $P(x_0, t_0, r_0, -r_0^2) = \{(x', t') \mid x' \in B_{t'}(x_0, r_0), t' \in [t_0 - r_0^2, t_0]\}$; we need to prove that $vol_{t_0}(B_{t_0}(x_0, r_0))/r_0^4$ can be bounded from below by some constant independent of $\tilde{r}$ (and of course r_0, t_0, x_0) provided that the surgeries are performed in sufficiently small scale, which may depend on $\tilde{r}$. As we explained above, we will divide the proof into two steps.

Step 1: In this step, we deal with the estimates when the scale r_0 is not too small compared with $\tilde{r}$. We assume that $r_0 \geq \frac{\tilde{r}}{C(\varepsilon)}$, where $C(\varepsilon)$ is some fixed constant (to be determined later in Step 2) depending only on ε. In this case, we adapt the proof of lemma 5.5 in [**8**] as follows.

Since the surgeries occur in places where the curvatures are bigger than $\delta^{-2}(t)\tilde{r}^{-2}$, which is much larger than $\tilde{r}^{-2}$, we first modify the argument of lemma 5.5 in [**8**] to show that any $\mathcal{L}$ geodesic $\gamma(\tau), \tau \in [0, \bar{\tau})$ $(\bar{\tau} \leq t_0 - (l-1)\varepsilon^2)$, starting from (x_0, t_0) with reduced length $\leq \varepsilon^{-1}$, stays far away from the places where surgeries occur. More precisely, we claim that if some $\gamma(\tau_0)$ is not far from some cap which is glued by surgery procedure at time $t = t_0 - \tau_0$, then the reduced length of γ defined by

$$\frac{1}{2\sqrt{\bar{\tau}}} \int_0^{\bar{\tau}} \sqrt{\tau}(R(\gamma(\tau), \tau) + |\dot{\gamma}(\tau)|^2) d\tau$$

is $\geq 25\varepsilon^{-1}$.

This estimate for the manifold case was established in (5.8) on page 238 of [**8**]. Let us first recall the idea of the proof there. Note that the place where a $\delta(t)$-cutoff surgery is performed is deep inside the horn (after normalization) and the parabolic region $P(x_0, t_0, r_0, -r_0^2)$ is far from it by the curvature estimates in the canonical neighborhood condition. Thus at time $t = t_0 - \tau_0$, the point $\gamma(\tau_0)$ lies deeply inside a very long tube and the segment $\gamma(\tau), \tau \in [0, \tau_0]$ tends to escape from the tube. If $\gamma(\tau)$ escapes from the very long tube within a short time, say $\leq CR(x_1, t_0 - \tau_0)^{-1}$, from τ_0, where C is some universal constant and x_1 is a point in the neck where surgery takes place, then $\int_0^{\bar{\tau}} |\dot{\gamma}(\tau)|^2 d\tau$ contributes a big quantity to the above integral since the tube is quite long. However, if $\gamma(\tau)$ stays a longer time, say $\geq CR(x_1, t_0 - \tau_0)^{-1}$, on the long tube, then $\int_0^{\bar{\tau}} R d\tau$ contributes a large quantity to the above integral, since for any $1 > \zeta > 0$, we have the estimate

$$R(x,t) \geq R(x_1, t_0 - \tau_0) \frac{Const.}{\frac{3}{2} - R(x_1, t_0 - \tau_0)(t - t_0 + \tau_0)} \tag{4.1}$$

on $\gamma \mid_{[\tau_0 - \frac{3}{2}(1-\zeta)R(x_1, t_0), \tau_0]}$, when $\delta(t)$ is small enough and $\gamma(\tau)$ stays not too far from the cap.

All the above arguments of [**8**] still work in our present orbifold case except the verification of the last statement on the estimate of the scalar curvature on the tube. In [**8**], the proof of the above estimate on the scalar curvature on the tube was given as follows: Rescale the solution with factor $R(x_1, t_0 - \tau_0)$ around $(x_0, t_0 - \tau_0)$. Since the necks in the manifold case of [**8**] are uniformly noncollapsing, we can extract a convergent limit as $\delta(t_0 - \tau_0) \to 0$. The limit, called standard solution, is rotationally symmetric. It exists exactly on the time interval $[0, \frac{3}{2})$ and has the curvature estimates $\frac{Const.}{\frac{3}{2}-s}$ at time s, which gives the estimate 4.1 when scaling back to the original solution. But in the current orbifold case, a priori, we do not know whether the necks in the canonical neighborhoods are collapsed or not.

To overcome this difficulty, we again use Hamilton's canonical parametrization as in Proposition 4.2. So we have $\Phi_\Gamma : \mathbb{S}^3/\Gamma \times (-L, L)_s \to H$, where H is a horn and we assume that the surgery is performed at the cross section corresponding to $\{s = 0\}$ (at time $t = t_0 - \tau_0$). Let $\Phi : \mathbb{S}^3 \times (-L, L) \to H$ be the natural lifting of Φ. The pullback metric on $\mathbb{S}^3 \times (-L, L)$ via Φ (after scaling) is very close to the standard cylinder. We perform a standard surgery on $\mathbb{S}^3 \times (-L, L)$ by cutting open the cylinder at $\{s = 0\}$ and gluing back two (smooth) caps. Denote the resulting space by Y. Clearly, we can require Φ to be extended and defined on Y to the space after surgery, and the pullback metric is close to the (two) standard capped infinite cylinders. Note that the gradient estimates in the canonical neighborhood condition (with accuracy ε) imply a curvature bound for the pullback solution of the Ricci flow on Y. Then as $\delta(t_0 - \tau_0) \to 0$, we can apply the uniqueness theorem [**7**] to show that the solutions on Y around points near the caps converge to a standard solution. So the above estimate 4.1 on the scalar curvature also holds in our present case.

After proving that any $\mathcal{L}$ geodesic of reduced length $< 25\varepsilon^{-1}$ does not touch the surgery region, one can apply the same argument of lemma 5.5 in [**8**] of using Perelman's Jacobian comparison to bound $vol_{t_0}(B_{t_0}(x_0, r_0))/r_0^4$ from below by constant depending only on $\varepsilon, \kappa_{l-1}, \tilde{r}_{l-1}$ (see [**8**], pages 238–241, for the details).

Step 2: In this step, we deal with the estimates on scales less than $\frac{\tilde{r}}{C(\varepsilon)}$. This case is easier in [**8**] because the space has no singularity and the canonical neighborhoods are not collapsed there. In our present orbifold case, a priori, the canonical neighborhoods in our orbifold may be sufficiently collapsed. So we need a new argument.

Clearly, we may assume $R(x', t') = r_0^{-2}$ for some point on $P(x_0, t_0, r_0, -r_0^2) = \{(x', t') \mid x' \in B_{t'}(x_0, r_0), t' \in [t_0 - r_0^2, t_0]\}$. Since $r_0 \leq \frac{\tilde{r}}{C(\varepsilon)}$, by the gradient estimates of curvature in the canonical neighborhood condition (with accuracy ε), we can choose $C(\varepsilon)$ large enough so that every point in $B_{t_0}(x_0, r_0)$ has curvature $\geq \tilde{r}^{-2}$. In particular, the point

x_0 at the time t_0 has a canonical neighborhood which is a strong ε-neck or ε-cap. For both cases, the canonical neighborhood contains an ε-neck N which is close to $(-\varepsilon^{-1}, \varepsilon^{-1}) \times (\mathbb{S}^3/\Gamma)$. Clearly, in order to get the uniform κ-noncollapsing, we only need to bound the order $|\Gamma|$ of the group Γ from above.

Now we consider one of the boundaries of N. Since the curvature is $\geq \tilde{r}^{-2}$ there, there is an ε-neck or ε-cap adjacent to N. If it is an ε-cap which is adjacent to N, we stop for this end and consider the other boundary of N. If it is an ε-neck (denoted by N') which is adjacent to N, and N' contains a point having curvature $\leq \tilde{r}^{-2}$, then we also stop and consider the other boundary of N. Otherwise, $N \cup N'$ forms a longer (topological) neck with curvature $\geq \tilde{r}^{-2}$ everywhere. We repeat the above process with N replaced by $N \cup N'$. Since there is a point $\bar{x}$ on the space such that $R(\bar{x}, t_0) \leq \tilde{r}^{-2}$ according to assumption (b) in the lemma, there must be an extension of one boundary of N such that the final adjacent neck or cap has a point with curvature $\leq \tilde{r}^{-2}$. By the gradient estimate of the canonical neighborhood condition (with accuracy ε), the curvature at the final neck or cap is $\leq C(\varepsilon)^2\tilde{r}^{-2}$ everywhere. We conclude that there is a tube T consisting of ε-necks such that T contains the initial neck N and another ε-neck N_1 with curvature $\leq C(\varepsilon)^2\tilde{r}^{-2}$ everywhere. By Step 1, we can bound

$$\frac{vol_{t_0}(N_1)}{\varepsilon^3 diam(N_1)^4} \geq \frac{1}{C(\varepsilon, \kappa_{l-1}, \tilde{r}_{l-1})}$$

from below uniformly. On the other hand, by using Hamilton's canonical parametrization $\Phi_\Gamma : \mathbb{S}^3/\Gamma \times (A, B) \to N_1$ to parametrize N_1, we get $|\Gamma| vol_{t_0}(N_1) \leq C(\varepsilon) diam(N_1)^4$. Combining the above two inequalities, we get the desired uniform upper bound of $|\Gamma|$.

The proof of the lemma is completed. q.e.d.

With Lemma 4.5, we can now justify the canonical neighborhood condition with accuracy ε and hence the construction of solution of the Ricci flow with surgery on our initial manifold M which becomes extinct.

Theorem 4.6. *Given a compact four-dimensional orbifold (M, g) with positive isotropic curvature and with at most isolated singularities, and given any fixed small constant $\varepsilon > 0$, one can find three non-increasing positive and continuous functions $\widetilde{\delta}(t)$, $\widetilde{r}(t)$ and $\widetilde{\kappa}(t)$ defined on whole $[0, +\infty)$ with the following properties: For arbitrarily given positive continuous function $\delta(t) \leq \widetilde{\delta}(t)$ on $[0, +\infty)$, the Ricci flow with $\delta(t)$-cutoff surgery, starting with g, admits a solution satisfying the pinching condition, the canonical neighborhood condition (with accuracy ε and scale $r = \widetilde{r}(t)$) and the κ-noncollapsing condition (with $\kappa = \widetilde{\kappa}(t)$) on a maximal time interval $[0, T)$ with $T < +\infty$ and becoming extinct* *at T. Moreover, the solution is obtained by performing at most a finite number of $\delta(t)$-cutoff surgeries on $[0, T)$.*

Proof. The pinching condition has already been justified in Lemma 4.3. To justify the canonical neighborhood condition with accuracy ε, we can apply the same argument as in [**8**] because we have all the ingredients we need to mimic the proof of Proposition 5.4 there. We note that the surgery does not occur on the place where the scalar curvature achieves its minimum. Then by applying the maximum principle to the scalar curvature equation $(\frac{\partial}{\partial t} - \triangle)R = 2|Ric|^2$, we conclude that the surgical solution must be extinct in finite time. Finally, the finiteness of the number of surgeries is guaranteed by the κ-noncollapsing condition of the solution. Therefore, the proof of the theorem is completed. q.e.d.

4.3. Recovering the topology as an orbifold connected sum. Now we are ready to prove Theorem 2.1.

Proof. Consider a surgical solution to the Ricci flow with surgery obtained by Theorem 4.6 on $[0, T)$ (where $T < +\infty$) which becomes extinct. Now we can recover the topology of the initial orbifold M as an orbifold connected sum of spherical orbifolds as follows.

Suppose our surgery times are $0 < t_1 < t_2 < \cdots < t_k < T$. For each $p \in \{1, 2, \dots, k\}$, right after the surgery performed at time t_p, we denote by $M_1^p, M_2^p, \dots M_{i_p}^p$ all the connected components of the surgical solution which either have positive curvature operator or contain no point of $\Omega_{\rho(t_p)}$, the remaining connected components are denoted by $N_1^p, \dots, N_{i'_p}^p$. Recall that our construction for the surgical solution is to stop the Ricci flow on M_l^p for $l = 1, \dots, i_p$ and to continue the Ricci flow on N_l^p for $l = 1, \dots, i'_p$. Thus, we also denote $N_1^k, \dots, N_{i'_k}^k$ by $M_1^{k+1}, M_2^{k+1}, \dots, M_{i_{k+1}}^{k+1}$ because our surgical solution becomes extinct at time T. Now, we collect all these M_j^i's in a set $\mathcal{S} = \{M_1^1, \dots, M_{i_{k+1}}^{k+1}\}$. On each M_j^i, we will mark a finite number of points $P_{j,l}^i$'s in the following inductive way.

At the first surgery time t_1, we perform a surgery on a δ-horn H; i.e., we cut open the δ-horn along a cross section of its neck N and glue back a (smooth or orbifold) cap to the finite part of the horn connected to $\Omega_{\rho(t_1)}$. Remember that we also glue back a cap to the infinite part of the horn (the so-called horn-shape end). We denote by P, $\bar{P}$ the tips of these two caps. P and $\bar{P}$ are the marked points alluded to above. Inductively, at the surgery time t_p for $p \in \{1, \dots, k\}$, we apply the same procedure as above to the N_l^{p-1}'s to obtain new marked points. Note that the points marked in previous surgeries may be separated to lie in different connected components. Also, once a component M_j^i is terminated at a surgery time t_i, then there is no more point marked on it in any later surgery time. At the end, we obtain a collection of marked points on the M_j^i's in $\mathcal{S}$.

Now we investigate the topology of each $M_j^i \in \mathcal{S}$. We know that M_j^i is either diffeomorphic to a spherical orbifold $\mathbb{S}^4/\Gamma$ with at most isolated singularities, or it is covered by ε-necks and ε-caps. Now we consider the latter case.

If M_j^i contains no cap, then M_j^i is diffeomorphic to smooth manifold $\mathbb{S}^3/\Gamma \times \mathbb{S}^1$ or $\mathbb{S}^3/\Gamma \times_f \mathbb{S}^1$.

If M_j^i contains caps, then M_j^i is diffeomorphic either to one of the smooth manifolds $\mathbb{S}^4$, $\mathbb{RP}^4$, $C_\Gamma^\sigma \cup_f C_{\Gamma'}^{\sigma'}$, or to one of the orbifolds $C_\Gamma^\sigma \cup_f C_{\Gamma'}$, $C_\Gamma \cup_f C_{\Gamma'}$, $\mathbb{S}^4/(x, \pm x')$, $\mathbb{S}^4/(x, \pm x')\#\mathbb{RP}^4$, $\mathbb{S}^4/(x, \pm x')\#\mathbb{S}^4/(x, \pm x')$.

So we conclude that each M_j^i is diffeomorphic to an orbifold connected sum of at most two spherical orbifolds $\mathbb{S}^4/\Gamma_{j,1}^i$ and $\mathbb{S}^4/\Gamma_{j,2}^i$ with at most isolated singularities. Now, reversing the surgery procedures is clearly equivalent to removing the open neighborhoods of suitable pairs of marked points above and gluing the corresponding boundaries by suitable diffeomorphisms. This amounts to taking orbifold connected sums of the spherical orbifolds with at most isolated singularities which form the M_j^i's. Thus, Theorem 2.1 is proved. q.e.d.

5. Proof of Main Theorem

The main purpose of this section is to deduce the Main Theorem from Theorem 2.1. For this, we need an algebraic lemma, Lemma 5.2, on the action of a finite group of isometries on the standard sphere $\mathbb{S}^{2n}$. We first prove a special case in Lemma 5.1 below.

Lemma 5.1. *Let $G \subset SO(2n+1)$ $(n \geq 2)$ be a finite subgroup such that each nontrivial element in G has exactly one eigenvalue equal to* 1. *Then there is a common nonzero vector $0 \neq v \in \mathbb{R}^{2n+1}$ such that for all $g \in G$ we have $g(v) = v$.*

Proof. The idea of the proof is similar to the classification of fixed point free finite subgroups of the isometry group of $\mathbb{S}^{2n+1}$ in [**28**]. We divide our argument into two cases.

Case 1): $|G|$ is even. In this case, there is an element of order 2 by Cauchy theorem. We denote this element by σ. We claim that σ is the unique element of order 2 in G. Indeed, suppose σ' is another distinct order 2 element. Note that, by our assumption, σ and σ' must have one eigenvalue equal to 1 and $2n$ eigenvalues equal to -1. Let E_1 and E_2 be the eigenspaces with eigenvalue -1 of σ and σ' respectively. Clearly, $G \ni \sigma\sigma'^{-1} = 1$ on $E_1 \cap E_2$. Since $n \geq 2$, the intersection $E_1 \cap E_2$ has dimension $\geq 2n-1 \geq 3$; this implies that $\sigma = \sigma'$ on the whole space. This is a contradiction.

By the uniqueness of σ, we know that $g^{-1}\sigma g = \sigma$ for any $g \in G$. Suppose $\sigma(v) = v$ for $|v| = 1$; then $\sigma g(v) = g(v)$. Hence $g(v) = v$ or $g(v) = -v$. We claim that $g(v) = -v$ cannot happen. The reason is as follows. Let $g(u) = u$ for $|u| = 1$; then $g^2(u) = u$. By combining with $g^2(v) = v$, we know that either $g^2 = 1$ or $v = \pm u$. If $g(v) = -v$, then v cannot be $\pm u$, so g has order 2, and is equal to σ by the uniqueness of the order 2 element, this contradicts $\sigma(v) = v$. So we have showed that $g(v) = v$ for any $g \in G$.

Case 2): $|G|$ is odd. First, we show that every subgroup of order p^2 (p is a prime number) of G is cyclic. Namely, we will show that G satisfies the p^2 condition.

Indeed, suppose H is a noncyclic subgroup of order p^2 for some prime number p. Since a group of order p^2 with p prime must be abelian, we can apply the same argument as in Case (1) to conclude that there is a unit vector v fixed by the whole group. Let $W \cong \mathbb{R}^{2n}$ be the orthogonal complement of v in $\mathbb{R}^{2n+1}$. Then H induces a fixed point free action on the unit sphere $\mathbb{S}^{2n-1}$ of W. So for any $v' \in \mathbb{S}^{2n-1}$, we have $0 = \sum_{g\in H} g(v')$, as $\sum_{g\in H} g(v')$ is fixed by all elements in H. On the other hand, since G is abelian and noncyclic, we conclude that each nontrivial element has order exactly p; the intersection of any two distinct order p groups contains only the identity. Let H_i, $i = 1, \dots, m$,$(m \geq 2)$ be the subgroups in H of order p; then for any $v' \in \mathbb{S}^{2n-1}$, we have

$$0 = \sum_{g\in G} g(v') = \sum_{i=1}^{m} \sum_{g\in H_i} g(v') - (m-1)v' = -(m-1)v',$$

where we have used the fact $\sum_{g\in H_i} g(v') = 0$, since H_i also acts freely on $\mathbb{S}^{2n-1}$. The contradiction shows that H is cyclic.

Now, the fact that G satisfies the p^2 condition implies that every Sylow subgroup of G is cyclic (see theorem 5.3.2 in [**28**]; note that since $|G|$ is odd, so must be p). By Burnside theorem (see theorem 5.4.1 in [**28**]), once we know that every Sylow subgroup of G is cyclic, then G is generated by two elements A and B with defining relations

$$\begin{aligned} &A^m = B^n = 1, \quad BAB^{-1} = A^r, \quad |G| = mn; \\ &((r-1)n, m) = 1, \quad r^n \equiv 1 (mod \;\; m). \end{aligned}$$

We may assume that both m and n are greater than 1; otherwise G is cyclic and the conclusion of our lemma clearly holds. Now, let $A(v) = v$ for $|v| = 1$. We will show that $B(v) = v$. Indeed, by the relation $BAB^{-1} = A^r$, we have $AB^{-1}(v) = B^{-1}(v)$. This implies $B^{-1}(v) = v$ or $B^{-1}(v) = -v$. $B^{-1}(v) = -v$ will not happen, because it implies $B^{-2} = 1$ by the argument in Case 1). This will imply the order of $|G|$ is even, which is a contradiction with our assumption. So v is fixed by the whole group G. q.e.d.

The following is the algebraic lemma needed to prove our Main Theorem; it is proved by reducing to the special case in Lemma 5.1.

Lemma 5.2. *Let $G \subset O(2n+1)(n \geq 2)$ be a finite group of orthogonal matrices such that each nontrivial element in G has at most one eigenvalue equal to* 1*. Then there is a finite group $G' \subset SO(2n)$ (which is isomorphic to G as an abstract group) acting freely on the sphere $\mathbb{S}^{2n-1}$ and a character $\chi : G' \to \{\pm 1\}$ such that after conjugation, the group $G = \{\begin{pmatrix} \chi(g) & 0 \\ 0 & g \end{pmatrix} : g \in G'\}$.*

Proof. Let $G_0 = G \cap SO(2n+1)$. If $G_0 = G$, every nontrivial element of G has exactly one eigenvalue equal to 1. By Lemma 5.1, there exists a nontrivial common fixed vector v of G. Now, let G' = the restriction of G on the orthogonal complement of v and $\chi \equiv 1$. We are done.

If $G_0 \neq G$, then G_0 is an index 2 normal subgroup of G. We again assume that v is a nontrivial common fixed vector of G_0. Then we claim that for any $g \in G \backslash G_0$, we have $g(v) = -v$. The argument is as follows. Since $g^2 \in G_0$, we have $g^2(v) = v$. Let $E = \mathrm{span}\{v, g(v)\}$. We will show $\dim E = 1$. Indeed, suppose $\dim E = 2$. Since $g(v+g(v)) = v+g(v)$ and $g(v-g(v)) = -(v-g(v))$, E is an invariant subspace of g, $\dim E^\perp$ is odd, and $\det(g\mid_{E^\perp}) = 1$. So g has another fixed nonzero vector in $E^\perp$. This contradiction shows that $\dim E = 1$ and hence $g(v) = v$ or $g(v) = -v$. If $g(v) = v$, then $\det(g\mid_{\{v\}^\perp}) = -1$, and hence g must have another fixed vector in $\{v\}^\perp$ since $\dim\{v\}^\perp$ is even; this again contradicts the assumption that g has at most one eigenvalue 1. This proves our claim.

Next, we show that G acts freely on the unit sphere of $\{v\}^\perp$. For this, we only need to check for any $g \in G \backslash G_0$; g has no nonzero fixed vector in $\{v\}^\perp$. But if this is not true, we have $g^2 = 1$; this implies that g has one eigenvalue 1 (by assumption) and $2n$ eigenvalues -1, which contradicts $\det(g) = -1$. To finish the proof, we only have to take G' = the restriction of G on $\{v\}^\perp$ and χ is the character which takes value 1 on G_0 and -1 otherwise. q.e.d.

In the following, we prove the Main Theorem by using Theorem 2.1 and Lemma 5.2. Note that, for the purpose of our Main Theorem, we will apply Theorem 2.1 where the initial space M is a **manifold**.

Proof. With the help of Lemma 5.2, we can describe the structure of the spherical orbifolds $\mathbb{S}^4/\Gamma$ appearing in Theorem 2.1. Since the resulting quotient space $\mathbb{S}^4/\Gamma$ has at most isolated singularities, each nontrivial element of Γ has at most a pair of antipodal fixed points, so the group Γ satisfies the assumption in Lemma 5.2.

There are three cases for the resulting space $\mathbb{S}^4/\Gamma$. The first case is that Γ acts on $\mathbb{S}^4$ freely; this can only occur when Γ is trivial or when Γ is equal to $\mathbb{Z}_2$ generated by the antipodal map. The resulting space is a smooth manifold diffeomorphic to $\mathbb{S}^4$ or $\mathbb{RP}^4$. The second case is that $\Gamma \neq \{1\}$ and $\Gamma \subset SO(5)$. Assume that $\mathbb{S}^4 \subset \mathbb{R}^5$ has equation $x_1^2+x_2^2+\cdots+x_5^5 = 1$. By Lemma 5.2, we may assume that the north pole $P = (0,0,0,0,1)$ and the south pole $-P = (0,0,0,0,-1)$ of $\mathbb{S}^4$ are the common fixed points of Γ. The resulting spherical orbifold $\mathbb{S}^4/\Gamma$ has two orbifold singularities at P and $-P$. Let $\mathbb{S}^3 = \mathbb{S}^4 \cap \{x_5 = 0\}$. Then Γ also acts on $\mathbb{S}^3$ without a fixed point. After removing suitable neighborhoods of P and $-P$ from the spherical orbifold $\mathbb{S}^4/\Gamma$, the resulting space is diffeomorphic to $\mathbb{S}^3/\Gamma \times [-1,1]$. Here we regard Γ as a fixed point free isometry subgroup of $\mathbb{S}^3$. We will call it $C(\Gamma)$ as in the introduction. The third case is that $\Gamma \neq \mathbb{Z}_2$ and $\Gamma \nsubseteq SO(5)$. We let $\Gamma_0 = \Gamma \cap SO(5)$, which is nontrivial and $\subsetneq \Gamma$. By Lemma 5.2, with the same notation as in the second case, we may assume that the north pole P and the south pole $-P$ of $\mathbb{S}^4$ are the fixed points of Γ_0. Also, any element in $\Gamma\backslash\Gamma_0$ exchanges P and $-P$. The resulting spherical orbifold $\mathbb{S}^4/\Gamma$ will thus have only one orbifold singularity, which is given by the Γ orbit of P. Note that Γ again acts on $\mathbb{S}^3$ as a fixed point free isometry subgroup. After removing suitable neighborhoods of the Γ orbit of P from the spherical orbifold $\mathbb{S}^4/\Gamma$, the resulting space is diffeomorphic to the quotient of $\mathbb{S}^3/\Gamma_0 \times [-1,1]$ by a group of order two generated by $\hat{\sigma} : (x,s) \mapsto (\sigma(x), -s)$ where σ is a fixed point free isometric involution on $\mathbb{S}^3/\Gamma_0$ induced by an element in $\Gamma\backslash\Gamma_0$. This is simply the smooth cap $C^{\sigma}_{\Gamma_0}$ in our previous notation.

Now let $X_1, \ldots, X_m$ be the spherical orbifolds with at most isolated singularities appearing in Theorem 2.1. The orbifold connected sum procedures there can be described in two steps. The first step is to resolve by orbifold connected sums all singularities of $X_1, \ldots, X_m$ which are introduced pairwise in the surgeries of the Ricci flow. The resulting space consists of a finite number of smooth closed connected manifolds, denoted by $Y_1, \ldots, Y_n$. The second step in the connected sum procedures corresponds to reversing the surgeries of the Ricci flow which do not introduce singularity, i.e., when smooth caps $\mathbb{B}^4$ are glued during the surgeries. For this, we perform the usual connected sums among the $Y_1, \ldots, Y_n$ and a finite number of $\mathbb{S}^4$, $\mathbb{RP}^4$, $\mathbb{S}^3 \times \mathbb{S}^1$, or $\mathbb{S}^3 \tilde{\times} \mathbb{S}^1$. Note that the last two manifolds occur because taking the usual connected sum through two embedded 3-spheres on a connected smooth manifold is equivalent to taking the usual connected sum of this manifold with $\mathbb{S}^3 \times \mathbb{S}^1$ or $\mathbb{S}^3 \tilde{\times} \mathbb{S}^1$.

Thus, to prove our Main Theorem, it remains to show that each Y_l is diffeomorphic to a cocompact quotient of the standard $\mathbb{S}^3 \times \mathbb{R}$ by a discrete isometry group. For this, we first note that each spherical orbifold X_i (with at most isolated singularities) falls into one of the three cases mentioned above. In particular, each Y_l is obtained by gluing a number of cylinders $C(\Gamma_k)$'s and caps $C_{\Gamma_j}^{\sigma_i}$'s along their common boundaries for suitable Γ_k's, Γ_j's, and σ_i's. Since there is only one end for a cap, at most two caps can occur. To proceed, we make the following observations:

1. Each of the two boundaries of $C(\Gamma)$ is diffeomorphic to $\mathbb{S}^3/\Gamma$ and the (only) boundary of C_Γ^σ is diffeomorphic to $\mathbb{S}^3/\Gamma$.

2. $\mathbb{S}^3/\Gamma_1$ is diffeomorphic to $\mathbb{S}^3/\Gamma_2$ if and only if Γ_1 and Γ_2 are conjugate in $SO(4)$.

3. Up to diffeomorphisms, the gluing of two cylinders $C(\Gamma)$'s or the gluing of a cylinder $C(\Gamma)$ with a cap C_Γ^σ is independent of the gluing diffeomorphism.

4. Both $C(\Gamma)$ and C_Γ^σ can be equipped with a metric which is locally isometric to $\mathbb{S}^3 \times \mathbb{R}$.

From observations 1 and 2, we see that Y_l is obtained by gluing a finite number of $C(\Gamma)$'s and $C_\Gamma^{\sigma_i}$'s along their common boundaries $\mathbb{S}^3/\Gamma$ for some Γ and σ_i's. Observation 3 then says that, up to diffeomorphisms, Y_l is one of the following types: the self-gluing of the two ends of a cylinder $C(\Gamma)$ or the gluing of two caps, C_Γ^σ and $C_\Gamma^{\sigma'}$, by suitable diffeomorphism on $\mathbb{S}^3/\Gamma$. Now, by observation 4 and the fact that any diffeomorphism on a three-dimensional spherical space form is isotopic to an isometry (see [**18**]), Y_l can be equipped with a metric which is locally isometric to $\mathbb{S}^3 \times \mathbb{R}$. This completes the proof of our Main Theorem. q.e.d.

Our proof of the Main Theorem actually works when the initial space M is an orbifold with at most isolated singularities. The result is given in the following

Corollary 5.3. *A compact 4-orbifold with at most isolated singularities and with positive isotropic curvature is diffeomorphic to the connected sum* $\#_i(\mathbb{S}^3 \times \mathbb{R}/G_i)\#_j(\mathbb{S}^4/\Gamma_j)$, *where* G_i *and* Γ_j *are standard group actions and the connected sum is in the usual sense.*

Proof. Since Theorem 2.1 already works for orbifold, the only modification of the proof of the Main Theorem we have to make is to allow the Y_l's to be orbifolds with at most isolated singularities. Thus, besides the smooth manifolds $\mathbb{S}^3/\Gamma \times_f \mathbb{S}^1$ and $C_\Gamma^\sigma \cup_f C_\Gamma^{\sigma'}$ obtained above, Y_l can also be given by $C_\Gamma^\sigma \cup_f C_\Gamma$ or $C_\Gamma \cup_f C_\Gamma$. Now, by [**18**], f is isotopic to an isometry f' of $\mathbb{S}^3/\Gamma$, which can be naturally extended to a diffeomorphism from C_Γ to itself. This gives a diffeomorphism from $\mathbb{S}^4/\{\Gamma, \hat{\sigma}\}$ or $\mathbb{S}^4/\Gamma$ to $C_\Gamma^\sigma \cup_{f'} C_\Gamma$ or $C_\Gamma \cup_{f'} C_\Gamma$ respectively. Thus, Y_l remains a cocompact quotient of $\mathbb{S}^3 \times \mathbb{R}$ (with isolated orbifold singularities). The remainder of the proof of our Main Theorem goes through. q.e.d.

6. Appendix

Let ε be a positive constant. We call an open subset $N \subset X$ in a metric space **GH ε-neck of radius** r if $r^{-1}N$ is homeomorphic to and Gromov-Hausdorff ε-close to a neck $S \times \mathbb{I}$ where S is some Alexandrov space with nonnegative curvature without boundary, $diam(S) \leq \frac{1}{\sqrt{\varepsilon}}$ and $\mathbb{I} = (-\varepsilon^{-1}, \varepsilon^{-1})$.

Proposition 6.1. *There exists a constant $\varepsilon_0 = \varepsilon_0(n) > 0$ such that for any complete noncompact n-dimensional intrinsic Alexandrov space X with nonnegative curvature, there is a positive constant $r_0 > 0$ and a compact set $K \subset X$ such that any GH ε-neck of radius $r \leq r_0$ on X with $\varepsilon \leq \varepsilon_0$ must be contained in K entirely.*

Proof. When the Alexandrov space is required to be smooth and the topology used to define the ε-neck is in $C^{[\frac{1}{\varepsilon}]}$, the proof is given in [**8**]. Now we modify the argument there to our present situation. The key observation is that we used essentially only the triangle comparison in [**8**]. Here we include the proof for completeness.

We argue by contradiction. Suppose that there exists a sequence of positive constants $\varepsilon^\alpha \to 0$ and a sequence of n-dimensional complete noncompact pointed Alexandrov spaces (X^α, P^α) with nonnegative curvature such that for each fixed α, there exists a sequence of GH ε^α-necks N_k of radius $r_k \leq 1/k$ on X^α with $N_k \subset X^\alpha \backslash B(P^\alpha, k)$. Recall that by the definition of Gromov-Hausdorff distance, there is a metric space Z_k containing isometric embeddings of $r_k^{-1}N_k$ and $S \times \mathbb{I}$ such that $S \times \mathbb{I} \subset B_{\varepsilon^\alpha}(r_k^{-1}N_k)$ and $r_k^{-1}N_k \subset B_{\varepsilon^\alpha}(S \times \mathbb{I})$. Let $P_k \in r_k^{-1}N_k$ be a point having distance $\leq \varepsilon^\alpha$ with $S \times \{0\}$ (in Z_k). Then we have $d(P^\alpha, P_k) \to \infty$ as $k \to \infty$.

Let α be fixed and sufficiently large, connecting each P_k to P^α by a minimizing geodesic γ_k. By passing to subsequence, we may assume that the angle θ_{kl} between the geodesics γ_k and γ_l at P^α is very small and tends to zero as $k, l \to +\infty$, and we may also assume that the length of γ_{k+1} is much bigger than the length of γ_k. Let us connect P_k to P_l by a minimizing geodesic η_{kl}.

For any three points $A, B, C \in X^\alpha$, we use $\bar{\Delta}_{\bar{A}\bar{B}\bar{C}}$ to denote the corresponding triangle in the Euclidean plane $\mathbb{P}$ with $d(A, B) = |\bar{A}\bar{B}|, d(A, C) = |\bar{A}\bar{C}|, d(B, C) = |\bar{B}\bar{C}|$, and we also use $\bar{\angle}\bar{A}\bar{B}\bar{C}$ to denote the angle of $\bar{\Delta}_{\bar{A}\bar{B}\bar{C}}$ at $\bar{B}$.

Clearly, for each $l > k$, $\bar{\angle}\bar{P}^\alpha\bar{P}_k\bar{P}_l$ is close to π by angle comparison. Let $P'_k \in \gamma_k \cap \partial N_k$ and $P''_k \in \eta_{kl} \cap \partial N_k$. It is clear that for any point $x \in \partial N_k$, either we have $\bar{\angle}\bar{P}'_k\bar{P}_k\bar{x}$ small and $\bar{\angle}\bar{P}''_k\bar{P}_k\bar{x}$ close to π or we have $\bar{\angle}\bar{P}'_k\bar{P}_k\bar{x}$ close to π and $\bar{\angle}\bar{P}''_k\bar{P}_k\bar{x}$ small. This depends on which connected component of ∂N_k it is that x lies on.

By using the above facts and the triangle comparison (see [**8**]), we can show that when k is large enough, each minimizing geodesic γ_l with $l > k$, connecting P^α to P_l, must go through the whole N_k.

Hence by taking a limit, we get a geodesic ray γ emanating from P^α which passes through all the necks N_k, $k = 1, 2, \dots$, except for a finite number of them. Throwing away this finite number of necks, we may assume that γ passes through all necks N_k, $k = 1, 2, \dots$. Denote the central cross section of N_k by S_k and their intersection points with γ by $p_k \in S_k \cap \gamma$, for $k = 1, 2, \dots$. Take a sequence of points $\gamma(m)$ with $m = 1, 2, \dots$. For each fixed neck N_k, choose an arbitrary point $q_k \in N_k$ near S_k and draw a geodesic segment γ^{km} from q_k to $\gamma(m)$. Now we can show by triangle comparison that for any fixed neck N_l with $l > k$, γ^{km} will pass through N_l for all sufficiently large m.

For any $s > 0$, choose two points $\tilde{p}_k$ on $\overline{p_k\gamma(m)} \subset \gamma$ and $\tilde{q}_k$ on $\overline{q_k\gamma(m)} \subset \gamma^{km}$ with $d(p_k, \tilde{p}_k) = d(q_k, \tilde{q}_k) = s$. By Toponogov comparison theorem, we have

$$\lim_{m\to\infty} \frac{d(\tilde{p}_k, \tilde{q}_k)}{d(p_k, q_k)} \geq 1.$$

Letting $m \to \infty$, we see that γ^{km} has a convergent subsequence whose limit γ^k is a geodesic ray passing through all N_l with $l > k$. Denote by $p_j = \gamma(t_j), j = 1, 2, \dots$. From the above computation, we deduce that

$$d(p_k, q_k) \leq d(\gamma(t_k + s), \gamma^k(s))$$

for all $s > 0$.

Let $\varphi(x) = \lim_{t\to+\infty}(t - d(x, \gamma(t)))$ be the Busemann function constructed from the ray γ. By the definition of Busemann function φ associated to the ray γ, we see that $\varphi(\gamma^k(s_1)) - \varphi(\gamma^k(s_2)) = s_1 - s_2$ for any s_1, $s_2 \geq 0$. Consequently, by investigating the value of φ on ∂N_l and from the linearity of $\varphi\mid_{\gamma^k}$, we know that for each $l > k$, we have $\gamma^k(t_l - t_k) \in \varphi^{-1}(\varphi(p_l)) \cap N_l$. This implies that the diameter of $\varphi^{-1}(\varphi(p_k)) \cap N_k$ is not greater than the diameter of $\varphi^{-1}(\varphi(p_l)) \cap N_l$ for any $l > k$, which is a contradiction as l is much larger than k. The proposition is proved. q.e.d.

Remark 6.2. Without introducing a compact set K, the conclusion of Proposition 6.1 may not be true. Counterexamples can be given by cones with small aperture.

References

[1] S. Brendle, *A general convergence result for the Ricci flow in higher dimensions*, Duke Math. J. **145** (2008), no. 3, 585–601, MR 2462114, Zbl 1161.53052.

[2] S. Brendle & R. Schoen, *Manifolds with* 1/4*-pinched curvature are space forms*, J. Amer. Math. Soc. **22** (2009), 287–307, MR 2449060, Zbl pre05859406.

[3] S. Brendle & R. Schoen, *Sphere theorems in geometry*, Surveys in Diff. Geom., volume XIII (2009), 49–84, International Press, Somerville, MA. MR 2537082, Zbl 1184.53037.

[4] S. Brendle & R. Schoen, *Riemannian manifolds of positive curvature*, Proceedings of ICM 2010, volume I, 449–475, MR 2827901, Zbl pre05971070.

[5] C. Böhm, & B. Wilking, *Manifolds with positive curvature operator are space forms*, Ann. of Math. (2) **167** (2008), no. 3, 1079–1097, MR 2415394, Zbl 1185.53073.

[6] H.D. Cao & X.P. Zhu, *A complete proof of Poincare and geometrization conjectures—application of Hamilton-Perelman theory of Ricci flow*, Asian J. Math. **10** 2 (2006), 165–492, MR 2233789, Zbl 1200.53057.

[7] B.L. Chen & X.P. Zhu, *Uniqueness of the Ricci flow on complete noncompact manifolds*, J. Diff. Geom. **74** (2006), 119–154, MR 2260930, Zbl 1104.53032.

[8] B.L. Chen & X.P. Zhu, *Ricci flow with surgery on four-manifolds with positive isotropic curvature*, J. Diff. Geom. **74** (2006), 177–264, MR 2258799, Zbl 1103.53036.

[9] D. DeTurck, *Deforming metrics in the direction of their Ricci tensors*, J. Diff. Geom. **18** (1983), 157–162, MR 0697987, Zbl 0517.53044.

[10] A.M. Fraser, *Fundamental groups of manifolds with positive isotropic curvature*, Ann. of Math. (2) **158** (2003), no. 1, 345–354, MR 1999925, Zbl 1044.53023.

[11] M. Gromov, *Positive curvature, macroscopic dimension, spectral gaps and higher signatures*, Functional analysis on the eve of the 21st century, Vol. II (New Brunswick 1993), 1–213, Progr. Math., **132**, Birkhäuser, Boston (1996), MR 1389019, Zbl 0945.53022.

[12] R.S. Hamilton, *Three manifolds with positive Ricci curvature*, J. Diff. Geom. **17** (1982), 255–306, MR 0664497, Zbl 0504.53034.

[13] R.S. Hamilton, *Four-manifolds with positive curvature operator*, J. Differential Geom. **24** (1986), 153–179, MR 0862046, Zbl 0628.53042.

[14] R.S. Hamilton, *Four-manifolds with positive isotropic curvature*, Commu. in Analysis and Geometry, **5** (1997), 1–92 (or see *Collected Papers on Ricci Flow*, edited by H.D. Cao, B. Chow, S.C. Chu and S.T. Yau, International Press 2002), MR 1456308, Zbl 0892.53018.

[15] R.S. Hamilton, *Three-orbifolds with positive Ricci curvature*, 521–524 *Collected Papers on Ricci Flow*, edited by H.D. Cao, B. Chow, S.C. Chu and S.T. Yau, International Press 2003, MR 2143256, Zbl 1108.53002.

[16] H. Izeki, *Limit sets of Kleinian groups and conformally flat Riemannian manifolds*, Invent. Math. **122** (1995), 603–625, MR 1359605, Zbl 0854.53035.

[17] P. Lu, *A compactness property for solutions of the Ricci flow on orbifolds* American Journal of Mathematics, **123** (2001), 1103–1134, MR 1867313, Zbl 1012.53030.

[18] D. Mccullough, *Isometries of elliptic 3-manifolds,* J. London Math. Soc. (2) **65** (2002), no 1, 167–182, MR 1875143, Zbl 1012.57023.

[19] M. Micallef & J.D. Moore, *Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes*, Ann. of Math. (2) **127** (1988) 199–227, MR 0924677, Zbl 0661.53027.

[20] M. Micallef & M. Wang, *Metrics with nonnegative isotropic curvatures*, Duke Math. J. **72** (1993), no. 3, 649–672, MR 1253619, Zbl 0804.53058.

[21] H. Nguyen, *Isotropic curvature and the Ricci flow,* Internat. Math. Res. Notices no. **3**, 536–558 (2010), MR 2587576, Zbl 1190.53068.

[22] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math.DG/0211159.

[23] G. Perelman, *Ricci flow with surgery on three manifolds,* arXiv: math. DG/0303109.

[24] J.G. Ratcliffe, Foundations of hyperbolic manifolds, G.T.M. **149**, Springer New York, 2006, MR 2249478, Zbl 1106.51009.

[25] R. Schoen & S.T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math. **92** (1988), 47–71, MR 0931204, Zbl 0658.53038.

[26] R. Schoen, *Open problems proposed in Pacific Northwest Geometry Seminar,* 2007-fall (at Univ. of Oregon).

[27] W. Thurston, *Geometry and topology of three manifolds*, Lecture notes, Princeton University, 1979.

[28] J. Wolf, *Spaces of constant curvature,* Publish or Perish, Inc., MR 2742530, Zbl 1216.53003.

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