J. DIFFERENTIAL GEOMETRY
90 (2012) 391-411

THE DONALDSON-THOMAS INVARIANTS UNDER BLOWUPS AND FLOPS

Jianxun Hu & Wei-Ping Li

Abstract

Using the degeneration formula for Donaldson-Thomas invariants [**L-W**, **MNOP2**], we proved formulae for blowing up a point, simple flops, and extremal transitions.

1. Introduction

Given a smooth projective Calabi-Yau 3-fold X, the moduli space of stable sheaves on X has virtual dimension zero. Donaldson and Thomas [**D-T**] defined the holomorphic Casson invariant of X which essentially counts the number of stable bundles on X. However, the moduli space has positive dimension and is singular in general. Making use of virtual cycle technique (see [**B-F**] and [**L-T**]), Thomas showed in [**Thomas**] that one can define a virtual moduli cycle for some X including Calabi-Yau and Fano 3-folds. As a consequence, one can define Donaldson-type invariants of X which are deformation invariant. Donaldson-Thomas invariants provide a new vehicle to study the geometry and other aspects of higher-dimensional varieties. It is important to understand these invariants.

Much studied Gromov-Witten invariants of X are the counting of stable maps from curves to X. In [**MNOP1**, **MNOP2**], Maulik, Nekrasov, Okounkov, and Pandharipande discovered relations between Gromov-Witten invariants of X and Donaldson-Thomas invariants constructed from moduli spaces of ideal sheaves of curves on X. They conjectured that these two invariants can be identified via the equations of partition functions of both theories. This suggests that many phenomena in Gromov-Witten theory have counterparts in Donaldson-Thomas theory.

Donaldson-Thomas invariants are deformation independent. In the birational geometry of 3-folds, we have blowups and flops. Donaldson-Thomas invariants cannot be effective in studying birational geometry unless we understand how invariants change under birational operations. Li and Ruan in [**L-R**] studied how Gromov-Witten invariants change under a flop for a Calabi-Yau 3-fold. They proved that one can identify the 3-point functions of X and the flop X^f of X up to some transformation

Received 10/15/2009.

of the q variables. They also studied how Gromov-Witten invariants change under an extremal transition. The same questions were also studied by Liu and Yau recently in [**L-Y**] using J. Li's degeneration formula from algebraic geometry. In [**Hu1**, **Hu2**], the first author studied the change of Gromov-Witten invariants under the blowup. In this paper, we will study how Donaldson-Thomas invariants in [**MNOP2**] change under the blowup of a point, some flops, and extremal transitions.

The method we use is the degeneration formula for Donaldson-Thomas invariants studied in [**L-W**, **MNOP2**]. The blowup of X has a description in terms of a degeneration of X. An extremal transition can also be described in terms of a semi-stable degeneration which relates the extremal transition of X, if it exists, with a blowup of X. Then we can apply the degeneration formula.

In the category of symplectic manifolds, one uses symplectic sum or symplectic cutting for the blowup and the extremal transition of X. The gluing formula for Gromov-Witten invariants in the symplectic setup is studied in [**I-P1**, **I-P2**, **L-R**]. Besides the difference of degeneration and symplectic cutting, the arguments used in [**L-R**, **Hu1**, **Hu2**, **L-Y**] rely on the fact that stable maps have connected domains, while the curves defined by ideal sheaves are in general not connected. Therefore the formulae for flops and extremal transitions are a bit different from those of Gromov-Witten invariants in [**L-R**].

We remark that a different method, the categorical method, is used by Toda [**Toda**] based on the Bridgeland stability conditions [**Bri**] and the wall-crossing formula developed by Joyce and Song [**J-S**] and Kontsevich and Soibelman [**K-S**] to study the change of Donaldson-Thomas invariants under the birational transformations.

The organization of the paper is as follows. In §2, we set up terminologies and notations, and list the basic results needed. The degeneration formula is discussed. In §3, using the degeneration formula, we prove a blowup formula for the blowup of X at a point. In §4, we prove the equality of Donaldson-Thomas partition functions under a flop. In §5, we establish a relation on the Donaldson-Thomas invariants between X and its extremal transition.

Acknowledgments. Authors would like to thank Jun Li, Miles Reid, Qi Zhang, Yongbin Ruan, and Zhenbo Qin for many helpful discussions. The second author would like to thank the Department of Mathematics at Sun Yat-sen University for the hospitality during his several visits in the spring semester of 2005. The first author would like to thank HKUST for the hospitality during his visit in January of 2005. Both authors would like to thank the ICCM held at the Chinese University of Hong Kong where they met and initiated the work. Finally, both authors would like to thank the referee for pointing out a mistake and for helpful comments. Hu was partially supported by the NSFC Grant 10825105. Li was partially supported by the grant HKUST6114/02P.

2. Preliminaries

In this section, we shall discuss the basic materials on Donaldson-Thomas invariants studied by Maulik, Nekrasov, Okounkov, and Pandharipande. For the details, one can consult [**D-T**, **L-R**, **I-P1**, **I-P2**, **Li1**, **Li2**, **L-W**, **MNOP1**, **MNOP2**, **Thomas**].

Let X be a smooth projective 3-fold and $\mathcal{I}$ be an ideal sheaf on X. Assume the sub-scheme Y defined by $\mathcal{I}$ has dimension ≤ 1. Here Y is allowed to have embedded points on the curve components. Therefore we have the exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

The 1-dimensional components, with multiplicities taken into consideration, determine a homology class

$$[Y] \in H_2(X, \mathbb{Z}).$$

Let $I_n(X, \beta)$ denote the moduli space of ideal sheaves $\mathcal{I}$ satisfying

$$\chi(\mathcal{O}_Y) = n, \quad [Y] = \beta \in H_2(X, \mathbb{Z}).$$

$I_n(X, \beta)$ is projective and is a fine moduli space. From the deformation theory, one can compute the virtual dimension of $I_n(X, \beta)$ to obtain the following result:

Lemma 2.1. *The virtual dimension of $I_n(X, \beta)$, denoted by vdim, equals $\int_\beta c_1(T_X)$.*

Note that the actual dimension of the moduli space $I_n(X, \beta)$ is usually larger than the virtual dimension.

Let $\mathfrak{I}$ be the universal family over $I_n(X, \beta) \times X$ and π_i be the projection of $I_n(X, \beta) \times X$ to the i-th factor. For a cohomology class $\gamma \in H^l(X, \mathbb{Z})$, consider the operator

$$ch_{k+2}(\gamma) : H_*(I_n(X, \beta), \mathbb{Q}) \longrightarrow H_{*-2k+2-l}(I_n(X, \beta), \mathbb{Q}),$$

$$ch_{k+2}(\gamma)(\xi) = \pi_{1*}(ch_{k+2}(\mathcal{J}) \cdot \pi_2^*(\gamma) \cap \pi_1^*(\xi)).$$

Descendent fields in Donaldson-Thomas theory are defined in [**MNOP2**], and denoted by $\tilde{\tau}_k(\gamma)$, which correspond to the operations $(-1)^{k+1} ch_{k+2}(\gamma)$. The descendent invariants are defined by

$$< \tilde{\tau}_{k_1}(\gamma_{l_1}) \cdots \tilde{\tau}_{k_r}(\gamma_{l_r}) >_{n,\beta} = \int_{[I_n(X,\beta)]^{vir}} \prod_{i=1}^{r} (-1)^{k_i+1} ch_{k_i+2}(\gamma_{l_i}),$$

where the latter integral is the push-forward to a point of the class

$$(-1)^{k_1+1} ch_{k_1+2}(\gamma_{l_1}) \circ \cdots \circ (-1)^{k_r+1} ch_{k_r+2}(\gamma_{l_r})([I_n(X, \beta)]^{vir}).$$

The Donaldson-Thomas partition function with descendent insertions is defined by

$$Z_{DT}(X;q \mid \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_{l_i}))_\beta = \sum_{n\in\mathbb{Z}} < \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_{l_i}) >_{n,\beta} q^n.$$

The degree 0 moduli space $I_n(X,0)$ is isomorphic to the Hilbert scheme of n points on X. The degree 0 partition function is $\mathsf{Z}_{DT}(X;q)_0$.

The reduced partition function is obtained by formally removing the degree 0 contributions,

$$Z'_{DT}(X;q \mid \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_{l_i}))_\beta = \frac{Z_{DT}(X;q \mid \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_{l_i}))_\beta}{Z_{DT}(X;q)_0}.$$

Relative Donaldson-Thomas invariants are also defined in [**L-W**, **MNOP2**]. Let S be a smooth divisor in X. An ideal sheaf $\mathcal{I}$ is said to be relative to S if the morphism

$$\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_S \to \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_S$$

is injective. A proper moduli space $I_n(X/S,\beta)$ of relative ideal sheaves can be constructed by considering the ideal sheaves relative to the expended pair $(X[k], S[k])$. For details, one can read [**Li1**, **L-W**, **MNOP2**].

Let Y be the subscheme defined by $\mathcal{I}$. The scheme theoretic intersection $Y \cap S$ is an element in the Hilbert scheme of points on S with length $[Y]\cdot S$. If we use $\mathrm{Hilb}(S,k)$ to denote the Hilbert scheme of points of length k on S, we have a map

$$\epsilon : I_n(X/S,\beta) \longrightarrow \mathrm{Hilb}(S, \beta\cdot[S]).$$

The cohomology of the Hilbert scheme of points of S has a basis via the representation of the Heisenberg algebra on the cohomologies of the Hilbert schemes.

Following Nakajima in [**Nakajima**], let η be a cohomology weighted partition with respect to a basis of $H^*(S,\mathbb{Q})$. Let $\eta = \{\eta_1,\ldots,\eta_s\}$ be a partition whose corresponding cohomology classes are $\delta_1,\ldots,\delta_s$, let

$$C_\eta = \frac{1}{\mathfrak{z}(\eta)} P_{\delta_1}[\eta_1]\cdots P_{\delta_s}[\eta_s]\cdot \mathbf{1} \in H^*(\mathrm{Hilb}(S,|\eta|),\mathbb{Q}),$$

where

$$\mathfrak{z}(\eta) = \prod_i \eta_i |\mathrm{Aut}(\eta)|,$$

and $|\eta| = \sum_j \eta_j$. The Nakajima basis of the cohomology of $\mathrm{Hilb}(S,k)$ is the set

$$\{C_\eta\}_{|\eta|=k}.$$

We can choose a basis of $H^*(S)$ so that it is self dual with respect to the Poincaré pairing, i.e., for any i, $\delta_i^* = \delta_j$ for some j. To each weighted partition η, we define the dual partition $\eta^\vee$ such that $\eta_i^\vee = \eta_i$ and the corresponding cohomology class to $\eta_i^\vee$ is δ_i^*. Then we have

$$\int_{\mathrm{Hilb}(S,k)} C_\eta \cup C_\nu = \frac{(-1)^{k-\ell(\eta)}}{\mathfrak{z}(\eta)} \delta_{\nu,\eta^\vee}$$

(see [**Nakajima**]).

The descendent invariants in the relative Donaldson-Thomas theory are defined by

$$< \tilde{\tau}_{k_1}(\gamma_{l_1}) \cdots \tilde{\tau}_{k_r}(\gamma_{l_r}) \mid \eta >_{n,\beta} = \int_{[I_n(X/S,\beta)]^{vir}} \prod_{i=1}^{r} (-1)^{k_i+1} ch_{k_i+2}(\gamma_{l_i}) \cap \epsilon^*(C_\eta).$$

Define the associated partition function by

$$Z_{DT}(X/S; q \mid \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_{l_i}))_{\beta,\eta} = \sum_{n\in\mathbb{Z}} < \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_{l_i}) \mid \eta >_{n,\beta} q^n.$$

The reduced partition function is obtained by formally removing the degree 0 contributions,

$$Z'_{DT}(X/S; q \mid \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_{l_i}))_{\beta,\eta} = \frac{Z_{DT}(X/S; q \mid \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_{l_i}))_{\beta,\eta}}{Z_{DT}(X/S; q)_0}.$$

In the remainder of the section, we shall discuss the degeneration formula due to B. Wu and J. Li. It is the main tool employed in the paper.

Let $\pi\colon \mathcal{X} \to C$ be a smooth 4-fold over a smooth irreducible curve C with a marked point denoted by $\mathbf{0}$ such that $\mathcal{X}_t = \pi^{-1}(t) \cong X$ for $t \neq \mathbf{0}$ and $\mathcal{X}_{\mathbf{0}}$ is a union of two smooth 3-folds X_1 and X_2 intersecting transversely along a smooth surface S. We write $\mathcal{X}_{\mathbf{0}} = X_1 \cup_S X_2$. Assume that C is contractible and S is simply connected.

Consider the natural maps

$$i_t\colon X = \mathcal{X}_t \to \mathcal{X}, \qquad i_{\mathbf{0}}\colon \mathcal{X}_{\mathbf{0}} \to \mathcal{X},$$

and the gluing map

$$g = (j_1, j_2)\colon X_1 \coprod X_2 \to \mathcal{X}_{\mathbf{0}}.$$

We have

$$H_2(X) \xrightarrow{i_{t*}} H_2(\mathcal{X}) \xleftarrow{i_{0*}} H_2(\mathcal{X}_{\mathbf{0}}) \xleftarrow{g_*} H_2(X_1) \oplus H_2(X_2),$$

where i_{0*} is an isomorphism since there exists a deformation retract from $\mathcal{X}$ to $\mathcal{X}_{\mathbf{0}}$ (see [**Clemens**]) and g_* is surjective from the Mayer-Vietoris sequence. For $\beta \in H_2(X)$, there exist $\beta_1 \in H_2(X_1)$ and $\beta_2 \in H_2(X_2)$ such that

$$i_{t*}(\beta) = i_{0*}(j_{1*}(\beta_1) + j_{2*}(\beta_2)). \tag{2.1}$$

For simplicity, we write $\beta = \beta_1 + \beta_2$ instead.

Lemma 2.2. *With the assumption as above, given $\beta = \beta_1 + \beta_2$. Let $d = \int_\beta c_1(X)$ and $d_i = \int_{\beta_i} c_1(X_i)$, $i = 1, 2$. Then*

$$d = d_1 + d_2 - 2\int_{\beta_1}[S], \qquad \int_{\beta_1}[S] = \int_{\beta_2}[S]. \tag{2.2}$$

Proof. The formulae (2.2) come from the adjunction formulae $K_{\mathcal{X}_t} = K_{\mathcal{X}}|_{\mathcal{X}_t}$ and $K_{X_i} = (K_{\mathcal{X}} + X_i)|_{X_i}$ for $i = 1, 2$, and $X_1 \cdot (X_1 + X_2) = X_1 \cdot \mathcal{X}_{\mathbf{0}} = 0$. q.e.d.

Similarly for cohomology, we have the maps

$$H^k(\mathcal{X}_t) \xleftarrow{i_t^*} H^k(\mathcal{X}) \xrightarrow{i_0^*} H^k(\mathcal{X}_{\mathbf{0}}) \xrightarrow{g^*} H^k(X_1) \oplus H^k(X_2),$$

where i_0^* is an isomorphism. Take $\alpha \in H^k(\mathcal{X})$ and let $\alpha(t) = i_t^*\alpha$.

There is a degeneration formula which takes the form

$$\begin{aligned} & Z'_{DT}(\mathcal{X}_t; q \mid \prod_{i=1}^r \tilde{\tau}_0(\gamma_{l_i}(t)))_\beta \\ = & \sum Z'_{DT}(X_1/S; q \mid \prod \tilde{\tau}_0(j_1^*\gamma_{l_i}(0)))_{\beta_1,\eta} \frac{(-1)^{|\eta|-\ell(\eta)}\mathfrak{z}(\eta)}{q^{|\eta|}} \\ & \cdot Z'_{DT}(X_2/S; q \mid \prod \tilde{\tau}_0(j_2^*\gamma_{l_i}(0)))_{\beta_2,\eta^\vee}, \end{aligned} \tag{2.3}$$

where the sum is over the splitting $\beta_1 + \beta_2 = \beta$, and cohomology weighted partitions η. γ_{l_i}'s are cohomology classes on $\mathcal{X}$. There is a compatibility condition

$$|\eta| = \beta_1 \cdot [S] = \beta_2 \cdot [S]. \tag{2.4}$$

For details, one can see [**Li1**, **Li2**, **L-W**, **MNOP2**].

3. Blowup at a point and a blowup formula

In [**MNOP1**, **MNOP2**], the authors discovered a correspondence between Gromov-Witten theories and Donaldson-Thomas theories. In [**Hu1**, **Hu2**], the first author studied the change of Gromov-Witten invariants under the blowup operation. In this section, we will study the change of Donaldson-Thomas invariants under the blowup along a point.

The key idea is that the blowup can be obtained via a semistable degeneration as follows. Let X be a smooth projective 3-fold and $\tilde{X}$ be the blowup of X at a general point x. Denote by $p : \tilde{X} \longrightarrow X$ the natural projection of the blowup. Let $\mathcal{X}$ be the blowup of $X \times \mathbb{C}$ at the point $(x, 0)$ and let π be the natural projection from $\mathcal{X}$ to $\mathbb{C}$. It is a semistable degeneration of X with the central fiber $\mathcal{X}_0$ being a union of $X_1 \cong \tilde{X}$ and $X_2 \cong \mathbb{P}^3$, which is the exceptional divisor in $\mathcal{X}$. X_1 and X_2 intersect transversely along $E \cong \mathbb{P}^2$, which is the exceptional divisor in $X_1 = \tilde{X}$. As a divisor in X_2, E is a hyperplane. $c_1(X_2) = 4E$.

There is an injection

$$\iota\colon H_2(X) \longrightarrow H_2(\tilde{X})$$

such that the image of ι is the set $\{\beta \in H_2(\tilde{X}) \,|\, \beta \cdot E = 0\}$.

Theorem 3.1. *Let X be a smooth projective 3-fold. Suppose that $\beta \in H_2(X,\mathbb{Z})$ and $\gamma_{l_i} \in H^*(X,\mathbb{R})$, $i = 1,\dots,r$. Then*

$$Z'_{DT}(X;q \mid \prod_{i=1}^{r} \tilde{\tau}_0(\gamma_{l_i}))_\beta = Z'_{DT}(\tilde{X};q \mid \prod_{i=1}^{r} \tilde{\tau}_0(p^*\gamma_{l_i}))_{\iota(\beta)}. \tag{3.1}$$

Proof. Choose the support of γ_{l_i} outside of x. Then we have $\gamma_{l_i} \in H^*(X_1)$ and no γ_{l_i}'s in $H^*(X_2)$. In fact, let $p_1\colon \mathcal{X} \to X$ be the composition of the blowing-down map $\mathcal{X} \to X \times \mathbb{C}$ with the projection $X \times \mathbb{C} \to X$. One can check that $i_t^* p_1^* \gamma_{l_i} = \gamma_{l_i}$ and $j_1^* i_0^* p_1^* \gamma_{l_i} = p^*\gamma_{l_i}$ and $j_2^* i_0^* p_1^* \gamma_{l_i} = 0$. We apply the degeneration formula (2.3) to the cohomology classes $p_1^*\gamma_{\ell_i}$ on $\mathcal{X}$.

By the degeneration formula (2.3), we may express the absolute Donaldson-Thomas invariants of X in terms of the relative Donaldson-Thomas invariants of (X_1, E) and (X_2, E) as follows:

$$\begin{aligned} Z'_{DT}(X;q \mid \prod_{i=1}^{r} \tilde{\tau}_0(\gamma_{l_i}))_\beta &= \sum_{\eta,\beta_1+\beta_2=\beta} Z'_{DT} \\ &\left(X_1/E;q \mid \prod_{i=1}^{r} \tilde{\tau}_0(p^*\gamma_{l_i})\right)_{\beta_1,\eta} \frac{(-1)^{|\eta|-\ell(\eta)}\mathfrak{z}(\eta)}{q^{|\eta|}} Z'_{DT}(X_2/E;q)_{\beta_2,\eta^\vee}. \end{aligned} \tag{3.2}$$

Now we need to compute the summands in the right-hand side of the degeneration formula. For this we have the following claim:

Claim: There are only terms with $\beta_2 = 0$.

In fact, if $|\eta| \neq 0$, then $\beta_2 \neq 0$ because $\beta_2 \cdot E = |\eta|$. By Lemma 2.1, we have

$$c_1(X_1)\cdot\beta_1 = \mathrm{vdim} I_n(X_1/E,\beta_1) = \sum_{i=1}^{r} \deg ch_2(\gamma_{l_i}) + \deg \epsilon_1^*(C_\eta),$$

where $\epsilon_1 : I_n(X_1/E,\beta_1) \longrightarrow \mathrm{Hilb}(E,|\eta|)$ is the canonical intersection map, and

$$\begin{aligned} c_1(X_2)\cdot\beta_2 &= \mathrm{vdim} I_n(X_2/E,\beta_2) = 4E\cdot\beta_2 = 4|\eta|, \\ c_1(X)\cdot\beta &= \mathrm{vdim} I_n(X,\beta) = \sum_{i=1}^{r} \deg ch_2(\gamma_{l_i}). \end{aligned}$$

We have the last equality above because, otherwise, the involved Donaldson-Thomas invariants of X and $\tilde{X}$ will vanish and the theorem holds.

By (2.2), we have

$$c_1(X)\cdot\beta = c_1(X_1)\cdot\beta_1 + c_1(X_2)\cdot\beta_2 - 2|\eta|.$$

Combining all the four equations above, we obtain

$$0 = \deg C_\eta + 2|\eta|.$$

This is a contradiction. Therefore $|\eta| = 0$. So the claim is proved.

Thus $\beta_2 \cdot E = 0$. Since E is the hyperplane in $X_2 \cong \mathbb{P}^3$, we must have $\beta_2 = 0$.

It is easy to see that $i_{t*} = i_{0*} \circ j_{1*} \circ \iota$ and $j_{1*} \circ \iota$ is injective. Since $\beta_1 \cdot E = 0$, we have $\beta_1 \in \iota(H_2(X))$. Let's write $\beta_1 = \iota(\alpha)$. Since $i_{t*}(\beta) = i_{0*} \circ j_{1*}(\beta_1)$ by definition, we get $\alpha = \beta$. Therefore $\beta_1 = \iota(\beta)$.

By the degeneration formula, we have

$$(3.3) Z'_{DT}(X; q \mid \prod_{i=1}^{r} \tilde{\tau}_0(\gamma_{l_i}))_\beta = Z'_{DT}(X_1/E; q \mid \prod_{i=1}^{r} \tilde{\tau}_0(p^*\gamma_{l_i}))_{\iota(\beta)}.$$

Now we want to use the degeneration formula one more time to study the Donaldson-Thomas invariants of $\tilde{X}$. We blow up $\tilde{X} \times \mathbb{C}$ along the surface $E \times 0$ to get a 4-fold $\tilde{\mathcal{X}}$. There is a projection $\tilde{\pi}\colon \tilde{\mathcal{X}} \to \mathbb{C}$. The central fiber is a union of $\tilde{X}_1 = \tilde{X}$ and $\tilde{X}_2 = \mathbb{P}(\mathcal{O}_E(-1) \oplus \mathcal{O}_E)$ intersecting transversely along a smooth surface Z, which is the surface E in $\tilde{X}_1$ and the infinite section D_∞ in the projective bundle $\tilde{X}_2$. Note that $\tilde{X}_2 - D_\infty$ is the line bundle $\mathcal{O}_E(-1)$, $\iota(\beta) \cdot E = 0$, and $PD(\gamma_{l_i}) \cap E = \emptyset$. Let $\tilde{p_1}$ be the composition of the map $\tilde{\mathcal{X}} \to \tilde{X} \times \mathbb{C}$ and the map $\tilde{X} \times \mathbb{C} \to \tilde{X}$. Applying the degeneration formula (2.3) to the cohomology classes $\tilde{p_1}^*(\gamma_{l_i})$, we have

$$Z'_{DT}(\tilde{X}; q \mid \prod_{i=1}^{r} \tilde{\tau}_0(\gamma_{l_i}))_{\iota(\beta)} = \sum_{\beta_1+\beta_2=\iota(\beta),\, \eta} Z'_{DT}(\tilde{X}_1/Z; q \mid \prod_{i=1}^{r} \tilde{\tau}_0(p^*\gamma_{l_i}))_{\beta_1,\eta} \frac{(-1)^{|\eta|-\ell(\eta)}\mathfrak{z}(\eta)}{q^{|\eta|}} Z'_{DT}(\tilde{X}_2/Z; q)_{\beta_2,\eta^\vee},$$

where $\beta_1 \cdot Z = |\eta|$.

Here we have the following claim as in the first part of our proof:

Claim: There are only terms with $\beta_2 = 0$ and no η.

It is easy to see that $\tilde{X}_2$ is the blowup $\tilde{\mathbb{P}^3}$ of $\mathbb{P}^3$ at a point p_0. Denote by $\rho : \tilde{\mathbb{P}^3} \longrightarrow \mathbb{P}^3$ the projection of the blowup. Let $\ell \subset \tilde{\mathbb{P}^3}$ be the strict transform of a line in $\mathbb{P}^3$ passing through the blown-up point p_0, and S be the exceptional surface of the blowup ρ. Thus we can write $\tilde{X}_2 = \mathbb{P}(\mathcal{O}_S(-1) \oplus \mathcal{O}_S)$. Denote by e a line in S which is an extremal ray. Since ℓ is a fiber of $\mathbb{P}(\mathcal{O}_S(-1) \oplus \mathcal{O}_S) \longrightarrow S$ which also is an extremal ray, by Mori's theory, we have $\beta_2 = a\ell + be$, $a \geq 0$, $b \geq 0$. Let H be the hyperplane class in $\mathbb{P}^3$. Since $\rho^*H \sim D_\infty$, we have $a = \rho^*H \cdot \beta_2 = |\eta|$. Consider the divisor $E \times \mathbb{C}$ in $\mathcal{X}$. The strict transform of $E \times \mathbb{C}$ in $\tilde{\mathcal{X}}$ is isomorphic to $E \times \mathbb{C}$, still denoted by $E \times \mathbb{C}$. Note that the intersection of $E \times \mathbb{C}$ with the central fiber $\tilde{\mathcal{X}}_0$ is the surface S in $\tilde{X}_2$. By taking the intersection of $E \times \mathbb{C}$ with the formula (2.1) with β being replaced by

$\iota(\beta)$, we get $S \cdot \beta_2 = E \cdot \iota(\beta) = 0$. Since $S \cdot e = -1$ and $S \cdot \ell = 1$, we have $0 = S \cdot (a\ell + be) = a - b$. Thus $a = b$.

Similar to arguments in the proof of the previous claim, we have

$$c_1(\tilde{X}_1) \cdot \beta_1 = \mathrm{vdim} I_n(\tilde{X}_1/Z, \beta_1) = \sum_{i=1}^{r} \deg ch_2(\gamma_{l_i}) + \deg \epsilon_2^*(C_\eta),$$

where $\epsilon_2 : I_n(\tilde{X}_1/Z, \beta_1) \longrightarrow \mathrm{Hilb}(Z, |\eta|)$ is the canonical intersection map, and

$$\begin{aligned} c_1(\tilde{X}_2) \cdot \beta_2 &= (4\rho^* H - 2S) \cdot \beta_2 = 4D_\infty \cdot \beta_2 = 4|\eta|, \\ c_1(\tilde{X}) \cdot \iota(\beta) &= \mathrm{vdim} I_n(\tilde{X}, \iota(\beta)) = \sum_{i=1}^{r} \deg ch_2(\gamma_{l_i}). \end{aligned}$$

By (2.2), we have

$$c_1(\tilde{X}) \cdot \iota(\beta) = c_1(\tilde{X}_1) \cdot \beta_1 + c_1(\tilde{X}_2) \cdot \beta_2 - 2|\eta|.$$

Combining all the four equations above, we obtain

$$0 = \deg C_\eta + 2|\eta|.$$

Therefore $|\eta| = 0$. Now we have $b = a = |\eta| = 0$. Thus $\beta_2 = 0$.

We can see similarly as above that $\beta_1 = \iota(\beta)$.

By the degeneration formula, we have

$$\begin{aligned} Z'_{DT}(\tilde{X}; q \quad &| \quad \prod_{i=1}^{r} \tilde{\tau}_0(p^* \gamma_{l_i}))_{\iota(\beta)} \qquad (3.4) \\ &= Z'_{DT}(\tilde{X}_1/Z; q \mid \prod_{i=1}^{r} \tilde{\tau}_0(p^* \gamma_{l_i}))_{\iota(\beta)} \cdot Z'_{DT}(\tilde{X}_2/Z; q)_0 \\ &= Z'_{DT}(\tilde{X}/E; q \mid \prod_{i=1}^{r} \tilde{\tau}_0(p^* \gamma_{l_i}))_{\iota(\beta)}. \qquad (3.5) \end{aligned}$$

Note that $\tilde{X}_1 \cong \tilde{X}$. Comparing (3.3) with (3.4), we proved the theorem. q.e.d.

4. Blowup of $(-1,-1)$-curves and a flop formula

In this section, we will study how Donaldson-Thomas invariants change under some flops. The materials related to the birational geometry of 3-folds can be found in [**Kollar**], [**Kawamata**], [**KMM**], [**K-M**], [**Matsuki**].

Let X be a smooth projective 3-fold, and D be an effective divisor on X. Suppose that X admits a contraction of an extremal ray with respect to $K_X + \epsilon D$, where $0 < \epsilon \ll 1$,

$$\varphi \colon X \longrightarrow Y.$$

Assume furthermore that the exceptional locus $Exc(\varphi)$ of φ consists of finitely many disjoint smooth rational $(-1,-1)$-curves $\Gamma_2, \dots, \Gamma_\ell$. Y is a normal projective variety, $-D$ is φ-ample, and all curves Γ_i are numerically equivalent. Let's use $[\Gamma]$ to denote the numerically equivalent classes Γ_i, $i = 2, \dots, \ell$. There exists a smooth projective 3-fold X^f and a morphism

$$\varphi^f \colon X^f \longrightarrow Y,$$

which is the flop of φ. X^f can be obtained as follows in our situation. We blow up X along all the curves Γ_i, $i = 2, \dots, \ell$ to get a smooth projective 3-fold $\widetilde{X}$ with the exceptional divisors $E_i \cong \Gamma_i \times \mathbb{P}^1$, $i = 2, \dots, \ell$. Let $\mu \colon \widetilde{X} \to X$ be the blowup map. We can blow down $\widetilde{X}$ along all the Γ_i-direction. The new 3-fold X^f is smooth, projective, and contains $(-1,-1)$-curves Γ_i^f for $i = 2, \dots, \ell$. Γ_i^f is the image of E_i under the blow down. X and X^f are birational and isomorphic in codimension one.

For any divisor B on X, let B^f be the strict transform of B in X^f. We have an isomorphism $N^1(X) \cong N^1(X^f)$ and

$$N^1(X) \cong \varphi^* N^1(Y) \oplus \mathbb{R}[D], \quad N^1(X^f) \cong (\varphi^f)^* N^1(Y) \oplus \mathbb{R}[D^f].$$

Similarly, we get an isomorphism $H_2(X) \to H_2(X^f)$, denoted by ϕ_*, such that $\phi_*([\Gamma_i]) = -[\Gamma_i^f]$ (see [**L-R**]). The map ϕ_* induces isomorphisms $\phi^* \colon H^{2i}(X^f) \to H^{2i}(X)$.

The map ϕ_* can also be seen as follows (see [**L-R**]). There is an injection ι from $H_2(X)$ to $H_2(\tilde{X})$ such that the image of ι is the set $\{\beta \in H_2(\tilde{X}) \,|\, \beta \cdot E = 0\}$ where E is the exceptional divisor of the blowup. Similarly, there is an injection ι^f from $H_2(X^f)$ to $H_2(\tilde{X})$ with the same image. In fact, $(\iota^f)^{-1} \circ \iota$ induces the isomorphism ϕ_*.

Let $\mathcal{X}$ be the blowup of $X \times \mathbb{C}$ along all the curves $\Gamma_i \times 0$. Let $\pi \colon \mathcal{X} \to \mathbb{C}$ be the natural projection. Thus we get a semi-stable degeneration of X whose central fiber is a union of $X_1 \cong \tilde{X}$ and $X_i = \mathbb{P}(\mathcal{O}_{\Gamma_i}(-1) \oplus \mathcal{O}_{\Gamma_i}(-1) \oplus \mathcal{O}_{\Gamma_i})$ for $i = 2, \dots, \ell$ with X_1 and X_i intersecting transversely along the smooth surface E_i.

Here is a technical lemma.

Lemma 4.1. *The power series $\sum\limits_{d>0} d^k x^d$ has an analytic continuation $f_k(x)$ in the domain $\mathbb{C} - \{1\}$ such that*

$$f_k(x^{-1}) = (-1)^{k+1} f_k(x).$$

Proof. From the geometric series formula $1 + x + \cdots + x^d + \cdots = (1-x)^{-1}$, we get

$$x + 2x^2 + \cdots + dx^d + \cdots = x \cdot (1 + x + \cdots + x^d + \cdots)' = \frac{x}{(1-x)^2}.$$

Let $f_1(x) = \dfrac{x}{(1-x)^2}$. One can check that $f_1(x^{-1}) = f_1(x)$.

Assume that the statement in the lemma holds for k. Then

$$x + 2^{k+1}x^2 + \cdots + d^{k+1}x^d + \cdots = x \cdot (x + \cdots + d^k x^d + \cdots)'$$

has an analytic continuation $f_{k+1}(x) = f_k'(x) \cdot x$. From the chain rule, one has $f_k'(x^{-1})(-x^{-2}) = (-1)^{k+1} f_k'(x)$. Therefore

$$f_{k+1}(x^{-1}) = x^{-1} f_k'(x^{-1}) = (-1)^{k+2} x f_k'(x) = (-1)^{k+2} f_{k+1}(x).$$

By mathematical induction, we proved the lemma. q.e.d.

From the proof, one can see that $f_k(x) = f_k(x^{-1})$ when k is odd.

Define a series $g(q, v, \Gamma)$ by

$$g(q, v, \Gamma) = exp\{u^{-2} \sum_{d>0} \frac{1}{d^3} v^{d\Gamma}\} \cdot \frac{1}{(1 - v^{\Gamma})^{1/12}}, \tag{4.1}$$

where $q = -e^{iu}$.

Theorem 4.2. *Suppose cohomology classes $\gamma_{l_i} \in H^{2k}(X^f)$, $i = 1, \ldots, r$, and $k = 1, 2, 3$, have supports away from all the exceptional curve Γ_i.*

(i) *If $\beta = m[\Gamma]$, we have*

$$Z'_{DT}(X; q)_\beta = Z'_{DT}(X^f; q)_{-\phi_*(\beta)}.$$

(ii) *There exist power series*

$$\begin{aligned} \Phi_X(q, v|\{\phi^* \gamma_{\ell_i}\}) &= \sum_{\beta \in \iota(H_2(X))} \Phi_X(q|\{\phi^* \gamma_{\ell_i}\})_\beta \cdot v^\beta, \\ \Phi_{X^f}(q, v|\{\gamma_{\ell_i}\}) &= \sum_{\beta \in \iota^f(H_2(X^f))} \Phi_{X^f}(q|\{\gamma_{\ell_i}\})_\beta \cdot v^\beta, \end{aligned}$$

and $G(q, v, \Gamma)$ such that

$$\Phi_X(q, v|\{\phi^* \gamma_{\ell_i}\}) = \Phi_{X^f}(q, v|\{\gamma_{\ell_i}\}),$$

$G(q, v, \Gamma_i)/g(q, v, \Gamma_i)$ and $G(q, v^{-1}, \Gamma_i^f)/g(q, v^{-1}, \Gamma_i^f)$ are equivalent under analytic continuation, and

$$Z'_{DT}(X; q, v \mid \prod_{i=1}^{r} \tilde{\tau}_0(\phi^* \gamma_{l_i})) = \Phi_X(q, v|\{\phi^* \gamma_{\ell_i}\}) \cdot \prod_{i=2}^{\ell} G(q, v, \Gamma_i), \tag{4.2}$$

$$Z'_{DT}(X^f; q, v \mid \prod_{i=1}^{r} \tilde{\tau}_0(\gamma_{l_i})) = \Phi_{X^f}(q, v|\{\gamma_{\ell_i}\}) \cdot \prod_{i=2}^{\ell} G(q, v, \Gamma_i^f). \tag{4.3}$$

Proof. There is a degeneration formula similar to (2.3) (see [**L-W**, **MNOP2**]) for the degeneration $\mathcal{X}$ described above. For simplicity, we shall prove the case when there is only one Γ_i, denoted by Γ. The proof for the general case is similar.

By the degeneration formula (2.3), we have

$$Z'_{DT}\big(X; q \mid \prod_{i=1}^{r} \tilde{\tau}_0(\phi^*\gamma_{l_i})\big)_\beta = \sum_{\eta, \beta_1+\beta_2=\beta} Z'_{DT}\big(X_1/E; q \mid \prod_{i=1}^{r} \tilde{\tau}_0(\mu^*\phi^*\gamma_{l_i})\big)_{\beta_1,\eta} \frac{(-1)^{|\eta|-\ell(\eta)}\mathfrak{z}}{q^{|\eta|}} Z'_{DT}(X_2/E; q)_{\beta_2,\eta^\vee},$$

where E is the intersection of X_1 with X_2, which is also the exceptional divisor in X_1.

Similar to the proof of Theorem 3.1, we need to study the summands in the right-hand side. Therefore, we also need to compute the virtual dimensions of involved moduli spaces. About the contributions of each term in the right-hand side, we have the following claim:

Claim: There are only terms without η.

In fact, suppose that $|\eta| \neq 0$. First of all, we want to compute the first Chern class of X_2.

Let $V = \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma$ and $p : \mathbb{P}(V) \longrightarrow \Gamma$ be the projection. $X_2 = \mathbb{P}(V)$. For this projective bundle, we have the Euler exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(V)} \longrightarrow p^*V \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \longrightarrow T_{\mathbb{P}(V)/\Gamma} \longrightarrow 0.$$

We also have

$$0 \longrightarrow p^*\Omega^1_\Gamma \longrightarrow \Omega^1_{\mathbb{P}(V)} \longrightarrow \Omega^1_{\mathbb{P}(V)/\Gamma} \longrightarrow 0.$$

Therefore, we have

$$\begin{aligned} c_1(\Omega^1_{\mathbb{P}(V)}) &= p^*c_1(\Omega^1_\Gamma) + c_1(\Omega^1_{\mathbb{P}(V)/\Gamma}) \\ &= p^*c_1(\Omega^1_\Gamma) - c_1(p^*V \otimes \mathcal{O}_{\mathbb{P}(V)}(1)) \\ &= p^*c_1(K_\Gamma) - p^*c_1(V) - 3c_1(\mathcal{O}_{\mathbb{P}(V)}(1)) \\ &= -3c_1(\mathcal{O}_{\mathbb{P}(V)}(1)), \end{aligned}$$

where $c_1(\mathcal{O}_{\mathbb{P}(V)}(1)) = [E]$ is the hyperplane at infinity in $\mathbb{P}(V)$ due to the inclusion $\mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma(-1) \longrightarrow \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma$. Therefore we have

$$c_1(X_2) \cdot \beta_2 = 3|\eta|.$$

By the definition of absolute Donaldson-Thomas invariants, we may assume that

$$c_1(X) \cdot \beta = \mathrm{vdim} I_n(X, \beta) = \sum_{i=1}^{r} \deg ch_2(\gamma_{l_i}).$$

Otherwise, the involved Donaldson-Thomas invariants of X and $\tilde{X}$ will vanish and the theorem holds.

We also have

$$c_1(X_1) \cdot \beta_1 = \text{vdim} I_n(X_1/E, \beta) = \sum_{i=1}^{r} \deg ch_2(\gamma_{\ell_i}) + \deg \epsilon_1^* \eta.$$

By Lemma 2.2, we have

$$c_1(X) \cdot \beta = c_1(X_1) \cdot \beta_1 + c_1(X_2) \cdot \beta_2 - 2|\eta|.$$

Combining all the four equalities above, we have

$$0 = \deg \epsilon_1^* C_\eta + |\eta|.$$

Hence $|\eta| = 0$.

(i) Suppose that $\beta = m[\Gamma]$. Notice that the virtual dimension of the moduli space will be zero since $c_1(X) \cdot \beta = 0$. Let Γ_∞ be the curve coming from the inclusion $\mathcal{O}_\Gamma \to V$, $F \cong \mathbb{P}^2$ be a fiber of p, f be a line in F. Then one can compute easily that

$$E \cdot \Gamma_\infty = 0, \quad F \cdot \Gamma_\infty = 1, \quad f \cdot F = 0, \quad f \cdot E = 1.$$

Therefore we can write $\beta_2 = af + m[\Gamma_\infty]$. Since $E \cdot \beta_2 = 0$, we have $a = 0$. Therefore $\beta_2 = m[\Gamma_\infty]$ for some $m \geq 0$. Under the morphism, σ

$$\sigma \colon \mathcal{X} \longrightarrow \mathcal{X} \times \mathbb{C} \longrightarrow X,$$

we have $\beta = \sigma(\beta_1) + m[\Gamma]$ in $NE(X)$. β_1 can only be a union of curves C_i not lying on E and curves D_j on E. Since $\mathbb{R}[\Gamma]$ is a ray, we must have $C_i = 0$. For effective curves D_j on E, $D_j \cdot E \neq 0$. However, since $\beta_1 \cdot E = 0$, we must have $D_j = 0$. Thus $\beta_1 = 0$. Therefore, by the degeneration formula, we have

$$\begin{aligned} (4.4) \qquad Z'_{DT}(X;q)_{m[\Gamma]} &= Z'_{DT}(\tilde{X}/E;q)_0 \cdot Z'_{DT}(X_2/E;q)_{m[\Gamma_\infty]} \\ &= Z'_{DT}(X_2/E;q)_{m[\Gamma_\infty]} \end{aligned}$$

$$\begin{aligned} Z'_{DT}(X^f;q)_{m[\Gamma^f]} &= Z'_{DT}(\tilde{X^f}/E;q)_0 \cdot Z'_{DT}(X_2^f/E;q)_{m[\Gamma_\infty^f]} \\ &= Z'_{DT}(X_2^f/E;q)_{m[\Gamma_\infty^f]}. \end{aligned}$$

Observe that $(\tilde{X}_2, E)$ and $(\tilde{X}_2^f, E)$ are isomorphic. Therefore, we have

$$Z'_{DT}(X;q)_{m[\Gamma]} = Z'_{DT}(X^f;q)_{m[\Gamma_f]}.$$

To write in another way for $\beta = m[\Gamma]$, we have

$$Z'_{DT}(X;q)_\beta = Z'_{DT}(X^f;q)_{-\phi_*(\beta)}.$$

To prove (ii), by a similar argument to that in (i), we have $\beta = \beta_1 + m[\Gamma_\infty]$ with $m \geq 0$ and $\beta_1 \cdot E = 0$.

Furthermore, by the degeneration formula, we have

$$Z'_{DT}\big(X;q \quad | \quad \prod_{i=1}^{r}\tilde{\tau}_0(\phi^*\gamma_{l_i})\big)_\beta = \sum_{\substack{\beta=\beta_1+m[\Gamma_\infty],\\ \beta_1\in\iota(H_2(X))}} Z'_{DT}(\tilde{X}/E;q$$

$$\text{(4.5)} \qquad | \quad \prod_{i=1}^{r}\tilde{\tau}_0(\mu^*\phi^*\gamma_{l_i}))_{\beta_1}\cdot Z'_{DT}(X_2/E;q)_{m[\Gamma_\infty]}.$$

Consider the map $c_*\colon H_2(X) = H_2(\mathcal{X}_t)\xrightarrow{i_{t*}}H_2(\mathcal{X})\xrightarrow{i_{0*}^{-1}}H_2(\mathcal{X}_0)$. From Lemma 2.11 in [**L-R**], c_* is injective. Therefore we have

$$\begin{aligned}
\text{(4.6)} \qquad & Z'_{DT}\left(X;q,v\mid\prod_{i=1}^{r}\tilde{\tau}_0(\phi^*\gamma_{l_i})\right)\\
&= \sum_{\beta\in H_2(X)} Z'_{DT}\left(X;q\mid\prod_{i=1}^{r}\tilde{\tau}_0(\phi^*\gamma_{l_i})\right)_\beta v^\beta\\
&= \sum_{\beta\in H_2(X)}\sum_{\substack{\beta=\beta_1+m[\Gamma_\infty],\\ \beta_1\in\iota(H_2(X))}} Z'_{DT}(\tilde{X}/E;q\mid\prod_{i=1}^{r}\tilde{\tau}_0(\mu^*\phi^*\gamma_{l_i}))_{\beta_1}v^{\beta_1}\\
&\quad\cdot Z'_{DT}(X_2/E;q)_{m[\Gamma_\infty]}v^{m[\Gamma_\infty]}\\
&= \left(\sum_{\beta_1\in\iota(H_2(X))} Z'_{DT}(\tilde{X}/E;q\mid\prod_{i=1}^{r}\tilde{\tau}_0(\mu^*\phi^*\gamma_{l_i}))_{\beta_1}v^{\beta_1}\right)\\
&\quad\cdot\left(\sum_{m\geq 0} Z'_{DT}(X_2/E;q)_{m[\Gamma_\infty]}v^{m[\Gamma_\infty]}\right).
\end{aligned}$$

Define a function $\Phi_X(q,v|\{\phi^*\gamma_{\ell_i}\})$ as follows:

$$\Phi_X(q,v|\{\phi^*\gamma_{\ell_i}\}) = \sum_{\beta_1\in\iota(H_2(X))} Z'_{DT}(\tilde{X}/E;q\mid\prod_{i=1}^{r}\tilde{\tau}_0(\mu^*\phi^*\gamma_{l_i}))_{\beta_1}v^{\beta_1}.$$

Applying the formula (4.4) to $X = X_2$, we get $Z'_{DT}(X_2/E;q)_{m[\Gamma_\infty]} = Z'_{DT}(X_2;q)_{m[\Gamma_\infty]}$. We define a function $G(q,v,\Gamma_\infty)$ as follows:

$$\begin{aligned}
G(q,v,\Gamma_\infty) &= \sum_{m\geq 0} Z'_{DT}(X_2/E;q)_{[m\Gamma_\infty]}v^{[m\Gamma_\infty]}\\
&= \sum_{m\geq 0} Z'_{DT}(X_2;q)_{[m\Gamma_\infty]}v^{[m\Gamma_\infty]}\\
\text{(4.7)} \qquad &= Z'_{GW}(\mathcal{O}_{\Gamma_\infty}(-1)\oplus\mathcal{O}_{\Gamma_\infty}(-1);u,v).
\end{aligned}$$

The last equality is from theorem 3 in [**MNOP1**] for local Calabi-Yau $\mathcal{O}_{\Gamma_\infty}(-1)\oplus\mathcal{O}_{\Gamma_\infty}(-1)$.

From [**MNOP1**], we have

$$Z'_{GW}(\mathcal{O}_{\Gamma_\infty}(-1)\oplus\mathcal{O}_{\Gamma_\infty}(-1);u,v)$$
$$= exp\{F'_{GW}(\mathcal{O}_{\Gamma_\infty}(-1)\oplus\mathcal{O}_{\Gamma_\infty}(-1);u,v)\},$$

$$F'_{GW}=\sum_{d>0}\sum_{g\geq 0}N_{g,d}u^{2g-2}v^{d[\Gamma_\infty]},$$

where $N_{g,d}$ is computed in [**F-P**]:

$$N_{0,d}=\frac{1}{d^3},\quad N_{1,d}=\frac{1}{12d},\quad N_{g,d}=\frac{|B_{2g}|d^{2g-3}}{2g\cdot(2g-2)!}\quad \text{for } g\geq 2.$$

Therefore, we have

$$\begin{aligned} F'_{GW} &= u^{-2}\sum_{d>0}\frac{1}{d^3}(v^{[\Gamma_\infty]})^d+\sum_{d>0}\frac{1}{12d}(v^{[\Gamma_\infty]})^d \\ &+ \sum_{g\geq 2}\frac{|B_{2g}|}{2g\cdot(2g-2)!}u^{2g-2}\sum_{d>0}d^{2g-3}(v^{[\Gamma_\infty]})^d. \end{aligned}$$

Now $G(q,v,\Gamma_\infty)/g(u,v,\Gamma_\infty)$ has the analytic continuation

$$exp\{\sum_{g\geq 2}\frac{|B_{2g}|}{2g\cdot(2g-2)!}u^{2g-2}f_{2g-3}(v^{[\Gamma_\infty]})\}$$

where $f_{2g-3}(x)$ is defined as in Lemma 4.1.

Applying the same argument to X^f, we also have

$$Z'_{DT}\big(X^f;q\mid\prod_{i=1}^r\tilde{\tau}_0(\gamma_{l_i})\big)_\beta=\sum_{\substack{\beta=\beta_1+m[\Gamma^f_\infty],\\ \beta_1\in\iota^f(H_2(X^f))}}Z'_{DT}(\tilde{X}/E;q$$

$$\mid\prod_{i=1}^r\tilde{\tau}_0(\nu^*\gamma_{l_i}))_{\beta_1}\cdot Z'_{DT}(X^f_2/E;q)_{m[\Gamma^f_\infty]}, \tag{4.8}$$

where $\nu\colon\widetilde{X}\to X^f$ is the blowup map, $\tilde{X}\cong\tilde{X^f}$.

Applying the same argument above for X to X^f, define a function $\Phi_{X^f}(q,v|\{\gamma_{\ell_i}\})$ as follows:

$$\Phi_X(q,v|\{\gamma_{\ell_i}\})=\sum_{\beta_1\in\iota(H_2(X))}Z'_{DT}(\tilde{X}/E;q\mid\prod_{i=1}^r\tilde{\tau}_0(\nu^*\gamma_{l_i}))_{\beta_1}v^{\beta_1}.$$

We have (4.3).

The function $G(q,v,\Gamma^f_\infty)/g(q,v,\Gamma^f_\infty)$ has the analytic continuation

$$exp\{\sum_{g\geq 2}\frac{|B_{2g}|}{2g\cdot(2g-2)!}u^{2g-2}f_{2g-3}(v^{[\Gamma^f_\infty]})\}.$$

From Lemma 4.1 and the fact that $\mu^*\phi^*=\nu^*$, we proved (ii). q.e.d.

One should compare Theorem 4.2 with definition 1.1, theorem A and corollary A.2 in [**L-R**]. There, Li and Ruan studied the question of naturality of quantum cohomology under birational operations such as flops. They observed that one must use analytic continuation to compare the quantum cohomology of two Calabi-Yau 3-folds which are flop equivalent. A similar phenomenon occurs for Donaldson-Thomas invariants. However, there is a slight complexity due to the function $g(q, v, \Gamma)$ coming from genus zero and genus one contributions. It is possible that genus zero and genus one create an anomaly.

5. Extremal transition

Let X_0 be a singular projective 3-fold whose singularities are ordinary double points $p_2, \dots, p_m$. Let X be a small resolution of X_0:

$$\rho\colon X \longrightarrow X_0$$

such that $C_i = \rho^{-1}(p_i)$ is a smooth rational curve with the normal bundle $\mathcal{O}_{C_i}(-1) \oplus \mathcal{O}_{C_i}(-1)$. Assume in addition that X is a Calabi-Yau 3-fold. It is known from [**Friedman**] and [**Tian**] that if $\sum_i \lambda_i [C_i] = 0$ in $H^2(X, \Omega_X^2)$ where $\lambda_i \neq 0$ for all i, there exists a global smoothing of X_0:

$$\pi\colon \mathcal{X} \longrightarrow \Delta$$

where Δ is a unit disc, $\pi^{-1}(0) = X_0$, and $\pi^{-1}(t)$ is nonsingular for $t \neq 0$. Assume furthermore that the morphism π is projective. By the semi-stable reduction theorem, there exists a semi-stable degeneration

$$\Phi\colon \mathcal{Y} \longrightarrow \Delta$$

via some base change and modifications on π and $\mathcal{X}$. $\mathcal{Y}_0 = \Phi^{-1}(0)$ is a union of $\widetilde{X}$ and $Q_2, \dots, Q_m$ where

$$\tau\colon \widetilde{X} \longrightarrow X$$

is the blowup of X along $C_2, \dots, C_m$ and Q_i is a smooth quadric in $\mathbb{P}^4$. Let $E_2, \dots, E_m$ be the exceptional divisors of the blowup $\widetilde{X} \longrightarrow X$ corresponding to $C_2, \dots, C_m$ respectively. Then $E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ is a hyperplane section of Q_i in $\mathbb{P}^4$. $\widetilde{X}$ and Q_i intersect transversally along E_i. Let $E = E_2 \cup \cdots \cup E_m$. $\mathcal{Y}_t = \Phi^{-1}(t)$ is a smooth Calabi-Yau 3-fold.

There exists a map (see [**L-R**, **L-Y**, **C-S**])

$$\varphi_e\colon H_2(X) \longrightarrow H_2(\mathcal{Y}_t).$$

The kernel of the map φ_e is generated by $C_2, \dots, C_m$.

Recall the formula (4.6):

$$\begin{aligned} Z'_{DT}(X;q,v) = \quad & \Big(\sum_{\beta_1\in\iota(H_2(X))} Z'_{DT}(\widetilde{X}/E;q)_{\beta_1}v^{\beta_1}\Big) \\ & \cdot\prod_{i=2}^{m}\Big(\sum_{k\geq 0} Z'_{DT}(X_i/E_i;q)_{k[C_{i\infty}]}v^{k[C_{i\infty}]}\Big). \end{aligned}$$

We used $G(q,v,C_{i\infty})$ in the formula (4.7) to denote $\sum_{k\geq 0} Z'_{DT}(X_i/E_i;q)_{k[C_{i\infty}]}v^{k[C_{i\infty}]}$. These functions are known explicitly as explained in the proof of Theorem 4.2.

Let's use $\Phi_X(q,v)$ to denote

$$\Phi_X(q,v) = \sum_{\beta_1\in\iota(H_2(X))} Z'_{DT}(\widetilde{X}/E;q)_{\beta_1}v^{\beta_1}. \tag{5.1}$$

Theorem 5.1. *With notations and assumptions as above, if we write*

$$\Phi_X(q,v) = \sum_{\alpha\in H_2(X)} \Phi_X(q)_\alpha v^{\iota(\alpha)},$$

then

$$\begin{aligned} Z'_{DT}(X;q,v) &= \Big(\sum_{\alpha\in H_2(X)} \Phi_X(q)_\alpha v^{\iota(\alpha)}\Big)\cdot\prod_{i=2}^{m} G(q,v,C_{i\infty}), \\ Z'_{DT}(\mathcal{Y}_t;q,v) &= \sum_{\alpha\in H_2(X)} \Phi_X(q)_\alpha v^{\varphi_e(\alpha)}. \end{aligned}$$

Proof. Applying the degeneration formula to the degeneration $\Phi\colon \mathcal{Y}\longrightarrow\Delta$, we get

$$\begin{aligned} Z'_{DT}(\mathcal{Y}_t;q)_\beta = & \sum_{\substack{\beta=\beta_1+\beta_2+\cdots+\beta_m,\\ \eta_2,\cdots,\eta_m}} Z'_{DT}(\widetilde{X}/E;q)_{\beta_1,\eta_2,\ldots,\eta_m} \\ & Z'_{DT}(Q_2/E_2;q)_{\beta_2,\eta_2^\vee}\ldots Z'_{DT}(Q_m/E_m;q)_{\beta_m,\eta_m^\vee}, \end{aligned}$$

where $|\eta_i| = \beta_i\cdot E_i$.

Let's explain the notations above. The expression $\beta=\beta_1+\beta_2+\cdots+\beta_m$ is defined in a way similar to (2.1). We have maps

$$\epsilon_i\colon I_n(\widetilde{X}/E,\beta_1)\longrightarrow \mathrm{Hilb}(E_i,\beta_1\cdot[E_i]),$$

$$Z'_{DT}(\widetilde{X}/E;q)_{\beta_1,\eta_2,\ldots,\eta_m} = \int_{[I_n(\widetilde{X}/E,\beta_1)]^{vir}} \epsilon_2^*(C_{\eta_2})\cup\cdots\cup\epsilon_m^*(C_{\eta_m}).$$

By the adjunction formula, we have

$$K_{Q_i} = (K_{\mathbb{P}^4}+Q_i)|_{Q_i} = \mathcal{O}_{Q_i}(-3), \quad \text{and } c_1(Q_i) = 3E_i.$$

By the dimension formula for moduli spaces, we have

$$\begin{aligned} c_1(\widetilde{X})\cdot\beta_1 &= \mathrm{vdim} I_n(\widetilde{X}/E,\beta_1) = \sum_{i=2}^{m} \deg \epsilon_i^*(C_{\eta_i}), \\ c_1(Q_i)\cdot\beta_i &= \mathrm{vdim} I_n(Q_i/E_i,\beta_i) = 3E_i\cdot\beta_i = 3|\eta_i|, \\ c_1(\mathcal{Y}_t)\cdot\beta &= \mathrm{vdim} I_n(\mathcal{Y}_t,\beta) = 0. \end{aligned}$$

Similar to Lemma 2.2, we have

$$\begin{aligned} c_1(\mathcal{Y}_t)\cdot\beta_1 = c_1(\widetilde{X})\cdot\beta_1 + c_1(Q_2) \\ \cdot\beta_2 + \cdots + c_1(Q_m)\cdot\beta_m - 2|\eta_2| - \cdots - 2|\eta_m|. \end{aligned}$$

Combining the formulae above, we get

$$\begin{aligned} 0 &= \sum_{i=2}^{m} \deg \epsilon_i^*(C_{\eta_i}) + 3|\eta_2| + \cdots + 3|\eta_m| - 2|\eta_2| - \cdots - 2|\eta_m|, \\ 0 &= \sum_{i=2}^{m} \deg \epsilon_i^*(C_{\eta_i}) + |\eta_2| + \cdots + |\eta_m|. \end{aligned}$$

Therefore we get $|\eta_2| = \cdots = |\eta_m| = 0$.

Since $E_i = c_1(\mathcal{O}_{Q_i}(1))$ and $\beta_i \cdot E_i = |\eta_i| = 0$, we must have $\beta_i = 0$ for $2 \le i \le m$.

Thus the degeneration formula becomes

$$Z'_{DT}(\mathcal{Y}_t;q)_\beta = \sum_{\beta_1} Z'_{DT}(\widetilde{X}/E;q)_{\beta_1}, \tag{5.2}$$

where β_1 satisfies the following conditions:

$$\beta_1\cdot E_i = 0, \quad \text{for } 2 \le i \le m, \tag{5.3}$$

$$c_*(\beta) = j_{1*}(\beta_1)$$

where c is the Clemens map inducing $c_*\colon H_2(\mathcal{Y}_t) \longrightarrow H_2(\mathcal{Y}_0)$ and

$$j_1\colon \widetilde{X} \longrightarrow \mathcal{Y}_0 = \widetilde{X} \cup Q_2 \cup \cdots \cup Q_m$$

is the natural inclusion.

The condition (5.3) implies β_1 lies in the image of $\iota\colon H_2(X) \longrightarrow H_2(\widetilde{X})$ defined in §4, i.e.,

$$\beta_1 = \iota(\alpha) \quad \text{for a unique } \alpha \in H_2(X).$$

One can check that

$$j_{1*}\circ\iota = c_*\circ\varphi_e,$$

i.e., the following diagram is commutative:

$$\begin{array}{ccc} H_2(X) & \xrightarrow{\varphi_e} & H_2(\mathcal{Y}_t) \\ \downarrow^{\iota} & & \downarrow^{c_*} \\ H_2(\widetilde{X}) & \xrightarrow{j_{1*}} & H_2(\mathcal{Y}_0) \end{array}$$

Since φ_e is surjective and c_* is injective, we have

$$c_*(\beta) = j_{1*}\beta_1 = j_{1*} \circ \iota(\alpha) \quad \text{iff } \beta = \varphi_e(\alpha).$$

Now we can rewrite the degeneration formula (5.2) as

$$Z'_{DT}(\mathcal{Y}_t; q)_\beta = \sum_{\substack{\alpha \in H_2(X), \\ \varphi_e(\alpha)=\beta}} Z'_{DT}(\widetilde{X}/E; q)_{\iota(\alpha)} \quad \text{for } \beta \neq 0. \tag{5.4}$$

We write

$$\begin{aligned} Z'_{DT}(\mathcal{Y}_t; q, v) &= 1 + \sum_{0 \neq \beta \in H_2(\mathcal{Y}_t)} Z'_{DT}(\mathcal{Y}_t; q) v^\beta \\ &= 1 + \sum_{0 \neq \beta \in H_2(\mathcal{Y}_t)} \Big(\sum_{\substack{\alpha \in H_2(X), \\ \varphi_e(\alpha)=\beta}} Z'_{DT}(\widetilde{X}/E; q)_{\iota(\alpha)} \Big) v^\beta \\ &= 1 + \sum_{\substack{\alpha \in H_2(X), \\ \varphi_e(\alpha)\neq 0}} Z'_{DT}(\widetilde{X}/E; q)_{\iota(\alpha)} v^{\varphi_e(\alpha)} \\ &= 1 + \sum_{\alpha \in H_2(X)} Z'_{DT}(\widetilde{X}/E; q)_{\iota(\alpha)} v^{\varphi_e(\alpha)}. \qquad (5.5) \end{aligned}$$

Note that if $\varphi_e(\alpha) = 0$, we have $Z'_{DT}(\widetilde{X}/E; q)_{\iota(\alpha)} = 0$. In fact, $\varphi_e(\alpha) = 0$ implies that α lies on the extremal face generated by $C_2, \dots, C_m$. Therefore $\iota(\alpha)$ cannot represent a curve in $\widetilde{X}$ as shown in the proof of (i) of Theorem 4.2.

If we replace v^{β_1} in $\Phi_X(q, v)$ by $v^{\varphi_e \circ \tau(\beta_1)} = v^{\varphi_e(\alpha)}$ where $\beta_1 = \iota(\alpha)$, from (5.1) and (5.5), we see that $\Phi_X(q, v)$ equals $Z'_{DT}(\mathcal{Y}_t; q, v)$.

This establishes the relation between the Donaldson-Thomas invariants of X and those of the extremal transition $\mathcal{Y}_t$ of X.

q.e.d.

References

[B-F] K. Behrend & B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997), 45–88, MR 1437495, Zbl 0909.14006.

[Bri] T. Bridgeland, *Stability conditions on triangulated categories*, Ann. of Math. (2) **166**, no. 2 (2007), 317–345, MR 2373143, Zbl 1137.18008.

[Clemens] H. Clemens, *Degeneration of Kähler manifolds*, Duke. Math. J. **44** (1977), 215–290, MR 0444662, Zbl 0353.14005.

[C-S] A. Corti & I. Smith, *Conifold transitions and Mori theory*, Math. Res. Lett. **12**, no. 5-6 (2005), 767–778, MR 2189237, Zbl 1093.14024.

[D-T] S. Donaldson & R. Thomas, *Gauge theory in higher dimensions*, in The Geometric Universe: Science, Geometry, and the Work of Roger Penrose, S. Huggett et. al eds., Oxford Univ. Press, (1998), MR 1634503, Zbl 0926.58003.

[F-P] C. Faber & R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, Invent. Math. **139** (2000), 173–199, MR 1728879, Zbl 0960.14031.

[Friedman] R. Friedman, *Simultaneous resolutions of threefold double points*, Math. Ann. **274** (1986), 671–689, MR 0848512, Zbl 0576.14013.

[Fulton] W. Fulton, *Intersection theory*, Ergebnisse der Math. und ihrer Grenzgebiete 3, Folge Band 2, (1984), MR 1644323, Zbl 0885.14002.

[G-P] T. Graber & R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), 487–518, MR 1666787, Zbl 0953.14035.

[G-V] T. Graber & R. Vakil, *Relative virtual localization and vanishing of tautological classes on moduli spaces of curves*, Duke Math. J. **130** no. 1 (2005), 1–37, MR 2176546, Zbl 1088.14007.

[Hu1] J. Hu, *Gromov-Witten invariants of blowups along points and curves*, Math. Z. **233** (2000), 709–739, MR 1759269, Zbl 0948.53046.

[Hu2] J. Hu, *Gromov-Witten invariants of blow-ups along surfaces*, Compositio Math. **125** (2001), 345–352, MR 1818985, Zbl 1023.14029.

[I-P1] E. Ionel & T. Parker, *Relative Gromov-Witten invariants*, Ann. of Math. **157** (2003), 45–96, MR 1954264, Zbl 1039.53101.

[I-P2] E. Ionel & T. Parker, *The symplectic sum formula for Gromov-Witten invariants*, Ann. of Math. **159** (2004), 935–1025, MR 1954264, Zbl 1039.53101.

[J-S] D. Joyce & Y. Song, *A theory of generalized Donaldson-Thomas invariants*, arXiv:0810.5645, to appear in Memoirs of the A.M.S.

[Kawamata] Y. Kawamata, *Crepant blowing ups of three dimensional canonical singularities and applications to degenerations of surfaces*, Ann. Math. **127** no. 1 (1988), 93–163, MR 0924674, Zbl 0651.14005.

[KMM] Y. Kawamata, K. Masuda & K. Matsuki, *Introduction to the minimal model problem*, Adv. Stud. Pure Math. **10**, Alg. Geom., Sendai, T. Oda ed. (1985), 283–360, MR 0946243, Zbl 0672.14006.

[Kollar] J. Kollár, *Flops*, Nagoya Math. J. **113** (1989), 15–36, MR 0986434, Zbl 0645.14004.

[K-M] J. Kollár & S. Mori, *Birational geometry of algebraic varieties* (with the collaboration of H. Clemens and A. Corti), Cambridge University Press, (1998), MR 1658959, Zbl 1143.14014.

[K-S] M. Kontsevich & Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, arXiv:0811.2435.

[Li1] J. Li, *Stable morphisms to singular schemes and relative stable morphisms*, JDG, **57** (2001), 509–578, MR 1882667, Zbl 1076.14540.

[Li2] J. Li, *A degeneration formula of GW-invariants*, JDG, **60** (2002), 199–293, MR 1938113, Zbl 1063.14069.

[L-R] A. Li & Y. Ruan, *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds I*, Invent. Math. **145** (2001), 151–218, MR 1839289, Zbl 1062.53073.

[L-T] J. Li & G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, JAMS **11** (1998), 119–174, MR 1467172, Zbl 0912.14004.

[L-W] J. Li & B. Wu, *Good degeneration of Quot-schemes and coherent systems,* arXiv:11110.0390.

[L-Y] C. H. Lui & S. T. Yau, *Transformation of algebraic Gromov-Witten invariants of three-folds under flops and small extremal transitions,* *with an appendix from the stringy and the symplectic viewpoint*, math.AG/0505084.

[Matsuki] K. Matsuki, *Introduction to the Mori program*, Universitext, Springer-Verlag, New York, (2002), MR 1875410, Zbl 0988.14007.

[MNOP1] D. Maulik & N. Nekrasov, A. Okounkov, R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory I*, Compositio Math. **142** (2006), 1263–1285, MR 2264664, Zbl 1108.14046.

[MNOP2] D. Maulik, N. Nekrasov, A. Okounkov & R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory II*, Compositio Math., **142** (2006), 1286–1304, MR 2264665, Zbl 1108.14047.

[M-P] D. Maulik & R. Pandharipande, *Foundations of Donaldson-Thomas theory*, in preparation.

[Nakajima] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, AMS, (1999), MR 1711344, Zbl 0949.14001.

[Thomas] R. Thomas, *A holomorphic Casson invariant for Calabi-Yau 3-folds and bundles on K3 fibrations*, JDG **53** (1999), 367–438, MR 1818182, Zbl 1034.14015.

[Tian] G. Tian, *Smoothing 3-folds with trivial canonical bundle and ordinary double points*, Essays on Mirror Manifolds, ed. by S.T. Yau, 459–479, MR 1191437, Zbl 0904.32022.

[Toda] Y. Toda, *Curve counting theories via stable objects II: DT/ncDT flop formula*, arXiv:0909.5129.

[Wilson1] P.M.H. Wilson, *Symplectic deformations of Calabi-Yau threefolds*, JDG, **45** (1997), 611–637, MR 1472891, Zbl 0885.32025.

[Wilson2] P.M.H. Wilson, *Flops, Type III contractions and Gromov-Witten invariants on Calabi-Yau threefolds,* New trends in algebraic geometry (Warwick, 1996), 465–484, London Math. Soc. Lecture Note Ser., 264, Cambridge Univ. Press, Cambridge, (1999), MR 1714834, Zbl 1077.14567.

Department of Mathematics
Sun Yat-sen University
Guangzhou 510275, P. R. China
E-mail address: stsjxhu@mail.sysu.edu.cn

Department of Mathematics
Hkust
Clear Water Bay
Kowloon, Hong Kong
E-mail address: mawpli@ust.hk